# SMA 6304 / MIT 2.853 / MIT 2.854
# Manufacturing Systems

## Lecture 10: Data and Regression Analysis

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# Agenda

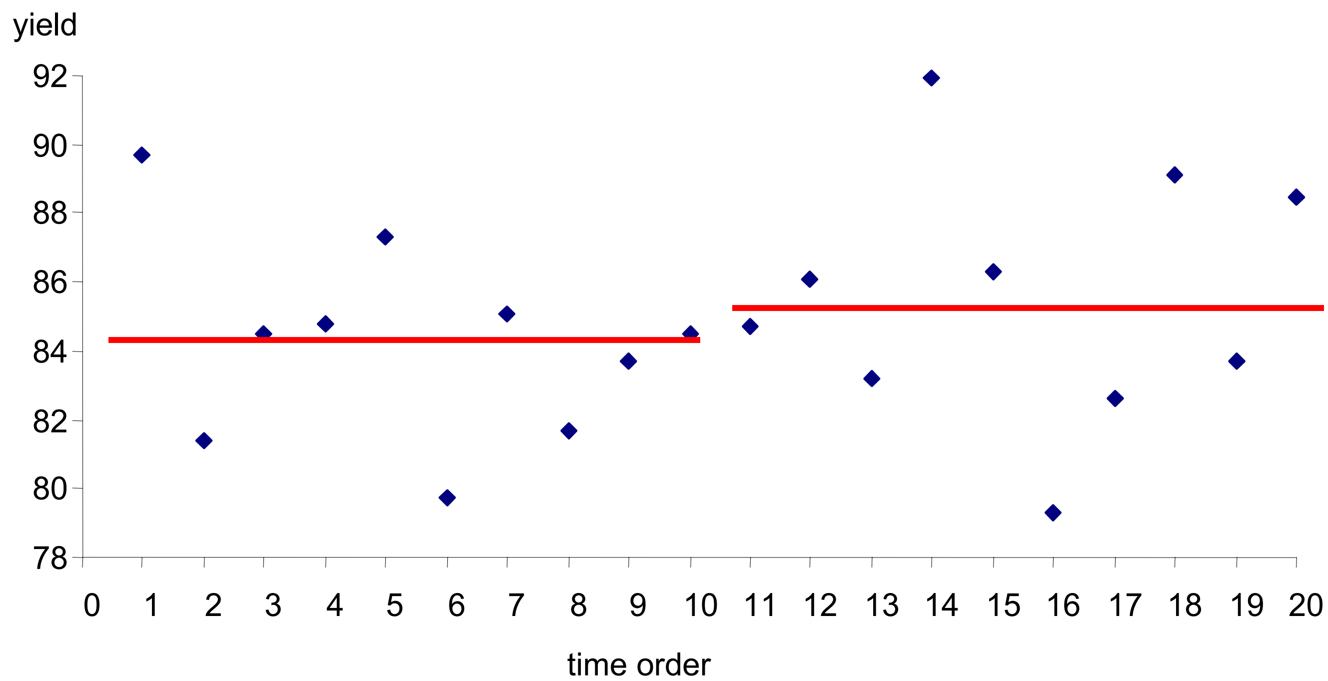
1. Comparison of Treatments (One Variable)
   - Analysis of Variance (ANOVA)

2. Multivariate Analysis of Variance
   - Model forms

3. Regression Modeling
   - Regression fundamentals
   - Significance of model terms
   - Confidence intervals

# Is Process B Better Than Process A?

| time order | method | yield |
|---|---|---|
| 1 | A | 89.7 |
| 2 | A | 81.4 |
| 3 | A | 84.5 |
| 4 | A | 84.8 |
| 5 | A | 87.3 |
| 6 | A | 79.7 |
| 7 | A | 85.1 |
| 8 | A | 81.7 |
| 9 | A | 83.7 |
| 10 | A | 84.5 |
| 11 | B | 84.7 |
| 12 | B | 86.1 |
| 13 | B | 83.2 |
| 14 | B | 91.9 |
| 15 | B | 86.3 |
| 16 | B | 79.3 |
| 17 | B | 82.6 |
| 18 | B | 89.1 |
| 19 | B | 83.7 |
| 20 | B | 88.5 |




$$\bar{y}_A = 84.24$$
$$\bar{y}_B = 85.54$$
$$\bar{y}_B - \bar{y}_A = 1.30$$

$$H_0 : \mu_A = \mu_B$$
$$H_1 : \mu_A < \mu_B$$

# Two Means with Internal Estimate of Variance

| Method A | | Method B | |
|---|---|---|---|
| count | $n_A = 10$ | count | $n_B = 10$ |
| sum | 842.4 | sum | 855.4 |
| average | $\bar{y}_A = 84.24$ | average | $\bar{y}_B = 85.54$ |
| sum squares | $\sum(y_A - \bar{y}_A)^2 = 75.784$ | sum squares | $\sum(y_B - \bar{y}_B)^2 = 119.924$ |

$$\bar{y}_B - \bar{y}_A = 1.30$$

Pooled estimate of σ[2] $\quad s^2 = \frac{75.784+119.924}{10+10-2} = \frac{195.708}{18} = 10.8727$ with ν=18 d.o.f

