SMA 6304 / MIT 2.853 / MIT 2.854
Manufacturing Systems
Lecture 7: Inventory I

Lecturer: Stanley B. Gershwin

Slides by Larry Wein and Stan Gershwin

Inventory Theory

3 Motives for Holding Inventory

1) seasonal inventory: due to seasonality (predictable)

2) buffer inventory (= safety stock): due to random fluctuations (unpredictable)

3) cycle inventory: due to economies of scale

Newsboy Model

- Buying a perishable item
- Given forecast of future sales
- Excess demand is lost
- Excess supply is salvaged

c = purchase cost per unit
r = revenue per unit
s = salvage value per unit
d = unknown demand

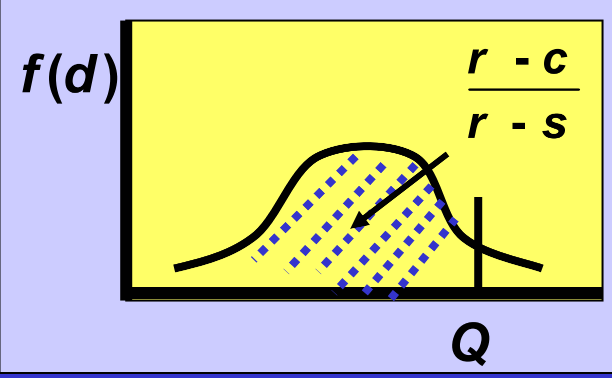
marginal cost = c

$$\text{marginal revenue} = \begin{cases} r & \text{if } d \geq Q \\ s & \text{if } d < Q \end{cases}$$

$$MC = E[MR]$$

$$c = r\Pr(d \geq Q) + s\Pr(d < Q)$$

Order Q, where $\Pr(d \leq Q) = \dfrac{r-c}{r-s}$



Newsvendor Problem

... formerly called the "Newsboy Problem".

Motive: randomness in demand.

Newsguy buys x newspapers at c dollars per paper. Demand for newspapers, at price $r > c$ is w. Unsold newspapers are redeemed at price $s < c$.

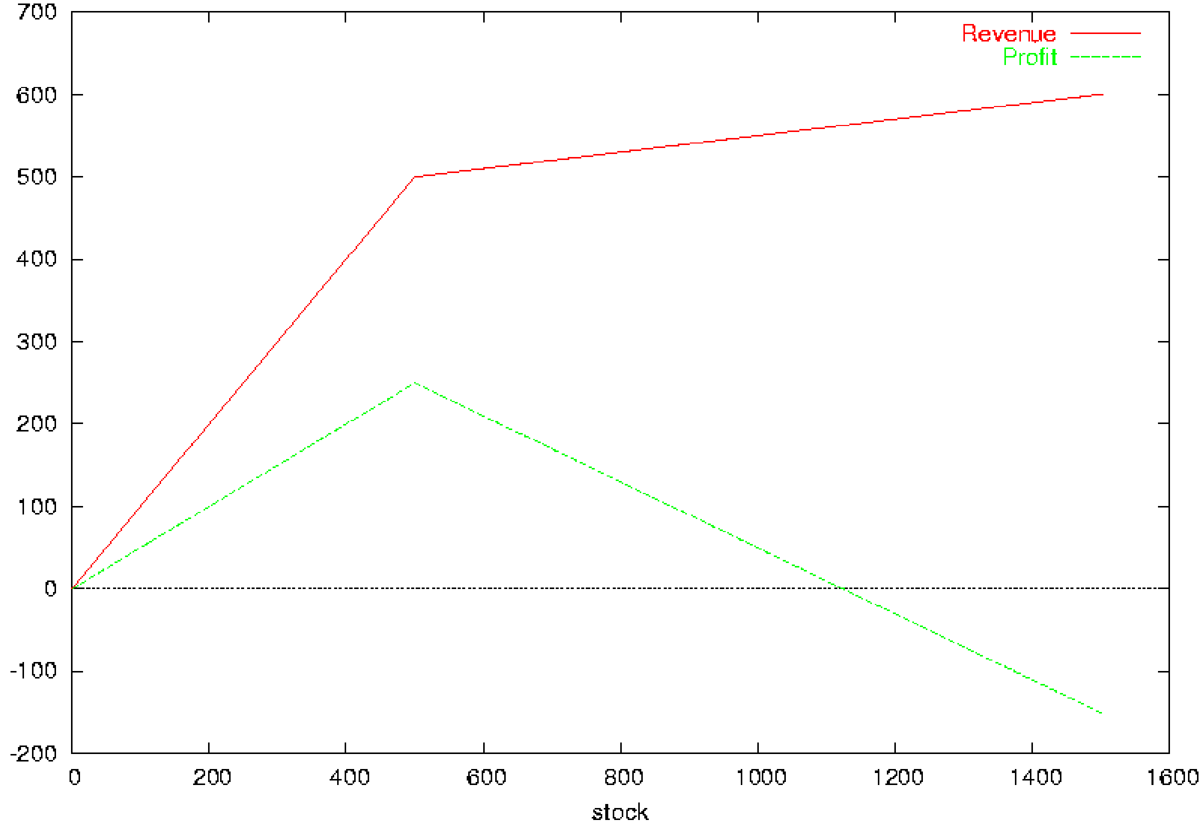
w is a continuous random variable with distribution function $F(W) =$ prob$(W \leq w)$; $f(w) = dF(w)/dw$. Note that $w \geq 0$ so $F(w) = 0$ for $w \leq 0$.

