Class 2 Outline

1. **Monte-Carlo:**
   - **Framework/Definition**
   - **Algorithm**
   - **Examples**

2. **Random Number Generation**

Monte-Carlo Framework

Estimate $\theta = E[\ h(\mathbf{X})\ ]$

where $\mathbf{X} = \{X_1,\ldots,X_m\}$ is a random vector in R^m,

h(.) is a function $R^m \rightarrow R$,

and $E[\ |\ h(\mathbf{X})\ |\ ] < \infty$

Monte-Carlo Algorithm

1. Generate n samples of **X**: $\mathbf{X}^1, \ldots, \mathbf{X}^n$
2. Compute $h(\mathbf{X}^1), h(\mathbf{X}^2), \ldots, h(\mathbf{X}^n)$
3. Estimate $\theta = E[\, h(\mathbf{X}) \,]$ with

$$\bar{\theta} = [h(\mathbf{X}^1)+h(\mathbf{X}^2)+\ldots+h(\mathbf{X}^n)]/n$$

- Why is $\bar{\theta}$ a good estimator?

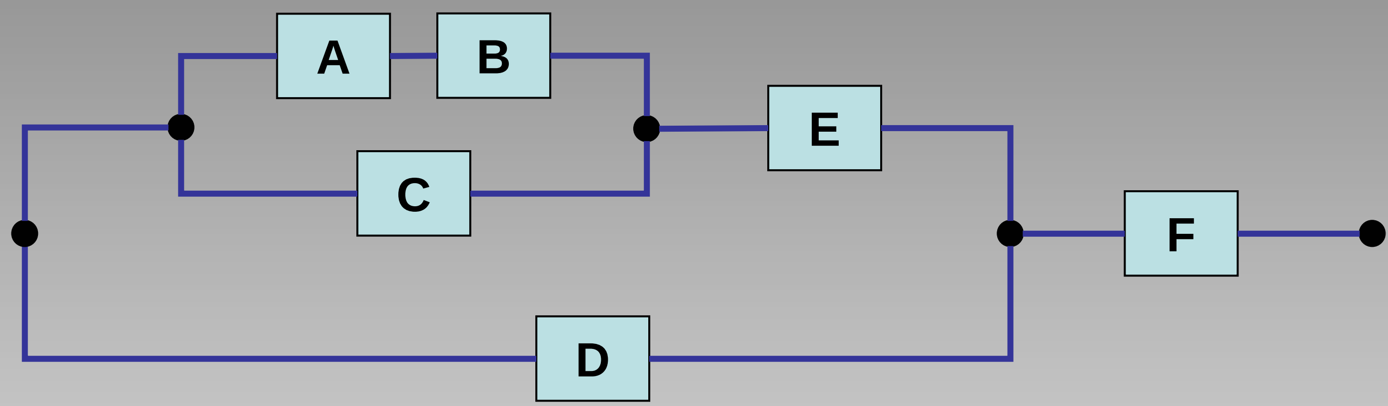
A Sales Incentive Plan

A company has a sales incentive plan with the following structure: In the months when sales are lower than $10,000, a salesrep get a base salary of $2000. If the monthly sales achieved are between $10,000 and $15,000, then he/she is paid 20% of the sales. Finally, if the monthly sales figure is larger than $15,000, the salesrep get 30% of the sales.

Assuming the monthly sales achieved by a salesman follows an exponential distribution with mean $15,000, what is the mean and standard deviation of the salesman's monthly salary under this compensation plan?

