TUBE WAVES, SEISMIC WAVES AND EFFECTIVE SOURCES

by

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ABSTRACT

A simple asymptotic analysis, based on the smallness of the ratio of the borehole radius to the wavelength, reveals the interaction between tube waves and seismic waves. The pressure field in a tube wave acts as a secondary source of seismic waves and conversely an incoming seismic wave excites a tube wave. The asymptotic analysis leads to a characterization of these sources in terms of the solution to two-dimensional elastostatic problems. These may be solved exactly when the borehole has an elliptical cross-section even in an anisotropic formation. Also the borehole need not be straight provided that its radius of curvature is large compared with a wavelength.

INTRODUCTION

The Aim of This Paper

This paper is concerned with acoustics in and around a narrow fluid-filled borehole in an anisotropic or isotropic solid. An important wave mode here is the tube wave, which is primarily a longitudinal vibration of the fluid. It propagates with a speed depending upon the properties of the fluid and, because of some compliance of the borehole wall, the geometry of the hole and the properties of the solid. If the solid were perfectly rigid the tube wave speed would be the characteristic acoustic speed in the fluid.

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Because of the compliance of the borehole wall a tube wave may act as a source of seismic waves in the surrounding solid. In a fast formation, where the tube wave is slower than the S wave in the solid, a steadily moving tube wave produces a disturbance which is confined to the neighborhood of the hole, but seismic waves are excited where the tube wave behaves discontinuously, such as near the source and as the wave passes through significant interfaces. In a slow formation, where the fluid wave speed is faster than the S wave in the solid, in addition, a propagating tube wave will continuously shed a conical shear wave as it travels.

Recently there has been much interest in cross-hole tomography in which sources are placed in one borehole and receivers in another. One may think of the sources acting on the solid only through a tube wave as an intermediary, and reciprocally, the receivers reponding to the seismic wave only through intermediate tube waves excited in the second well. Thus it is important to understand the interaction between the tube wave and the seismic wave. Moreover, the recent emphasis on anisotropy requires that the solid not be restricted to isotropy. Also, the existence of horizontal and deviated wells means that we cannot assume that the borehole has any particularly symmetrical orientation with respect to the symmetry planes of the anisotropic medium or that the borehole has a circular cross-section, especially along highly curved sections.

Most previous related work has been confined to circular holes in isotropic solids because wave problems in such a structure can be solved exactly in terms (of infinite sums and integrals) of Bessel functions and exponentials. See, for instance, Lee and Balch (1982), where the authors obtain an exact solution and then consider the low-frequency régime, where the borehole radius is much less than a wavelength.

In view of the difficulty (perhaps impossibility) of solving analytically any but the most symmetrical dynamical problems, and the difficulty at this time of obtaining three-dimensional numerical solutions, we have developed a method of obtaining the low-frequency (or narrow-borehole) approximation directly without first obtaining an exact solution. The method consists of introducing the ratio of borehole radius to wavelength ϵ as a small parameter and then obtaining an asymptotic solution in ascending powers of ϵ. In this way we obtain a sequence of problems which are solvable.

The first problem is a two-dimensional static elastic problem for the inflation of a hole in an arbitrary anisotropic solid. The relevant theory was developed to study stress concentrations around holes in plates under tension. See Lekhnitskii (1963) and Savin (1961). In those works the solution is obtained in terms of stress functions. For variety, in this paper, we present the theory in terms of displacement, since this relates more closely to the physics under consideration. The next problem arising in the asymptotic approach involves a one-dimensional hyperbolic system of equations for pressure and longitudinal particle velocity in the fluid. The coefficients in this system involve the solution to the static elastic problem just mentioned.

The next and final step is to find a source of seismic waves in the solid, which is equivalent to the traveling tube wave. That is, we replace the solid with a hole by an intact solid and construct a moving system of body-force dipoles concentrated along the location of the centerline of the hole, which generates the same seismic radiation as the propagating tube wave in the actual hole. As final generalizations we allow the cross-section of the borehole to be elliptical and allow the borehole centerline to be curved, provided that the radius of curvature is long compared with a wavelength. We assume the ratio is $O(1/\epsilon)$.

Other Previous Work

White and Sengbush (1963) computed the low-frequency far-field radiation from a point source in a fluid-filled borehole by integrating the contribution from the pressure wave propagating along the borehole at the tube wave speed. In doing this they used a result obtained by Heelan (1953) who computed the far-field low-frequency displacements due to a transient pressure applied to a short length of an empty cylindrical borehole. The far-field low-frequency displacements thus obtained are a generalization of the results obtained by Lee and Balch (1982) who derived them by a stationary phase approximation to the exact solution for a point source in a fluid-filled borehole. White and Sengbush (1963) addressed also the case where the shear wave speed of the formation was slower than the tube wave speed.

Based on the results of Lee and Balch (1982), Ben-Menahem and Kostek (1991) recognized that under certain circumstances the far-field low-frequency radiation from a point source in a fluid-filled borehole was equivalent to that generated by a suitable combination of a monopole source and a vertical dipole source localized in the formation in the absence of the borehole. Kurkjian et al. (1992) showed that the exact radiation pattern was obtained if the mechanism above was allowed to move up and down the borehole at the tube wave speed.

Outline of the Paper

In Section 2 we define borehole-centered coordinates, which are a curvilinear orthogonal system in which one coordinate is arc-length s along the centerline of the hole and the other two are cartesian coordinates in the plane perpendicular to the borehole at the point specified by s. As s varies this system does not rotate around the centerline. The system is singular at the center of curvature of the centerline. The acoustic equations in the fluid, the elastodynamic equations in the solid, the appropriate interface conditions on the surface of the borehole, and the conditions at infinity are stated first in cartesian coordinates and then in the borehole centered coordinates.

In Section 3 we introduce the small parameter ϵ, assuming the radius of the hole is $O(1)$, the wavelength is $O(1/\epsilon)$, and the radius of curvature of the borehole centerline is $O(1/\epsilon^2)$. We then specify that the solution should depend on time t and s only through $T = \epsilon t$ and $S = \epsilon s$ and assume an asymptotic power series in ϵ for each dependent variable. The leading-order equations tell us that to this order the fluid motion is longitudinal (along the well) and that the displacement in the solid is related to the pressure in the fluid by a two-dimensional elastostatic problem. The equations involving terms of the next order yield the one-dimensional acoustic system for pressure and longitudinal particle velocity in the fluid, showing how the coefficients depend upon the solution of the elastostatic problem which arose at the leading order. This one-dimensional system is then solved for a volume-injection source concentrated at a point in the borehole.

In Section 4 we derive the distribution of body force which, when acting in the intact solid, would produce the same radiation as the tube wave in the borehole. This turns out to be a line distribution of dipoles, concentrated along the borehole centerline, which depends upon linear operators defined by the elastostatic problem. In Section 5 we use a complex variable technique to solve that elastostatic problem and so obtain the linear operators which occur in the expression for the body force distribution. It is interesting that the linear operator involves the product of three matrices, one depends upon the elastic constants of the solid, one on the fluid pressure and any incident stress field, and the third on the profile of the borehole cross-section. The calculations are analytical except that a sextic equation has to be solved for an eigensystem.

In Section 6 the configuration is specialized so that a plane perpendicular to the borehole axis is a plane of symmetry for the anisotropic medium. When this is so the 3×3 elastostatic problem of Section 4 decouples into a 2×2 and a 1×1 system which may be solved analytically. Further specialization to transverse isotropy with axis parallel or perpendicular to the borehole, and finally to isotropy, enables one to calculate analytically the operators giving the coefficients in the one-dimensional acoustic system and the body-force distribution.

In Section 7 we quote a form for the far-field Green's function in an anisotropic medium and then combine it with the body force distribution to calculate the far field. Results are obtained for situations where the tube wave is either faster or slower than the quasi-shear waves. The case of two boreholes with a source in one well and a receiver in the other well is also analysed. The solution is obtained in a form which clearly exhibits reciprocity.

In Section 8 the far field is specialized first to a circular borehole in an isotropic medium, and shown to be identical to those of Lee and Balch (1982) and Ben-Menahem and Kostek (1991). Then, the nontrivial case of a borehole in a transversely isotropic medium is considered, where we show the radiation patterns and wavefront surfaces for various situations involving media with symmetry axis parallel or perpendicular to the borehole axis, and also with triplication of the quasi-shear wavefront.