Newsvendor Problem

Revenue = $R = \begin{cases} rx & \text{if } x \leq w \\ rw + s(x - w) & \text{if } x > w \end{cases}$

Profit = $P = \begin{cases} (r - c)x \text{ if } x \leq w \\ rw + s(x - w) - cx \\ \qquad = (r - s)w + (s - c)x \text{ if } x > w \end{cases}$

Newsvendor Problem



$$r = 1.; c = .5; s = .1; w = 500.$$

Newsvendor Problem

Expected Profit $= EP(x) =$

$$\int_{-\infty}^{x} [(r-s)w + (s-c)x] f(w) dw$$

$$+ \int_{x}^{\infty} (r-c)x f(w) dw$$

Newsvendor Problem

or $EP(x)$

$$= (r-s)\int_{-\infty}^{x} wf(w)dw + (s-c)x\int_{-\infty}^{x} f(w)dw$$
$$+ (r-c)x\int_{x}^{\infty} f(w)dw$$

$$= (r-s)\int_{-\infty}^{x} wf(w)dw$$
$$+ (s-c)xF(x) + (r-c)x(1-F(x))$$

Newsvendor Problem

The first term is independent of x. The remainder of the expression can be written

$$x((s-c)F(x)+(r-c)(1-F(x))$$

which is 0 when $x = 0$. When $x \to \infty$, the last term goes to 0 and the remaining term,

$$x(s-c)F(x) \to -\infty.$$

Newsvendor Problem

$$\frac{dEP}{dx} = (r-s)xf(x) + (s-c)F(x) + (s-c)xf(x)$$
$$+(r-c)(1-F(x)) - (r-c)xf(x)$$

$$= xf(x)(r-s+s-c-r+c)$$
$$+r-c+(s-c-r+c)F(x)$$

$$= r-c+(s-r)F(x)$$

Newsvendor Problem

Note that $dEP/dx > 0$ when $x = 0$. Therefore EP has a maximum which is greater than 0.

The $x = x^*$ that maximizes EP satisfies

$$\frac{dEP}{dx}(x^*) = 0.$$

Therefore $F(x^*) = \dfrac{r - c}{r - s}$.

Newsvendor Problem

This can also be written

$$F(x^*) = \frac{r - c}{(r - c) + (c - s)}$$

$r - c > 0$ is the marginal profit when $x < w$.

$c - s > 0$ is the marginal loss when $x > w$.

Choose x^* so that the fraction of time you do not buy too much is

$$\frac{\text{marginal profit}}{\text{marginal profit} + \text{marginal loss}}$$

DDI Example

$d =$ *Demand ~ Normal*

$\mu = 150{,}000$

$\sigma = 45{,}000$

$r = \$150$

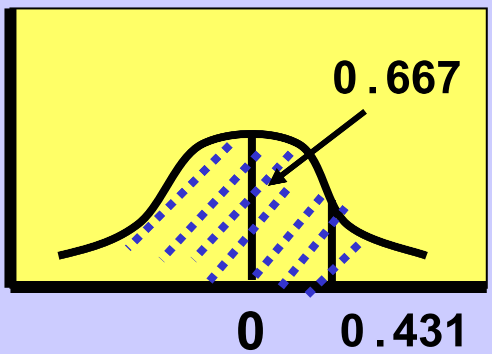
$c = \$50$

$s = \$0$

$$\Pr(d \le Q) = \frac{r-c}{r-s} = .667$$

$$\Pr(d \le Q) = \Pr\left(\frac{d-150{,}000}{45{,}000} \le \frac{Q-150{,}000}{45{,}000}\right)$$

$$= \Pr\left(Z \le \frac{Q-150{,}000}{45{,}000}\right) = 0.667$$



$$\Pr(Z \le 0.431) = 0.667$$

$$\frac{Q-150{,}000}{45{,}000} = 0.431$$

$$Q = 150{,}000 + \underbrace{0.431(45{,}000)}_{\text{safety stock}} = 169{,}395$$