Estimated variance of $\bar{y}_B - \bar{y}_A$ $\quad s^2\left(\frac{1}{n_A}+\frac{1}{n_B}\right) = \frac{2s^2}{10} = \frac{s^2}{5}$

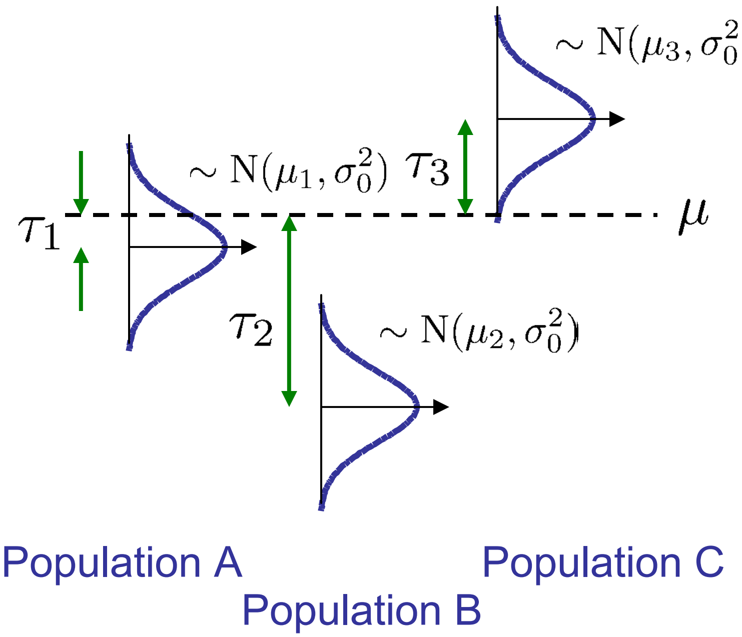
Estimated standard error of $\bar{y}_B - \bar{y}_A$ $\quad \sqrt{\frac{s^2}{5}} = \sqrt{\frac{10.8727}{5}} = 1.47$

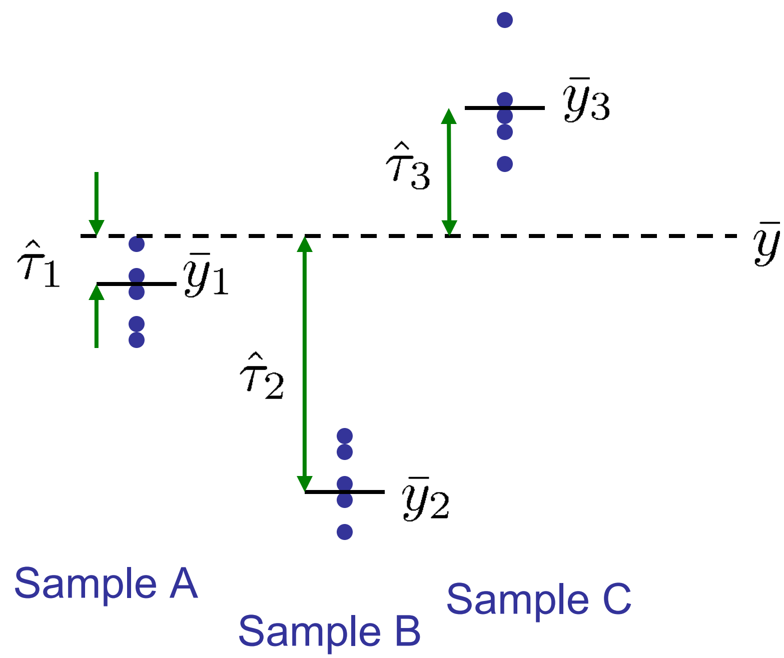
$$t_0 = \frac{(\bar{y}_B-\bar{y}_A)-(\eta_B-\eta_A)_0}{s\sqrt{1/n_A+1/n_B}}$$

For $(\eta_B - \eta_B)_0 = 0, t_0 = \frac{1.30}{1.47} = 0.88$ with $\nu = 18$ degrees of freedom.

$\Pr(t \geq 0.88) = 0.195$ So only about 80% confident that mean difference is "real" (signficant)

# Comparison of Treatments




- Consider multiple conditions (treatments, settings for some variable)
  - There is an overall mean $\mu$ and real "effects" or deltas between conditions $\tau_i$.
  - We observe samples at each condition of interest
- Key question: are the ***observed*** differences in mean "significant"?
  - Typical assumption (should be checked): the underlying variances are all the same – usually an unknown value ($\sigma_0^2$)

# Steps/Issues in Analysis of Variance

1. Within group variation
   – Estimates underlying population variance

2. Between group variation
   – Estimate group to group variance

3. Compare the two estimates of variance
   – If there is a difference between the different treatments, then the between group variation estimate will be ***inflated*** compared to the within group estimate
   – We will be able to establish confidence in whether or not observed differences between treatments are significant

