LAMINAR FREE CONVECTION HEAT TRANSFER IN A VERTICAL CHANNEL WITH LINEAR WALL TEMPERATURES

by

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Cambridge, Massachusetts
19 May 1961

Professor Philip Franklin
Secretary of the Faculty
Massachusetts Institute of Technology
Cambridge, Massachusetts

Dear Professor Franklin:

In partial fulfillment of the requirements for the degree of Bachelor of Science in Chemical Engineering, we herewith submit this thesis entitled "LAMINAR FREE CONVECTION HEAT TRANSFER IN A VERTICAL CHANNEL WITH LINEAR WALL TEMPERATURES".

Very truly yours,

Nelson E. Stefany

Paul H. Fricke

ACKNOWLEDGEMENT

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I. SUMMARY

The large scale use of nuclear reactors presents a serious problem as to the disposal of the radioactive waste created in the fission process. These wastes require immobilization for several hundred years for the radioactivity to become innocuous. Goldman et al., (7) have developed a method whereby the fission products are incorporated into a solid vitreous matrix, cylindrical in shape. It is presently proposed that these cylinders be stored in an underground cavern in some form of geometrical arrangement to allow air to circulate and cool by natural convection.

Flat heated, vertical and parallel, plates, similar to an arrangement of these cylinders stacked on top and alongside of each other, were used to study the ultimate storage problem of the cylinders with cooling by the natural convection of air.

A theoretical analysis of this particular system was done by Kim (9) which yielded an equation describing the heat transfer to the air flowing in the channel between the two plates:

$$Nu^* = \sqrt{2} Ra^{*1/4} . \qquad (1)$$

Heat transfer to cooling air, experimentally determined at plate separations up to 2 1/4 inches, determined that for the experimental system investigated, the correct form of the equation is

$$Nu^* = 1.1 Ra^{*\,1/4}. \qquad (2)$$

Equation (1) is satisfactory for design purposes, and the value of the constant should be checked on a system of larger magnitude.

II. INTRODUCTION

A. Origin of Problem

There are relatively few operational nuclear power reactors in the world today, but in the foreseeable future dwindling coal and oil resources may demand the use of nuclear power on a large scale in many parts of the world. A serious problem inherent in the use of nuclear power is how to dispose of the large quantities of radioactive fission products and isotopes which are produced in the fission process. Although methods of using these fission products are under development, no efficient utilization of all of these products is conceived within the present scope of technology. Thus, man is faced with the problem of protecting himself and his environment from the harmful effects of these radioactive waste products for time spans that measure over a thousand years.

Currently the fission products are separated from reclaimable material by solvent extraction and are stored as liquids in underground tanks. However, this is only a temporary solution since the highly corrosive nature of these liquids prevents their immobilization in tanks for sufficient time to render them harmless. Several alternate methods have been suggested, but technological and economic factors have made them unattractive.

Recently (February 1960) work by Goldman et al., (7) at M.I.T. has resulted in an attractive method for the disposal of radioactive wastes by converting their liquid solution into a relatively insoluble vitreous solid in the shape of a cylinder. Since these cylinders will contain radioactive materials of long half life, they must be isolated, and as a large amount of heat is generated internally from this radioactivity, they must be cooled. Consequently it was proposed that the cylinders be stacked in underground concrete vaults and be cooled by the natural convection of atmospheric air flowing between stacks of cylinders. Air was selected over more efficient liquid coolants for two reasons: (1) liquids tend to leach out the radioactive materials from the glass, presenting a safety problem; and (2) the cylinders are to be stored for a long period of time, and any mechanical means to stimulate circulation of the liquid would make this design economically unattractive.

Goldman et al., established the maximum axial temperature of the cylinders and the rate of heat generation per unit mass. Then they made preliminary calculations to show that cooling the solid suspension of radioactive wastes by the natural convection of atmospheric air was possible. The actual detailed design of how the cylinders should be stacked and air channel size and placement was not attempted. This is the more difficult part of the design problem; to find a method of choosing the optimum geometric parameters for stacking the cylindrical blocks.

B. Progress on Convection Cooling of Parallel Plates

Assuming, as Kim (10) suggested, that the cylinders will be stacked in parallel rows, whose length is much greater than height, the problem, ideally, can be reduced to one of cooling vertical, parallel heated plates by the natural convection flow of air. Extensive research has been done on the transfer of heat from one vertical heated plate of various temperatures and dimensions to free air (3,5,12,15). The general result has been that the heat transfer from the plate to air can be described by

$$Nu = C\ (Gr\ Pr)^n, \qquad (3)$$

where C and n are constants between zero and one, depending on the range of values for (Gr Pr)(12).

Previous experimentation with vertical parallel heated plates is limited and was done with relatively small plates. Elenbaas (6) experimented with various heights, distances between, and constant surface temperatures of heated parallel plates, with a maximum height of 24cm and a maximum temperature difference ($T_{surface} - T_{air}$) of 325°F. The experimental data were correlated by the equation,

$$Nu = 1/24\ b/h\ GrPr \left[1 - e^{-(35h/GrPrb)^{C_2}}\right]^{C_3}$$

Elenbaas, adjusting this equation for semi-infinite parallel plates, arrived at a relationship for optimum heat transfer from the plates to air,

$$\frac{(b - .3d)^4\ g\rho_w^2\ T_{plate}}{h\,\eta^2\ T_{air}} = 63 \qquad (4)$$

where b is the distance between, and h the height of plates.

This equation yielded values of b slightly higher than experimental values as radiation effects were not accounted for.

Kim (9) performed a mathematical analysis on the system which was herein investigated. He reduced the system to two parallel vertical slabs, infinite in the horizontal direction, z, (Fig. I). The velocity and temperature distributions of the flow field were obtained by solving the conservation laws subject to the following assumptions:

$$\frac{\delta u}{\delta z} = 0, \tag{5}$$

$$\frac{\delta u}{\delta y} = 0, \tag{6}$$

$$\frac{\delta T}{\delta y} = A, \tag{7}$$

i.e., the flow is a fully developed laminar flow in two dimensions and the vertical temperature gradient of the flow field and the slab is a constant. The assumption of linear temperature variation has been used by several investigators (13,11,2) in analysis of free convection flow and experimentally justified for slabs with uniform heat flux (2,11). Since slabs with a uniform distribution of radioactive waste will have a uniform surface heat flux, the assumption of a linear temperature variation is used in this work.

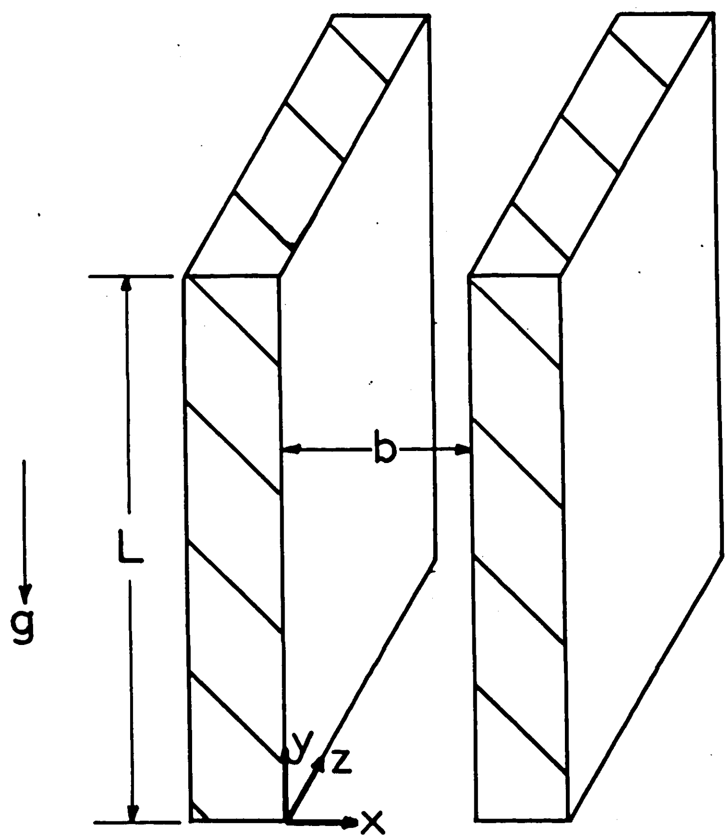


FIGURE I CHANNEL GEOMETRY

C. Object of Investigation

The object of the investigation is to determine the heat transfer rates from two parallel plates to free air with a constant temperature gradient imposed on the plates. These data, in addition to plate and air temperatures and physical constants, are used to calculate Nu* and Ra*. If ln Nu* is plotted versus ln Ra*, and the theoretical equation

$$Nu^* = \sqrt{2}\ Ra^{*1/4} \qquad (1)$$

developed by Kim for the heat transfer (Appendix A-2) is correct, the slope of the line should be 1/4 and its intercept $\sqrt{2}$.

