SMA 6304 / MIT 2.853 / MIT 2.854
Manufacturing Systems
Lecture 19-20: Single-part-type, multiple stage systems

Lecturer: Stanley B. Gershwin

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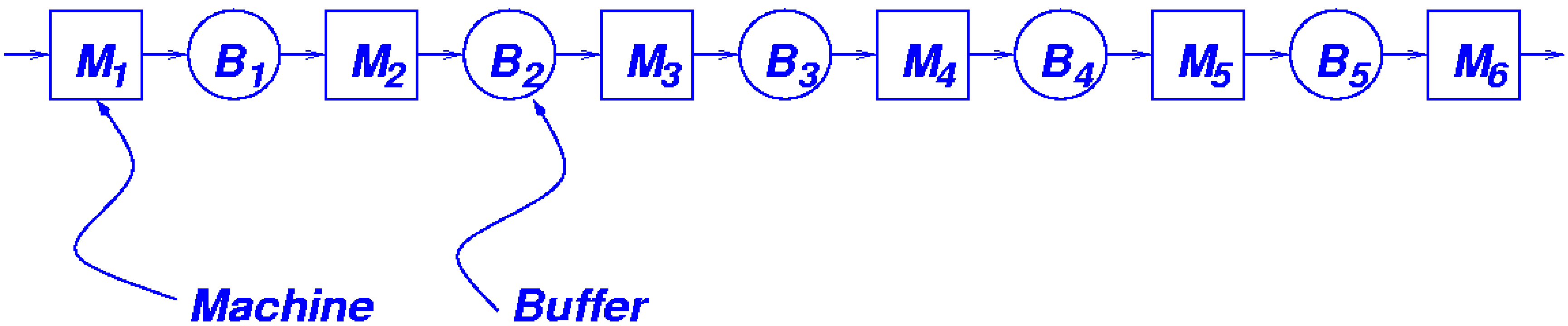
Flow Line

... also known as a Production or Transfer Line.



- Machines are unreliable.
- Buffers are finite.

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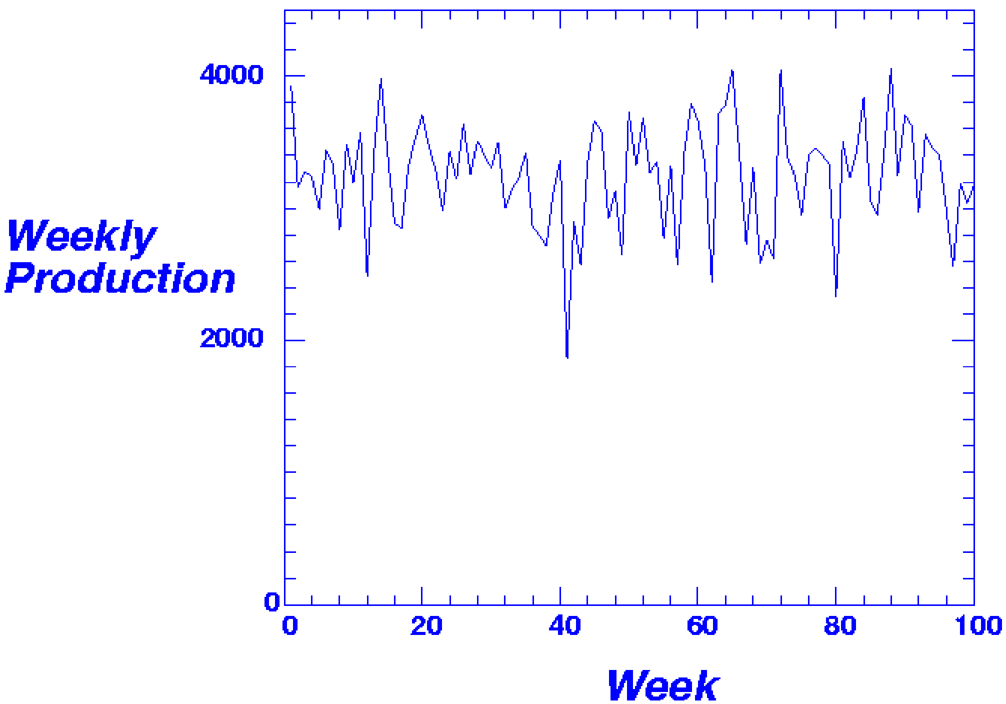
Flow Line

Output Variability



Production output from a simulation of a transfer line.

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Single Reliable Machine

- If the machine is perfectly reliable, and its average operation time is τ, then its maximum production rate is $1/\tau$.
- *Note:*
  - ⋆ Sometimes *cycle time* is used instead of *operation time* , but *BEWARE:* cycle time has two meanings!
  - ⋆ The other meaning is the time a part spends in a system. If the system is a single, reliable machine, the two meanings are the same.

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Single Reliable Machine

- Operation-Dependent Failures
  - ⋆ A machine can only fail while it is working.
  - ⋆ *IMPORTANT!* MTTF *must* be measured in working time!
  - ⋆ This is the usual assumption.
- *Note:* MTBF = MTTF + MTTR

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Single Reliable Machine

Production rate

- If the machine is unreliable, and
  - ⋆ its average operation time is τ,
  - ⋆ its mean time to fail is MTTF,
  - ⋆ its mean time to repair is MTTR,

then its maximum production rate is

$$\frac{1}{\tau}\left(\frac{\text{MTTF}}{\text{MTTF}+\text{MTTR}}\right)$$

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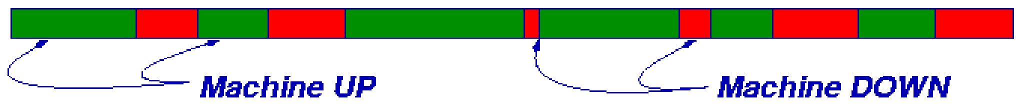
Single Reliable Machine

Production rate

Proof



- Average production rate, while machine is up, is $1/\tau$.
- Average duration of an up period is MTTF.
- Average production during an up period is MTTF/τ.
- Average duration of up-down period: MTTF + MTTR.
- Average production during up-down period: MTTF/τ.
- Therefore, average production rate is
  (MTTF/τ)/(MTTF + MTTR).

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- *Assumptions:* Operation time is constant (τ). Failure and repair times are *geometrically* distributed.
- Let p be the probability that a machine fails during any given operation. Then $p = \tau/\text{MTTF}$.

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Single Reliable Machine

Geometric Up- and Down-Times

- Let r be the probability that M gets repaired in during any operation time when it is down. Then $r = \tau/\text{MTTR}$.
- Then the *average production rate* of M is

$$\frac{1}{\tau}\left(\frac{r}{r+p}\right).$$

- *(Sometimes we forget to say "average.")*

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- So far, the machine really has *three* production rates:
  - $1/\tau$ when it is up *(short-term capacity)* ,
  - 0 when it is down *(short-term capacity)* ,
  - $(1/\tau)(r/(r+p))$ on the average *(long-term capacity)* .

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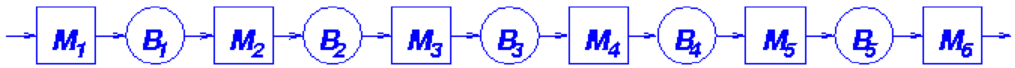
Infinite-Buffer Line



Assumptions:

- A machine is not idle if it is not starved.
- The first machine is never starved.

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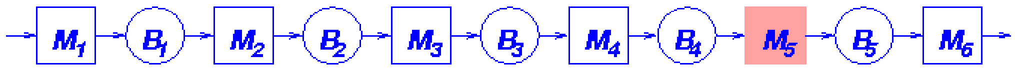
Infinite-Buffer Line



- The production rate of the line is the production rate of the *slowest* machine in the line — called the *bottleneck* .
- *Slowest* means least average production rate, where average production rate is calculated from one of the previous formulas.

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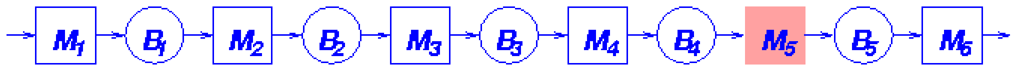
Infinite-Buffer Line



- Production rate is therefore

$$P = \min_i \frac{1}{\tau_i} \left(\frac{\text{MTTF}_i}{\text{MTTF}_i + \text{MTTR}_i} \right)$$

- and M_i is the bottleneck.

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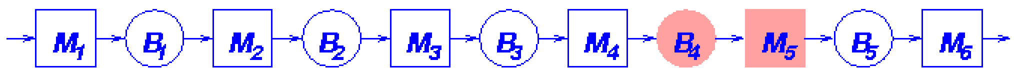
Infinite-Buffer Line



- The system is not in steady state.
- An infinite amount of inventory accumulates in the buffer upstream of the bottleneck.
- A finite amount of inventory appears downstream of the bottleneck.

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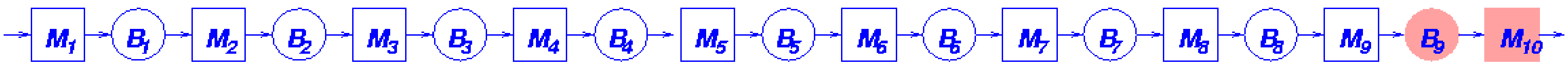
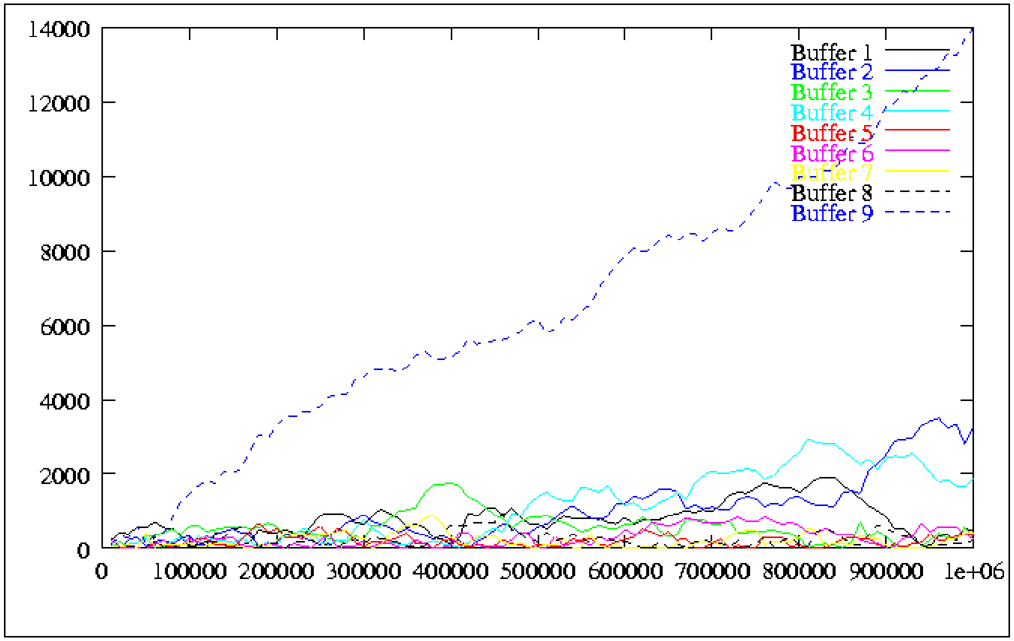
Infinite-Buffer Line





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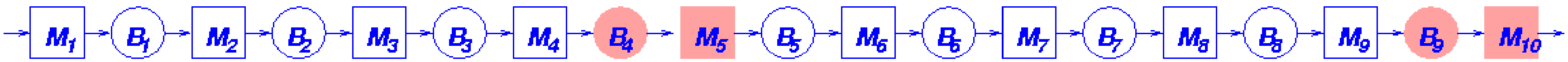
Infinite-Buffer Line



- The *second bottleneck* is the slowest machine upstream of the bottleneck. An infinite amount of inventory accumulates just upstream of it.
- A finite amount of inventory appears between the second bottleneck and the machine upstream of the first bottleneck.
- Et cetera.