Hint: we'll be using *F* tests to look at ratios of variances

# (1) Within Group Variation

- Assume that each group is normally distributed and shares a common variance $\sigma_0^2$
- $SS_t$ = sum of square deviations within $t^{th}$ group (there are k groups)

$$SS_t = \sum_{j=1}^{n_t} (y_{tj} - \bar{y}_t)^2 \text{ where } n_t \text{ is number of samples in treatment } t$$

- Estimate of within group variance in $t^{th}$ group (just variance formula)

$$s_t^2 = SS_t/\nu_t = \frac{SS_t}{n_t - 1} \quad \text{where } \nu_t \text{ is d.o.f. in treatment } t$$

- Pool these (across different conditions) to get estimate of common within group variance:

$$s_R^2 = \frac{\nu_1 s_1^2 + \nu_1 s_1^2 + \cdots + \nu_k s_k^2}{\nu_1 + \nu_2 + \cdots + \nu_k} = \frac{SS_R}{\nu_R} = \frac{SS_R}{N - k}$$

- This is the within group "mean square" (variance estimate)

$$MS_R = \frac{SS_R}{\nu_R} = s_R^2$$

# (2) Between Group Variation

- We will be testing hypothesis $\mu_1 = \mu_2 = \ldots = \mu_k$
- If all the means are in fact equal, then a 2nd estimate of $\sigma^2$ could be formed based on the observed differences between group means:

$$s_T^2 = \frac{\sum_{t=1}^{k} n_t(\bar{y}_t - \bar{y})^2}{k-1}$$

where $n_t$ is number of samples in treatment $t$ and $k$ is the number of different treatments

- If all the treatments in fact have different means, then $s_T^2$ estimates something larger:

$$s_T^2 \simeq \sigma_0^2 + \frac{\sum_{t=1}^{k} n_t \tau_t^2}{k-1}$$

where $\tau_t$ is the (real) difference between group $t$ mean and the grand mean $\mu$

Variance is "inflated" by the real treatment effects $\tau_t$

# (3) Compare Variance Estimates

- We now have two different possibilities for $s_T^2$, depending on whether the observed sample mean differences are "real" or are just occurring by chance (by sampling)
- Use *F* statistic to see if the ratios of these variances are likely to have occurred by chance!
- Formal test for significance:

$$\text{Reject } H_0 \text{ (no mean difference) if } \frac{s_T^2}{s_R^2} \text{ is significantly greater than 1.}$$

# (4) Compute Significance Level

- Calculate observed *F* ratio (with appropriate degrees of freedom in numerator and denominator)
- Use *F* distribution to find how likely a ratio this large is to have occurred by chance alone
  - This is our "significance level"
  - If $F_0 = s_T^2 / s_R^2 \; > \; F_{\alpha,k-1,N-k}$
    then we say that the mean differences or treatment effects are significant to (1-$\alpha$)100% confidence or better

# (5) Variance Due to Treatment Effects

- We also want to estimate the sum of squared deviations from the grand mean among all samples:

$$SS_D = \sum_{t=1}^{k} \sum_{i=1}^{n_t} (y_{ti} - \bar{y})^2$$

$$s_D^2 = SS_D / \nu_D = \frac{SS_D}{N - 1} = MS_D$$

where $N$ is the total number of measurements

# (6) Results: The ANOVA Table

| source of variation | sum of squares | degrees of freedom | mean square | $F_0$ | $\Pr(F_0)$ |
|---|---|---|---|---|---|
| Between treatments | $SS_T$ | $k-1$ | $s_T^2 = \frac{SS_T}{k-1}$ | $\frac{s_T^2}{s_R^2}$ | table |
| Within treatments | $SS_R$ | $N-k$ | $s_R^2 = \frac{SS_R}{N-k}$ | | |
| Total about the grand average | $SS_D$ | $N-1$ | $s_D^2 = \frac{SS_D}{N-1}$ | | |

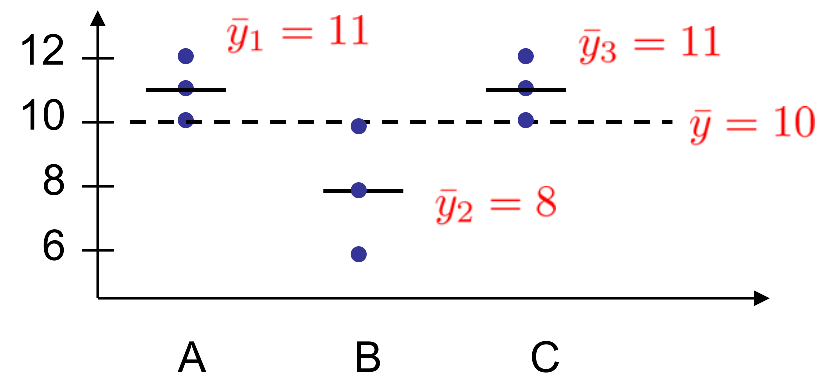
Also referred to as "residual" SS (for $SS_R$)