The data were obtained by using two, 3 feet by 3 feet, 1/2 inch thick glass plates uniformly heated at one face and mounted vertically and parallel to each other. Glass plates were used in order to simulate the actual temperature distribution in a slab of vitreous solid containing radioactive wastes and self-heated by decay heat. The plates were heated electrically, and power input, plate and air temperatures were measured. The result obtained is in accord with Kim's correlation equation (1).

III. PROCEDURE

Standard data for copper-constantan thermocouples versus ice water reference potential were obtained from the Department of Chemical Engineering. Readings of the thermocouples in the apparatus at standard temperatures, ice and water and boiling water, came within $\pm$ 0.005 millivolts of the supplied data, and therefore it was accepted to convert millivolt readings to temperatures throughout the experiment.

Initially, the two plates were placed together, face to face, to determine the power lost through the apparatus as a function of center slab surface temperature. Power readings were recorded after steady state had been achieved over a range of center temperatures of 280 to 350°F. They were plotted as a function of center slab temperature minus the ambient air temperature.

The procedure for all trials was identical. Eight trials were made in all where the plates were separated by distances of 1/2 inch to 2 1/4 inches at 1/4 inch increments. At each trial, power was adjusted to maintain the center slab temperature at approximately 350°F. Voltage, amperage, and temperature measurements were made frequently during each trial and after about thirty hours at the desired setting, steady state was achieved, i.e., temperatures showed no increasing or decreasing trends. Concurrently, ambient and exit air temperatures were taken from thermomenters.

For each plate setting, the proper thickness of insulation was cut, wrapped in aluminum foil, and inserted in the vertical opening formed by separating the frames. This was to prevent any lateral heat loss through radiation and convection, and to permit air to enter the channel only from the bottom.

Equations (31) and (33) were used to compute Nu* and Ra*. The product of the voltage and the amperage for each plate gave the power input to that plate. The thermocouple readings supplied temperatures to determine the remaining parameters involved in the calculations.

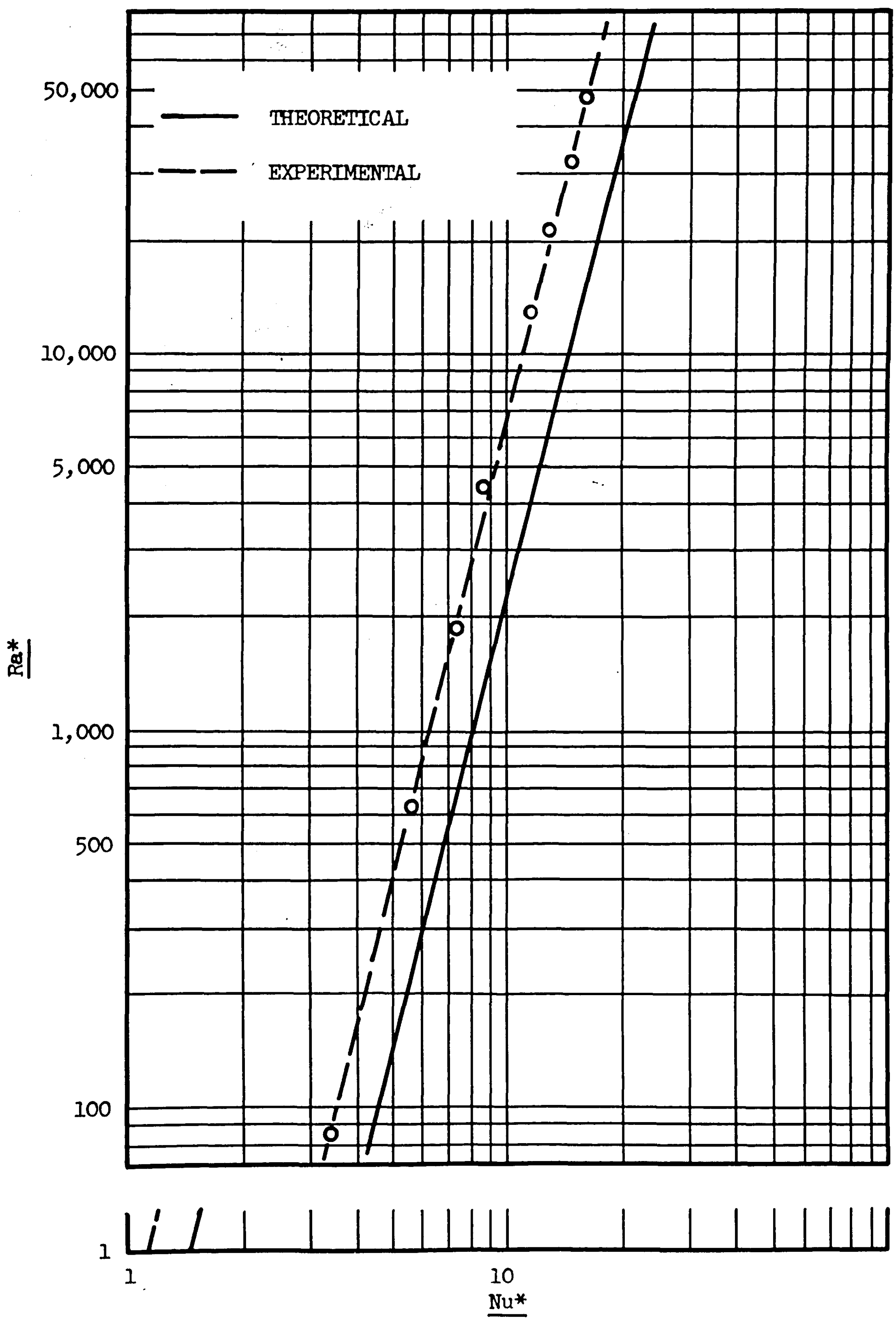
IV. RESULTS

The maximum experimental error was estimated as 10%, due mainly to the graphical method of determining A. Figure II is a plot of ln Nu* versus ln Ra*, showing the experimental and theoretical correlation. The slope and intercept of the experimental line are 0.24 and 1.1, respectively, compared to 0.25 and 1.41 predicted by equation (1). The slope of the experimental plot is within experimental error of the theoretical value, but the experimental intercept is not within 10% of the theoretical 1.41. The empirical results for the system investigated indicate that equation (1) should be

$$Nu^* = 1.1\ Ra^{*1/4} \qquad (2)$$

for an L of 3 feet and a b of 1/2 to 2 1/4 inches.

FIGURE II HEAT TRANSFER CORRELATION



V. DISCUSSION OF RESULTS

A. Explanation of Results

This investigation was done to determine the validity of the equation,

$$Nu^* = \sqrt{2}\ Ra^{*1/4} \qquad (1)$$

which was derived by Kim (Appendix A-2) to describe the heat transfer when opposing sides of two vertical, parallel, uniformly heated plates with a linear temperature variation are cooled by the natural convection of air flowing between them. The broken line in Figure II shows a plot of the experimental values determined for ln Nu* versus ln Ra* and the solid line represents the theoretical equation (1). The slopes of the two lines, which correspond to the exponent of Ra*, are .24 for the experimental and .25 for the theoretical. This deviation of 4% is well within the accuracy of the experimental numbers. Thus the derived exponent of 1/4 in equation (1) must be very nearly correct and should be satisfactory for the design purposes for which the equation is to be used.

The intercepts of the lines in Figure II correspond to the constant, which is $\sqrt{2}$ in equation (1). The experimental value of this constant is 1.1 or 21% lower than the theoretical. This experimental value is beyond the limits of a variable experimental errors and indicates that the derived value will not fit empirical systems, or that there was some consistent error made in determining the values for the experimental plot, which was undeterminable if it existed. According to the results, equation (1) should be adjusted to

$$Nu^* = 1.1\ Ra^{*1/4} \quad (2)$$

to be empirically correct for the system investigated for an h of 3 feet and a b of 1/2 to 2 1/4 inches.