EQUATIONS OF MOTION IN BOREHOLE-CENTERED COORDINATES

Borehole-Centered Coordinates

Let $\mathbf{x} = (x_1, x_2, x_3)$ be cartesian coordinates and let

$$\mathbf{x} = \mathbf{X}(s) \tag{1}$$

be the equation of the borehole axis parametrized by arclength s. Then

$$\mathbf{e}_3(s) = \mathbf{X}'(s) \tag{2}$$

is a unit tangent vector to this curve. We complete $\mathbf{e}_3$ into an orthonormal triple $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ by requiring that as s varies, this frame does not rotate about $\mathbf{e}_3(s)$; that is, the component of its angular velocity about $\mathbf{e}_3(s)$ is zero. To do this we regard $\mathbf{e}_3(s)$ and $\mathbf{e}_3'(s)$ as given and choose $\mathbf{e}_1(s)$, $\mathbf{e}_2(s)$ so that

$$\begin{aligned} \mathbf{e}_1'(s) &= -[\mathbf{e}_3'(s)\cdot\mathbf{e}_1(s)]\mathbf{e}_3(s), \\ \mathbf{e}_2'(s) &= -[\mathbf{e}_3'(s)\cdot\mathbf{e}_2(s)]\mathbf{e}_3(s). \end{aligned} \tag{3}$$

Then

$$\begin{aligned} \mathbf{e}_1'(s) &= -\alpha_1\mathbf{e}_3(s), \\ \mathbf{e}_2'(s) &= -\alpha_2\mathbf{e}_3(s), \\ \mathbf{e}_3'(s) &= \alpha_1\mathbf{e}_1(s) + \alpha_2\mathbf{e}_2(s), \end{aligned} \tag{4}$$

where

$$\alpha_1 = \mathbf{e}_3'(s)\cdot\mathbf{e}_1(s), \quad \alpha_2 = \mathbf{e}_3'(s)\cdot\mathbf{e}_2(s). \tag{5}$$

We set

$$\mathbf{x} = \mathbf{X}(s) + q_1\mathbf{e}_1(s) + q_2\mathbf{e}_2(s). \tag{6}$$

Then q_1, q_2, $s = q_3$ are borehole-centered coordinates.

$$\begin{aligned} \mathrm{d}\mathbf{x} &= [\mathbf{X}'(s) + q_1\mathbf{e}_1'(s) + q_2\mathbf{e}_2'(s)]\,\mathrm{d}s + \mathrm{d}q_1\mathbf{e}_1(s) + \mathrm{d}q_2\mathbf{e}_2(s) \\ &= \mathrm{d}q_1\mathbf{e}_1 + \mathrm{d}q_2\mathbf{e}_2 + (1 - \alpha_1 q_1 - \alpha_2 q_2)\,\mathrm{d}q_3\mathbf{e}_3, \end{aligned} \tag{7}$$

and so

$$|\,\mathrm{d}\mathbf{x}|^2 = \mathrm{d}q_1^2 + \mathrm{d}q_2^2 + h^2\,\mathrm{d}q_3^2, \tag{8}$$

where

$$h(q_1, q_2, q_3) = 1 - \alpha_1(q_3)q_1 - \alpha_2(q_3)q_2. \tag{9}$$

Thus (q_1, q_2, q_3) form an orthogonal curvilinear coordinate system. We shall next state the equations of motion in cartesian coordinates and then rewrite them in borehole-centered coordinates.

The Equations of Motion in Cartesian Coordinates

The momentum and constitutive equations in the borehole fluid are

$$\rho_f \mathbf{v}_{,t} + \nabla p = \mathbf{0}, \tag{10}$$

$$\sigma p_{,t} + \nabla \cdot \mathbf{v} = G_{,t}. \tag{11}$$

Here ρ_f and σ are the density and bulk compliance of the fluid, p and $\mathbf{v}$ are the pressure and velocity, respectively, and G is a source term representing the production of volume. Let ρ be the density, $\mathbf{u}$ the particle displacement, $\mathbf{w} = \mathbf{u}_{,t}$ the particle velocity, and $\boldsymbol{\tau}$ the stress in the solid. Then the momentum and constitutive equations in the solid are

$$\rho \mathbf{w}_{,t} - \nabla \cdot \boldsymbol{\tau} = \mathbf{0}, \tag{12}$$

$$\boldsymbol{\tau} = \mathbf{c} : \nabla \mathbf{u}, \tag{13}$$

where $\mathbf{c}$ is the fourth rank tensor of elastic constants (stiffnesses). Equation (13) may be differentiated with respect to t to obtain

$$\boldsymbol{\tau}_{,t} = \mathbf{c} : \nabla \mathbf{w}, \tag{14}$$

Together with (12) they form the elastodynamic equations as a first-order hyperbolic system.

On the interface between the fluid and the solid (the borehole wall) the normal particle velocity and traction are continuous:

$$\mathbf{n} \cdot (\mathbf{w} + \mathbf{w}^I) = \mathbf{n} \cdot \mathbf{v}, \tag{15}$$

$$(\boldsymbol{\tau} + \boldsymbol{\tau}^I) \cdot \mathbf{n} = -p\mathbf{n}. \tag{16}$$

Here $\mathbf{n}$ is the unit normal to the interface pointing into the solid; $\mathbf{w}^I$ and $\boldsymbol{\tau}^I$ are the particle velocity and stress fields due to an incident wave. They satisfy the homogeneous elastodynamic equations outside the borehole and enter our equations only through the interface conditions (15) and (16). To complete the specification of the problem we require the boundary condition at infinity that $\mathbf{u}$, $\mathbf{w}$,and $\boldsymbol{\tau}$ tend to zero as the field point recedes from the borehole.

The Equations of Motion in Borehole-Centered Coordinates

Using the formulae presented above we rewrite the equations of motion as follows. Equation (10) becomes

$$\begin{aligned} \rho_f v_{1,t} + p_{,1} &= 0, \\ \rho_f v_{2,t} + p_{,2} &= 0, \\ \rho_f v_{3,t} + h^{-1} p_{,3} &= 0. \end{aligned} \tag{17}$$

Equation (11) becomes

$$\sigma p_{,t} + v_{1,1} + v_{2,2} + h^{-1}(v_{3,3} - \alpha_1 v_1 - \alpha_2 v_2) = G_{,t}. \tag{18}$$

The interface equations (15) and (16) become

$$n_k(w_k + w_k^I) = n_k v_k, \tag{19}$$

and

$$(\tau_{ij} + \tau_{ij}^I)n_j = -pn_i. \tag{20}$$

The boundary condition at infinity becomes

$$\mathbf{u},\ \mathbf{w},\ \tau \to 0 \ \text{ as } \ \sqrt{q_1^2 + q_2^2} \to \infty. \tag{21}$$

The momentum equation in the solid becomes

$$\begin{aligned}
\rho w_{1,t} &= \tau_{11,1} + \tau_{21,2} + h^{-1}\tau_{31,3} + h^{-1}(-\alpha_1\tau_{11} - \alpha_2\tau_{21} + \alpha_1\tau_{33}), \\
\rho w_{2,t} &= \tau_{12,1} + \tau_{22,2} + h^{-1}\tau_{32,3} + h^{-1}(-\alpha_1\tau_{12} - \alpha_2\tau_{22} + \alpha_2\tau_{33}), \\
\rho w_{3,t} &= \tau_{13,1} + \tau_{23,2} + h^{-1}\tau_{33,3} + 2h^{-1}(-\alpha_1\tau_{13} - \alpha_2\tau_{23}).
\end{aligned} \tag{22}$$

The differentiated constitutive law becomes

$$\tau_{,t} = \mathbf{c} : \nabla\mathbf{w}, \tag{23}$$

where

$$\nabla\mathbf{w} = \begin{pmatrix} w_{1,1} & w_{1,2} & h^{-1}(w_{1,3} + \alpha_1 w_3) \\ w_{2,1} & w_{2,2} & h^{-1}(w_{2,3} + \alpha_2 w_3) \\ w_{3,1} & w_{3,2} & h^{-1}(w_{3,3} - \alpha_1 w_1 - \alpha_2 w_2) \end{pmatrix}. \tag{24}$$

Substituting (24) into (23) and writing the result in full, one obtains

$$\begin{aligned}
\tau_{ij,t} = {} & c_{ijk\delta} w_{k,\delta} + h^{-1} c_{ij\gamma 3}(w_{\gamma,3} + \alpha_\gamma w_3) \\
& + h^{-1} c_{ij33}(w_{3,3} - \alpha_1 w_1 - \alpha_2 w_2),
\end{aligned} \tag{25}$$

where the repeated Greek subscripts are summed over the values 1,2.

