TOPOLOGICAL MODELING OF ENVIRONMENTAL PSYCHOLOGY

by

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IV. ABSTRACT

In this study, the possibilities of using general and algebraic topology as a metaphor and perhaps even as a model to probe relationships in the fields of spatial perception and meaning were investigated. Some interesting connections emerged and many possible avenues of both further modeling and empirical testing are left to be explored.

The body of the work includes a presentation of the intuitive concepts of topology, a sort of annotated guide to topology for environmental psychologists, and an attempt, through many fragmentary investigations, to seek out possible applications of the topological concepts to the study of environmental psychology. The last two examples developed tell of ways to systematize the Gestalt rules of good configuration and explore how topology might describe how people perceive space through feelings of enclosure, separation of objects, patterns and systems of movement.

No empirical study in the field of environmental psychology has gone into the findings of this paper, though several avenues of such study are recommended. Rather, the hypothesis here is that the forms of topology are tools that can suggest relationships and functions of the human mind and human perception.

V. PREFACE

Though this study should be judged on its merits as a self-contained piece, there were problems and limitations encountered along the way which affect the final product and which the reader ought to understand.

First, there was a deadline which kept me from exploring a much broader range of topics in the fields related to topology, any of which might have turned up significant results. Also, this prevented any good testing of some of the forms hypothesized relevant to environmental psychology. Particularly unfortunate was the impossibility of including extensive findings of my own ongoing research in child care centers which employs some of the concepts of "spatial gestalt" and graph theory.

The mathematics presented is, of course, the work of an amateur; some statements may be so condensed as to not be quite technically correct. This is not, however, important to my purposes here.

A third constraint was a lack of communication with others who might be involved in related work. No doubt, the section on the history of the "topology metaphor" is already obsolete.

Finally, one serious failure altered the final status of this paper greatly. Initially, it was my goal to discover something new about perception of the environment or to clarify possible contradictions which may exist in our understanding of such perception. The latter is the role usually assumed by analytic philosophers toward a branch of science in order to reveal meaningless statements and inconsistencies in research through the rigorous application of logical, systematic models. This kind of effort might be relevant in environmental psychology, particularly in the study of perception (Piaget, after all, was an epistemologist), yet I was not, from the start, clear on what kind of effort and restrictions this would entail and was never really prepared to perform such a task. The first goal, to discover new knowledge about psychology, is quite impossible without some kind of experimentation or ovservation of people in real or artificial environments. A body of mathematical knowledge can only be used as a model or a metaphor for a real-world situation; in my final effort, I have endeavored to use topology as a metaphor.

VI. MODELING BY METAPHOR

It is fitting to begin by discussing the use of a branch of mathematics as a metaphor. What does such a use mean? Of what value is it? What are its limitations? In this section, I shall address these three questions.

The metaphor is to artistry as the scientific method is to science. In the ways it occurs in various art forms, the metaphor is a juxtaposition of forces that seem to have little relation to one another. This process may be thought of as putting on a strange set of glasses to look at an object just to see it in a new way. If artists are, as Marshall McLuhan says, the "antennae of the race" and if one accepts that we are utterly surrounded by a mystery of which we know very little, then perhaps such a far-flung idea as looking at the spatial environment through the "glasses" of topology takes on the form of a potentially useful probe.

Of course, I don't pretend to justify this effort on purely artistic merits. There are precedents for using such a method to achieve new ideas that might eventually become more grounded in scientific fact. For example, Edward C. Tolman, in adopting a sociological model for general behavior, calls this type of approach a <u>sui generis</u> model which invents

> "a set of explanatory structures and processes (hypothetical constructs) which draw on analogies from whatever other disciplines---mathematics, physics, mechanics, physiology, etc.--- as may be deemed useful. Freud's water-reservoir concept of the 'libido' and Lewin's 'topological and vector' psychology belong primarily in this...category."*

*Tolman, Edward, "A Psychological Model", p. 283.

One justification for choosing to use this metaphorical model is that the seemingly "random" choice of a modeling system is perhaps not so random at all. After all, we are limited creatures and perhaps there are more unconscious correlations between seemingly disparate fields of our knowledge than would at first appear obvious. In comparing the Twentieth Century breakdown of atomistic physics to the discoveries of Freud, L. L. Whyte states,

> "We hear of unstable particles in physics and of unconscious mind in psychology. Is this a mere chance or a sign of a parallel between the two sciences? Is there some common factor which leads both to name a basic idea in this backhanded manner? I believe there is..."*

In the case of the subjects of this paper, it seems quite plausible that the men who originated the concepts of topology were unconsciously influenced by the forms they experienced in dealing with their spatial environment.

Still another reason for such a probe is the potential value to the modeling system itself, gained through its application to the modeled structure. The psychology of perception, even if it does not benefit from being modeled by a deductive system, may still provide a concrete example of the reality of topological forms. This could benefit the teaching of topology as well as possibly advancing topological horizons. Since Piaget has suggested that the primacy of topological principles in perception indicates that it might be taught much earlier in the mathematics program,

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*Whyte, L.L., Essay on Atomism, pp. 4-5.

I hope that this paper can be of some assistance to one interested in developing such a curriculum.

In adopting a metaphorical method one also must accept some rules of operation. Tolman states these well:

> "Such a model can be defended only insofar as it proves helpful in explaining and making understandable already observed behavior and insofar as it also suggests new behaviors to be looked for. And any such model must, of course, be ready to undergo variations and modifications to make it correspond better with new empirical findings. Finally, insofar as such a model holds up and continues to have pragmatic value, it must be assumed that eventually more and more precise and intelligible correlations will be discovered between it and underlying...structures and processes..."*

These are the tests, then which will ultimately determine the validity and usefulness of the ideas presented here.

Now that some of the reasons underlying a metaphorical approach have been noted, I would like to mention some further reasons for attempting such a connection in the particular cases of topology and environmental psychology, reasons which are to be found in the specifid natures of these two fields.

Topology, as a branch of mathematics, is a relatively recent addition to the study of geometry, originating in the mid-nineteenth century. In the past several decades, its relationships to other fields of mathematics have been established, thus bringing topology into its own in importance. What is important about it is that, though more recent than geometry, it is also more fundamental in its theory and axiomatic basis. General topology does not concern itself with angles, straight lines, size or shape. The concepts

*Tolman, Edward, loc. cit.

of topology---continuity, connectedness, separation, order, denseness, proximity and enclosure, to name a few---are much more intuitive to picture and describe. In actuality, geometry in the classical Euclidean sense is a special form of topology, one with projective concepts (straight lines and perspective), affine concepts (parallelism) and metric concepts (distance) added.

It is in this sense that topology, or analysis situs, as it is sometimes called, is fundamental---a very general theory of what space is, though this does not imply that the study of topology is simple. The nature of this study is to find ways to classify and categorize in very basic ways as well as in more refined distinctions.

Man's experience in physical space is likewise a very basic phenomenon. By basic, I mean to differentiate the awareness (conscious or unconscious) of being in a room or being exposed to a wide open field on one side from the knowledge of a conceptual relationship or one's memory of a remote person. This latter type of experience is largely detached from a spatial milieu, yet it is my belief that our ability to perform these latter, more abstract functions, derives from the archetypal forms we learn from perceiving and interacting with physical space. It is the belief that lines, angles, metrics and similar concepts cannot adequately represent these psychological archetypes that leads me to the study of topology.

VII. ESSENTIALS OF TOPOLOGY

1. PROCEDURES

Notation

$\forall$ = for all or for every

$\exists$ = there is

$\Rightarrow$ = logically implies

$\Leftrightarrow$ or iff = is logically equivalent to; if and only iff

$\ni$ = such that or so that

E^1 is the real number line, i.e. all rational and non-rational numbers from negative infinity to positive infinity.

$[a,b]$ is the inclusive interval between a and b (all $x \ni a \leqq x \leqq b$)

(a,b) is the exclusive interval between a and b (all $x \ni a < x < b$)

I = 0,1 the real inclusive interval between zero and one

E^2 is the set of all points on the real plane, represented as the set of all ordered pairs, e.g. $(3.45, -2\frac{1}{4})$ = the element 3.45 as the first member and $-2\frac{1}{4}$ as the second member. This is not the same point as $(-2\frac{1}{4}, 3.45)$. (Unfortunately, we use the same notation for ordered pairs as we do for exclusive intervals. One must judge which is meant from context.)

E^3 is the real Euclidean 3-space, represented as the set of all ordered triplets of real numbers.

In general, we let E^n be real Euclidean n-space, all ordered n-tuples of real numbers, e.g. $(x_1, x_2, \ldots, x_n)$.

Other procedures

Key concept names are underlined when first defined.

In the glossary one can find references to the section in which any mathematical term is first defined.

SET THEORY AND ORDER

1. Sets and operations

1.1 Points are a fundamental accepted notion in mathematics. A set is represented as a collection of points by brackets, e.g. $B = \{x, y, z\}$. One can alxo say $x \in B$ to mean x is an element of B or x belongs to B. Sets can also be called classes---we say $A \subset B$ (or $B \supset A$) to mean A is contained in B, i.e. $\forall x,\ x \in A \Rightarrow x \in B$. It is also proper to say $A = B$ if $A \subset B$ and $B \subset A$ ($\forall x,\ x \in A \Leftrightarrow x \in B$). Because of logical stickiness (the Russell antimony), not all classes can be considered sets, while sets and points are the only things that have the right to belong to ($\in$) another entity, but we need not worry about this problem.

There is no formal distinction between points and sets; we just call something a point in a circumstance if we want it to be thought of as an uncuttable object. Any set can be identified to a point (see 1.8) and treated as one from them on. It is all a matter of perspective.

1.2 The null set ($\emptyset$) is the set with no members. All null sets are identical. The universe is the whole set of points or sets to which we restrict our discussion at a given time.

1.3 Union ($\cup$): $A \cup B =$ all x which belong to either A or B or both.

Intersection($\cap$): $A \cap B =$ all x which belong to both A and B.

Complement ($'$): $A' =$ all x which do not belong to A.

Difference ($-$): $A - B = A \cap B'$.

1.4 Cartesian product ($\times$): $A \times B =$ the set of all ordered pairs $(a, b) \ni a \in A$ and $b \in B$. An example is E^2 which $= E^1 \times E^1$. As mentioned before, order matters in the formation of ordered pairs. Another example of a Cartesian product: Let M = the set of males in a group of people and let F = the set of females in the same group. Then $M \times F$ = the set of all possible couples in the group. Cartesian products can also be extended beyond two-set products to include any finite number of sets (e.g. E^n).

1.5 A <u>function</u> or <u>map</u> assigns to <u>each</u> member of one set a <u>unique</u> member of another set. We say f "takes" X into Y and write $f: X \rightarrow Y$. Note that a point in y can be the map of more than one point in X, just one point in X or no points at all, but every point in X has exactly one point in Y to which it is mapped. A function can also be seen as a set of ordered pairs in $X \times Y$. For a subset $A \subset X$, we say $f(A) = B$ iff B = all y which have at least one $x \in A$ mapped into them. Every function has an <u>inverse function</u>, f^{-1}, not necessarily a function itself. For any set $B \subset Y$, $f^{-1}(B)$ = all x in $X \ni f(x) \in B$.

1.6 For an $f: X \rightarrow Y$, not every element of Y must necessarily $\in f(x)$. If this condition does hold, the map f is called <u>surjective</u> or <u>onto</u>. Also, a y which is the function of some $x \in X$ can also be the function of another x_1 in X. If this is not the case for any y in Y, the map f is called <u>injective</u> or <u>one-to-one</u>. If a map is injective and surjective at the same time, it is called <u>bijective</u>. A function is bijective iff f^{-1} is also a function. (See Fig. 1.)

1.7 A <u>binary relation</u>, R, in a set X, is a subset of $X \times X$, i.e. a set of ordered couples. if $(x,y) \in R$, we can also write xRy or "x has relation R to y". There are several special types of relations:

Reflexive relation: $\forall x \in X$, xRx. E.g. "lives with".

Symmetry relation: xRy $\Leftrightarrow$ yRx. E.g. "is married to".

Antisymmetry relation: xRy and yRx $\Rightarrow$ x=y. E.g. "is the father of".

Transitive relation: xRy and yRz $\Rightarrow$ xRz. E.g. in E^1, "is greater than".

Complete relation: Every pair is related in some way---$\forall x,y \in X$, xRy or yRx or both. E.g. "$\leq$" in E^1.

1.8 An equivalence relation is a relation which is reflexive, symmetric and transitive. Every equivalence relation breaks a set into equivalence classes such that any two members of one class are related while any two members of distinct classes are not and each element belongs to exactly one equivalence class. "Lives with" is an equivalence relation; "=" in E^1 is a trivial example of a relation in which each point is its own equivalence class. An equivalence relation, R, in a set X, creates a new set, the quotient set or X/R, whose points are the equivalence classes created by R. (See Fig. 2.) For a subset $A \subset X$, if we let (x,y) R iff x=y or both x and $y \in A$, then X/R (sometimes called the quotient set of A, or X/A) consists of $X - A$ plus the set A identified to a point.

1.9 A partition of a set X is a group of disjoint sets which cover X, i.e. each x in X belongs to one and only one set of the partition. A set of equivalence classes in X is always a partition.

2. Orderings and order

2.1 A weak ordering or preordering is a relation R on a set X which is transitive and reflexive. If a preordering is also

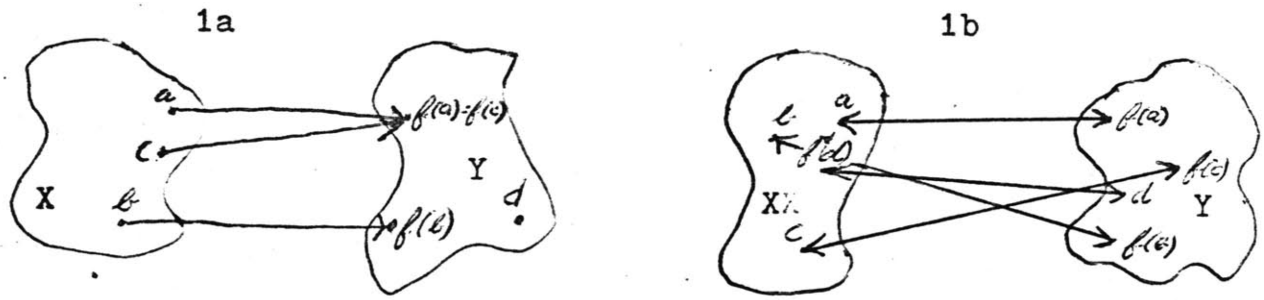


Figure 1. Types of functions. The function in 1a is neither surjective (because d Y has no inverse) nor injective (because f(a) = f(c). The function in 1b as shown is both injective and surjective; thus, it is bijective.

