SOME PROPERTIES OF THE FINITE TIME SAMPLE AUTOCORRELATION OF THE ELECTROENCEPHALOGRAM

by

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ABSTRACT

One goal of quantitative studies of physical phenomena consists in transforming a set of measured variables into another set that will describe the phenomenon under investigation in terms of meaningful parameters. Most analyses of brain waves by means of autocorrelation functions that have been carried out seem to have been based on two implicit assumptions: (1) that frequency-emphasizing transformations (such as autocorrelation) are relevant to the study of the EEG and (2) that probabilistic models (inherent in the use of autocorrelation analysis) are applicable. Both these assumptions were examined in the present investigation which concerned itself with the problem of estimating the autocorrelation function of the EEG from a finite sample of the EEG time series. A narrow-band, Gaussian noise model was assumed in order to study the errors that arise from the estimation of the autocorrelation function on the basis of a finite sample of the time series. A measure of both the magnitude and form of these errors is derived and verified experimentally. The EEG time series is then discussed in the light of this noise model. Some estimate of the distribution of amplitudes is computed. The results obtained showed in particular that the cyclic activity exhibited by EEG correlograms for "long delays" may derive from such errors of truncation.

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Title: Assistant Professor of Electrical Engineering

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CHAPTER 1 INTRODUCTION

The electroencephalogram (EEG)* is an electrical potential measurable at the surface of the scalp of man. It is of the order of magnitude of 50 microvolts peak to peak, and spectral analysis indicate a range of frequencies from 2 or 3 cycles/sec to 35 cycles/sec. At the present time little is known concerning the specific origin of this electrical activity from the microscopic structure of the brain. It has been known for some time, however, that the units of the nervous system (neurons) exhibit spontaneous or background activity (8, 19, 22, 48, 66, 67, 81, 85, 99) that is unrelated to any known stimulus. It is clear that the gross electrical activity as reflected by the EEG is some function of this unit activity. Various sources have suggested that the EEG is a summation of the classical action potentials of the single units. Others claim that slower dendritic potentials contribute to the gross potentials. The question of origin is by no means settled at this time. (6, 16, 18, 21, 23, 51, 52, 59)

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*The term "EEG" will be used here to apply to that brain potential that is measured extra-cranially when no known, externally-applied stimulus is present.

Equally important is the question of the physiological significance of this electrical potential. Lindsley, (68) among others, claims that some of the components in the EEG (alpha rhythm) reflect an excitability cycle in the cortex. That is, some basic metabolic and respiratory rhythm of the organism's nervous system is responsible for the rhythmic components of the EEG. This question of rhythms and synchronous activity is one which will be returned to later.

Another view of the EEG is its interpretation as a reflection of the state of the organism. This view stems from a considerable amount of evidence of the sensitivity of the temporal patterns of the EEG to the internal and external environment of the organism. The effect of the consciousness of the subject upon these patterns is marked. EEG patterns show slower activity as the subject becomes dormant and exhibit high frequency components as the subject becomes attentive. (14, 29). Furthermore, the effects of anaesthesia are as marked as those of sleep and wakefulness. The concept of state is given more meaning by some results of research done in the reticular formation. (70) It has been found that a non-specific pathway to the cortex through the reticular formation of the brain stem has much to do with the receptivity of the cortex to sensory information. Furthermore, changes in the electrical potentials of the reticular formation have resulted in concomittant changes in both the behavior of the animal and the electrical potentials of the brain.

Indications of the interpretation of the EEG as a reflection of physiological state also come from studies of reaction time versus EEG (40, 62) and effects of visual attention (10, 13, 91) upon EEG patterns. Here again, there is evidence that some connection exists, although detailed knowledge of the relation is unknown.

From results gleaned from the literature*, it can be concluded that the EEG is of neural origin and bears some relationship to important physiological processes and also correlates with behavioral changes in the organism. It is also apparent that the EEG is a highly labile phenomenon, varying in its patterns from individual to individual, varying as a function of the state of a given individual and as a function of the positions of recording electrodes on the scalp of an individual. In fact, determining and controlling the many sources of variation of the EEG is one of the chief problems confronting the research worker today.

In the past much of the research in the EEG field

*For review articles of EEG literature see references 11, 72, and 95.

has resulted in the collection of a vast amount of data. Experiments have depended to a large extent on ill-defined criteria and human judgements. In recent years, however, there has been an effort to get away from these kinds of experiments. More emphasis is being placed on asking specific questions and attempting to find answers to these questions by quantitative and objective methods. One of the major efforts has been to find some transformation of the EEG patterns of voltage versus time that will yield a new variable or set of variables that will be easier to interpret. Easier to interpret in the sense that the new variables will be insensitive to those changes in the experiment that the experimenter is not concerned with and yet sensitive to those changes that the experimenter is studying.

One such technique (37, 38) was designed to emphasize the rhythmic burst activity (alpha rhythm) that is prominent in the EEG of many subjects when they are asked to relax in a particular kind of environment. This environment is one in which the subject is deprived of all auditory and visual stimulation. The aim of the project was to determine some measure of the variability of the amount of rhythmic burst activity in the records of four normal subjects when the conditions of the experiment were controlled as best as they could be. The ultimate goal was to determine if these statistics were stable enough under the conditions of the experiment to make them useful variables for a study of the effects of a changing environment upon the EEG of a subject or group of subjects. Essentially, the experiment was a beginning in the search for methods of characterizing the physiological state of a subject. Since all the EEG data used in this thesis were recorded in the same manner as in this experiment, a more detailed discussion of the project will be given here.

Subjects who were instructed to keep their eyes closed were seated in an anechoic chamber with the lights off. The EEG data of these subjects were recorded from standard electrode positions (7, 9) (left parieto-occipital area) with gross, wire electrodes. The experiments consisted of 13 minute recording runs followed by a 3 minute intermission in which the lights were turned on and the subject was allowed to chat with the experimenters. At the end of this intermission the dark, quiet environment was restored and another 4 minutes of resting EEG data were recorded. This procedure was followed for 4 subjects on 6 different occasions. These experiments covered a period of approximately two months and were generally run once a week at approximately the same time of day.

The data were recorded on magnetic tape at a tape speed of 1.2 inches per second after amplification by low noise amplifiers. The analysis was performed on 3 minutes of data at a time. This length of data was sampled at 300 samples per second and read into the memory of the TX-0 Computer through an analog to digital converter. Once in the memory of the computer, the data were analyzed by a program that essentially marked those intervals of the record that contained bursts of rhythmic alpha activity. The criteria for this determination were based on amplitude, zero-crossing intervals and succession of intervals of the proper interval length. All three of these criteria could be varied at the programmer's desire. Furthermore, the resulting statistics of this analysis were independent of gain and time base. The two statistics that were of particular interest were the number of bursts in a particular length of record and the percentage of time in which there was alpha activity. The only results to be even schematically mentioned here are those to which there will be some reference later.

It was found that in successive three minute intervals of EEG record, the total activity (percent time during which there was alpha activity in the record) decreased in a statistically significant manner. After the three minute intermission, the activity tended to increase again for the next interval. These were by far the most significant data produced concerning the resting alpha activity.

Aside from this effort at quantification of EEG data, most of the other methods used to date can be categorized as harmonic analysis methods. Two different techniques that emphasize essentially the frequency components of the EEG have been in use. The first method consists of filtering the EEG with a series of narrow band filters and thus determining an estimate of how much energy is contained in various frequency bands. In general these frequency spectra are very complex except for the case of a pronounced alpha activity, in which instance there is often a sharp peak at 10 cps.

The second and theoretically equivalent, although computationally quite different, method coming under the general heading of harmonic analysis is the correlation analysis approach. This method will be discussed in detail in the ensuing chapters.

The correlation analysis technique has been used relatively successfully in the case of the EEG when exhibiting alpha activity. Successfully, in terms of the relative simplicity of the derived data (autocorrelograms). Examples of data taken from four different subjects exhibiting varying amounts of alpha rhythm in their EEGs is shown in figure 1.1. As can be seen from the data, three of the subjects exhibit a marked amount of roughly 10 cps activity. The persistance of this activity in the EEG of many of the subjects has led many researchers to feel that this relatively simple-looking phenomenon is more readily quantifiable. Figures 4.41 and 4.42 show the kind of correlograms that are machine calculated from this kind of data. Note that the correlograms exhibit some of the important temporal characteristics of the signals. For a simple sinusoid, the correlogram looks like the bottom curve of figure 3.12. It is itself a sinusoid.

The problem with which this thesis is conerned is the behavior of the autocorrelograms of EEG when characterized by large amounts of alpha activity. It has been noted for some time now, that these correlograms exhibit a damped sinusoidal behavior. There has been a considerable amount of discussion concerning the significance of the fact that the autocorrelogram (machine calculated, finite time sample autocorrelation function) exhibits rhythmic 10 cps activity at relatively large values of the delay parameter (τ). This type of behavior can be noted in figure 3.24 as opposed to the activity in figure 3.26.

The problem has come about from the interpretation of this phenomenon. As will be discussed more fully in Chapter 2, a correlation function whose decrement for an interval τ is small indicates that the signal at two points in time separated by τ are strongly related. In fact, their mean-square linear relationship is given by the correlation function. Thus a long term cyclic activity in the autocorrelogram might be interpreted as showing a strong relationship between the values of the EEG at time intervals separated by as much as several seconds. This interpretation has led to the formulation of a clock hypothesis. That is, the rhythmic alpha activity has been assumed to be the manifestation of some very precise timing mechanism in the nervous system. The major support for this hypothesis is the above-mentioned feature of the autocorrelogram of EEG.

One facet of this hypothesis makes itself clear. The concept of an autocorrelation function, which is a mathematically defined but operationally useless concept, has been used interchangeably with the concept of an autocorrelogram. A correlogram yields an estimate

Figure 1.1

Samples of the EEG of Four Resting Subjects

Sample length - 20 seconds
Location - left parieto-occipital area

Figure 1.2

Typical Movement and Muscle Potential Artifacts

Sample length - 20 seconds

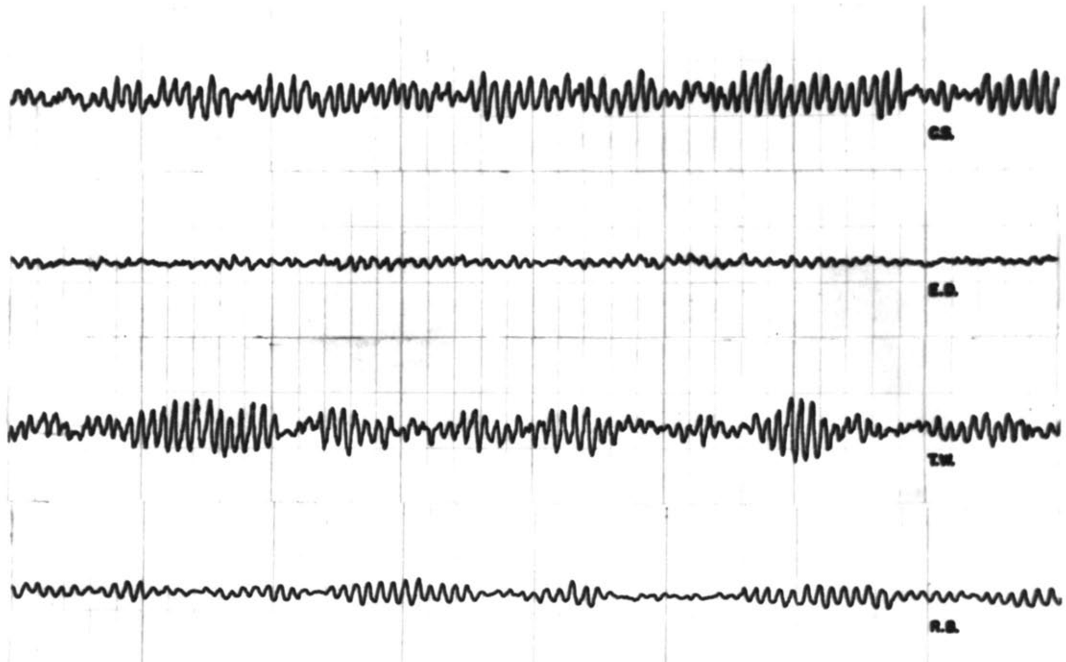


FIGURE 1. 1 RESTING EEG

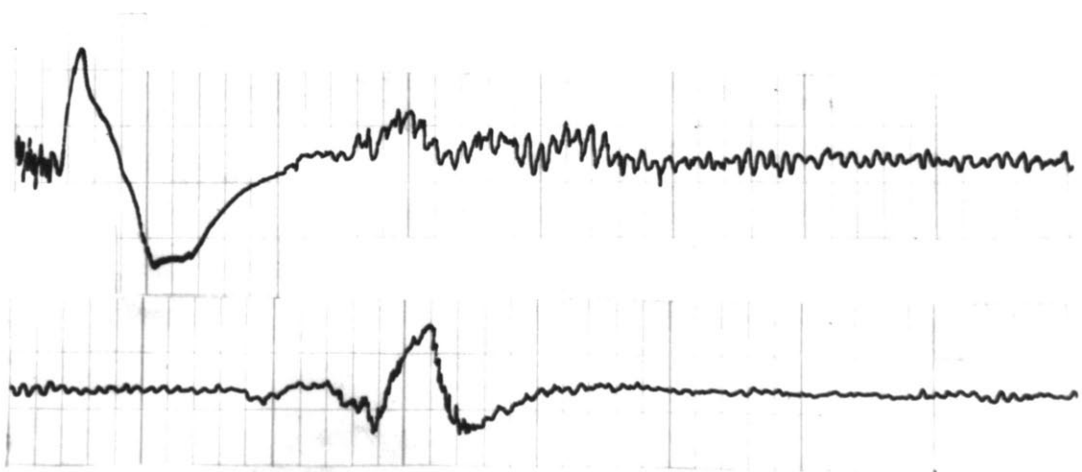


FIGURE 1. 2 MOVEMENT AND MUSCLE POTENTIAL ARTEFACT

of the autocorrelation function of a process. The autocorrelation function is the statistical function defined for infinite time samples of data while the correlogram is the estimate of the correlation function computed by instruments of finite resolution and from finite sample lengths of data. A probabilistic model is suitable for the analysis of these data and indeed the probabilistic model is at the heart and core of the definition of an autocorrelation function. But, the bridge between the correlation function and the correlogram is not at all obvious. Certainly the two are not identical. Thus the correlogram's behavior must be examined in the light of the probabilistic model that has been used in the definition of the autocorrelation function.'

It is the purpose of this thesis to investigate the behavior of correlograms computed from finite lengths of time series and to determine if the type of behavior exhibited at large values of delay can be explained by the probabilistic model used. In this light Chapter 2 is an effort to introduce the probabilistic model and to investigate the estimation of the autocorrelation function of a process from a finite sample of time series. For this purpose a narrow-band noise signal is used both for theoretical and experimental work. This is done for several reasons. First, this type of signal is easily characterizeable analytically and second, it bears some resemblance to EEG, at least in a very gross way. For instance, some of the correlograms of EEG and narrow band noise appear indistinguishable to the naked eye.

Upon investigation of this narrow band noise problem, the implications of the results of this work upon EEG analysis are studied.

CHAPTER 2 ESTIMATION OF THE AUTOCORRELATION FUNCTION

2.1 Introduction and Background in Probability Theory

A probabilistic model of a physical process is often a useful approach to its description when all the causal effects upon the phenomenon are not known or when these effects are too complex to be analyzed on a microscopic scale. In this thesis a probabilistic model is presumed for the study of the EEG. A particular EEG record is visualized as a finite piece of a sample function of a random process. Consider, therefore, a universe of EEG records taken under the same conditions and interpret a particular piece of finite data as being a piece of one of the sample functions of the random process (the sample functions being defined for all time). With this sort of model of the physical process, the autocorrelation of the EEG can be investigated using the mathematical tools that have been developed in the field of probability theory.

The first question that might be raised about this model is, what is meant by the term "same conditions" in reference to the ensemble? The point to be emphasized here is that the ensemble of sample functions of EEG is an abstraction. It is a mathematical ensemble and the tacit assumption that the experiment could be repeated many times under precisely the same conditions is made.

In reality, the experiment is, of course, not repeatable in exactly the same way, but this is of no consequence here. The only thing that is demanded of the model is that it describe the data in some way. The point of how well the EEG actually fits this model will be returned to in Chapter 4.

Accepting this model for the time being, the question that may now be asked is what does a finite piece of data recorded in the laboratory say about the model? What can be inferred about the statistics of the model from the observed phenomenon? This is a very different question from, what can be inferred from the model about the neurophysiological process involved? The second question is by far the more interesting and the ultimately important one, but the first question must be answered before the second can be approached. This paper is an attempt at answering one aspect of the first question. In order to pursue the question of what the observations of the EEG determine about the statistical model, the language of probability theory must be introduced. It will be assumed throughout this paper that the reader has some familiarity with probability theory* and this introductory section is

*See references (3), (25), and (28) for a treatment of basic probability theory and the theory of stochastic processes.

intended merely as a systematic and convenient mechanism for the introduction of the notation to be used.

The discussion in this chapter will be concerned with real stochastic processes whose sample functions have time as a parameter. Thus, the random variable defined over an ensemble of sample functions is denoted as x_t. The subscript t denoting the time index of the random process x_t. A particular sample function of the ensemble is denoted as x(t). The probability density function $p(x_t)$ is defined as the probability that at any time t the random variable will lie between the values x and $x + \delta x$, where δx can be chosen arbitrarily small. Defined in this way, the probability density function has the following properties:

$$\int_{-\infty}^{\infty} p(x_t)\, dx_t = 1 \qquad \text{and}$$

$$p(x_t) \geq 0$$

The probability distribution function is defined as the probability that the random variable x_t is less than some value X or:

$$P(x_t \leq X) = \int_{-\infty}^{X} p(x_t)\, dx_t$$

Since the probability density function is defined as being non-negative, the probability distribution function must be monotonically increasing with values 0 at $-\infty$ and 1 at $+\infty$.

These definitions for the univariate case can be extended to the multivariate case by defining the joint density function of n random variables, $x_1, x_2, \ldots x_n$, as $p(x_1, x_2, \ldots x_n)$. Here the subscript t has been dropped since it will be assumed that time is a parameter for all the random processes discussed here (unless otherwise stated). The subscript then serves to differentiate the random variables. In a similar fashion, the joint distribution function becomes $P(x_1 \leq X_1, x_2 \leq X_2, \ldots x_n \leq X_n)$.

In addition to the joint density functions, it is convenient to use the concept of the conditional density function in the multivariate case. The notation used for the bivariate conditional density function is $p(x_1/x_2)$, which is to be read as the probability of the occurrence of x_1 given the occurrence of x_2. The conditional probability notation can be extended to the more general multivariate case and can also be defined for the probability distribution function.

This brief outline should suffice to explain the notation to be used with respect to random variables and their associated probability functions. The next step is to define various statistical averages that may be of interest. The mean or expectation of the random variable x_t is defined as:

$$E[x_t] = m_x = \int_{-\infty}^{\infty} x_t p(x_t)\, dx_t$$

This definition can be extended to functions of the random variable by defining the mean of $f(x_t)$ as :

$$E[f(x_t)] = \int_{-\infty}^{\infty} f(x_t)p(x_t)\,dx_t$$

For example, the n^{th} moment of the random variable x_t is defined as:

$$E[x_t^{\,n}] = \int_{-\infty}^{\infty} x_t^{\,n} p(x_t)\,dx_t$$

and the n^{th} central moment of x_t is defined as:

$$E[(x_t - m_x)^n] = \int_{-\infty}^{\infty} (x_t - m_x)^n p(x_t)\,dx_t$$

Of these higher moments, the second central moment or variance is of particular interest and is defined as:

$$\sigma_x^2 = E[(x_t - m_x)^2]$$

All of these statistical expectations, defined above for the uni-variate case, can be generalized for the multi-variate case. For the bivariate case in particular, the joint first moment of the two random variables x_1 and x_2 is:

$$E[x_1x_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2p(x_1,x_2)\,dx_1dx_2$$

and is given the name of covariance function. This covariance function has a number of very interesting properties. It can be shown, for instance, that this function is proportional to the slope of the regression line that is the best linear mean-square fit of the joint occurrences of

x_1 and x_2. That is, if one wanted to predict, say x_2 for a particular value of x_1 using a least-mean-square error criterion that was linear, the covariance function would be proportional to the slope of the predictor. This assumes that the experiment of finding the joint occurrences of x_1 and x_2 has been repeated many times and has formed a part of the history of the prediction problem.

For the very important case of the Gaussian Distribution, it can further be shown that the optimum mean square estimate is also the optimum linear mean square estimate. The covariance function, therefore, takes on a particularly important meaning in this case. In fact, if all the covariance functions and means of the random variables $x_1, x_2, \ldots x_n$ are known then the joint density function, $p(x_1, x_2, \ldots x_n)$ is also know, if it is jointly Gaussian.

The concept of a covariance function is also a very useful one in the study of some stochastic processes if the subscripts 1,2,...n are interpreted as different points in time $t_1, t_2, \ldots t_n$. Under these conditions the random process is discussed at these various times and the covariance function becomes a very useful concept. It gives a relationship between the values of the random variables at two instants in time. A more thorough investigation of the properties of this function are made in the succeeding sections.

Before this introductory section is complete for our purposes, one more very useful statistic is introduced. This is $M_{x_t}(jv)$, the characteristic function of the random process x_t and is defined as:

$$M_{x_t}(jv) = E\left[e^{jvx_t}\right] = \int_{-\infty}^{\infty} p(x_t) e^{jvx_t} dx_t$$

Under the usual conditions for which $p(x_t)$ is well-behaved, it forms a Fourier Transform pair with $M_{x_t}(jv)$ and the inverse transform can be defined. The multivariate case of the characteristic function again follows by analogy.

The working language of probability theory is now defined for the purposes to be used here and all further definitions will be made as they are needed.

2.2 Correlation as a Time Average Process

For a real, stochastic process the covariance function of the random variables x_{t_1} and x_{t_2} is defined as:

$$R_x(t_1,t_2) = E[x_1,x_2] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1x_2p(x_1,x_2)\, dx_1dx_2$$

where x_1 is the value of x_t at t_1 and x_2 is the value of x_t at t_2. Under conditions of strict sense stationarity, the probability density function, $p(x_{t_1+u},x_{t_2+u},\ldots x_{t_n+u})$ is independent of u, and it can be seen quite readily that the covariance function $R_x(t_1,t_2)$ becomes a function of τ, the time difference t_1-t_2. Thus,

$$R_x(\tau) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_tx_{t+\tau}p(x_tx_{t+\tau})\, dx_tdx_{t+\tau}$$

If the further assumptions of ergodicity are invoked then there are more powerful statements that can be made about the process. Ergodic ensembles are formed by taking one sample function x(t), defined for all time, and generating the entire ensemble (except for pathological sample functions) by merely shifting the time origin of the original sample function. Thus any finite piece of a particular sample function is assured of appearing identically in some part of all the other sample functions with probability one. The probability one statement allows the occurrence of a finite number of pathological cases in the infinite ensemble. If a particular ensemble is ergodic, therefore, then looking at any one sample function for all time must be equivalent to looking at the whole ensemble at any one time with probability one. Thus it is possible to define time average statistics of the process that are entirely equivalent to ensemble average statistics. In particular, the autocorrelation function of the sample function x(t) is defined as:

$$\phi_x(\tau) = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)\, dt$$

For the ergodic ensemble $\phi_x(\tau) = R_x(\tau)$ with probability one. Heuristically, it can easily be seen that these two are equivalent for this case since for fixed τ they each average the occurrences of all possible products $x_t x_{t+\tau}$ and this average is then performed for all values of τ.