EOQ Model

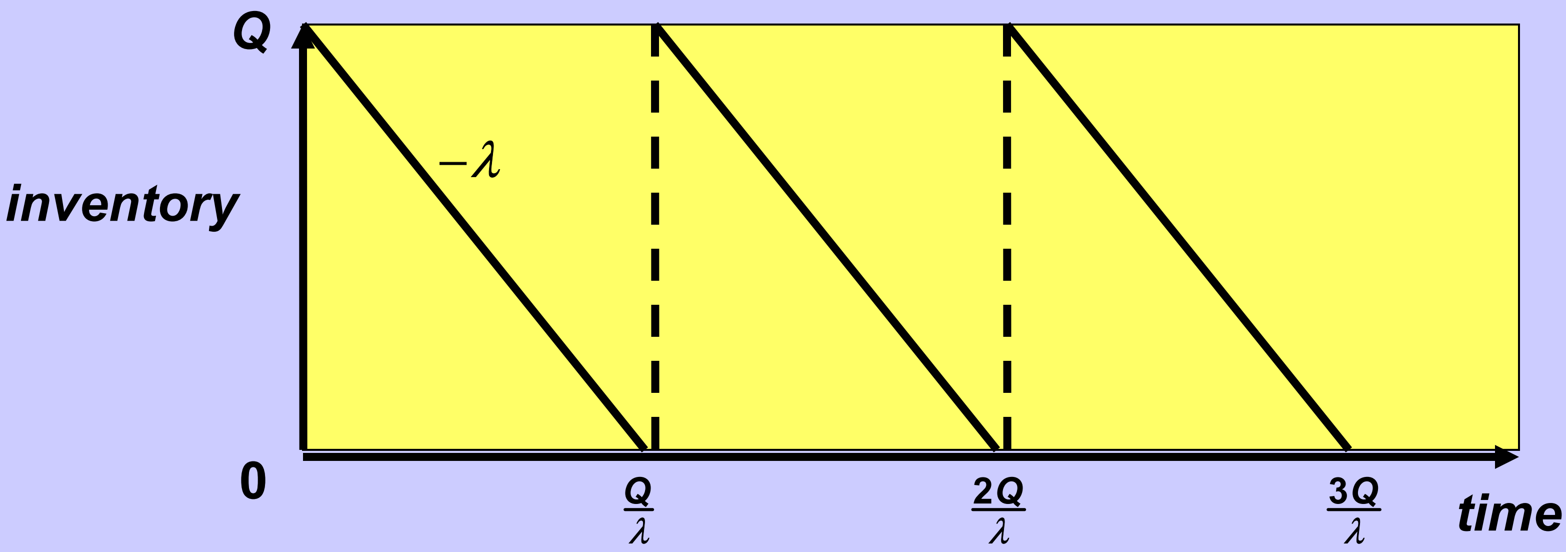
tradeoff of holding cost vs. ordering cost

λ = annual demand rate
c = variable cost per unit
k = inventory carrying cost per year (%)
ck = holding cost per unit per year
s = fixed ordering cost
Q = order quantity (decision)

Assume

1) No statistical uncertainty
2) instantaneous replenishment
3) no shortages allowed

EOQ Derivation



$$\text{cost / year} = \frac{s\lambda}{Q} + \frac{ckQ}{2} + c\lambda$$

$$\frac{d(\text{cost})}{dQ} = 0 \Rightarrow Q^* = \sqrt{\frac{2\lambda s}{ck}} = EOQ$$

Comments

1) *EOQ* is robust $\quad Cost\left(\frac{Q^*}{2}\right) = \text{cost}\,(2Q^*) = 1.25\ \text{cost}(Q^*)$

2) $s \uparrow \Rightarrow Q^* \uparrow$

3) $s \downarrow \Rightarrow Q^* \downarrow$

4) $\lambda \uparrow \Rightarrow Q^* \uparrow$

5) $c \uparrow \Rightarrow Q^* \downarrow$

Dellpaq Example

$$\lambda = 300,000 \text{ per year}$$

$$s = \$100,000 + \$50 = \$100,050$$

$$c = \$3,000 + \$25 + \$1 + \$5 + \$0.50 = \$3,031.50$$

$$k = 0.20$$

$$Q^* = \sqrt{\frac{2s\lambda}{ck}} = 9950.4$$

What is re-order point?