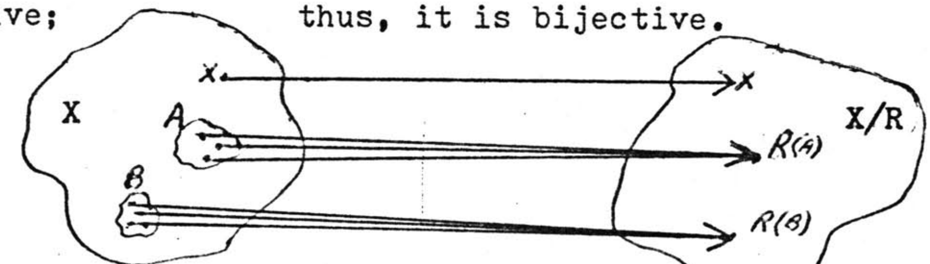


Figure 2. Quotient set. The points of X/R represent equivalence classes of X with respect to R. A, B and the point x are each identified to a point in the quotient set X/R.

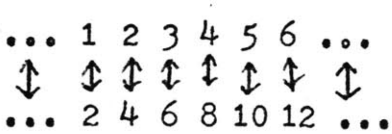


Figure 3. Cardinality. If a simple bijection can be established between two sets such as the integers and the even integers, the sets must be of the same cardinality (i.e. equipotent), in this case $\aleph_0$. Other bijections, such as that between E^1 and E^2 are harder to find.

symmetric, it is an equivalence relation. If it is anti-symmetric, it is called a *partial order* or *order relation*. (Example: If we allow a person to be his own descendant, the relationship "descendant of" is a partial order of the human race; also, in any set X, the relation of inclusion, "$\leq$", among the set of all subsets of X is a partial order.) I can think of few examples of a preordering which are not also either equivalence or order relations. In an order relation, it is not necessary that every two elements be related (i.e. that the relation be complete); if this condition does hold, the set is called a *totally ordered* set, e.g. the relation "$\leq$" in E^1. (In fact we must use this total order on E^1 before we can talk about Euclidean spaces.)

2.2 A *well-ordered set* or *ordinal* is a totally ordered set in which every non-empty subset A has a first element, i.e. $\exists a_i \in A \ni \forall a \in A,\ a_i R a$. A totally ordered set need not be well-ordered, e.g. for the relation "$\leq$" in E^1, there is no first element for any exclusive interval (a,b), since a does not belong to the interval and every element in the interval has lesser elements. The relation "$\leq$" is well-ordered for the set of non-negative integers, but again not for the set of all integers.

2.3 Remarkably enough, every set has some relation in it which is an ordinal! This allows one to "count" the elements of any set and compare them with elements in any other set, since the well-ordering relation dictates a first element to the whole set, a second element, etc. Thus one can assign a *cardinal number* to each set and say that two sets have the same cardinal number (or are <u>equipotent</u>) iff $\exists$ a bijective map between them. (See Fig. 3.) So two finite sets are equipotent if they have the same number of elements in them. The concept of equipotence is only interesting when we consider non-finite sets. The first non-finite cardinal is called $\aleph_0$ ("Aleph-zero"). If the cardinal number associated with a set X (written card X) is $\aleph_0$, we also say that X is <u>countable</u>. Examples include the set of even integers, the set of integers and the set of rational numbers. The next highest cardinal (we assume) and the first uncountable one is $\aleph_1$ and in general one can create higher cardinals by considering the set of all subsets of a set X, call it P(X). Then card X $<$ card P(X). Card E^1 = card $E^n = \aleph_1$. I think that we need only worry about the finite cardinals, $\aleph_0$ and $\aleph_1$.

FUNDAMENTALS OF GENERAL TOPOLOGY

<u>1. Topological spaces</u>

1.1 <u>Open sets</u>---We define a <u>topology</u>, T, on a set X as a collection of subsets of X $\ni$ all intersections of a finite number of members of T belongs to T and all unions of any number of sets in T also belongs to T. Each set in the topology is called an <u>open set</u>. (See Fig. 4.) $\emptyset$ and X both T. A set and a topology on it form a <u>topological space</u>, (X,T). Note that there is no intrinsic meaning to the term "open set" other than that any finite intersection or arbitrary union of ope sets is open. An example of a very strange topological space, let X = $\{0,1\}$ and let T consist of $\{0\}$, $\{0,1\}$ and $\emptyset$, but <u>not</u> $\{1\}$. This space, called <u>Sierpinski space,</u>

is a perfectly legitimate one.

1.2 Examples: In E^1, let all exclusive intervals and all unions of exclusive intervals be open. This is called the Euclidean topology for E^1. (Notice that if we didn't restrict our topology to <u>finite</u> intersections of open sets, then $(a-1,b+1)\cap(a-\frac{1}{2},b+\frac{1}{2})\cap(a-\frac{1}{4},b+\frac{1}{4})\cap\ldots=[a,b]$ would also be open, therefore, every subset of E^1 would be open.) Similarly, in E^2 let the open sets be the interiors of any closed curve, that is all the points inside, not including the points on the curve itself, and all unions of these sets. We can likewise define the Euclidean topology for any E^n and from now on, when I speak of a <u>Euclidean space</u>, I mean a set with the Euclidean topology. Intuitively speaking, because we allow all unions and only finite intersections of the open sets, open sets in Euclidean spaces are usually "fuzzy"---that is, one cannot pinpoint where they end because their "borders" are not part of them. In the discussion here, I think it is worthwhile to understand the mathematical definition of openness, instead of just the specific Euclidean applications.

1.3 We use the word "space" to refer to any topological space. The word "set" is usually used to refer to a set which is not topologized. Thus, the same set X can form many topologies, e.g. (X,T_1) and (X,T_2). For example, consider E^1 with the topology described above (called the Euclidean topology). Another topology could be defined as all intervals which are exclusive at their upper end and all unions of these intervals. Thus both (a,b) and $[a,b)$ are open. Clearly, this topology is larger than the <u>Euclidean topology</u> in the sense that it includes all of the

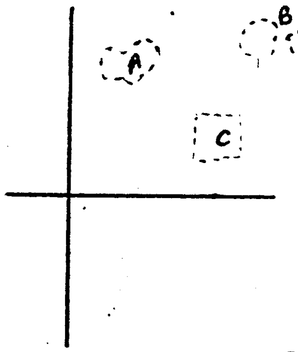


Figure 4. Open sets in E^2. The three sets shown are represented by dotted lines to show that they do not include their borders. This representation, the Euclidean topology is only one possible topology on the set E^2.

Figure 5. Two ways to form closed sets. The shaded area in 5a is the complement of the open set A. The shaded area in 5b is the closure of the open set A.

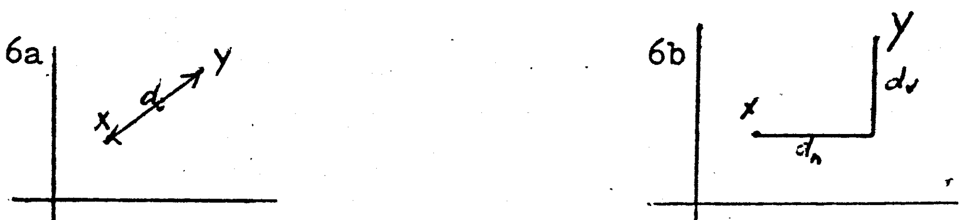


Figure 6. Metrics on E^2. The natural Euclidean metric is shown in 6a. $d(x,y) = d_o$. The metric $d(x,y) = d_h + d_v$, is shown in 6b.

Figure 7. Formation of metric spaces. 7a and 7b show examples of open balls for the metrics of 6a and 6b, respectively. Though each of these different metrics forms a different basis, the two bases generate the same topology.

open sets of the Euclidean topology and more.

1.4 A neighborhood of a point x, U(x) is an open set $\ni x \in U$, i.e. any open set which contains that point. One can also talk about neighborhoods of a set. In Sierpinski space, the only neighborhood of $\{1\}$ is the whole space, while $\{0\}$ has itself as well as the whole space as neighborhoods.

1.5 Closed sets can be defined in two ways, either one necessarily implying the other. First, a set $C \subset X$ with a topology T is closed iff C' is open. In other words, the closed sets in a topological space are the complements of the open sets. The second definition requires the definition of boundary or fringe. For any $A \subset X$, Fr(A) = all $x \in X \ni$ any neighborhood of x intersects both A and A', i.e. $\forall U(x)$, $U \cap A \neq \emptyset$ and $U \cap A' \neq \emptyset$. For example, in the Euclidean space E^1 the fringe of the set (a,b) is the points a and b. This is also the fringe of the set [a,b]. The second definition for a closed set is thus: A set A is closed iff it contains its own boundary, i.e. $Fr(A) \subset A$. In E^1, [a,b] is closed. It is possible though, in some topologies, for a set to be both open and closed. (See Fig. 5.) Other concepts are:

Interior: $Int(A) = A - Fr(A)$ = all points in A that are not on the fringe. Int(A) is the largest open set contained in A.

Closure: Closure of $A = \bar{A} = A \cup Fr(A)$ = the smallest closed set which contains A. Thus, for any space and any set A, Int(A) and Fr(A) form a partition of $\bar{A}$ and Int(A), Fr(A) and Int(A') form a partition of the whole space. A subset of a space X is dense if $\bar{D} = X$. For example, with the Euclidean topology, the set of rational numbers is dense in E^1.

1.6 A <u>metric</u>, d, on a set X is a map $d: X \times X \to E^1$ assigning a non-negative real number (called the <u>distance</u>) to each pair of points in X, under the following conditions:

1) $d(x,y) = 0$ iff $x = y$, i.e. two distinct points are a positive distance apart.

2) $d(x,y) = d(y,x)$ i.e. distance is symmetrical.

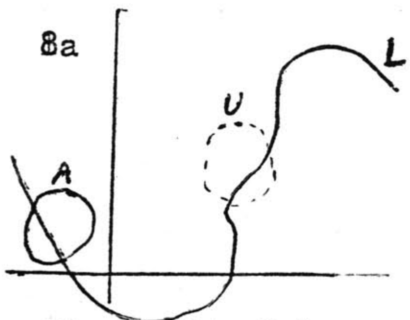
3) $d(x,y) + d(y,z) \geq d(x,z)$ i.e. the "direct" distance between two points cannot be greater than any indirect distance.

A metric set is said to be <u>bounded</u> iff d has a finite maximum, M, i.e. for any x and y, $d(x,y) < M$. (See Fig. 6.)

An <u>open ball</u>, written $B(x;r)$ where x is a point in a set X and r is a positive real number, is a subset of X with a metric d, such that all points in B are "closer" than r to x. Formally, $B(x;r) = \text{all } y \in X \ni d(x,y) < r$. In E^1 with the metric $d(x,y) = |x-y|$, the open ball $B(x;r)$ is just the set of all points in the interval $(x-r, x+r)$. For E^3 with the metric corresponding to our traditional notion of distance, any open ball, $B(x;r)$, is just the set of all points in the interior of a sphere centered around x with radius r. Another example would be a four-dimensional space with three physical coordinates and one temporal coordinate. To determine the distance between two events or points of this space, one must find some way to combine temporal duration with physical length to form a single metric. Special relativity goes beyond our definition of a metric in this space since it states that the distance between two points is not uniquely determined by the position of those two points; in addition, there are "subjective" factors referring to the measurer. A metric must be an objective phenomenon. However there are many other metrics in Euclidean spaces besides the traditional one (the straight line distance); some of these may even be based upon objective psychological states, such as common ways in which humans distort duration and distance because of environmental cues. The traditional distance is called the natural Euclidean metric.

1.7 A basis for a topology is a collection of open sets any open set can be expressed as the union of sets from the basis. Thus, the Euclidean topology for E^1 has as a basis the set of all exclusive intervals, (x,y). In general, we can create a topology on a set by establishing a basis, rather than dealing with the whole topology. For example, for any set, X, with a metric defined on it, we can create a space by letting the set of all open balls in X be the basis for a topology. The last example of a basis for E^1 is such a basis. A space created in such a way is called a metric space. Any metric on a set thus defines a unique space, but a topological space may not be metric or may have several different metrics. (See Fig. 7.)

1.8 A subspace is a subset $Y \subset$ a space (X, T_X) with its own topology, $T_Y \ni V \in T_Y$ iff $V = Y \cap U$ for some $U \in T_X$. This topology is called the relative topology, denoted $T(Y)$. It is important if, for example, we want to discuss topologies of a curve on a surface, the latter of which has already been topologized. (See Fig. 8.) Obviously, if Y is open in (X, T_X), $T(Y) =$ all open U in $X \ni U \subseteq Y$.



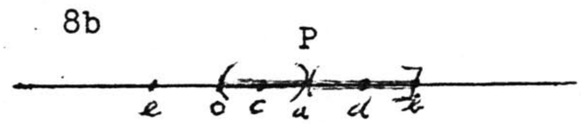


Figure 8. Subspaces. Consider a curve, L, in E^2 as shown in 8a. Although L and any subsets of L are closed in E^2 with the Euclidean topology, as a subspace, L contains many open and closed sets. The subset of L within the dotted lines is open because it can be represented as $L \cap U$, where U is open in E^2. By the same reasoning, $A \cap L$ is a closed set in the subspace. Let $P = (0,a) \cup (a,b]$ be a subspace of E^1 as in 8b. The subset $(0,a)$ is both open and closed in P: $(0,a) = (e,d) \cap P = [e,d] \cap P$. Also, $(d,b]$ is open and $(0,c]$ is closed.