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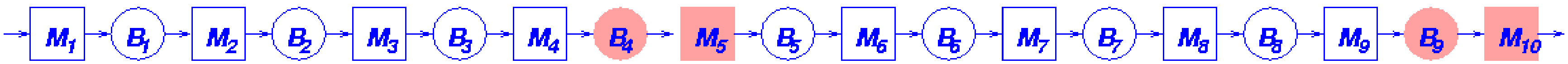
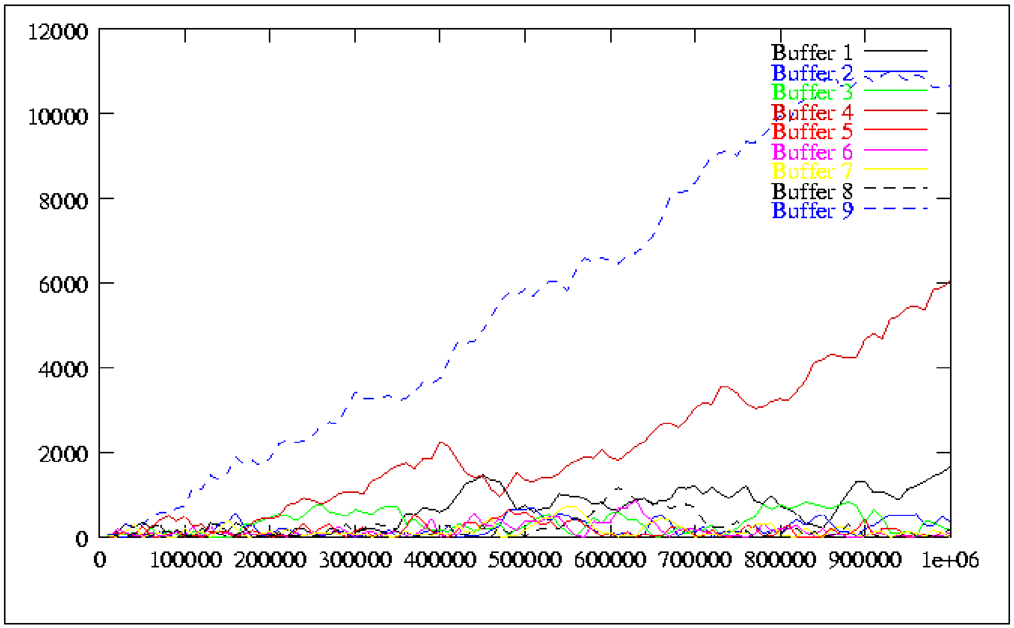
Infinite-Buffer Line





A 10-machine line with bottlenecks at Machines 5 and 10.

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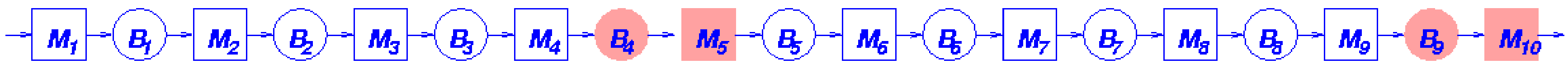
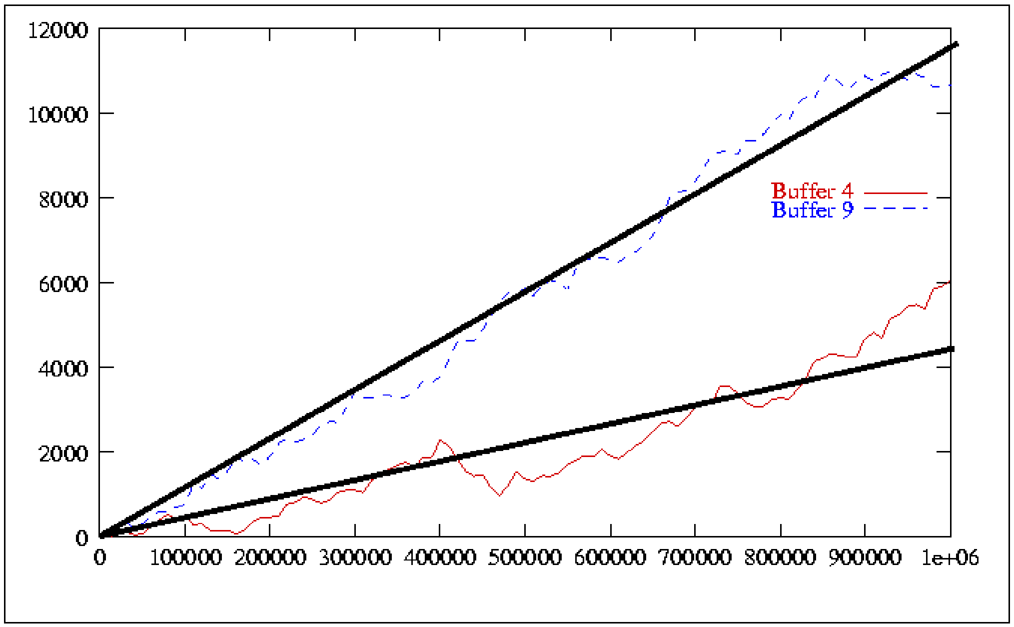
Infinite-Buffer Line





Question:

- What are the slopes *(roughly!)* of the two indicated graphs?

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Infinite-Buffer Line

Questions:

- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?

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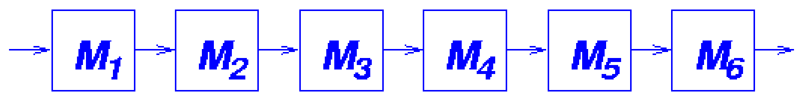
Zero-Buffer Line



- If any one machine fails, or takes a very long time to do an operation, *all* the other machines must wait.
- Therefore the production rate is usually less — possibly much less – than the slowest machine.

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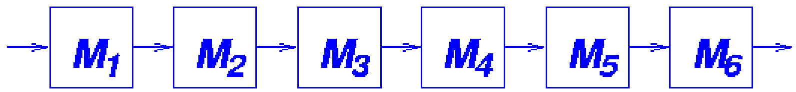
Zero-Buffer Line



- *Example:* Constant, unequal operation times, perfectly reliable machines.
  - ⋆ The operation time of the line is equal to the operation time of the slowest machine, so the production rate of the line is *equal to* that of the slowest machine.

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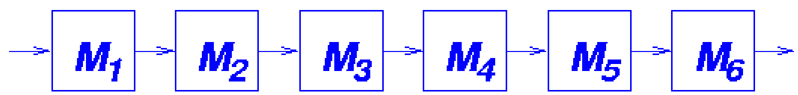
Zero-Buffer Line

Constant, equal operation times, unreliable machines



- *Assumption:* Failure and repair times are *geometrically* distributed.
- Define $p_i = \tau/\text{MTTF}_i$ = probability of failure during an operation.
- Define $r_i = \tau/\text{MTTR}_i$ probability of repair during an interval of length τ when the machine is down.

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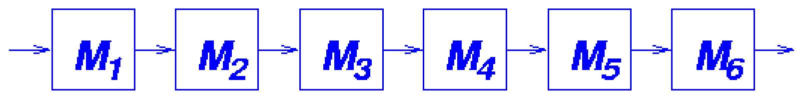
Zero-Buffer Line

Constant, equal operation times, unreliable machines



Buzacott's Zero-Buffer Line Formula:

Let k be the number of machines in the line. Then

$$P = \frac{1}{\tau} \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$

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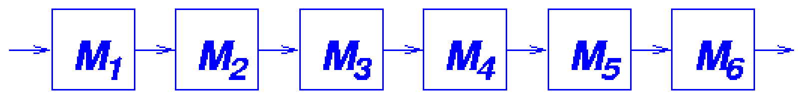
Zero-Buffer Line

Constant, equal operation times, unreliable machines



- Same as the earlier formula (page 6, page 9) when $k = 1$. The *isolated production rate* of a single machine M_i is

$$\frac{1}{\tau}\left(\frac{1}{1+\frac{p_i}{r_i}}\right) = \frac{1}{\tau}\left(\frac{r_i}{r_i+p_i}\right).$$

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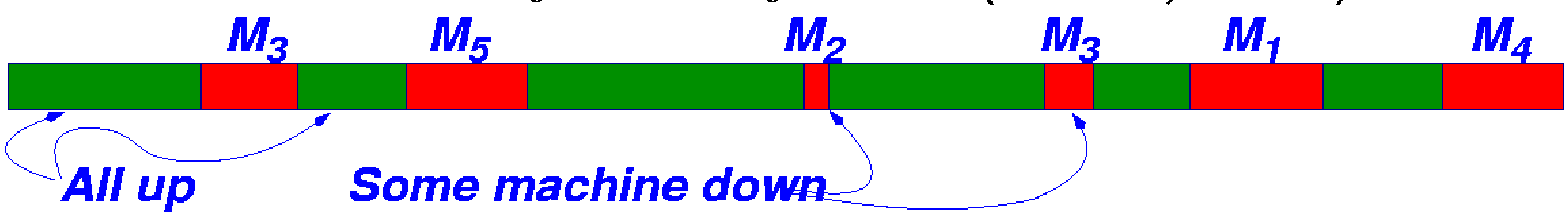
Zero-Buffer Line

Proof of formula

- Let τ (the operation time) be the time unit.
- *Assumption:* At most, <u>one</u> machine can be down.
- Consider a long time interval of length $T\tau$ during which Machine M_i fails m_i times ($i = 1, \ldots k$).



- Without failures, the line would produce T parts.

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Zero-Buffer Line

Proof of formula

- The average repair time of M_i is τ / r_i each time it fails, so the total system down time is close to

$$D\tau = \sum_{i=1}^{k} \frac{m_i \tau}{r_i}$$

where D is the number of operation times in which a machine is down.

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Zero-Buffer Line

Proof of formula

- The total up time is approximately

$$U\tau = T\tau - \sum_{i=1}^{k} \frac{m_i \tau}{r_i}.$$

- where U is the number of operation times in which all machines are up.

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Zero-Buffer Line

- Since the system produces one part per time unit while it is working, it produces U parts during the interval of length $T\tau$.
- Note that, approximately,

$$m_i = p_i U$$

because M_i can only fail while it is operational.

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Zero-Buffer Line

Proof of formula

- Thus,

$$U\tau = T\tau - U\tau \sum_{i=1}^{k} \frac{p_i}{r_i},$$

or,

$$\frac{U}{T} = E_{ODF} = \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$

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Zero-Buffer Line

Proof of formula

and

$$P = \frac{1}{\tau} \frac{1}{1 + \sum_{i=1}^{k} \frac{p_i}{r_i}}$$

- Note that P is a function of the *ratio* p_i/r_i and not p_i or r_i separately.
- The same statement is true for the infinite-buffer line.
- However, the same statement is *not* true for a line with finite, non-zero buffers.

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Zero-Buffer Line

Questions:

- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?

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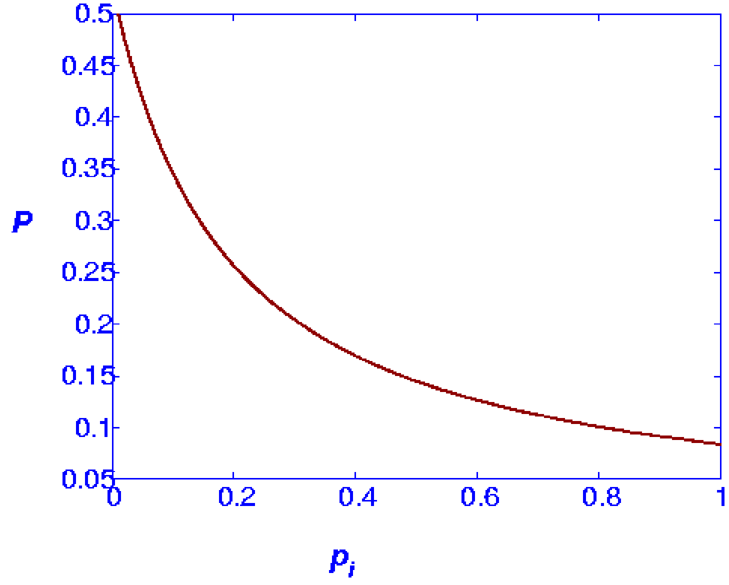
Zero-Buffer Line

P as a function of p_i

All machines are the same except M_i. As p_i increases, the production rate decreases.