Some of the important properties of correlation and covariance functions can be demonstrated by doing a simple example. Consider the case of an ensemble of random-phased, equal amplitude and equal frequency sinusoids. Thus a typical sample function might be:

$x(t) = \sin(\omega t+\theta)$ where θ is a random variable and has a uniform distribution between 0 and 2π. From the definition of the covariance function:

$$R_x(\tau) = \int_{-1}^{1}\int_{-1}^{1} x_t x_{t+\tau} p(x_t, x_{t+\tau})\, dx_t dx_{t+\tau}$$

The limits are imposed since the amplitude of the functions x_t and $x_{t+\tau}$ are limited by unity. Now $p(x_t,x_{t+\tau})$ is found by using as extension of Bayes' Theorem which says:

$$p(x_t,x_{t+\tau}) = p(x_{t+\tau}/x_t)p(x_t)$$

But, $p(x_{t+\tau}/x_t)$ is a degenerate conditional probability density function since if the value of x_t is known then the value of $x_{t+\tau}$ is known with unit probability. The probability density function, $p(x_{t+\tau}/x_t)$ is then a unit impulse occurring at $x_{t+\tau}$ or:

$$p(x_{t+\tau}) = \mu_o(x_{t+\tau}-\sin(\sin^{-1}x_t+\omega\tau) \quad \text{where}$$

$\mu_o(x)$ is the unit impulse function having infinite height and unit area at x=0 and being zero elsewhere. The density function, $p(x_t)$ can be found quite simply to be:

$$p(x_t) = \frac{1/\pi}{1-x_t^2} \qquad -1\le x_t\le 1$$

The covariance function then becomes:

$$R_x(\tau) = 1/\pi\int_{-1}^{1}\frac{x_t}{1-x_t^2}\,dx_t\int_{-1}^{1}x_{t+\tau}\mu_o(x_{t+\tau}-\sin(\sin^{-1}x_t+\omega\tau))dx_{t+\tau}$$

$$R_x(\tau) = 1/\pi\int_{-1}^{1}\frac{x_t}{1-x_t^2}\,\sin(\sin^{-1}x_t+\omega\tau)dx_t$$

$$R_x(\tau) = 1/\pi\cos\omega\tau\int_{-1}^{1}\frac{x_t^2}{1-x_t^2}dx_t+1/\pi\int_{-1}^{1}\sin\omega\tau\;\;x_tdx_t$$

The second integral is seen to be zero and the first integral is found to give the result:

$$R_x(\tau) = 1/2 \cos\omega\tau$$

Before this result is discussed at any length, the same problem is solved by taking the time average or correlation function. Since the function is periodic, the limiting operation can be eliminated and the integral can be evaluated for one period.

$$\phi_x(\tau) = \frac{1}{2\pi/\omega} \int_0^{2\pi/\omega} \sin(\omega t+\theta)\ \sin(\omega t+\omega\tau+\theta)dt$$

$$\phi_x(\tau) = \frac{\omega}{2\pi} \int_0^{2\pi/\omega} \left\{\cos\omega\tau-\cos(2\omega t+\omega\tau+2\theta)\right\}dt$$

The second term contributes nothing to the integration and the first term yields the result:

$$\phi_x(\tau) = 1/2\cos\omega\tau$$

Several important properties of covariance and correlation functions have been illustrated by this simple example. First, it is seen that time and ensemble averages are actually equal for the ergodic model used here. Secondly, the correlation function of any arbitrarily-phased sinusoid is a cosinusoid (with the same frequency as the original sinusoid) and furthermore, the phase of the original sample function is not expressed at all in the autocorrelation function. It should be noted in this connection that the same correlation function results from an infinite number of sinusoids, all differing in phase. Finally, the value of the autocorrelation function for $\tau=0$ is the mean-square value of the sinusoid which can be shown to be equal to the variance of the distribution of the sinusoid.

The results of this problem can be extended to the more general case of any random process by expanding the process into a Fourier Series or by transforming the process by a Fourier Integral. The results show that in general the correlation function has the same frequency components as the time series and all the frequency components are in cosine phase. Thus the correlation function is an even function with all the phase information of the time series destroyed. Furthermore, each frequency component has an amplitude that is its mean-square value. Sums of statistically independent non-periodic components also have the property of additive correlation functions. In addition, it can be shown that for $\tau=0$,

$$R_x(0) = \phi_x(0) = \sigma_x^2 + m_x^2$$

and for $\tau\to\infty$,

$$R_x(\tau\to\infty) = \phi_x(\tau\to\infty) = m_x^2 + \text{periodic components},$$

where x_t is any random process that has a correlation function.

Because of these properties, it is clear that the correlation function is not a unique function. That is, there are many different random processes that have the same correlation function. Furthermore, the correlation function does not uniquely define the probability distribution function of a process in general. This can be seen by expanding the multivariate characteristic function in a Taylor's Series and noting that the coefficients of the terms are the moments of the distribution. The correlation function is just one of the coefficients in this expansion. For the particular case of the Gaussian distribution, the correlation function uniquely specifies the distribution. Since this is the case, then it is also clear that for any random process (that has a first and second moment) there is a Gaussian random process with the same autocorrelation function.

Thus it is concluded, that correlation can be viewed as an extension of frequency analysis to stochastic processes. With this view in mind Wiener (96) has lumped all the various frequency-emphasizing-transformation methods (for periodic, non-periodic and random time series) under the title of General Harmonic Analysis.

2.3 Convergence and Estimation of the Autocorrelation Function

As the Introduction indicated, the problem with which this chapter is concerned is the estimation of the autocorrelation function of a stochastic process from a finite sample of time series. This problem has been attacked in the past by a number of authors. (7, (26, 27, 30, 60, 88) Most of these efforts, however, have been directed at getting general results. In this paper the effort is directed more at getting specific results that can be related to the problem of estimating the autocorrelation function of the EEG.

The function with which this paper is concerned

(2.31) is:

$$\phi_x(T,\tau) = 1/T \int_0^T x(t)x(t-\tau)dt$$

The most important point to note about this function is that it is itself a discrete random variable with parameters T and τ, where T is the sample length and τ is the delay or shift parameter. The desired solution to the problem is the probability density function of $\phi_x(T,\tau)$ in terms of the probability density function of x_t. This function could then be studied for convergence as T approached infinity. That this function, $\phi_x(T,\tau)$ is a consistant estimate of the autocorrelation function and converges to $\phi_x(\tau)$ in the limit, follows from the following argument given by Davenport, Johnson and Middleton. (27)

Consider the general problem of a finite moving average:

$$y(T) = 1/T \int_0^T z(t)dt$$

where z(t) is some sample function of an ergodic random process z_t.

The mean of the random variable y(T) is:

$$E[y(T)] = E\left[1/T \int_0^T z(t)dt \right]$$

$$= 1/T \int_0^T E[z(t)]\, dt$$

$$= E[z(t)]$$

Thus the mean of y(T) is the mean of z(t). The variance of y(T) can be gotten by the following argument:

$$E[y^2(T)] = E\left[1/T^2 \int_0^T\int_0^T z(t_1)z(t_2)dt_1dt_2\right]$$

$$= 1/T^2 \int_0^T\int_0^T E[z(t_1)z(t_2)]\, dt_1dt_2$$

$$= 1/T^2 \int_0^T\int_0^T R_z(t_1-t_2)dt_1dt_2$$

Using the change of variables:

$t_\tau - t_2 = \tau_0$ and

$t_1 + t_2 = U$ the result is:

$$E[y^2(T)] = 2/T^2 \int_0^T \int_{\tau_0}^{2T-\tau_0} R_z(\tau_0)dUd\tau_0 \, |J|$$

where J is the Jacobian

$$J = \begin{vmatrix} \frac{\delta t_1}{\delta \tau_o} & \frac{\delta t_2}{\delta \tau_o} \\ \frac{\delta t_1}{\delta U} & \frac{\delta t_2}{\delta U} \end{vmatrix} = \begin{vmatrix} 1/2 & -\ 1/2 \\ 1/2 & 1/2 \end{vmatrix} = 1/2$$

$$E\left[y^2(T)\right] = 1/T^2 \int_0^T \int_{\tau_o}^{2T-\tau_o} R_z(\tau_o)\,dU\,d\tau_o$$

$$E\left[y^2(T)\right] = 2/T \int_0^T (1 - \tau_o/T)\ R_z(\tau_o)\,d\tau_o$$

$$\begin{aligned} \sigma_y^2(T) &= E\left[y^2(T)\right] - E^2\left[y(T)\right] \\ &= E\left[y^2(T)\right] - E^2\left[z(t)\right] \\ &= 2/T \int_0^T (1 - \tau_o/T)\left(R_z(\tau_o) - E^2\left[z(t)\right]\right) d\tau_o \end{aligned}$$

Now,

$$\left| \int_0^T (1-\tau_o/T) \left(R_z(\tau_o) - E^2\left[z(t)\right]\right) d\tau_o \right| \leq$$

$$\int_0^T \left|1-\tau_o/T\right| \left|R_z(\tau_o) - E^2\left[z(t)\right]\right| d\tau_o \leq$$

$$\int_0^T \left|R_z(\tau_o) - E^2\left[z(t)\right]\right| d\tau_o$$

If $$\int_{-\infty}^{\infty} \left|R_z(\tau_o) - E^2\left[z(t)\right]\right| d\tau_o < \infty$$

then $\sigma_y{}^2(T)$ approaches zero as T approaches infinity.

The conclusion of this argument is that the finite moving average as defined, converges in the mean to the expectation of the process as the sample length is allowed to increase to an unbounded value, provided that the condition that

$$\int_{-\infty}^{\infty} \left| R_z(\tau_o) - E^2[z(t)] \right| \, d\tau_o \quad < \infty$$

is met. If the z(t) function is now defined as,

$$z(t) = x(t)x(t-\tau)$$

then it has been proven that $\phi_x(T,\tau)$ converges in the mean to $\phi_x(\tau)$ as T approaches infinity. This is an encouraging thought in the estimation problem, since it says that if longer and longer record lengths are taken, eventually the computeable, finite sample length autocorrelation function converges to the theoretical autocorrelation function. The manner in which $\phi_x(T,\tau)$ converges to $\phi_x(\tau)$ is, however, unknown at this point. It is conceivable that $\phi_x(T,\tau)$ converges to $\phi_x(\tau)$ in some oscillatory manner and there exists an optimum length T or a set of optimum lengths T_k for which the estimates of the autocorrelation function are best. On the other hand the convergence might be uniform and the estimate get better continuously as the sample length is increased. This question about the manner of convergence can best be settled by finding the probability density function of $\phi_x(T,\tau)$ and examining its behavior as T. is increased. Unfortunately this problem is a very complex one and to date there is no general solution.

Some measure of the deviation of $\phi_x(T,\tau)$ from its mean, $\phi_x(\tau)$, can be gotten, however, by calculating the variance of $\phi_x(T,\tau)$. This can be done by starting with the equation for the second moment of y(T) as previously derived:

$$E\left[y^2(T)\right] = 2/T\int_0^T(1-\tau_o/T)R_z(\tau_o)d\tau_o$$

This equation now becomes:

(2.32) $$E\left[\phi_x^2(T,\tau)\right] = 2/T\int_0^T(1-\tau_o/T)E\left[x(t)x(t-\tau)x(t-\tau_o)x(t-\tau-\tau_o)\right]d\tau_o$$

If no further assumptions are made about the x_t process then the above equation is the best that can be done in terms of estimating the second moment of $\phi_x(T,\tau)$. This is an unahppy situation, for in order to estimate the errors incurred in the finite autocorrelation function of some process, knowledge of the fourth-order moments must be available. Thus a higher-order statistic is needed. One might ask the question at this point that if indeed the higher-order statistic were known, for what reason would one need to know the estimate of errors of a lower order statistic? One would presumably already know it.

To get some estimate of the truncation errors, the assumption that x_t is a Gaussian process is made. This allows for a simplification of the above expressions and a variance term can then be calculated. This assumption of Normality is not as severe as one might think. First of all, for the very important case of Gaussian random variables the expression for the variance is exact, and secondly, for non-Gaussian processes some estimate of errors can still be achieved through this approximation. As has been pointed out by Tukey (87), (88) the errors of finiteness of record length are very much dependent upon the distribution of the particular random variable in question. The estimate for the Gaussian process, however, is neither the worst nor the best. As Tukey has pointed out, some random processes will yield good estimates while others will not. For instance, consider a process that consists of long bursts of constant frequency sinusoids. The frequency of the bursts is abruptly changed at random. It can be seen, heuristically, that a particular finite autocorrelation function will yield a very poor estimate of the autocorrelation function of the process and furthermore, neither will yield much information about the actual process. Thus the estimate of the autocorrelation function of such a process will be exceedingly poor. Similarly, random processes can be constructed for which the estimate of the autocorrelation function can be better than for the Guassian case.

In any event, to carry out the derivation of the variance of the finite sample autocorrelation function in this chapter, normality is assumed since it simplifies the mathematics. Thus, for the Gaussian case, with zero mean for the random variable x_t:

$$\begin{aligned} E[x(t)x(t-\tau)x(t-\tau_o)x(t-\tau_o-\tau)] = {} & E[x(t)x(t-\tau_o)]\ E[x(t-\tau)x(t-\tau_o-\tau)] + \\ & E[x(t)x(t-\tau)]\ E[x(t-\tau_o)x(t-\tau_o-\tau)] + \\ & E[x(t)x(t-\tau_o-\tau)]E[x(t-\tau)x(t-\tau_o)] \end{aligned}$$

This result follows from the expansion of the characteristic function of the random variables $x_t, x_{t-\tau}$, $x_{t-\tau}, x_{t-\tau-\tau}$. Therefore,

$$E[x(t)x(t-\tau)x(t-\tau_o)x(t-\tau_o-\tau)] = \phi_x^2(\tau_o)+\phi_x^2(\tau)+\phi_x(\tau+\tau_o)\phi_x(\tau_o-\tau)$$

and,

$$\sigma_\phi^2(T,\tau) = 2/T^2 \int_0^T (T-\tau_o) \left\{ \phi_x^2(\tau_o)+\phi_x(\tau+\tau_o)\phi_x(\tau_o-\tau)\right\} d\tau_o \tag{2.33}$$

Equation 2.33 is the final result if no further assumptions are made about the random process x_t. To summarize the results so far:

The mean of the random variable $\phi_x(T,\tau)$, defined in equation 2.31, is $\phi_x(\tau)$ and the variance is given in equation 2.33. The assumptions that have been made are that the joint fourth order distribution of x_t is Gaussian and that further the mean of x_t is zero.

2.4 The Finite-Sample Autocorrelation Function of the Narrow-Band Gaussian Process*

Equation 2.33, although still quite general, reveals very little about the finite autocorrelation of the EEG. For reasons that might become clearer in Chapter 4, a further assumption about the nature of the random variable x_t is made. It is assumed here that x_t is a narrow-band Gaussian process. It is to be emphasized, that this is in no way to be construed as an assumption upon the nature of the EEG signal. The discussion in this section is concerned only with the mean and variance of the finite sample autocorrelation of narrow-band Gaussian noise.

The narrow band noise signal is derived by filtering white Gaussian noise with a linear, quadratic filter whose impulse respone is:

(2.41) $$h(t) = Ae^{-\alpha t}\sin(\omega_d t - \phi) \quad \text{for } t \geq 0$$

where,

$$A = \omega_o \Big/ \sqrt{\left(\frac{\omega_o}{\alpha}\right)^2 - 1}$$

$$\omega_d = \sqrt{\omega_o^2 - \alpha^2}$$

$$\phi = \sin^{-1} \frac{\sqrt{\omega_o^2 - \alpha^2}}{\omega_o}$$

---

*The expression given here is derived independently of a similar expression given in Bendat. (7)

The Spectral Density of the input noise is N watts/cps. The autocorrelation function of the output of this system is computed in Appendix 1 and the normalized result is:

(2.42) $$R_x(\tau) = e^{-\alpha|\tau|}\cos\omega_o\tau$$

which is valid for the assumption that $\omega_o >> \alpha$.

Now the expression for the autocorrelation function of narrow-band noise (equation 2.42) is substituted into equation 2.33.

(2.43)
$$\sigma_\phi^2(T,\tau) = 2/T^2\int_0^T (T-\tau_o)\Big\{e^{-2\alpha|\tau_o|}\cos^2\omega_o\tau_o + e^{-\alpha|\tau_o+\tau|}\cos\omega_o(\tau_o+\tau)e^{-\alpha|\tau_o-\tau|}\cos\omega_o(\tau_o-\tau)\Big\}d\tau_o$$

$$= 2/T^2\int_0^T (T-\tau_o)\, e^{-2\alpha\tau_o}\cos^2\omega_o\tau_o d\tau_o +$$

$$2/T^2\int_\tau^T (T-\tau_o)\, e^{-2\alpha\tau_o}\cos\omega_o(\tau+\tau_o)\cos\omega_o(\tau-\tau_o)\, d\tau_o +$$

$$2/T^2\int_0^\tau (T-\tau_o)\, e^{-2\alpha\tau}\cos\omega_o(\tau+\tau_o)\cos\omega_o(\tau-\tau_o)\, d\tau_o$$

These three integrals are evaluated in Appendix 2 and the result in simplified form is:

$$
\begin{aligned}
(2.44)\quad \sigma_{\phi}^{2}(T,\tau) = {} & 1/T^{2}e^{-2\alpha T}\left\{ 1/4\alpha^{2} - \frac{\alpha\omega_o}{(\alpha^{2}+\omega_o^{2})^{2}}\sin 2\omega_o T \right. \; + \\
& \left. \frac{\alpha^{2}-\omega_o^{2}}{2(\alpha^{2}+\omega_o^{2})^{2}}\cos 2\omega_o T + \frac{1}{4\alpha^{2}}\cos 2\omega_o\tau \right\} \; + \\
& 1/T^{2}\left\{ \frac{\omega_o^{2}-\alpha^{2}}{4(\alpha^{2}+\omega_o^{2})^{2}} + \frac{1}{4\omega_o^{2}}\, e^{-2\alpha\tau} \right. \; + \\
& e^{-2\alpha\tau}\sin 2\omega_o\tau\left(\frac{\alpha\omega_o}{2(\alpha^{2}+\omega_o^{2})^{2}}\right) \; + \\
& e^{-2\alpha\tau}\cos 2\omega_o\tau\left(\frac{\omega_o^{2}-\alpha^{2}}{4(\alpha^{2}+\omega_o^{2})^{2}} - \frac{1}{4\alpha^{2}} - \frac{1}{4\omega_o^{2}}\right) \; + \\
& \tau e^{-2\alpha\tau}\sin 2\omega_o\tau\left(\frac{\omega_o}{2(\alpha^{2}+\omega_o^{2})} - \frac{1}{2\omega_o}\right) \; + \\
& -\tau e^{-2\alpha\tau}\cos 2\omega_o\tau\left(\frac{\alpha}{2(\alpha^{2}+\omega_o^{2})} + \frac{1}{2\alpha}\right) \; + \\
& \left. -\frac{\tau^{2}}{2}\, e^{-2\alpha\tau}\cos 2\omega_o\tau \right\} \; + \\
& 1/T\left\{ \frac{1}{2\alpha}\;\frac{2\alpha^{2}+\omega_o^{2}}{\alpha^{2}+\omega_o^{2}} \right. \; + \\
& e^{-2\alpha\tau}\sin 2\omega_o\tau\left(\frac{1}{2\omega_o} - \frac{\omega_o}{2(\alpha^{2}+\omega_o^{2})}\right) + \\
& e^{-2\alpha\tau}\cos 2\omega_o\tau\left(\frac{\alpha}{2(\alpha^{2}+\omega_o^{2})} + \frac{1}{2\alpha}\right) + \\
& \left. \tau e^{-2\alpha\tau}\cos 2\omega_o\tau \right\}
\end{aligned}
$$

To make sense of these cumbersome expressions a series of reasonable approximations are made. The first approximation is one that already has been made, namely that $\omega_o >> \alpha$. This assumption is due to the statement that the noise signal considered is narrow-band. It is further assumed that the record length, T, of the sample function is much larger than the maximum value of delay for which the autocorrelogram will be examined. This is done to prevent rather obvious truncation errors from destroying the entire meaning of the correlogram. The last assumption is that the maximum value of delay of the correlogram is larger than the longest time constant present in the autocorrelation function. This assumption simply says that the correlogram will be computed to a sufficiently long enough value of delay so that the phenomenon to be studied can be examined.

To summarize, the assumptions made are the following:

$$\omega_o >> \alpha$$

$$T >> \tau_{max}$$

$$\tau_{max} >> 1/\alpha$$

Making use of these approximations in equation 2.44, the following results:

(2.45)
$$\sigma_\phi^2(T,\tau) \doteq 1/T^2 \left\{ \frac{1}{4\omega_o^2} + \frac{1}{4\omega_o^2} e^{-2\alpha\tau} + \frac{\alpha}{2\omega_o^3} e^{-2\alpha\tau} \sin 2\omega_o\tau + \right.$$
$$- \frac{1}{4\alpha^2} e^{-2\alpha\tau} \cos 2\omega_o\tau - \frac{\tau}{2\alpha} e^{-2\alpha\tau} \cos 2\omega_o\tau +$$
$$\left. - \frac{\tau^2}{2} e^{-2\alpha\tau} \cos 2\omega_o\tau \right\} +$$
$$1/T \left\{ \frac{1}{2\alpha} + \frac{1}{2\alpha} e^{-2\alpha\tau} \cos 2\omega_o\tau + \right.$$
$$\left. \tau\, e^{-2\alpha\tau} \cos 2\omega_o\tau \right\}$$

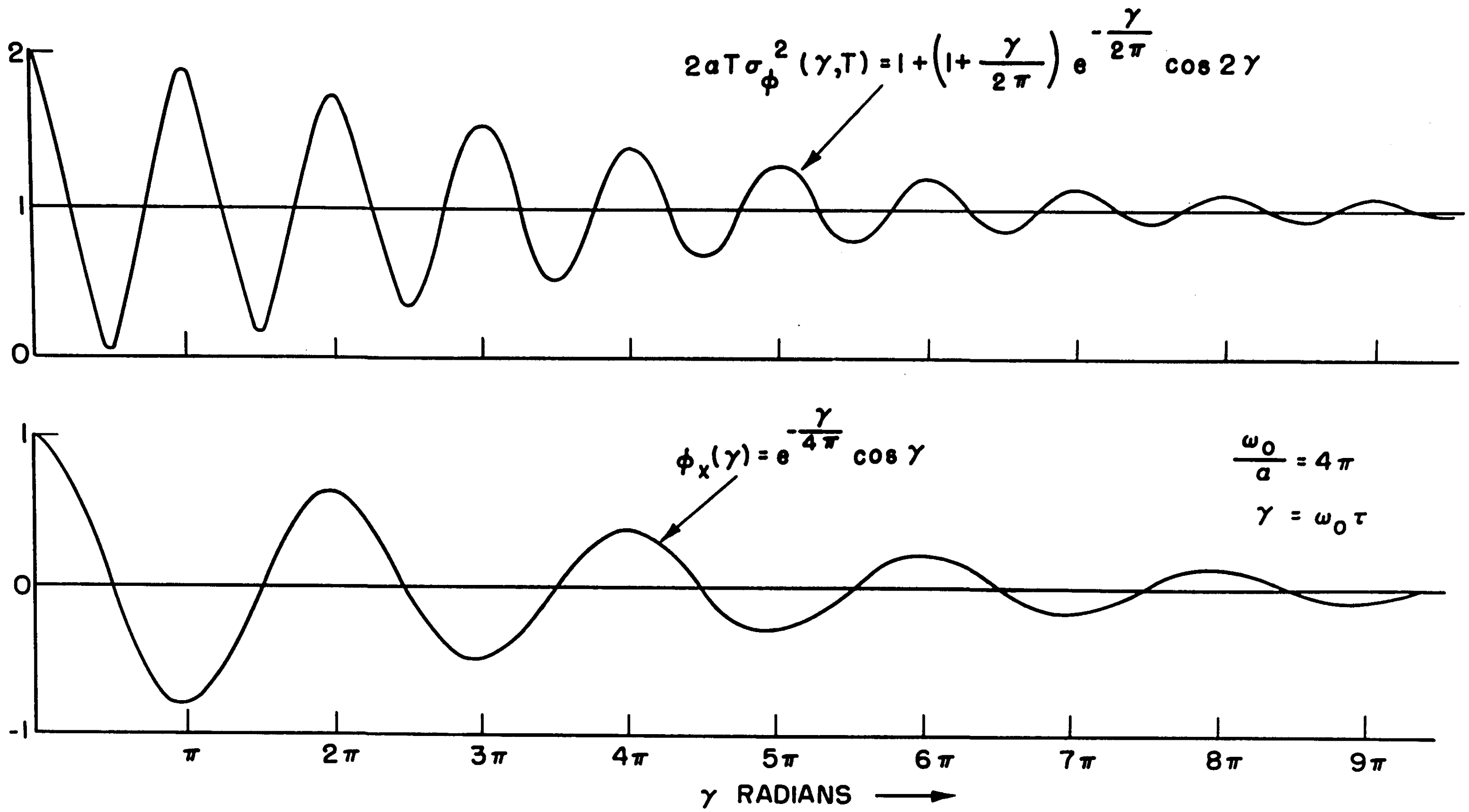
A further simplification can be made in equation 2.45 by noting three facts. First, the sum of the maxima of terms that comprise the coefficients of the $1/T^2$ term, are small when compared to the coefficients of the $1/T$ term. Secondly, the terms in the coefficient of $1/T^2$ tend to cancel. Thirdly, the assumption of dropping the $1/T^2$ terms becomes more valid for large values of τ since for these values, one term, $1/2\alpha T$, predominates in the evaluation of the variance. For a first order result, therefore, the following is offered:

(2.46)
$$\sigma_\phi^2(T,\tau) \doteq 1/2\alpha T \left[1 + (1+2\alpha\tau) e^{-2\alpha\tau} \cos 2\omega_o\tau\right]$$

This result is shown plotted in normalized form in figure 2.41 In addition, the theoretical autocorrelation function of narrow-band Gaussian noise is also plotted in the same figure.