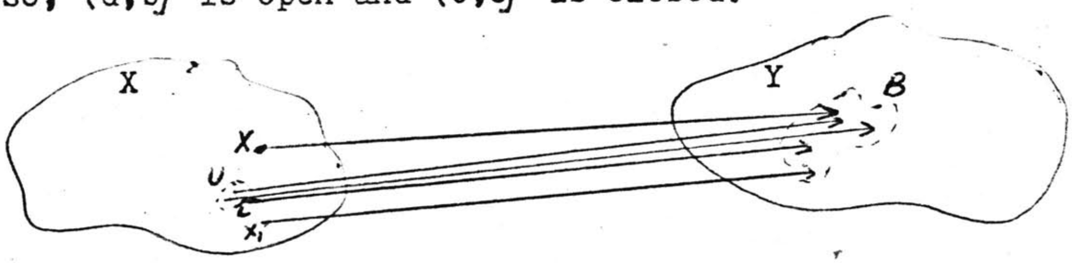


Figure 9. Continuity. For the f: X→Y illustrated, f is not continuous since $f^{-1}(B)$ (with B open in Y) equals $x \cup x_1 \cup U \cup L$, which is not open in X.

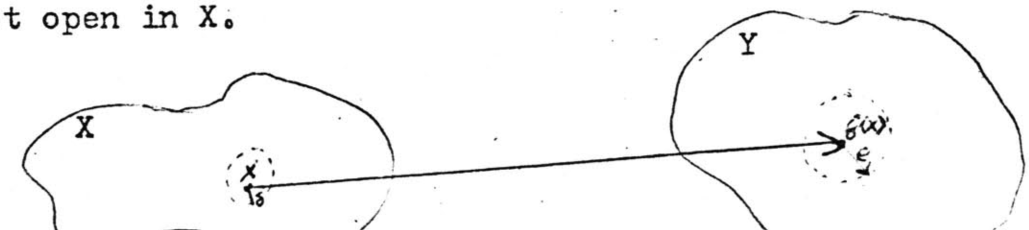


Figure 10. Metric continuity. For f: X→Y, where X and Y are both metric spaces, f is continuous iff $\forall e > 0$, however small, there is some $\delta > 0 \ni f(B_X(x,\delta))$ lies entirely within $B_Y(f(x),e)$.

1.9 A <u>continuous function</u> $f: (X,T_X) \rightarrow (Y,T_Y)$ maps a topological space (not just a set but a space) into another $\ni \forall V$ open in Y ($V \in T_Y$), $f^{-1}(V)$ is open in X (i.e. $f^{-1}(V) \in T_X$). (See Fig. 9.) The conventional idea of continuity in Euclidean space in delta-epsilon terms (I use "e" instead of "ϵ" to avoid confusion) is just a specific application of topological continuity to metric spaces, that is, for any x and f(x), $\forall e > 0$, no matter how small, $\exists \delta > 0 \ni f(B_X(x,\delta)) \subset B_Y(f(x),e)$. (See Fig. 10.)

1.10 A <u>homeomorphism</u> between two topological spaces, X and Y is a bijective, continuous map, f, $\ni f^{-1}$(which is also a map, due to bijectivity) is continuous. Thus, a homeomorphism is a bijective, bicontinuous map, written $f: X \cong Y$. This also means that $f^{-1}: X \cong Y$. If any such map exists between two spaces, we can also say that the two spaces are homeomorphic to one another. A homeomorphic relation sets up a one-to-one correspondence both between the points of each space and between the open sets of each topology. We say two homeomorphic spaces are <u>topologically equivalent</u> (that homeomorphism is indeed an equivalence relation is easily verifiable) and the qualities in which the study of topology is most interested are <u>topological invariants</u>, that is, qualities for which, if the quality applies to a space X, it applies to any space homeomorphic to X. Examples: 1) E^1 is not homeomorphic to E^n, nor, in general, is $E^m \cong E^n$ unless m=n. 2) A sphere $\cong$ a cube $\cong$ any polyhedron or ellipsoid or any enclosing surface in E^3 as long as all these figures are given the relative topology as subspaces of E^3 with the Euclidean topology. Such a surface may

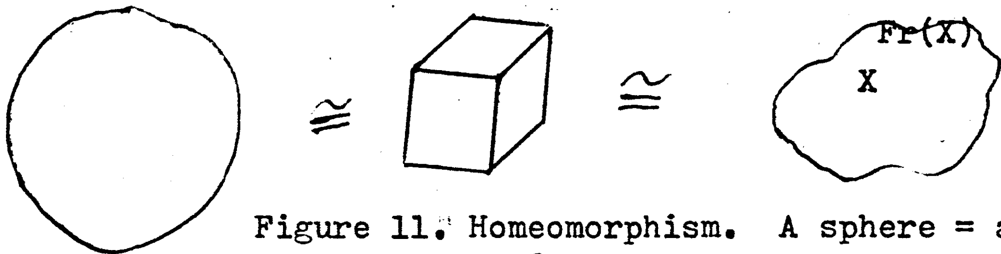


Figure 11. Homeomorphism. A sphere = a cube = any closed, bounded subset of E^2, call it X, for which the quotient set, X/Fr(X), has beentaken.

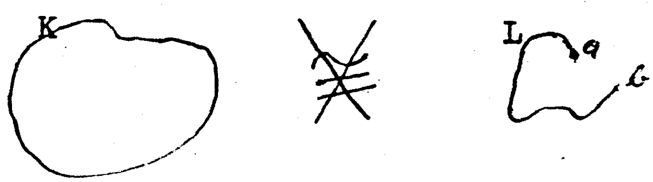


Figure 12. Simple nonhomeomorphic curves. These curves must be non-homeorphic since any continuous f: K→L would have to have $f^{-1}(a) = f^{-1}(b)$.

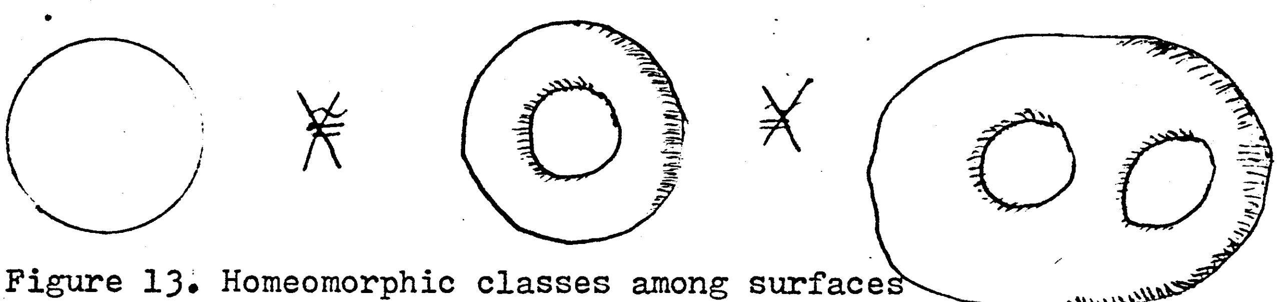


Figure 13. Homeomorphic classes among surfaces in E^3. The sphere, the 1-fold torus (just called torus), and the two-fold torus are examples of three different homeomorpic classes among the surfaces possible in E^3. In fact, all other physically real homeomorphic classes of this type surface are exemplified by all the n-fold toruses, where n is any positive integer.

be pictured as a closed, bounded sheet in a plane with all the boundary points identified to form one point. (See Fig. 11 and 12.) Yet none of these surfaces is homeomorphic to the surface of a 1-fold torus (or simply called a torus) or donut-shaped figure. (See Fig. 13.) Because of the equivalence classes created by homeomorphisms, topology is said to be geometry without size or shape.

2. Separation

2.1 There are several axioms of separability referred to as T_0, T_1, T_2, etc. (I only know of those up to T_5 through my reading). Each successive one is more restrictive than its predecessors. We will only look at T_2 and T_4, the most important ones, although for my purposes, many possible psychological spaces are less clearly separated than the ones that topologists study, so the lower axioms become more relevant.

2.2 We call a space T_0 iff for any two points, at least one of them has a neighborhood which does not contain the other point. One might say that at least one point can be separated from the other. A space is T_1 iff for any two points, each has a neighborhood not containing the other, i.e. each point can be separated from the other. A space is T_2, also called Hausdorff, iff, for any two points, x and y, there are neighborhoods U(x) and V(y) $\ni$ $U \cap V = \emptyset$, in other words, if any two points possess disjoint neighborhoods.

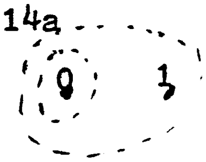
2.3 Examples: All metric spaces (including the Euclidean spaces) are Hausdorff. Since most spaces we can intuitively grasp are

Hausdorff, I will provide some mathematical examples that are not Hausdorff. (See Fig. 14.) Sierpinski space is T_0 but not T_1 or T_2. The only points to consider are 0 and 1; 0 can be separated from 1, but 1 has no neighborhood which does not contain 0. Another example of a T_0 but not T_1 (or higher) space is the space $X = [0,1) \subset E^1$ (the interval which includes 0 but not 1) with $T_X =$ all $B \subset X \ni B = [0,k)$, for any k greater than 0 and ≤ 1. Any neighborhood of $x \in X$ contains all $y < x$ so the lesser of two points can be separated from the greater one, but not vice-versa. A final example is provided by any infinite set with the topology consisting of all sets of the form X minus a finite number of points. This space is T_1 but not Hausdorff: for two points x and $y \in X$ the open set X-y is a neighborhood of x which does not contain y and likewise the set X-x separates y from x, but these two sets (and any other two sets) cannot satisfy the Hausdorff condition since they intersect.

2.4 All points and all finite sets are closed in a Hausdorff space (though they may also conceivably be open). A subspace of a Hausdorff space is also Hausdorff if it has the relative topology. And the Hausdorff quality is a topological invariant (i.e. any space which is homeomorphic to a Hausdorff space is Hausdorff).

2.5 A space X is <u>T_4</u> or <u>normal</u> iff $\forall$ disjoint closed sets A and B in X, $\exists$ open $U \supset A$ and an open $V \supset B \ni U \cap V = \emptyset$. The requirement for normality just replaces both points in the Hausdorff condition with closed sets. Another, intermediate condition, called <u>T_3</u> or <u>regular</u> requires that there be nonintersecting neighborhoods for

14a



14b

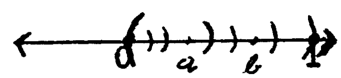
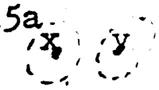


Figure 14. Spaces of little separation (T_0). Sierpinski space, whose open sets are shown in 14a, is T_0: 0 can be separated from 1, but not vice-versa. The interval (0,1) with open sets all of the form (0,x), $0 < x \leq 1$, is also T_0 since any neighborhood of b also contains a, though a can be separated from b.

15a



15b

15c

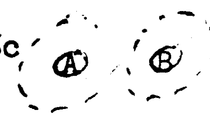


Figure 15. Separation axioms. Separation asks if nonintersecting neighborhoods canbe found for: 15a) distinct points, Hausdorff; 15b) a closed set and a distinct point, regularity; 15c) non-intersecting closed sets, normality.

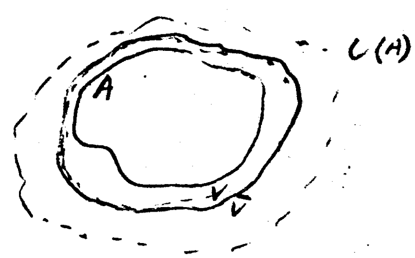


Figure 16. An alternate characterization for normality. For any closed set A and any neighborhood U(A), is there an open V such that $A \subset V \subset \bar{V} \subset U$?

any point and any closed set which does not contain that point. For any space, normal implies regular which implies Hausdorff and normality is the strongest criterion I will cover. (See Fig. 15.)

2.6 There is another easy way to define normal: A space X is normal iff $\forall$ closed A and any open neighborhood $U \supset A$, $\exists$ an open $V \ni A \subset V \subset \overline{V} \subset U$. (See Fig. 16.) Normality is a topological invariant, but a subspace of a normal space need not be normal, unless it is a closed subspace.

3. Connectedness

3.1 A space is connected iff there exists no partition of it into two or more open sets. An equivalent way to state this is that there can be no subset which is both closed and open (other than $\emptyset$ and the space itself). Any E^n with Euclidean topology is connected; in fact for $n > 1$, E^n minus any countable subset is connected. Connectedness is invariant under any continuous mapping.

3.2 A subset of a space is connected iff it is connected as a subspace. (This is a definition, not a derived fact.) Thus, for example, though E^1 is connected, the set $D = E^1$ minus any point x, is not, since $\{\text{all } y < x\} \cap D = \{\text{all } y \leq x\} \cap D$ is both open and closed by the relative topology. (Fig. 8b provides another similar example of a disconnected subspace of E^1.) In general, the only connected subsets of E^1 are individual points, intervals and E^1 itself.

3.3 A component, C(x), of a point in a space is the largest connected subset of the space which contains x. Examples: In the example given above in 3.2, there are just two components:

$\{$all $y < x\}$ and $\{$all $y > x\}$. In the set of all rational numbers considered as a subspace of E^1, each point is its own component because no two points belong to a connected subset. In a connected space there is only one component. The relation "belongs to the same component as" is an equivalence relation, so the components of any space form a partition of that space. Every component is a closed set.

3.4 Intuitively, the presence of disconnection in a space means that there is at least one subset, A, Int $A = A = \overline{A}$. Thus, A has no border outside itself though it is open or "fuzzy". Usually, I picture disconnection as breaking a space into two (or more) totally disjoint subspaces with no "interesting" relationships between individual members or subsets across the break(s).

3.5 A path is intuitively a curve in a space which connects two points, formally $p: I \rightarrow X$, is a continuous map, with p(0) called the initial point of the path and p(1) called the terminal point. (Fig. 19 incidentally shows some paths.) The concept of path is important in many fields of topology.

3.6 A path-connected space is a space in which every two points are connected by a path. This is a more intuitive notion of connectedness than the first one and it is also stronger, though in most cases comes out to the same thing. The path-connectedness idea also leads to path-components in an anlogous way.

4. Compactness

4.1 A covering of a set is a collection of sets $\ni$ every $x \in X$ belongs to at least one member of the covering. An <u>open covering</u> of a space is (obviously) a covering in which all the sets are open, e.g. any basis. There are both <u>finite coverings</u> and <u>infinite coverings</u> depending on the size (i.e. cardinality) of the covering. A <u>subcovering</u> is a covering which is a subset of the origninal covering, i.e. is a refinement of the covering in that it eliminates some redundant sets.