B. Accuracy of Results

The nearness of the experimental points to lying in an exact straight line in Figure II indicates, as expected, that the standard deviations in the experimental readings and measurements are extremely small and thus negligible. The only significant uncertainty in the results is due to the method of choosing A, the vertical temperature gradient. The actual vertical temperature gradient in the air could not be measured for two reasons. First, conduction and radiation of heat from the ends of the heated plates caused the gradient in the plate to be non-linear at the ends, and this heat lost by radiation and conduction was not transferred directly to the air as was assumed in choosing a constant A. Secondly, it was not possible to get an accurate value for the exit air temperature from the experimental system due to wide fluctuations in this temperature. These fluctuations were due to ambient air currents in the room where the experiment was performed.

Since A of the air could not be measured directly, Kim's assumption that the temperature gradient on the plate is equal to the temperature gradient in the air stream and that they are both constant was employed. Thus A was found by drawing a temperature profile of the vertical plate temperature, and using the slope of the linear portion of the profile as A (see Figures III and IV). The uncertainty involved in this method evolves from the fact that vertical temperature gradients on the experimental plates were not linear due to heat losses at the top and bottom of the system. The top and bottom plate temperatures are lower than if the temperature gradient were linear. Thus the value of A used has an estimated uncertainty of 10% when determined in the manner described for this system. If further work is to be done along these lines, the experimental system should be modified so that the vertical temperature gradient is more nearly linear and the uncertainty in the results reduced. This could be accomplished by increasing the vertical length of the system so that end effects would become negligible. Other solutions would be to wrap the heating elements around the ends of the plates to decrease heat losses, or use much better insulation on the ends of the plates.

It should be noted that the temperatures of the experimental plates varied a few degrees along horizontal lines, but these variations had negligible effect on the results.

The radiant heat losses from the top and bottom ends of the experimental system ranged from less than 10 percent of the total heat transferred for small channel widths to over 20 percent for the 2 1/4 inch channel. Therefore, these radiant heat losses, in addition to the losses by conduction through the apparatus, were subtracted from the total heat input to determine the actual amount of heat removed by air convection. Due to the wide variation of temperatures on the radiating surfaces, radiating temperatures were approximated by choosing the temperatures on the vertical center line of the plates, 2 inches from the top and bottom for determination of radiant heat losses out of the top and bottom annuli, respectively. This introduced an estimated uncertainty of less than 5 percent in Nu*.

The deviation of the vertical temperature profile from linearity is due mainly to the heat losses from the ends of the system. This is shown quite well in Figures III and IV which are the experimental vertical temperature profiles at channel widths of 3/4 and 2 1/4 inches, respectively. The radiant heat loss for the 3/4 inch channel is small, therefore, the temperature profile is only slightly curved at the ends. However, at the channel width of 2 1/4 inches, the radiant heat loss is large, especially at the top of the plate. For this case, the temperature at the top is somewhat lower than the temperature farther down the plate. This factor is responsible for the negative slope at the upper end of the temperature profile in Figure IV.

FIGURE III VERTICAL TEMPERATURE PROFILE AND GRADIENT ON PLATE 1 FOR 3/4" SPACING

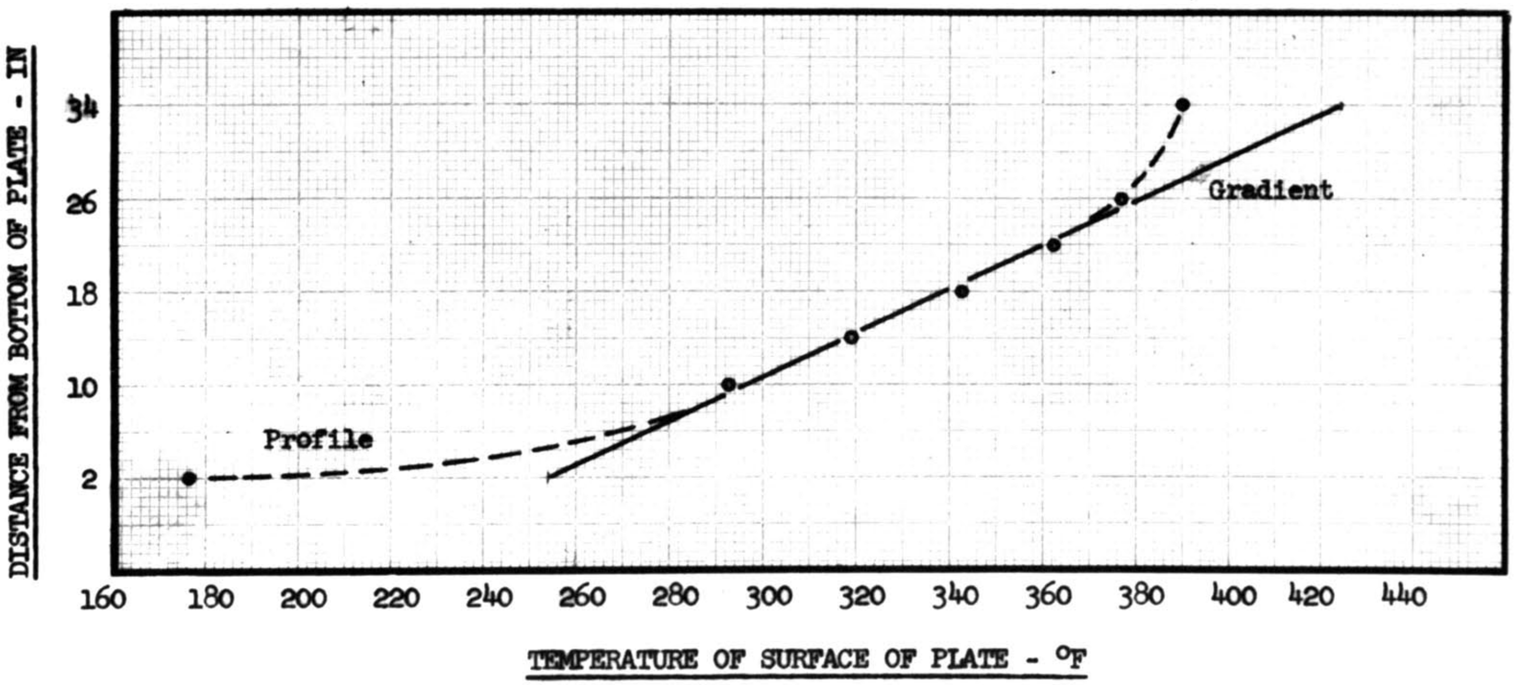
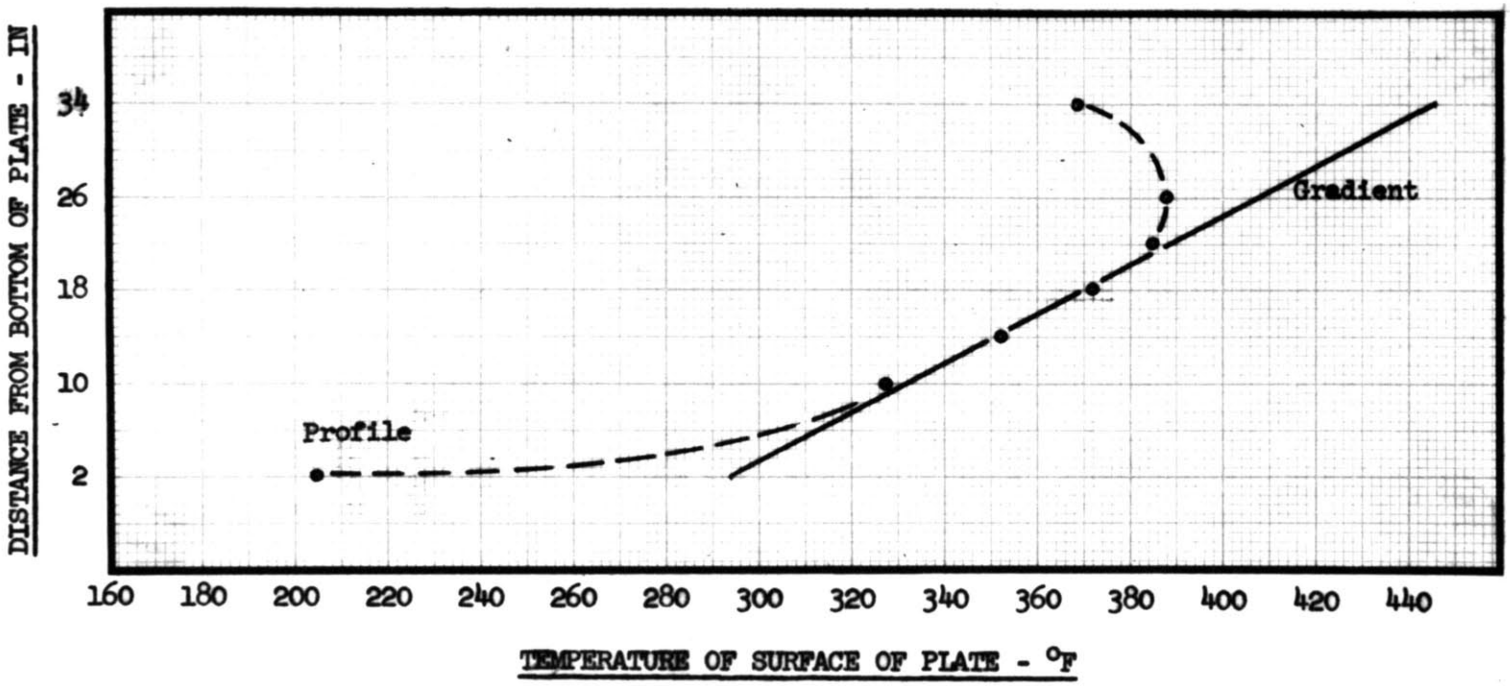