$SS_D = SS_T + SS_R$

$\nu_D = \nu_T + \nu_R$

# Example: Anova

| A | B | C |
|---|---|---|
| 11 | 10 | 12 |
| 10 | 8 | 10 |
| 12 | 6 | 11 |




Excel: Data Analysis, One-Variation Anova

| Anova: Single Factor | | | | | | |
|---|---|---|---|---|---|---|
| | | | | | | |
| SUMMARY | | | | | | |
| *Groups* | *Count* | *Sum* | *Average* | *Variance* | | |
| A | 3 | 33 | 11 | 1 | | |
| B | 3 | 24 | 8 | 4 | | |
| C | 3 | 33 | 11 | 1 | | |
| | | | | | | |
| | | | | | | |
| ANOVA | | | | | | |
| *Source of Variation* | *SS* | *df* | *MS* | *F* | *P-value* | *F crit* |
| Between Groups | 18 | 2 | 9 | 4.5 | 0.064 | 5.14 |
| Within Groups | 12 | 6 | 2 | | | |
| | | | | | | |
| Total | 30 | 8 | | | | |

$$F = \frac{S_T^2}{S_R^2} = \frac{9}{2} = 4.5$$

$$F_{0.05,2,6} = 5.14$$

$$F_{0.10,2,6} = 3.46$$

$$SS_1 = (12-11)^2 + (11-11)^2 + (10-11)^2 = 2$$
$$SS_2 = 2^2 + 0^2 + 2^2 = 8$$
$$SS_3 = 1^2 + 0^2 + 1^{=}2$$

$$s_1^2 = MS_1 = SS_1/2 = 2/2 = 1$$
$$s_2^2 = MS_2 = 8/2 = 4$$
$$s_3^2 = MS_3 = 2/2 = 1$$

$$s_R^2 = \frac{SS_1 + SS_2 + SS_3}{N-k} = \frac{12}{6} = 2$$

$$s_T^2 = \frac{3(11-10)^2+3(8-10)^2+3(11-10)^2}{3-1} = \frac{SS_T}{\nu_T} = \frac{18}{2} = 9$$

# ANOVA – Implied Model

- The ANOVA approach assumes a simple mathematical model:

$$\begin{aligned} y_{ti} &= \mu + \tau_t + \epsilon_{ti} \\ &= \mu_t + \epsilon_{ti} \end{aligned}$$

- Where $\mu_t$ is the treatment mean (for treatment type t)
- And $\tau_t$ is the treatment effect
- With $\varepsilon_{ti}$ being zero mean normal residuals $\sim N(0,\sigma_0^2)$
- Checks
  - Plot residuals against time order
  - Examine distribution of residuals: should be IID, Normal
  - Plot residuals vs. estimates
  - Plot residuals vs. other variables of interest

# MANOVA – Two Dependencies

- Can extend to two (or more) variables of interest. MANOVA assumes a mathematical model, again simply capturing the means (or treatment offsets) for each discrete variable level:

$$
\begin{array}{rcccccccc}
y_{ti} & = & \mu & + & \tau_t & + & \beta_i & + & \epsilon_{ti} \\
\hat{y}_{ti} & = & \hat{\mu} & + & \hat{\tau}_t & + & \hat{\beta}_i & & \\
 & & & & & & & & \\
\#\text{ model coeffs} & = & 1 & + & k & + & n & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\#\text{ independent model coeffs} & = & 1 & + & (k-1) & + & (n-1) & &
\end{array}
$$

Recall that our $\hat{\tau}_t$ are *not* all independent model coefficients, because $\sum \tau_t = 0$. Thus we really only have $k-1$ independent model coeffs, or $\nu_t = k-1$.

- Assumes that the effects from the two variables are ***additive***

# Example: Two Factor MANOVA

- Two LPCVD deposition tube types, three gas suppliers. Does supplier matter in average particle counts on wafers?
  - Experiment: 3 lots on each tube, for each gas; report average # particles added

Factor 1: Gas

| Factor 2: Tube | A | B | C | |
|---|---|---|---|---|
| 1 | **7** | **36** | **2** | 15 |
| 2 | **13** | **44** | **18** | 25 |
| | 10 | 40 | 10 | |

**Analysis of Variance**

| Source | DF | Sum of Squares | Mean Square | F Ratio |
|---|---|---|---|---|
| Model | 3 | 1350.00 | 450.0 | 32.14 |
| Error | 2 | 28.00 | 14.0 | Prob > F |
| C. Total | 5 | 1378.00 | | 0.0303 |