4.2 A space is <u>compact</u> iff every open covering contains a finite subcovering. (See Fig. 17.) Compactness is invariant under all continuous mappings. Also, any closed subspace of a compact space is compact.

4.3 Examples: Consider the interval (0,1) as a subspace of E^1 and the infinite open covering including all sets of the form $(0,1-\frac{1}{n})$ for each $n > 1$. This is indeed a covering---every point belongs to some interval, no matter how close to 1 it is---yet there is no finite refinement of this covering which still covers the whole interval; hence (0,1) is not compact. The unit interval $I = [0,1]$, on the other hand, is compact. For instance the infinite open covering which includes every $[0,1-\frac{1}{n})$ plus $(1-\frac{1}{m},1]$ for some fixed $m > 1$ can reduced by eliminating all but a necessary number of sets, e.g. $[0,1-\frac{1}{m+1})$ and $(1-\frac{1}{m}]$.

4.4 Metric spaces provide a more understandable meaning for compactness. A compact subspace of a metric space is necessarily closed and bounded and, in particular, for any subspace of a Euclidean space E^n, compactness closed and bounded.

4.5 A Hausdorff space is <u>2^o countable</u> (read: "second-degree countable") iff it has a countable basis. For a space to be 2^o

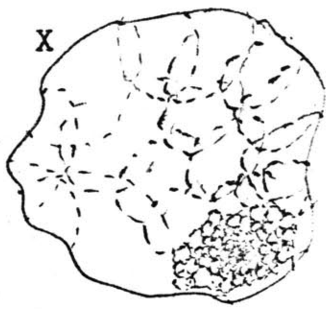


Figure 17. Compactness. The illustration symbolically shows a possible infinite open covering of a space, X. Can a finite sub-collection of these sets be chosen which will still cover X? If so, X is compact. Closed and bounded subspaces of Euclidean spaces are always compact.

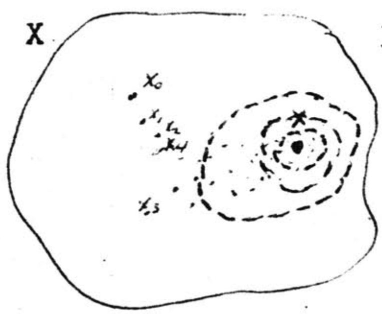


Figure 18. Convergence. An infinite sequence of points, x_n (n ranging from 0 to infinity) in a space, X, is said to converge to a point, x, iff every neighborhood of x contains all but at most a finite subset of the x_n. The point x is called a limit point of the sequence. In fact, this concept is akin to the idea of a fringe.

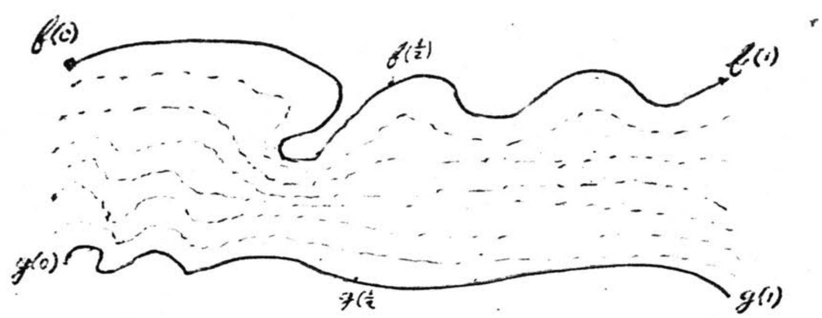


Figure 19. Homotopy of paths. Functions f and g are mappings of $I = 0,1 \quad E^1$ into a space, X. We say f is homotopic to g iff there is a continuous deformation of f into g, i.e. a continuous set of paths with f as the first path and g as the last. Unlike homeomorphism, with homotopy, the deformation must take place within a space (in this case X) that both functions relate to.

countable every open covering has a countable subcovering. So 2^{o} countable is a weaker property than compactness. For example, E^1 is 2^{o} countable. Consider all open balls, $B(x,r)$ both x and r are rational numbers. This set is countable and is a basis. Yet E^1 is not compact.

5. Convergence and completeness

5.1 A sequence is an infinite set of points, $x_1, x_2, \ldots$ called "the sequence x_n." A sequence x_n is said to converge to a point x a space X iff for any neighborhood $U(x)$, an integer N n N x_n U. (See Fig. 18.)

5.2 In a Hausdorff space, a sequence can converge to, at most, one point.

5.3 The Cauchy criterion for convergence in a metric space is this: For any positive number e (no matter how small), $\exists$ an integer $N, \ni \forall n$ and m both greater than N, $d(x_n, x_m) < e$, i.e. the members of the sequence grow closer and closer to each other as the sequence progresses. (This is not quite the same as saying that the members grow closer and closer to a limit point, which is the criterion for convergence in a metric space.) If a sequence in a metric space converges, it meets the Cauchy criterion; the converse is not necessarily true, but if it is, the metric space is called complete. Note that a space may be complete with one metric, but not with another. In general we call a space complete if it has at least one complete metric.

5.4 Examples: All Euclidean spaces are complete. The interval $(0,1]$ as a subspace of E^1 is not complete, for the Cauchy sequence

$x_n = \frac{1}{n}$ converges to a point outside of the space, namely 0. The rational numbers are similarly incomplete since a sequence of rational numbers may converge to an irrational. Though the irrationals are also incomplete with the natural Euclidean metric, there are other metrics for which the irrational numbers are complete.

5.5 Oddly enough, completeness is not a topological invariant.

6. Homotopy

6.1 Homotopy ($\simeq$) is a relation between two continuous functions that take a given topological space into another. We say that given $f: X \to Y$ and $g: X \to Y$ (or simply $f,g: X \to Y$), $f \simeq g$ iff $\exists$ a continuous $F: X \times I \to Y \ni F(X,0) = f(X)$ and $F(X,1) = g(X)$. Intuitively, if we let $F_t = F(X,t)$ for $0 \leq t \leq 1$, then the set of all the F_t represents a continuous deformation between f and g, so that $F_0 = f$ and $F_1 = g$. (See Fig. 19.)

6.2 A constant map, $f: X \to Y$ is a map for which $\forall x \in X$, $f(x) = y_0$. In other words, f takes the whole space X into one point of Y. Such a map is always continuous. Any function homotopic to a constant function is called nullhomotopic.

6.3 Y^X is the set of all continuous functions from X into Y, called the function set. Homotopy is an equivalence relation on this set and divides it into homotopy classes. For example in a connected space there is one and only one nullhomotopic class, i.e. set of functions which can be deformed into a constant map. If a space is disconnected, there are as many nullhomotopic classes as there are components.

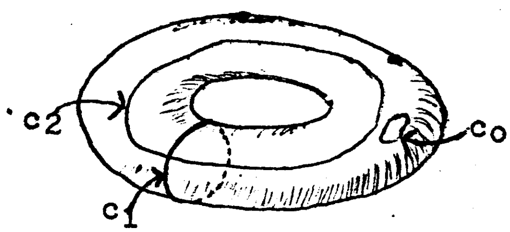
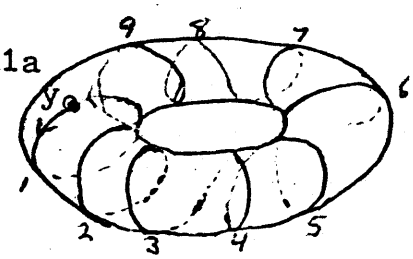


Figure 20. Basic homotopy classes of loops on a torus. c_0 is a nullhomotopic loop and thus belongs to the null class. c_1 is called a meridian and c_2 is called a parallel.

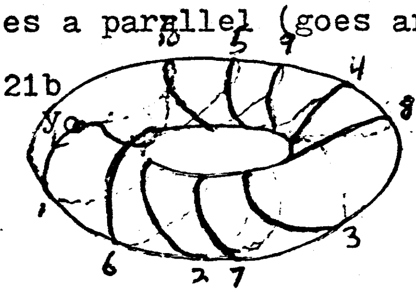
21a



r = 9
s = 1

Figure 21. Homotopy classes of loops on a torus. All classes of loops on a torus can be described as a combination of two integers---the number of times the loop crosses the meridian (or goes around a parallel), called r, and the number of times it crosses a parallel (goes around in the direction of a meridian) called s.

21b



r = 10
s = 2

6.4 Paths are a particular type of function, p: $I \rightarrow Y$ and among these there is a subset of closed paths, called loops or Jordan curves, consisting of those paths for which the initial point and terminal point are the same, i.e. $p(0) = p(1) = y_0$. This set of loops is called the fundamental group at y_0 when its homotopy classes are identified. If the space, Y is connected, then all fundamental groups have the same form, regardless of their base point y_0.

6.5 Examples: On the surface of a sphere, any closed path is continuously deformable into a point, i.e. all loops are null-homotopic, so the fundamental group has only one member. Whenever the fundamental has only one meber, the space is called simply-connected; thus, all closed surfaces homeomorphic to a sphere are simply-connected. All Euclidean spaces are also simply-connected. For the 1-fold torus, the situation is more complex. (See Figs. 20 and 21.) And for toruses with more "donut-holes" the complexity multiplies considerably.

6.6 The unit square is $I^2 = I \times I$. Generally, the unit n-cube = I^n = the set of all n-tuples $(u_1, u_2, \ldots, u_n)$ with $0 \leq u_i \leq 1$ for any i from 1 to n. J^n is the boundary of I^n defined to be all points with at least one co-ordinate equal to 0 or 1. (See Fig. 22.) If we have a space Y and consider all continuous maps p: $I^n \rightarrow Y$ $\ni p(J^n) = y_0$ and thence the homotopy classes thus formed, we find we have an n-dimensional homotopy group. This whole concept is nearly impossible to visualize, but it becomes easier if you realize that the fundamental group is just the one-dimensional homotopy group and try to extend it from there.

6.7 Homotopy bears a peculiar relationship to homeomorphism. Homeomorphism compares the forms of spaces in a very fundamental way, while homotopy compares various maps between two spaces, and thus is usually a way to measure the relative complexity of spaces. In many cases, comparing maps to see if they are deformable into one another is the same as comparing those maps considered as spaces. Yet in other cases, there is a difference e.g. loops which are all homeomorphic to one another as spaces but not always homotopic as maps. Replacing one of the two spaces in a function set by a space homeomorphic to it will not change the structure of the homotopy classes; we have already used this fact to assert that all spaces homeomorphic with the sphere are simply-connected.

GLIMPSES OF ALGEBRAIC TOPOLOGY

This section may seem contrived and quite distant from the last two. This is partly because its paractical applications are already so well developed that I have directed my choice of what to cover towards those known uses and also partly because, in ignoring some of the difficult theoretical material, particularly the concept of homology, I have cut out many of the means by which one could have seen how it all fits together. Nevertheless, this section will stand up on its own and suit our purposes well.

1. Simplicial complexes

1.1 In E^n, a set, P, of m points ($m \leq n$) is <u>linearly independent</u>, iff, intuitively speaking, no subset consisting

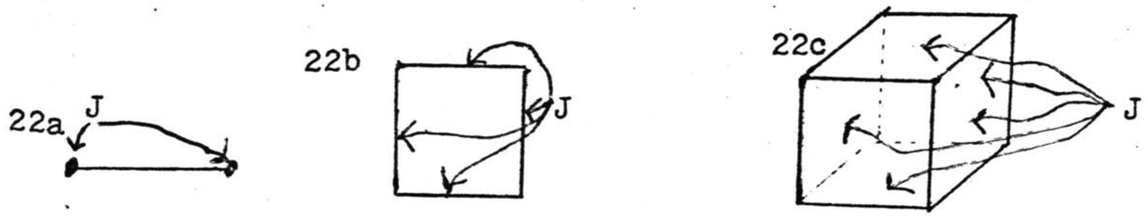


Figure 22. The unit n-cube. 22a-c show the unit n-cubes for n = 1, 2 and 3. In each case, J represents the set of all points on the boundary of I (where J is just two points), I^2 (J is all points on the four edges) and I^3 (J is all the points on faces, edges or vertices. If we form the homotopy classes of the mappings of I^n/J onto any selected space, X, we obtain the n-dimensional homotopy group. For n=1, I/J is a loop and the group obtained is the fundamental group.

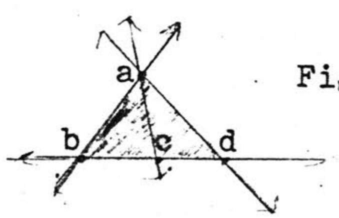


Figure 23. Linear subspaces of Euclidean spaces. Because points b, c and d are not linearly independent, the dimension of $P = \{a,b,c,d\}$ is only 2. The subspace spanned by $\{a,b,c,d\}$ is E^2, the plane of this sheet of paper. The convex hull formed is shown by the shaded area.

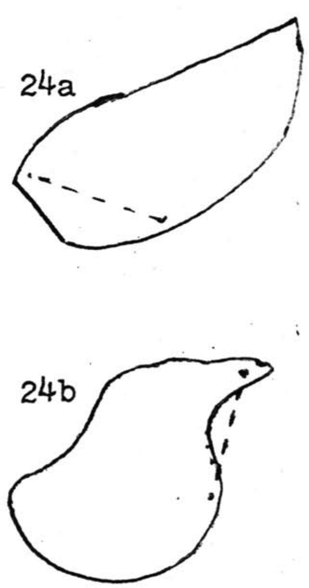


Figure 24. Convexity. A figure is convex (in a Euclidean space) iff any line segment connecting two points of the figure is totally contained within the figure. 24a is convex in E^2, 24b is not.

iff, intuitively speaking, no subset consisting of 3 points lie on a line, no 4 points lie in the same plane and, in general, no i+2 points lie in the same i-space. The __dimension__ of P is m if P is linearly independent; otherwise the dimension is the dimension of the largest linearly independent subset contained in P. For any $P \subset E^n$, the dimension of $P \leqq n$. (See Fig. 23.)

1.2 A set of points, P, forms a linear subspace of E^n iff all the points on any line determined by two points in P also belong to P. If the word "line" in this definition is restricted to the "line segment" between two points in P, we have a __convex linear subspace__. In general, any figure for which any line segment between two points is contained in the figure is called __convex__. (See Fig. 24.) For example, three linearly independent points will determine a traiangle as its convex subspace. A set of m+1 independent points (or any m-dimensional set) will determine the interior of an m-dimensional polyhedron, with the points of the set as vertices. The points of the set are said to __span__ the subspace.