ASYMPTOTIC ANALYSIS

The Small Parameter ϵ

In this section we shall make some assumptions about the relative sizes of the borehole radius, the wavelength, the radius of curvature of the centerline of the borehole, and the length scale on which the material properties vary, in terms of a small parameter ϵ. We shall assume that, in the units of the coordinates q_i, the borehole radius is $O(1)$ as

$\epsilon \to 0$. We shall assume that the wavelength is $O(1/\epsilon)$. The scale on which the medium properties vary and the radius of curvature are both $O(1/\epsilon^2)$. Under these assumptions α_1 and α_2 are $O(\epsilon^2)$. So, let us write

$$\alpha_i = \epsilon^2\beta_i, \quad \beta_i = O(1), \tag{26}$$

and

$$h^{-1} = (1-\epsilon^2\beta_1 q_1 - \epsilon^2\beta_2 q_2)^{-1} = 1 + O(\epsilon^2). \tag{27}$$

Let us introduce new "slow" variables T and S defined by

$$T = \epsilon t, \quad S = \epsilon s. \tag{28}$$

Then

$$\partial_t = \epsilon\partial_T, \quad \partial_s = \epsilon\partial_S. \tag{29}$$

We shall suppose that all quantities depend upon t and s only though T and S. Thus we may rewrite (17) to (20) as

$$\begin{aligned} \epsilon\rho_f v_{\alpha,T} + p_{,\alpha} &= 0, \\ \rho_f v_{3,T} + p_{,S} &= O(\epsilon^2). \end{aligned} \tag{30}$$

$$\epsilon\sigma p_{,T} + \epsilon v_{3,S} + v_{1,1} + v_{2,2} - \epsilon G_{,T} = O(\epsilon^2), \tag{31}$$

$$n_\alpha(w_\alpha + w_\alpha^I) + \epsilon m_3(w_3 + w_3^I) = n_\alpha v_\alpha + \epsilon m_3 v_3, \tag{32}$$

$$\begin{aligned} (\tau_{\alpha\beta} + \tau_{\alpha\beta}^I)n_\beta + \epsilon m_3(\tau_{\alpha 3} + \tau_{\alpha 3}^I) &= -p n_\alpha, \\ (\tau_{3\beta} + \tau_{3\beta}^I)n_\beta + \epsilon m_3\tau_{33} &= -\epsilon p m_3, \end{aligned} \tag{33}$$

where the Greek subscripts range over 1,2 only. Because the equation of the borehole wall depends upon q_3 only through S, we have written

$$n_3 = \epsilon^2 m_3. \tag{34}$$

Next (22) becomes

$$\begin{aligned} \epsilon\rho w_{1,T} &= \tau_{11,1} + \tau_{21,2} + \epsilon\tau_{31,S} - \epsilon^2\beta_\alpha\tau_{\alpha 1} + O(\epsilon^3), \\ \epsilon\rho w_{2,T} &= \tau_{12,1} + \tau_{22,2} + \epsilon\tau_{32,S} - \epsilon^2\beta_\alpha\tau_{\alpha 2} + O(\epsilon^3), \\ \epsilon\rho w_{3,T} &= \tau_{13,1} + \tau_{23,2} + \epsilon\tau_{33,S} - 2\epsilon^2\beta_\alpha\tau_{\alpha 3} + O(\epsilon^3), \end{aligned} \tag{35}$$

and (23) and (24) become

$$\tau_{,t} = \mathbf{c} : \nabla\mathbf{w}, \tag{36}$$

and

$$\nabla\mathbf{w} = \begin{pmatrix} w_{1,1} & w_{1,2} & w_{1,S} + \epsilon^2\beta_1 w_3 \\ w_{2,1} & w_{2,2} & w_{2,S} + \epsilon^2\beta_2 w_3 \\ w_{3,1} & w_{3,2} & w_{3,S} - \epsilon^2\beta_1 w_1 - \epsilon^2\beta_2 w_2 \end{pmatrix} + O(\epsilon^3). \tag{37}$$

Then (25) yields

$$\begin{aligned}\epsilon\tau_{ij,T} &= c_{ijk\delta}w_{k,\delta} + \epsilon c_{ijk3}w_{k,S} \\ &\quad + \epsilon^2 c_{ij\gamma 3}\beta_\gamma w_3 - \epsilon^2 c_{ij33}\beta_\alpha w_\alpha + O(\epsilon^3).\end{aligned} \tag{38}$$

We shall assume that the field quantities can be expanded in powers of ϵ. However, they do not all start at the same power. Thus let

$$\begin{aligned}\mathbf{v} &= \mathbf{v}^{(0)} + \epsilon\mathbf{v}^{(1)} + O(\epsilon^2), \\ p &= p^{(0)} + \epsilon p^{(1)} + O(\epsilon^2), \\ \mathbf{u} &= \mathbf{u}^{(0)} + \epsilon\mathbf{u}^{(1)} + O(\epsilon^2), \\ \mathbf{w} &= \epsilon\mathbf{w}^{(0)} + \epsilon^2\mathbf{w}^{(1)} + O(\epsilon^3), \\ \tau &= \tau^{(0)} + \epsilon\tau^{(1)} + O(\epsilon^2), \\ \mathbf{u}^I &= \mathbf{u}^{I(0)} + \epsilon\mathbf{u}^{I(1)} + O(\epsilon^2), \\ \mathbf{w}^I &= \epsilon\mathbf{w}^{I(0)} + \epsilon^2\mathbf{w}^{I(1)} + O(\epsilon^3), \\ \tau^I &= \tau^{I(0)} + \epsilon\tau^{I(1)} + O(\epsilon^2).\end{aligned} \tag{39}$$

Also

$$\mathbf{u}, \mathbf{w}, \tau \to 0 \text{ as } \sqrt{q_1^2 + q_2^2} \to \infty. \tag{40}$$

The Leading Terms in ϵ and the Elastostatic Problem

We next write (30) to (38) to leading order in ϵ, using (39):

$$\begin{aligned}p^{(0)}_{,\alpha} &= 0, \\ \rho_f v^{(0)}_{3,T} + p^{(0)}_{,S} &= 0.\end{aligned} \tag{41}$$

$$v^{(0)}_{\alpha,\alpha} = 0. \tag{42}$$

$$0 = n_\alpha v^{(0)}_\alpha. \tag{43}$$

$$\begin{aligned}(\tau^{(0)}_{\alpha\beta} + \tau^{I(0)}_{\alpha\beta})n_\beta &= -p^{(0)}n_\alpha, \\ (\tau^{(0)}_{3\beta} + \tau^{I(0)}_{3\beta})n_\beta &= 0.\end{aligned} \tag{44}$$

$$0 = \tau^{(0)}_{\alpha k,\alpha}. \tag{45}$$

$$\tau^{(0)}_{ij,T} = c_{ijk\gamma}w^{(0)}_{k,\gamma}, \tag{46}$$

which may be integrated in T to get

$$\tau^{(0)}_{ij} = c_{ijk\gamma}u^{(0)}_{k,\gamma}. \tag{47}$$

The boundary condition gives

$$\mathbf{u}^{(0)} \to 0 \text{ as } \sqrt{q_1^2 + q_2^2} \to \infty. \tag{48}$$

Equation (41) implies that $p^{(0)}$ is independent of q_1 and q_2. Thus

$$p^{(0)} = p^{(0)}(S,T). \tag{49}$$

Assuming irrotational fluid motion,

$$\nabla \times \mathbf{v} = \mathbf{0}, \tag{50}$$

equations (42) and (43) imply that

$$v_1^{(0)} = 0, \quad v_2^{(0)} = 0. \tag{51}$$

Equations (50) and (51) imply that

$$v_{3,\alpha}^{(0)} = 0, \tag{52}$$

so that

$$v_3^{(0)} = v_3^{(0)}(S,T). \tag{53}$$

Equations (44), (45), (47), and (48) form a two-dimensional elastostatic problem for $\mathrm{u}^{(0)}$ and $\tau^{(0)}$ with forcing terms $\tau^{I(0)}$ and $p^{(0)}$ in (44). Thus

$$\mathbf{u}^{(0)} \quad \text{and} \quad \mathbf{w}^{(0)} \quad \text{depend linearly upon} \quad \tau_T^{I(0)} + p_T^{(0)}\mathbf{I}. \tag{54}$$

Terms of the Next Order in ϵ

We proceed to the next order in ϵ in (31) and (32), making use of (39),

$$\sigma p_{,T}^{(0)} + v_{3,S}^{(0)} + v_{1,1}^{(1)} + v_{2,2}^{(1)} - G_{,T} = 0. \tag{55}$$

$$n_\alpha(w_\alpha^{(0)} + w_\alpha^{I(0)}) = n_\alpha v_\alpha^{(1)}. \tag{56}$$