**Effect Tests**

| Source | Nparm | DF | Sum of Squares | F Ratio | Prob > F |
|---|---|---|---|---|---|
| Tube | 1 | 1 | 150.00 | 10.71 | 0.0820 |
| Gas | 2 | 2 | 1200.00 | 42.85 | 0.0228 |

$$y_{ti} = \mu + \tau_t + \beta_i + \epsilon_{ti}$$

$$\hat{y}_{ti} = \bar{y} + (\bar{y}_t - \bar{y}) + (\bar{y}_i - \bar{y}) + (y_{ti} - \bar{y}_t - \bar{y}_i + \bar{y})$$

$$\begin{bmatrix} 7 & 36 & 2 \\ 13 & 44 & 18 \end{bmatrix} \quad \begin{bmatrix} 20 & 20 & 20 \\ 20 & 20 & 20 \end{bmatrix} + \begin{bmatrix} -10 & 20 & -10 \\ -10 & 20 & -10 \end{bmatrix} + \begin{bmatrix} -5 & -5 & -5 \\ 5 & 5 & 5 \end{bmatrix} + \begin{bmatrix} 2 & 1 & -3 \\ -2 & -1 & 3 \end{bmatrix}$$

$$SS = SS_A + SS_T + SS_B + SS_R$$

# MANOVA – Two Factors with Interactions

- May be interaction: not simply additive – effects may depend synergistically on both factors:

$$y_{tij} = \mu_{ti} + \epsilon_{tij}$$

IID, ~N(0,$\sigma^2$)

An effect that depends on both t & i factors simultaneously

t = first factor = 1,2, … k (k = # levels of first factor)
i = second factor = 1,2, … n (n = # levels of second factor)
j = replication = 1,2, … m (m = # replications at t, jth combination of factor levels

- Can split out the model more explicitly…

$$y_{tij} = \mu + \tau_t + \beta_i + \omega_{ti} + \epsilon_{tij}$$

Estimate by:

$$\hat{y}_{tij} = \bar{y} + (\bar{y_t} - \bar{y}) + (\bar{y_i} - \bar{y}) + (\bar{y_{ti}} + \bar{y_t} - \bar{y_i} + \bar{y})$$

$$\omega_{ti} = \text{interaction effects} = (\bar{y_{ti}} + \bar{y_t} - \bar{y_i} + \bar{y})$$

$$\tau_t, \beta_i = \text{main effects}$$

# MANOVA Table – Two Way with Interactions

| source of variation | sum of squares | degrees of freedom | mean square | $F_0$ | Pr($F_0$) |
|---|---|---|---|---|---|
| Between levels of factor 1 (T) | $SS_T$ | $k-1$ | $s_T^2$ | $s_T^2/s_E^2$ | table |
| Between levels of factor 2 (B) | $SS_B$ | $n-1$ | $s_B^2$ | $s_B^2/s_E^2$ | table |
| Interaction | $SS_I$ | $(k-1)(n-1)$ | $s_I^2$ | $s_I^2/s_E^2$ | table |
| Within Groups (Error) | $SS_E$ | $nk(m-1)$ | $s_E^2$ | | |
| Total about the grand average | $SS_D$ | $nkm-1$ | | | |

# Measures of Model Goodness – $R^2$

- Goodness of fit – $R^2$
  - Question considered: how much better does the model do that just using the grand average?

$$R^2 = \frac{SS_T}{SS_D}$$

  - Think of this as the fraction of squared deviations (from the grand average) in the data which is captured by the model
- Adjusted $R^2$
  - For "fair" comparison between models with different numbers of coefficients, an alternative is often used

$$R^2_{\text{adj}} = 1 - \frac{SS_R/\nu_R}{SS_D/\nu_D} = 1 - \frac{s_R^2}{s_D^2}$$

  - Think of this as (1 – variance remaining in the residual). Recall $\nu_R = \nu_D - \nu_T$

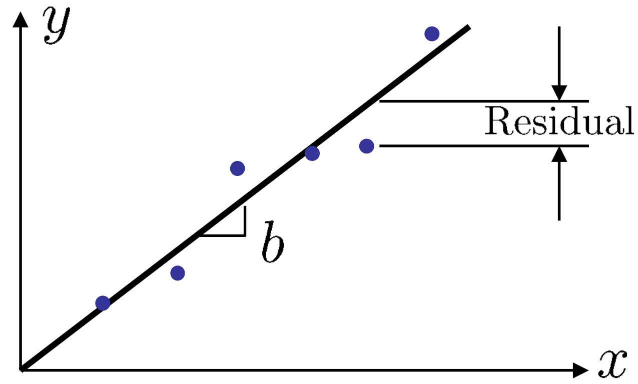
# Regression Fundamentals

- Use least square error as measure of goodness to estimate coefficients in a model
- One parameter model:
  - Model form
  - Squared error
  - Estimation using normal equations
  - Estimate of experimental error
  - Precision of estimate: variance in b
  - Confidence interval for $\beta$
  - Analysis of variance: significance of b
  - Lack of fit vs. pure error
- Polynomial regression

# Least Squares Regression

- We use ***least-squares*** to estimate coefficients in typical regression models
- One-Parameter Model:

$$\begin{aligned} y_i &= \beta x_i + \epsilon_i, \quad i = 1, 2, ..., n \\ \hat{y}_i &= b x_i \end{aligned}$$