1.3 An __open m-simplex__ (σ^m) is a convex linear subspace spanned by m+1 linearly independent points. The union of an open m-simplex and its boundary, the surface of its m-polyhedron, is called a __closed m-simplex__ ($\overline{\sigma^m}$). A __q-face__ of an m-simplex ($q \leq m$) is a q-simplex whose spanning set of q+1 points is a subset of the spanning set of the m-simplex. Thus, a 0-face is called a __vertex__, a 1-face is called an __edge__ and a 2-face is usually just called a __face__.

1.4 A <u>simplicial complex</u> or just <u>complex</u> is a set of simplexes of (possibly) various dimensions with the only stipulation being that all the q-faces ($0 \leq q < m$) of any m-simplex are also members of the complex. (See Fig. 25.) The <u>dimension</u> of a complex K is the dimension of the largest simplex in K, i.e. the maximum m for which $\exists$ an m-simplex in K. A <u>q-section</u> of a complex K is the subcomplex consisting of all m-simplexes $\ni m \leq q$.

1.5 <u>Triangulation</u> is a process of finding for a given subspace of an E^n, a complex K, so that for any point in the subspace, there is a simplex K that contains the point. To extend triangulation beyond just polyhedra, we must define a new simplex = any set which is homeomorphic to our previous definition of simplex. (Some triangulations of two-dimensional spaces are illustrated in Figures 26 and 27.)

1.6 The <u>Euler characteristic</u> of a triangulation is a function of the number of simplexes of each dimension that belong to the triangulating complex K. Each subspace of E^n has an Euler characteristic which does not change for different triangulations; moreover, the Euler characteristic (I will call it P) is a topological invariant. The formula for any two-dimensional figure is P = the number of 2-faces minus the number of 1-faces (edges) plus the number of 0-faces (vertices) = $F - E + V$. P for any simply-connected, bounded space $\cong$ a sphere = 2. P for any space $\cong$ a 1-fold torus = 0. P for any space $\cong$ a disk = 1...

2. Graphs

2.1 A <u>graph</u> is a set of points, A, with a relation $P \subset A \times A$. In

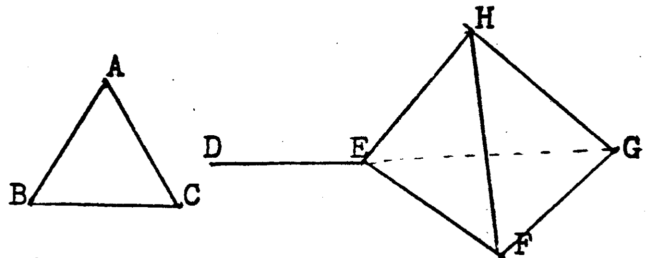


Figure 25. Simplicial complexes. Illustrated is a simplicial complex with one 3-face (EFGH), five 2-faces (ABC, EFG, EFH, EGH and FGH), ten 1-faces (or edges) and eight 0-faces (or vertices). This complex is not connected.

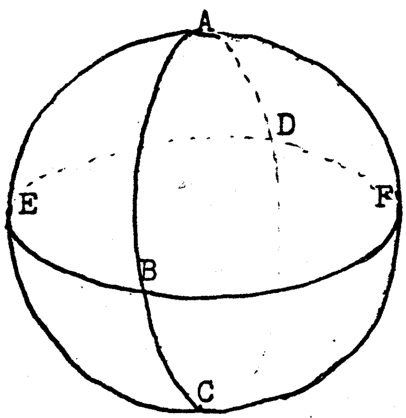


Figure 26. Triangulation of a sphere. One possible triangulation of a sphere into eight curvilinear faces, 12 edges, and six vertices. The Euler characteristic, P = 2.

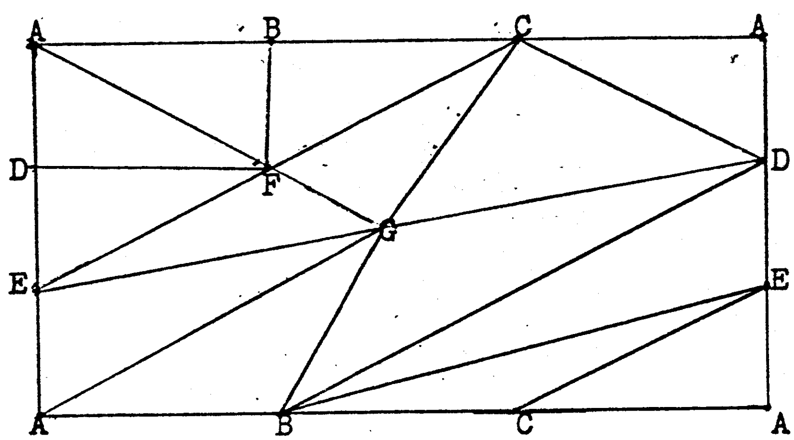


Figure 27. Triangulation of a torus. The illustration represents a triangulation of a torus into 14 curvilinear faces, 21 edges and seven vertices. P = 0. (The representation is formed into a torus by joining the upper and lower edges of the rectangle together (ABCA) to form a cylinder and then joining the two bases of the cylinder (ADEA) together to obtain a torus.)

simpler terms, a graph is a set of points with some pairs of them connected by arrows, called arcs. We write (x,y) ∈ P iff there is an arc from x to y. The basic things we seek to ask about a graph deal with the form or pattern of the existing arcs. (See Fig. 28a.)

2.2 A graph is reflexive ⇔ all possible loops are included, i.e. iff ∀x, (x,x) P.

A graph is transitive ⇔ if (a,b) and (b,c) are arcs, then so is (a,c)

A graph is symmetric ⇔ if (a,b) is an arc, so is (b,a). A symmetric graph usually uses a non-directed line, called an edge, to represent any arc-pair or loop. (See Fig. 28b.)

A graph is anti-symmetric ⇔ if (a,b) is an arc and a ≠ b, then (b,a) is not an arc.

A graph is complete ⇔ every pair of points is connected by at least one arc.

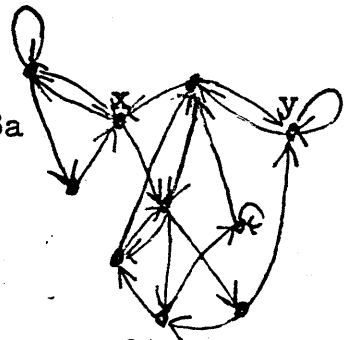
An equivalence graph is a symmetric, reflexive and transitive graph.

A preordered graph is transitive and reflexive.

An ordered graph is anti-symmetric, transitive and reflexive.

A totally-ordered graph is complete and ordered.

2.3 A path (or chain, in a symmetric graph) is an ordered set of arcs (or edges) ∋ each arc (or edge) except the last leads into the same point that its successor leads out of. (In Fig. 29a, abcdefg is a path.) A circuit (or cycle, in a symmetric graph) is a specific kind of path (or chain) which ends at the same point it begins.

28a


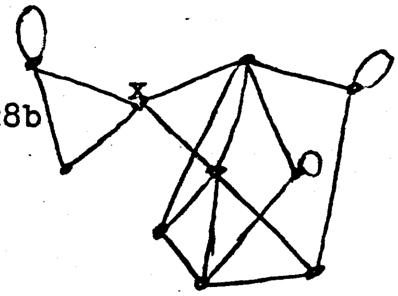
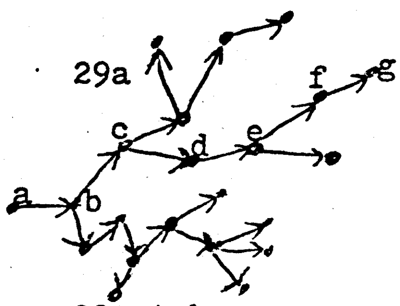
28b


Figure 28. Graphs. 28a shows a graph and 28b shows the same graph represented as a symmetrical graph. In either case, the point x is an articulation point.

29a


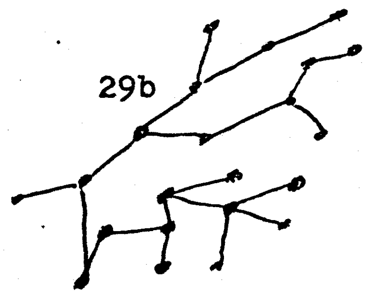
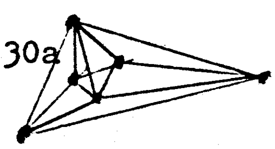
29b


Figure 29. Arborescences and trees. 29a illustrates an arborescence and 29b shows the same figure represented as a tree. Note that, while every arborescence has a unique root (in this case at a), a tree need not have a unique root.

30a


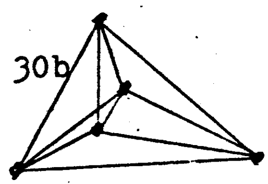
30b


Figure 30. Simply-representable graphs. Both of the figures shown above are not simply-representable in two dimensions.

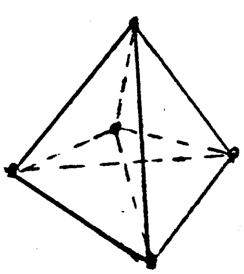


Figure 31. A simply-representable configuration in three dimensions for 30b. Note that the cyclomatic here is 10.

2.4 One often wants to know the qualities and "density" of the overall connection in a graph. A graph is <u>connected</u> iff there is no way to divide the graph into two parts without at least one arc joining them. A graph is <u>strongly connected</u> if there is a path leading from any point to any other point. A connected, symmetric graph is automatically strongly connected. A point in a connected graph, the loss of which point would mean a disconnection is called an <u>articulation point</u>. (See Fig. 28.)

2.5 In a symmetric graph, there are several indexes used to relate the number of edges, E, and the number of points, V, to tell us something about the "density" of the graph. $\beta = \frac{E}{V}$ is the simplest, but there are many others.

2.6 Another key concept in graph theory is distance, though it rarely meets the requirements of a metric in the sense we used it previously. The <u>length</u> of a path or chain is equal to the number of arcs or edges in it. The <u>distance</u> from x to y, d(x,y) is the length of the shortest path (or chain) from x to y. Example: In Fig. 28a, d(x,y) = 3 and d(y,x) = 2. Only in a symmetric graph does d(x,y) = d(y,x) always.

2.7 In a strongly connected graph, the <u>diameter</u> is the maximum distance found between any two points in the graph. In Fig. 28b, the diameter = 3. For a given point in a strongly connected graph, its <u>accessibility</u> = the sum of the distances from it to all other points. In Fig. 28b, the accessibility of x is 14. The <u>Konig number</u> for a point is the maximum distance to any point. In the whole graph, the <u>mean dispersion</u> is the average accessibility for all the points in the graph.

2.8 An arborescence with a root r (one of the points in the graph) is a graph with every point having only one arc leading into it, except r which has none. A tree is a symmetric representation of an arborescence, with arcs replaced by edges. (See Fig. 29.) Arborescences and trees have no circuits (or cycles). Another way to think of a tree is as a minimally-connected symmetric graph, $E = V - 1$. A tree is one of very few graphs for which distance is a true metric.

2.9 I will call a symmetrical graph simply-representable iff it can be represented on a plane without any arcs or edges intersecting one another. More generally, I will assign a dimension n to a graph (not to be confused with previous definitions of dimension) iff E^n is the smallest dimension Euclidean space in which the graph can be represented without intersection. A graph of dimension 1 is no more than a chain. The largest complete graph which is simply-representable (dimension 2) has only four points. (See Fig. 30 and Fig. 31.)

2.10 Any graph in simply-represented form on a plane can be seen as a degeneralte form of a triangulation if we expand the concept of 2-face (or just face) to include any cycle, no matter how many members, that is not subdivided into other cycles. If we allow this system, we find that any such graph has an Euler Characteristic of 1, i.e. $F-E+V = 1$. Another way to express it is in terms of the cyclomatic or number of cycles in a graph. Since this is the same as F, we say $C = E-V+1$ for any graph of dimension 2.

SUMMARY

The point of this summary is to pull some of the major points of this mathematical outline together in a concise fashion and to emphasize to the reader those concepts of topology which I feel are most applicable to environmental science and psychology. These are the concepts which will be explored in a rather disconnected fashion in the next sections.

Set theory and order---Set and operation ideas are very general ones which form the foundation of any mathematical study. They are all that we have, so it is difficult to say anything good or bad about them. Order is a similarly fundamental concept, but in it we find the beginning of the topologist's desire to categorize sets. I am led to believe that in the mathematician's few methods for ordering a space---whether it be into a partition of equivalence classes or a comparative relation which imposes a linear order of some sort and whether the points are perceived as a continuum ($\aleph_1$), as discretely countable, but still infinite ($\aleph_0$) or as finite---there may be a clue to that fiels of psychoogy that deals with"pattern recognition".

General topology---Topology, it seems to me, has grown through several negations of restrictive previous notions of mathematics, thus leading it to seek broader formulations and categories. A couple of mathematicians may state that the known Euclidean spaces all exhibit the trait that any covering has a finite subcovering, which leads someone to ask what spaces could be like which do not have this quality. Hence, the study of compactness begins its development.

The concepts of continuity, convergence, homeomorphism, separation, connectedness, compactness, completeness and homotopy are qualities of spaces and mappings of spaces which have had limited application so far and all of these in the realm of psychology. I feel that exploration of the basic structures of the mind is the best area in which these properties will find further application. Perhaps the computer programs written for the study of artificial intelligence can also utilize the structures involved int these properties, though this is a field of which I know next to nothing.

Separation and connectedness offer formal means of dissecting a space. Separation gives us a means of seeing a part of a space as an object unto itself while connectedness allows us to relate any part to any other part. They are not opposites though. A space can have both, or neither or just one of these characteristics.

Convergence is a formal way to characterize proximity. By isolating a sequence of points and an "ideal" to which these points grow closer and closer we have discovered a convergent sequence. If the space is not Hausdorff separable, then this ideal is not unique; if the space is metric in some fashion, then we can have sequences which "ought" to converge (i.e. Cauchy sequences), but may not coverge if the metric space is incomplete.