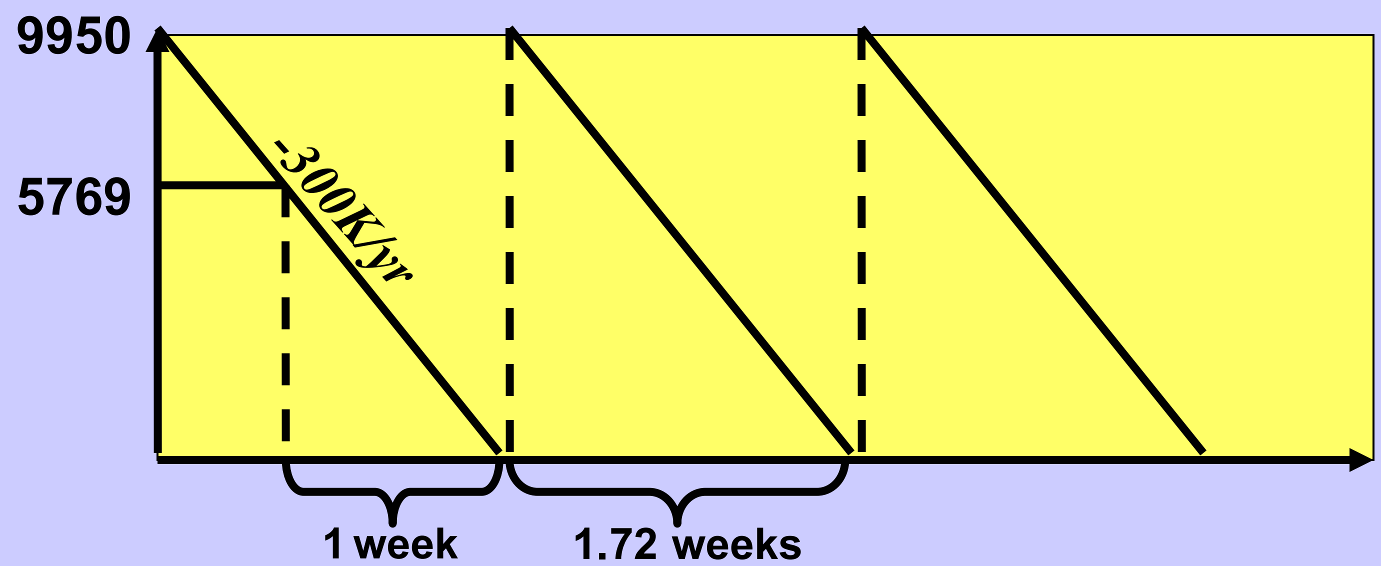
$$\text{Lead time } = 1 \text{ week}$$

$$\left(\frac{9950.4 \text{ units}}{300,000 \text{ units/year}}\right)\frac{52 \text{ weeks}}{\text{year}} = 1.72 \text{ weeks}$$

Dellpaq Example Cont.

Lead time $= 1$ week

$$\left(\frac{9950.4 \text{ units}}{300,000 \text{ units/year}}\right)\frac{52 \text{ weeks}}{\text{year}} = 1.72 \text{ weeks}$$



Reorder Point $=$ inventory level at time of ordering

$=$ demand during lead time

$$= \left(\frac{300,000 \ \textit{units/year}}{52 \ \textit{weeks/year}}\right) 1 \ \textit{week} = 5769 \text{ units}$$

