Approach a wall, stopping a desired distance $d_i$ in front of it.

What causes these different types of responses? Is there a systematic way to optimize $K$?

Response of system is concisely represented with difference equation.

```
X ----------> Y
```

proportional controller: $v[n] = Ke[n] = K(d_i[n] - d_o[n])$

locomotion: $d_o[n] = d_o[n-1] - Tv[n-1]$

sensor with no delay: $d_i[n] = d_i[n]$

The difference equations provide a concise description of behavior.

$\begin{align*}
    d_i[n] &= d_i[n-1] - Tv[n-1] = d_i[n-1] - TK(d_i[n-1] - d_o[n-1]) \\
    \text{However it provides little insight into how to choose the gain} \ K.
\end{align*}$
**Analysis of wallFinder System: Block Diagram**

A block diagram for this system reveals two feedback paths.

\[ d_i = \text{desiredFront} \quad d_o = \text{distanceFront} \]

proportional controller: \( v[n] = K_e[n] = K (d_i[n] - d_o[n]) \)

locomotion: \( d_o[n] = d_o[n-1] - T v[n-1] \)

sensor with no delay: \( d_s[n] = d_o[n] \)

\[ D_i \quad - \quad K \quad -T \quad + \quad R \quad D_o \]

**Analysis of wallFinder System: System Functions**

Simplify block diagram with \( R \) operator and system functions. Start with accumulator.

\[ D_i \quad + \quad K \quad -T \quad + \quad R \quad D_o \]

What is the input/output relation for an accumulator?

\[ Y = RW = R(X + Y) \]

\[ Y = \frac{R}{1 - R} \]

This is an example of a recurring pattern: Black's equation.

**Black's Equation**

The system function for a feedback system is given by Black's equation.

\[ X \quad + \quad F \quad G \quad Y \]

**Check Yourself**

Determine the system function \( H = \frac{Y}{X} \).

\[ X \quad + \quad F \quad G \quad Y \]

1. \( \frac{F}{1 - FG} \)
2. \( \frac{F}{1 + FG} \)
3. \( F + \frac{1}{1 - G} \)
4. \( F \times \frac{1}{1 - G} \)

**Black's Equation**

Black's equation has two common forms.

\[ X \quad + \quad W \quad F \quad G \quad Y \quad X \quad + \quad W \quad F \quad G \quad Y \]

Difference: equivalent to changing sign of \( G \).

Right form is useful in most control applications where the goal is to make \( Y \) converge to \( X \).

**Analyzing wallFinder: System Functions**

Simplify block diagram with \( R \) operator and system functions.

\[ D_i \quad + \quad K \quad -T \quad + \quad R \quad D_o \]

Replace accumulator with equivalent block diagram.

\[ D_i \quad + \quad K \quad -T \quad + \quad R \quad D_o \]

Now apply Black's equation a second time:

\[ \frac{D_o}{D_i} = \frac{1 - R}{1 + \frac{-KTR}{1 - R}} = \frac{-KTR}{1 - R - KTR} = \frac{-KTR}{1 - (1 + KT)R} \]
Analyzing wallFinder: System Functions

We can represent the entire system with a single system function.

\[ D_i \rightarrow -K \rightarrow T \rightarrow R \rightarrow D_o \]

Replace accumulator with equivalent block diagram.

\[ D_i \rightarrow -K \rightarrow T \rightarrow \frac{R}{1-R} \rightarrow D_o \]

Equivalent system with a single block:

\[ D_i \rightarrow -\frac{KTR}{1-(1+KT)R} \rightarrow D_o \]

Modular! But we still need a way to choose \( K \).

---

Analyzing wallFinder: Poles

The system function contains a single pole at \( z = 1 + KT \).

\[ \frac{D_o}{D_i} = \frac{-KTR}{1-(1+KT)R} \]

The numerator is just a gain and a delay.

The whole system is equivalent to the following:

\[ D_i \rightarrow \frac{1-p_0}{p_0} \rightarrow \frac{R}{1-R} \rightarrow D_o \]

where \( p_0 = 1 + KT \). Here is the unit sample response for \( KT = -0.2 \):

\[ h[n] \]

0.2

0

\( n \)

---

Analyzing wallFinder

We are often interested in the step response of a control system.

\[ d_i = \text{desiredFront} \rightarrow d_o = \text{distanceFront} \]

Start the output \( D_o \) at zero while the input is held constant at one.

---

Step Response

The response of a system (represented by \( H \)) to the unit step signal is equal to the accumulated responses to the unit sample signal.

\[ x[n] = \delta[n] \rightarrow u[n] \rightarrow H \rightarrow y_1[n] = s[n] \]

\[ x[n] = \delta[n] \rightarrow h[n] \rightarrow y_2[n] = s[n] \]

\( y_1[n] = y_2[n] \) because these systems are commutative (provided each starts at rest).

---

Analyzing wallFinder

The step response of the wallFinder system is slow because the unit sample response is slow.