FIGURE IV VERTICAL TEMPERATURE PROFILE AND GRADIENT ON PLATE 1 FOR 2 1/4" SPACING



Another suggestion to make the experimental system more nearly like the system to be designed is to suspend metal particles in the glass plates and subject them to induction heating. This would compare with internal heat generation by radioactive particles.

VI. CONCLUSIONS

1. The exponent 1/4 in the heat transfer equation for two parallel plates,

$$Nu^* = \sqrt{2}\ Ra^{*1/4} \qquad (1)$$

is correct within experimental limits.

2. The constant $\sqrt{2}$ should be replaced by 1.1

$$Nu^* = 1.1\ Ra^{*1/4} \qquad (2)$$

to make an empirically correct heat transfer equation for the system investigated.

3. Equation (1) will be satisfactory for the design purposes it was developed for if the proper empirical constants are verified for a system of the magnitude proposed.

VII. RECOMMENDATIONS

1. A larger experimental system could be built to minimize the relative effects of heat losses from the top and bottom of the system, and thus decrease uncertainty in further investigations.

2. The magnitude of heat losses from the top and bottom of a similar experimental system could be reduced by heating slab ends and using better insulation against radiation and conduction losses.

3. Further investigations should be made to determine an empirical constant so that equation (1) will be suitable for design purposes.

4. Induction heating of metal particles suspended in glass could be used to simulate the internal heat generation as there would be when radioactive particles are suspended in glass.

VIII. APPENDIX

A. SUPPLEMENTARY DETAILS

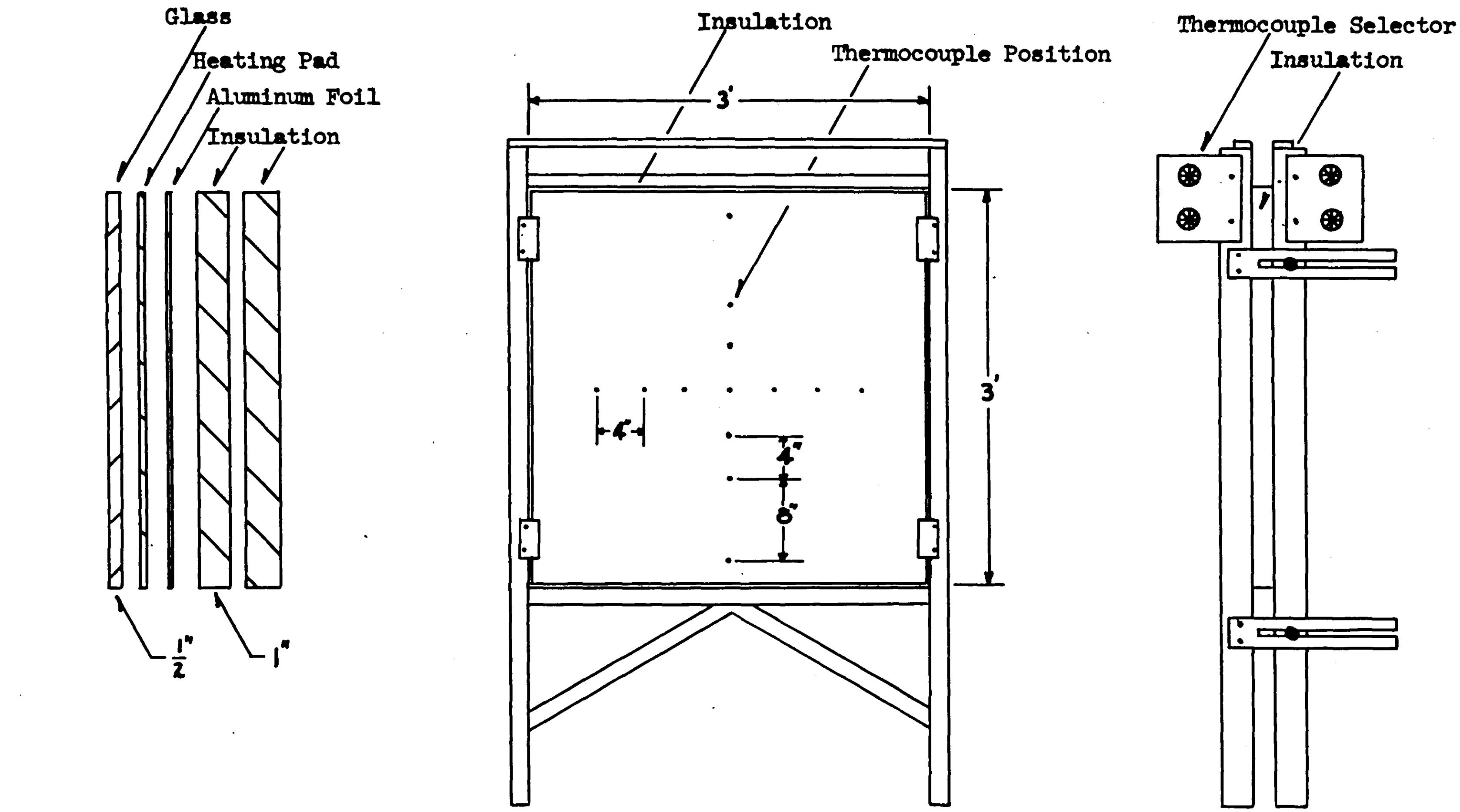
A-1. Details of Apparatus

Glass was chosen for the plates as it has a thermal conductivity and heat transfer coefficient almost identical to that of the vitreous cylinders themselves. The flat surface of the plate can be visualized easily to represent the face of a bank of cylinders stacked on top of and along side each other. Thus the channel between two vertical and parallel plates represents the channel between two of these banks of columns.

a. Mounting of Glass Plates

A 3' x 3' x 1/2" polished glass plate was secured vertically in each of two wooden frames by metal clips, Fig. V. The surface of the glass was allowed to protrude 1/8" from the plane of the frame so that the screws holding the clips to the frame would not interfere with bringing the faces of the glass together for heat loss tests. Frame 1 was plumbed and secured in place. Brackets were then installed connecting frame 2 to frame 1 so that frame 2 could be adjusted to any desired distance from frame 1 and be fastened firmly in place. In adjusting the width of the channel, the distance between glass faces was obtained by moving the adjustable plate up against metal blocks, whose dimensions corresponded to the desired separation. This method ensured the plumbness of the adjustable plane relative to the fixed one.

FIGURE V DETAILS OF APPARATUS



a.
EXPLODED CROSS SECTION OF GLASS PLATE AND BACKING

b.
GLASS PLATE MOUNTED IN STAND

c.
END VIEW OF STANDS PLACED TOGETHER

b. Heating Apparatus and Insulation

Prior to installation of the glass in the wooden stand, three 1' x 3' Electroflux heating pads were glued to the back of each plate with resin glue. The pads were purchased from the Electroflux Company, Hartford, Connecticut, and specified to produce 500 watts at 220 volts AC. The surfaces of the wooden frames contacting the ends of the glass plate were lined with 3/16" asbestos tape, and the plates were secured in place. Insulation was then applied to the heating pad side of the plates as follows:

1. layer of aluminum foil
2. Two layers of 1" B. E. H. spun fiberglass insulation

The layers of insulation were held firmly in place in the frames by metal straps across the back of the frames.

c. Thermocouples and Temperature Measurement

Thirty gauge copper-constantan thermocouples were emplaced on the surfaces of both plates according to the pattern in Fig. V. Small niches were ground out of the plate and the bead glued in place with ceramic cement so that it was flush with the plate surface. The wires were run straight up the surface, tacked at intervals to the plate with cement. Additional thermocouples were between the glass and heating pad as the temperature of the glue could not exceed 400°F.