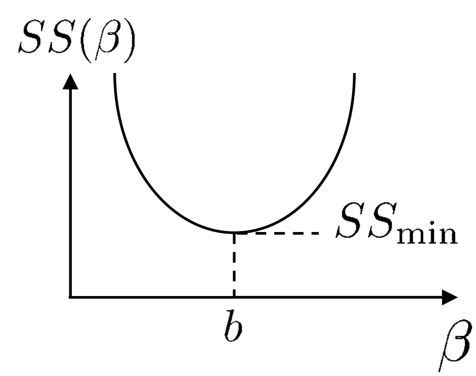



- Goal is to estimate β with "best" b
- How define "best"?
  - That b which minimizes sum of squared error between prediction and data

$$SS(\beta) = \sum_{i=1}^{n} (y_i - \hat{y}_i)^2 = \sum_{i=1}^{n} (y_i - \beta x_i)^2$$

  - The residual sum of squares (for the best estimate) is

$$SS_{\min} = \sum_{i=1}^{n} (y_i - b x_i)^2 = SS_R$$



# Least Squares Regression, cont.

- Least squares estimation via normal equations
  - For linear problems, we need not calculate SS($\beta$); rather, direct solution for b is possible
  - Recognize that vector of residuals will be normal to vector of x values at the least squares estimate

$$\begin{aligned} \sum (y - \hat{y})x &= 0 \\ \sum (y - bx)x &= 0 \\ \sum xy &= \sum bx^2 \end{aligned}$$

$$\Rightarrow \quad b = \frac{\sum xy}{\sum x^2}$$

- Estimate of experimental error
  - Assuming model structure is adequate, estimate $s^2$ of $\sigma^2$ can be obtained:

$$s^2 = \frac{SS_R}{n-1}$$

# Precision of Estimate: Variance in b

- We can calculate the variance in our estimate of the slope, b:

$$\hat{V}(b) = \frac{s^2}{\sum x_i^2}$$

$$\text{s.e.}(b) = \sqrt{\hat{V}(b)}$$

$$b \pm \text{s.e.}(b)$$

- Why?

$$\begin{aligned} b &= \frac{x_1}{\sum x^2} \cdot y_1 + \frac{x_2}{\sum x^2} \cdot y_2 + \cdots \frac{x_n}{\sum x^2} \cdot y_n \\ &= a_1 y_1 + a_2 y_2 + \cdots + a_n y_n \end{aligned}$$

$$\begin{aligned} V(b) &= (a_1^2 + a_2^2 + \cdots + a_n^2)\sigma^2 \\ &= \left[ (\frac{x_1}{\sum x^2})^2 + \cdots + (\frac{x_n}{\sum x^2})^2 \right] \sigma^2 \\ &= \frac{\sum x^2}{(\sum x^2)^2} \sigma^2 \\ &= \frac{\sigma^2}{\sum x^2} \end{aligned}$$

# Confidence Interval for β

- Once we have the standard error in b, we can calculate confidence intervals to some desired (1-α)100% level of confidence

$$\frac{b-\beta}{\text{s.e.}(b)} \sim t \qquad \Rightarrow \qquad \beta = b \pm t_{\alpha/2} \cdot \text{s.e.}(b)$$

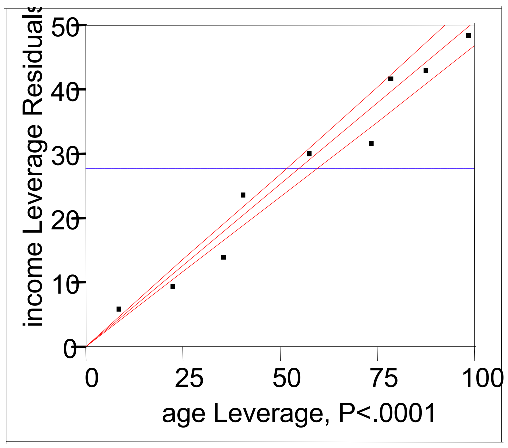
- Analysis of variance
  - Test hypothesis: $H_0 : \beta = b = 0$
  - If confidence interval for β includes 0, then β not significant
  - Degrees of freedom (need in order to use t distribution)

$$\begin{array}{ccccc} \sum y_i^2 & = & \sum \hat{y}_i^2 & + & \sum (y_i - \hat{y}_i)^2 \\ n & = & p & + & n-p \end{array}$$

p = # parameters estimated
by least squares

# Example Regression

| Age | Income |
|---|---|
| 8 | 6.16 |
| 22 | 9.88 |
| 35 | 14.35 |
| 40 | 24.06 |
| 57 | 30.34 |
| 73 | 32.17 |
| 78 | 42.18 |
| 87 | 43.23 |
| 98 | 48.76 |




**Whole Model**

**Analysis of Variance**

| Source | DF | Sum of Squares | Mean Square | F Ratio |
|---|---|---|---|---|
| Model | 1 | 8836.6440 | 8836.64 | 1093.146 |
| Error | 8 | 64.6695 | 8.08 | Prob > F |
| C. Total | 9 | 8901.3135 | | <.0001 |