Integrating (55) across the borehole, and using the two-dimensional divergence theorem with (56), we get

$$\begin{aligned} \sigma A p_{,T}^{(0)} + A v_{3,S}^{(0)} - \int_\Sigma G_{,T}\,\mathrm{d}A &= -\int_{\partial\Sigma} n_\alpha v_\alpha^{(1)}\,\mathrm{d}s \\ &= -\int_{\partial\Sigma} n_\alpha(w_\alpha^{(0)} + w_\alpha^{I(0)})\,\mathrm{d}s. \end{aligned} \tag{57}$$

It follows from (54) that $\int_{\partial\Sigma} n_\alpha w_i^{(0)}\,\mathrm{d}s$ depends linearly upon $\tau_{,T}^{I(0)} + p_{,T}^{(0)}\mathbf{I}$. Thus

$$\int_{\partial\Sigma} n_\alpha u_i^{(0)}\,\mathrm{d}s = A N_{i\alpha pq}(\tau_{pq}^{I(0)} + p^{(0)}\delta_{pq}), \tag{58}$$

say. Then

$$\int_{\partial\Sigma} n_\alpha u_\alpha^{(0)}\,\mathrm{d}s = AN(\boldsymbol{\tau}^{I(0)} + p^{(0)}\mathbf{I}), \tag{59}$$

where $N(\boldsymbol{\tau}) = N_{\alpha\alpha pq}\tau_{pq}$. Also differentiating (59) with respect to T gives

$$\int_{\partial\Sigma} n_\alpha w_\alpha^{(0)}\,\mathrm{d}s = AN(\boldsymbol{\tau}_{,T}^{I(0)} + p_{,T}^{(0)}\mathbf{I}). \tag{60}$$

In (57) to (60) Σ is the right cross-section of the borehole at $\mathbf{X}(S)$, A is its area, and $\partial\Sigma$ its boundary. Hence we have

$$(\sigma + N(\mathbf{I}))p_{,T}^{(0)} + v_{3,S}^{(0)} = \frac{1}{A}\int_\Sigma G_{,T}\,\mathrm{d}A - \frac{1}{A}\int_\Sigma w_{\alpha,\alpha}^{I(0)}\,\mathrm{d}A - N(\boldsymbol{\tau}_{,T}^{I(0)}). \tag{61}$$

Because $\tau_{pq}^{I(0)} = c_{pqij}u_{i,j}^{I(0)}$ we may write (61) as

$$(\sigma + N(\mathbf{I}))p_{,T}^{(0)} + v_{3,S}^{(0)} = \frac{1}{A}\int_\Sigma G_{,T}\,\mathrm{d}A - [\delta_{ij} - t_it_j + (\delta_{mn} - t_mt_n)N_{mnpq}c_{pqij}]u_{i,jT}^{I(0)}. \tag{62}$$

The One-Dimensional Acoustic System in the Fluid

Equation (61) together with the third equation of (41):

$$\rho_f v_{3,T}^{(0)} + p_{,S}^{(0)} = 0, \tag{63}$$

form a one-dimensional hyperbolic system for $v_3^{(0)}$ and $p^{(0)}$. We should remember that $\mathbf{v}^{(0)}$, $p^{(0)}$, $\mathbf{w}^{(0)}$, $\boldsymbol{\tau}^{(0)}$ are related to $\mathbf{v}$, p, $\mathbf{w}$, $\boldsymbol{\tau}$ by (39), and that these are evaluated in the coordinates q_1, q_2, q_3, which have the same physical units as x_1, x_2, x_3. Transforming (61) and (63) back to the variables s, t using (28) and (29) we get

$$(\sigma + N(\mathbf{I}))p_{,t}^{(0)} + v_{3,s}^{(0)} = \frac{1}{A}\int_\Sigma G_{,t}\,\mathrm{d}A - \frac{1}{A}\int_\Sigma u_{\alpha,\alpha t}^{I(0)}\,\mathrm{d}A - N(\boldsymbol{\tau}_{,t}^{I(0)}), \tag{64}$$

$$\rho_f v_{3,t}^{(0)} + p_{,s}^{(0)} = 0. \tag{65}$$

The One-Dimensional Acoustic Solution

We restate the system (64), (65):

$$\rho_f v_{3,t}^{(0)} + p_{,s}^{(0)} = 0, \tag{66}$$

$$(\sigma + N(\mathbf{I}))p_{,t}^{(0)} + v_{3,s}^{(0)} = \frac{1}{A}\int G_{,t}\,\mathrm{d}A = \frac{V_0}{A}g'(t)\delta(s), \tag{67}$$

specializing to a source concentrated at the origin and supplying accumulated volume $V_0 g(t)$ up to time t, and assuming that there is no incident stress field: $\tau^I(0) = \mathbf{0}$. If the coefficients vary slowly, as we assume, we may solve (66), (67) by the method of generalized progressing waves and so we assume a solution in the form

$$\begin{pmatrix} v_3^{(0)} \\ \\ p^{(0)} \end{pmatrix} = \sum_{n=0}^{\infty} \begin{pmatrix} \bar{v}_n(s) \\ \\ \bar{p}_n(s) \end{pmatrix} f_n(t - \tau(s)) \tag{68}$$

away from the source. Here each f_n is one integration step smoother than the previous one:

$$f_n = f'_{n+1}, \tag{69}$$

and travel time $\tau(s)$, and coefficients $\bar{v}_n$, and $\bar{p}_n$ are to be determined.

On substituting (68) into (66), (67), and then equating the coefficients of the successive f_n to zero, we get for $n = -1$

$$\begin{pmatrix} \rho_f & -\tau' \\ \\ -\tau' & \sigma + N(\mathrm{I}) \end{pmatrix} \begin{pmatrix} \bar{v}_0 \\ \\ \bar{p}_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}, \tag{70}$$

and for $n = 0$

$$\begin{pmatrix} \rho_f & -\tau' \\ \\ -\tau' & \sigma + N(\mathrm{I}) \end{pmatrix} \begin{pmatrix} \bar{v}_1 \\ \\ \bar{p}_1 \end{pmatrix} + \begin{pmatrix} \bar{p}'_0 \\ \\ \bar{v}'_0 \end{pmatrix} = \begin{pmatrix} 0 \\ \\ 0 \end{pmatrix}. \tag{71}$$

Equation (70) implies

$$\tau' = \pm\gamma_T, \tag{72}$$

where γ_T is the tube wave slowness given by

$$\gamma_T = \sqrt{\rho_f(\sigma + N(\mathrm{I}))}. \tag{73}$$

Then

$$\frac{\bar{p}_0}{\bar{v}_0} = \pm\zeta, \tag{74}$$

where ζ is the characteristic impedance given by

$$\zeta = \sqrt{\frac{\rho_f}{\sigma + N(\mathrm{I})}}. \tag{75}$$

So we may write

$$\begin{pmatrix} \bar{v}_0 \\ \\ \bar{p}_0 \end{pmatrix} = a_0(s) \begin{pmatrix} \pm\zeta^{-\frac{1}{2}} \\ \\ \zeta^{\frac{1}{2}} \end{pmatrix}, \tag{76}$$

where a_0 is a coefficient to be determined. Using (76) in (71) and multiplying on the left by $(\pm\zeta^{-\frac{1}{2}} \quad \zeta^{\frac{1}{2}})$ we find that

$$a_0'(s) = 0. \tag{77}$$

Thus, to a first approximation,

$$\begin{pmatrix} v_3^{(0)} \\ \\ p^{(0)} \end{pmatrix} = a_0^{\pm} \begin{pmatrix} \pm\zeta^{-\frac{1}{2}} \\ \\ \zeta^{\frac{1}{2}} \end{pmatrix} f_0^{\pm}[t - \tau_{\pm}(s)], \tag{78}$$

where $a_0^{\pm}$ are constants and

$$\tau_{\pm}' = \pm\gamma_T. \tag{79}$$

We may now determine $a_0^{\pm} f_0^{\pm}$ by expressing (66) and (67) near the source as jump conditions:

$$\begin{aligned} p^{(0)}(+0,t) - p^{(0)}(-0,t) &= 0, \\ v_3^{(0)}(+0,t) - v_3^{(0)}(-0,t) &= \frac{V_0}{A} g'(t). \end{aligned} \tag{80}$$

Since causality requires the waves to propagate away from the source, we shall assume that the + or − sign in (78) is the same as the sign of s. Using (78) in (80) we see that

$$\begin{aligned} a_0^+ f_0^+(t) - a_0^- f_0^-(t) &= 0, \\ a_0^+ f_0^+(t) - a_0^+ f_0^-(t) &= \frac{V_0 \zeta^{\frac{1}{2}}(0) g'(t)}{A(0)}, \end{aligned} \tag{81}$$

so that

$$a_0^+ f^+(t) = a_0^- f^-(t) = \frac{V_0 \zeta^{\frac{1}{2}}(0) g'(t)}{2A(0)}. \tag{82}$$

