SMA 6304 / MIT 2.853 / MIT 2.854
Manufacturing Systems
Lecture 3′: Some Discrete Distributions

Lecturer: Stanley B. Gershwin

Bernoulli

Flip a biased coin. If X^B is Bernoulli, then

Prob($X^B = 1$) = p.

Prob($X^B = 0$) = $1 - p$.

The sum of n Bernoulli random variables X_i^B is a binomial random variable X^b.

$$X^b = \sum_{i=0}^{n} X_i^B$$

$$\text{Prob}(X^b = x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{(1-x)}$$

Geometric

The number of Bernoulli random variables X_i^B tested until the first 1 appears is a geometric random variable X^g.

$$X^g = \min_i \{X_i^B = 1\}$$

To calculate Prob$(X^b = x)$,

Prob$(X^b = 1) = p$; Prob$(X^b > 1) = 1 - p$

$$\text{Prob}(X^b > x) = \text{Prob}(X^b > x | X^b > x - 1)\text{Prob}(X^b > x - 1)$$

$$= (1 - p)\text{Prob}(X^b > x - 1), \text{ so}$$

Prob$(X^b > x) = (1 - p)^x$ and Prob$(X^b = x) = (1 - p)^{x-1}p$

Poisson Distribution

$$\text{Prob}(X^P = x) = e^{-\lambda} \frac{\lambda^x}{x!}$$

Discussion later.

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Lecture 4: Continuous Random Variables

Lecturer: Stanley B. Gershwin

Slides by Larry Wein and Stan Gershwin

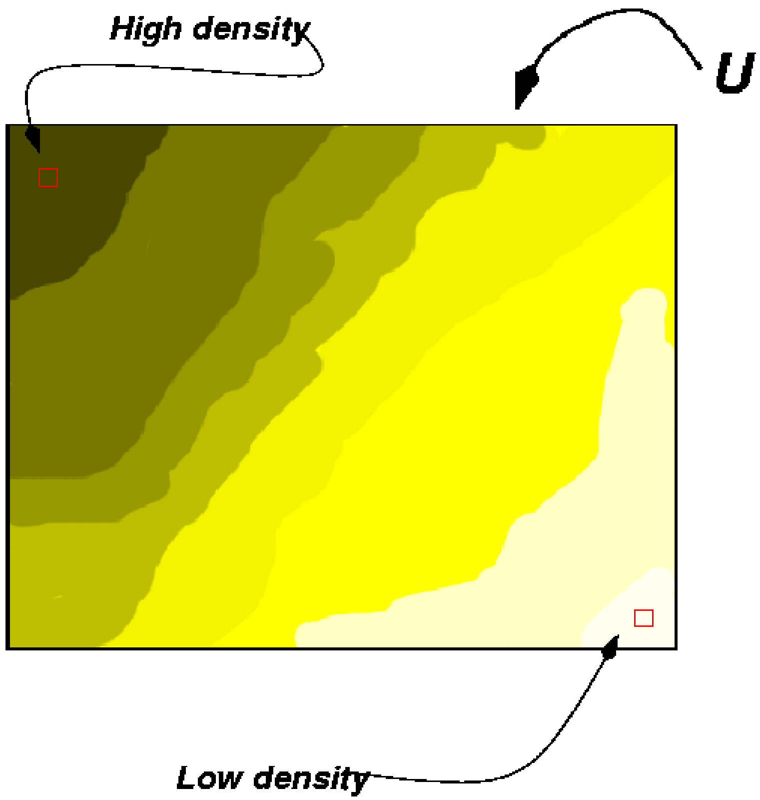
Philosophical issues

1. Mathematically, continuous and discrete random variables are very different.
2. *Quantitatively* , however, some continuous models are very close to some discrete models.
3. Therefore, which kind of model to use for a given system is a matter of *convenience* .

Example: The production process for small metal parts (nuts, bolts, washers, etc.) might better be modeled as a continuous flow than a large number of discrete parts.