Product Reliability

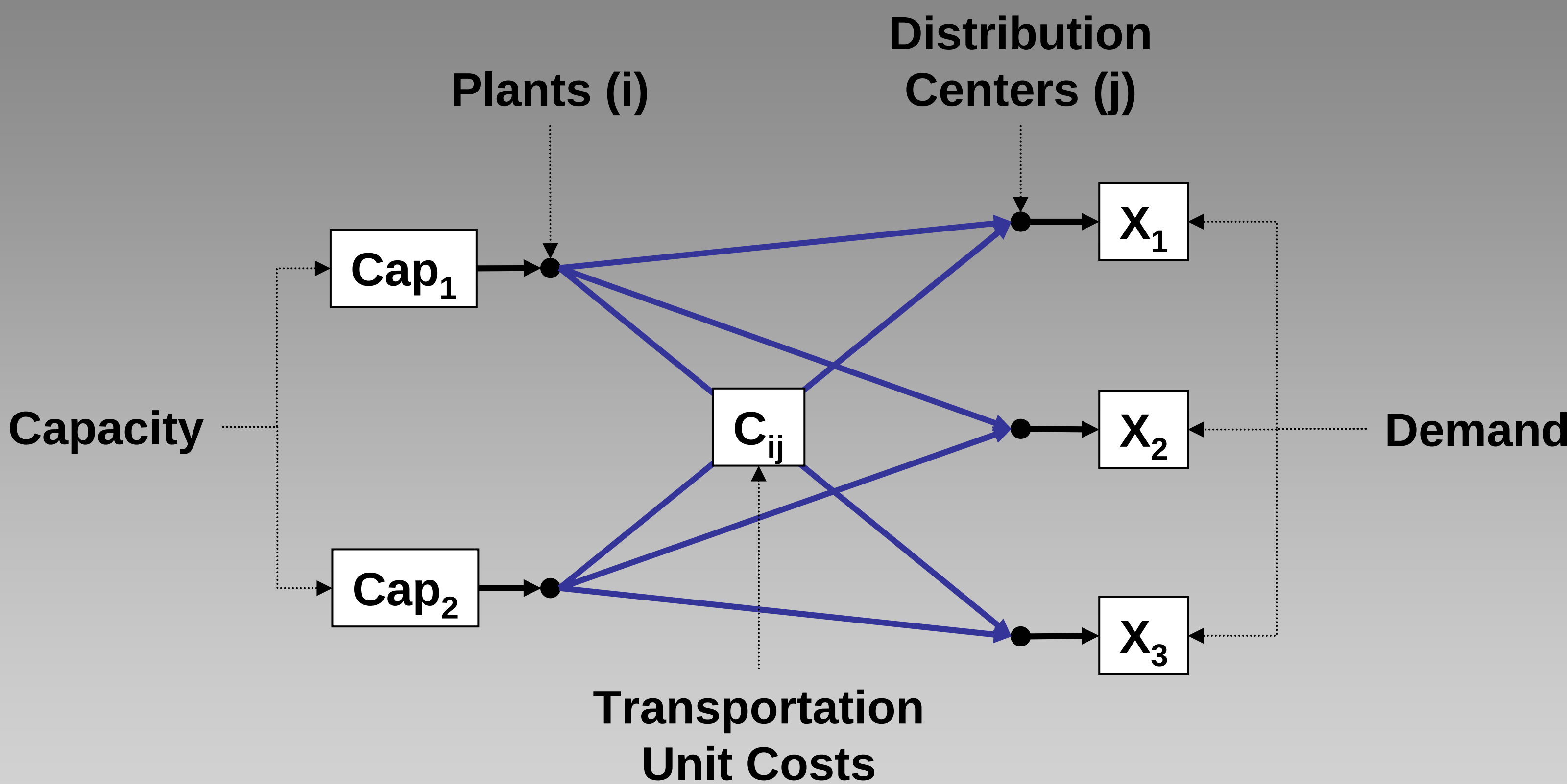
- **Consider a product reliability model given by:**



- **Component life lengths:** **$X_A \sim \exp(5.1)$; $X_B \sim N(4.5,1)$; $X_C \sim \exp(6.5)$**
  $X_D \sim \exp(5.4)$; $X_E \sim \exp(6.4)$; $X_F \sim N(5.5,0.8)$

- ***What is the probability that the product will function for at least 3 years (warranty period)?***

Stochastic Programming



Stochastic Demand $\begin{cases} X_1 \sim N(1500,350) \\ X_2 \sim N(2000,600) \\ X_3 \sim N(1500,450) \end{cases}$

Random Number Generation

- **The key in a Monte-Carlo simulation is to generate sample values drawn from a known probability distribution. How is this done?**

Generate Sample Value from Uniform[0,1]

Discrete Distribution

- **Inverse transform method**
- **Rejection method**

Continuous Distribution

- **Inverse transform method**
- **Rejection method**
- **Composition method**
- **Polar method**

U[0,1] Sample Generation

1. **Calculate Z_1, Z_2,… given by the Linear Congruential Generator:**

$$Z_i = (aZ_{i-1} + c) \textit{ modulo } m$$

where m, a, c and Z_0 are non-neg. integers

modulus **seed**

2. **Take $U_i = Z_i / m$ as a proxy to an independent sequence of U[0,1] random variates (note: $0 \leq U_i \leq 1$ for all i)**

Linear Congruential Generator

- **Example:** $Z_i = (11Z_{i-1}) \textit{ modulo } 16$

  $Z_0 = 1$
  $Z_1 = (11) \bmod 16 = 11$
  $Z_2 = (121) \bmod 16 = 9$
  $Z_3 = (99) \bmod 16 = ?$
  $Z_4 = (33) \bmod 16 = ?$