Finally we get

$$\begin{pmatrix} v_3^{(0)} \\ \\ p^{(0)} \end{pmatrix} = \frac{V_0}{2A_0(0)} \begin{pmatrix} \pm\zeta^{\frac{1}{2}}(0)\zeta^{-\frac{1}{2}}(s) \\ \\ \zeta^{\frac{1}{2}}(0)\zeta^{\frac{1}{2}}(s) \end{pmatrix} g'[t - \operatorname{sgn}(s)\int_0^s \gamma_T(s')\,\mathrm{d}s']. \tag{83}$$

Isolating $p^{(0)}$ we have

$$p^{(0)}(s,t) = \frac{V_0 \zeta^{\frac{1}{2}}(0)\zeta^{\frac{1}{2}}(s)}{2A(0)} g'[t - \operatorname{sgn}(s)\int_0^s \gamma_T(s')\,\mathrm{d}s']. \tag{84}$$

This pressure distribution, which is a generalization of a standard result (White, 1983, p. 147), radiates seismic waves into the formation and will be used later to calculate an equivalent source body-force distribution in the formation.

THE LINE DENSITY OF DIPOLES

Let us extend the definition of $\mathbf{u}$ and $\mathbf{w}$ to the interior of the borehole, defining them to be zero there. Consider f_i defined by

$$\rho u_{i,tt} - (c_{ijk\ell} u_{k,\ell})_{,j} = f_i. \tag{85}$$

Then f_i will be an effective source field. To evaluate f_i, let us introduce a vector test function ϕ_i and integrate using the distributional definition of the derivatives:

$$I = \int_{-\infty}^{+\infty} \mathrm{d}t \int_V \phi_i f_i \,\mathrm{dx} \;=\; \int_{-\infty}^{+\infty} \mathrm{d}t \int_V [\rho\phi_{k,tt} - (\phi_{i,j} c_{ijk\ell})_{,\ell}] u_k \,\mathrm{dx}. \tag{86}$$

This is an ordinary integral since the integrand is finite everywhere. Also the integrand vanishes inside the borehole. Thus

$$I = \int_{-\infty}^{+\infty} \mathrm{d}t \int_{V_\epsilon} \phi_i f_i \,\mathrm{dx} \;=\; \int_{-\infty}^{+\infty} \mathrm{d}t \int_{V_\epsilon} [\rho\phi_{k,tt} - (\phi_{i,j} c_{ijk\ell})_{,\ell}] u_k \,\mathrm{dx}, \tag{87}$$

where V_ϵ is the region exterior to the borehole.

Now let us use the divergence theorem in V_ϵ to get

$$\begin{aligned} I \;=\; & \int_{-\infty}^{+\infty} \mathrm{d}t \int_{V_\epsilon} [-\rho\phi_{i,t} u_{i,t} + \phi_{i,j} c_{ijk\ell} u_{k,\ell}] \,\mathrm{dx} \\ & + \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\partial V_\epsilon} \phi_{i,j} c_{ijk\ell} u_k n_\ell \,\mathrm{d}A, \end{aligned} \tag{88}$$

where ∂V_ϵ is the boundary of V_ϵ, unit normal $\mathbf{n}$ points towards the exterior of the borehole, and $\mathrm{d}A$ indicates an element of area. Integrating once more by parts we get

$$\begin{aligned} I \;=\; & \int_{-\infty}^{+\infty} \mathrm{d}t \int_{V_\epsilon} \phi_i [\rho u_{i,tt} - (c_{ijk\ell} u_{k,\ell})_{,j}] \,\mathrm{dx} \\ & + \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\partial V_\epsilon} \phi_{i,j} c_{ijk\ell} u_k n_\ell - \phi_i c_{ijk\ell} u_{k,\ell} n_j \,\mathrm{d}A \\ =\; & \int_{-\infty}^{+\infty} \mathrm{d}t \int_{\partial V_\epsilon} \phi_{i,j} c_{ijk\ell} u_k n_\ell - \phi_i c_{ijk\ell} u_{k,\ell} n_j \,\mathrm{d}A, \end{aligned} \tag{89}$$

because u_i satisfies the homogeneous equation (85) in V_ϵ. Since the borehole is narrow

$$\mathrm{d}A = \mathrm{d}s\,\mathrm{d}s', \tag{90}$$

to leading order in ϵ, where s is arclength along the borehole and s' is arclength around the circumference $\partial\Sigma_s$ of the right cross-section at s. Let us also write for $\mathbf{x}$ on the borehole wall

$$\begin{aligned} \phi_i(\mathbf{x}) &= \phi_i(\mathbf{X}(s)) + [x_m - X_m(s)]\phi_{i,m}(\mathbf{X}(s)) + O(\epsilon^2), \\ c_{ijk\ell}(\mathbf{x}) &= c_{ijk\ell}(\mathbf{X}(s)) + O(\epsilon^2), \end{aligned} \tag{91}$$

where $c_{ijk\ell}(\mathbf{X}(s))$ is defined by the elastic constants of the material drilled out to form the borehole. Then to leading order

$$\begin{aligned} I &= \int_{-\infty}^{+\infty} \mathrm{d}t \int \mathrm{d}s \int_{\partial\Sigma_s} \Big\{ \phi_{i,m}(\mathbf{X}(s),t) \Big[c_{imk\ell} u_k n_\ell \\ &\qquad - c_{ijk\ell} u_{k,\ell} n_j (x_m - X_m(s)) \Big] - \phi_i(\mathbf{X}(s),t) c_{ijk\ell} u_{k,\ell} n_j \Big\} \, \mathrm{d}s'. \end{aligned} \tag{92}$$

But

$$c_{ijk\ell} u_{k,\ell} n_j = -p n_i, \tag{93}$$

where p is the pressure in the borehole fluid at $\mathbf{X}(s)$, which is independent of s', $p = p^{(0)}(s,t)$, to leading order in ϵ. Thus the last term in (92) vanishes and (92) becomes

$$I = \int_{-\infty}^{+\infty} \mathrm{d}t \int \mathrm{d}s \int_{\partial\Sigma_s} \phi_{i,m}(\mathbf{X}(s),t) [c_{imk\ell} u_k n_\ell + p^{(0)} (x_m - X_m(s)) n_i] \, \mathrm{d}s', \tag{94}$$

where in the integrand $c_{imk\ell}$ is taken to be $c_{imk\ell}(\mathbf{X}(s))$.

Let

$$\begin{aligned} g_{im}(s,t) &= \int_{\partial\Sigma_s} [c_{imk\ell}(\mathbf{X}(s)) u_k(\mathbf{X}(s),t) n_\ell(s,s') \\ &\qquad + p^{(0)}(\mathbf{X}(s),t)(x_m - X_m(s)) n_i(s,s')] \, \mathrm{d}s'. \end{aligned} \tag{95}$$

We now apply Stokes's theorem to (95). But first we write

$$\mathbf{n} = \mathbf{t}' \times \mathbf{t}, \quad n_p = \epsilon_{pqr} t'_q t_r, \tag{96}$$

where $\mathbf{t} = \mathbf{e}_3$ is the unit tangent to the borehole axis, and $\mathbf{t}'$ is the unit tangent to $\partial\Sigma_s$. Then

$$\begin{aligned} p^{(0)} \int_{\partial\Sigma_s} (x_m - X_m(s)) n_i \, \mathrm{d}s' &= p^{(0)} t_r \int_{\partial\Sigma_s} (x_m - X_m(s)) \epsilon_{iqr} t'_q \, \mathrm{d}s' \\ &= p^{(0)} t_r \int_{\Sigma_s} \epsilon_{qnp} (x_m - X_m(s))_{,n} \epsilon_{ipr} t_q \, \mathrm{d}A \\ &= p^{(0)} A (\delta_{qr}\delta_{ni} - \delta_{qi}\delta_{nr}) t_r t_q \delta_{mn} \\ &= p^{(0)} A (\delta_{im} - t_i t_m). \end{aligned} \tag{97}$$

The other term in (95) must be treated separately since u_k is not continuous up to the boundary in Σ_s. Let us extend $c_{ijk\ell}$ smoothly and define the fictitious displacement field u_k^* *inside* the borehole so that

$$c_{ijk\ell}u^*_{k,\ell j} = 0 \text{ in } \Sigma_s, \quad u^*_k = u_k \text{ in } \partial\Sigma_s. \tag{98}$$