Continuous random variables

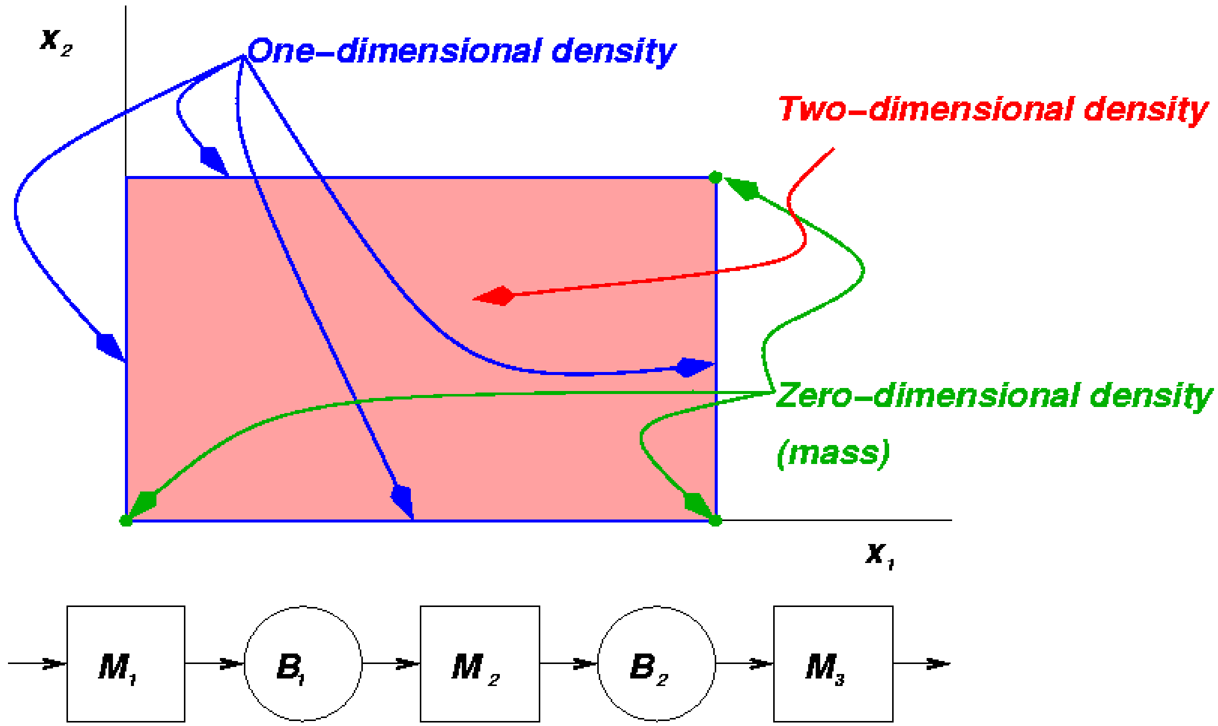
Probability density



The probability of a two-dimensional random variable being in a small square is the *probability density* times the area of the square. (Actually, it is more general than this.)

Probability density

- Continuous random variables can be defined
  - ⋆ in one, two, three, ..., infinite dimensional spaces;
  - ⋆ in finite or infinite regions of the spaces.
- Continuous random variables can have
  - ⋆ probability measures with the same dimensionality as the space;
  - ⋆ lower dimensionality than the space;
  - ⋆ a mix of dimensions.