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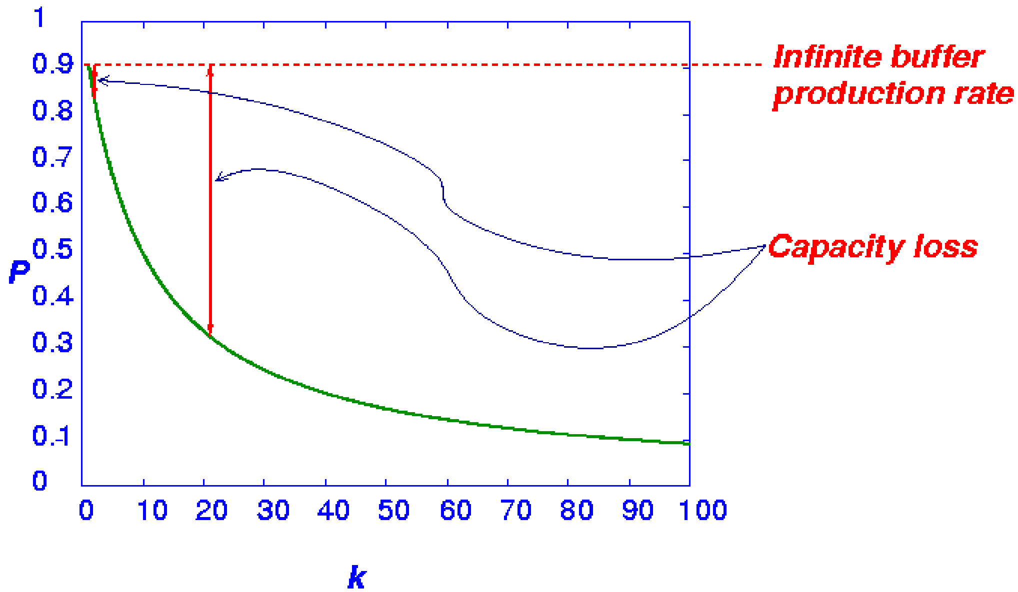
Zero-Buffer Line

P as a function of k

All machines are the same. As the line gets longer, the production rate decreases.



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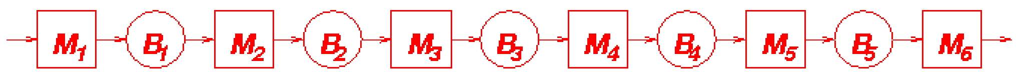
Finite-Buffer Lines



- Motivation for buffers: recapture some of the lost production rate.
- Cost
  - ⋆ in-process inventory/lead time
  - ⋆ floor space
  - ⋆ material handling mechanism

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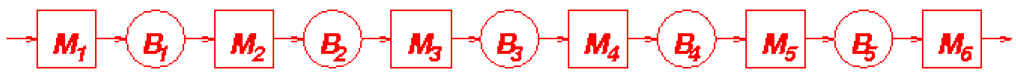
Finite-Buffer Lines



- Infinite buffers: no propagation of disruptions.
- Zero buffers: instantaneous propagation.
- Finite buffers: delayed propagation.
  - ★ New phenomena: *blockage* and *starvation* .

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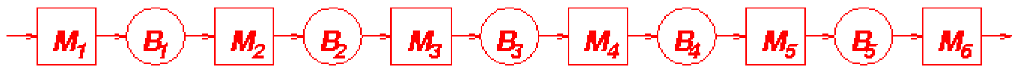
Finite-Buffer Lines



- Difficulty:
  - ★ No simple formula for calculating production rate or inventory levels.
- Solution:
  - ★ Simulation
  - ★ Analytical approximation

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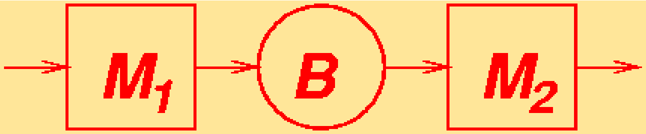
Two Machine, Finite-Buffer Lines



- Exact solution *is* available to Markov process model.
- *Discrete time-discrete state Markov process:*

$$\text{prob}\{X(t+1) = x(t+1) | X(t) = x(t),$$

$$X(t-1) = x(t-1), | X(t-1) = x(t-1), ...\} =$$

$$\text{prob}\{X(t+1) = x(t+1) | X(t) = x(t)\}$$

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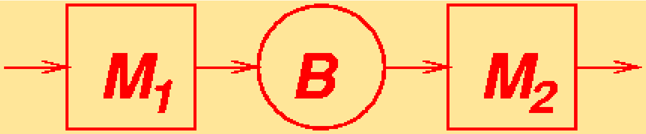
Two Machine, Finite-Buffer Lines



Here, $X(t) = (n(t), \alpha_1(t), \alpha_2(t))$, where

- n is the number of parts in the buffer; $n = 0, 1, ..., N$.
- α_i is the repair state of M_i; $i = 1, 2$.
  - $\star$ $\alpha_i = 1$ means the machine is *up* or *operational*;
  - $\star$ $\alpha_i = 0$ means the machine is *down* or *under repair*.

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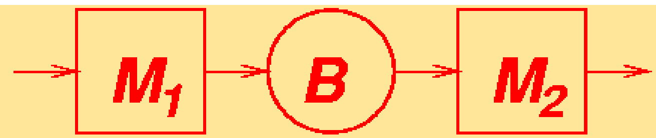
Two Machine, Finite-Buffer Lines



Several models available:

- *Deterministic processing time* , or *Buzacott model:* deterministic processing time, geometric failure and repair times; discrete state, discrete time.

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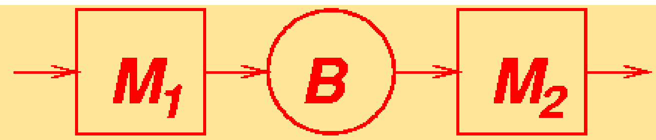
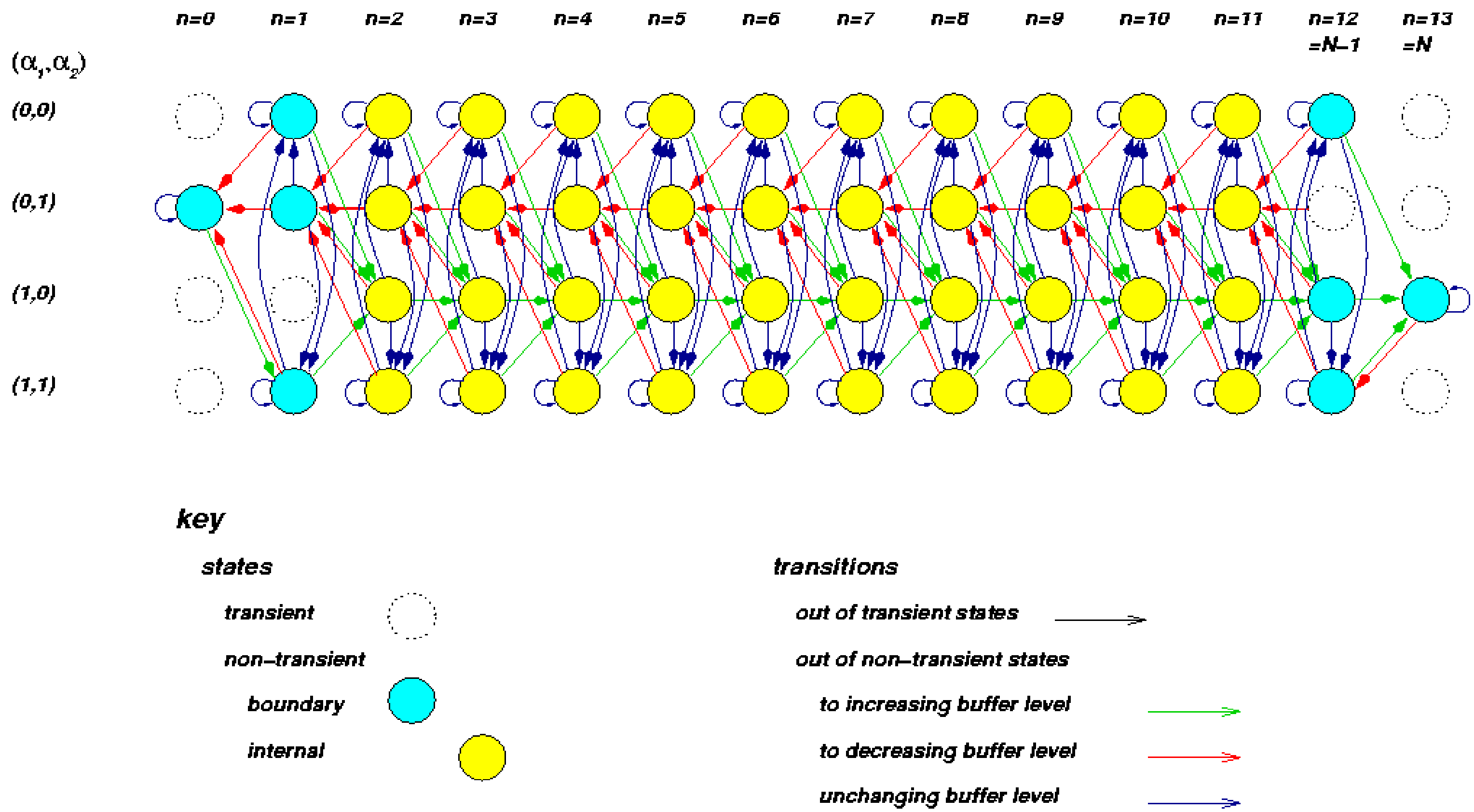
Two Machine, Finite-Buffer Lines





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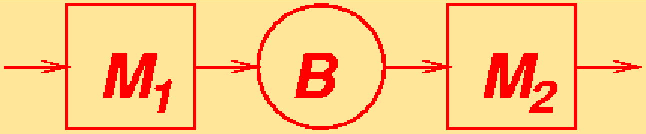

- *Exponential processing time:* exponential processing, failure, and repair time; discrete state, continuous time.
- *Continuous material,* or *fluid:* deterministic processing, exponential failure and repair time; mixed state, continuous time.

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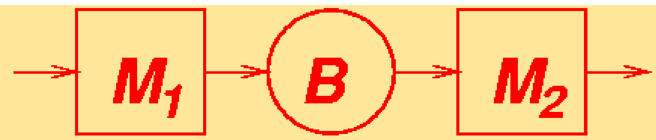
Two Machine, Finite-Buffer Lines