Thus $\mathbf{u}^*$ satisfies the elastic equilibrium equations inside the borehole and has the same displacement on the boundary as the external field. Hence using Stoke's theorem we have

$$\begin{aligned}
\int_{\partial\Sigma_s} c_{imk\ell}u_k n_\ell \, \mathrm{d}s' &= \int_{\partial\Sigma_s} c_{imk\ell}u^*_k \epsilon_{\ell qr} t'_q t_r \, \mathrm{d}s' \\
&= \int_{\Sigma_s} \epsilon_{qpn} c_{imk\ell} u^*_{k,p} \epsilon_{\ell nr} t_q t_r \, \mathrm{d}A \\
&= \int_{\Sigma_s} (\delta_{qr}\delta_{p\ell} - \delta_{q\ell}\delta_{pr}) c_{imk\ell} u^*_{k,p} t_q t_r \, \mathrm{d}A \\
&= \int_{\Sigma_s} (t_r t_r c_{imk\ell} u^*_{k,\ell} - t_\ell t_p c_{imk\ell} u^*_{k,p}) \, \mathrm{d}A \\
&= \int_{\Sigma_s} c_{imk\ell} u^*_{k,p} (\delta_{\ell p} - t_\ell t_p) \, \mathrm{d}A \\
&= \int_{\Sigma_s} \tau^*_{im} \, \mathrm{d}A,
\end{aligned} \tag{99}$$

where τ^* is the stress field belonging to the displacement $\mathbf{u}^*$ inside Σ_s. We have used the fact that the tangential s-derivative $t_p u^*_{k,p}$ is $O(\epsilon^2)$ at most, and so the term $t_\ell t_p c_{imk\ell} u^*_{k,p}$ does not contribute. When the borehole is elliptical in cross-section τ^* is constant and we may write

$$\int_{\partial\Sigma_s} c_{imk\ell} u^*_k n_\ell \, \mathrm{d}s' = A\tau^*_{im}. \tag{100}$$

Thus, using (97) and (100) in (95) we obtain

$$\begin{aligned}
g_{im} &= A[\tau^*_{im} + p^{(0)}(\delta_{im} - t_i t_m)] \\
&= Ap^{(0)}[c_{impq}N_{pqrs}(\delta_{rs} - t_r t_s) + \delta_{im} - t_i t_m] \\
&= Ap^{(0)}N^*_{im},
\end{aligned} \tag{101}$$

say. Notice the similarities between the terms in square brackets in (62) and (101). In the Appendix it is shown that $N_{pqrs} = N_{rspq}$, and so the square bracketed expressions are indeed identical. Comparing (86) and (94) using (95) and (101) we find that

$$\begin{aligned}
\int \phi_i f_i \, \mathrm{d}\mathbf{x} &= \int \phi_{i,m}(\mathbf{X}(s) g_{im} \, \mathrm{d}s \\
&= \int \mathrm{d}\mathbf{x} \int \phi_{i,m}(\mathbf{x}, t)\delta(\mathbf{x} - \mathbf{X}(s)) g_{im}(s,t) \, \mathrm{d}s \\
&= \int \mathrm{d}\mathbf{x} \int -\phi_i(\mathbf{x}, t)\delta_{,m}(\mathbf{x} - \mathbf{X}(s)) g_{im}(s,t) \, \mathrm{d}s,
\end{aligned} \tag{102}$$

and since ϕ_i is an arbitrary test function we have

$$\begin{aligned} f_i(\mathbf{x}, t) &= -\int g_{im}(s,t)\delta_{,m}(\mathbf{x} - \mathbf{X}(s))\,\mathrm{d}s \\ &= -\int A(s)N^*_{im}(s)p^{(0)}(s,t)\delta_{,m}(\mathbf{x} - \mathbf{X}(s))\,\mathrm{d}s. \end{aligned} \tag{103}$$

The integrand above is a superposition of nine body force dipoles of differing strengths $g_{im}(s,t)$ depending on position and time. In symmetrical configurations the off diagonal terms are zero, thus reducing the force system to three mutually orthogonal dipoles moving along the borehole (see Figure 1). As will be shown later, these dipoles move away from the source at the tube wave speed.

THE BOREHOLE COMPLIANCE: THE ELASTOSTATIC PROBLEM

The Complex Variable Method

In this section we set up the machinery for calculating the operator N and N^*_{im} of (59) and (103). Thus we must solve the elastostatic problem (44), (45) and (46), which we restate using the displacement $\mathbf{u}^{(0)}$ instead of the particle velocity $\mathbf{w}^{(0)}$. The constitutive equation is

$$\tau^{(0)}_{ij} = c_{ijk\gamma}u^{(0)}_{k,\gamma}. \tag{104}$$

The equilibrium equation is

$$\tau^{(0)}_{i\alpha,\alpha} = 0, \tag{105}$$

where to this order of approximation the solution is (locally) independent of x_3, the coordinate parallel to the borehole axis. The boundary condition on the borehole wall is

$$\tau^{(0)}_{i\alpha}n_\alpha = -(\tau^{I(0)}_{i\alpha}n_\alpha + p^{(0)}n_i), \tag{106}$$

where $\mathbf{n}$ is normal to the borehole wall and $n_3 = 0$. In equations (104), (105), (106) Roman subscripts range over 1,2,3, whereas Greek subscripts range over 1,2 only, and the summation convention applies. In solving this system of equations we use the complex variable method of Lekhnitskii (1963), Savin (1961) and Muskhelishvili (1953). These authors set up their equations in terms of stress functions, which leads naturally to the use of elastic compliances, and this is most direct when one is concerned primarily with stress concentration. But we are concerned primarily with displacements, and in the following development we take the elastic stiffnesses and displacement components as basic. However, a form of stress function will play a minor role in imposing the boundary conditions. This complex-variable method is also known as the *Stroh formalism* and was used by Stroh (1958) and (1962). See also Ting (1990) for a modern review.

The method requires that we seek the displacement as an analytic function of the combination

$$x_1 + \eta x_2. \tag{107}$$

Substituting (104) into (105) we get

$$c_{i\alpha k\beta} u_{k,\alpha\beta} = 0, \tag{108}$$

where we have dropped the superscript $^{(0)}$ on $\mathbf{u}$; we shall continue to drop it on $\mathbf{u}$, $\boldsymbol{\tau}$ and p in this section. Then, setting

$$u_k = u_k(z) = u_k(x_1 + \eta x_2), \tag{109}$$

equation (108) leads to

$$\mathbf{P}(\eta)\mathbf{u}'' = 0, \tag{110}$$

where

$$P_{ik}(\eta) = c_{i1k1} + (c_{i1k2} + c_{i2k1})\eta + c_{i2k2}\eta^2. \tag{111}$$

Thus

$$\det(\mathbf{P}(\eta)) = 0. \tag{112}$$

But this is a sextic in η with real coefficients. So the roots are real or occur as complex conjugate pairs. Actually it follows from energy considerations that there are no real roots. Let the six roots be

$$\eta_n, \quad \bar{\eta}_n, \quad \Im\{\eta_n\} > 0, \quad n = 1, 2, 3. \tag{113}$$

From (110) it follows that $\mathbf{u}''$ is a null vector of $\mathbf{P}(\eta_n)$, and therefore a multiple of a standard null vector $\mathbf{U}_n$.

$$u_k'' = A_n(z_n) U_{kn}, \tag{114}$$

where $\mathbf{U}_n$ is a function only of the $c_{ijk\ell}$ and not of the coordinates, and $A_n(z_n)$ is a scalar coefficient depending on x_1, x_2 through

$$z_n = x_1 + \eta_n x_2, \quad n = 1, 2, 3, \tag{115}$$

on which it depends analytically. Because u_k is real, we seek u_k as the real part of a superposition of such solutions

$$u_k = \Re\{\sum_{n=1}^{3} U_{kn} A_n(z_n)\}, \quad \mathbf{u} = \Re\{\mathbf{UA}\}. \tag{116}$$

Here $\mathbf{A}$ is the column vector with components $A_n(z_n)$, $n = 1, 2, 3$ and $\mathbf{U}$ is the matrix whose columns are the null vectors $\mathbf{U}_n$, $n = 1, 2, 3$. $A_n(z_n)$ is an analytic function of its argument. Since η_n and $\mathbf{U}_n$ are independent of x_1 and x_2, depending only on the elastic constants $c_{i\alpha k\beta}$, (104) and (116) give

$$\tau_{ij} = \Re\{\sum_{n=1}^{3}(c_{ijk1} + c_{ijk2}\eta_n)U_{kn}A'_n(z_n)\}. \tag{117}$$