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Analyzing wallFinder

The step response is faster if \( KT = -0.8 \) (i.e., \( p_0 = 0.2 \)).
Analyzing wallFinder: Poles

The poles of the system function provide insight for choosing $K$.

$$\frac{D_0}{D_1} = \frac{-KTR}{1 - (1 + KT)R} = \frac{(1 - p_0)R}{1 - p_0R} : p_0 = 1 + KT$$

- $0 < p_0 < 1$: monotonic converging
- $-1 < KT < 0$: alternating converging
- $-2 < KT < -1$: alternating diverging
- $p_0 < -1$: $KT < -2$

Check Yourself

Find $KT$ for fastest convergence of unit sample response.

1. $KT = -2$
2. $KT = -1$
3. $KT = 0$
4. $KT = 1$
5. $KT = 2$
6. none of the above

Analyzing wallFinder

The optimum gain $K$ moves robot to desired position in one step.

$$d_i = \text{desiredFront} = 1 \text{ m}$$
$$d_o = \text{distanceFront} = 2 \text{ m}$$

$$KT = -1$$

$$K = \frac{1}{T} = -1 \times \frac{1}{10} = -10$$

$$v[n] = K(d_i[n] - d_o[n]) = -10(1 - 2) = 10 \text{ m/s}$$

exactly the right speed to get there in one step!

Analyzing wallFinder: Space-Time Diagram

The optimum gain $K$ moves robot to desired position in one step.

Analysis of wallFinder System: Adding Sensory Delay

Adding delay tends to destabilize control systems.


Locomotion: $d_o[n] = d_o[n - 1] - Tv[n - 1]$

Sensor with delay: $d_s[n] = d_o[n - 1]$
Analysis of wallFinder System: Block Diagram

Incorporating sensor delay in block diagram.

proportional controller: \( v[n] = K e[n] = K (d_i[n] - d_s[n]) \)
locomotion: \( d_o[n] = d_o[n - 1] - T v[n - 1] \)
sensor with no delay: \( d_s[n] = d_o[n - 1] \)

Check Yourself

Find the system function \( H = \frac{D_o}{D_i} \).

1. \( \frac{KTR}{1 - R} \)
2. \( \frac{-KTR}{1 + R - KTR^2} \)
3. \( \frac{KTR}{1 - R} - KTR \)
4. \( \frac{-KTR}{1 - R - KTR^2} \)
5. none of the above

Analyzing wallFinder: System Functions

We can represent the entire system with a single system function.

Analyzing wallFinder: Poles

Substitute \( R = \frac{1}{2} \) in the system functional to find the poles.

The poles are then the roots of the denominator.

\[
z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT}
\]

Feedback and Control: Poles

If \( KT \) is small, the poles are at \( z \approx 0 \) and \( z \approx 1 \).

\[
z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2 + KT} \approx \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2} = 0.1
\]

Pole near 0 generates fast response.
Pole near 1 generates slow response.
Slow mode (pole near 1) dominates the response.
**Feedback and Control: Poles**

As $KT$ becomes more negative, the poles move toward each other and collide at $z = \frac{1}{2}$ when $KT = -\frac{1}{4}$.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2} + KT = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2} - \frac{1}{4} = \frac{1}{2}$$

Persistent responses decay. The system is stable.

---

**Feedback and Control: Poles**

If $KT < -\frac{1}{4}$, the poles are complex.

$$z = \frac{1}{2} \pm \sqrt{\left(\frac{1}{2}\right)^2} + KT = \frac{1}{2} \pm j\sqrt{-KT - \left(\frac{1}{2}\right)^2}$$

Complex poles → oscillations.

---

**Check Yourself**

What is the period of the oscillation?

1. 1 2. 2 3. 3 4. 4 5. 6 0. none of above

---

**Feedback and Control: Poles**

The closed loop poles depend on the gain.

If $KT: 0 \to -\infty$: then $z_1, z_2: 0, 1 \to \frac{1}{2}, \frac{1}{2} \to \frac{1}{2} \pm j\infty$

---

**Check Yourself**

Find $KT$ for fastest response.

1. 0 2. $-\frac{1}{4}$ 3. $-\frac{1}{4}$ 4. $-1$ 5. $-\infty$ 0. none of above
Destabilizing Effect of Delay

Adding delay in the feedback loop makes it more difficult to stabilize.

Ideal sensor: \( d_s[n] = d_o[n] \)

More realistic sensor (with delay): \( d_s[n] = d_o[n - 1] \)

Fastest response without delay: single pole at \( z = 0 \).
Fastest response with delay: double pole at \( z = \frac{1}{2} \) much slower!

Designing Control Systems: Summary

System Functions provide a convenient summary of information that is important for designing control systems.

The long-term response of a system is determined by its dominant pole — i.e., the pole with the largest magnitude.

A system is unstable if the magnitude of its dominant pole is > 1.
A system is stable if the magnitude of its dominant pole is < 1.

Delays tend to decrease the stability of a feedback system.

Check Yourself

How many of the following statements are true?

1. This system has 3 poles.
2. Unit sample response is the sum of 3 geometric sequences.
3. Unit-sample response is \( y[n] : 0, 0, 0, 1, 0, 0, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots \)
4. Unit-sample response is \( y[n] : 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1, 0, 0, 1 \ldots \)
5. One of the poles is at \( z = 1 \).