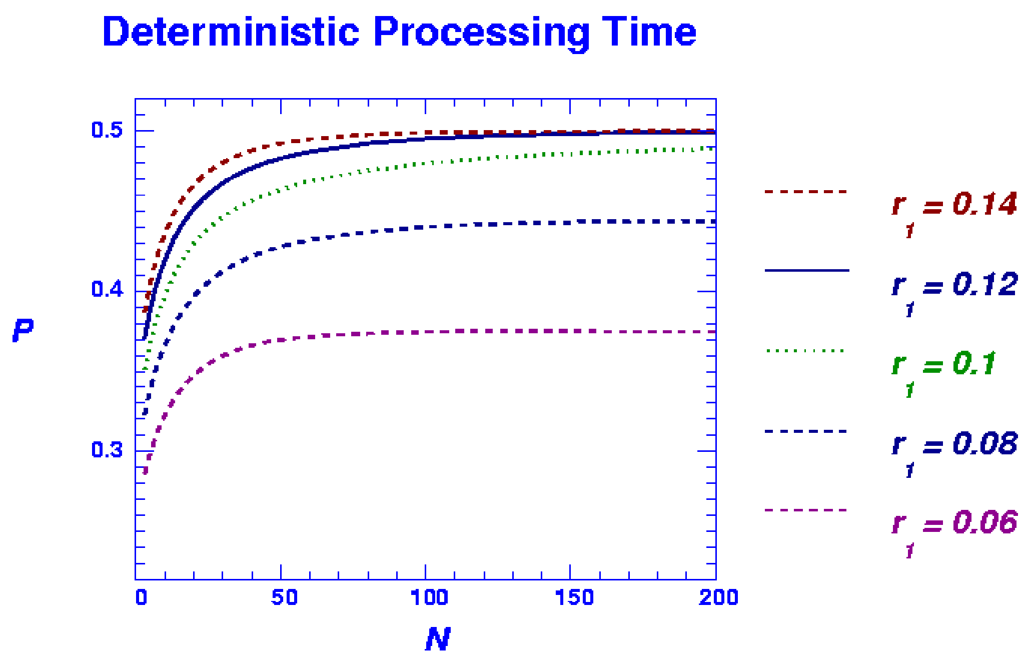


$\tau = 1.$

$p_1 = .1$

$r_2 = .1$

$p_2 = .1$



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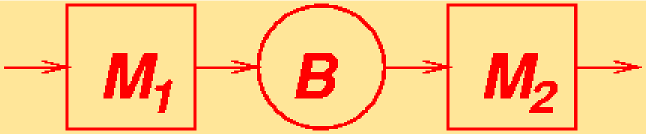
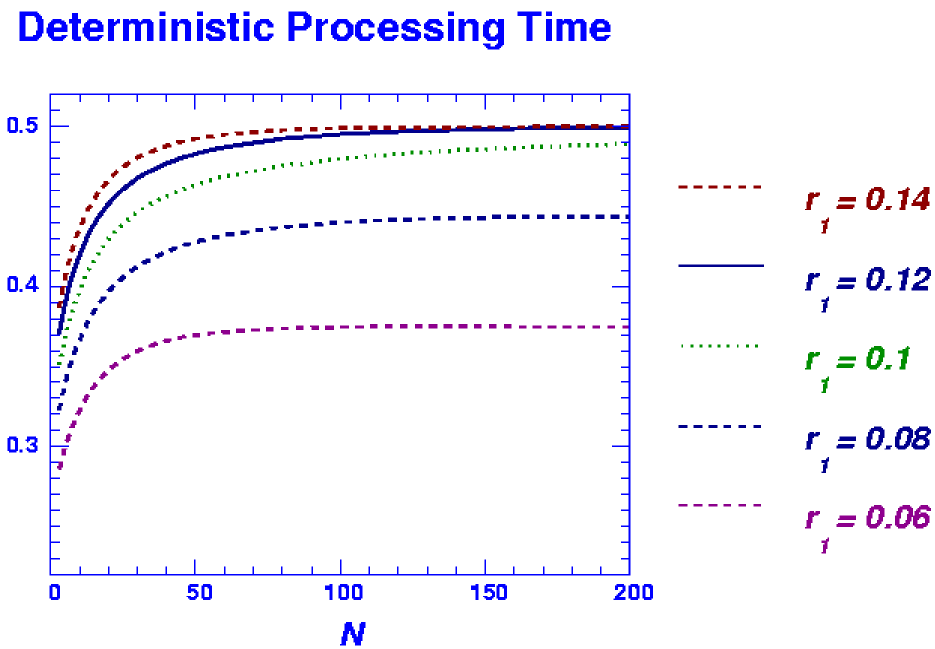


Discussion:

- Why are the curves increasing?
- Why do they reach an asymptote?
- What is P when $N = 0$?
- What is the limit of P as $N \to \infty$?
- Why are the curves with smaller r_1 lower?



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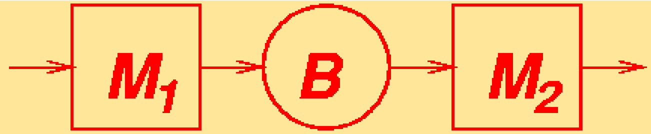
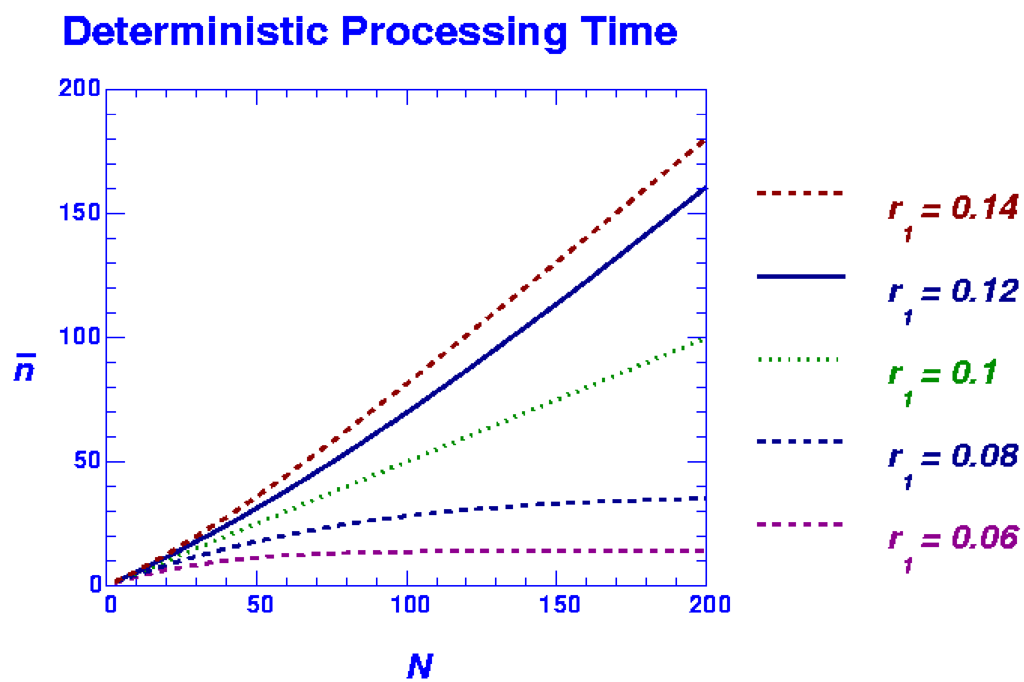
Two Machine, Finite-Buffer Lines



Discussion:

- Why are the curves increasing?
- Why *different* asymptotes?
- What is $\bar{n}$ when $N = 0$?
- What is the limit of $\bar{n}$ as $N \to \infty$?
- Why are the curves with smaller r_1 lower?



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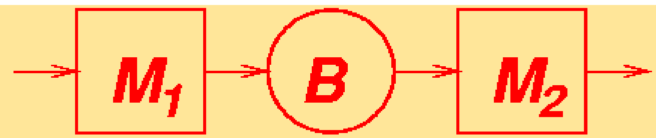
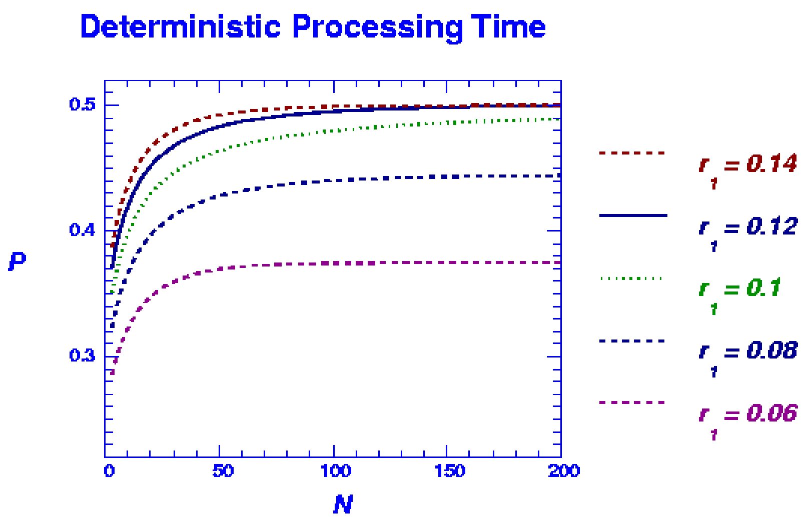
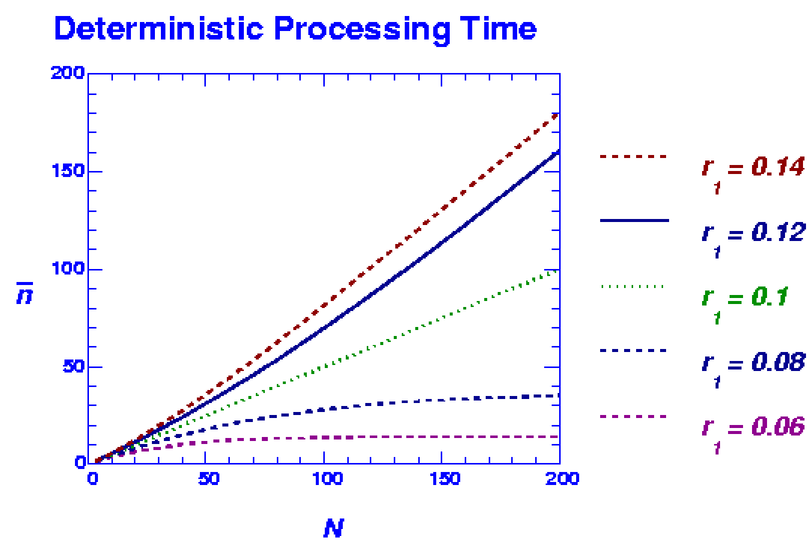
Two Machine, Finite-Buffer Lines







- *What can you say about the optimal buffer size?*
- *How should it be related to r_i, p_i?*

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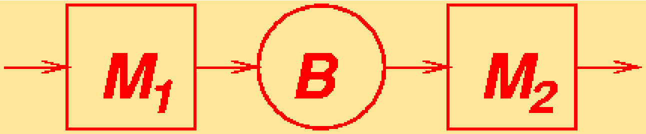
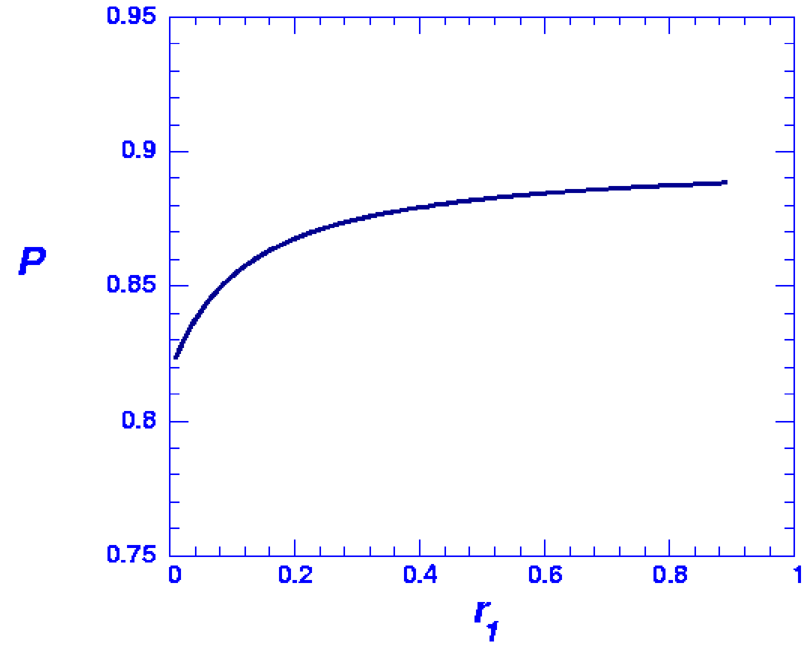
Two Machine, Finite-Buffer Lines



Should we prefer short, frequent, disruptions or long, infrequent, disruptions?

- $r_2 = 0.8$, $p_2 = 0.09$, $N = 10$
- r_1 and p_1 vary together and $\frac{r_1}{r_1+p_1} = .9$
- *Answer:* evidently, short, frequent failures.
- *Why?*



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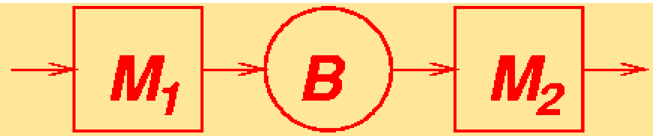
Two Machine, Finite-Buffer Lines



Questions:

- If we want to increase production rate, which machine should we improve?
- What would happen to production rate if we improved any other machine?