Before imposing the boundary conditions (106) we shall introduce the stress functions T_i such that

$$\tau_{i1} = -T_{i,2}, \quad \tau_{i2} = T_{i,1}, \tag{118}$$

which exist by virtue of (105). Let us now consider the x_1x_2-plane. The cylindrical borehole wall cuts this plane in a curve, which we shall call the borehole profile, with tangent $\mathbf{t}$ related to x_α and n_α by

$$x_{1,s} = t_1 = -n_2, \quad x_{2,s} = t_2 = n_1, \tag{119}$$

where $x_{1,s}$ and $x_{2,s}$ are the derivatives of x_1 and x_2 with respect to arclength s along the borehole profile. Then using (118), (119) in the left member of (106) we get

$$\tau_{i\alpha}n_\alpha = \tau_{i1}t_2 - \tau_{i2}t_1 = -T_{i,2}x_{2,s} - T_{i,1}x_{1,s} = -T_{i,s}, \tag{120}$$

and in the right member

$$\begin{array}{rcl} -(\tau^I_{1\alpha}n_\alpha + pn_1) & = & -\tau^I_{11}x_{2,s} + \tau^I_{12}x_{1,s} - px_{2,s}, \\ -(\tau^I_{2\alpha}n_\alpha + pn_2) & = & -\tau^I_{21}x_{2,s} + \tau^I_{22}x_{1,s} + px_{1,s}, \\ -\tau^I_{3\alpha}n_\alpha & = & -\tau^I_{31}x_{2,s} + \tau^I_{32}x_{1,s}. \end{array} \tag{121}$$

Using (120), (121) in (106), with the fact that the τ^I_{ij} are constant, we obtain

$$\begin{array}{rcl} T_1 & = & -\tau^I_{12}x_1 + (\tau^I_{11} + p)x_2, \\ T_2 & = & -(\tau^I_{22} + p)x_1 + \tau^I_{21}x_2, \\ T_3 & = & -\tau^I_{32}x_1 + \tau^I_{31}x_2. \end{array} \tag{122}$$

Let us summarize (122) as

$$T_i = (\tau^I_{ik} + p\delta_{ik})\epsilon_{k\alpha}x_\alpha, \tag{123}$$

where

$$\epsilon = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{124}$$

From (117) and (118) we have

$$\tau_{i2} = T_{i,1} = \Re\{\sum_{n=1}^{3}(c_{i2k1} + c_{i2k2}\eta_n)U_{kn}A'_n(z_n)\}, \tag{125}$$

and so integrating and using (123) we get

$$\Re\{\sum_{n=1}^{3}(c_{i2k1}+c_{i2k2}\eta_n)U_{kn}A_n(z_n)\} = (\tau^I_{ik}+p\delta_{ik})\epsilon_{k\alpha}x_\alpha \tag{126}$$

on the borehole profile. Let us define the matrices $\mathbf{K}_{21}$, $\mathbf{K}_{22}$, and $\boldsymbol{\Lambda}$ as follows:

$$(\mathbf{K}_{21})_{ik} = c_{i2k1}, \quad (\mathbf{K}_{22})_{ik} = c_{i2k2}, \quad \boldsymbol{\Lambda} = \mathrm{diag}\{\eta_1,\eta_2,\eta_3\}. \tag{127}$$

Then (126) may be rewritten

$$\Re\{(\mathbf{K}_{21}\mathbf{U}+\mathbf{K}_{22}\mathbf{U}\boldsymbol{\Lambda})\mathbf{A}\} = (\boldsymbol{\tau}^I+p\mathbf{I})\boldsymbol{\epsilon}\mathbf{x} \tag{128}$$

on the borehole profile.

The Elliptical Borehole

Before proceeding further we shall specialize to a borehole having elliptical profile, with major semi-axis r_1, minor semi-axis r_2, and whose major axis makes an angle α with the 1-axis. Then the elliptical profile may be parameterized by r_1, r_2, and angle θ where

$$\begin{aligned} z = x_1 + \mathrm{i}x_2 &= \mathrm{e}^{\mathrm{i}\alpha}(r_1\cos\theta + \mathrm{i}r_2\sin\theta) \\ &= \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\alpha}[(r_1+r_2)\mathrm{e}^{\mathrm{i}\theta}+(r_1-r_2)\mathrm{e}^{-\mathrm{i}\theta}]. \end{aligned} \tag{129}$$

Setting $\zeta = \mathrm{e}^{\mathrm{i}\theta}$, (129) becomes

$$z = x_1+\mathrm{i}x_2 = \tfrac{1}{2}\mathrm{e}^{\mathrm{i}\alpha}[(r_1+r_2)\zeta+(r_1-r_2)\tfrac{1}{\zeta}] \quad \text{on} \quad |\zeta|=1. \tag{130}$$

But if we allow $|\zeta|>1$, (130) is a conformal mapping of the exterior of the unit circle in the ζ plane onto the exterior of the ellipse in the z plane, and, because $\Im\{\eta_n\}>0$, the exterior of the unit circle in ζ plane is mapped onto the exteriors of ellipses in the z_n-planes. Thus the $A_n(z_n(\zeta))$ may be regarded as functions of ζ analytic outside the unit circle, constrained by (126) on the unit circle. So, rewriting (126)

$$\Re\{\sum_{n=1}^{3}(c_{i2k1}+c_{i2k2}\eta_n)U_{kn}A_n(z_n(\zeta))\} = (\tau^I_{ik}+p\delta_{ik})\epsilon_{k1}x_1(\zeta)+(\tau^I_{ik}+p\delta_{ik})\epsilon_{k2}x_2(\zeta), \tag{131}$$

where from (129)

$$\begin{aligned} x_1(\zeta) &= \tfrac{1}{2}[(r_1\cos\alpha+\mathrm{i}r_2\sin\alpha)\zeta+(r_1\cos\alpha-\mathrm{i}r_2\sin\alpha)\tfrac{1}{\zeta}], \\ x_2(\zeta) &= \tfrac{1}{2}[(r_1\sin\alpha-\mathrm{i}r_2\cos\alpha)\zeta+(r_1\sin\alpha+\mathrm{i}r_2\cos\alpha)\tfrac{1}{\zeta}]. \end{aligned} \tag{132}$$

and $|\zeta|=1$.

We may now use the Schwartz formula for a function $F(\zeta)$, analytic outside the unit circle, whose real part is given on the unit circle:

$$F(\zeta) = -\frac{1}{2\pi \mathrm{i}} \oint \Re\{F(\sigma)\} \frac{\sigma+\zeta}{\sigma-\zeta} \frac{\mathrm{d}\sigma}{\sigma}, \tag{133}$$

where the integral is once around the unit circle $|\sigma| = 1$, $|\zeta| > 1$, and $F(\zeta)$ is analytic in $|\zeta| > 1$. Applying this to the function in braces on the left of (131) we obtain

$$\begin{aligned} \sum_{n=1}^{3} (c_{i2k1} + c_{i2k2}\eta_n) U_{kn} A_n(z_n(\zeta)) &= -\frac{1}{2\pi \mathrm{i}} \oint (D_i \sigma + E_i \frac{1}{\sigma}) \frac{\sigma+\zeta}{\sigma-\zeta} \frac{\mathrm{d}\sigma}{\sigma} \\ &= \frac{2E_i}{\zeta}, \end{aligned} \tag{134}$$

where D_i and E_i are the coefficients of ζ and $\frac{1}{\zeta}$ on the right of (131) after (132) is used for $x_1(\zeta)$ and $x_2(\zeta)$. From (134) we see that $A_n(z_n(\zeta))$ is proportional to $\frac{1}{\zeta}$, and referring back to (128), we have

$$\mathbf{A} = (\mathbf{K}_{21}\mathbf{U} + \mathbf{K}_{22}\mathbf{U}\mathit{\Lambda})^{-1}\mathbf{E}\frac{2}{\zeta}. \tag{135}$$

Explicitly

$$\begin{aligned} 2\mathbf{E} &= (\tau^I + p\mathbf{I})\epsilon \begin{pmatrix} r_1 \cos\alpha - \mathrm{i} r_2 \sin\alpha \\ r_1 \sin\alpha + \mathrm{i} r_2 \cos\alpha \\ 0 \end{pmatrix} \\ &= (\tau^I + p\mathbf{I}) \begin{pmatrix} r_1 \sin\alpha + \mathrm{i} r_2 \cos\alpha \\ r_1 \cos\alpha - \mathrm{i} r_2 \sin\alpha \\ 0 \end{pmatrix}. \end{aligned} \tag{136}$$