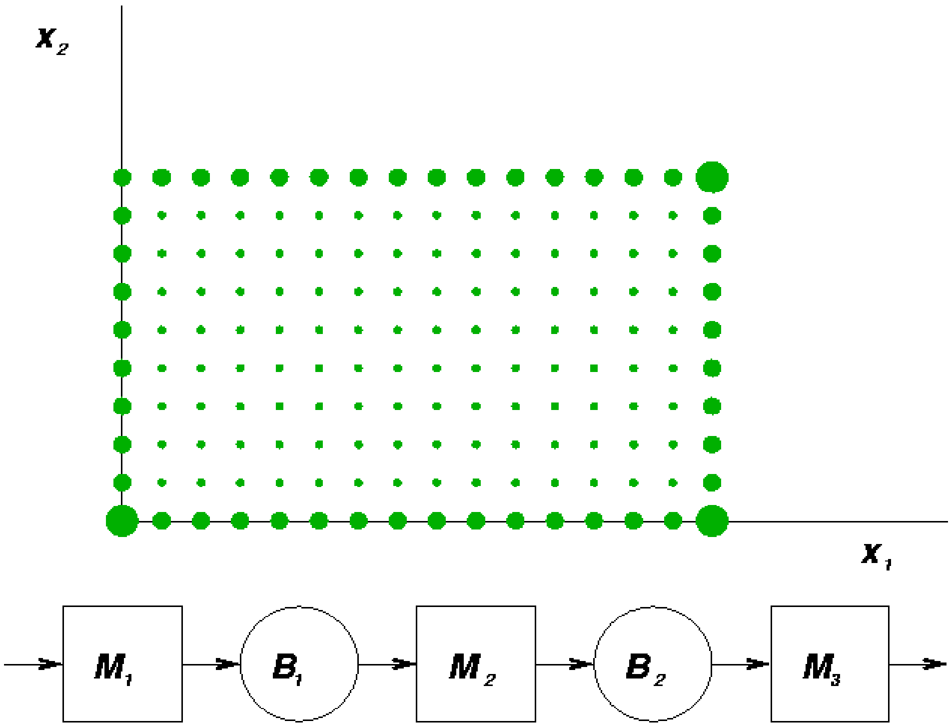


Probability distribution of the amount of material in each of the two buffers.

Continuous random variables

Spaces

Discrete approximation



Probability distribution of the amount of material in each of the two buffers.

Continuous Random Variables

cumulative distribution function (cdf) is

$$F(t) = P(X \le t) \text{ for all } t$$

probability density function (pdf) is

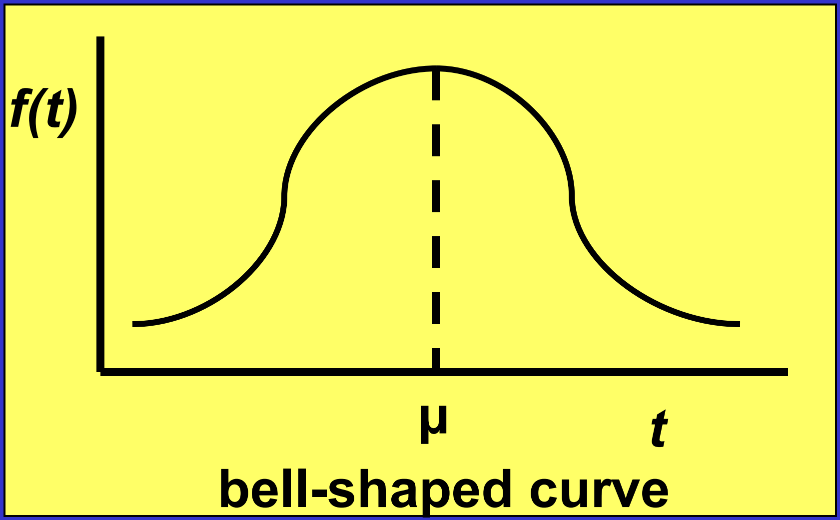
$$f(t) = \frac{dF(t)}{dt}$$

$$\Pr(a \le X \le b) = \int_a^b f(t)\,dt = F(b) - F(a)$$

E[X] and *VAR(X)* are similar to discrete case, except you replace sums by integrals

$$E(X) = \int x f(x)\,dx$$

Normal (or Gaussian) Distribution



$$f(t) = \frac{1}{\sqrt{2\pi\sigma}} e^{-(t-\mu)^2 / 2\sigma^2}$$

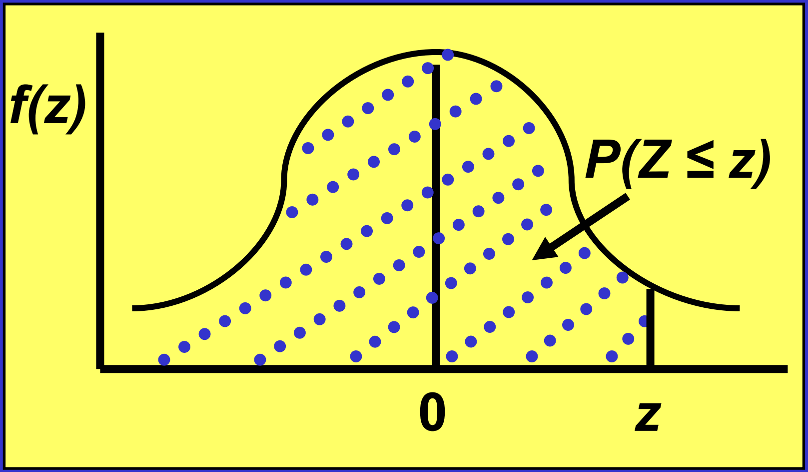
mean = μ **variance = σ^2**

X **is a** ***N(μ,σ)*** **random variable**

Fact: if *X* and *Y* are normal, so is *aX+bY+c*

Statistics books have tables for *Z = N(0,1)*

Fact: if X is *N(μ,σ)*, then $\frac{X-\mu}{\sigma}$ is *N(0,1)*



Example:

IQ test scores are normally distributed with $\mu = 100$ and $\sigma = 10$

What is $P(X > 125)$?