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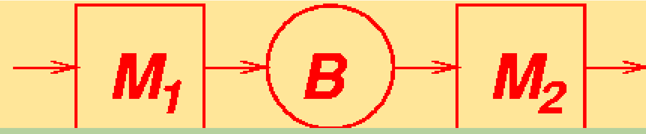
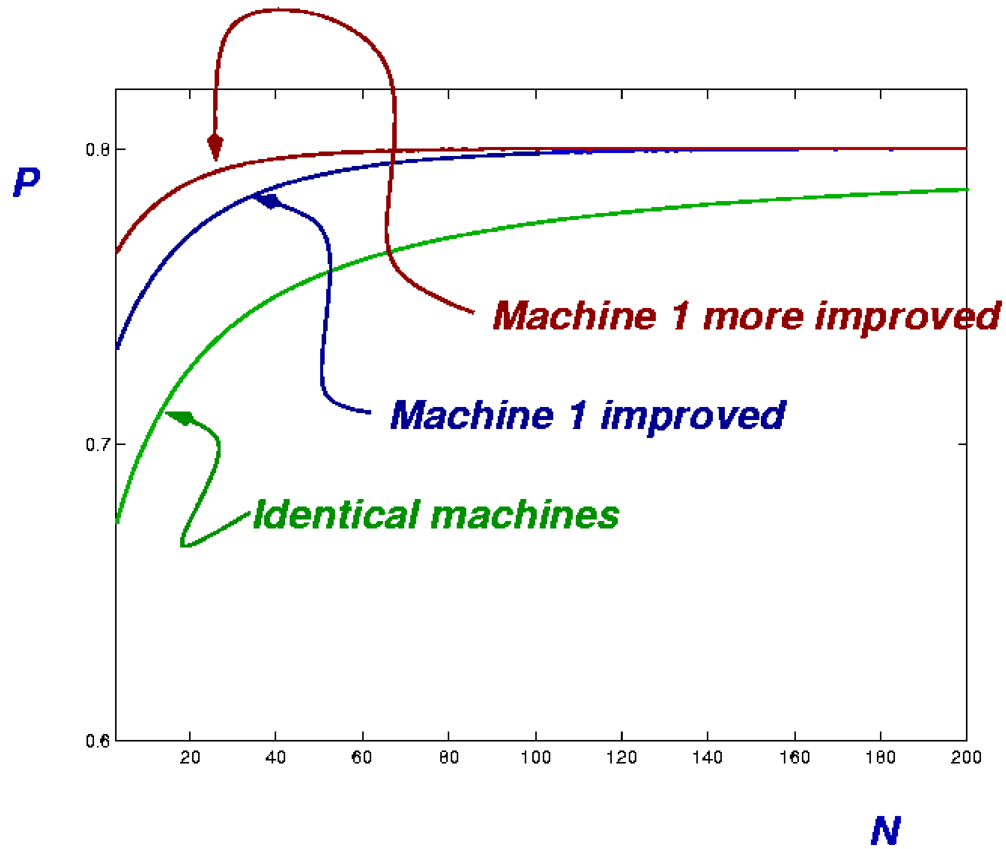
Two Machine, Finite-Buffer Lines



Production rate vs. storage space

Improvements to *non-bottleneck* machine.



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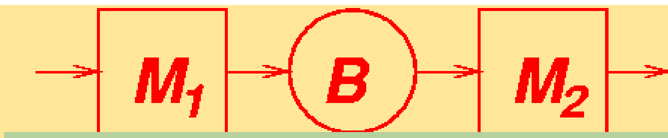
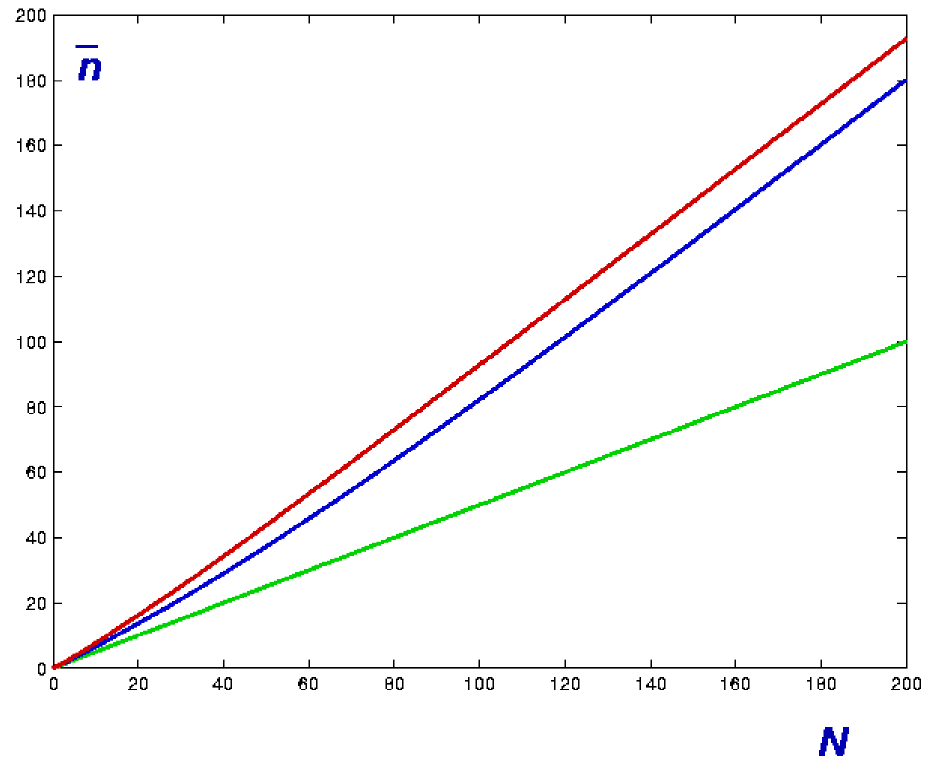
Two Machine, Finite-Buffer Lines



Avg. inventory vs. storage space

- Inventory increases as the (non-bottleneck) *upstream* machine is improved and as the buffer space is increased.
- If the *downstream* machine were improved, the inventory would be less and it would increase much less as the space increases.



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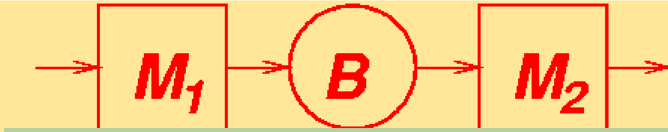
Two Machine, Finite-Buffer Lines



Other models

Exponential — discrete material, continuous time

- $\mu_i \delta t$ = the probability that M_i completes an operation in $(t, t + \delta t)$;
- $p_i \delta t$ = the probability that M_i fails during an operation in $(t, t + \delta t)$;
- $r_i \delta t$ = the probability that M_i is repaired, while it is down, in $(t, t + \delta t)$;

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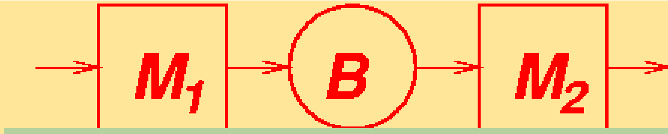
Two Machine, Finite-Buffer Lines



Other models

Continuous — continuous material, continuous time

- $\mu_i \delta t$ = the amount of material that M_i processes, while it is up, in $(t, t + \delta t)$;
- $p_i \delta t$ = the probability that M_i fails, while it is up, in $(t, t + \delta t)$;
- $r_i \delta t$ = the probability that M_i is repaired, while it is down, in $(t, t + \delta t)$;

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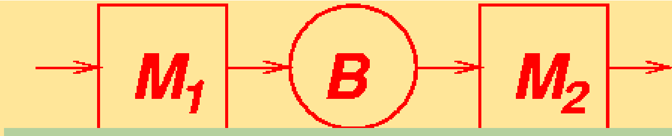
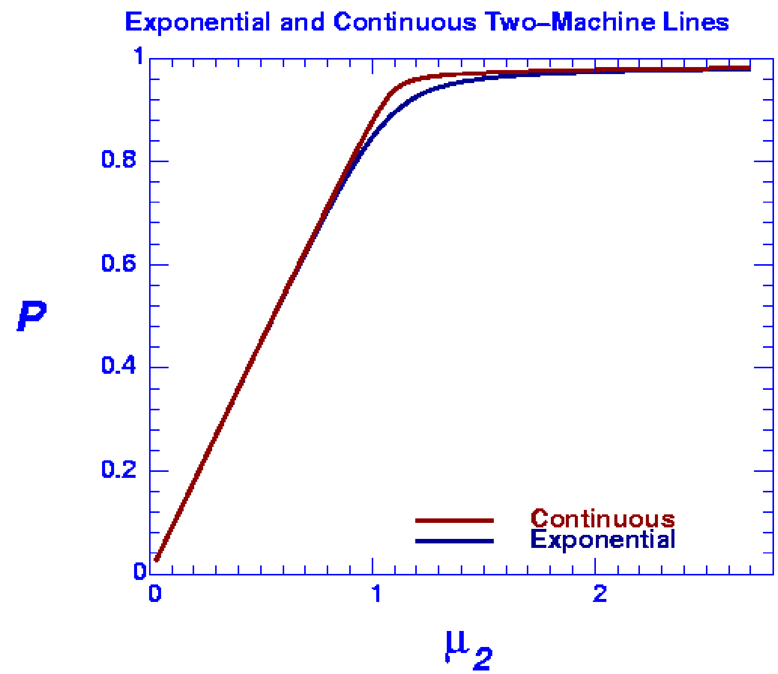
Two Machine, Finite-Buffer Lines



Other models

- $r_1 = 0.09$, $p_1 = 0.01$, $\mu_1 = 1.1$
- $r_2 = 0.08$, $p_1 = 0.009$
- $N = 20$
- *Explain the shapes of the graphs.*



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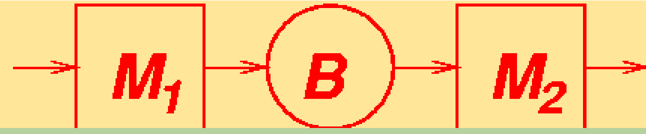
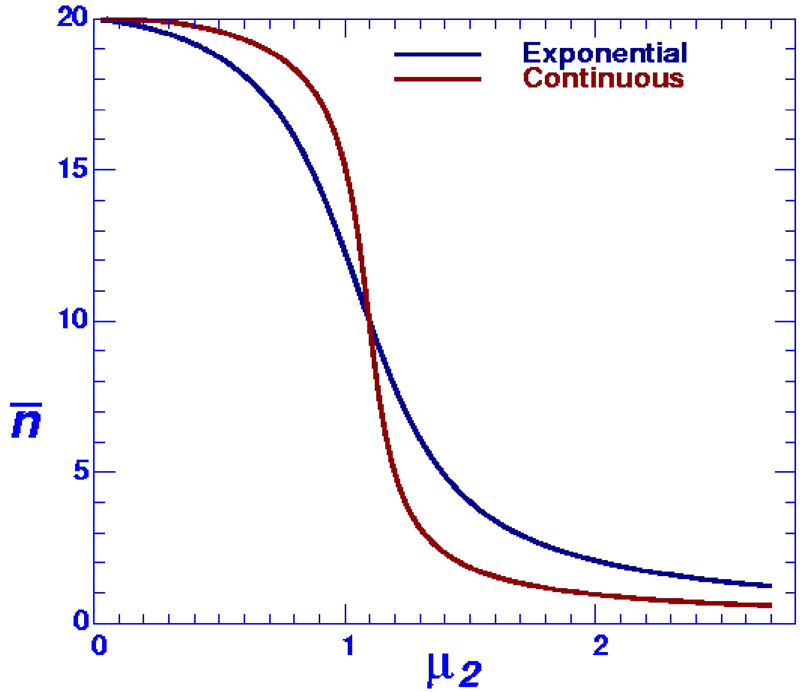