Compactness and 2^o countable are topological concepts which measure the complexity and depth of a topology in much the same way that cardinality measures any set. In a psychological space the absence of compactness means that the mind has no method of analysing the space into a finite set of complexly related (i.e. open) overlapping components. To say a subset of a space is compact is a much more detailed characterization of "fuzziness" than openness, for compactness measures this fuzziness or unfathomableness throughout a set while openness is only at the borders.

Homeomorphism and homotopy are equivalence relations which compare structures of spaces and functions between spaces, respectively. I will discuss possible applications of these tools soon, but I should note here that examples of their use so far are almost entirely as explanatory and clarifying metaphors, not as models which can generate possible new data.

Algebraic topology---With the application of purely topological concepts to structures of graphs and simplicial complexes, much of the broadness and complexity of these general categorizations is lost, or at least reduced to an algebraic situation with highly precise numerical indexes substituted for depth. The notions about algebraic topology presented in this paper are just a small sample of what is possible---this is because my emphasis is on the possibilities of application to psychological problems, for which the broad concepts of general topology, including non-Euclidean spaces, may be found to be most useful.

VIII. APPLICATIONS OF TOPOLOGY TO ENVIRONMENTAL PSYCHOLOGY

PAST EFFORTS*

Topology, despite its youth, has already earned quite an illustrious history of use as a modeling system. Basically, most of these applications have occurred ourside the field of general topology, using principles of graph or network theory, a branch of algebraic topology, the development of which has consisted greatly in the work of applied mathematicians and specific practitioners.

Network theory has had broad applications to many purely physical systems, such as transportation systems, other urban service delivery systems, electronic circuitry and certain aspects of economics (e.g. imports and exports among a set of nations). Part of the appeal of graphs to these studies is that they are easily adapted to a simple level of computer programs, including the use of numerical weighting of the connections between points. Such systems will not be discussed here, since they are in no way psychological; however they are useful since they demonstrate the complex development of which network theory is capable.

Another major use of network theory, more directly related to human psychology, is the sociogram, a tool of analytic sociology in which points represent people in a group and arcs or edges represent relations between them such as communications relayed (letters, phone calls, conversations, etc.) or relations of authority, kinship or friendship. This analysis, with much

*Any reference to authors of books or papers is listed in the bibliography.

clearer definitions of certain concepts than a purely intuitive approach is an excellent tool for observing leadership qualities, bureaucracies and hon-hierarchical social structures, cliques and strengths of connection in a group, to name just a few. Of course, there is a limitation of such a reduction of individuals to total equals, or, possibly with the aid of some weighting system, equals who differ only in one linear value in addition to the differences in the position of each individual in the graph relative to the graphed relation ship. This reduction hampers the adaptation of this sociometric model to individual or interpersonal psychology as well as to man-environment relationships, as we shall observe.

An interesting application of a network model to city systems is noted by Christopher Alexander in "A City Is Not a Tree." Actually, this "tree" is an arborescence. In his model of the city, the root point is the entire city, possibly represented by its government, and the other points signify systems or elements of various systems of services throughout the city. The idea is that an individual or group may be served by more than one element of a system (e.g. he may visit and use several libraries outside of his neighborhood branch) and also served by several different systems which overlap to meet his (or their) integrated needs. Alexander takes municipal planners to task for ignoring this overlap of elements and overlapping needs of individuals. They rely on a bureaucratic (tree) model of districts and sub-districts and neighborhoods and segmentation of individual needs which not only ignores the human psychological situation in cities, but also the fact that cities are primarily formed so as to provide increased mobility for residents to enlarge choice and possible configurations. The planning connections ought to be multiplied to conform to the realities of city living.

Alexander has generally been interested in applying mathematical models to the architectural and planning processes, yet most of them have been only tangential to perception of environments. An application of the network approach which is much closer to the mark is a paper by Ranko Ban, which looked at the micro-environment of single-family dwellings at a scale with considerable direct relevance to humans. By using points to represent rooms and edges to symbolize doorways and other connections, he was able to classify various possible configurations of a house, and particularly to investigate forms of cycles such as the living room-dining room-kitchen-hall cycle which was most common in frequency. Ban has done little to investigate the effects of such configurations on the people who live in them; thus the psychological import of such a model is, as yet, unknown.

We shall find, in turning away from networks for the moment, that applications to sociology and psychology in the field of general topology are much rarer, much less versatile, and thus much more difficult to draw broad conclusions from.

Kurt Lewin, the well-known psychologist, looked at various forms of mathematically modeling psychology, topology among them. In his book, Principles of Topological Psychology, he endeavored to construct an entire framework, however rudimentary, in which psychological behavior can be placed and observed. Though he did not specifically discuss perception, and dealt very little with spatial experience, Lewin's effort is not just the simple network theory of most sociologists, and thus is of great interest to this study.

Lewin's basic notion is that of the region, physical, social or conceptual in which an individual may exist and possibly move via paths to and from other regions. Barriers and boundaries of varying difficulties are covered (though varying strength of boundary is not a strictly topological concept) and examples are given of structures in which an individual may exist for a time, illustrating certain configurational qualities such as possible paths, regions of free access and structures built around a goal (also represented as a region). Lewin concentrates a great deal of his attention on topologizing conceptual processes. The development of any of these concepts into theories with meaningful implications is avoided; his purpose here is just to provide a system for psychologists to use in representing behavior. Of course, in encouraging such a system, one is also encouraging a sort of world-view that accompanies this system, a world-view which, in this case, asserts the psychological reality of the configuration of these regions, boundaries and contiguities in determining possibilities for an individual in a situation. As such, I find this model useful for integrating the dynamic concepts of spatial perception and meaning.

Undoubtedly, the author of the most significant studies done using general topological concepts to model experience with space is the Swiss philosopher and psychologist, Jean Piaget. Combining a habit of rigorous analytic thinking in philosophy and skill in psychological experimentation, he restricted his efforts to studying the growth of intelligent behavior in children. Child study, more and more, seems to be a necessary prerequisite for the study of any fundamental psychological issues, since questions about the nature of a phenomenon almost inevitably raise questions about its evolution. (Kurt Lewin was also primarily based in child psychology.)

In a book co-authored by Barbel Inhelder, The Child's Conception of Space, Piaget documents his findings in the development of conception (as studied through representation) of space in children. His thesis is that before a child is able to conceive of Euclidean space with definite sizes, shapes, angles, straight lines and parallelism, he conceives of space topologically and evolves through intermediate stages, most notably a projective stage. There are but two related limitations to Piaget's work which I hope can be overcome in the future. One is that, in the author's words, "The subject of the present work is not the development of space in general, but only that of representational space, and, therefore the analysis of perceptual space goes beyond our set limits." (p. 5.) The second limitation is the application of topological constructs only to objects as more-or-less sparate from a spatial surrounding. Perhaps this is a result of the emphasis on cognition and conception as opposed to perception and spatial feelings such as sense of place. The authors assert that, "This primitive, topological space is purely internal to the particular figure whose intrinsic properties it expresses, as opposed to spatial relationships of the kind which enable it to be related to other figures. Thus it has none of the features possessed by a space capable of embracing all possible figures, and the only relation between two or more figures comprehended by topological operations is that of simple one-one and bi-continuous correspondence, the basis of 'homeomorphism' or structural equivalence between figures." I would argue that this is a limitation of the authors' perspective, not of psychological reality, and that topological notions can address problems of total relationships. Aside from these limitations (which are, after all, limitations and not errors), the effort is a significant contribution to the psychology of space, particularly in the creative and extensive set of detailed experiments conducted.

A study by Kevin Lynch also applied a concept of general topology, in this case, homotopy, to people's representations of spatial form in a city-scale environment. Experiments showed that persons having a reasonable familiarity with an area mapped that area by a map which was always a continuous deformation of the real map. Though often quite distorted, these maps would contain all the correct topological relationships of paths to regions and of regions to other regions. Furthermore, those whose maps were "torn" renditions, i.e. non-continuous deformations, were found to have a basic misunderstanding of the area. As in Piaget's studies, this topological construct was applied only to representation.

The last field of research to be included in a certainly incomplete list is the topological study of regular figures. In a sense, the various developments in the field of patterns, symmetrical repetitions and modules are no more significant as integrative modeling than any other <u>purely</u> mathematical research. However, the study of regular figures as a mathematical phenomenon is so closely tied to the realities of the empirical world of architecture and design, that it seems to afford a ready-made springboard for psychological research and testing. Sadly, this is the last mention of the subject of regular figures that I will make, except for one reference to be found in the bibliography.

To summarize this section, it is obvious that I am not first in calling for the application of topological concepts in the social sciences (though spatial perception and meaning have only been investigated tangentially so far), and indeed, it seems that often the convenience of mathematical models guides the research more perhaps than it ought to, particularly in the field of sociology. So the main difficulty is, in Kurt Lewin's words, "the dealing with problems which lie, so to say, <u>between</u> sociology and mathematics."

POSSIBLE FUTURE APPLICATIONS

<u>General Notions about Set Theory and Order</u>

The following is a random compendium of possible "meanings" and philosophical points to consider about set theory and order.

The idea of sets and points with a general undifferentiated belonging-to relationship is basic to set theory, to any human language and, apparently, to human thinking. One may talk about a set of many elements, perhaps the city of Boston, and by changing the perspective or scale of the discussion, create a quotient set, with "the city of Boston" and other sets identified to points. Conceptually, we make these changes quite easily, and as our civilization has progressed, we have increased our ability to perform these changes of scale and broadened the range of scales in which we can operate.

Similar archetypal forms are partitioning and ordering. The arborescence, an order relationship, is the form of all bureaucracies and hierarchies. We also frequently use real-valued functions, i.e. maps of a space into E^1, as indexes to totally order a set. More common and closely related to the shape of things in physical space is partitioning. Socially, we construct groupings and "types" often to make our treatment of others easier.

In architecture, we create rooms to serve the partition function. In a sense, one is put into an equivalence class according to what room he or she is in. The implication of such an architectural ordering is that one's relations to others are symmetric, i.e. one is related to those people and objects that are in the room and not related to anything outside the room. Of course the whole situation in any architectural system is much more flexible, and rooms constitute one factor out of many which affect us, spatially and temporally---so partitioning should be evaluated as part of a complex.

In some cases, symmetry is taken for granted. We generally assume that most relations between two people or two places are symmetric, especially for places, yet the possibilities for accomodating asymmetrical relationships have not been investigated enough perhaps. Certainly there are ways in which authority is identified with certain places, making the connections between those places and other places antisymmetric. The transportation between places by automobile, on the other hand, may be too asymmetric, so that some trips and some particular roads are not recognized when reversed. Perhaps in many cases there ought to be some landmarks or other key elements that are perceived in about the same way from either side, so that paths can be easily identifiable, even if most of the elements are asymmetric.

An analogous situation exists for transitivity. The assumption that friendship and kinship webs are transitive may be inconsistent, and destructive of some social groupings, such as communes.

In thinking about order, I feel that there must be an assignment of cardinalities to places and objects so that the various cardinal numbers---finite, countably infinite, uncountable---have different psychological meanings. (Probably density would have a similar role.) Many people believe that the quality of being planned destroys the infinite complexity of the "organic city". It seems to me particularly that the difference between $\aleph_1$, the cardinal of the continuum, and $\aleph_0$, the countable infinity, is significant; for example, in the case of packaged do-it-yourself handiwork kits of many types which are fashionable now, I would set as a criterion for a kit's value as a creative medium whether or not it allows a continuum, $\aleph_1$, of choices for the user to determine the final product. A finite, or even $\aleph_0$ number of choices, even though the latter would almost insure that every individual's product would be unique, would still leave the entire creative process in the hands of the kit manufacturer. Similarly, politics, with its polls and images, and other sociological and psychological institutions draw distinctions by reducing continuums to a countable or finite number of possibilities. Obviously, all such evaluations are not purely mathematical ones.

Topological Spaces

In this section, I will attempt to formulate the basic rules for at least two topologies in physical space, or more precisely in the space in which people perceive physical space; this is physical space as we know it, or perceive it.

The simplest way to look at physical space is as a pure Euclidean three-space where an open set may be thought of as the interior of any closed, two-dimensional surface homeomorphic to a sphere or any union of these sets.

This space is pure in the sense that it is unaware of any form that exists within it, whether it be people or objects or atoms. It is a homogeneous, dense collection of locations (though the positioning of the origin is unimportant), much in the sense in which Newtonian physics liked to picture space. (Perhaps the relativistic model could be expressed as Euclidean four-space with a strangely curved metric.) It is not useful for our purposes except as other spaces are mapped into or from it. We also may want to use the natural Euclidean metric as a foundation for other more relative and personal metrics.

There are other ways of portraying topologies and open and closed sets. In Marshall McLuhan's hot and cool media we find an analogy for closed and open sets. (This idea was brought to my mind by Ken Kesey, who used the names "closed circle" and "open circle" or "trip" to describe much the same thing.) Cool media, or open sets, lack something which makes them incomplete (in a topological sense), thus encouraging participation. We can consider that any open set can be closed by filling in the missing fringe, and that any closed set has an open set as its interior or "content". (This differs from McLuhan's system, for which every medium, hot or cool, has an interior.) Similarly, we can use neighborhoods of sets as cool media which use those sets as parts of their content. It may then be possible to place the study of media in a framework which includes separation, continuity of maps between media or specific objects in a medium, connectedness of spaces, denseness (one might assert that television is dense within the space of social affairs, which may in turn be a subspace of one's total life-space), compactness, etc.

In general, topologizing by use of open and closed sets may be possible in many situations where a dialectic tension occurs.

<u>Topologizing the good configuration</u>---One of the most

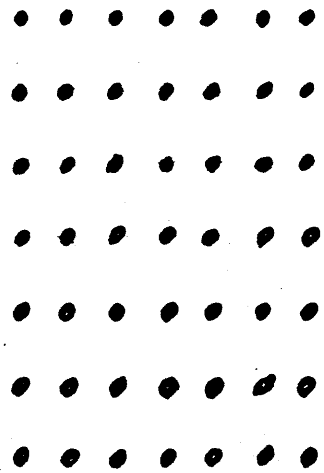
important theories developed by the Gestalt school of psychology was the rules of organization for the visual perception field. The principal precept of this theory is the figure-ground relationship. Simply stated, it says that at various times, particularly at the first moment we see a visual (mostly two-dimensional) field, we perceive an undifferentiated mass of data without definition, as in a mist. We then proceed to create order and configuration in the field, to distinguish objects, or figures into a "good" pattern, one with usually just enough order to suit our needs. Of course the ways in which this field is organized will depend on specific unique qualities it posseses, on cultural and psychological forces operating on the individual, on particular needs the observer may have at that instant, and on a host of other factors, yet through a great deal of experimental research (most of it, admittedly, utilizing quite abstract material), the Gestalt psychologists have produced a pretty convincing set of rules for how a "good" configuration is perceived. Although some of these rules were based indirectly on ideas of form selected from different fields of mathematics, and also in spite of work done by Piaget which, in some sense, extended those concepts in a topological direction, I feel some value is to be had in exploring topological meanings of these concepts, whether original to me or suggested by others. At least one reason for doing this is to introduce the succeding discussion of place or shell-perception.