Figure 2.41

Mean and Variance of Estimate of the Autocorrelation Function of Narrow Band, Gaussian Noise



$2aT\sigma_\phi^2(\gamma,T) = 1 + \left(1 + \frac{\gamma}{2\pi}\right) e^{-\frac{\gamma}{2\pi}} \cos 2\gamma$
$\phi_x(\gamma) = e^{-\frac{\gamma}{4\pi}} \cos \gamma$
$\frac{\omega_0}{a} = 4\pi$
$\gamma = \omega_0 \tau$
2
1
0
-1
π
2π
3π
4π
5π
6π
7π
8π
9π
γ RADIANS

Several features of this function, $\sigma_{\emptyset}^{2}(T,\tau)$, should be noted. As a first approximation it varies inversely with T for a fixed value of τ. Thus if longer and longer sample lengths are taken, the variance decreases to zero. Furthermore, for a fixed value of T, the variance approaches a constant as τ is increased. However, the signal level is decreasing exponentially. Thus the signal-to-noise ratio is decreasing approximately exponentially for large values of τ. This is seen from the following definition of the signal-to-noise ratio:

$$\frac{E[\emptyset_x(T,\tau)]}{\sigma_{\emptyset}(T,\tau)} \doteq \frac{e^{-\alpha\tau}\cos\omega_o\tau}{\left[1/2\alpha T\left\{1+(1+2\alpha\tau)e^{-2\alpha\tau}\cos 2\omega_o\tau\right\}\right]^{1/2}}$$

For large values of τ the approximation becomes:

(2.47) $$\frac{E\left[\emptyset_x(T,\tau)\right]}{\sigma_{\emptyset}(T,\tau)} \doteq \sqrt{2\alpha T}\; e^{-\alpha\tau}\cos w_o\tau \qquad \tau \gg 0$$

This expression can be shown to be a valid expression for the signal-to-noise ratio at large τ, even with the condition that $T \gg \tau_{max}$ dropped. The new conditions become:

$$\tau \gg 1/\alpha$$

$$T \gg 1/\alpha$$

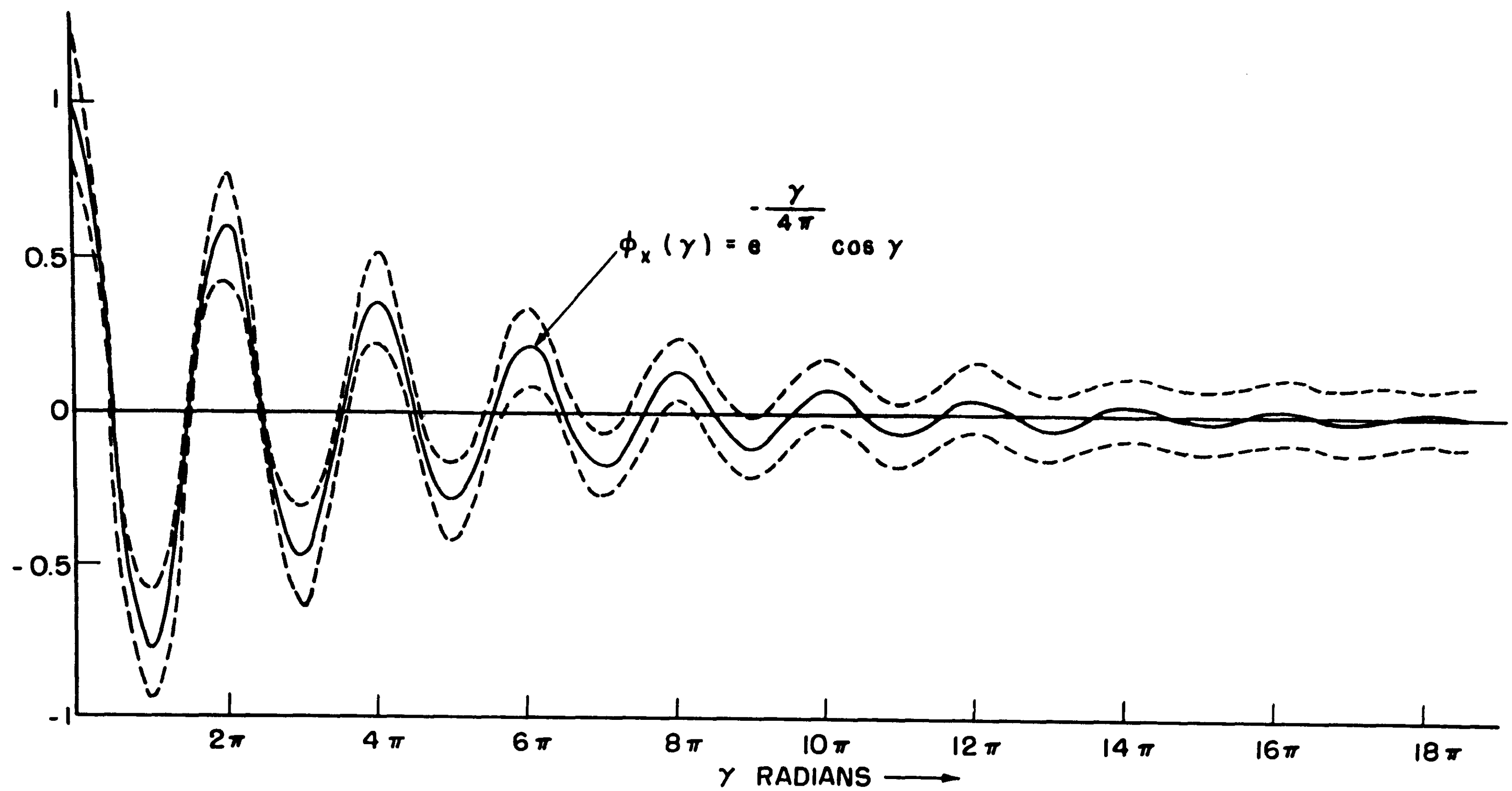
$$\omega_o \gg \alpha$$

Another way to viualize the phenomenon described is to plot the theoretical autocorrelation function and to superimpose upon it a 3σ confidence limit (shown schematically in figure 2.42). Since the distribution of $\phi_x(T,\tau)$ is not known, some of the significance of this tolerance band is lost. That is to say, a probability of the signal lying outside of this range cannot be calculated. Nevertheless, this sort of display is useful for visualizing the effect of the errors of estimation introduced by the finiteness of the record.

The results of this section so far have given some estimate of the errors introduced into the computation of the autocorrelation function of narrow-band, Gaussian noise due to the finiteness of the record. Since the form of these errors can be found experimentally, it is of some interest to see if the theory can predict something about their form. If this form were simply random in amplitude, with no correlation between successive points, then it might be expected that the estimation of the autocorrelation function would be a simple process. Only two parameters are required to determine the autocorrelation function of narrow-band noise, the central frequency and the bandwidth. Since the signal-to-noise ratio is very good for small values of τ and if the signal and noise are readily separable to the eye, then estimation of

Figure 2.42

Estimated Autocorrelation Function of Narrow-band, Gaussian Noise with Confidence Limits



1
0.5
0
- 0.5
-1
$\phi_x(\gamma) = e^{-\frac{\gamma}{4\pi}} \cos\gamma$
2π
4π
6π
8π
10π
12π
14π
16π
18π
γ RADIANS

of the necessary parameters is simple. It is the purpose of this last section of this chapter to show that this is indeed not the case, and in fact, the errors due to finiteness or record look like the signal (theoretical autocorrelation) for large values of τ.

To examine this problem, consider the cross-correlation of two samples of the finite sample auto-correlation function for large values of delay. The function to be studied is:

$$\mu = E\left[\phi_x(T,\tau)\ \phi_x(T,\tau+\tau^1)\right]$$

Proceeding in the computation of this function in a similar manner as before, the following results:

$$\mu = 1/T^2 \int_0^T\int_0^T E\left[x(t_1)x(t_1-\tau)x(t_2)x(t_2-\tau-\tau^1)\right]dt_1dt_2$$

The joint distribution of the variables x_{t_1}, $x_{t_1-\tau}$, x_{t_2} and $x_{t_2-\tau-\tau^1}$ is again assumed to Gaussian with zero mean and, therefore, factors to give:

$$\mu = 1/T^2 \int_0^T\int_0^T \Big\{E\left[x(t_1)x(t_1-\tau)\right] E\left[x(t_2)x(t_2-\tau-\tau^1)\right] + E\left[x(t_1)x(t_2)\right]\ E\left[x(t_1-\tau)x(t_2-\tau-\tau^1)\right] + E\left[x(t_1)x(t_2-\tau-\tau^1)\right] E\left[x(t_1-\tau)x(t_2)\right]\Big\}\ dt_1dt_2$$

It is now assumed that the value of the delay, τ, is large enough to consider the variables $x(t)$ and $x(t-\tau)$ statistically independent. That is to say, the phenomenon to be studied occurs at values of τ for which the theoretical autocorrelation function is reduced essentially to its baseline. With this assumption of independence the joint second moment $E[x(t)x(t-\tau)]$ factors into $E[x(t)]\,E[x(t-\tau)]$ and this term is dropped due to the assumption that the process has zero mean. Equation 2.49 then becomes:

$$\mu = 1/T^2 \int_0^T\int_0^T \Big\{ R_x(t_1-t_2)R_x(t_1-t_2+\tau^1) \quad + \quad R_x(t_1-t_2+\tau+\tau^1)R_x(t_1-t_2-\tau) \Big\}\, dt_1 dt_2$$

Performing the change of variable $\tau_o = t_1 - t_2$ and $\mu = t_1 + t_2$ as previously shown, the following results:

$$\mu = 2/T^2 \int_0^T (T-\tau_o) \Big\{ R_x(\tau_o)R_x(\tau_o+\tau^1) \quad + \quad R_x(\tau_o+\tau+\tau^1)R_x(\tau_o-\tau) \Big\}\, d\tau_o$$

The narrow-band assumption is again made **at** this point and μ then becomes:

$$\mu = 2/T^2 \int_0^T (T-\tau_o) \Big\{ e^{-\alpha|\tau_o|}\cos\omega_o\tau_o\, e^{-\alpha|\tau_o+\tau^1|}\cos\omega_o(\tau_o+\tau^1) \quad + \quad e^{-\alpha|\tau_o+\tau+\tau^1|}\cos\omega_o(\tau_o+\tau+\tau^1)\, e^{-\alpha|\tau_o-\tau|}\cos\omega_o(\tau_o-\tau)\, d\tau_o$$

$$\mu = 2/T^2 e^{-\alpha\tau^1} \int_0^T (T-\tau_o) e^{-2\alpha\tau_o} \cos\omega_o\tau_o \cos\omega_o(\tau_o+\tau^1)\, d\tau_o \quad +$$

$$2/T^2 e^{-\alpha\tau^1} \int_\tau^T (T-\tau_o) e^{-2\alpha\tau_o} \cos\omega_o(\tau_o+\tau+\tau^1)\cos\omega_o(\tau_o-\tau)\, d\tau_o \quad +$$

$$2/T^2 e^{-\alpha\tau^1} e^{-2\alpha\tau} \int_0^\tau (T-\tau_o)\cos\omega_o(\tau_o+\tau+\tau^1)\cos\omega_o(\tau_o-\tau)\, d\tau_o$$

These three integrals are evaluated in Appendix 3. Making the assumptions that:

$$\omega_o >> \alpha$$
$$\tau >> 1/\alpha$$
$$T >> 1/\alpha$$

the result, as shown in Appendix 3, is:

(2.410) $$\mu = 1/2\alpha T e^{-\alpha\tau^1} \cos\omega_o\tau^1 \qquad \tau^1 \geq 0$$

This result can be checked against the variance in equation 2.46. If this equation is evaluated for large τ, then $\sigma^2(T,\tau) = 1/2\alpha T$

Evaluating equation 2.410 for τ^1 equal to zero gives the same result.

Now what does the result obtained for the cross-correlation of the samples of the finite sample autocorrelation function mean? It says essentially that the errors of estimation that result from the finite sample computation have a form that has the same basic temporal characteristics as the autocorrelation function and the original narrow-band process. That is, successive samples separated by a value of delay τ^1 are correlated in the same manner as the signals $x(t)$ and $x(t+\tau^1)$. This is unfortunate since if the signal (theoretical autocorrelation function) and the noise (errors due to the truncation of the time series) look alike, then how are they to be told apart? The answer is that the noise decreases as the sample length (T) is increased while the signal is not decreased. This is really the only way the two can be told apart unless more is known about the signal.

CHAPTER 3 - EXPERIMENTAL RESULTS OF THE AUTOCORRELATION OF A FINITE-TIME SAMPLE OF NARROW BAND, GAUSSIAN NOISE

3.1 Introduction and Description of Correlator and Machine Correlation Method

This chapter presents the results of an experimental investigation of the effects of truncation of a time series on the autocorrelation function of that time series. In particular the narrow-band process studied in Chapter 2 is investigated. For this purpose, the Analog Correlator (2) of the Communications Biophysics Laboratory was used for computing the correlograms (machine-calculated finite time sample autocorrelation functions). The Correlator is a device that calculates the value of the integral

$$K\int_0^T x_1(t)x_2(t-\tau)dt$$

for discrete intervals of τ. The schematic diagram of this device is shown in figure 3.11.

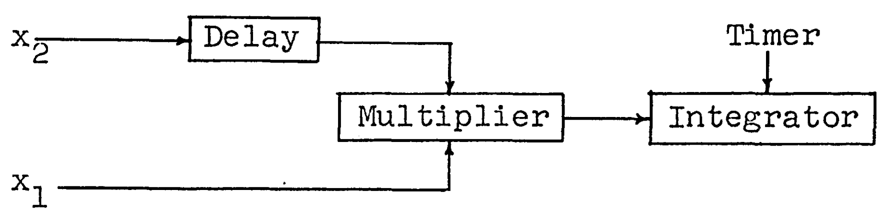


figure 3.11
Schematic Diagram of Correlator

The delay is achieved by the use of a magnetic drum. The two signals to be correlated are recorded on the drum in adjacent tracks. The delay is achieved by reading the two signals from read heads that are displaced along the circumference by a distance proportional to the value of delay. This value of delay can be stepped incrementally in values of τ from 0.05 milliseconds to 5.0 milliseconds. The maximum total value of delay is 185 milliseconds.

The multiplier is a quarter-square device and the integrator is a simple Miller integrator. Thus, to get a correlogram by this method, it is necessary to record the data on magnetic tape. Reflectors are then taped onto the magnetic tape separated by a distance along the tape proportional to the sample length (T). The correlator can then be set to automatically make one pass over the data (from reflector to reflector) for each point on the correlogram. The beginning and end of the sample are sensed by shining a light on the tape. When the reflector is reached, a photo-electric cell produces a pulse that triggers control relays that rewind the tape, start the correlator, and stop the correlator.

Before the narrow-band noise data is dealt with it might be propitious to present some control runs on the correlator in order to demonstrate that the errors of extimation to be encountered with the data are not machine artifact. In this light, the top correlogram of figure 3.12 shows the correlogram that results from cross-correlating zero with a narrow-band noise signal. This control is a check on the correlator balance, which is seen to be good for the gain settings employed. These gain settings are the same as those used for most of the correlograms of narrow-band noise. The second correlogram of the same figure shows the amount of zero drift for inputs constrained to zero. The last correlogram shows the autocorrelogram of a 250 cps sinusoid. There is no significant change in the period of the sinusoid as a function of delay. Thus the effects of tape stretch and wow are seen to be negligible.

For the sake of completeness it might be added that the auxilliary equipment (amplifiers, tape recorder, etc.) have bandwidths that are more than adequate to reliably reproduce both the narrow-band noise and the EEG signals to be studied here. Thus it can be concluded that, for the purpose of studying the statistical errors of finite sample correlation, as defined in Chapter 2, the Analog Correlator and associated equipment appear to be more than adequate. The machine artifacts can be assumed to be second-order effects.

Figure 3.12

Correlograms of Control Signals

Top
Crosscorrelation of Narrow-Band Noise with
Zero Input

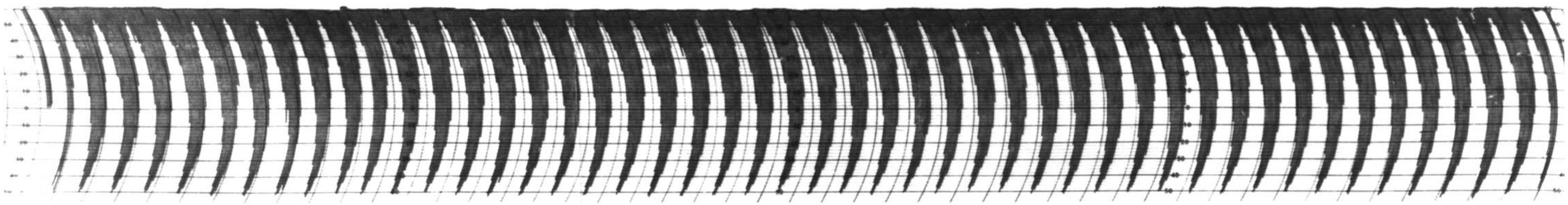
T (sample length) = 7.5 seconds
$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 185 milliseconds

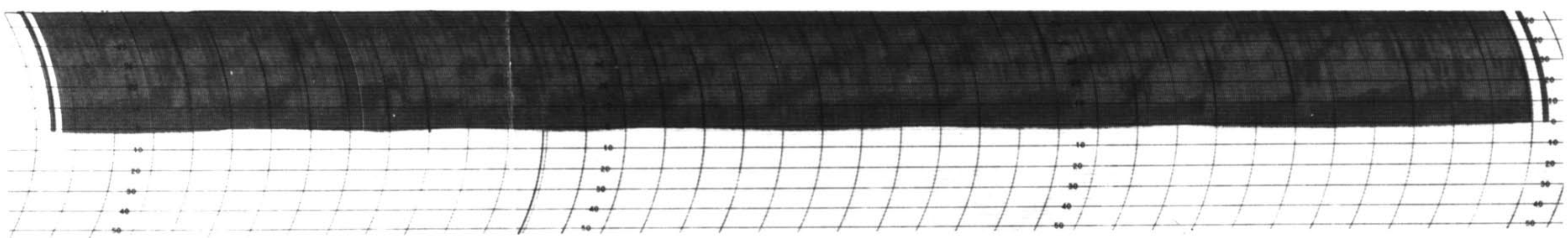
Center
Autocorrelation of Zero (Inputs Shorted)

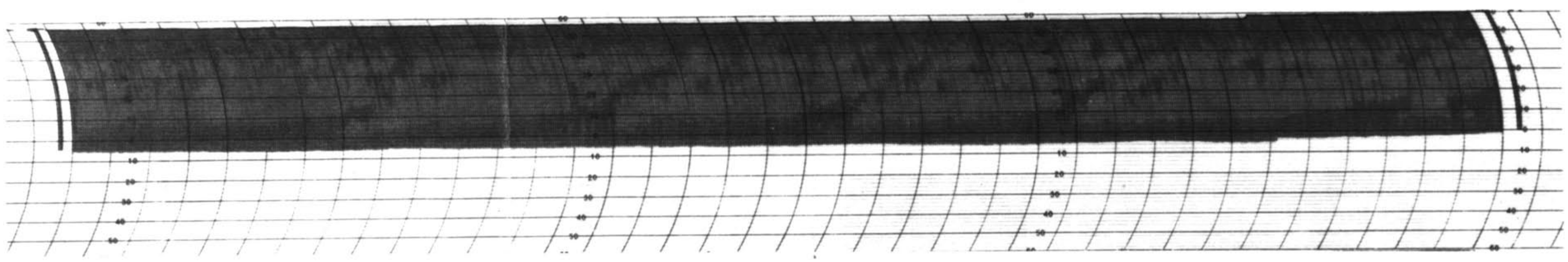
T (sample length) = 7.5 seconds
$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 185 milliseconds

Bottom
Autocorrelation of 250 cps Sinusoid

T (sample length) = 7.5 seconds
$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 185 milliseconds







3.2 Machine Correlation of Narrow-Band, Gaussian Noise

The narrow-band signal to be studied here was obtained by filtering wide-band (20 kc) noise with a narrow-band, quadratic filter. The schematic circuit diagram is shown in figure 3.21.

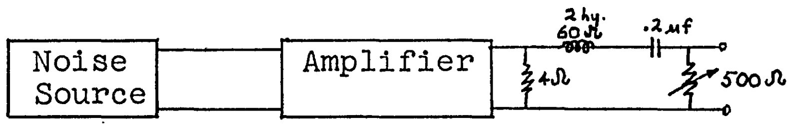


Figure 3.21
Schematic of System to Generate
Narrow-Band Noise

The 500 ohm potentiometer was used to vary the Q of the circuit, Q being defined as:

$$Q = \frac{\omega_o}{2\alpha} = \frac{\text{central frequency}}{\text{band width}}$$

The central frequency (ω_o) and the Q of the circuit were picked such that convenient circuit parameters could be used for the filter and so that the phenomenon to be studied would be easily discernable. A value of Q exceeding ten was desirable to make the approximations of a narrow-band process, made in Chapter 2, valid. Too large a value of Q would, however, make circuit parameters for the filter inconvenient and would make the finite sample errors impossible to study. Due to the physical limitations of the correlator's ability to carry out long delay correlograms, a value of Q is needed such that the correlogram will reduce to its theoretical baseline in about 1/3 of the maximum delay available. This would allow the examination of the record for values of delay beyond the point where the correlation function should theoretically be reduced to zero.