Other models

- *Explain the shapes of the graphs.*



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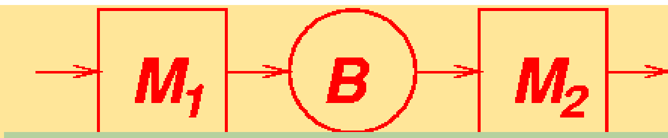
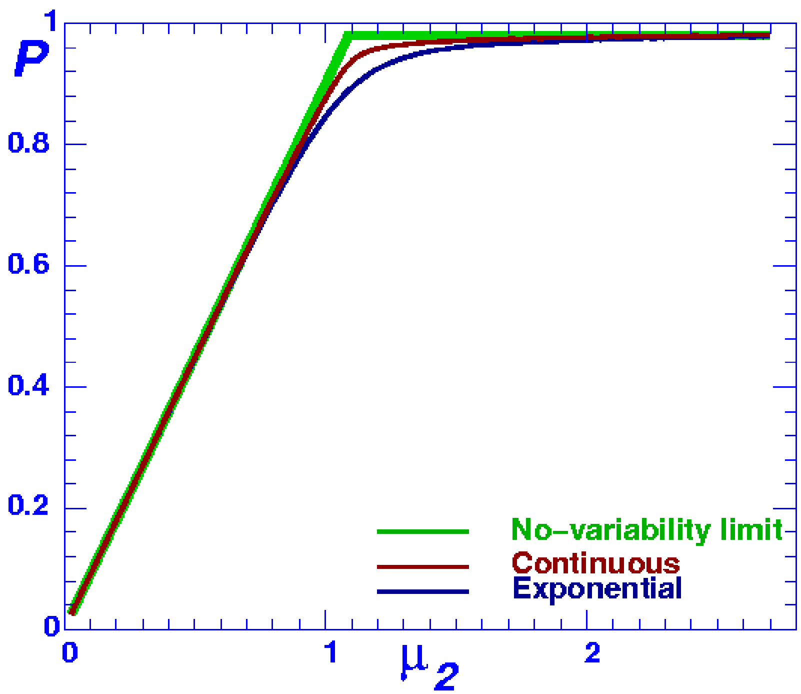
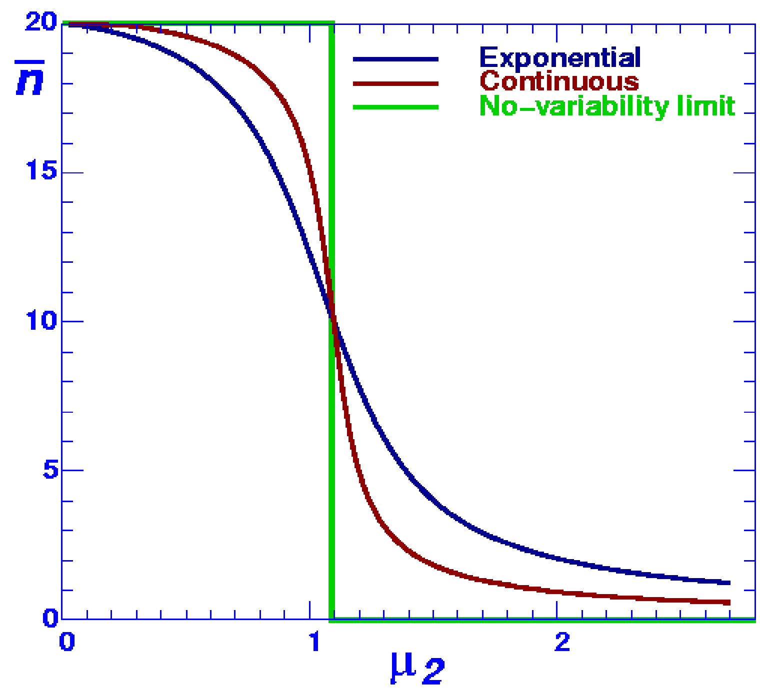
Two Machine, Finite-Buffer Lines



Other models





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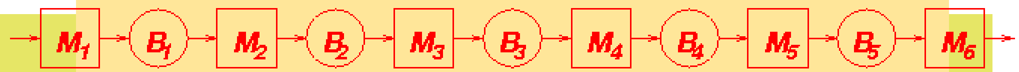
Long Lines



- Difficulty:
  - ⋆ No simple formula for calculating production rate or inventory levels.
  - ⋆ State space is too large for exact numerical solution.
    - ∗ If all buffer sizes are N and the length of the line is k, the number of states is $S = 2^k(N+1)^{k-1}$.
    - ∗ if $N = 10$ and $k = 20$, $S = 6.41 \times 10^{25}$.
  - ⋆ *Decomposition* seems to work successfully.

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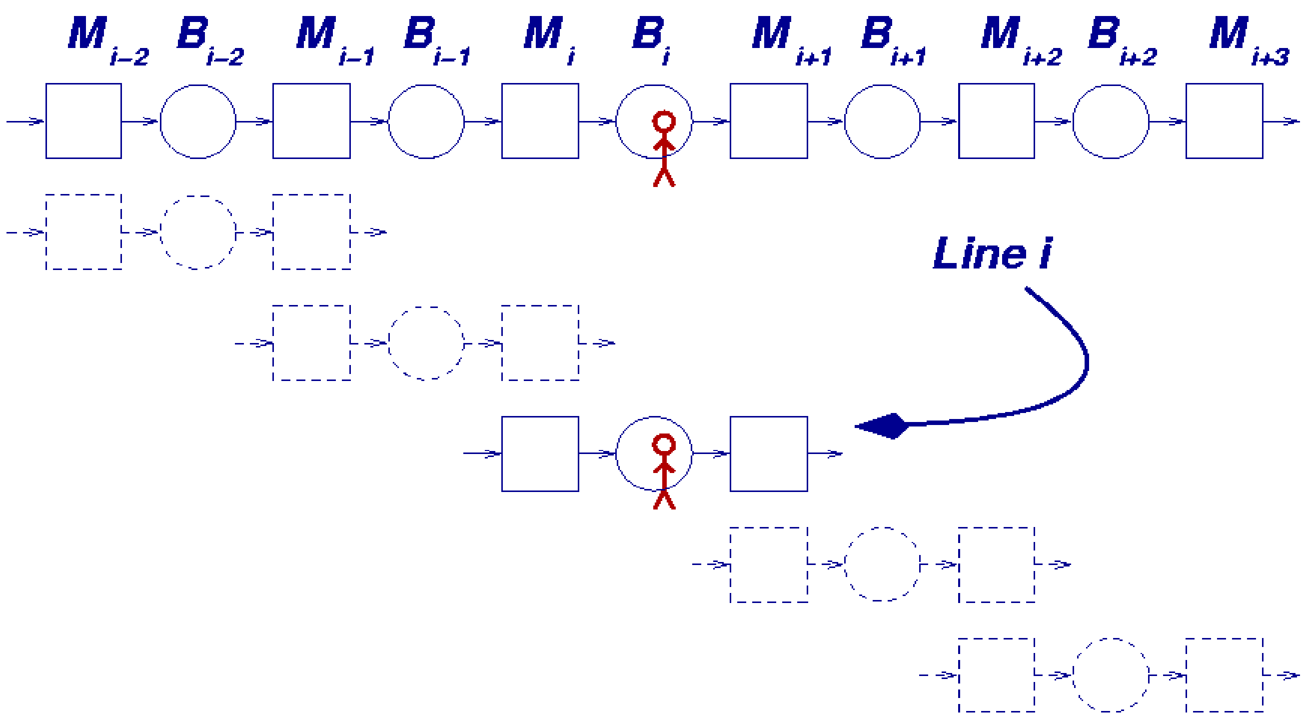

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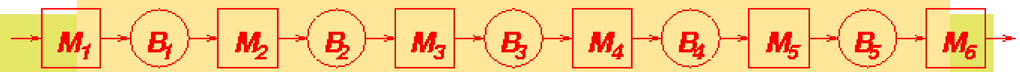

Decomposition

- Consider an observer in Buffer B_i.
  - ⋆ Imagine the material flow that the observer sees *entering* and *leaving* the buffer.
- We construct a two-machine line (ie, we find r_1, p_1, r_2, p_2, and N) such that an observer in its buffer will see almost the same thing.
- The parameters are chosen as functions of the behaviors of the other two-machine lines.

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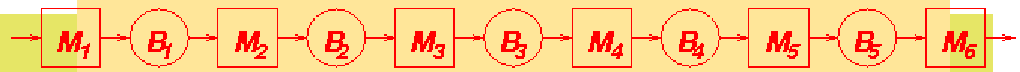
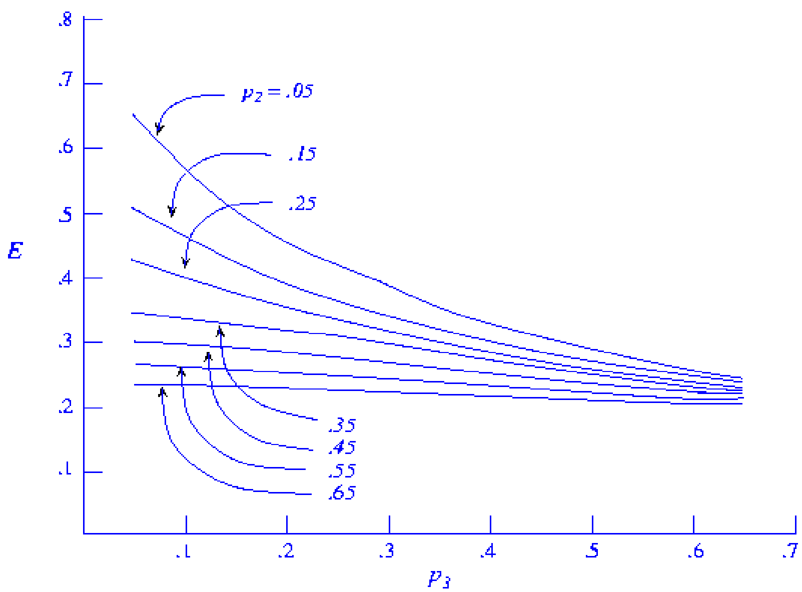


Three-machine line – production rate.



$$r_1 = r_2 = r_3 = .2$$

$$p_1 = .05$$

$$N_1 = N_2 = 5$$

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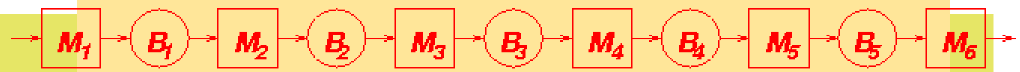
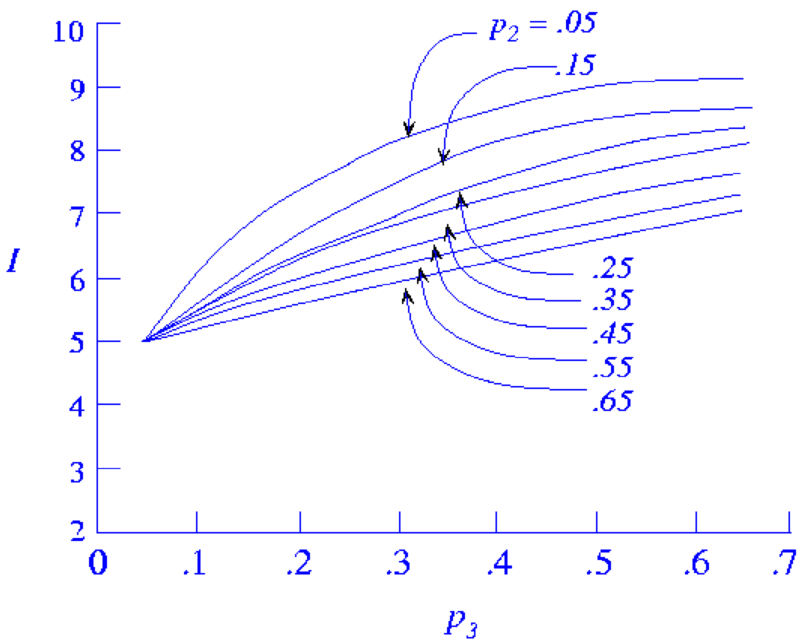