The figure-ground relation itself seems quite analogous to a closed set-open set complementary pair. Thus the forming of a Gestalt, a whole perception, represents the creation of a closed set or figure. The following is a list of five of the most relevant rules by which we perceive figures:

1) The primary phenomenon in a group of organizational principles is proximity. Figure 32 is an illustration of proximity in action; proximity seems from this example to be primarily a grouping based on the natural Euclidean metric, yet in other instances, it may be related to less metric qualities. Piaget, for example uses proximity in close association with separation, that is, two figures form a proximity if they are not separable by the presence of non-intersecting, surrounding neighborhoods. Intuitively, we think of there being a "misty" space between the two figures. In many cases, proximity depends on the relations of the other rules.

2) Similarity is almost as important as proximity and stands a good deal more on its own merits, rather than as a relative quality. Figure 33 illustrates how similarity of objects in a matrix causes them to be grouped together. In cases like this, this rule can be stronger than proximity. Similarity deals with the entire study of regular figures and also with the homeomorphism concept. As an example of the latter, the importance of the human face is very easily learned by infants, according to Piaget, despite the various positions, types, expressions and perspective views which faces can provide. The reason behind this is that however different two faces are and however different a similing face may be from a frowning face, they are all continuous

32a



32b

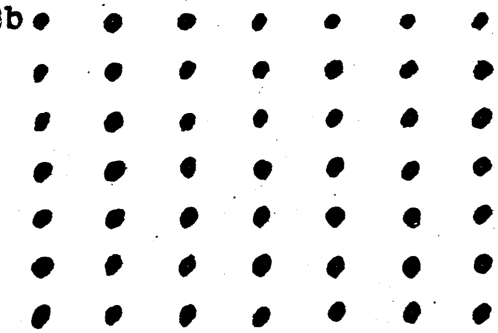
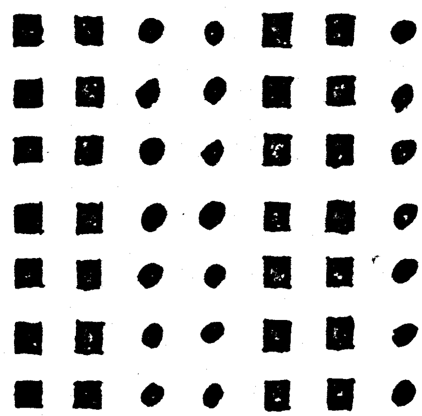


Figure 32. Proximity. Because of the different intervals between dots in the two matrices, 32a is perceived as horizontal groupings of dots while 32b is seen as a vertical grouping.

Figure 33. Similarity. This illustration is perceived as blocks of vertical groupings and probably would still be seen as that even if the horizontal rows were further apart.



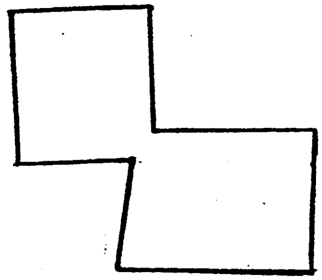


Figure 34. Closure. The drawing on the left is perceived almost as two separate figures because the whole figure is not convex but each of the two parts is in itself convex.

deformations of each other with the relationships of eyes, ears, nose, mouth, hair, etc. to one another always the same.

3) Closure is another Gestalt principle, shown in Figure 34, but topologically speaking, the phenomenon should be called convexity. Notice that, although its analytic definition involves straight lines, there are really few size or shape restrictions imposed on the possible convex figure; for example, any ellipse or rectangle, whether regular (circles or squares) or very elongated, is convex.

The well-known illusion of Figure 35, as well as many other optical illusions are based on convexity. The figure that appears shorter does so because it is nearly defined as a part of a closed figure which is implied by the direction of the surrounding segments. The other figure is left hanging in between two slightly-defined, or at least implied, convex closures.

4) Continuity (see Figure 36) may mean topologic continuity, or possibly connectedness or convergence, all of which are related. In the illustration, what forms the continuity is a set of vectors between each pair of adjacent points. These form a kind of Cauchy sequence---as one's eyes move along the sequence one more and more expects the direction of the next vector to be neighborhood-close to the direction of the last few. Thus certain configurations are avoided.

5) A final general phenomenon which is closely tied in with convexity is boundedness. Boundedness is independent of convexity, but the principle of convexity allows one to infer a bounded figure from just a few elements. Figure 37 is a reversible figure---

35a

35b

35c

35d

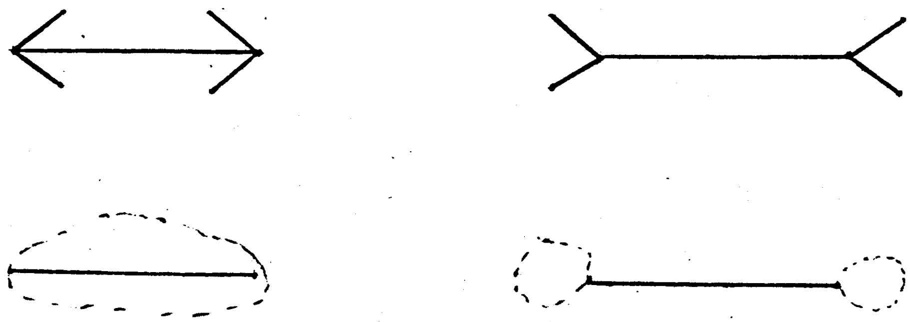


Figure 35. An Optical Illusion Based on Closure. The horizontal line in 35a appears to be shorter than the one in 35b, even though they are the same length. The two lower diagrams suggest that impled convex sets influence this illusion.

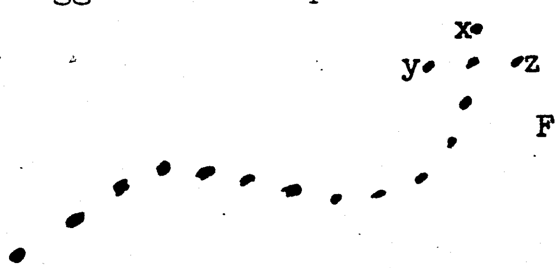
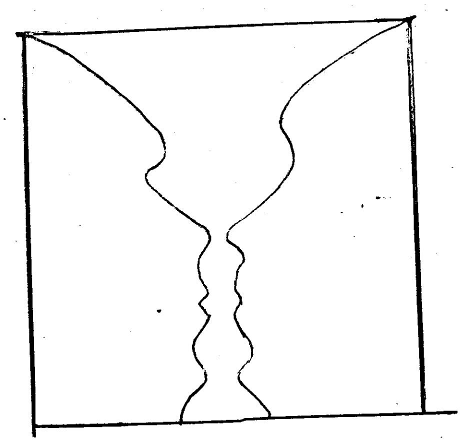


Figure 36. Continuity. If you had a choice of adding one of the three labeled points to the rest of them, you'd immediately choose point x, generally disregarding rules of proximity.

Figure 37. Boundedness. Either the vase or the two facing profiles may be seen in this configuration since they are both equally "good".



one which can be perceived in two ways each being about equal in its "goodness" to an observer. What occurs if one looks at a reversible while relaxing, yet fixing one's stare for awhile, is that the two possible figures altbrnate roles, one playing figure and the other being the ground and vice-versa. In topological terms, we say that the two figures share a common boundary (which is already the case between <u>any</u> figure and ground), and, moreover, each figure is bounded, while a ground is generally not bounded. The importance of this difference lies in the fact that a closed, bounded space is usually compact, a quality which gives a figure finitude (or "fathomableness" as I said before).

Note that this topologizing of perceptual patterns or configuration is Euclidean and spatial in some respects, yet also involves a mixture of other qualities (e.g. facial homeomorphs) which are also topological in nature. What has been done in this section, then, is to fit the topological concepts to the facts of empirically-derived reality, not the other way around. If such a model is to be of more than mere illuminatory value, it must bear fruit as a predictor or aid in understanding phenomena.

<u>Topologizing sense of place</u>---Can we carry this knowledge of the topological rules for forming good Gestalt over into people's perception of physical space around them as a shell or place? I believe there are several possible analogies to at least be investigated. Let me begin the discussion of these possible analogies with an anecdotal example.

My cat, who is considerably less prone to depend on his eyes and more prone to use his senses of touch, taste and smell than humans, is lying comfortably curled up on a kitchen chair. Pick him up for any reason and you will probably receive a dirty look and a cold reaction for having destroyed his peaceful equilibrium. Yet if you pick up the whole chair without disturbing his relationship to it, he will not seem the least bit flustered, though you may move him to another room this way and even though he is not totally unaware of your intervention.

What are the cat's figure and ground? Is he the figure and the chair the ground? This is a possibility, yet it seems most analogous to the idea of the figure-ground relationship to call the cat and chair together a figure, since they form in the cat's perception a good Gestalt. Thus the background out of which such a figure emerged, a background which is apparently still vaguely in the cat's conscious or unconscious awareness, can be called the ground.

From this example we can see some of the relationships between visual perception of objects and patterns and total perception of shell. The most important difference is that in the latter case, the perceiver is always a part of the figure. This is really part of my definition of sense of place, not an empirical fact, but I think it is an accurate assumption because it places the phenomenon in a highly functional position, that is, one feels a sense of place because an environment and a subjective set of needs and expectations one might have at a particular moment all come together to form the best "spatial Gestalt" at that moment. Thus one's subjective state is a vital part of this configuration. In addition, this perception is not merely visual though the visual contribution is still quite significant. Thus, by stepping into a physically sealed room, we may no longer be able to see the ground at all, though it still presumably has some importance for us. Finally, as in the example of my cat, formation of a good spatial Gestalt does not preclude movement or change, but these temporal dimensions do, of course, complicate the matter.

With these guiding principles in mind and conscious of the similarities and differences that the perception of a place has to the perception of an object in a visual field, we can look at some possible topological qualities of shell-perception space. Of course, unlike the last topic, this one is operating in a relative vacuum of established empirical data.

Obviously, convexity plays a great role in aiding one's feeling of place. Is this true convexity, or does it just mean that anything which can be connected to the perceiver by a straight line is considered available for inclusion into the figure? (See Fig. 38.) I sense through observation that even

Figure 38. Convexity in spatial perception. Does the perceiver at x sense himself as part of all the space within his visual field, as in 38a, or does a more objective sense of convexity limit his sense of place, as shown in 38b?

though the latter criterion may be of precedence on the most immediate level, the former seems to become just as important through one's continual practice and increasing experience with good spatial Gestalt. It also may be more favored by senses other than vision.

In looking at a fluid environment, one where a wealth of factors other than "just walls" contributes to the "mist" from which one will choose a "good shell" at a particular instant, it seems that many of the same rules that apply to the two-dimensional visual field may go into determining our choice of figure. Certain rules of similarity and homeomrphism may help us to differentiate potential places, as may the presence of elements in proximity to one another and perhaps some aspects of continuity. However, none of these constructs are yet developed to the point where I can discuss them systematically and intelligently. The study of such "fluid environments" has importance not only for some indoor spaces like day care centers, but also for many outdoor spaces and the highway route. Topology can suggest some factors that could be important and, if these factors test out, topology can provide a formal framework in which these phenomena can be discussed and articulated.

The question of separation between spaces seems more ambiguous in perception of place than in perception of object configuration, mostly because the three-dimensional quality of the former prevents vision from having absolute mastery over whether two spaces are separated. Since most walls are not often perceived as being two-sided entities, because we rarely are able to make that connection, it might seem that walls "de-normalize" space because they make it difficult to perceive disjoint open fields which contain our objects, i.e. shells. This is not meant to cast aspersions on our use of walls to divide space; rather it is to open the possibility that this separation has some psychological importance which can thus be tested in regard to any other spatial arrangement. There are as well, many types of walls, such as temporary room-dividers, partitions, etc. which encourage perception as "normal" space.

Mappings between spaces---Once again, I must present a bare skeleton of a possible use for topological concepts. This is the idea of mappings between spaces to check for homeomorphism or to test for various homotopy classes. Here are some fragments:

1) Sense-spaces: Most studies in perception have tended to segment the senses and study them one at a time. Very little has been done to relate any two senses, really. Research has been performed to test the development of eye-hand coordination in children and to experiment with the effects of hampering one sense on the functioning of another. As more of this sort of thing is done, and more integrative knowledge of the senses is acqquired, systematization will be necessary. A topological mapping can compare whether two sense-spaces might be homeomorphic, or if they are not, whether there are continuous mappings in either direction. Or one could compare two actions such as eating and reading to see if there are different types of mappings between

a dominant sense and a somewhat passive sense in various functions. Likewise, inter-cultural mappings might turn up similar structures. We might find, for example, that our sense-space is less prone to formation of Gestalten in many situations than the space of a primitive society. This is to say that mappings from our sense-space are continuous in cases where a primitive's may not be. (These formulations are identical since they both assert that our topology is larger, i.e. contains more open sets, than theirs.)

2) A possible group of mappings which strikes me as having a more natural derivation consists of mappings between spatial relations and social or psychological systems, or between spatial relations and conceptual frameworks. This sort of mapping would seem to be a logical follow-up to Lewin's work, since he divided the life-space into three planes---physical, social and conceptual---without much discussing integration of them. By mapping the regions involved in a psychological space, one might find it homeomorphic to another space from another field. An example of this being accomplished in a non-rigorous way was the analogous carryover of the Gestaltist theories of perception into the theory of Gestalt therapy.

