MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Electrical Engineering and Computer Science

6.241: Dynamic Systems—Fall 2003

Midterm Exam Solutions

Problem 1 (i) $\|Ax-b\|^2 = 3x^2 - 8x + 6$, a quadratic with imaginary roots (hence positive minimum). Differentiating, $\frac{\mathrm{d}}{\mathrm{d}x}(3x^2-8x+6) = 6x-8$. Set equal to zero and solve for x, $x = 4/3$.

(ii) $\begin{bmatrix} \Delta^* & e^* \end{bmatrix} = \text{argmin}\{\|\begin{bmatrix} \Delta & e \end{bmatrix}\|_F \mid \begin{bmatrix} A & -b \end{bmatrix} + \begin{bmatrix} \Delta & e \end{bmatrix} \text{is singular}\}$; $\|\begin{bmatrix} \Delta^* & e^* \end{bmatrix}\|_F = \sigma_2(\begin{bmatrix} A & -b \end{bmatrix})$.

(iii) $x(t) = e^{At}x_o$, where $e^{At} = I + At + A^2t^2/2 + A^3t^3/6$ since $A^4, A^5, \ldots$ are zero. Now, $At = \begin{bmatrix} 0 & t & 0 & 0 \\ 0 & 0 & t & 0 \\ 0 & 0 & 0 & t \\ 0 & 0 & 0 & 0 \end{bmatrix}$, $A^2t^2/2 = \begin{bmatrix} 0 & 0 & t^2/2 & 0 \\ 0 & 0 & 0 & t^2/2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$, and $A^3t^3/6 = \begin{bmatrix} 0 & 0 & 0 & t^3/6 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$. So, $x(t) = \begin{bmatrix} 1+t+t^2/2+t^3/6 \\ 1+t+t^2/2 \\ 1+t \\ 1 \end{bmatrix}$.

(iv) $A = \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}$, $B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}$, $C = \begin{bmatrix} C_1 & C_2 \end{bmatrix}$, and $D = \begin{bmatrix} D_1 & D_2 \end{bmatrix}$, where $A_1 = 1$, $B_1 = 1$, $C_1 = 1$, and $D_1 = 0$ from $\frac{1}{s-1}$ and $A_2 = -2$, $B_2 = -2$, $C_2 = 1$, and $D_2 = 1$ from $\frac{s}{s+2} = 1 - \frac{2}{s+2}$.

(v) Note that any $(x_1, 0)$ is an equilibrium point. Since there are no isolated equilibrium points, the system is not asymptotically stable. However, it is stable i.s.L., since $V = x_1^2 + x_2^2$ is a Lyapunov function for which $\dot{V} = -8x_2^2(x_1^2+x_2^2)$ is negative semidefinite.

Problem 2 (i) The system becomes unstable when the eigenvalues of $A + \Delta$ cross the imaginary axis, i.e. $\det(jw_oI - (A+\Delta)) = 0$ for some $w_o \in \mathbb{R}$. Now, the Δ with the smallest 2-induced norm that results in $\det((jwI_o - A) - \Delta) = 0$ has size greater or equal to $\sigma_{min}(jw_oI - A)$ for a particular w_o. To find the smallest, we minimize over all $w \in \mathbb{R}$, so, $\min_\Delta\{\|\Delta\|_2 \mid \text{system is unstable}\} = \min_{\Delta,w}\{\|\Delta\|_2 \mid jwI - (A+\Delta)\text{is singular}\} \geq \min_w \sigma_{min}(jwI - A)$ (*). Now, since A is real and symmetric, $A = U\Lambda U'$ where Λ is a diagonal matrix with the eigenvalues of A on the diagonal, and U is orthonormal. So, $jwI - A = U(jwI - \Lambda)U'$, and $(\sigma(jwI-A))^2 = \text{eig}((jwI-A)(jwI-A)') = \text{eig}(U(jwI-\Lambda)(jwI-\Lambda)'U') = \text{eig}(U(jwI-\Lambda)(-jwI-\Lambda)U')$ =the diagonal elements of $w^2I + \Lambda^2$. So, $\min_w \sigma_{min}(jwI - A) = \min_w(w^2 + \lambda_{min}^2(A))^{\frac{1}{2}} = |\lambda_{min}(A)|$. Finally, we note that the inequality in (*) is actually an equality, i.e., $\min_\Delta\{\|\Delta\|_2 \mid \text{system is unstable}\} = |\lambda_{min}(A)|$ since we can find a real Δ for which the minimum is achieved, namely, $\Delta = -\sigma_n(A)u_nv_n'$ where n corresponds to the index of the smallest singular value of A and u_n and v_n come from the svd of A and are real since A is symmetric and real.