Three-machine line – total average inventory



$$r_1 = r_2 = r_3 = .2$$

$$p_1 = .05$$

$$N_1 = N_2 = 5$$

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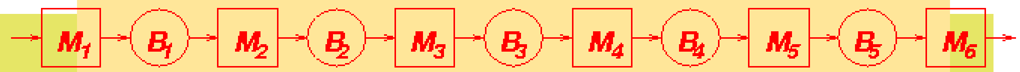
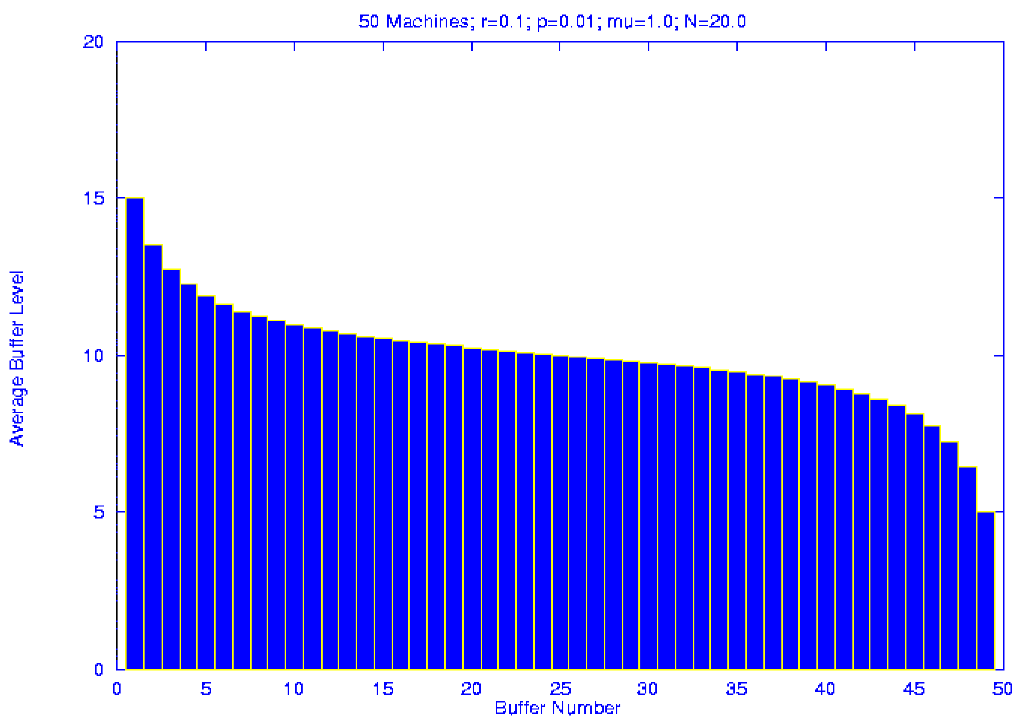
Long Lines



Examples



Distribution of material in a line with identical machines and buffers. *Explain the shape.*

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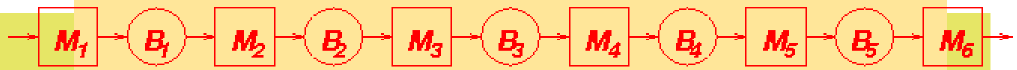

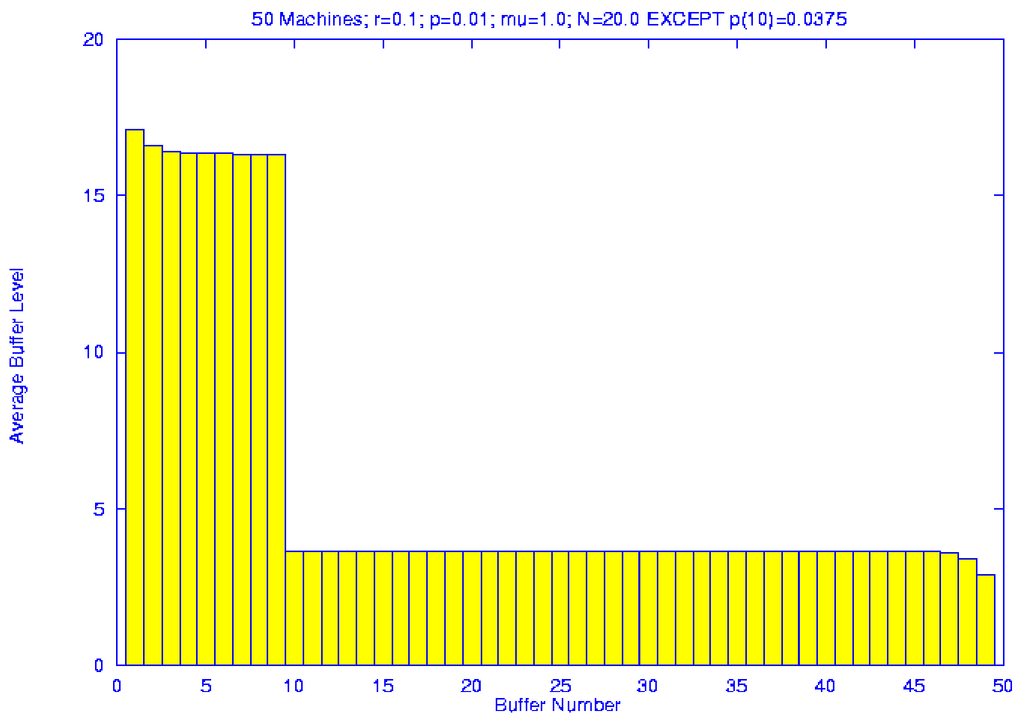

Effect of a bottleneck. Identical machines and buffers, except for M_{10}.

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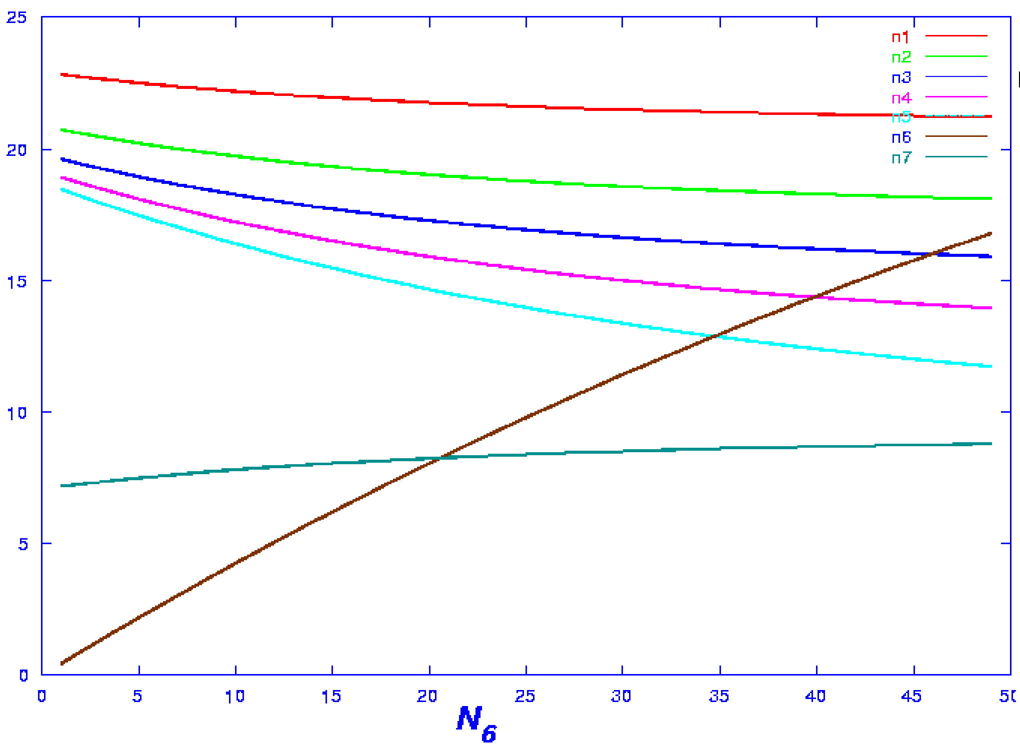

Continuous material model.

- Eight-machine, seven-buffer line.
- For each machine, $r = .075$, $p = .009$, $\mu = 1.2$.
- For each buffer *(except Buffer 6)*, $N = 30$.

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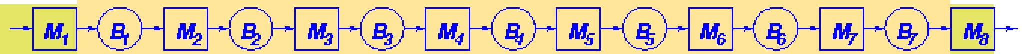
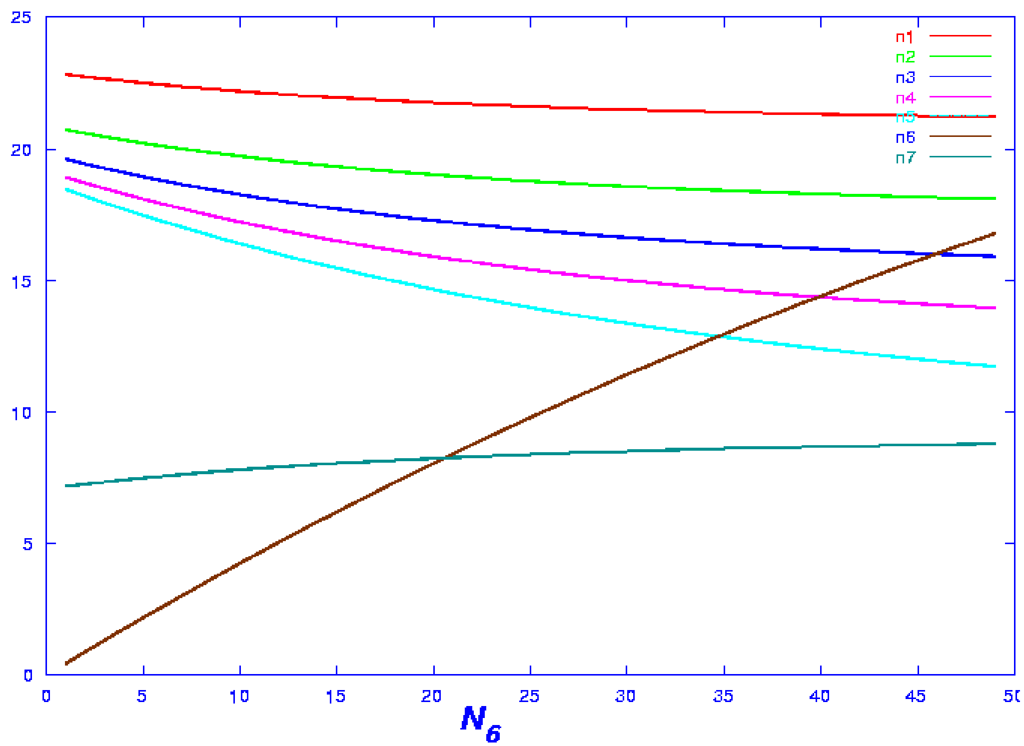


Long Lines

Examples



- Which $\bar{n}_i$ are decreasing and which are increasing?
- Why?

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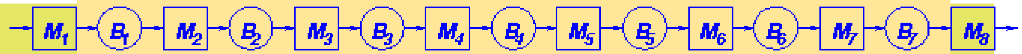
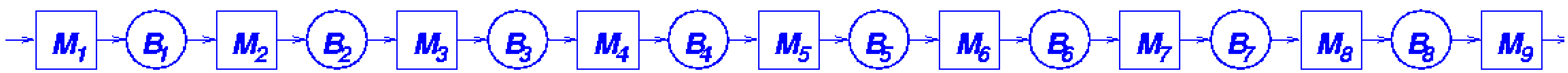
Examples

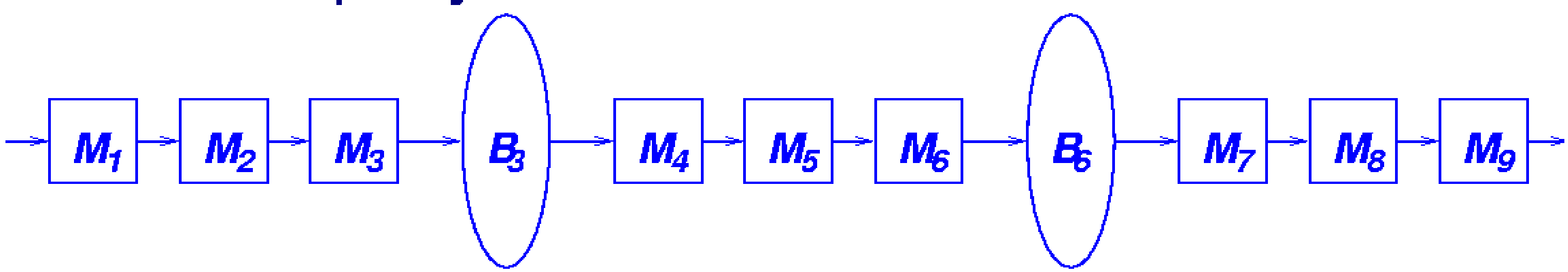


Which has a higher production rate?

- 9-Machine line with two buffering options:
  - ★ 8 buffers equally sized; and



  - ★ 2 buffers equally sized.