Tested against reduced model: Y=0

**Parameter Estimates**

| Term | | Estimate | Std Error | t Ratio | Prob>\|t\| |
|---|---|---|---|---|---|
| Intercept | Zeroed | 0 | 0 | . | . |
| age | | 0.500983 | 0.015152 | 33.06 | <.0001 |

**Effect Tests**

| Source | Nparm | DF | Sum of Squares | F Ratio | Prob > F |
|---|---|---|---|---|---|
| age | 1 | 1 | 8836.6440 | 1093.146 | <.0001 |

- Note that this simple model assumes an intercept of zero – model must go through origin
- We will relax this requirement soon

# Lack of Fit Error vs. Pure Error

- Sometimes we have replicated data
  - E.g. multiple runs at same x values in a designed experiment

- We can decompose the residual error contributions

$$SS_R = SS_L + SS_E$$

Where
$SS_R$ = residual sum of squares error
$SS_L$ = lack of fit squared error
$SS_E$ = pure replicate error

- This allows us to TEST for lack of fit
  - By "lack of fit" we mean evidence that the linear model form is inadequate

$$\frac{s_L^2}{s_E^2} \sim F_{\nu_L, \nu_E}$$

# Regression: Mean Centered Models

- Model form $\eta = \alpha + \beta(x - \bar{x})$
- Estimate by $\hat{y} = a + b(x - \bar{x}), \quad y_i \sim \mathrm{N}(\eta_i, \sigma^2)$

$$\text{Minimize } SS_R = \sum (y_i - \hat{y}_i)^2 \text{ to estimate } \alpha \text{ and } \beta$$

$$a = \bar{y}$$

$$\mathrm{E}(a) = \alpha$$

$$\mathrm{Var}(a) = \mathrm{Var}\left[\frac{\sum y_i}{k}\right] = \frac{\sigma^2}{k}$$

$$b = \frac{\sum (x_i - \bar{x})(y_i - \bar{y})}{\sum (x_i - \bar{x})^2}$$

$$\mathrm{E}(b) = \beta$$

$$\mathrm{Var}(b) = \frac{\sigma^2}{\sum (x_i - \bar{x})^2}$$

# Regression: Mean Centered Models

- Confidence Intervals

$$\begin{aligned} \hat{y}_i &= \bar{y} + b(x_i - \bar{x}) \\ \mathrm{Var}(\hat{y}_i) &= \mathrm{Var}(\bar{y}) + (x_i - \bar{x})^2 \mathrm{Var}(b) \\ &= \frac{s^2}{n} + \frac{s^2 (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2} \end{aligned}$$

- Our confidence interval on y widens as we get further from the center of our data!

$$\hat{y}_i \pm t_{\alpha/2} \sqrt{\mathrm{Var}(\hat{y}_i)}$$

$$\hat{y}_i \pm t_{\alpha/2} \sqrt{\frac{s^2 (x_i - \bar{x})^2}{\sum (x_i - \bar{x})^2}}$$

# Polynomial Regression

- We may believe that a higher order model structure applies. Polynomial forms are also linear in the coefficients and can be fit with least squares

$$\eta = \beta_0 + \beta_1 x + \beta_2 x^2$$

Curvature included through $x^2$ term

- Example: Growth rate data

# Regression Example: Growth Rate Data

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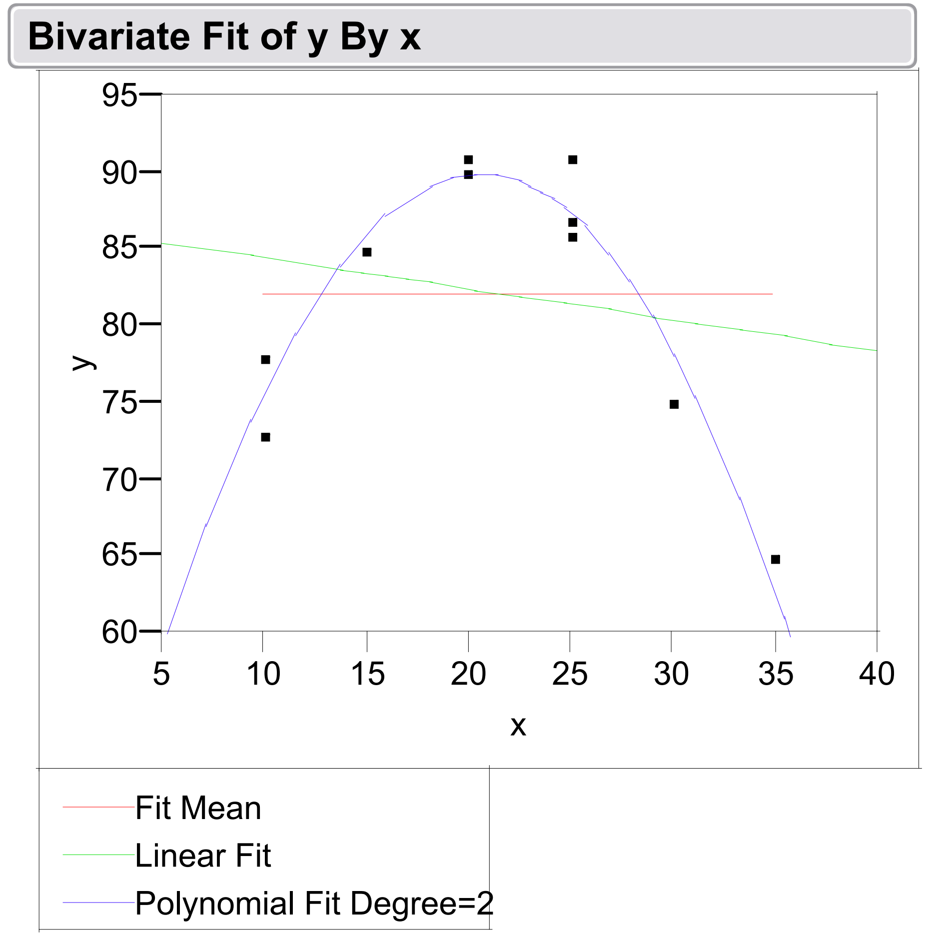