The frequency characteristic of the filter and associated equipment is shown in figure 3.22. The central frequency is 237 cps with a value of Q of 13.2. The transfer function, impulse response and autocorrelation function of the output of the filter are given below with normalized gains:

$$H(s) = \frac{s}{s^2+113s+2.21\times10^6}$$

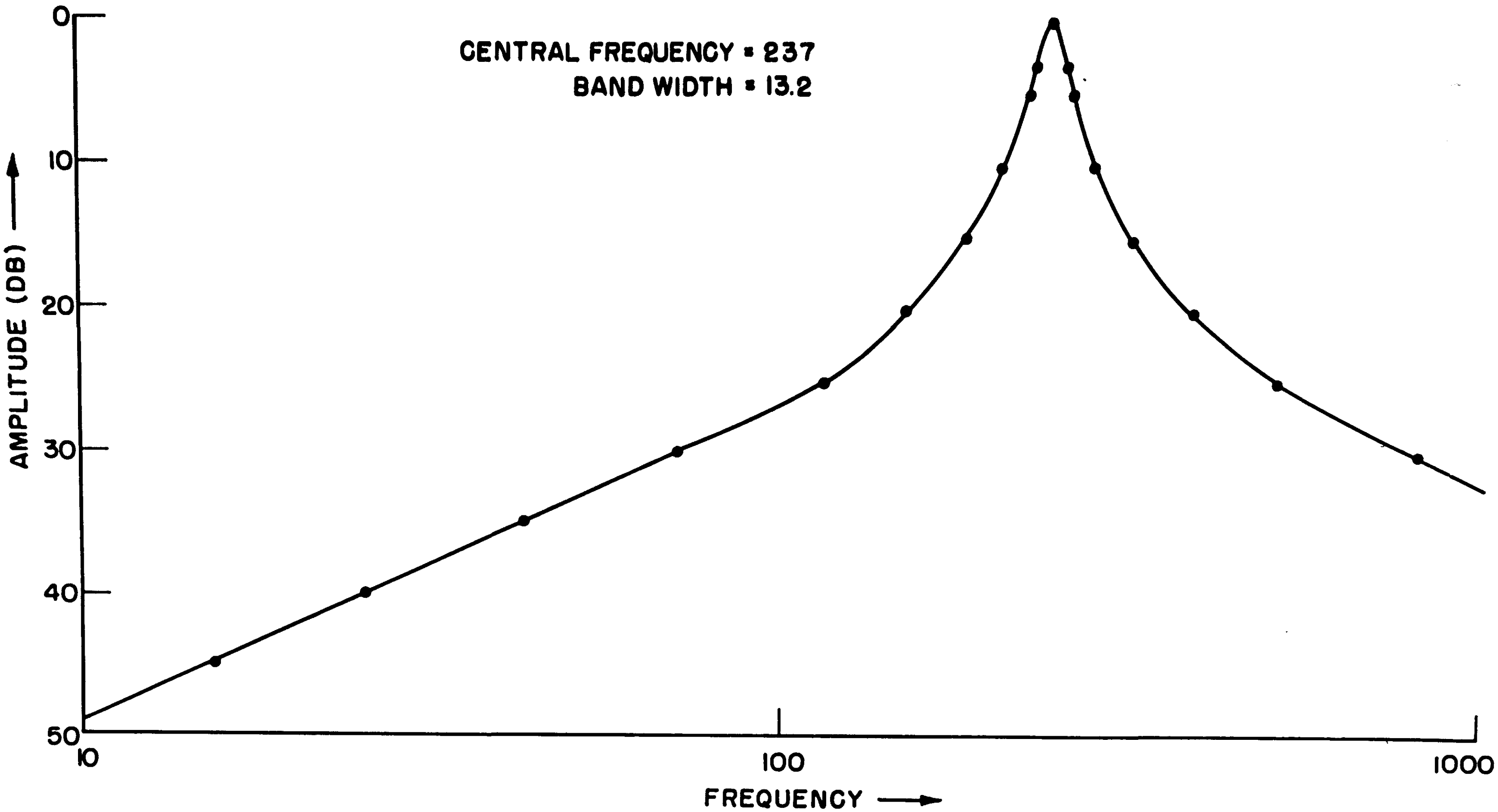
$$h(t) = e^{-113t}\cos 1488t \qquad t \geq 0$$

$$\phi_x(\tau) = e^{-113|\tau|}\cos 1488\tau$$

The left-hand presentation of figure 3.23 shows a sample of the wide-band noise, above which is shown a histogram of the amplitudes of this signal. The center curve shows the same information for the narrow-band noise. In each case the noise was sampled at 5 kc and the histogram represents a total of 262,000 samples. The right most curve is a control run of a 250 cps sinusoid that was randomly sampled. Each of these histograms was computed on the Average Response Computer.(24) This device is a digital computer that operates in its histogram mode by sampling a quantized signal and adding

Figure 3.22

Frequency Characteristic of Narrow-Band,
Quadratic Filter



CENTRAL FREQUENCY = 237
BAND WIDTH = 13.2
AMPLITUDE (DB)
0
10
20
30
40
50
10
100
1000
FREQUENCY

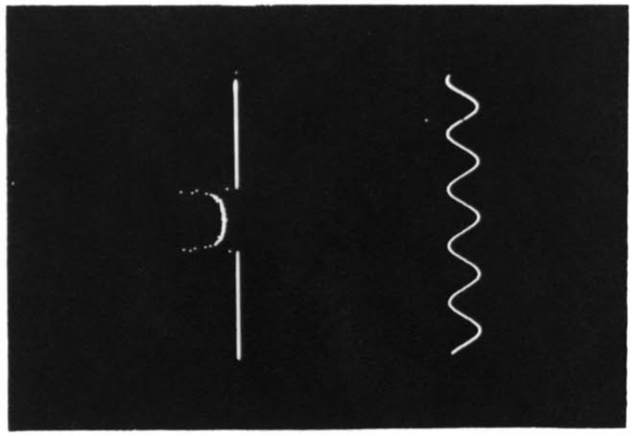
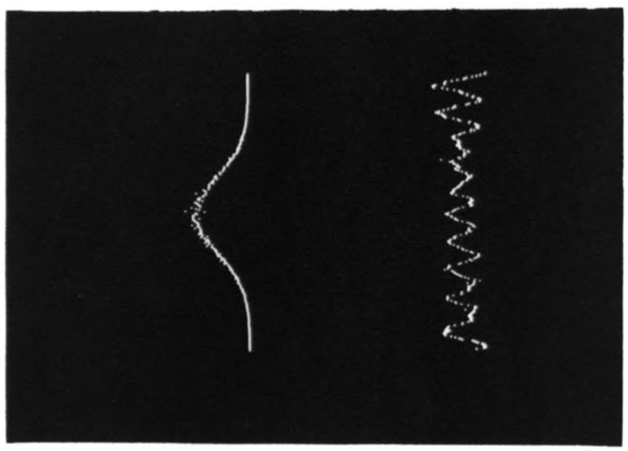
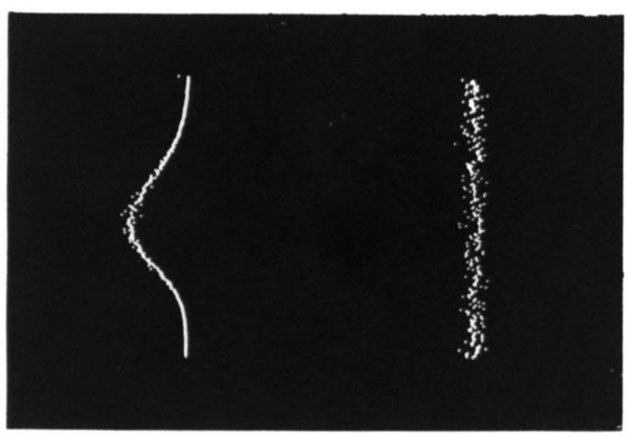
Figure 3.23

Histograms of Amplitudes of Known Signals

Left
Wide-Band Noise (Bandwidth = 20 Kc)
Number of samples = 262,000
Sampling frequency = 5 Kc

Center
Narrow-Band Noise
(Central frequency = 237 cps
Bandwidth = 13.2 cps)
Number of samples = 262,000
Sampling frequency = 5 Kc

Right
Sinusoid (250 cps)
Number of samples = 262,000
Random sampling

the number of times the amplitude of the samples reaches each quantization level.

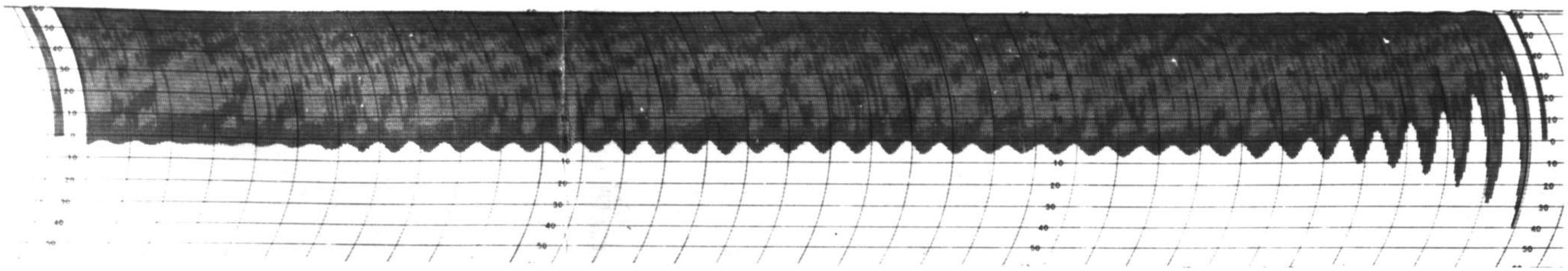
This narrow-band noise signal, recorded on magnetic tape, was then autocorrelated by the Analog Correlator. A series of ten correlograms were run, each of sample length of 7.5 seconds with a maximum delay of 185 milliseconds in incremental steps of 0.25 milliseconds. Figures 3.24 and 3.25 show six of the ten correlograms. It is to be noted that the correlograms decrease uniformly in each case, but in some cases they start to increase again. This characteristic "waxing and waning" of the envelope of the correlogram for large values of delay (it was evident in nine out of the ten correlograms) is the phenomenon that is to be studied experimentally in this section.

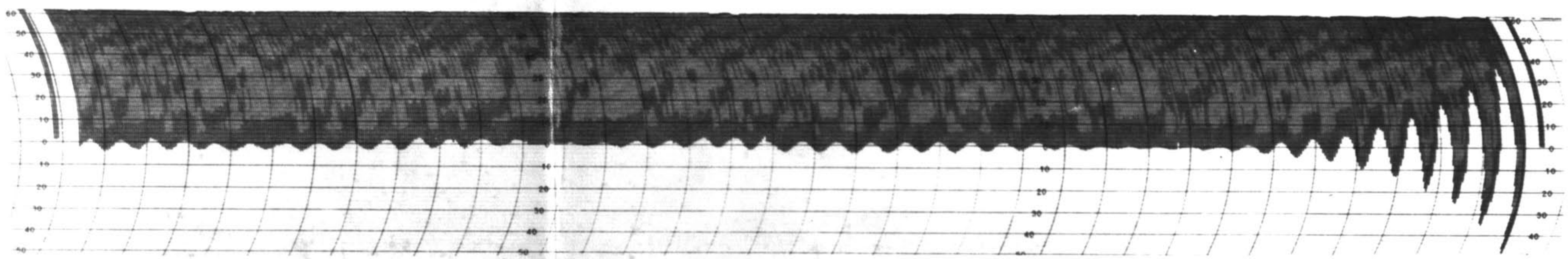
Two interpretations can be given for the effects seen in figures 3.24 and 3.25. Either the "waxing and waning" of the envelope of the correlogram indicates the presence of a periodic signal plus some corrupting signal or this effect is an error of the finite sample correlation process. If the former is the case, then it must be assumed that there is some strong in-phaseness of the 237 cps activity in the narrow-band signal. That is to say, the signal $x(t)$ is strongly correlated with the signal $x(t-\tau)$ for τ equal to as much as 185 milliseconds.

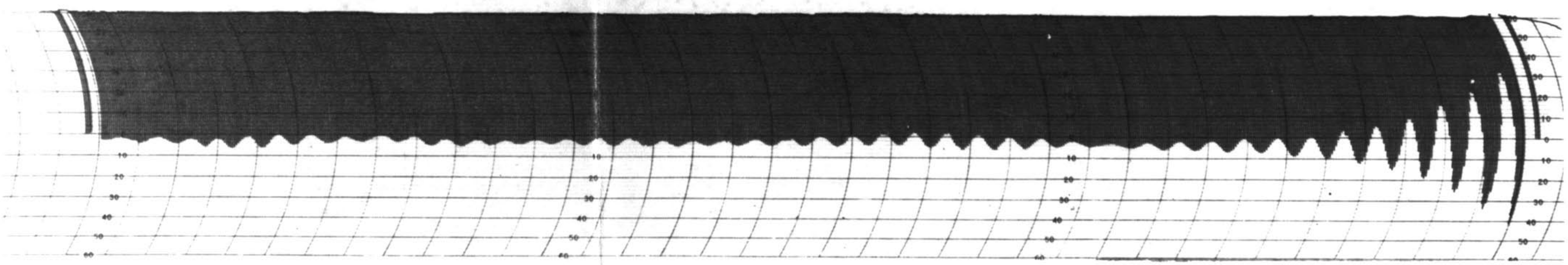
Figures 3.23 and 3.24

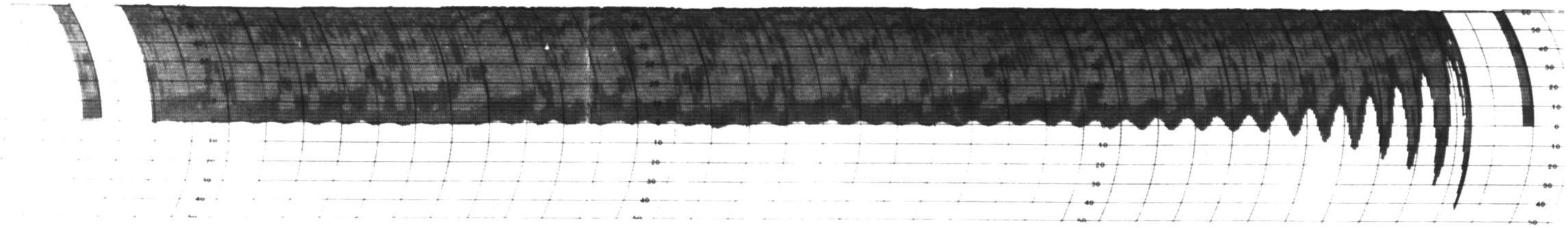
Six Correlograms of Narrow-Band, Gaussian Noise

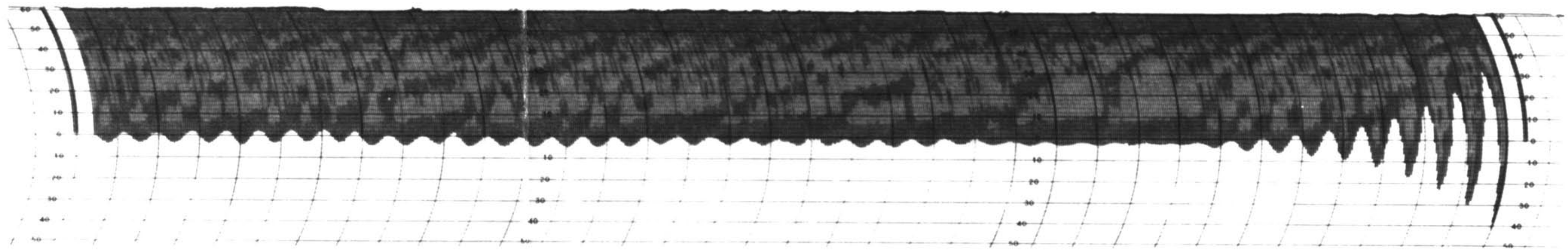
T	(sample length)	=	7.5 seconds
$\delta\tau$	(delay increment)	=	.25 milliseconds
τ_{max}	(maximum delay)	=	185 milliseconds

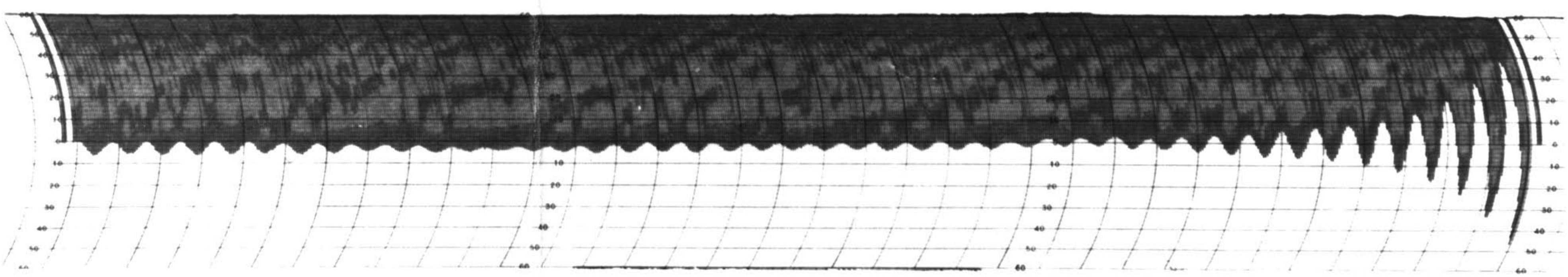












But, it is known that this should not be the case since the theoretical autocorrelation function of the narrow-band noise indicates that in approximately 30 milliseconds the autocorrelation function of the noise should be reduced by about 95 per cent of its peak value at $\tau=0$. It is known, therefore, that for the narrow-band noise process that "waxing and waning" effect is a statistical error. To show that it is, in fact, the error calculated in Chapter 2, the following series of experimental results are offered.

a) The first experiment is an effort to show that the long-delay oscillatory behavior does not exhibit a marked in-phaseness, and that instead the "waxing and waning" of the envelope is random. Note in figures 3.24 and 3.25, that the six correlograms do not all show the same behavior at large values of delay. This emphasizes the point made in Chapter 2, that the finite time sample autocorrelation function is itself a random variable. In any case, note that the center correlogram of figure 3.24 is an excellent example of the "waxing and waning" effect while the last correlogram shows very little of this effect. The top correlogram of the same figure shows an example of the type of phase changes that are encountered. Note the distance between peaks near the twelfth peak is a little longer than the average period of the rest of the signal. The central correlogram of figure 3.25 also shows this kind of phase reversal.

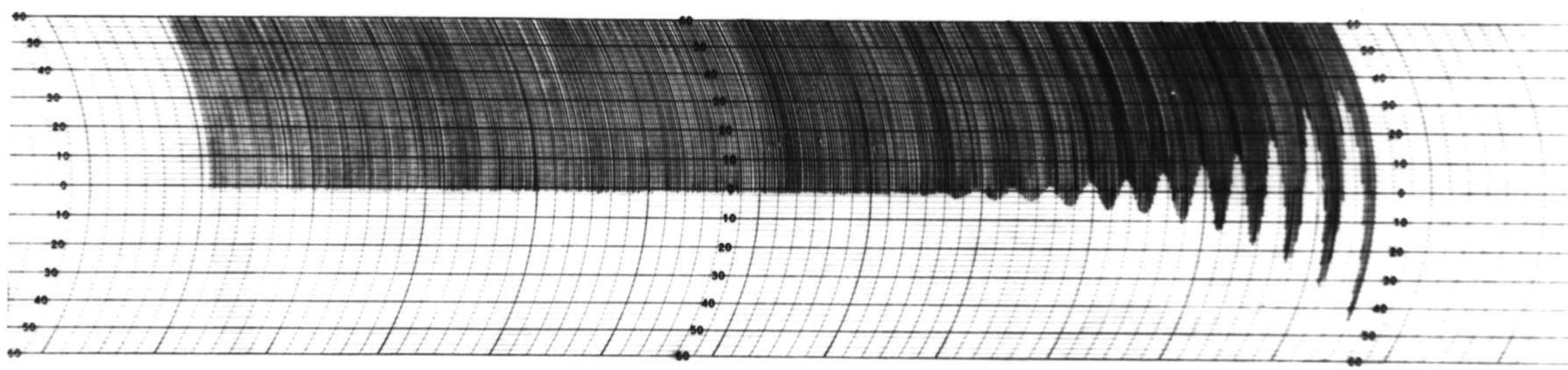
To show that the long delay behavior is not phase-locked to the short delay behavior of the correlograms, the mean of ten correlograms was computed and is shown in figure 3.26. This mean was computed by adding up the ten correlograms, point by point. That is to say, each point in figure 3.26 represents the summation of ten points for a fixed value of delay. The results show that the oscillatory behavior at long delay has tended to cancel. In other words, the long-delay oscillatory behavior is random-phased. Furthermore, the three-time constant level of decrement of the auto-correlogram is reached in about 30 to 40 milliseconds, as predicted theoretically.

b) The second experiment is an effort to show the same phenomenon from a slightly different point of view. It is to be recalled from Chapter 2 that successive samples of the finite sample autocorrelation function separated by some delay τ^1 are correlated in the same manner as are two samples of the signal separated by the same delay. To check that this is indeed the case, an experiment was done that autocorrelated narrow-band noise but used statistically independent samples of time series for each point in the correlogram. As shown in the schematic diagram (figure 3.27) the noise source

Figure 3.26

Point-by-point Sum of Ten Correlograms of Narrow-Band, Gaussian Noise

T	(sample length)	=	7.5 seconds
$\delta\tau$	(delay increment)	=	.25 milliseconds
τ_{max}	(maximum delay)	=	125 milliseconds



was kept running continually into the Analog Correlator. The Correlator's delay mechanism was indexed 0.25 milliseconds. Samples 7.5 seconds in length were taken at regular intervals with more than 10 seconds between samples and the signal was then autocorrelated. The 10 second interval insures statistical independence of the noise samples, since it is known from theoretical considerations of the autocorrelation function that at about 40 milliseconds the signals are linearly independent. In the Gaussian case linear independence implies statistical independence and therefore the noise samples can be assumed to be statistically independent.

The schematic of figure 3.27 shows a crystal oscillator used to trigger a multiple synch pulse generator. This device was used to pulse the Correlator such that it would start and stop the correlate cycle, index the delay mechanism of the correlator and restart the crystal oscillator timing mechanism. Thus a very accurate integrating time for the correlator was achieved.

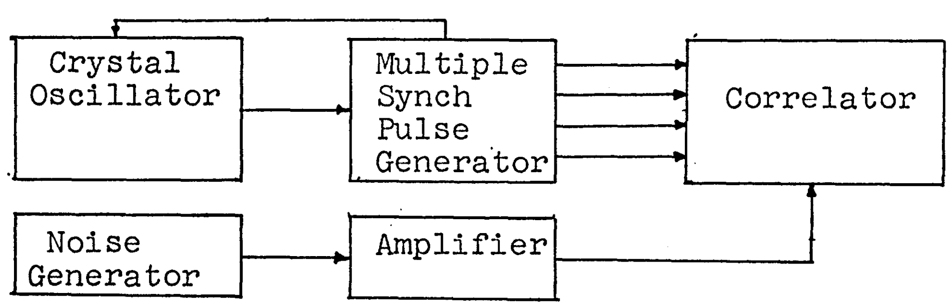


Figure 3.27
Schematic for Determining Autocorrelograms That Result from Independent Samples of Signal

Figure 3.28 is the result of the above procedure. From top to bottom the correlograms represent sample lengths of 7.5, 15 and 30 seconds. It is to be noted that in each case, the correlogram has traces of some oscillatory behavior for about 50 milliseconds and from there on appears to be just random noise. The "waxing and waning" phenomenon is gone. Therefore, it has been shown that if successive samples of time series used for the finite sample correlation are independent from point to point then the errors of estimation reduce to random noise. This noise should have variance that is equal to the one calculated in Chapter 2, namely $\sigma_{\phi}^{2}(T,\tau) = \frac{1}{2\alpha T}$. It is a little difficult to estimate the mean square value of the noise from figure 3.28, but it should be noted that there is a decided decrease of the amplitude of the noise as the sample length is increased in the three runs. It must be remembered that for comparison of the three runs, the noise amplitude must be normalized by dividing it by the value of the correlogram at zero delay.

Figure 3.29 shows two control runs to check both equipment and signal. The top correlogram shows the result of correlating the unfiltered wide-band noise. Note here, that the delay increment is 0.05 milliseconds and that the correlogram is essentially zero after approximately 0.3 milliseconds. This insures that the noise is white when compared to the filter characteristics. The remainder of the same correlogram shows the extent of correlator drift.

The second correlogram is a control run on a sinusoid to show frequency stability. No significant change in the period of the sinusoid can be seen as a function of delay.