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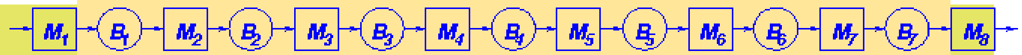
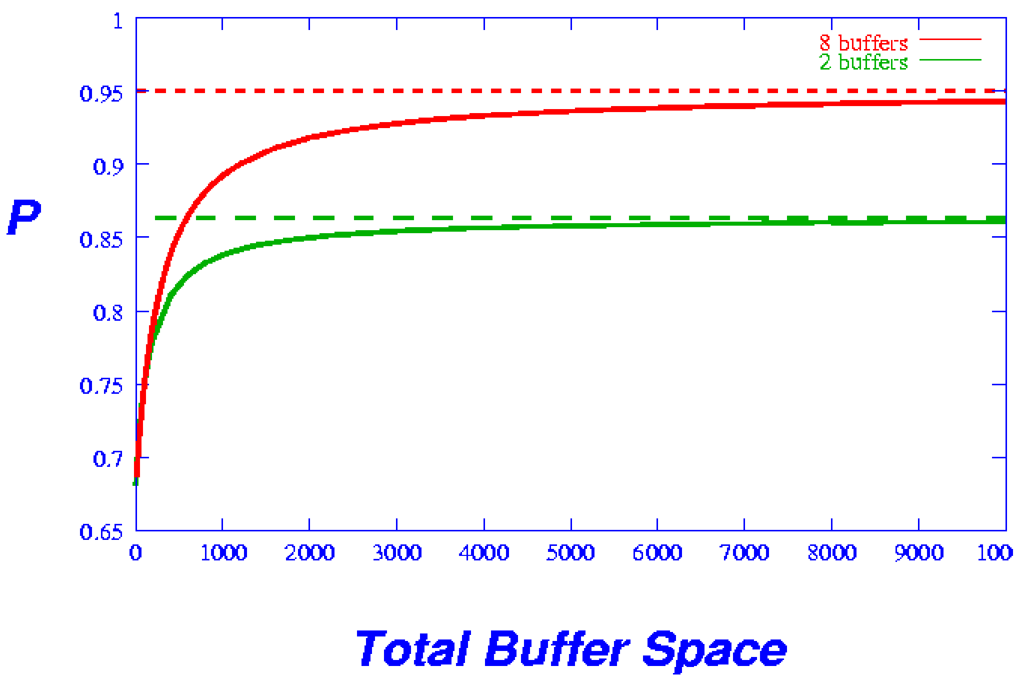


Examples



- Continuous model; all machines have $r = .019$, $p = .001$, $\mu = 1$.
- What are the asymptotes?
- Is 8 buffers *always* faster?

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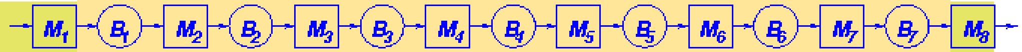
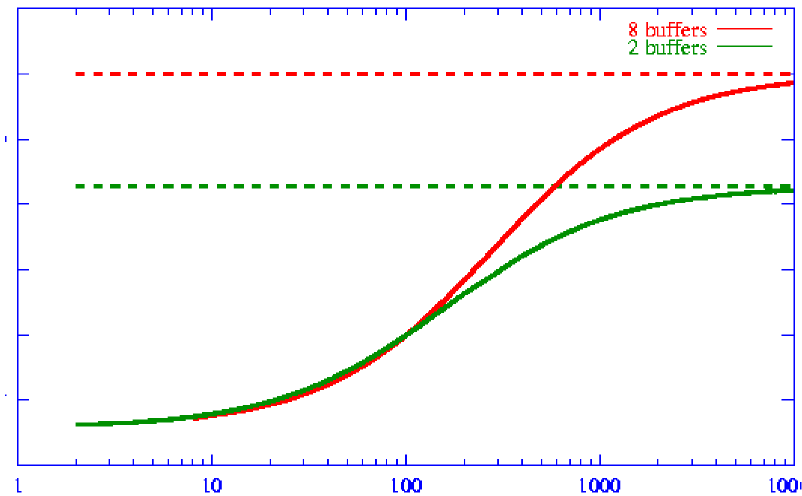


Examples



- *Is 8 buffers always faster?*
- Perhaps not, but difference is not significant in systems with very small buffers.

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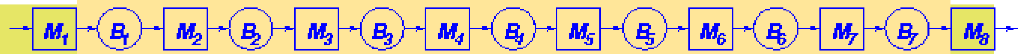

Optimal buffer space distribution.

- Design the buffers for a 20-machine production line.
- The machines have been selected, and the only decision remaining is the amount of space to allocate for in-process inventory.
- *The goal is to determine the smallest amount of in-process inventory space so that the line meets a production rate target.*

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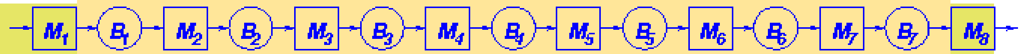

Optimal buffer space distribution.

- The common operation time is one operation per minute.
- The target production rate is .88 parts per minute.

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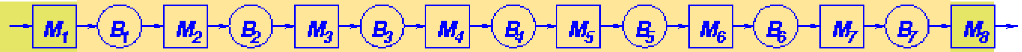

Optimal buffer space distribution.

- *Case 1* MTTF= 200 minutes and MTTR = 10.5 minutes for all machines ($\boldsymbol{P} = \mathbf{.95}$ parts per minute).

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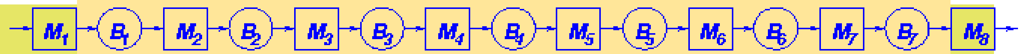

Optimal buffer space distribution.

- *Case 1* MTTF= 200 minutes and MTTR = 10.5 minutes for all machines ($P = .95$ parts per minute).
- *Case 2* Like Case 1 except Machine 5. For Machine 5, MTTF = 100 and MTTR = 10.5 minutes ($P = .905$ parts per minute).

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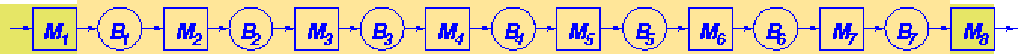

Long Lines

Optimal buffer space distribution.

- *Case 1* MTTF= 200 minutes and MTTR = 10.5 minutes for all machines ($P = .95$ parts per minute).
- *Case 2* Like Case 1 except Machine 5. For Machine 5, MTTF = 100 and MTTR = 10.5 minutes ($P = .905$ parts per minute).
- *Case 3* Like Case 1 except Machine 5. For Machine 5, MTTF = 200 and MTTR = 21 minutes ($P = .905$ parts per minute).

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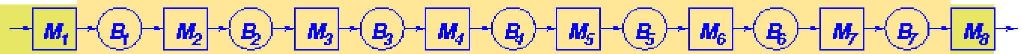

Optimal buffer space distribution.

Are buffers really needed?

Line	Production rate with no buffers, parts per minute
Case 1	.487
Case 2	.475
Case 3	.475

Yes. *How were these numbers calculated?*

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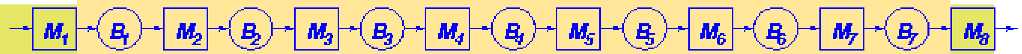
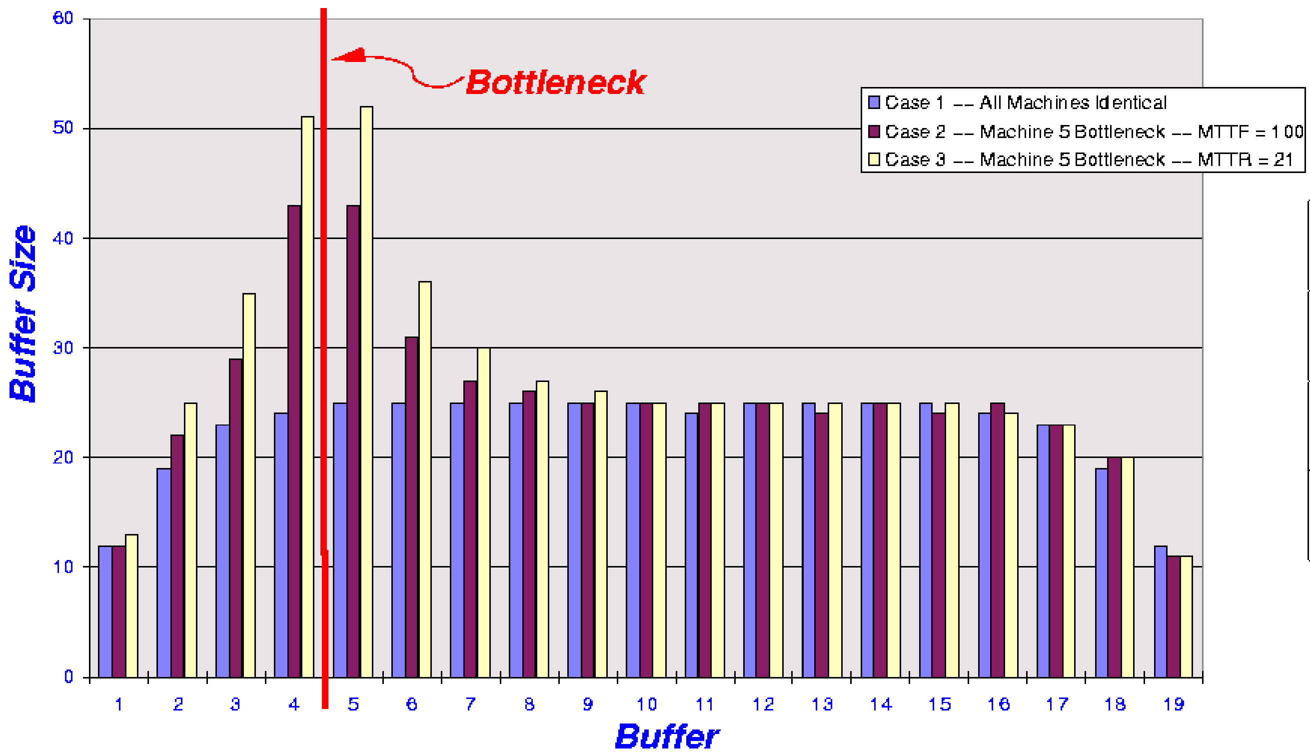


Optimal buffer space distribution.

Solution



Line	Space
Case 1	430
Case 2	485
Case 3	523

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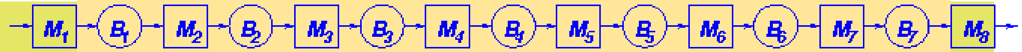

Optimal buffer space distribution.

- Observation from studying buffer space allocation problems:
  - ⋆ *Buffer space is needed most where buffer level variability is greatest!*

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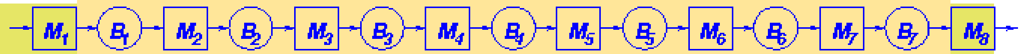
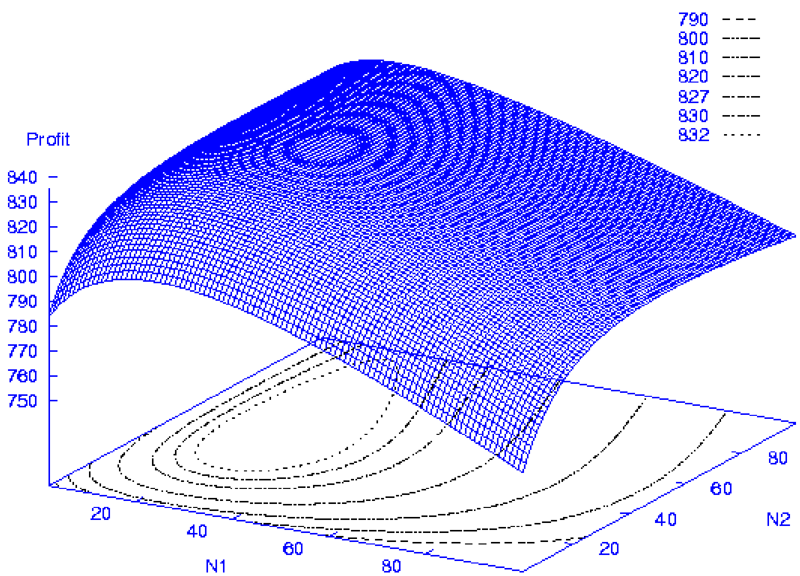


Profit as a function of buffer sizes



- Three-machine, continuous material line.
- $r_i = .1,\ p_i = .01, \mu_i = 1.$
- $\Pi = 1000P(N_1, N_2) - (\bar{n}_1 + \bar{n}_2).$

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