(ii) First, since Q is positive definite, $Q = F'F$, and F is invertible. Let $z = Fx$, we have $\max_x\{x'Px \mid x'Qx \leq 1\} = \max_z\{z'(F^{-1})'PF^{-1}z \mid z'z \leq 1\} = \max_z\{z'(F^{-1})'PF^{-1}z \mid z'z = 1\} = \lambda_{max}((F^{-1})'PF^{-1}) = \lambda_{max}(F^{-1}(F^{-1})'P) = \lambda_{max}(Q^{-1}P)$, where the third equality follows because $(F^{-1})'PF^{-1}$ is symmetric and the fourth equality is due to the fact that the eigenvalues of $(F^{-1})'PF^{-1}$ are the same as the eigenvalues of $F^{-1}(F^{-1})'P$: $(F^{-1})'PF^{-1}x = \lambda x \iff F^{-1}(F^{-1})'P(F^{-1}x) = \lambda(F^{-1}x)$. In general, we can relate the eigenvalues of AB to the eigenvalues of BA by the following claim.

Suppose $A \in \mathbb{R}^{m\times n}$ and $B \in \mathbb{R}^{n\times m}$ with $m \geq n$, AB has m eigenvalues, BA has n eigenvalues, and m of the eigenvalues of AB are those of BA, including multiplicity, and the remaining eigenvalues of AB are at zero. The proof of the statement is similar the proof of $det(I - AB) = det(I - BA)$, but has a small modification, thanks to which the statement immediately follows. Consider

$$\begin{bmatrix} I & A \\ B & \lambda I \end{bmatrix}\begin{bmatrix} I & -A \\ 0 & I \end{bmatrix} = \begin{bmatrix} I & 0 \\ B & \lambda I - BA \end{bmatrix}$$

and

$$\begin{bmatrix} I & A \\ B & \lambda I \end{bmatrix}\begin{bmatrix} \lambda I & 0 \\ -B & I \end{bmatrix} = \begin{bmatrix} \lambda I - AB & A \\ 0 & \lambda I \end{bmatrix},$$

and let $P = \begin{bmatrix} I & A \\ B & \lambda I \end{bmatrix}$, then $det(P) = det(\lambda I - BA)$, $\lambda^m det(P) = \lambda^n det(\lambda I - AB)$, so $det(\lambda I - AB) = \lambda^{m-n} det(\lambda I - BA)$.

(iii) The statement is false, let $r \geq 2$, $C = \begin{bmatrix} I_r & 0 \end{bmatrix}$ and $A = \begin{bmatrix} J_r & 0 \\ 0 & I_{n-r} \end{bmatrix}$, where $J_r = \begin{bmatrix} 0 & 1 & \dots & 0 \\ & \ddots & \ddots & \\ & & \ddots & 1 \\ & & & 0 \end{bmatrix}$, and $J_r \in \mathbb{R}^{r\times r}$.

Problem 3 (a)(i) vw' is $n \times n$ matrix that has rank 1, so, $n - 1$ of the eigenvalues are at the origin. For stability, we require that the remaining eigenvalue is inside the unit disk. Now, $det(\lambda I - vw') = \lambda^{n-1} det(\lambda - w'v) = \lambda^{n-1}(\lambda - w'v) = 0$ implies $\lambda = w'v$. So, the condition for stability is $|w'v| < 1$.