For convenience in reading the thirty-four thermocouples, they were connected to selector switches located on the ends of the frames, Fig. V, from which a single wire ran to the potentiometer. Potentials were measured directly against an ice water reference.

Thermometers were used to measure the ambient and exit air temperatures. An aluminum shield was installed on the latter to protect it from radiation effects of the plates. A thermocouple with a similar shield was also installed next to the exit air thermometer, but potentials fluctuated so widely that it was not reliable.

d. Power Supply

The circuit is outlined in Fig. VI. Voltages were adjusted equally across each plate by variacs. Power was determined from the voltmeter and ammeter readings.

FIGURE VI POWER CIRCUIT

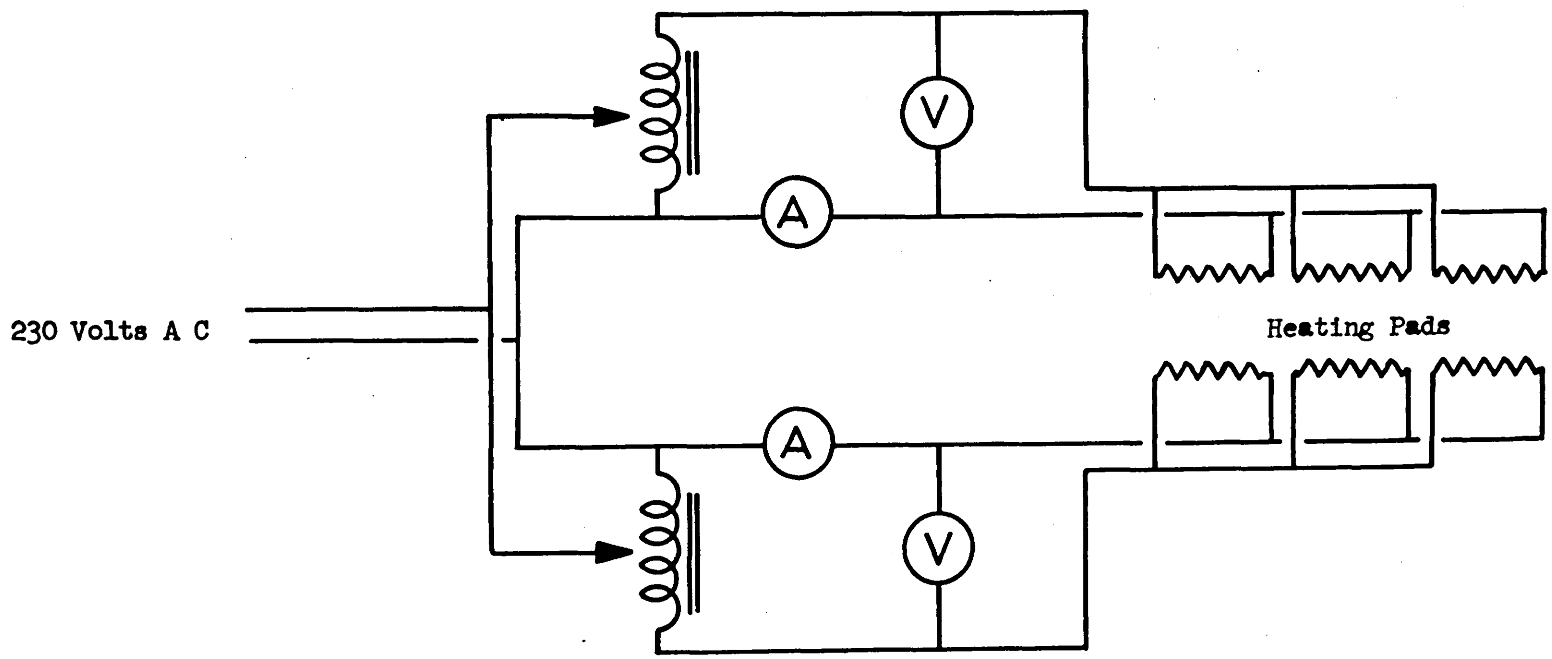
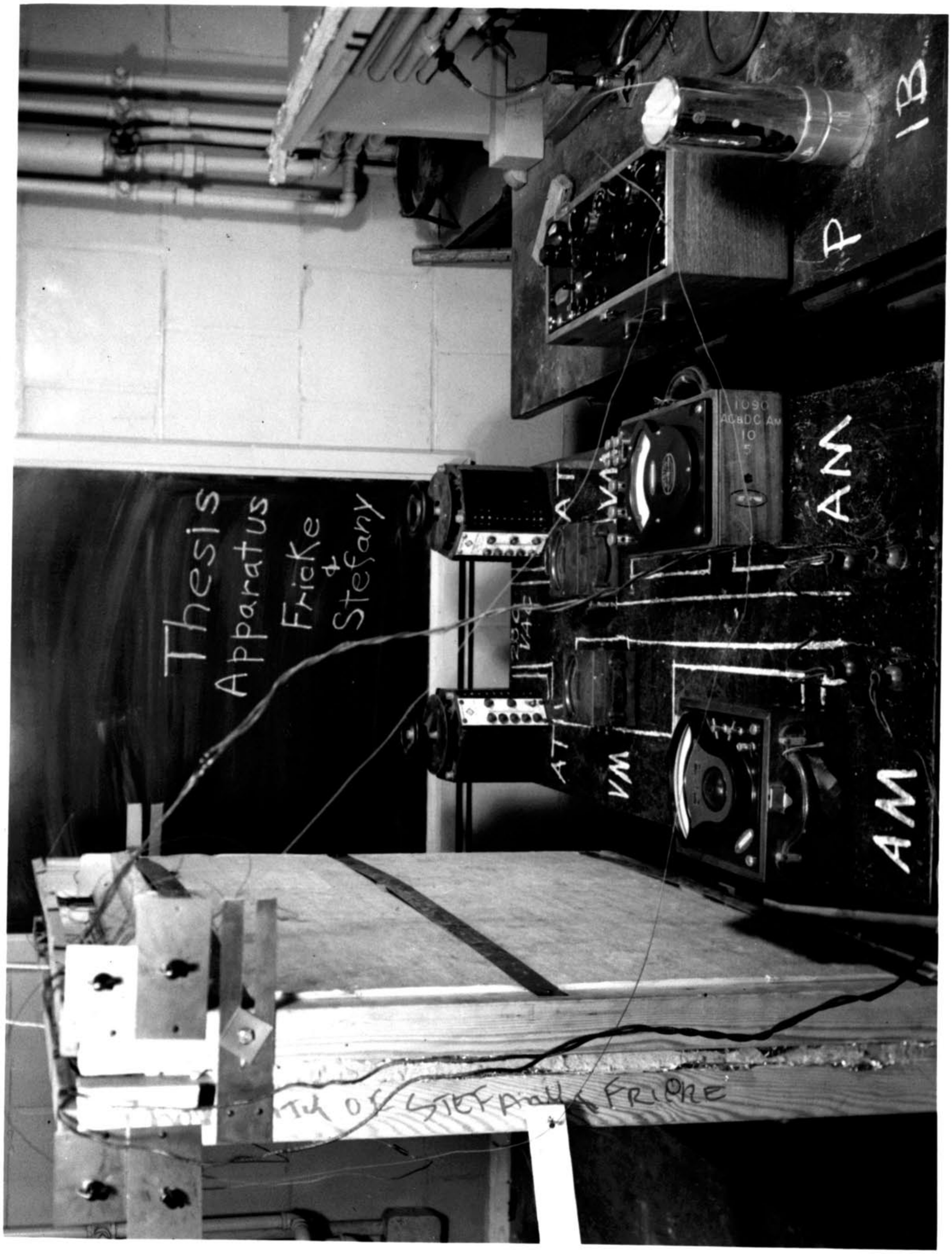
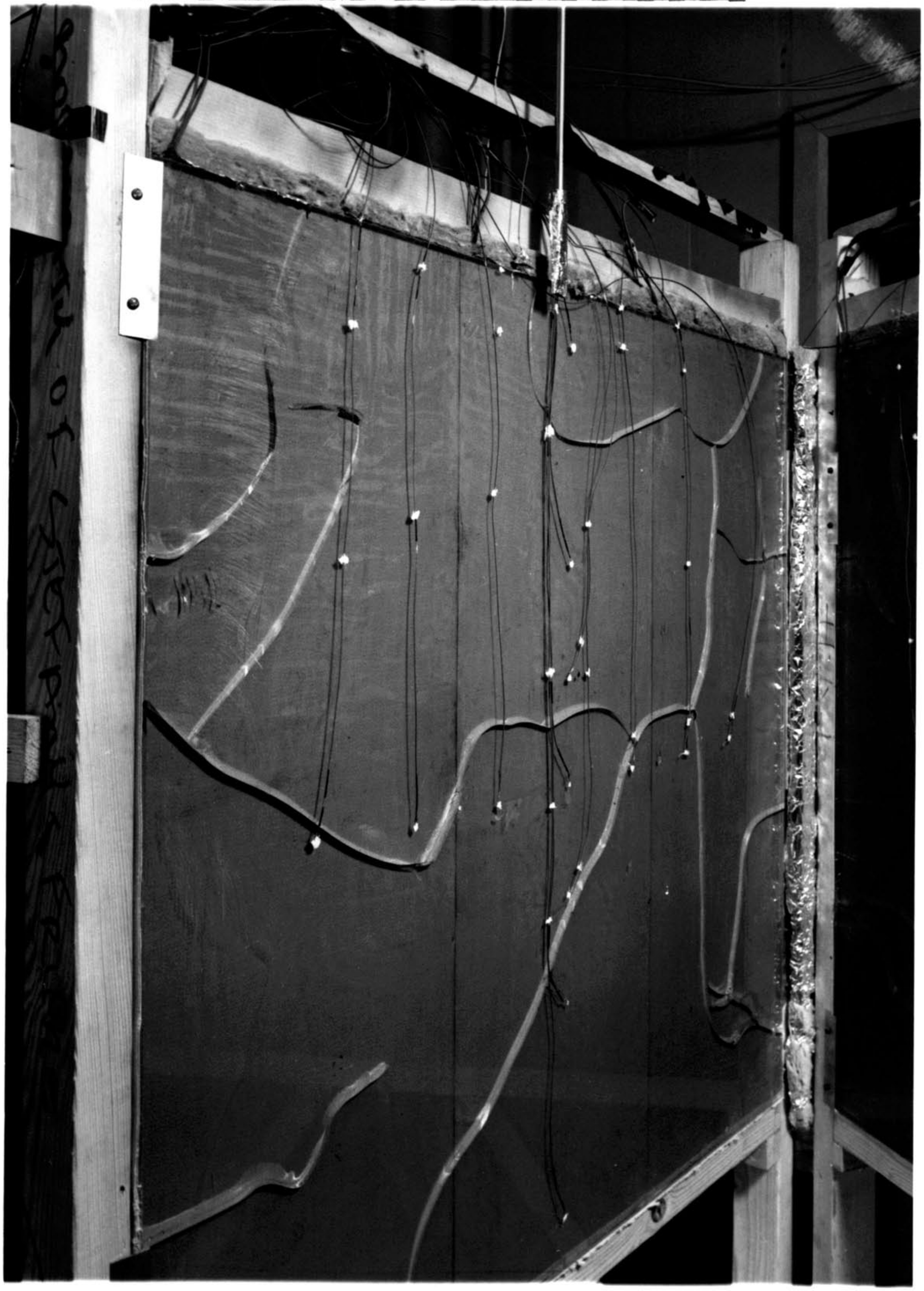