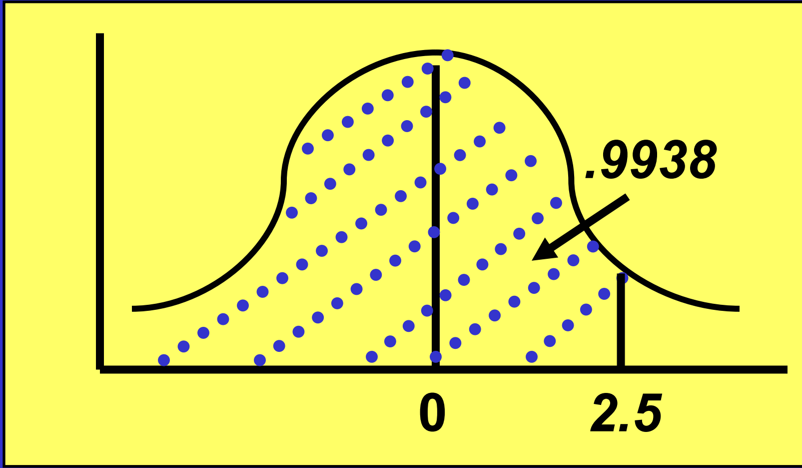
$$= P\left(\frac{X - 100}{10} > \frac{125 - 100}{10}\right)$$

$$= P(Z > 2.5)$$

$$= 1 - P(Z \leq 2.5)$$

$$= 1 - .9938$$

$$= .0062$$



Example:

Manufacturing cycle times are normal with μ = *100* days and σ = *10* days

You want to quote delivery lead times (= delivery date - current date) so that you achieve 90% on-time delivery

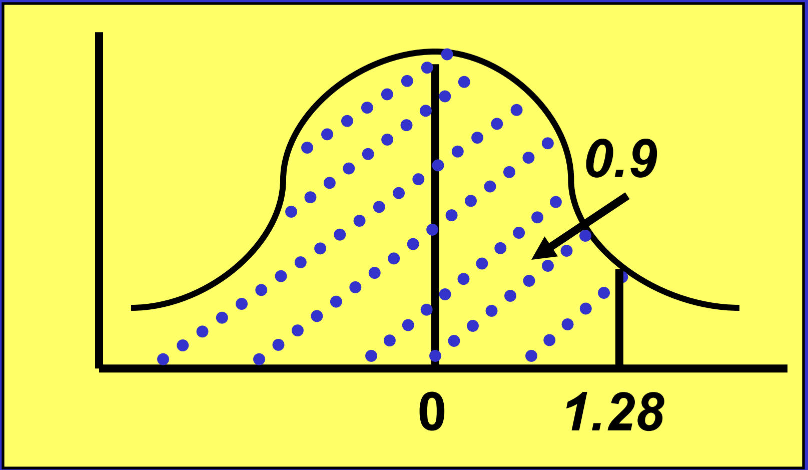
Q: What delivery lead time should you quote?

Choose *x* so that *P(X > x) = .1*

or $$P\left(\frac{X-100}{10} > \frac{x-100}{10}\right) = .1$$

$$P\left(Z > \frac{x-100}{10}\right) = .1$$

or $$P\left(Z \leq \frac{x-100}{10}\right) = .9$$



$P(Z \leq 1.28) = .9$ so $\frac{x-100}{10} = 1.28$ ***X= 112.8***

Central Limit Theorem

$X_1, \ldots, X_n$ are independent and identically distributed with

$E[X_i] = \mu \quad VAR(X_i) = \sigma^2$

X_i's are not normal!