If one believes that our mental processes develop in close correlation with our experience in spatial perception and interaction, than it is natural to expect close correspondence between some spatial constructs and analogous conceptual patterns. It has struck me that Mircea Eliade's discussion of "sacred space" in <u>The Sacred and the Profane</u> provides a start toward this type of mapping. Why is heaven up and not out? Is this because our eyes are horizontal or because gravity is vertical or are the two related? If the idea of a pole or hole in a house was so important for many primitive men to communicate with the spiritual plane, how would the same man react on the second story of a high-rise apartment building? The answers to such questions require a system for relating mappings of sense-spaces into both Euclidean "pure" space and so-called religious space.

3) As a concrete example of a mapping between spaces, consider the way I (and, I assume, many others) read a map. Because of the disconnected, four-component space that I try to map any connected real area into, I am inevitably susceptible to confusion. These four components are the four points of a compass. If a route that is close enough to North for me to label it such gradually (connectedly) changes to more-or-less East, my sense of direction is thrown off. We really want a homeomorphism in this situation. A system of keeping track of all right-angled turns and separately imagining the summary effect of all curves might work. Or perhaps each section of a region could post signs stating what kind of space it is!

4) The notions of homotopy and homeomorphism classes may find use in categorizing different spatial patterns of urban and non-urban life. The outdoor system of a city may be characterized in the roughest sense as a flattened m-fold torus, that is, a connected figure with a hole in it for every place where there is a building or the plane is broken up by some indoor use. This

representation is extremely crude, ignoring all the previously mentioned subtleties of spatial Gestalt, metric factors, and much, much more besides. Yet even this simplified map and its fundamental group can be used to classify various paths by their homotopy type (which may also be somewhat partially ordered) and to test for correlations between these classes of path and various psychological relations such as perceived time, memory of paths and reasons for choosing a path. On a small scale, it might be possible to combine this crude system with the ideas of spatial Gestalt including a metric relation (but not necessarily the natural Euclidean metric).

In Manhattan, where every block is a huge development, few closed paths are null-homotopic and it seems that, perhaps because of this fact, most everyone might return from a place in the same way they came, for such a return is the only way to make the closed path nullhomotopic. In the heart of Boston, which has a more delicate fiber, with a greater possibility of null-homotopic paths (in the sense that a null-homotopic path is a loop whose interior is easily understood), the whole feel of the city is different.

It is my feeling that looking at movement through an environment as a continuing series of changing figure-ground relationships could lead to some kind of increased understanding of perceived time (e.g. perhaps perceived time is proportional to the number of figures that occur to a perceiver in a given journey). But this experimentation, after the general topological structure of the spatial Gestalt is discovered, might work better using graphs in a similar fashion as the method to be expounded for child care study in the next section.

Graphs

Even though they entail a reduction of a system by uniform application of a few abstract principles, graphs probably provide the most useful possible systematic use of topology in the study of environmental psychology. Their application to transportation is already well-developed; their application to human-scale environments is just beginning.

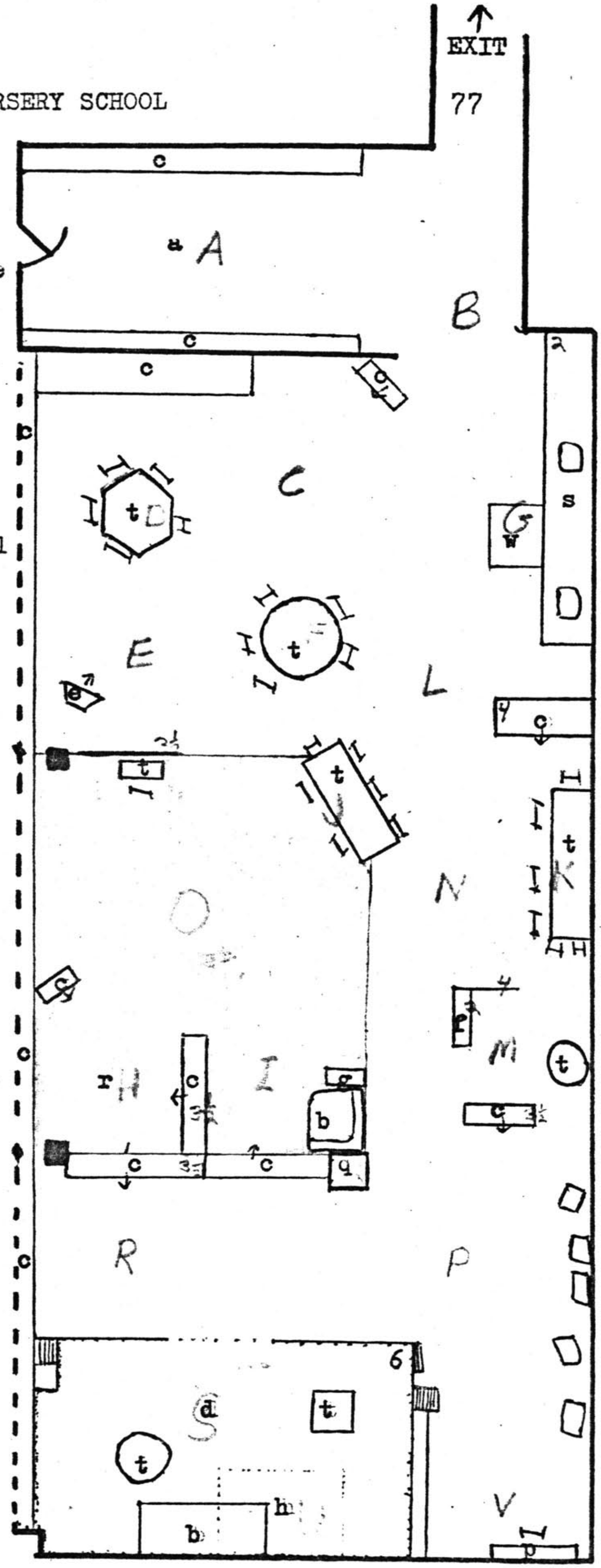
To perform a representation of a spatial configuration by a graph, one must first select the appropriate set of elements to be the points. This could be buildings in a town or land use zones or some form of place, but quite obviously one must believe that the relationships between these points is, for one reason or another, not terribly metric (in the Euclidean sense) and, moreover, that the connections between points are such that it is usually only important to know whether or not a connection exists between two points, not what the quality of that relationship is. If your system can be reduced that much, and still retain meaning, you have an excellent network.

To provide an example, I have chosen to discuss a network I drew to use in analysing a two-year old room in the Eliot Pearson Nursery School in Medford, Massachusetts. (See Fig. 39.)

The network was derived in two stages. First, by observation and interaction with the children I determined a set of points which represent the most likely occurrences of spatial Gestalt

Figure 39. ELIOT PEARSON NURSERY SCHOOL

2-year olds



LEGEND

a = Coat room
b = Couch or soft seat
c = Cabinets, book cases or storage
d = Domestic play area
e = Art easel
f = Counter with play cash register
g = Guinea pig cage
h = Cardboard play house on upper level
p = Piano
q = Pretend telephone booth
r = Reading corner
s = Sinks and counter space
t = Table
w = Water play area

⊢⊣ = Chairs
▤ = Steps or stairs
→ = Indicates the direction that an easel or cabinet faces
···· = Indicates top level of structure
▬ ▬ = Window
▒ = Rug

Numbers indicate approximate heights of various objects.

Captial letters indicate a potential place.

due to design, attitudes encouraged by the program and needs and desires of the children. Second, I determined which places were connected directly to which others, i.e. I would draw a line between two points if I perceived that a child could fairly directly move (physically and psychologically) from one place to the other. Unfortunately, the graph was always considered to be symmetric and thus places of privelege, places closer to the outside, and other places which could cause anti-symmetric relations were disregarded. Moreover, it is quite obvious to me now, that my ability to determine what "places" should be represented as points in a graph lags far behind my ability to build points into a graph and manipulate graph theory. Because of this, it might be better to provide as an example a less fluid environment such as a conventional house. Nevertheless, this graph should illustrate to the reader what the graph system can do.

The system shows a diameter of seven, a relatively long one for such a small child care room. The diameter will depend, not only on size and elongation of a space, but, perhaps more fundamentally, on how complex and subdivided the space is. Points P and N are significant articulation points though there are several other points (T,R,O) that, if removed would cut the system off from one other point, and point B controls the access of the system. Point J is the most accessible point of the graph, i.e. its accessibility is numerically less than that of any other point. Other measures given in the foundation section on graphs might provide other bases of comparison with other systems.

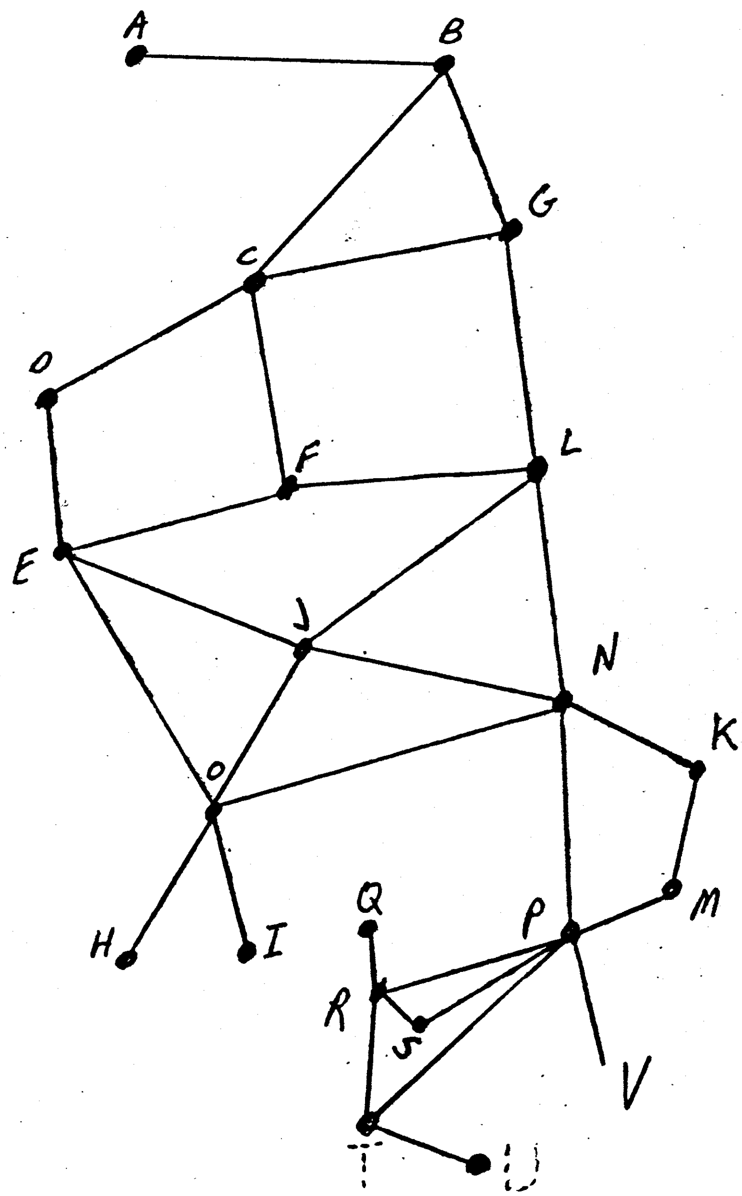


Figure 40. Graph of a child-care room. The above is a graphic representation of the places and connectors in the space of Figure 39. (T and U are elevated above the floor level.) The graph is shown in simply-representable form (no intersections of any connectors), but, as in any graph, the relative positions of points or lengths of connectors are irrelevant.

(See Fig. 40.)

In my initial use of this system I drew quite clear distinctions between certain kinds of place connectors, never intending to treat them equally. Furthermore, observation of a few child care centers has led me to find further distinctions in the type and strength of connectors possible. In addition, there are distinctions in importance between types of places. Hence, it would seem reasonable that in a system like this one, one might want to weight the connectors (the stronger the connection, the lower the weight) and, to a lesser extent (because they are less quantifiable in their differences), the places.

The use of graphs for systems like child care centers or houses or systems of buildings also raises the question of how to measure dimension. Does the presence of cycles constitute a raising of dimension (from one to two) by destroying linear order? Does simple-representability mean <u>psychologically</u> that a system is only two-dimensional, even though there may be three-dimensional relationships in the Euclidean sense? (This is the case in the example I provided.) It would be interesting to find an existing real example of a non-simply-representable system and observe its effects (if any). Or wouldn't it be interesting to build a house modeled after the graph in Figure 31?

CONCLUDING REMARK

As a possibly useful system for <u>modeling</u> phenomena occurring in environmental psychology, one must be very sceptical of topology. Perhaps this is because environmental psychology is still quite a young science. Certainly, it is beyond my scope of thinking here to speculate on the future of this science, if indeed it does have a future.

Throughout this effort I have only been able to produce "possible future applications" because empirical knowledge is not really ready for such systematization. And there is most certainly the possibility that it will never be. Still, I recommend the study of topological concepts to those interested in psychology and design for the possibilities of illuminated understanding, for much the same reasons that designers seriously study art, and---who knows?---for the possible applications of topology in modeling.

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X. GLOSSARY

The following alphabetical listing refers the concepts covered in the Essentials of Topology section.to the place in which it was first defined. P = Procedures; S = Set Theory and Order; T = Fundamentals of General Topology; A = Glimpses of Algebraic Topology. If a term is referred to another term ("See ...") this means that the two terms are identical.

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accessibility	A2.7
anti-symmetric graph	A2.2
anti-symmetric relation	S1.7
arborescence	A2.8
arc	A2.1
articulation point	A2.4
basis	T1.7
Boundary	T1.5
boundedness	T1.6
cardinal number	S2.3
Cartesian product	S1.4
Cauchy criterion	T5.3
chain	A2.3
circuit	A2.3
class	S1.1
closed set	T1.5
closed m-simplex	A1.3
closure	T1.5
compactness	T4.2
complement	S1.3
complete graph	A2.2
complete relation	S1.7
completeness	T5.3
complex See simplicial complex	
component	T3.3
connectedness	T3.1
connected graph	A2.4
constant map	T6.2
continuity	T1.9
convergence	T5.1
convexity	A1.2
countable ($\aleph_0$)	S2.3
countably infinite See countable	
covering	T4.1
cycle	A2.3
cyclomatic	A2.10
denseness	T1.5
diameter	A2.7
difference	S1.3
dimension of a complex	A1.4
dimension of a graph	A2.9