(ii) $y_k = (w'v)^{k-1} c'vw'x_o$, for $k \geq 1$ and $y_o = c'x_o$.

(iii) First, we note that $v(w + \delta)'$ is rank 1, so any δ will result in $n - 1$ eigenvalues at the origin. Furthermore, since we may assume $v(w + \delta)'$ is real, the remaining eigenvalue, $(w + \delta)'v$, must be real (if a real matrix has a complex eigenvalue, then the complex conjugate is also an eigenvalue). So, we seek to find the smallest δ such that $(w + \delta)'v = 1$ or $(w + \delta)'v = -1$, i.e., find the δ for either case and take the smaller of the two. That is, $\min_\delta\{\|\delta\|_2 \mid (w + \delta)'v \in \{1, -1\}\} = \min\{\min\{\|\delta\|_2 \mid (w + \delta)'v = 1\}, \min\{\|\delta\|_2 \mid (w + \delta)'v = -1\}\}$. Note, we need to find the solutions for two underdetermined systems of equations, $\min\{\min\{\|\delta\|_2 \mid (w + \delta)'v = 1\}, \min\{\|\delta\|_2 \mid (w + \delta)'v = -1\}\} = \min\{\min\{\|\delta\|_2 \mid v'\delta = 1 - w'v\}, \min\{\|\delta\|_2 \mid v'\delta = -1 - w'v\}\} = \min\{(1 - w'v)(v/\|v\|^2), (-1 - w'v)(v/\|v\|^2)\} = \min\{(1 - w'v), (-1 - w'v)\}(v/\|v\|^2)$.

(b)(i) $w'v = 1$ so the system is not asymptotically stable.

(ii) First, note that v is a right eigenvector of vw' and $w/(w'v)$ is a left eigenvector and $w'v$ is the eigenvalue. Since $c'v = 0$ and $w'b = 0$, v is a right eigenvector of $vw' + \alpha bc'$ and $w/(w'v)$ is a left eigenvector and $w'v$ is the eigenvalue. Similarly, b is a right eigenvector of bc' and $c/(c'b)$ is a left eigenvector and $c'b$ is the eigenvalue. Furthermore, b is a right eigenvector of $vw' + \alpha bc'$ and $c/(c'b)$ is a left eigenvector and $c'b$ is the eigenvalue. Thus,

$$vw' + \alpha bc' = \begin{bmatrix} v & b \end{bmatrix} \begin{bmatrix} w'v & \\ & \alpha c'b \end{bmatrix} \begin{bmatrix} w'/(w'v) \\ c'/(c'b) \end{bmatrix},$$

and

$$(vw' + \alpha bc')^n = \begin{bmatrix} v & b \end{bmatrix} \begin{bmatrix} (w'v)^n & \\ & (\alpha c'b)^n \end{bmatrix} \begin{bmatrix} w'/(w'v) \\ c'/(c'b) \end{bmatrix}.$$

So, $\Phi(k,l) = \Phi(k-l) = v(w'v)^{k-l-1}w' + b(\alpha c'b)^{k-l-1}c'$ for $k > l$ and $\Phi(0) = I$.

(iii) One way is to note that v and b are eigenvectors of $vw' + \alpha bc'$ since $c'v = 0$ and $w'b = 0$, and v is an eigenvector of vw', and b is an eigenvector if bc'. That is, $(vw' + \alpha bc')v = \lambda_1 v$ and $(vw' + \alpha bc')b = \lambda_2 b$, from which we have that $\lambda_1 = w'v = 1$ is an eigenvalue of the sytem no matter what α we use. Equivalently, you may have calculated $det(\lambda I - (vw' + \alpha bc')) = \lambda^2(\lambda - 1)(\lambda - \alpha)$, from which it is clear that 1 is an eigenvalue of the system.