Let $S_n = X_1 + \ldots + X_n$

Central Limit Theorem for sum: if n is large, then S_n is approximately normal with mean $n\mu$ and standard deviation $\sigma\sqrt{n}$

Let $m_n = \dfrac{X_1 + \ldots + X_n}{n}$

Central Limit Theorem for mean: if n is large, then m_n is approximately normal with mean μ and standard deviation $\dfrac{\sigma}{\sqrt{n}}$

Dice Graphs

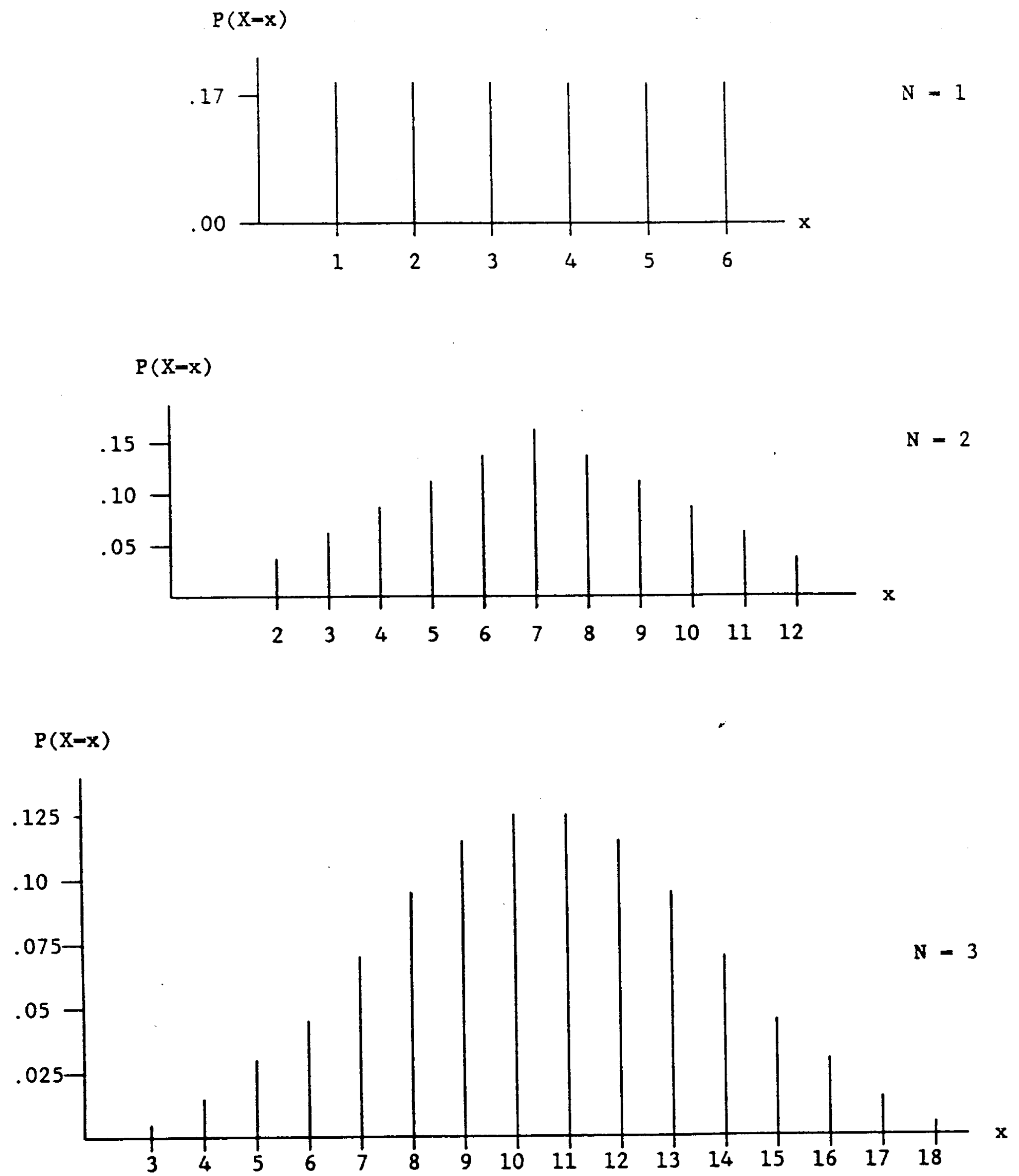
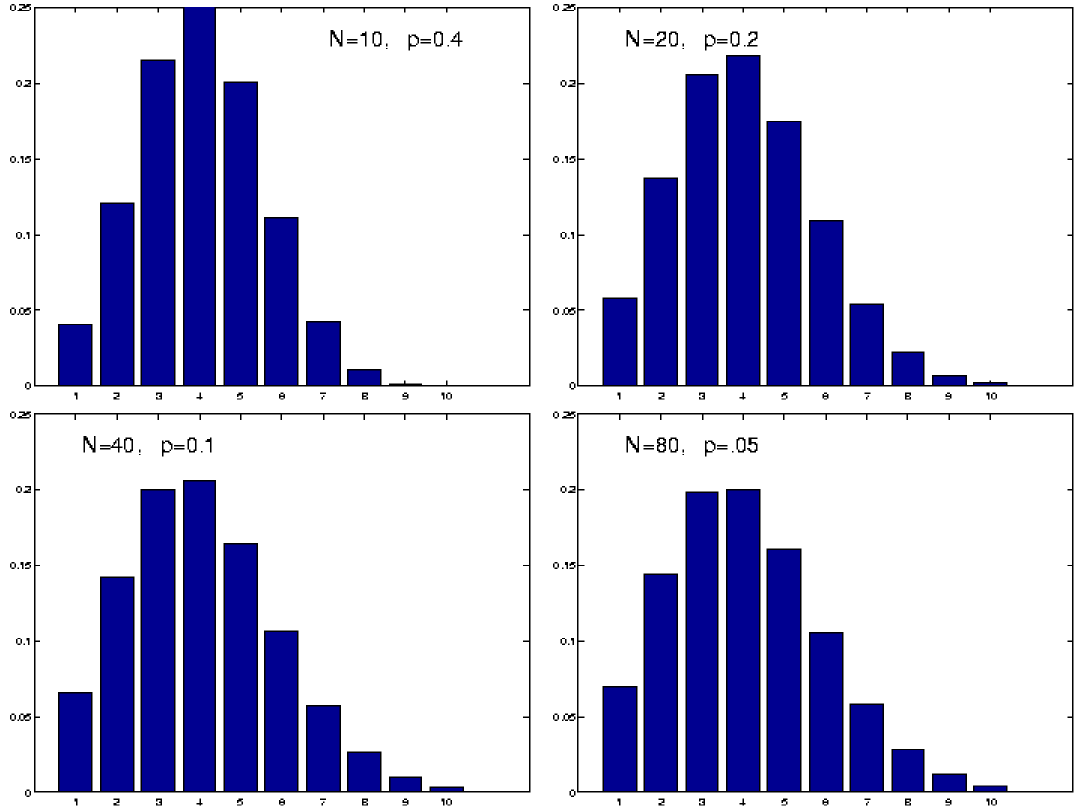