FIGURE VII PHOTOGRAPH OF APPARATUS



AT- Adjustable Transformer
VM- Volt Meter
AM- Ammeter
P- Potentiometer
IB- Ice Bath

FIGURE VIII PHOTOGRAPH OF PLATE SURFACE



A-2.

Determination of Temperature and Velocity Distributions and the Heat Transfer Relationship

I. Analysis

A plate was chosen to serve as an idealized model for a row cylinders. Heat is to be removed from two separated parallel plates by cooling at the surface through natural convection of air. As the flow velocity is ordinarily low in this type of convection, any pressure drop due to friction against the surfaces is negligible. Thus, the system can be considered to be surrounded by ambient air. The rate of heat removal by air passing through the channel can be determined when the velocity and temperature distribution in the flow field are known.

As the velocity and temperature distributions are interdependent, their solutions are obtained by solving simultaneously the Navier-Stokes equations of motion and the energy equation. This was done by the method of parametric linearization, whereby the properties of the fluid flowing, viscosity, density, thermal conductivity, are assumed independent of temperature, except where the variation of density is taken into account in the formulation of the body force term in equation (9). This method of attack has been accepted since the work of Polhousen in 1921 and the results of such an analysis are, in cartesian tensor-notation, (10)

$$\frac{\delta U_j}{\delta x_j} = 0 \qquad (8)$$

$$\rho U_j \frac{\delta U_j}{\delta x_j} = \rho F_i - \frac{\delta P}{\delta x_i} + \mu \left[\frac{\delta}{\delta x_j} \left(\frac{\delta U_i}{\delta x_j} + \frac{\delta U_j}{\delta x_i} \right) - \frac{2}{3} \frac{\delta}{\delta x_i} \left(\frac{\delta U_j}{\delta x_j} \right) \right] \qquad (9)$$

$$\rho C_v U_j \frac{\delta T}{\delta x_j} = k \frac{\delta}{\delta x_j} \left(\frac{\delta T}{\delta x_j} \right) \qquad (10)$$

In order to obtain analytical solutions, these equations must be modified further due to the non-linear terms present, and the interdependency of the equations of motion and velocity. Kim (9), in analyzing the flow between the two parallel slabs, Fig. 1, proposed the following assumptions for modification:

1. If the dimension of the slab in the z-direction is infinitely greater than the other two directions, then the temperature and velocity do not vary in that direction, *i.e.*,

$$\frac{\delta u}{\delta z} = 0 \qquad (5)$$

$$\frac{\delta T}{\delta z} = 0 \qquad (11)$$

2. If the length of the channel is large compared to the width, the flow will be fully developed laminar flow and consequently the components of velocity normal to the direction of flow vanish

identically. Then by the continuity equation, velocity is independent of the y direction, and only a function of the horizontal direction, x,

$$\frac{\delta u}{\delta y} = 0 \qquad (6)$$

Equations (9) and (10) can be rewritten with more familar notations for force and direction, incorporating assumptions 1 and 2.

$$\mu \frac{\delta^2 u}{\delta x^2} - \rho g - \frac{\delta P}{\delta y} = 0 \qquad (12)$$

$$u \frac{\delta T}{\delta y} = \frac{k}{\rho C_v} \left(\frac{\delta^2 T}{\delta x^2} + \frac{\delta^2 T}{\delta y^2} \right) \qquad (13)$$

As equation (13) still contains a non-linear term, a further assumption is needed.

3. Neglecting end effects, when a solid with internal heat generation is cooled by laminar flow, the temperature gradient is constant in the direction of flow. Similarly, the gradient in the flow field itself becomes constant over the major portion of the channel. Thus,

$$\frac{\delta T}{\delta y} = A. \qquad (7)$$

While a solution of equations (12) and (13) is for a particular value of y, this solution when combined with equations

(6) and (7) will provide solutions for any value of y. Let y = 0. Incorporating assumption 3 into equation (13) and combining the body force term with the pressure gradient, in equation (12) by introducing the coefficient of volumetric expansion, these equations are:

$$u = \frac{k}{\rho C_v A} \; \frac{\delta^2 T}{\delta x^2} \; (x,0) = 0 \qquad (14)$$

$$\frac{\delta^2 u}{\delta x^2} + \frac{\rho\beta g}{\mu} \Big(T(x,0) - T_{oo} \Big) = 0 \qquad (15)$$

Eliminating T(x,0) between equations (14) and (15) gives

$$\frac{\delta^4 u}{\delta x^4} + \left(\frac{2r}{\sqrt{2}} \right)^4 u = 0 \qquad (16)$$

where

$$r^4 = \frac{C_v \rho^2 \beta g A}{2 \mu k} \qquad (17)$$

The solution of equation (16) has the form

$$u = C_1 \sin rx \sinh rx + C_2 \cos rx \sinh rx + C_3 \sin rx \cosh rx + C_4 \cos rx \cosh rx. \qquad (18)$$

The constants can be evaluated by subjecting the equation to the following boundary conditions:

1. $u(0) = 0$
2. $u(b) = 0$

3. $T(0,0) = T(b,0)$ or $u''(0) = u''(b)$

4. $- k \frac{\delta T}{\delta x} (x=0) = q$ or $- u'''(0) = C_5 \frac{q}{k}$.