The third correlogram of figure 3.29 shows a 7.5 second sample of noise, correlated by the independent sample method again. In this case the Q of the filter has been set to 6.4. It is seen that the correlogram now reduces to the noise level in about 30 milliseconds. This again checks with the theoretical computation of the autocorrelation function.

c) The last experiment that was done to demonstrate that the "waxing and waning" effect is an error due to the finite sample length of the time series was designed to show that the error decreases as the sample length increases. This is most easily shown by cross-correlating two statistically independent samples of narrow-band noise. The Correlator actually computes the function: $\phi_x(T,\tau) = K\int_0^T x_1(t)x_2(t-\tau)\,dt$. In the case of the autocorrelation function ($x_1 = x_2$), normalization is achieved by noting the peak value of this function relative to

Figure 3.28

Autocorrelograms Computed from Successively Independent Samples of Narrow-Band, Gaussian Noise as a Function of Sample Length

Top

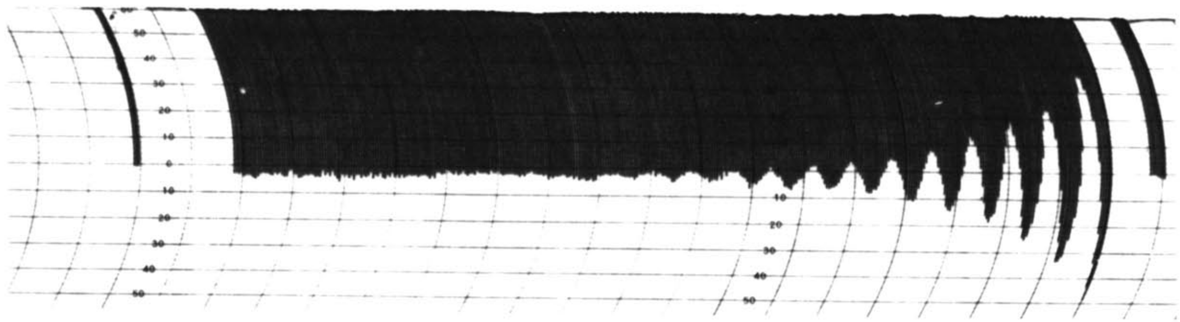
T (sample length) = 7.5 seconds
$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 93 milliseconds

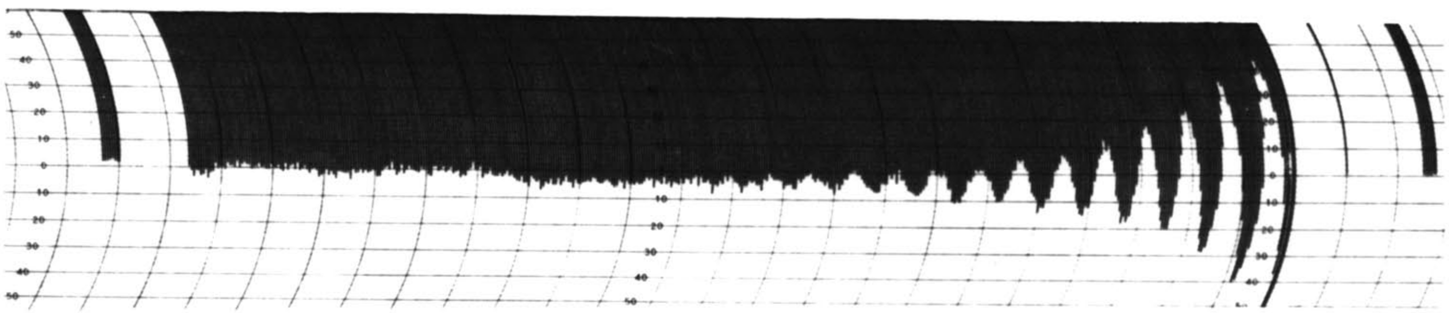
Center

T (sample length) = 15 seconds
$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 108 milliseconds

Bottom

T (sample length) = 30 seconds
$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 86 milliseconds





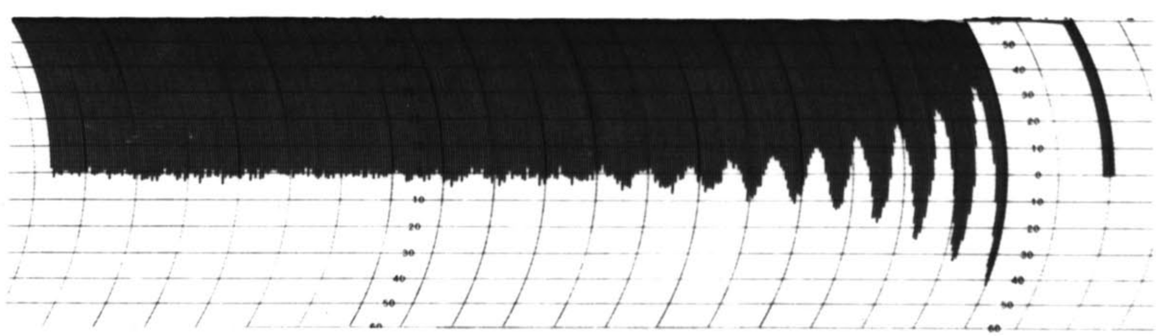


Figure 3.29

Autocorrelograms Computed from Successively Independent Samples of Control Signals

Top

Wide band noise (bandwidth - 20kc)

T (sample length) = 7.5 seconds

$\delta\tau$ (delay increment) = 0.05 milliseconds

τ_{max} (maximum delay) = 11 milliseconds

Center

Sinusoid (250 cps)

T (sample length) = 7.5 seconds

$\delta\tau$ (delay increment) = 0.25 milliseconds

τ_{max} (maximum delay) = 63 milliseconds

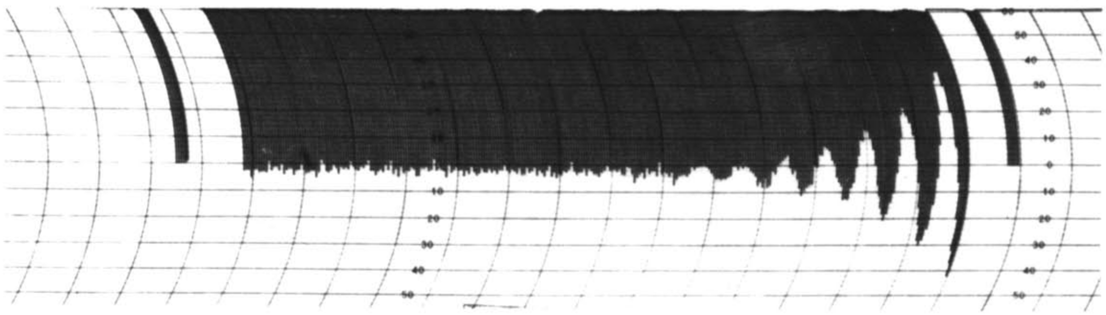
Bottom

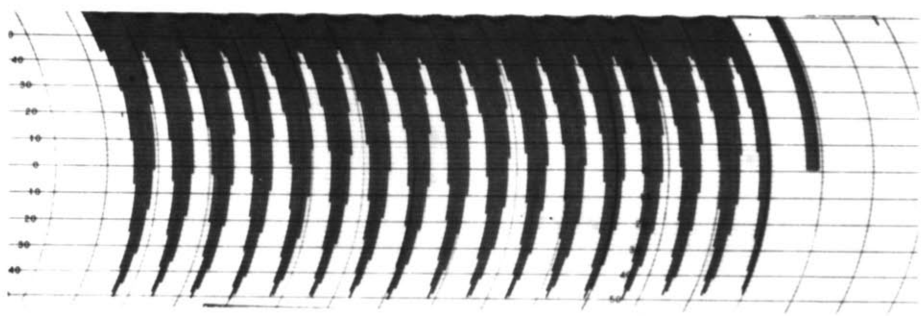
Narrow-band, Gaussian noise

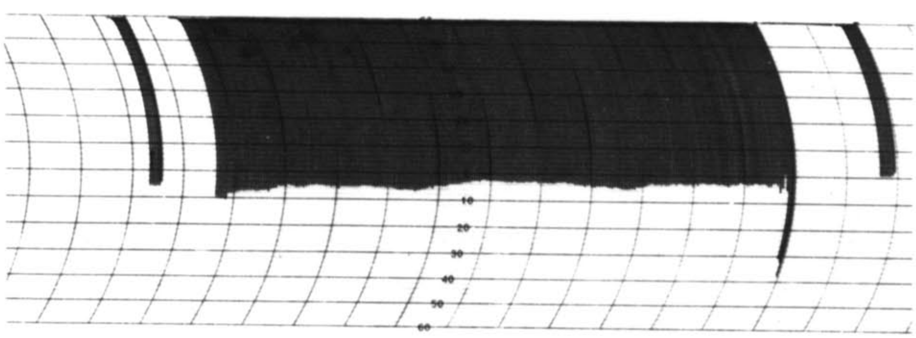
T (sample length) = 7.5 seconds

$\delta\tau$ (delay increment) = 0.25 milliseconds

τ_{max} (maximum delay) = 71 milliseconds







its value at long delays. For the case of cross-correlations, however, the constant $1/T$ must be included in calculating $\phi_x(T,\tau)$ so that comparisons of runs at different sample lengths can be made. This was done in the series of correlograms shown in figures 3.210 and 3.211 by halving the gain of the correlator each time the sample length was doubled. The figures represent sample lengths of 4 to 32 seconds starting at the top of figure 3.210 and ending at the bottom of figure 3.211. It is clear from these correlograms that the amplitude of the error decreases as the sample length (T) is increased. To get some quantitative measure of this decrease, the root-mean square values of the peaks of the correlograms were computed. The results, along with the theoretical values, are shown in Table 3.21. The agreement of experimental results to theoretically expected values is seen to be good, considering the difficulty of estimating $\sigma_\phi(T,\tau)$.

Table 3.21

Normalized Root-Mean-Square Height of Peaks of Crosscorrelograms of Narrow-Band, Gaussian Noise as a Function of Sample Length

Sample length (seconds)	Estimate of Normalized Root-Mean-Square Value	Normalized, Theoretical Root-Mean-Square Value
4	5.1	2.8
8	2.1	2.0
16	1.6	1.4
32	1.0	1.0

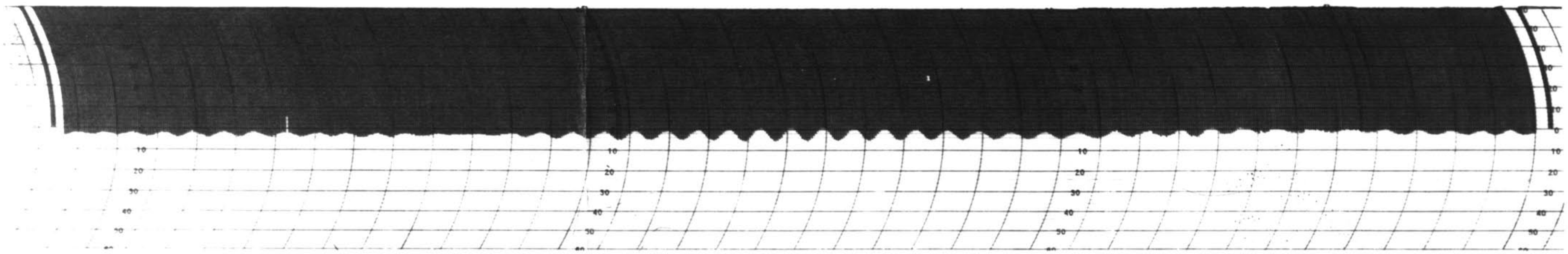
Figure 3.210

Crosscorrelograms of Independent Samples of Narrow-band, Gaussian noise as a Function of Sample Length

$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 185 milliseconds

Top
T (sample length) = 4 seconds

Bottom
T (sample length) = 8 seconds



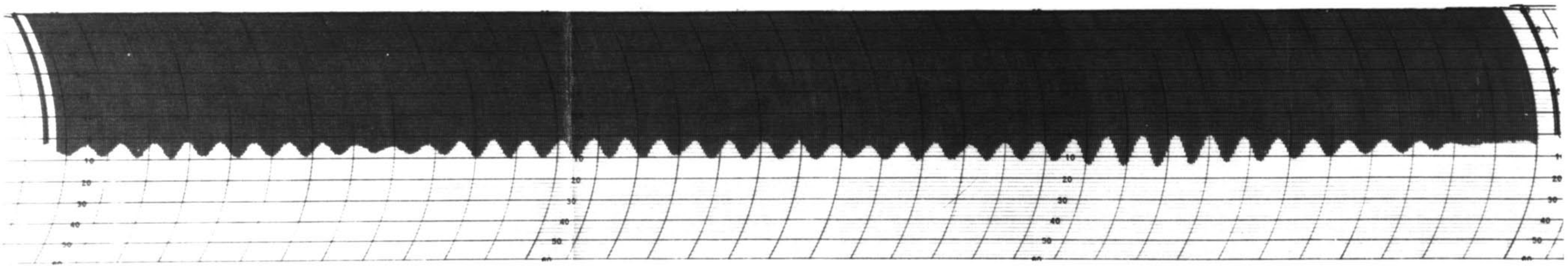


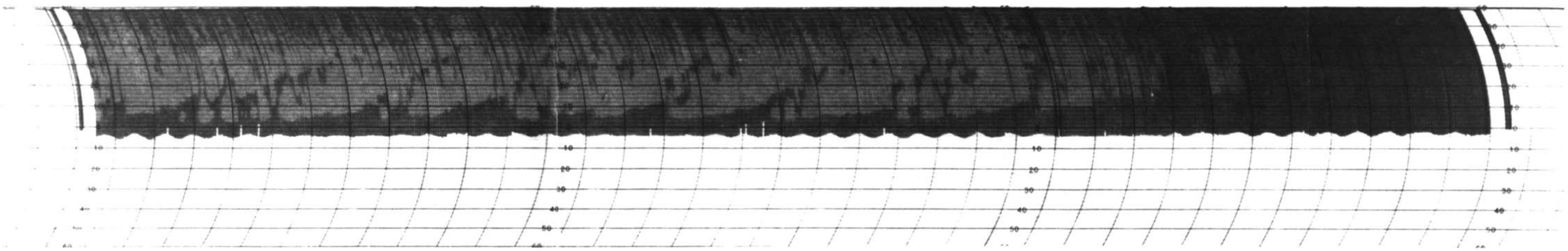
Figure 3.211

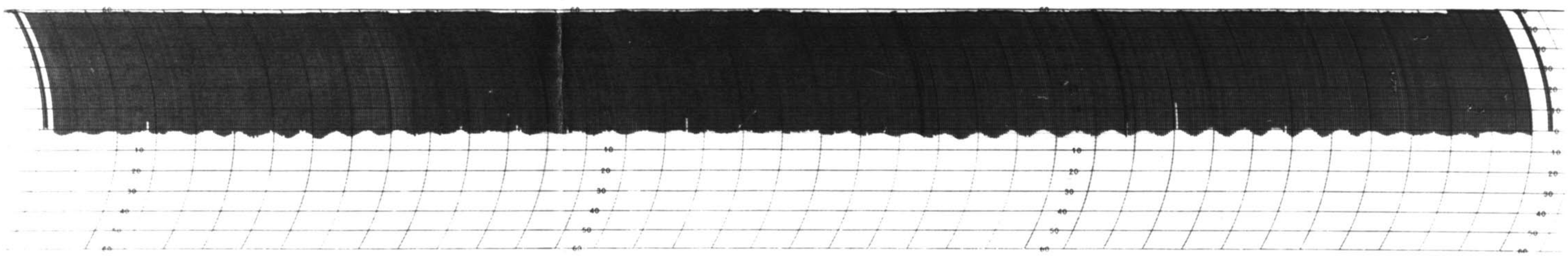
Crosscorrelograms of Independent Samples of Narrow-band, Gaussian Noise as Function of Sample Length

$\delta\tau$ (delay increment) = 0.25 milliseconds
τ_{max} (maximum delay) = 185 milliseconds

Top
T (sample length) = 16 seconds

Bottom
T (sample length) = 32 seconds





3.3 Summary of Experimental Work on the Finite Time Sample Autocorrelation Function of Narrow-Band, Gaussian Noise

In this chapter, the theoretical results of Chapter 2 have been verified. It has been shown that the interpretation of the finite time sample correlogram of narrow-band, Gaussian noise must be approached with some caution. In particular, the "waxing and waning" of the envelope of the correlogram for large values of delay has been shown to be a statistical estimation error due to the finite sample process and not an indication of marked in-phaseness in the time series. This confirms the suspicions of Frishkopf (39) with respect to the narrow-band process.

In addition to the mathematical interpretation of these results, a physical interpretation of this error is offered here. The narrow-band noise time series can be vizualized as pieces of essentially oscillatory time series (to be referred to as bursts from now on) separated by other non-oscillatory random time series. Thsee bursts, as shown in figure 3.23, occur at random and are not constant in amplitude, but their maxima occur at an average number of times that is proportional to the bandwidth of the filter. It is proposed, therefore, that in the finite sample correlation of this signal, the correlogram gets contributions from only bursts that are shifted with respect to themselves for small values of delay (small with respect to the time constant of the exponential decay of the correlogram). As the delay is increased, the bursts interact with each other in a random way. For a short sample of time series, this random interaction of neighboring bursts may contribute considerably to the correlogram and the result is the waxing and waning phenomenon. As the sample length is increased, however, the random interactions of neighboring bursts contribute proportionately less to the correlogram when compared to the interaction of the bursts with themselves (an interaction which is not random). Thus as the sample length is increased, the waxing and waning decreases. Furthermore, it has been shown that if the samples of time series are chosen independently for each point in the correlogram, then the smooth appearance of the waxing and waning phenomenon also disappears, since the systematic interactions of neighboring bursts is eliminated.

In conclusion, it has also been shown that useful information can be extracted from the finite time sample correlograms of narrow-band processes. The important parameters of bandwidth and central frequency can be estimated reasonably accurately by any of the procedures discussed in section 3.2. A further word of caution must be offered, however, with regard to the estimation of these parameters. In particular, frequency is determined experimentally by measuring some reasonably stable time interval such as the time between peaks or the time between the zero crossings of a waveform. It must be recalled from Chapter 2 that this time interval is itself a random variable. Thus all that can be determined is an estimated average period and it is hoped that this estimate converges rapidly to the period of the process as the sample length is increased. That it does indeed converge is assured since it has been shown that $\phi_x(T,\tau)$ converges to $\phi_x(\tau)$ as T approaches infinity.

A few words must still be said about sample lengths and delay increments at this point. In general, the correlogram is computed in incremental steps of delay. That is to say, the correlation function is evaluated at discrete intervals of delay and what results is really the finite time sample autocorrelation function sampled at the sampling frequency that corresponds to the inverse of the delay increment. If a certain frequency w radians/second is to be recovered from the data then the sampling frequency of the correlogram must obey Nyquist's Sampling Theorem and be at least equal to 2w radians/second.

This procedure insures that a given high frequency will be detectable by the correlation method. For the detection of the presence of some low frequency component, the maximum value of delay must be made large enough to show as many cycles of the low frequency as are desirable. If the maximum value of the delay is increased then the sample length must be increased proportionately to keep the signal to noise ratio the same. As has been shown in Chapter 2 and 3, the longer T, the better. No general rule about the length of sample required can really be made as this depends on the statistics of the time series in question. For a particular time series, however, these parameters can be calculated.

CHAPTER 4 INTERPRETATION OF THE AUTOCORRELOGRAM OF OF THE ELECTROENCEPHALOGRAM

4.1 A Statistical Model of the EEG

The problem of estimating the autocorrelation function of a process from a finite sample of time series has been discussed with respect to the model established in Chapter 2. This is a classical model in the study of stochastic processes. The observed data is assumed to be a finite length of a sample function that is visualized as coming from some ensemble of sample functions. This ensemble is a mathematical reality and can be defined quite precisely. However, the definition of the ensemble from which the observed sample function is drawn is left at the discretion of the analyst. That is to say, the process of finding an ensemble to which a given sample function will fit is by no means a unique one. The criteria for picking a particular ensemble or model is dictated only by what useful predictive or descriptive function the model will have with respect to the phenomenon under investigation. Once the ensemble is chosen, with the above criteria as a guide, its definition can be made precise.

Before the specific problem of the estimation of the autocorrelation function of the EEG is approached, such concepts as stationarity, ergodicity and normality must be discussed to determine how far (if at all) the results of the estimation of the autocorrelation function of narrow-band, Gaussian noise can be applied to the EEG.

4.2 The Concept of Stationarity as Applied to EEG

Stationarity is given a very precise mathematical definition with respect to a random process x_t. A stochastic process is defined as being strictly stationary if its n^{th} order distribution is independent of the time origin; the multivariate distribution of $x_{t_1+\tau}, x_{t_2+\tau}, \ldots x_{t_n+\tau}$ is independent of τ. A wide sense stationary process is one for which $E\left[|x_t|^2\right] < \infty$ and $E[x_t x_{t+\tau}]$ is independent of t. For the particular, but very important, case of the Gaussian random process, wide sense stationarity implies strict sense stationarity.

Precise as this definition is, it must be approached with caution if it is to be made use of to study some physical process. The dilemma occurs since one is usually confronted with data that consists of some function defined at a discrete and finite number of points. But the question of stationarity makes sense only in the context of an ensemble. It has already been shown, however, that the ensemble or model is dictated by extra-mathematical considerations. The concept of stationarity, therefore, makes sense only after the model of a physical process has been chosen. To make this point a little clearer, consider the following example:

Given a sample function that consists of a series of randomly presented rectangular pulses. In an interval, L, the function can be either zero or unity with equal probability. In the next adjoining interval, the function is zero. This pattern is repeated so that a particular sample function might look like the one depicted in figure 4.21.

One might now ask the question, "Is this process stationary?" But this question does not make sense unless something more is either known or assumed about the process. If the ensemble from which this sample function comes is visualized as the one in figure 4.22, then the answer is no. This can be seen by looking at the distribution of amplitudes of x_t at t_1 and at $t_1+\delta t$. At t_1 the distribution is such that the value of zero or unity are equally likely, while at $t_1+\delta t$, right after the discontinuity the probability of having zero amplitude is unity. Thus the distribution of x_t depends very much on the time origin and the process is defined as non-stationary.

Now suppose that the same sample function comes from an ensemble of random-phased functions as depicted in figure 4.23. Now the probability distribution is the same at each point in time. The probability of unity is 1/4 and probability of zero amplitude is 3/4. The process is, therefore, judged to be stationary.

This example further supports the contention that the question of stationarity is an improper one when applied to only a single member of the ensemble of functions defining a random process. In the case of EEG there are still further problems in a statistical characterization. In the first place, the EEG is a dynamic function that may reflect something as changeable as physiological state. It would be absurd as this stage of the science to talk about an all inclusive statistical model of the EEG; one that would adequately describe the EEG in all of its many ramifications. As a logical start, some limitations can be made on a permissible class of EEG functions for which a statistical characterization will be attempted. The constraint of studying the EEG when the subject is in a "relaxed" state has already been discussed in Chapter 1. Assume, therefore, that the characterization is attempted on the EEG of a subject who is in a relaxed state; that state in which large percentage of the subjects exhibit rhythmic bursts of alpha activity.

In Chapter 1, it was pointed out that in successive three minute intervals, the amount of alpha activity tended to decrease, for four consecutive such intervals. At least this trend was shown to be statistically significant. Two interpretations are possible now. Either the EEG of a "relaxed subject" is a function whose alpha content decreases on the average or it must be assumed that some constraint of the experiment upon the subject has affected the results. The first assumption leads to

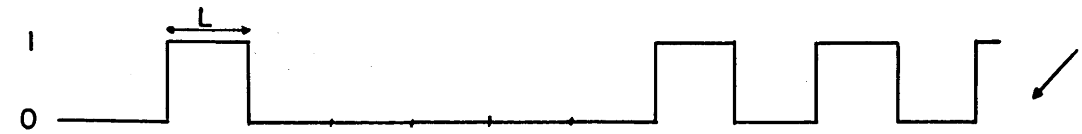


FIGURE 4.21 SAMPLE FUNCTION

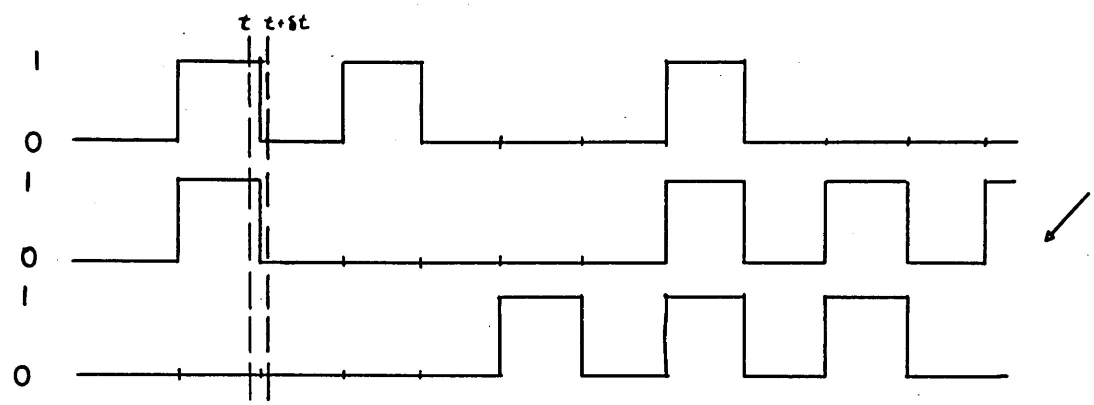


FIGURE 4.22 FIXED-PHASE ENSEMBLE

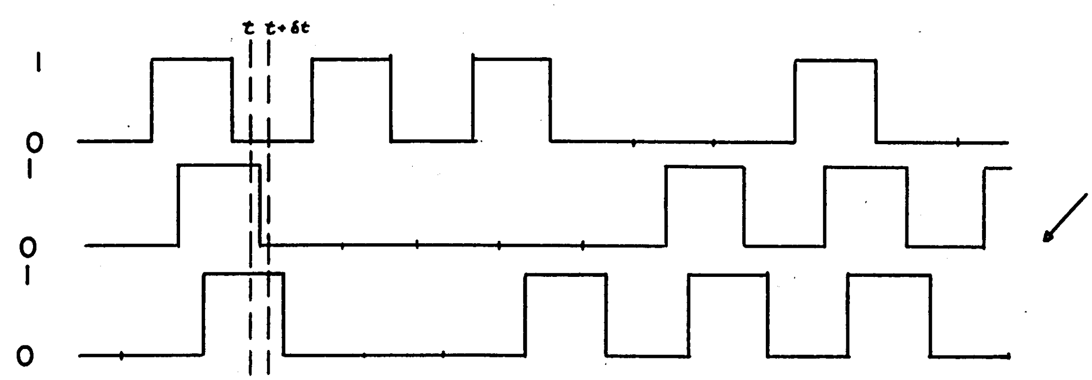


FIGURE 4.23 RANDOM-PHASE ENSEMBLE

obvious absurdities and it is reasonable, therefore, to assume that the latter is the case. This interpretation does not help in deciding how the experimental results can be used to estimate important statistics of the EEG. Before this can be done, more will have to be known about the important physiological variables that affect the EEG. The question of stability of criteria, discussed in Chapter 1, is intimately tied to this idea.

Unless the important physiological variables can be controlled, an ergodic model will not be applicable to the study of the EEG. Unfortunately, non-ergodic models are difficult to work with and there is the hope that EEG can be adequately described by an ergodic model if the effect of time can be abated in the course of the experiments. This is not to imply that the effects of particular experiments (such as the change in the amount of activity with time) are not useful in terms of knowledge gained of some specific physiological question. These particular results have interest in their own right, however, it would be valuable to obtain a more general characterization of the EEG; one that would explain the results of many experiments.

Until experiments can be done with the above goal in mind, the only way that the data gotten in connection with this thesis can be handled is to assume the results gotten are essentially time averaged results. Thus any statistic of the EEG process computed here is considered as an estimate of the time average statistic defined over the interval for which it was taken. This point will hopefully be clarified with the discussion of particular concepts such as histograms and correlograms.