Figure 1. Probability distribution function for the sum of numbers, X , obtained when rolling N dice [N = 1, 2, 3] .

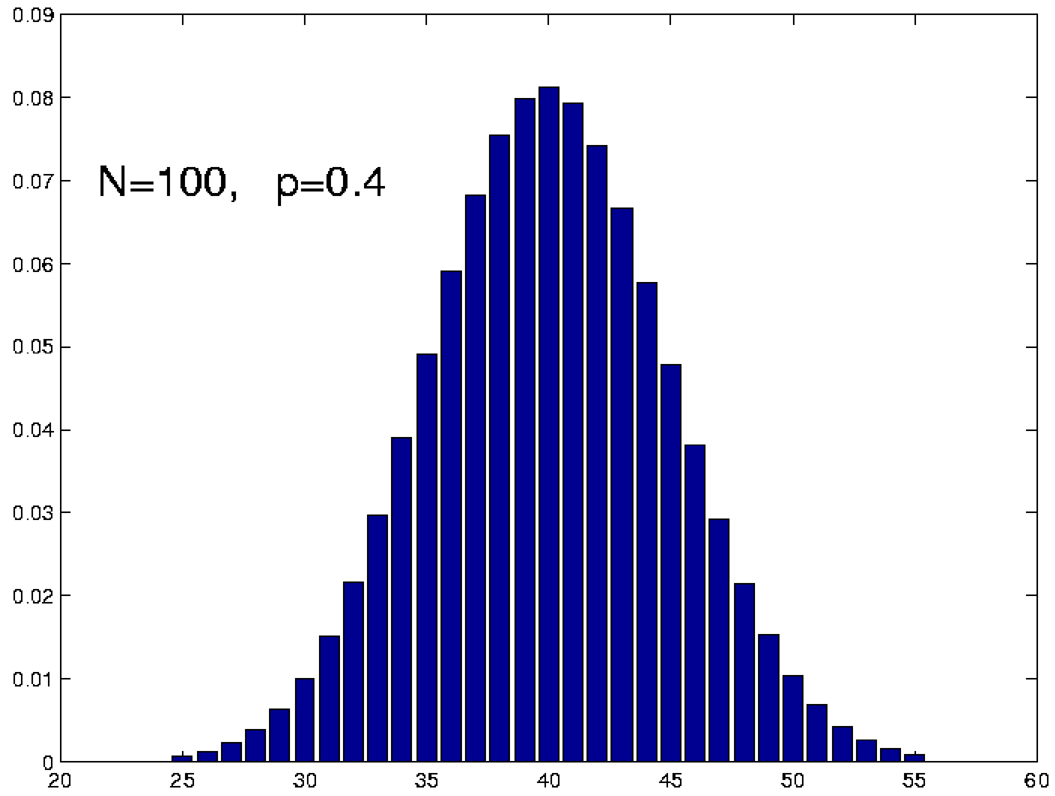
Binomial distributions

Why are these distributions so similar?



Binomial distributions

Binomial for large N approaches normal.

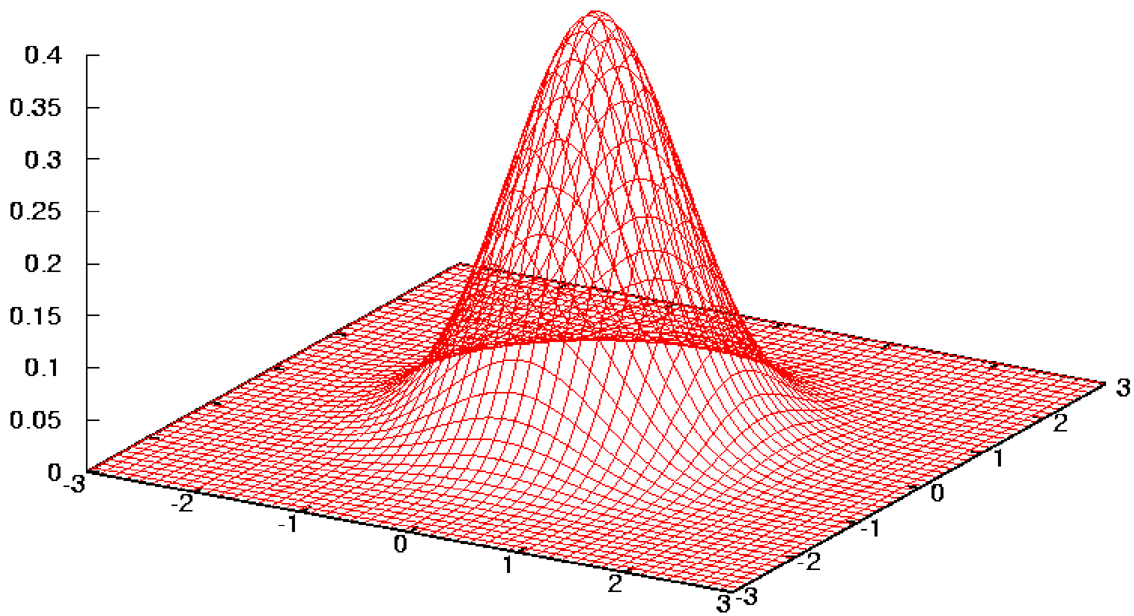


Normal distribution has 3 uses

1) Models many physical processes
2) Sum of normal random variables
3) Sum or mean of many iid random variables

Normal Density Function

... in Two Dimensions



Uniform

$$f(x) = \frac{1}{b-a} \text{ for } a \leq x \leq b$$

$$f(x) = 0 \text{ otherwise}$$

Exponential

- $f(t) = \lambda e^{-\lambda t}$ for $t \geq 0$; $f(t) = 0$ otherwise;
  Prob($T > t$) = $e^{-\lambda t}$ for $t \geq 0$
- Same as the geometric distribution but for continuous time.
- *Very* mathematically convenient. Often used as model for the first time until an event occurs.
- Memorylessness:
  Prob($T > t + x | T > x$) = Prob($T > t$)

$$\text{Prob}(X^P = x) = e^{-\lambda t}\frac{(\lambda t)^x}{x!}$$

is the probability that x events happen in $[0, t]$ if the events are independent and the times between them are exponentially distributed with parameter λ.

Typical examples: arrivals and services at queues.