By inspection, boundary condition 1 sets the value of C_4 as zero. Applying conditions 2, 3, and 4 yields three simultaneous equations:

$$C_1 \sin rb \sinh rb + C_2 \cos rb \sinh rb + C_3 \sinh rb \cosh rb = 0 \quad (19)$$

$$C_1(\cos rb \cos rb - 1) + C_2 \sin rb \cosh rb + C_3 \cos rb \sinh rb = 0 \quad (20)$$

$$- C_2 + C_3 = C_5 \frac{q}{k} \quad (21$$

which can be solved for values of C_1, C_2, and C_3 in terms of C_5 by employing determinants.

The value of C_5 is obtained implicitly. Rearranging equation (15) and taking the partial derivative with respect to x, gives

$$\frac{\delta T}{\delta x}(x,0) = - \frac{\mu}{\rho\beta g} \frac{\delta^3 u}{\delta x^3} \quad (22)$$

which at x = 0 according to boundary condition 4, is equal to q/k. Thus,

$$\frac{\mu}{\rho\beta g} \frac{\delta^3 u}{\delta x^3} = \frac{q}{k} . \quad (23)$$

Substituting

$$\frac{\delta^3 u}{\delta x^3} = 2r^3(- C_2 + C_3) \quad (24)$$

into equation (23) and rearranging, gives

$$\frac{\rho\beta g}{\mu 2r^3}\;\frac{q}{k} = (- C_2 + C_3) \; . \qquad (25)$$

On comparison of equation (25) with equation (21), it can be seen that the group $\frac{\rho\beta g}{\mu 2r^3}$ is C_5.

II. Velocity Distribution

Substituting the values for the constants in equation (18) gives the completed velocity distribution, independent of y, in dimensionless form.

$$\frac{u}{\frac{q\rho\beta g}{2\mu k r^3}} = \frac{(\cos^2 rb \sin h^2 rb + \sin^2 rb \cosh^2 rb)}{(\cos rb - \cosh rb)(\sinh rb - \sin rb)} (\sin rx \sinh rx)$$

$$+ \frac{\sin rb}{\sinh rb - \sin rb} (\cos rx \sinh rx)$$

$$+ \frac{\sinh rb}{\sinh rb - \sin rb} (\sin rx \cosh rx) \qquad (26)$$

III. Temperature Distribution

To satisfy equation (7), equation (15) is modified to give the following solution for any y:

$$T(x,y) - T_{oo} = Ay + \frac{\delta^2 u}{\delta x^2} \; . \qquad (27)$$

Rearranging and operating on the velocity gives

$$\frac{T(x,y) - T_{oo} - Ay}{\frac{q}{kr}} = \frac{\cos^2 rb \sinh^2 rb + \sin^2 rb \cosh^2 rb}{(\cosh rb - \cos rb)(\sinh rb - \sin rb)} (\cos rx \cosh rx)$$

$$+ \frac{\sin rb}{\sinh rb - \sin rb} (\sin rx \cosh rx) - \frac{\sinh rb}{\sinh rb - \sin rb} (\cos rx - \sinh rx) \quad (28)$$

for the completed temperature distribution in dimensionless form.

IV. Heat Transfer Relationship

For values of rb much greater than one, which is the case, the right side of equation (28) approaches a value of one. Consequently, rearranging terms and substituting equation (17) for r, yields,

$$\frac{q}{k(T - T_{oo} - Ay)} = \sqrt{2} \left(\frac{C_v \rho^2 \beta g A}{\mu k} \right)^{1/4} \quad (29)$$

A value for y of L/2 was chosen as this distance represents an average temperature difference between the plate and air over the vertical dimension of the slab.

Both sides of equation (29) are multiplied by an appropriate dimension of length, b, to make the expression dimensionless. The right hand side was divided by γ for arrangement of the terms into recognizable dimensionless groups. Defining the dimensionless groups as:

$$\frac{qb}{k\left(T - T_{oo} - A\,\frac{L}{2}\right)} = \sqrt{2}\left[\left(\frac{\mu C_p}{k}\right)\left(\frac{\rho^2 \beta g A b^4}{\mu^2}\right)\frac{1}{\gamma}\right]^{1/4} \qquad (30)$$

Defining the dimensionless groups as:

$$Nu* = \frac{qb}{k\left(T - T_{oo} - A\,\frac{L}{2}\right)} \qquad (31)$$

$$Gr* = \frac{\rho^2 \beta g A b^4}{\mu^2} \qquad (32)$$

$$Ra* = \frac{Pr\ Gr*}{\gamma} \qquad (33)$$

then,

$$Nu* = \sqrt{2}\left[\frac{Pr\ Gr*}{\gamma}\right]^{1/4} \qquad (34)$$

and

$$Nu* = \sqrt{2}\ Ra*^{1/4}. \qquad (1)$$

B. SUMMARY OF DATA AND CALCULATED VALUES

Figures III and IV are the temperature profiles and gradients for three-quarter and two and a quarter inch spacing, respectively. The profiles for other spacings are similar. The top and bottom temperatures were not considered in determining the gradient. The gradient was calculated directly from the graph and expressed in units of °F/ft for use in subsequent calculations.

Figure IX shows power loss through the apparatus and is used in determining the rate of heat transfer to air by convection.

Tables I and II contain data for the calculation of the final modified Nusselt and Rayleigh numbers.