4.3 Estimation of the Distribution of Amplitudes of the EEG in the Resting State

The estimation of the distribution of amplitudes of the EEG is again wedded to the ensemble idea. Only in the context of an ensemble does a probability distribution make complete sense. In the case of many observed signals the assumptions of an ergodic model are justified on the basis of a knowledge of the underlying microscopic structure of the process (the shot noise model for instance) and the task of finding a distribution to describe the data is simplified. If the ergodic assumption is made a histogram of amplitudes can be taken on the finite piece of data. The Law of Large Numbers can be called upon to insure that if a sufficient number of statistically independent samples of a well-behaved function are taken then the cummulative histogram of amplitudes will approach the theoretical distribution of the amplitudes of the sample function. Ergodicity further insures that with probability one this distribution of amplitudes of the sample function will equal the distribution of amplitudes of the random variable. With this theoretical background the thing that is actually measured, the histogram of amplitudes of a finite piece of a sample function, makes mathematical sense.

With EEG the problem is more complex. Very little is known about the underlying microscopic structure that gives rise to this signal. No model that has yet been proposed for the microscopic structure has been both physiologically sound and mathematically reasonable. The only thing one has to work with, therefore, is the gross phenomenon itself. Furthermore, a model for this phenomenon is complicated by the effects of the particular experimental conditions. In particular, an ergodic model can not be strictly justified for the data taken in connection with this thesis, due to the change of some statistics of the EEG with time. Thus, the meaning of an estimation of the probability distribution gotten by sampling the EEG amplitude, loses some of its validity. The assumptions of time average statistics is forced upon the investigation once again. Results presented in this section are then to be interpreted in this light and are to be accepted as preliminary results in the investigation of some statistical characteristics of the EEG.

Amplitude histograms (estimates of the time average probability density function) are shown in figures 4.31 and 4.32. A short sample of the EEG of the subject is shown below each histogram. The subjects range from low to high alpha subjects.

Figure 4.31

Amplitude Histograms of EEG

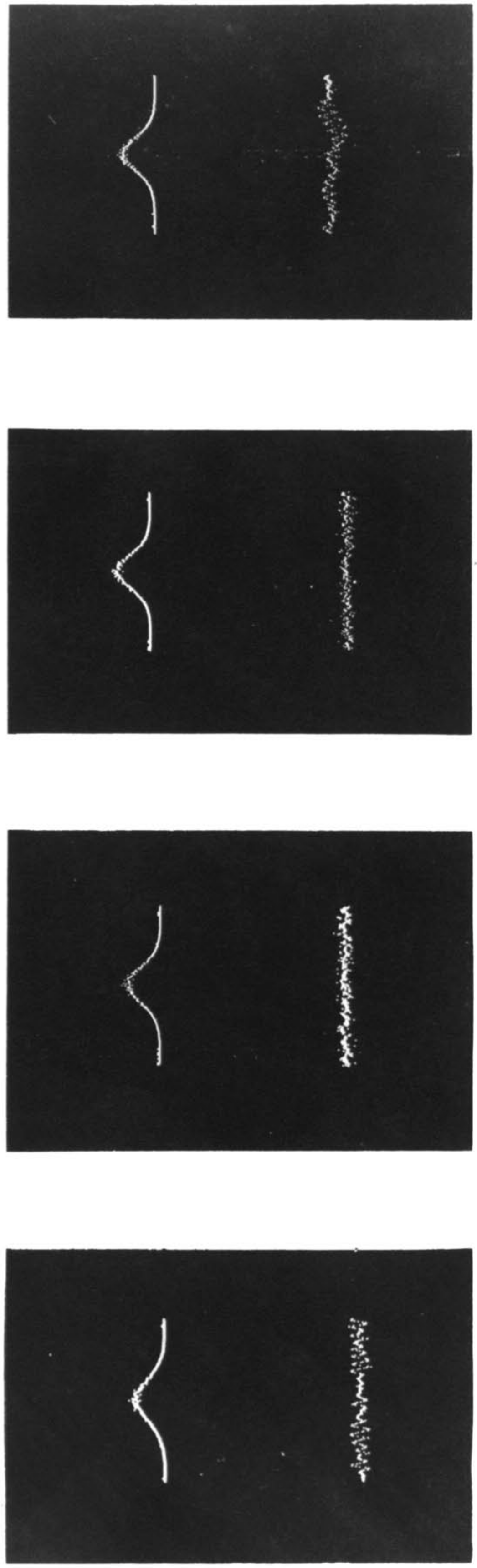
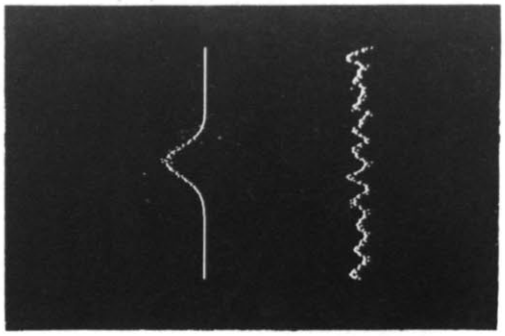
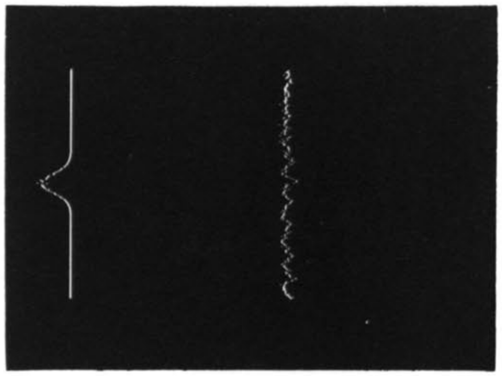
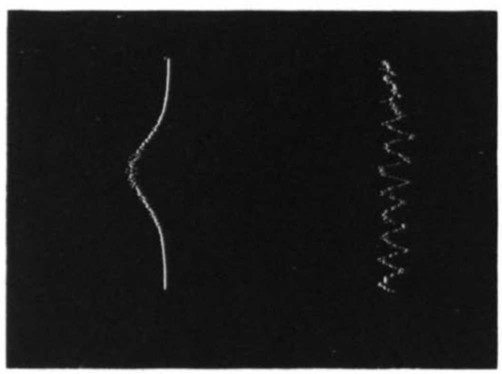


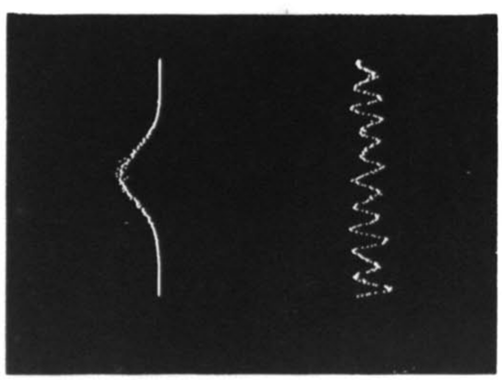
Figure 4.32

Amplitude Histograms of EEG







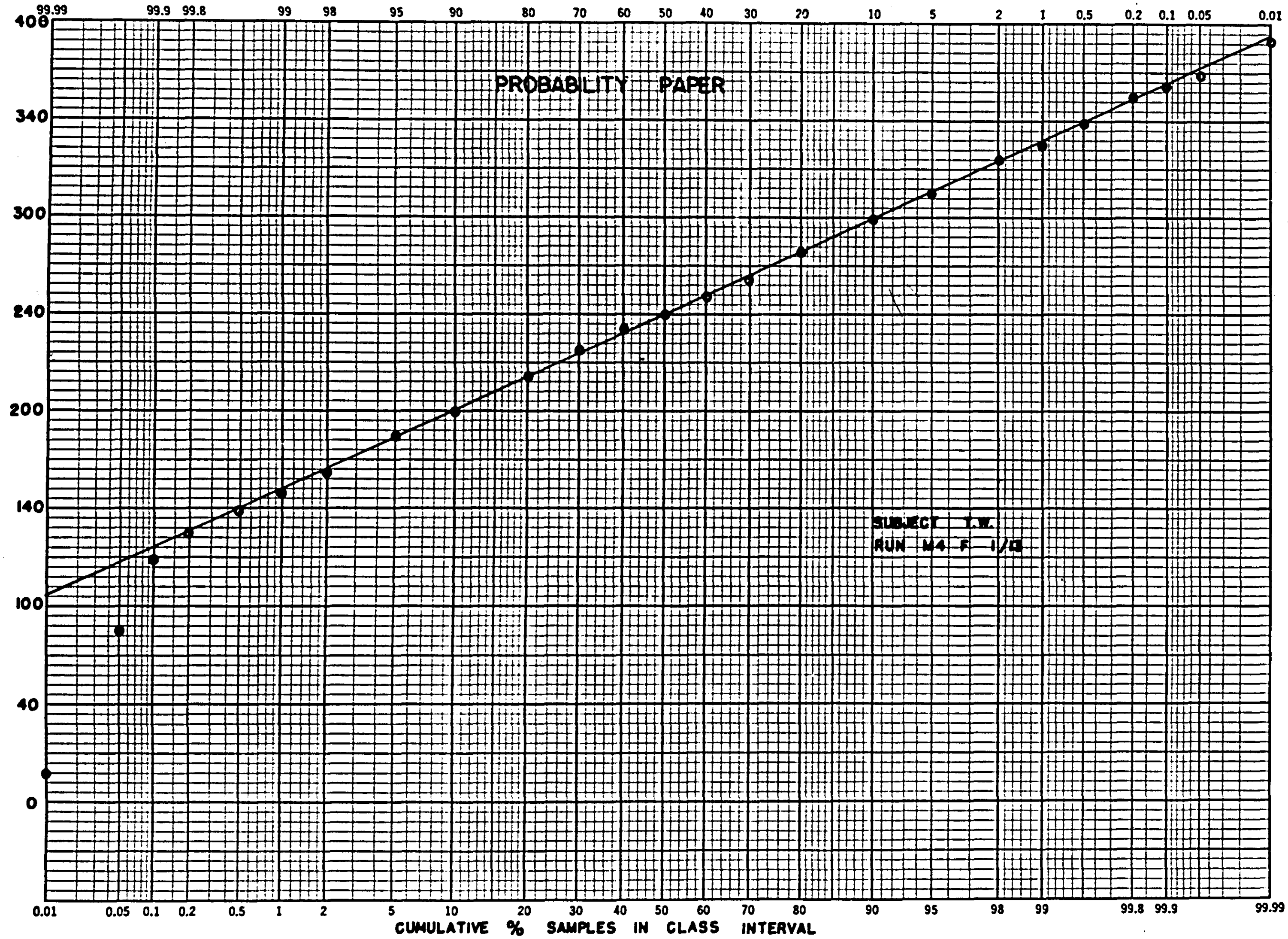


Preliminary tests on the nature of these histograms are presented graphically in figures 4.33 to 4.36. In these curves, the cummulative histograms are plotted on probability paper with a straight line approximation to the points. This type of graph paper plots Gaussian probability distributions as straight lines. The experimental points are seen to fit the straight line quite well for the center of the distribution, but the tails do not appear to fall off as fast as the Gaussian distribution. This is just what would be predicted for data that contains some muscle potentials and movement artifacts. These artifacts (as shown in figure 1.2) introduce large voltage changes and thus contribute more samples at large negative and large positive values. Note that very few samples are involved at the points of deviation.

It is certainly not presumed here that the probability paper plots represent a very sensitive test of the Gaussian hypothesis. On the basis of these results, however, the Gaussian hypothesis can not be discarded.

Figure 4.33

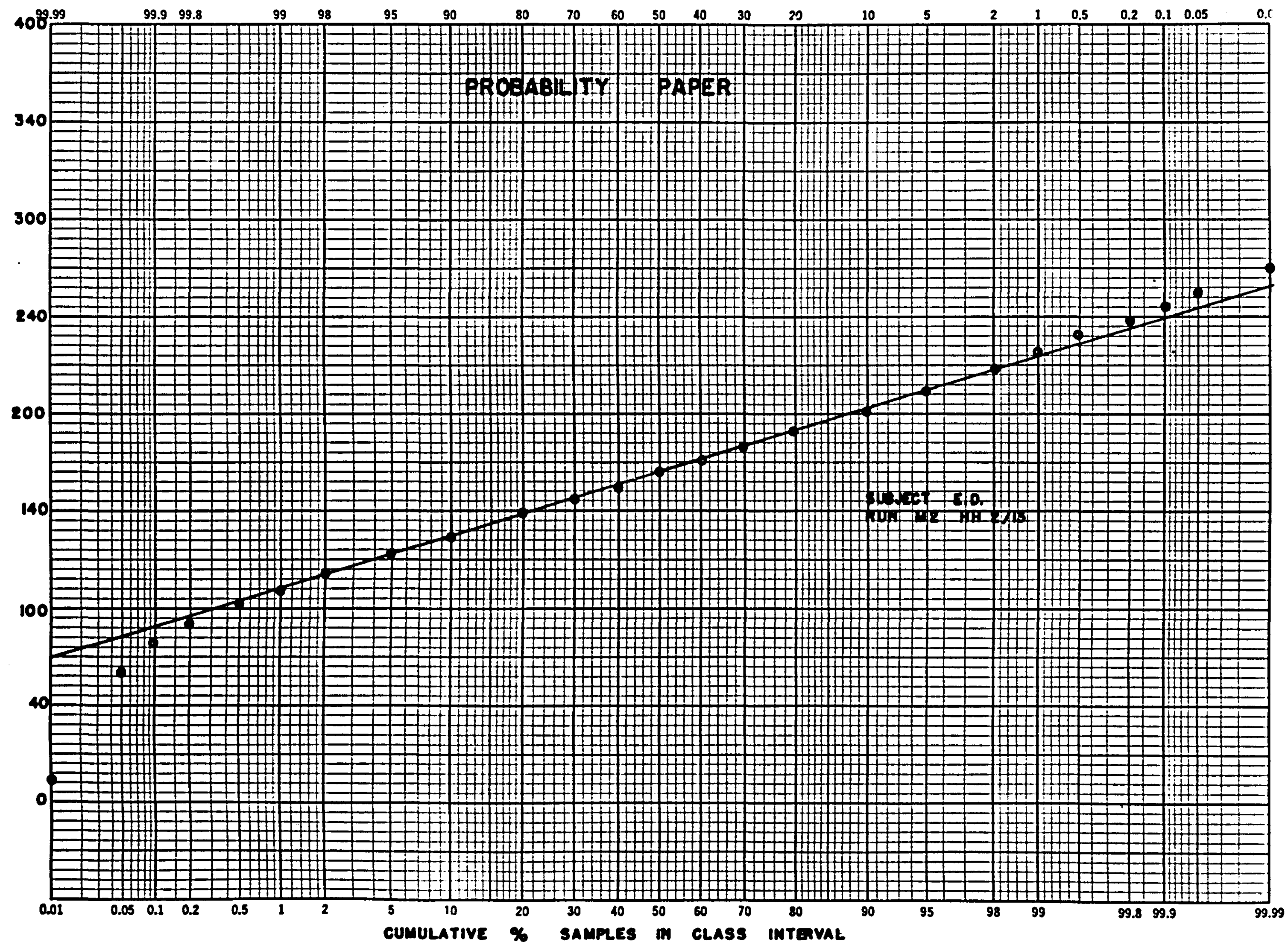
Cummulative Histograms of EEG
Plotted on Probability Paper



PROBABILITY PAPER
SUBJECT T.W.
RUN M4 F 1/12
CUMULATIVE % SAMPLES IN CLASS INTERVAL

Figure 4.34

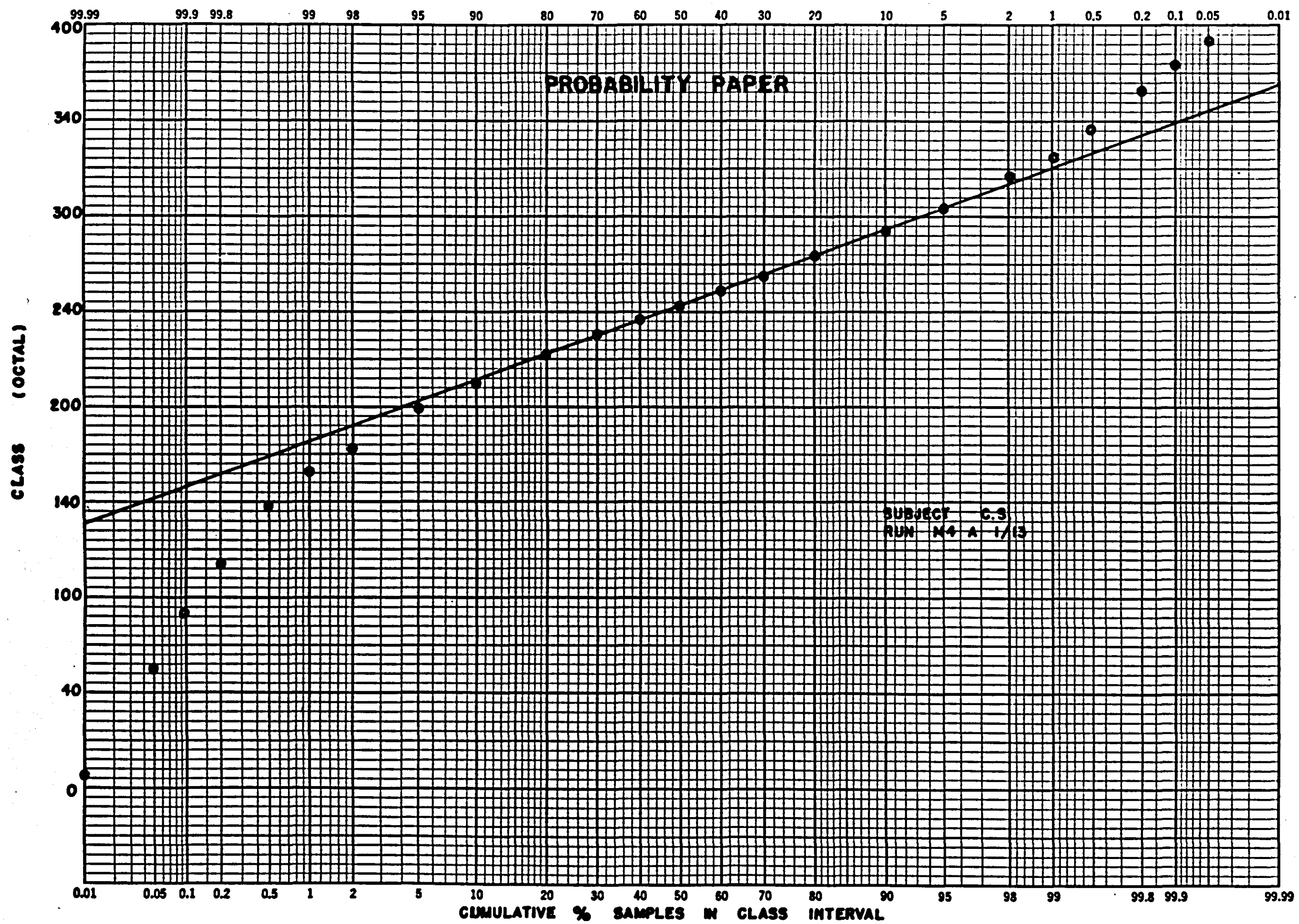
Cummulative Histograms of EEG
Plotted on Probability Paper



PROBABILITY PAPER
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.0
400
340
300
240
200
140
100
40
0
SUBJECT E.O.
RUN M2 HH 2/15
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.8 99.9 99.99
CUMULATIVE % SAMPLES IN CLASS INTERVAL

Figure 4.35

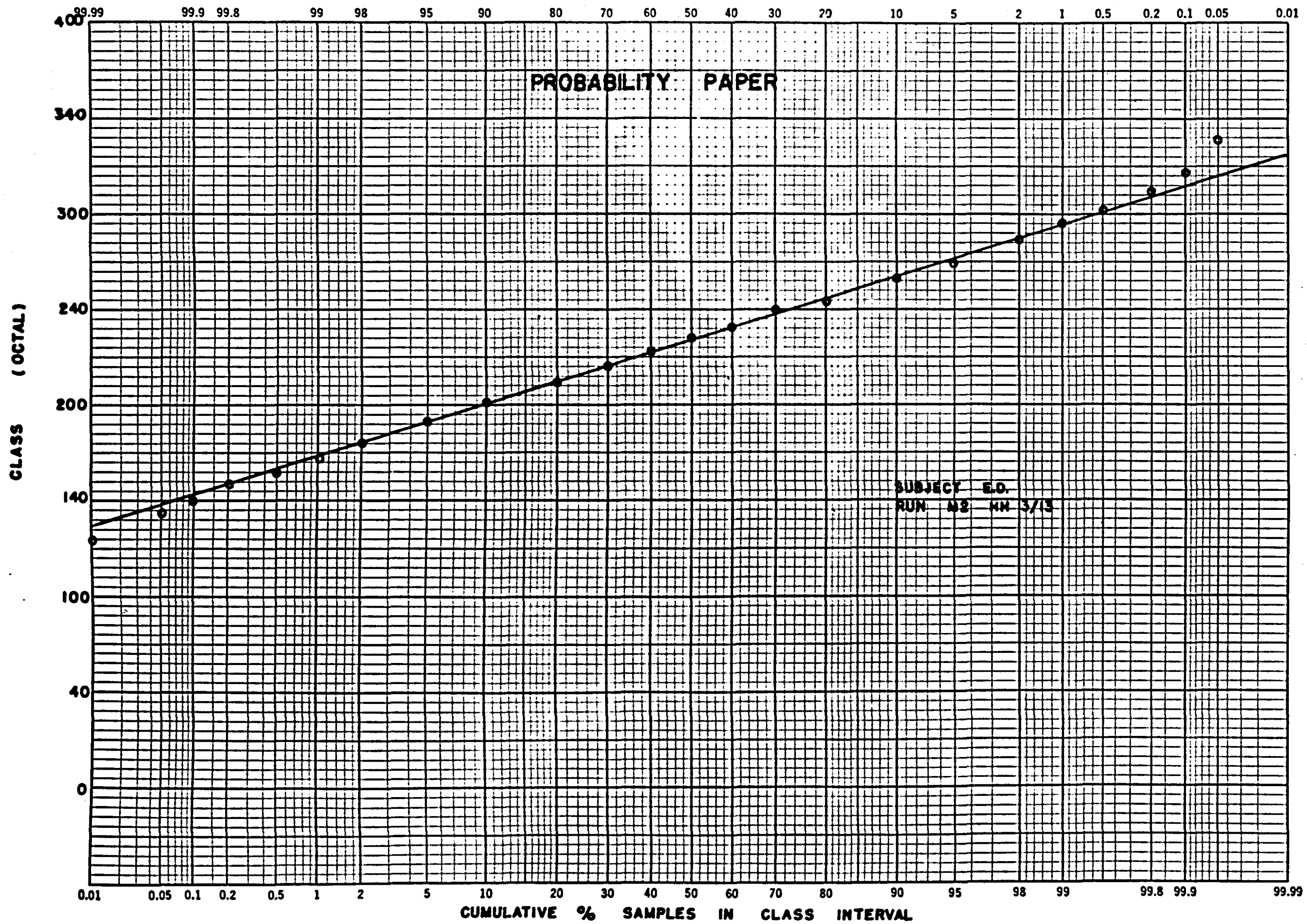
Cummulative Histograms of EEG
Plotted on Probability Paper



PROBABILITY PAPER
SUBJECT C.S
RUN N4 A 1/13
CLASS (OCTAL)
CUMULATIVE % SAMPLES IN CLASS INTERVAL
400
340
300
240
200
140
100
40
0
0.01
0.05
0.1
0.2
0.5
1
2
5
10
20
30
40
50
60
70
80
90
95
98
99
99.8
99.9
99.99

Figure 4.36

Cummulative Histograms of EEG
Plotted on Probability Paper


PROBABILITY PAPER
SUBJECT E.O.
RUN M2 HH 3/13
CLASS (OCTAL)
400
340
300
240
200
140
100
40
0
99.99 99.9 99.8 99 98 95 90 80 70 60 50 40 30 20 10 5 2 1 0.5 0.2 0.1 0.05 0.01
0.01 0.05 0.1 0.2 0.5 1 2 5 10 20 30 40 50 60 70 80 90 95 98 99 99.8 99.9 99.99
CUMULATIVE % SAMPLES IN CLASS INTERVAL

4.4 The Estimation of the Autocorrelation Function of the EEG

The implications of all the work done to this point are now directed at the problem of interest; the autocorrelation function of the EEG. It has already been shown that the long delay cyclic activity of the autocorrelogram of narrow band noise is a statistical error of estimation. Since this fact is now established then it follows that it can be offered as an alternate hypothesis to the explanation of the long delay cyclic activity as resulting from a very narrow band spectral component in the EEG. This is the weakest case that can be made here. It is really the only one that can be completely justified on mathematical grounds. This alternate hypothesis places the burden of proof on those that choose to interpret the long delay cyclic activity of the autocorrelogram of EEG as evidence for the existance of a physiological clock.