- Replicate data provides opportunity to check for lack of fit

# Growth Rate – First Order Model

- Mean significant, but linear term not
- Clear evidence of lack of fit

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# Growth Rate – Second Order Model

- No evidence of lack of fit
- Quadratic term significant

Image removed due to copyright considerations.

# Polynomial Regression In Excel

- Create additional input columns for each input
- Use "Data Analysis" and "Regression" tool

| x | x^2 | y |
|---|---|---|
| 10 | 100 | 73 |
| 10 | 100 | 78 |
| 15 | 225 | 85 |
| 20 | 400 | 90 |
| 20 | 400 | 91 |
| 25 | 625 | 87 |
| 25 | 625 | 86 |
| 25 | 625 | 91 |
| 30 | 900 | 75 |
| 35 | 1225 | 65 |

| *Regression Statistics* | |
|---|---|
| Multiple R | 0.968 |
| R Square | 0.936 |
| Adjusted R Square | 0.918 |
| Standard Error | 2.541 |
| Observations | 10 |

ANOVA

| | *df* | *SS* | *MS* | *F* | *Significance F* |
|---|---|---|---|---|---|
| Regression | 2 | 665.706 | 332.853 | 51.555 | 6.48E-05 |
| Residual | 7 | 45.194 | 6.456 | | |
| Total | 9 | 710.9 | | | |

| | *Coefficients* | *Standard Error* | *t Stat* | *P-value* | *Lower 95%* | *Upper 95%* |
|---|---|---|---|---|---|---|
| Intercept | 35.657 | 5.618 | 6.347 | 0.0004 | 22.373 | 48.942 |
| x | 5.263 | 0.558 | 9.431 | 3.1E-05 | 3.943 | 6.582 |
| x^2 | -0.128 | 0.013 | -9.966 | 2.2E-05 | -0.158 | -0.097 |

# Polynomial Regression

## Analysis of Variance

| Source | DF | Sum of Square | Mean Squar | F Ratio |
|---|---|---|---|---|
| Model | 2 | 665.70617 | 332.853 | 51.5551 |
| Error | 7 | 45.19383 | 6.456 | Prob > F |
| C. Total | 9 | 710.90000 | | <.0001 |

- Generated using JMP package

## Lack Of Fit

| Source | DF | Sum of Square | Mean Squar | F Ratio |
|---|---|---|---|---|
| Lack Of Fit | 3 | 18.193829 | 6.0646 | 0.8985 |
| Pure Error | 4 | 27.000000 | 6.7500 | Prob > F |
| Total Error | 7 | 45.193829 | | 0.5157 |
| | | | | Max RSq |
| | | | | 0.9620 |

## Summary of Fit

| | |
|---|---|
| RSquare | 0.936427 |
| RSquare Adj | 0.918264 |
| Root Mean Sq Error | 2.540917 |
| Mean of Response | 82.1 |
| Observations (or Sum Wgts) | 10 |

## Parameter Estimates

| Term | Estimate | Std Error | t Ratio | Prob>\|t\| |
|---|---|---|---|---|
| Intercept | 35.657437 | 5.617927 | 6.35 | 0.0004 |
| x | 5.2628956 | 0.558022 | 9.43 | <.0001 |
| x*x | -0.127674 | 0.012811 | -9.97 | <.0001 |

## Effect Tests

| Source | Nparm | DF | Sum of Squares | F Ratio | Prob > F |
|---|---|---|---|---|---|
| x | 1 | 1 | 574.28553 | 88.9502 | <.0001 |
| x*x | 1 | 1 | 641.20451 | 99.3151 | <.0001 |

# Summary

- Comparison of Treatments – ANOVA
- Multivariate Analysis of Variance
- Regression Modeling

# Next Time

- Time Series Models
- Forecasting