- ***Huge* literature on LCG periods, parameter choice, statistical tests for randomness…**
- **Crystal Ball uses $Z_i = (630{,}360{,}016\ Z_{i-1}) \bmod (2^{31}-1)$**

Discrete Distributions

- **We want to simulate a random variable X defined by $P(X = x_j) = p_j, \quad j=0,1,\ldots,m$**

1. **Generate a sample U from U[0,1]**
2. **if $U < p_0$ then $X = x_0$**

   if $p_0 < U < p_0 + p_1$ then $X = x_1$

   …..

   if $p_0 + \ldots + p_{j-1} < U < p_0 + \ldots + p_j$ then $X = x_j$

- **Why does this work?**

Solution

1. Generate a sample U from U[0,1]
2. if $U < p_0$ then $X = x_0$

   if $p_0 < U < p_0 + p_1$ then $X = x_1$

   …..

   if $p_0 + \ldots + p_{j-1} < U < p_0 + \ldots + p_j$ then $X = x_j$

- $P(X = x_j) = P(p_0 + \ldots + p_{j-1} < U < p_0 + \ldots + p_j)$
  $= p_j$

Continuous Distributions

- **We want to simulate a random variable X defined by $P(X < t) = F(t)$ (F inversible)**

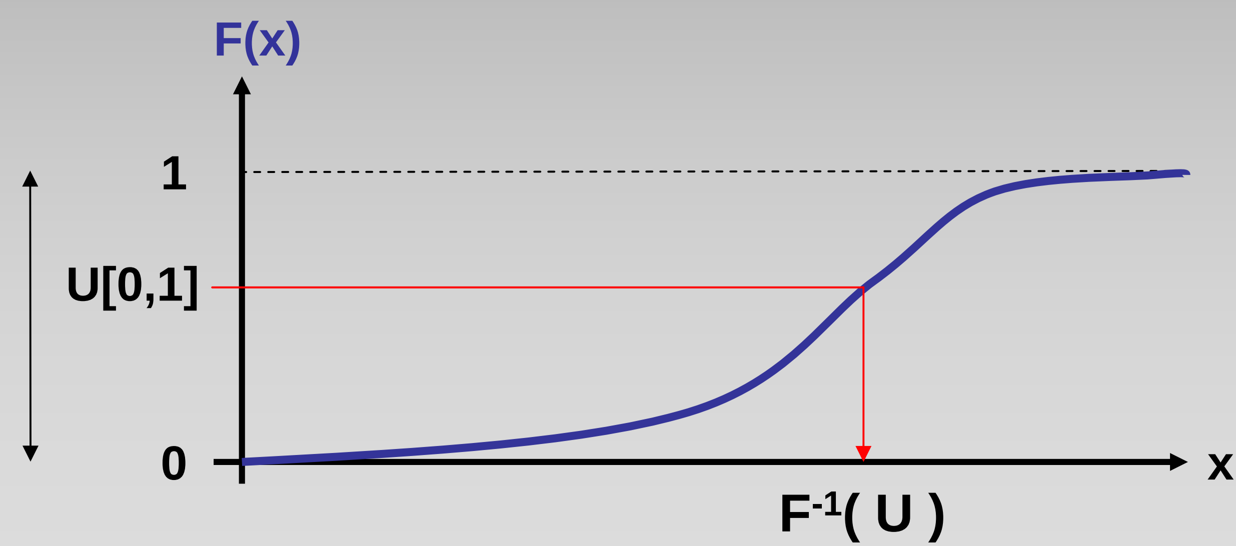
1. **Generate a sample U from U[0,1]**
2. **Set $X = F^{-1}(U)$**

- **Why does this work?**

Solution

1. Generate a sample U from U[0,1]
2. Set $X = F^{-1}(U)$

- $P(X < t) = P(F^{-1}(U) < t) = P(U < F(t)) = F(t)$



Exponential Distribution

- **Variable exp(λ) has cdf. $F(t) = 1 - \exp(-\lambda t)$; $F^{-1}(y) = -\log(1-y) / \lambda$**

1. **Generate U from U[0,1]**
2. **Compute $X = -\log(1-U) / \lambda$**
3. **The distribution of X is exp(λ) !**

Class 2 Wrap-Up

1. **Monte-Carlo Framework, Algorithm & Examples**

2. **Crystal Ball Modeling**

3. **Theory of Random Number Generation**