If a stronger statement of the results of this thesis is desired then the statistics of the EEG must be examined more carefully. The problems of an ergodic model have already been cited and the way out of the delimma mentioned; consider the results of the autocorrelation process as essentially time-averaged. Figures 4.41 to 4.42 show samples of the estimated time average autocorrelograms of EEG for several subjects. Note

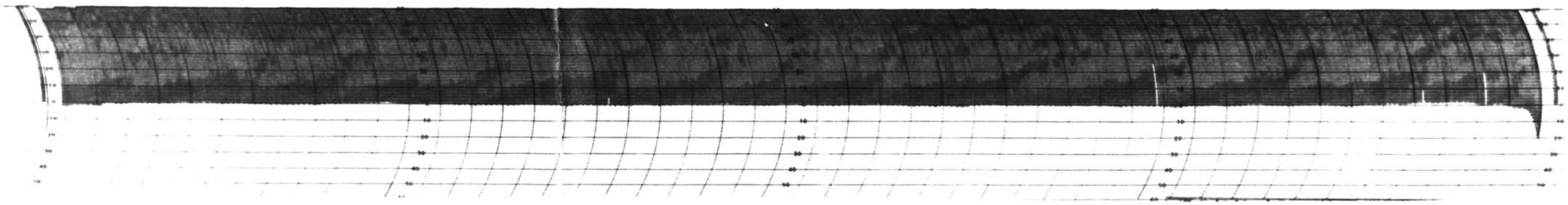
Figure 4.41

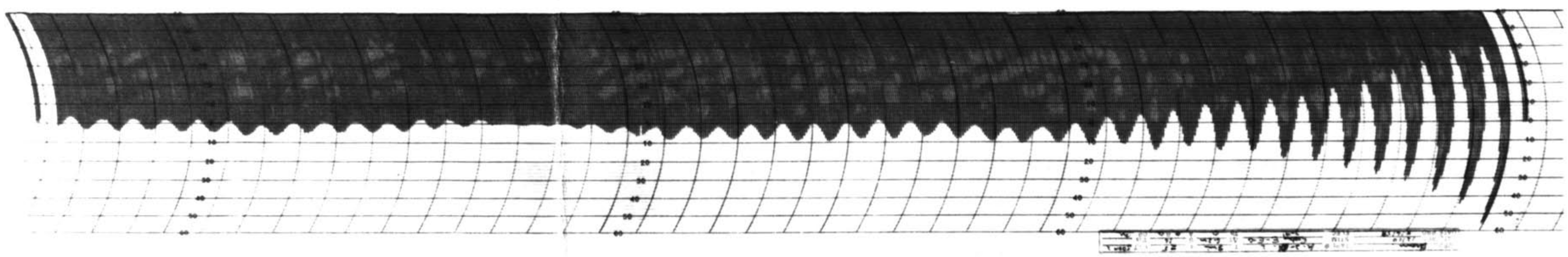
Autocorrelograms of EEG

T (sample length) = 3 minutes

$\delta\tau$ (delay increment) = 6.25 milliseconds

τ_{max} (maximum delay) = 4.6 seconds





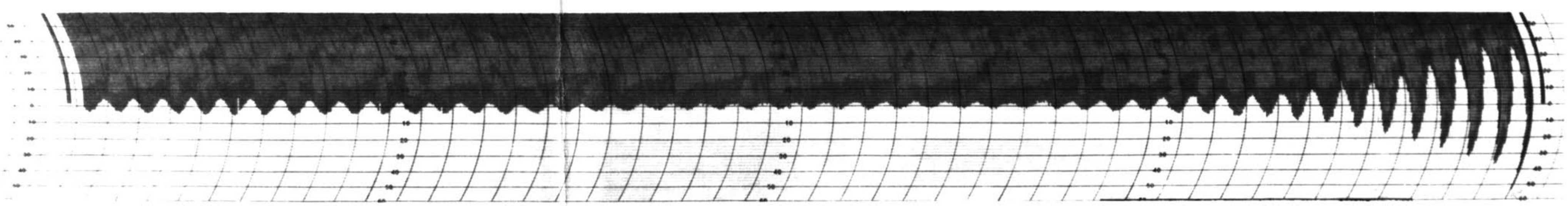


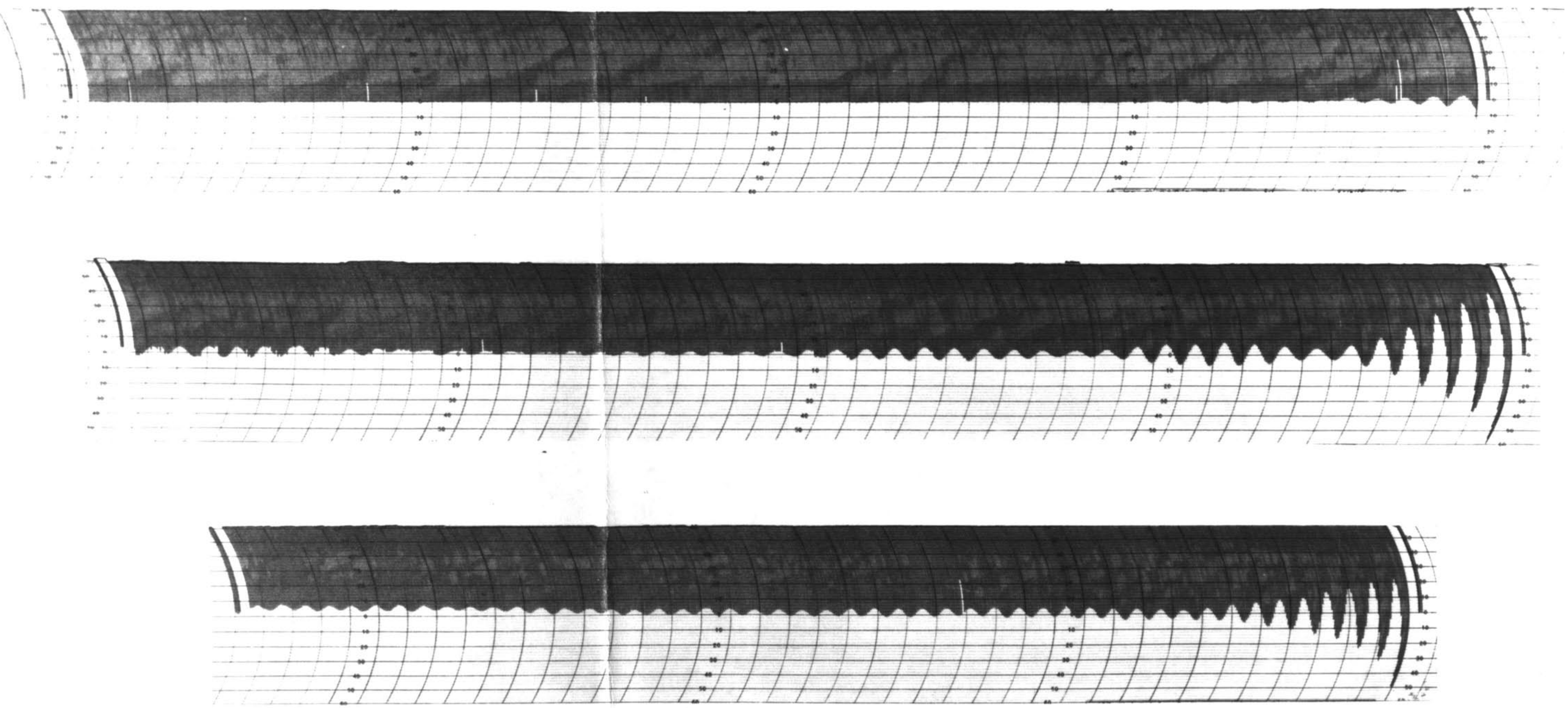
Figure 4.42

Autocorrelograms of EEG

T (sample length) = 3 minutes

$\delta\tau$ (delay increment) = 6.25 milliseconds

τ_{max} (maximum delay) = 4.6 seconds



also the resemblance of these correlograms to the correlograms of narrow-band noise (figures 3.24 and 3.25).

Now consider how well the EEG data fits the narrow-band model. The first problem is the time-dependence problem. It has been stated that this time dependence is a statistically significant trend. That is, some records did not exhibit this time dependence. If the percent time that there is alpha activity in the record is accepted as an indication of time dependence, then records can be found that exhibit little such time dependence. Table 4.41 gives a summary of the change in alpha content with time for one such record of EEG.

Time Interval (minutes)	o/o Alpha Activity	o/o Change
0 to 3	88.5	--
3 to 6	86.6	-1.9
6 to 9	88.3	-0.2
9 to 12	85.5	-3.0

Table 4.41 Change in Alpha Activity as a Function of Time

Subject: TW
Run: M-3 aa

This is the only record that could be found that contains very little variation of alpha activity with time. However, it can now be tested to see if the correlogram exhibits long delay cyclic activity that decreases with sample length. In other words, for this particular sample of data a stationary model would be a reasonable assumption. Figure 4.43 shows the correlograms of this data as a function of sample length. The first correlogram represents 1 minute and 40 seconds of data. The next one is twice that length and the last correlogram is four times that length. It is clear that the long delay cyclic activity decreases with sample length. Therefore, it must be an error of estimation. To get some measure of how fast it decreases, the root-mean square value of the peaks of the cyclic activity is computed and shown in table 4.42. The computation was done on data after 1.1 seconds of delay to insure that the first decay of the correlogram is not included in the computation.

Sample Length (seconds)	Average Amplitude of Peaks (arbitrary units)	(normalized)
100	2.2	2.0
200	1.4	1.4
400	1	1.0

Table 4.42
Normalized Root Mean Square Height of Peaks of Autocorrelogram of EEG as a Function of Sample Length
Subject: TW
Run:

It must be recalled that the function computed in Table 4.42 is an estimate of the root-mean-square value of the error and that this may account for some of the discrepancy. It is seen that the cyclic activity decreases considerably with sample length

Figure 4.43

Autocorrelograms of EEG as a Function of

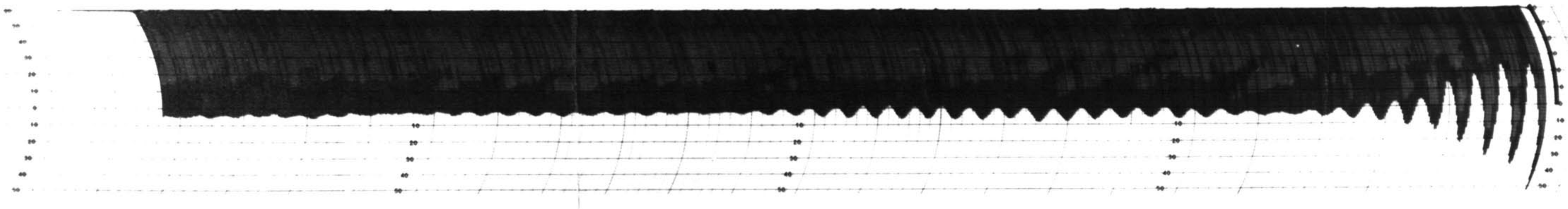
Sample Length

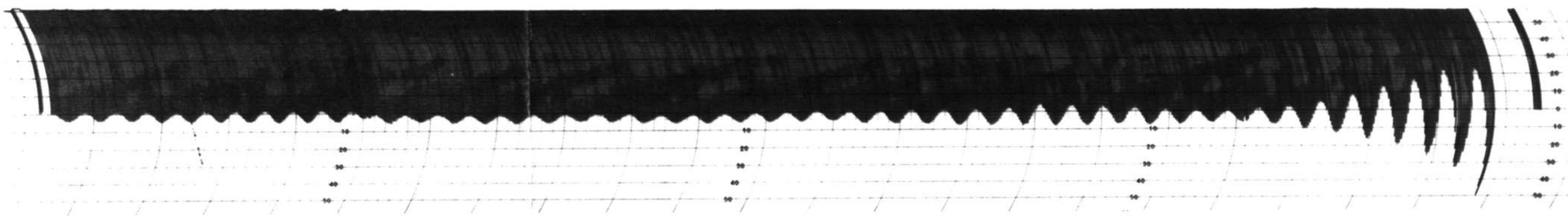
$\delta\tau$ (delay increment) = 6.25 milliseconds
τ_{max} (maximum delay) = 4.6 seconds

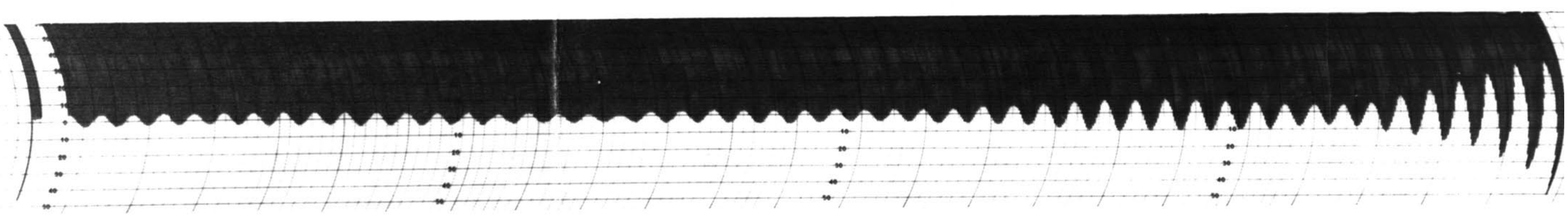
Top
T (sample length) = 100 seconds

Center
T (sample length) = 200 seconds

Bottom
T (sample length) = 400 seconds







and that the narrow-band noise model gives a good estimate of this variation; at least for this one sample of data.

An estimate of the central frequency and bandwidth of the sample shown in figure 4.43 yields:

$$\omega_o = 11 \text{ cps}$$

$$\text{band width} = .9 \text{ cps}$$

The other assumption of the narrow-band model is the Gaussian nature of the probability distribution. It has been shown that the relatively insensitive test of the probability paper has not contradicted this assumption. Assume for the moment, however, that a more sensitive test would discard the Gaussian hypothesis for the particular piece of data chosen for the correlogram in figure 4.43. It is clear that any non-Gaussian character of this sample of data has not seriously impaired the estimation of the error. It is entirely possible, therefore, that a Gaussian hypothesis be invalid in general and yet a Gaussian model be valid for the estimation of some particular statistic of the process. For this reason no great effort has been made to justify the Gaussian hypothesis by more sensitive statistical tests. This model appears to give a reasonably good prediction of the phenomenon in question.

CHAPTER 5 CONCLUSION AND SUGGESTIONS FOR FURTHER STUDY

5.1 Conclusions

Motivation for the quantitative study of the EEG phenomenon has been given by citing experimental evidence for its connection with important physiological and behavioral variables. Assuming, therefore, that quantitative studies of the EEG are desirable, a particular effort at quantification has been examined to investigate its relevance.

The reason for this study of the estimation of the autocorrelation of EEG is simply due to the fact that it is a widely used technique. It has been assumed in the past that some sort of frequency-emphasizing transformation is desirable for this kind of data. There is no reason to suppose that this is in any way the optimum transformation. It is not assumed here that autocorrelation is the best way of studying EEG, but it is certainly a way. Furthermore, much work has been done in this connection, and hypotheses concerning the nature of the EEG have been made on the basis of "evidence" obtained by correlation techniques.

One such hypothesis is the "physiological clock" hypothesis that stems from the long-delay cyclic activity noted in the correlograms of EEG that contain some alpha activity. This phenomenon has been studied here by making a narrow-band, Gaussian noise model of the EEG and examining the behavior of this model. Chapter 2 has shown that the finite-sample length of time series leads to a prediction of a long-term cyclic activity in the correlogram. Chapter 3 verifies the computation by exhibiting this behavior and by showing that it is indeed the finiteness of the record length that yields this statistical error.

Finally, EEG is discussed to see how well it fits the narrow-band model. Estimations of the time average probability distribution of the EEG are made and a reasonably good fit to a Gaussian distribution is indicated over the central range of the distribution. No presumptions of a proof of this hypothesis are made (if indeed this were possible at all).

In any case, it is seen that in a particular case, the narrow-band, Gaussian noise model predicts the behavior of the finite-sample autocorrelation function of EEG.

As a result of this research, the hypothesis that the long-delay cyclic behavior of the finite-sample autocorrelation function is a statistical error of estimation is offered.

5.2 Suggestions for Further Study

A start has been made in the direction of determining a reasonable model of the distribution of amplitudes of some EEG records. For this purpose the methods of statistical hypothesis testing are a propos. Preliminary tests were made using both the Chi-square Test and a test of third and fourth moments of the distributions. These tests essentially give the deviation of data points from an assumed distribution. In each case, however, the parameters of the hypothetical distribution are estimated from the data points. For the particular data used in connection with this thesis, these tests indicated a significant deviation from the Gaussian hypothesis. This result can be explained in part by the effects of the deviations of the tails of the distributions of the experimental data. It would be of considerable interest to modify the data to eliminate the effects of the pathological tails and then test for significance on several pieces of data. For this purpose, a general purpose digital computer could be used to do the data reduction.

In addition to a test of the distribution of amplitudes of the EEG, some further statistics might be tested to see how well the narrow-band noise model fits the data. For instance, the distribution of the amplitudes of the envelope of narrow-band, Gaussian noise is well-known (the Rayleigh distribution) (76) and this could be tested. The average time between successive maxima of the noise is also well known and is related to the bandwidth of the noise (76). This is in turn related to the time constant of the decrement of autocorrelation function. There are,some indications that a similar relationship may exist for the EEG. It if were found that the narrow-band model were a good representation of the alpha-activity-exhibiting EEG time series then a particular method of quantification for this phenomenon might be simply to estimate the variance, bandwidth and central frequency. In principle, all other statistics of the time series could then be computed.

To generalize the results of this thesis it might be of interest to study the effects of perturbations in the spectrum of the narrow-band noise upon the results. For this thesis a quadratic spectrum was assumed.

Appendix I - Calculation of the Autocorrelation Function of Narrow Band Noise

$$R_o(\tau) = \int_0^\infty h(t_1)\,dt_1 \int_0^\infty h(t_2)\,R_i(\tau - t_2 + t_1)\,dt_2 \qquad \text{for } \begin{array}{l} t_1 \geq 0 \\ t_2 \geq 0 \end{array}$$

but, $\qquad R_i(\tau) = N\mu_o(\tau)$

where $\mu_o(\tau)$ is the unit impulse function.

Therefore, substituting from equation 2.41:

$$R_o(\tau) = \int_0^\infty A\,e^{-\Delta\omega t_1} \sin(\omega_d t_1 - \psi)\,dt_1 \int_0^\infty A\,e^{-\Delta\omega t_2} \sin(\omega_d t_2 - \psi) \cdot N\mu_o(\tau - t_2 + t_1)\,dt_2$$

$$= NA^2 \int_0^\infty \left\{ e^{-\Delta\omega t_1} \sin(\omega_d(t_1 - \psi) \right\} \cdot$$

$$\left\{ e^{-\Delta\omega(\tau + t_1)} \sin(\omega_d(\tau + t_1) - \psi) \right\} dt_1$$

$$= \frac{NA^2}{2} e^{-\Delta\omega\tau} \int_0^\infty e^{-2\Delta\omega t_1} \left\{ \cos\omega_d\tau - \right.$$

$$\left. \cos(\omega_d\tau + 2\omega_d t_1 - 2\psi) \right\} dt_1$$

$$= \frac{NA^2}{2} e^{-\Delta\omega\tau} \left[\cos\omega_d\tau \int_0^\infty e^{-2\Delta\omega t_1} dt_1 - \right.$$

$$\left. \int_0^\infty e^{-2\Delta\omega t_1} \cos(\omega_d\tau + 2\omega_d t_1 - 2\psi)\,dt_1 \right]$$

Breaking up $R_o(\tau)$ into the two integrals and operating on them separately:

(1) $\int_0^\infty e^{-2\Delta\omega t_1}\,dt_1 = \frac{1}{2\Delta\omega}$

(2) let $\theta = \omega_d\tau - 2\psi$

then the second integral becomes

$$I_2 = \int_0^\infty e^{-2\Delta\omega t_1}\cos(2\omega_d t_1 + \theta)\,dt_1$$

$$= \int_0^\infty e^{-2\Delta\omega t_1}\cos(2\omega_d t_1)\cos\theta\,dt_1 -$$

$$\int_0^\infty e^{-2\Delta\omega t_1}\sin(2\omega_d t_1)\sin\theta\,dt_1$$

$$= \frac{-e^{-2\Delta\omega t_1}}{4\Delta^2\omega + 4\omega_d^2}\cos\theta\left[2\omega_d\sin 2\omega_d t_1 - 2\Delta\omega\cos 2\omega_d t_1\right]\Bigg|_0^\infty +$$

$$\frac{e^{-2\Delta\omega t_1}}{4\Delta^2\omega + 4\omega_d^2}\sin\theta\left[2\Delta\omega\sin 2\omega_d t_1 + 2\omega_d\cos 2\omega_d t_1\right]\Bigg|_0^\infty$$

$$= \frac{2\Delta\omega\cos\theta}{4\Delta^2\omega + 4\omega_d^2} - \frac{2\omega_d\sin\theta}{4\Delta^2\omega + 4\omega_d^2}$$

$$= \frac{1}{2}\,\frac{1}{\Delta^2\omega + \omega_d^2}\left\{\Delta\omega\cos\theta - \omega_d\sin\theta\right\}$$

$$= \frac{1}{2\sqrt{\omega_d^2 + \Delta^2\omega}}\sin\left(\theta - \tan^{-1}\frac{\Delta\omega}{\omega_d}\right)$$

Combining terms and substituting:

$$R_o(\tau) = \frac{NA^2}{2} e^{-\Delta\omega\tau}\left[\frac{1}{2\Delta\omega}\cos\left(\sqrt{\omega_o^2-\Delta^2\omega}\,\tau\right) - \frac{1}{2\omega_o}\sin\left(\sqrt{\omega_o^2-\Delta^2\omega}\,\tau - 2\sin^{-1}\frac{\sqrt{\omega_o^2-\Delta^2\omega}}{\omega_o} - \tan^{-1}\frac{\Delta\omega}{\sqrt{\omega_o^2-\Delta^2\omega}}\right)\right]$$

Using the approximation that $\omega_o >> \Delta\omega$ and simplifying the above, the result is:

$$R_o(\tau) = \frac{N}{4}\,\frac{\Delta\omega}{\omega_o}\,e^{-\Delta\omega|\tau|}\cos\omega_o\tau$$

Appendix II - Evaluation of the Variance of the Finite-Sample Autocorrelation Function of Narrow Band, Zero Mean, Gaussian Noise

$$\sigma_\phi^2(T,\tau) = \frac{2}{T^2}\int_0^T (T-\tau_0)\,e^{-2\alpha\tau_0}\cos^2\omega_0\tau_0\,d\tau_0 \;+$$

$$\frac{2}{T^2}\int_\tau^T (T-\tau_0)\,e^{-2\alpha\tau_0}\cos\omega_0(\tau+\tau_0)\cos\omega_0(\tau-\tau_0)\,d\tau_0 \;+$$

$$\frac{2}{T^2}\int_0^\tau (T-\tau_0)\cos\omega_0(\tau+\tau_0)\cos\omega_0(\tau-\tau_0)\,e^{-2\alpha\tau}\,d\tau_0$$