FIGURE IX POWER LOSS

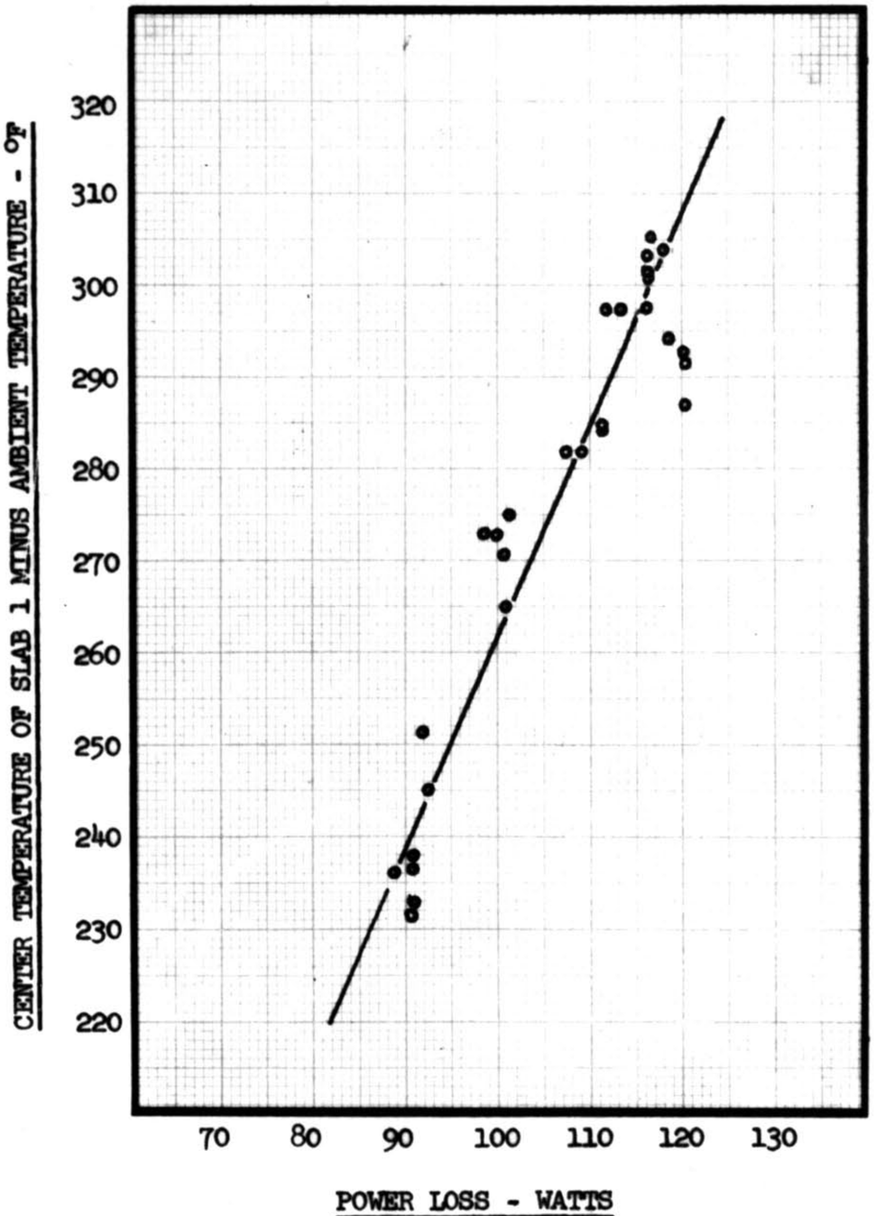


TABLE I

Data For Calculation of Modified Nusselt Number

Trial	b	T_c	T_{oo}	A	P_T	Q_r	P_L	k^+	Nu*
1	0.0417	313.5	77.0	63.3	644	101	89	0.0179	3.27
2	0.0625	343.0	78.5	63.3	818	159	101	0.0169	5.61
3	0.0833	341.0	76.0	59.1	828	209	102	0.0164	7.27
4	0.104	354.5	75.8	49.9	905	257	108	0.0164	8.51
5	0.125	354.0	69.0	62.2	940	288	110	0.0160	11.4
6	0.146	363.0	75.0	57.8	956	357	113	0.0160	12.7
7	0.167	360.5	71.8	50.0	995	398	112	0.0160	14.2
8	0.187	372.0	77.1	56.8	1000	453	114	0.0160	15.9

+ McAdams, W. H., "Heat Transmission", 3rd Ed., p.483, McGraw-Hill, New York (1954).

TABLE II

Data For Calculation of Modified Rayleigh Number

Trial	T_a	$\frac{\rho^2 B g C_p}{\mu k}$ +	b^4	Ra*
1	190	0.64×10^6	2.95×10^{-6}	84.8
2	157	0.89×10^6	1.52×10^{-5}	610
3	138	0.94×10^6	4.75×10^{-5}	1870
4	127	1.02×10^6	1.21×10^{-4}	4380
5	110	1.15×10^6	2.49×10^{-4}	12700
6	112	1.15×10^6	4.50×10^{-4}	21400
7	111	1.15×10^6	7.61×10^{-4}	31100
8	110	1.15×10^6	1.03×10^{-3}	47800

+ McAdams, W. H., op.cit., p.483.

C. SAMPLE CALCULATIONS

The calculations for all trials are identical. Data from plate number one were used consistently.

Consider trial 2 where the plates were spaced at 0.0625 feet (3/4 inch). The temperature profile and gradient for trial 2 are shown in Figure IV.

Total power input recorded at steady state,

$$P_T = 818 \text{ watts}$$

a. Determination of Heat Transferred to Air Flowing Through Channel

Center slab temperature obtained from Figure IV,

$$T_c = 343.0 \text{ °F.}$$

Corresponding power lost through apparatus from Figure IX,

$$P_L = 101 \text{ watts.}$$

Heat transferred to air by slab exclusive of radiation losses

$$Q_c \text{ (BTU/hr)} = \frac{(818 - 101) \text{ watts } (60) \text{ min/hr}}{(17.58) \text{watts/BTU/min}}$$

$$Q_c = 2420 \text{ BTU/hr, from plate 1.}$$

Radiation losses were considered out of the top and bottom annuli of the channel according to the formula,

$$Q_r = \sigma A_r F e (T_{t \text{ or } b}^4 - T_{oo}^4).$$

Applying this equation to the top annulus, for a 3/4 inch spacing,

$$A_r = \frac{3/4 \times 36}{144} = .1875 \text{ ft}^2$$

$$Q_{rt}(\text{BTU/hr}) = .173 \times .1875 \times 1 \times .963 \left[\frac{849}{100}^4 - \frac{539}{100}^4 \right]$$

$$Q_{rt} = 130 \text{ BTU/hr.}$$

The total heat loss due to radiation is then

$$Q_r = Q_{rt} + Q_{rb}$$

$$Q_r = 159 \text{ BTU/hr.}$$

Finally, the heat transferred to the air is

$$Q_t = Q_c - Q_r$$

$$Q_t = 2261 \text{ BTU/hr}$$

or,

$$q_c(\text{BTU/hr - ft}^2) = \frac{Q_t}{A}$$

$$q_c = 251 \text{ BTU/hr - ft}^2$$

b. Calculation of Modified Nusselt Number

$$Nu^* = \frac{bq_t}{k\theta}$$

where

$$\theta = T_c - T_{oo} - A \frac{L}{2}.$$

A value of k was obtained (12) corresponding to the average air temperature and A was calculated from Fig. IV. Substituting,

$$Nu^* = \frac{0.0625 \times 251}{0.0169 \times (343.0 - 78.5 - 63.3\ 3/2}$$

$$Nu^* = 5.61$$

c. Calculation of Modified Rayleigh Number

$$Ra^* = \frac{Pr\ Gr^*}{\gamma}$$

$$= \frac{\frac{\mu C_p}{k} \times \frac{\rho^2 \beta g A b^4}{\mu^2}}{\gamma}$$

$$= \frac{\rho^2 \beta g C_p A b^4}{\gamma \mu k}$$

A values for the group $\frac{\rho^2 \beta g C_p}{\mu k}$ were obtained (12) for corresponding average air temperatures. γ is relatively unaffected by temperature and was held constant throughout. Substituting

$$Ra^* = 0.89 \times \frac{10^6 \times 1.52}{1.405} \times 10^{-5} \times 63.3$$

$$Ra^* = 610$$

D. LOCATION OF ORIGINAL DATA

The original data may be found in the standard M. I. T. research notebook, pages 1-63, which is now in the possession of the Department of Civil and Sanitary Engineering.

E. NOMENCLATURE

a. General

A	longitudinal gradient of temperature	°F/ft
A_r	area of slab	ft^2
$C_1 C_2 C_3 C_4 C_5$	constants	
C_p	specific heat at constant pressure	BTU/(lb)(°F)
C_v	specific heat at constant temperature	BTU/(lb)(°F)
d,R	thickness of slab	
F	view factor for radiation	
F_i	body force	lb-force
h	coefficient of heat transfer	BTU/(hr)(ft^2)(°F)
k	thermal conductivity of air	BTU/(hr)(ft^2)(°F)/(ft)
k_s	thermal conductivity of slab	BTU/(hr)(ft^2)(°F)/(ft)
L	length of channel or slab	ft
P_L	power lost through apparatus	watts
P_T	total power input to apparatus	watts
q, q_c	rate of heat transfer to air by convection	BTU/(hr)(ft^2)
Q	heating density of slab	BTU/(hr)(ft^2)
Q_c	rate of heat transfer to air including radiation losses	BTU/hr
Q_r	Total rate of heat transfer to air by radiation	BTU/hr
Q_{rb}	rate of heat transfer to air by radiation out of the bottom of the channel	BTU/hr
Q_{rt}	rate of heat transfer to air by radiation out of the top of the channel	BTU/hr

General

Q_t	Rate of heat transfer to air by convection	BTU/hr
T(x,y)	temperature of air	°F
T_b	temperature of bottom of slab	°F
T_c	temperature of center of slab	°F
T_s	temperature of slab	°F
T_t	temperature of top of slab	°F
u,U	velocity of air flow	ft/hr
x,y,z	coordinate axes	

b. Dimensionless Numbers

Gr	Grashof number	$\rho^2\beta g\ \Delta T\ L^3/\mu^2$
Gr*	modified Grashof number	$\rho^2\beta g\ b^4 A/\mu^2$
Nu	Nusselt number	hL/k
Nu*	modified Nusselt number	$bq/k(T_c - T_{oo} - A\frac{L}{2})$
Ra*	modified Rayleigh number	$PrGr^*/\gamma$

c. Greek Letters

β	coefficient of volumetric expansion	$°F^{-1}$
γ	ration of specific heats, C_p/C_v	
ϵ	emissivity of glass	
θ	modified temperature, $T_c - T_{oo} - A\ \frac{L}{2}$	°F
u,η	viscosity of air	lb/(sec)(ft)
ρ,ρ_w	density of air	$lb/(ft^3)$
σ	Stefan-Boltzmann constant	

d. Subscripts

i,j	rectangular Cartesian tensor and summation subscripts

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