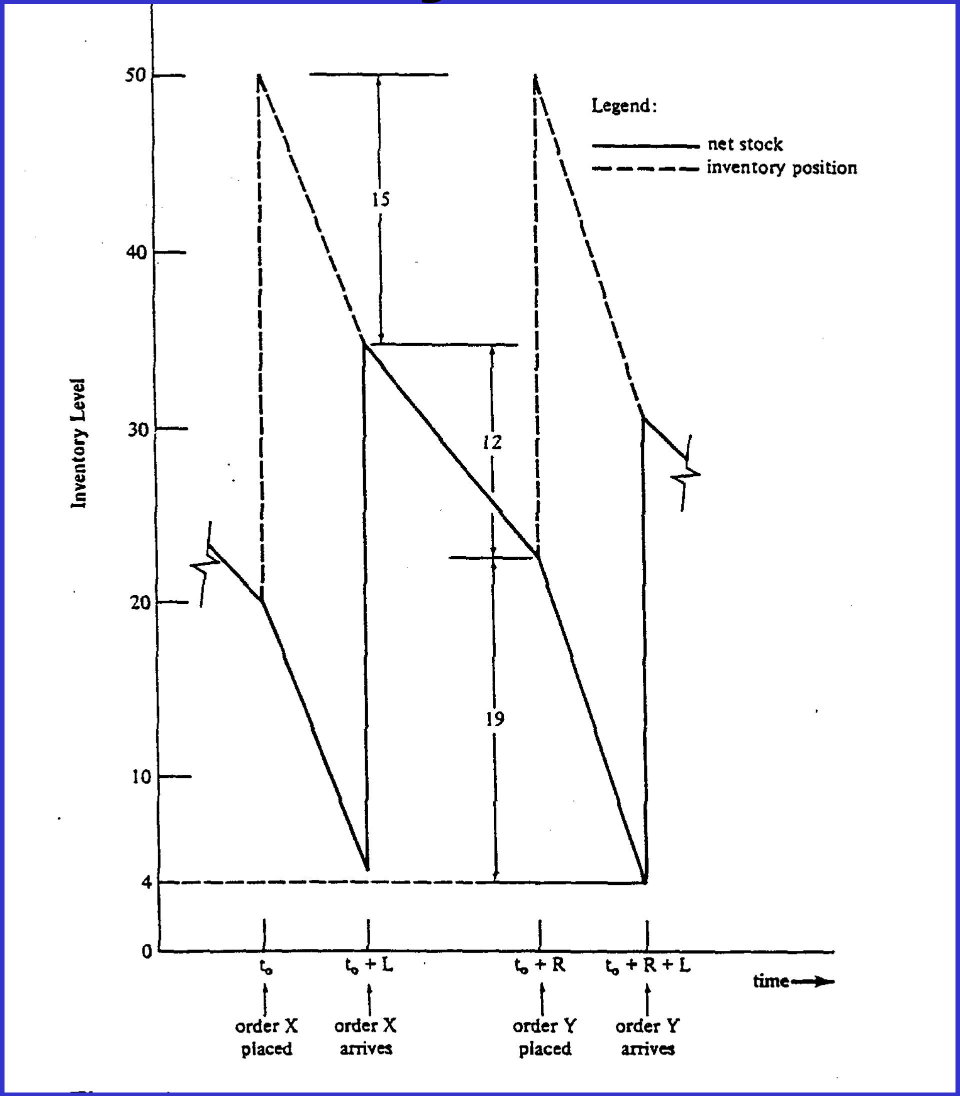
Reorder Point Policy $=$ when inventory level drops to 5769 units, order 9950 units

Inventory Position

Q: What if lead time = 3 weeks?

- **Then you make an order before the previous order has arrived**
- **Inventory position = inventory on hand + inventory on order**
- **You track the inventory position, and use a reorder point policy with respect to the inventory position**

Inventory Position



Key insight: the current order must satisfy demand until the next order arrives

(R,Q) Policy

Q: What if demand is uncertain?

***(R,Q)* policy: when inventory position drops to R, order Q**

Simple Heuristic:

Set $Q = EOQ$

Let = *DDLT* = demand during the lead time

We need R units to satisfy *DDLT*

Use newsboy model

Set R so that $\Pr(DDLT \le R) = \frac{r - c}{r - s}$

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Example

L = deterministic lead time

D_i = iid demand in each period

with mean $E[D]$ and variance $\sigma^2[D]$

$$DDLT = D_1 + \ldots + D_L$$

$$E[DDLT] = LE[D]$$

$$Var[DDLT] = L\sigma^2[D]$$

set R so that $\Pr(DDLT \le R)$

$$= \Pr\left(\frac{DDLT - LE[D]}{\sqrt{L}\sigma[D]} \le \frac{R - LE[D]}{\sqrt{L}\sigma[D]}\right)$$

$$= \frac{r - c}{r - s}$$