These three integrals will be evaluated separately. First:

$$I_1 = \int_0^T (T-\tau_0)\,e^{-2\alpha\tau_0}\cos^2\omega_0\tau_0\,d\tau_0$$

$$= \frac{1}{2}\int_0^T (T-\tau_0)\,e^{-2\alpha\tau_0}\,(1+\cos 2\omega_0\tau_0)\,d\tau_0$$

$$= \frac{1}{2}\int_0^T T e^{-2\alpha\tau_0}\,d\tau_0 + \frac{1}{2}\int_0^T T e^{-2\alpha\tau_0}\cos 2\omega_0\tau_0\,d\tau_0$$

$$-\frac{1}{2}\int_0^T \tau_0 e^{-2\alpha\tau_0}\,d\tau_0 - \frac{1}{2}\int_0^T \tau_0 e^{-2\alpha\tau_0}\cos 2\omega_0\tau_0\,d\tau_0$$

$$I_1 = \frac{T}{2}\left[\frac{e^{-2\alpha\tau_0}}{-2\alpha}\right]_0^T +$$

$$\frac{T}{2}\left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha\cos 2\omega_0\tau_0 + 2\omega_0 \sin 2\omega_0\tau_0\right)\right]_0^T +$$

$$-\frac{1}{2}\left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2}\left(-2\alpha\tau_0 - 1\right)\right]_0^T +$$

$$-\frac{1}{2}\left[\frac{\tau_0\, e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha\cos 2\omega_0\tau_0 + 2\omega_0\sin 2\omega_0\tau_0\right)\right.$$

$$\left. - \frac{e^{-2\alpha\tau_0}}{(4\alpha^2+4\omega_0^2)^2}\left\{(4\alpha^2-4\omega_0^2)\cos 2\omega_0\tau_0 - 8\alpha\omega_0\sin 2\omega_0\tau_0\right\}\right]_0^T$$

$$I_1 = \frac{T}{2}\left(\frac{1}{2\alpha}\right)\left[1-e^{-2\alpha T}\right] +$$

$$\frac{T}{4}\left(\frac{1}{\alpha^2+\omega_0^2}\right)\left[e^{-2\alpha T}\left(\omega_0\sin 2\omega_0 T - \alpha\cos 2\omega_0 T\right) + \alpha\right] +$$

$$\frac{1}{8\alpha^2}\left[e^{-2\alpha T}(2\alpha T+1) - 1\right] +$$

$$-\frac{1}{4}\left(\frac{1}{\alpha^2+\omega_0^2}\right)\left[T e^{-2\alpha T}\left(\omega_0\sin 2\omega_0 T - \alpha\cos 2\omega_0 T\right)\right] +$$

$$\frac{1}{8}\ \frac{1}{(\alpha^2+\omega_0^2)^2}\left[e^{-2\alpha T}\left\{(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0\sin 2\omega_0 T\right\}\right.$$

$$\left. - (\alpha^2-\omega_0^2)\right]$$

$$I_1 = \frac{T}{4\alpha}\left[1 - e^{-2\alpha T}\right] +$$

$$\frac{T}{4}\left(\frac{1}{\alpha^2+\omega_0^2}\right) e^{-2\alpha T}\left(\omega_0 \sin 2\omega_0 T - \alpha \cos 2\omega_0 T\right) +$$

$$\frac{T}{4}\left(\frac{\alpha}{\alpha^2+\omega_0^2}\right) +$$

$$\frac{1}{8\alpha^2}\left[(2\alpha T+1)e^{-2\alpha T}\right] - \frac{1}{8\alpha^2} +$$

$$-\frac{T}{4}\left(\frac{1}{\alpha^2+\omega_0^2}\right)e^{-2\alpha T}\left(\omega_0 \sin 2\omega_0 T - \alpha \cos 2\omega_0 T\right) +$$

$$\frac{1}{8}\frac{1}{(\alpha^2+\omega_0^2)^2} e^{-2\alpha T}\left((\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0 \sin 2\omega_0 T\right) +$$

$$-\frac{1}{8}\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2}$$

$$I_1 = \frac{T}{4\alpha} - \frac{T}{4\alpha}e^{-2\alpha T} + \frac{T}{4}\left(\frac{\alpha}{\alpha^2+\omega_0^2}\right) + \frac{T}{4\alpha}e^{-2\alpha T} +$$

$$\frac{1}{8\alpha^2}e^{-2\alpha T} - \frac{1}{8\alpha^2} - \frac{1}{8}\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2} +$$

$$\frac{1}{8}\left(\frac{1}{(\alpha^2+\omega_0^2)^2}\right)e^{-2\alpha T}\left[(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0 \sin 2\omega_0 T\right]$$

$$I_1 = \frac{1}{8\alpha^2}\left[e^{-2\alpha T} - 1\right] + \frac{T}{4\alpha}\left[1 + \frac{\alpha^2}{\alpha^2+\omega_0^2}\right] +$$

$$-\frac{1}{8}\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2} +$$

$$\frac{1}{8}\frac{1}{(\alpha^2+\omega_0^2)^2}e^{-2\alpha T}\left[(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0 \sin 2\omega_0 T\right]$$

The final form of I_1 is:

$$I_1 = \frac{1}{8\alpha^2}\left[e^{-2\alpha T} - 1\right] + \frac{T}{4\alpha}\left(\frac{2\alpha^2+\omega_0^2}{\alpha^2+\omega_0^2}\right) - \frac{1}{8}\left(\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2}\right) +$$

$$\frac{1}{8}\,\frac{1}{(\alpha^2+\omega_0^2)^2}\,e^{-2\alpha T}\left[(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0 \sin 2\omega_0 T\right]$$

The second integral I_2 is considered next:

$$I_2 = \int_{\tau}^{T} (T-\tau_0)\,e^{-2\alpha\tau_0} \cos\omega_0(\tau+\tau_0)\cos\omega_0(\tau-\tau_0)\,d\tau_0$$

$$= \frac{1}{2}\int_{\tau}^{T}(T-\tau_0)\,e^{-2\alpha\tau_0}\cos 2\omega_0\tau_0\,d\tau_0 \quad +$$

$$\frac{1}{2}\cos 2\omega_0\tau \int_{\tau}^{T}(T-\tau_0)\,e^{-2\alpha\tau_0}\,d\tau_0$$

$$I_2 = \frac{T}{2}\int_{\tau}^{T} e^{-2\alpha\tau_0}\cos 2\omega_0\tau_0 \, d\tau_0 \; +$$

$$-\frac{1}{2}\int_{\tau}^{T} \tau_0 e^{-2\alpha\tau_0}\cos 2\omega_0\tau_0 \, d\tau_0 \; +$$

$$\frac{T}{2}\cos 2\omega_0\tau \int_{\tau}^{T} e^{-2\alpha\tau_0}\, d\tau_0 \; +$$

$$-\frac{1}{2}\cos 2\omega_0\tau \int_{\tau}^{T} \tau_0 e^{-2\alpha\tau_0}\, d\tau_0$$

$$I_2 = \frac{T}{2}\left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha\cos 2\omega_0\tau_0 + 2\omega_0\sin 2\omega_0\tau_0\right)\right]_{\tau}^{T} +$$

$$-\frac{1}{2}\left[\frac{\tau_0 e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha\cos 2\omega_0\tau_0 + 2\omega_0\sin 2\omega_0\tau_0\right)\right]_{\tau}^{T} +$$

$$\frac{1}{2}\left[\frac{e^{-2\alpha\tau_0}}{(4\alpha^2+4\omega_0^2)^2}\left((4\alpha^2-4\omega_0^2)\cos 2\omega_0\tau_0 - 8\alpha\omega_0\sin 2\omega_0 T\right)\right]_{\tau}^{T} +$$

$$\frac{T}{2}\cos 2\omega_0\tau \left[\frac{e^{-2\alpha\tau_0}}{-2\alpha}\right]_{\tau}^{T} +$$

$$-\frac{1}{2}\cos 2\omega_0\tau \left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2}\left(-2\alpha\tau_0 - 1\right)\right]_{\tau}^{T}$$

$$I_2 = -\frac{T}{4}\,\frac{e^{-2\alpha T}}{\alpha^2+w_0^2}\left(w_0 \sin 2w_0\tau - \alpha\cos 2w_0\tau\right) +$$

$$\frac{\tau}{4}\,\frac{e^{-2\alpha\tau}}{\alpha^2+w_0^2}\left(w_0 \sin 2w_0\tau - \alpha\cos 2w_0\tau\right) +$$

$$\frac{1}{8}\,\frac{e^{-2\alpha T}}{(\alpha^2+w_0^2)^2}\left[(\alpha^2-w_0^2)\cos 2w_0T - 2\alpha w_0 \sin 2w_0T\right] +$$

$$-\frac{1}{8}\,\frac{e^{-2\alpha\tau}}{(\alpha^2+w_0^2)^2}\left[(\alpha^2-w_0^2)\cos 2w_0\tau - 2\alpha w_0 \sin 2w_0\tau\right] +$$

$$\frac{T}{4\alpha}\cos 2w_0\tau\left[e^{-2\alpha\tau} - e^{-2\alpha T}\right] +$$

$$\frac{1}{8\alpha^2}\cos 2w_0\tau\left[e^{-2\alpha T}(2\alpha T+1) - e^{-2\alpha\tau}(2\alpha\tau+1)\right]$$

$$I_2 = -\frac{T}{4}\,\frac{e^{-2\alpha T}}{\alpha^2+w_0^2}\left(w_0 \sin 2w_0T - \alpha\cos 2w_0\tau\right) +$$

$$\frac{\tau}{4}\,\frac{e^{-2\alpha\tau}}{\alpha^2+w_0^2}\left(w_0 \sin 2w_0T - \alpha\cos 2w_0\tau\right) +$$

$$\frac{1}{8}\,\frac{e^{-2\alpha T}}{(\alpha^2+w_0^2)^2}\left[(\alpha^2-w_0^2)\cos 2w_0T - 2\alpha w_0 \sin 2w_0T\right] +$$

$$-\frac{1}{8}\,\frac{e^{-2\alpha\tau}}{(\alpha^2+w_0^2)^2}\left[(\alpha^2-w_0^2)\cos 2w_0\tau - 2\alpha w_0 \sin 2w_0\tau\right] +$$

$$\frac{1}{4\alpha}\cos 2w_0\tau\, e^{-2\alpha\tau}\left[T - \tau\right] +$$

$$\frac{1}{8\alpha^2}\cos 2w_0T\left[e^{-2\alpha T} - e^{-2\alpha\tau}\right]$$

The last of the three integrals, I_3, is considered next:

$$I_3 = \int_0^{\tau} (T - \tau_0) \cos \omega_0 (\tau + \tau_0) \cos \omega_0 (\tau - \tau_0) \, d\tau_0$$

$$= \frac{1}{2} \int_0^{\tau} (T - \tau_0) \cos 2\omega_0 \tau_0 \, d\tau_0 \; +$$

$$\frac{1}{2} \cos 2\omega_0 \tau \int_0^{\tau} (T - \tau_0) \, d\tau_0$$

$$= \frac{T}{2} \int_0^{\tau} \cos 2\omega_0 \tau_0 \, d\tau_0 - \frac{1}{2} \int_0^{\tau} \tau_0 \cos 2\omega_0 \tau_0 \, d\tau_0 \; +$$

$$\frac{1}{2} \cos 2\omega_0 \tau \int_0^{\tau} (T - \tau_0) \, d\tau_0$$

$$= \frac{T}{2} \left[\frac{\sin 2\omega_0 \tau_0}{2\omega_0} \right]_0^{\tau} +$$

$$- \frac{1}{2} \left[\frac{1}{4\omega_0^2} \cos 2\omega_0 \tau_0 + \frac{1}{2\omega_0} \tau_0 \sin 2\omega_0 \tau_0 \right]_0^{\tau} +$$

$$\frac{1}{2} \cos 2\omega_0 \tau \left[T \tau_0 - \frac{\tau_0^2}{2} \right]_0^{\tau}$$

$$= \frac{T}{4\omega_0} \sin 2\omega_0 \tau + \frac{1}{4\omega_0} \left[\frac{1}{2\omega_0} \right] +$$

$$- \frac{1}{4\omega_0} \left[\frac{1}{2\omega_0} \cos 2\omega_0 \tau + \tau \sin 2\omega_0 \tau \right] +$$

$$\frac{1}{2} \cos 2\omega_0 \tau \left[T \tau - \frac{\tau^2}{2} \right]$$

$$I_3 = \frac{1}{4\omega_0}\sin 2\omega_0\tau\,[T-\tau] - \frac{1}{8\omega_0^2}\cos 2\omega_0\tau +$$

$$\frac{1}{8\omega_0^2} + \frac{1}{2}T\tau\cos 2\omega_0\tau - \frac{\tau^2}{4}\cos 2\omega_0\tau$$

The result for the variance term becomes:

$$\sigma_\phi^2(T,\tau) = \frac{2}{T^2}I_1 + \frac{2}{T^2}I_2 + \frac{2}{T^2}e^{-2\alpha\tau}I_3$$

$$= \frac{1}{4\alpha^2T^2}\left[e^{-2\alpha T}-1\right] + \frac{1}{2\alpha T}\left(\frac{2\alpha^2+\omega_0^2}{\alpha^2+\omega_0^2}\right) +$$

$$-\frac{1}{4T^2}\left(\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2}\right) +$$

$$\frac{1}{2T^2}\frac{1}{(\alpha^2+\omega_0^2)^2}e^{-2\alpha T}\left[(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0\sin 2\omega_0 T\right] +$$

$$-\frac{1}{2T}\frac{e^{-2\alpha\tau}}{\alpha^2+\omega_0^2}\left(\omega_0\sin 2\omega_0\tau - \alpha\cos 2\omega_0\tau\right) +$$

$$\frac{\tau}{2T^2}\frac{e^{-2\alpha\tau}}{\alpha^2+\omega_0^2}\left(\omega_0\sin 2\omega_0\tau - \alpha\cos 2\omega_0\tau\right) +$$

$$-\frac{1}{4T^2}\frac{e^{-2\alpha\tau}}{(\alpha^2+\omega_0^2)^2}\left[(\alpha^2-\omega_0^2)\cos 2\omega_0\tau - 2\alpha\omega\sin 2\omega_0\tau\right] +$$

$$\frac{1}{2\alpha T^2}\cos 2\omega_0\tau\, e^{-2\alpha\tau}[T-\tau] +$$

$$\frac{1}{4\alpha^2T^2}\cos 2\omega_0\tau\left[e^{-2\alpha T} - e^{-2\alpha\tau}\right] +$$

$$\frac{1}{2\omega_0 T} e^{-2\alpha\tau} \sin 2\omega_0\tau +$$

$$-\frac{1}{2\omega_0 T^2} e^{-2\alpha\tau}\left[\frac{1}{2\omega_0}\cos 2\omega_0\tau + \tau\sin 2\omega_0\tau\right] +$$

$$\frac{1}{4\omega_0^2 T^2} e^{-2\alpha\tau} +$$

$$\frac{1}{T^2} e^{-2\alpha\tau}\cos 2\omega_0\tau\left[T\tau - \frac{\tau^2}{2}\right]$$

Appendix 3 - Evaluation of the Expression for the Cross Correlation of Samples of the Finite-Sample Autocorrelation Function of Narrow Band, Gaussian Noise for Large Values of Delay

$$\mu = \frac{2}{T^2} e^{-\alpha\tau'} \int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \cos w_0\tau_0 \cos w_0(\tau_0+\tau') d\tau_0 \; +$$

$$\frac{2}{T^2} e^{-\alpha\tau'} \int_\tau^T (T-\tau_0) e^{-2\alpha\tau_0} \cos w_0(\tau_0+\tau+\tau') \cos w_0(\tau_0-\tau) d\tau_0 \; +$$

$$\frac{2}{T^2} e^{-\alpha\tau'} e^{-2\alpha\tau} \int_0^\tau (\tau-\tau_0) \cos w_0(\tau_0+\tau+\tau') \cos w_0(\tau_0-\tau) d\tau_0$$

These three integrals are evaluated separately.

$$I_1 = \int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \cos w_0\tau_0 \cos w_0(\tau_0+\tau') \, d\tau_0$$

$$= \cos w_0\tau' \int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \cos^2 w_0\tau_0 \, d\tau_0 \; +$$

$$- \sin w_0\tau' \int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \cos w_0\tau_0 \sin w_0\tau_0 \, d\tau_0$$

$$= \cos w_0\tau' \int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \cos^2 w_0\tau_0 \, d\tau_0 \; +$$

$$- \frac{1}{2} \sin w_0\tau' \int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \sin 2w_0\tau_0 \, d\tau_0$$

The first of these integrals has already been evaluated in Appendix 2.

$$\int_0^T (T-\tau_0) e^{-2\alpha\tau_0} \cos 2\omega_0\tau_0 \, d\tau_0 = \frac{1}{8\alpha^2}\left[e^{-2\alpha T} - 1\right] +$$

$$\frac{T}{4\alpha}\left(\frac{2\alpha^2+\omega_0^2}{\alpha^2+\omega_0^2}\right) - \frac{1}{8}\left(\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2}\right) +$$

$$\frac{1}{8}\,\frac{1}{(\alpha^2+\omega_0^2)^2}\, e^{-2\alpha T}\left[(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0 \sin 2\omega_0 T\right]$$

The second part of the expression for I_1 is:

$$I_{12} = \int_0^T (T-\tau_0)\, e^{-2\alpha\tau_0} \sin 2\omega_0\tau_0 \, d\tau_0$$

$$= T\left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha \sin 2\omega_0\tau_0 - 2\omega_0 \cos 2\omega_0\tau_0\right)\right]_0^T +$$

$$-\left[\frac{\tau_0\, e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha \sin 2\omega_0\tau_0 - 2\omega_0 \cos 2\omega_0\tau_0\right) +\right.$$

$$\left. - \frac{e^{-2\alpha\tau_0}}{(4\alpha^2+4\omega_0^2)^2}\left\{(4\alpha^2 - 4\omega_0^2)\sin 2\omega_0\tau_0 + 8\alpha\omega_0 \cos 2\omega_0\tau_0\right\}\right]_0^T$$

$$I_{12} = \frac{T}{2} \frac{\omega_0}{\alpha^2+\omega_0^2} - \frac{1}{2} \frac{\alpha\omega_0}{(\alpha^2+\omega_0^2)^2} +$$

$$\frac{e^{-2\alpha T}}{4(\alpha^2+\omega_0^2)^2} \left\{ (\alpha^2-\omega_0^2) \sin 2\omega_0 T + 2\alpha\omega_0 \cos 2\omega_0 T \right\}$$

Therefore

$$I_1 = \cos\omega_0\tau' \left[\frac{1}{8\alpha^2}\left(e^{-2\alpha T}-1\right) + \frac{T}{4\alpha}\left(\frac{2\alpha^2+\omega_0^2}{\alpha^2+\omega_0^2}\right) + \right.$$

$$- \frac{1}{8}\left(\frac{\alpha^2-\omega_0^2}{(\alpha^2+\omega_0^2)^2}\right) +$$

$$\frac{1}{8} \frac{1}{(\alpha^2+\omega_0^2)^2} e^{-2\alpha T} \left\{ (\alpha^2-\omega_0^2) \cos 2\omega_0 T + \right.$$

$$\left.\left. - 2\alpha\omega_0 \sin 2\omega_0 T \right\} \right] +$$

$$-\frac{1}{2} \sin\omega_0\tau' \left[\frac{T}{2} \frac{\omega_0}{\alpha^2+\omega_0^2} - \frac{1}{2} \frac{\alpha\omega_0}{(\alpha^2+\omega_0^2)^2} + \right.$$

$$\frac{1}{4} \frac{e^{-2\alpha T}}{(\alpha^2+\omega_0^2)^2} \left\{ (\alpha^2-\omega_0^2) \sin 2\omega_0 T + \right.$$

$$\left.\left. 2\alpha\omega_0 \cos 2\omega_0 T \right\} \right]$$

The integral I_2 is evaluated next:

$$I_2 = \int_\tau^T (T-\tau_0)\, e^{-2\alpha\tau_0} \cos\omega_0(\tau_0+\tau+\tau') \cos\omega_0(\tau_0-\tau)\, d\tau_0$$

$$= \frac{1}{2} \cos\omega_0(2\tau+\tau') \int_\tau^T (T-\tau_0)\, e^{-2\alpha\tau_0}\, d\tau_0 +$$

$$\frac{1}{2} \cos\omega_0\tau' \int_\tau^T (T-\tau_0)\, e^{-2\alpha\tau_0} \cos 2\omega_0\tau_0\, d\tau_0 +$$

$$-\frac{1}{2} \sin\omega_0\tau' \int_\tau^T (T-\tau_0)\, e^{-2\alpha\tau_0} \sin 2\omega_0\tau_0\, d\tau_0$$

$$I_{21} = \int_{\tau}^{T} (T-\tau_0)\, e^{-2\alpha\tau_0}\, d\tau_0$$

$$= T\left[\frac{e^{-2\alpha\tau_0}}{-2\alpha}\right]_{\tau}^{T} - \left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2}(-2\alpha\tau_0 - 1)\right]_{\tau}^{T}$$

$$= \frac{T}{2\alpha}\left[e^{-2\alpha\tau} - e^{-2\alpha T}\right] + \left[\frac{e^{-2\alpha T}}{4\alpha^2}(2\alpha T+1) - \frac{e^{-2\alpha\tau}}{4\alpha^2}(2\alpha\tau+1)\right]$$

$$= \frac{T}{2\alpha} e^{-2\alpha\tau} + \frac{1}{4\alpha^2} e^{-2\alpha T} - \frac{\tau}{2\alpha} e^{-2\alpha\tau} - \frac{1}{4\alpha^2} e^{-2\alpha\tau}$$

$$= \frac{T-\tau}{2\alpha} e^{-2\alpha\tau} + \frac{1}{4\alpha^2}\left[e^{-2\alpha T} - e^{-2\alpha\tau}\right]$$

$$I_{22} = \int_{\tau}^{T} (T-\tau_0)\, e^{-2\alpha\tau_0} \cos 2\omega_0\tau_0\, d\tau_0$$

$$= T\left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha\cos 2\omega_0\tau_0 + 2\omega_0 \sin 2\omega_0\tau_0\right)\right]_{\tau}^{T}$$

$$- \left[\frac{\tau_0 e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha\cos 2\omega_0\tau_0 + 2\omega_0 \sin 2\omega_0\tau_0\right) + \right.$$

$$\left. - \frac{e^{-2\alpha\tau_0}}{(4\alpha^2+4\omega_0^2)^2}\left\{(4\alpha^2 4\omega_0^2)\cos 2\omega_0\tau_0 - 8\alpha\omega_0 \sin 2\omega_0\tau_0\right\}\right]_{\tau}^{T}$$

$$= \left(\frac{\tau-T}{2}\right)\left\{\frac{e^{-2\alpha\tau}}{\alpha^2+\omega_0^2}\left(\omega_0 \sin 2\omega_0\tau - \alpha\cos 2\omega_0\tau\right)\right\} +$$

$$\frac{1}{4}\,\frac{e^{-2\alpha T}}{(\alpha^2+\omega_0^2)^2}\left\{(\alpha^2-\omega_0^2)\cos 2\omega_0 T - 2\alpha\omega_0 \sin 2\omega_0 T\right\} +$$

$$-\frac{1}{4}\cdot\frac{e^{-2\alpha\tau}}{(\alpha^2+\omega_0^2)^2}\left\{(\alpha^2-\omega_0^2)\cos 2\omega_0\tau - 2\alpha\omega_0 \sin 2\omega_0\tau\right\}$$

$$I_{23} = \int_{\tau}^{T} (T-\tau_0)\, e^{-2\alpha\tau_0} \sin 2\omega_0\tau_0 \; d\tau_0$$

$$= T\left[\frac{e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha \sin 2\omega_0\tau_0 - 2\omega_0 \cos 2\omega_0\tau_0\right)\right]_{\tau}^{T} +$$

$$-\left[\frac{\tau_0\, e^{-2\alpha\tau_0}}{4\alpha^2+4\omega_0^2}\left(-2\alpha \sin 2\omega_0\tau_0 - 2\omega_0 \cos 2\omega_0\tau_0\right)\right]_{\tau}^{T} +$$

$$\left[\frac{e^{-2\alpha\tau_0}}{(4\alpha^2+4\omega_0^2)^2}\left\{(4\alpha^2-4\omega_0^2)\sin 2\omega_0\tau_0 + 8\alpha\omega_0 \cos 2\omega_0\tau_0\right\}\right]_{\tau}^{T}$$

$$= \frac{T-\tau}{2}\left[\frac{e^{-2\alpha\tau}}{\alpha^2+\omega_0^2}\left(\alpha \sin 2\omega_0\tau + \omega_0 \cos 2\omega_0\tau\right)\right] +$$

$$\frac{1}{4}\frac{e^{-2\alpha T}}{(\alpha^2+\omega_0^2)^2}\left[(\alpha^2-\omega_0^2)\sin 2\omega_0 T + 2\alpha\omega_0 \cos 2\omega_0 T\right] +$$

$$-\frac{1}{4}\frac{e^{-2\alpha\tau}}{(\alpha^2+\omega_0^2)^2}\left[(\alpha^2-\omega_0^2)\sin 2\omega_0\tau + 2\alpha\omega_0 \cos 2\omega_0\tau\right]$$

The final result for I_2 is:

$$I_2 = \frac{1}{2}\cos\omega_0(2\tau+\tau')\, I_{21} +$$

$$\frac{1}{2}\cos\omega_0\tau'\, I_{22} +$$

$$-\frac{1}{2}\sin\omega_0\tau'\, I_{23}$$

The I_3 integral will not be evaluated since it has the term $2/T^2 e^{-\alpha\tau^1} e^{-2\alpha\tau}$ as a coefficient and it has already been assumed that τ is large in comparison to $1/\alpha$.

The final result therefore can be evaluated by recalling the approximations made 2.4 plus the additional approximation mentioned.

$$\omega_o \gg \alpha$$

$$\tau \gg 1/\alpha$$

I_1 then becomes:

$$I_1 \doteq T/4\alpha\cos\omega_o\tau^1$$

All the I_2 terms have factors of either $e^{-2\alpha\tau}$ or $e^{-2\alpha T}$. Therefore, under the assumptions made, these terms can be considered negligible.

The final result for μ is:

$$\mu \doteq 1/2\alpha T e^{-\alpha\tau^1}\cos\omega_o\tau^1$$

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