Then from (116)

$$\mathbf{u} = \Re\{\mathbf{U}\mathbf{A}\} = \Re\{(\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\mathit{\Lambda}\mathbf{U}^{-1})^{-1}\mathbf{E}\frac{2}{\zeta}\}, \tag{137}$$

or

$$\mathbf{u} = \Re\{2(\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\mathit{\Lambda}\mathbf{U}^{-1})^{-1}\mathbf{E}(\cos\theta - \mathrm{i}\sin\theta)\} \tag{138}$$

at the point with parameter θ according to (129), i.e.,

$$\begin{aligned} \mathbf{u} = \Re\Big\{ & 2(\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\mathit{\Lambda}\mathbf{U}^{-1})^{-1}\mathbf{E} \\ & \Big[x_1\Big(\frac{\cos\alpha}{r_1} + \frac{\mathrm{i}\sin\alpha}{r_2}\Big) + x_2\Big(\frac{\sin\alpha}{r_1} - \frac{\mathrm{i}\cos\alpha}{r_2}\Big)\Big]\Big\}. \end{aligned} \tag{139}$$

Thus, using (136), we may write the displacement gradient as

$$\nabla\mathbf{u}^* = \Re\{\mathbf{L}(\tau^I + p\mathbf{I})\mathbf{M}\}, \tag{140}$$

with corresponding strain

$$\varepsilon^* = \frac{1}{2}(\nabla \mathbf{u}^* + \mathbf{u}^*\nabla) = \Re\{\mathbf{L}(\tau^I + p\mathbf{I})\mathbf{M} + \mathbf{M}^T(\tau^I + p\mathbf{I})^T\mathbf{L}^T\}, \tag{141}$$

where

$$\mathbf{L} = (\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\Lambda\mathbf{U}^{-1})^{-1}, \tag{142}$$

and, from (136) and (139), we see that

$$\mathbf{M} = \frac{1}{r_1 r_2}\begin{pmatrix} \mathrm{i}(r_1^2\sin^2\alpha + r_2^2\cos^2\alpha) & r_1r_2 - \mathrm{i}(r_1^2 - r_2^2)\sin\alpha\cos\alpha & 0 \\ -r_1r_2 - \mathrm{i}(r_1^2 - r_2^2)\sin\alpha\cos\alpha & \mathrm{i}(r_1^2\cos^2\alpha + r_2^2\sin^2\alpha) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{143}$$

Equation (141) may also be written as

$$\varepsilon_{ij} = N_{ijk\ell}(\tau^I_{k\ell} + p\delta_{k\ell}), \tag{144}$$

where

$$N_{ijk\ell} = \frac{1}{2}\Re\{L_{ik}M_{\ell j} + L_{jk}M_{\ell i}\}. \tag{145}$$

The three matrix factors in (140) depend respectively on, the elastic constants of the material, the incident stress field (including the pressure p), and the geometry of the borehole profile. The constant tensor $\nabla \mathbf{u}^*$ is the uniform displacement gradient belonging to the interior problem mentioned after (99). The corresponding stress τ^*_{ij} is given by

$$\tau^*_{ij} = c_{ijk\gamma}u^*_{k,\gamma}. \tag{146}$$

Thus the operator N of (59) is given by

$$N(\tau) = \Re\{(\mathbf{L}\tau\mathbf{M})_{\alpha\alpha}\}, \tag{147}$$

and

$$N(\mathbf{I}) = \Re\{L_{\alpha\gamma}M_{\gamma\alpha}\} = \Re\{\mathrm{tr}(\mathbf{LM})\}. \tag{148}$$

Taking the borehole axis parallel to the 3-direction, we may now also write down the operator N^*_{im} of (101) and (103). Thus

$$\begin{aligned} \tau^*_{im} + p^{(0)}(\delta_{im} - \delta_{3i}\delta_{3m}) &= c_{imk\gamma}u^*_{k\gamma} + p^{(0)}(\delta_{im} - \delta_{3i}\delta_{3m}) \\ &= p^{(0)}[c_{imk\gamma}L_{k\alpha}M_{\alpha\gamma} + \delta_{im} - \delta_{3i}\delta_{3m}], \end{aligned} \tag{149}$$

and so

$$N^*_{im} = \Re\{c_{imk\gamma}L_{k\alpha}M_{\alpha\gamma} + \delta_{im} - \delta_{3i}\delta_{3m}\}. \tag{150}$$

SPECIAL EXPLICITLY SOLVABLE CASES

Orthorhombic Symmetry with the Borehole Parallel to an Axis

The matrix $(\mathbf{K}_{21} + \mathbf{K}_{22}\mathrm{U}\Lambda\mathrm{U}^{-1})^{-1}$ is generally too complicated to evaluate in a perspicuous form as a function of the elastic constants. However, when the coordinate planes are planes of material symmetry, the 3×3 matrices split into a 2×2 and a 1×1 diagonal block so that u_3 decouples from u_1 and u_2. The calculation is then manageable analytically. The c_{ijkl} are zero if any index appears an odd number of times. Thus P of (111) reduces to

$$\mathbf{P}(\eta) = \begin{pmatrix} a + c\eta^2 & (c+d)\eta & 0 \\ (c+d)\eta & c + b\eta^2 & 0 \\ 0 & 0 & e + f\eta^2 \end{pmatrix}, \tag{151}$$

where

$$\begin{aligned} a &= c_{1111}, \quad b = c_{2222}, \quad c = c_{1212}, \\ d &= c_{1122}, \quad e = c_{1313}, \quad f = c_{2323}. \end{aligned} \tag{152}$$

We see that $\det(\mathbf{P}(\eta))$ factors as

$$\det(\mathbf{P}(\eta)) = Q(\eta)R(\eta), \tag{153}$$

where

$$\begin{aligned} Q(\eta) &= (a + c\eta^2)(c + b\eta^2) - (c+d)^2\eta^2, \\ R(\eta) &= e + f\eta^2. \end{aligned} \tag{154}$$

We may set

$$\mathbf{U} = \begin{pmatrix} -(c+d)\eta_1 & -(c+d)\eta_2 & 0 \\ a + c\eta_1^2 & a + c\eta_2^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{155}$$

where η_1 and η_2 are the zeros of Q, and η_3 the zero of R, having positive imaginary parts. Thus

$$\eta_3 = \mathrm{i}\sqrt{\frac{e}{f}}. \tag{156}$$

We shall not need the explicit forms of η_1 and η_2. Inverting (155) we get

$$\mathbf{U}\Lambda\mathbf{U}^{-1} = \begin{pmatrix} \dfrac{a(\eta_1+\eta_2)}{a-c\eta_1\eta_2} & \dfrac{(c+d)\eta_1\eta_2}{a-c\eta_1\eta_2} & 0 \\ -\dfrac{(a+c\eta_1^2)(a+c\eta_2^2)}{(c+d)(a-c\eta_1\eta_2)} & \dfrac{-c\eta_1\eta_2(\eta_1+\eta_2)}{a-c\eta_1\eta_2} & 0 \\ 0 & 0 & \eta_3 \end{pmatrix}. \tag{157}$$

Since the matrix splits it is advantageous to deal separately with the 2×2 and the 1×1 blocks.

The 2×2 Block

We rewrite the 2×2 of (157) as

$$\mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{1}{a-c\eta_1\eta_2}\begin{pmatrix} a(\eta_1+\eta_2) & (c+d)\eta_1\eta_2 \\ -\dfrac{(a+c\eta_1^2)(a+c\eta_2^2)}{c+d} & -c\eta_1\eta_2(\eta_1+\eta_2) \end{pmatrix}. \tag{158}$$

Let us reduce the 21 entry by setting

$$a + c\eta^2 = \xi. \tag{159}$$

Then $Q(\eta) = 0$ implies

$$\xi[c + \frac{b(\xi-a)}{c}] - \frac{(c+d)^2(\xi-a)}{c} = 0. \tag{160}$$

The product of the roots of this quadratic equation in ξ is

$$\xi_1\xi_2 = (a+c\eta_1^2)(a+c\eta_2^2) = \frac{a(c+d)^2}{b}. \tag{161}$$

Thus (158) may be rewritten

$$\mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{1}{a-c\eta_1\eta_2}\begin{pmatrix} a(\eta_1+\eta_2) & (c+d)\eta_1\eta_2 \\ -\dfrac{a(c+d)}{b} & -c\eta_1\eta_2(\eta_1+\eta_2) \end{pmatrix}. \tag{162}$$

From (127)

$$\mathbf{K}_{21} = \begin{pmatrix} 0 & c & 0 \\ d & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad \mathbf{K}_{22} = \begin{pmatrix} c & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & f \end{pmatrix}. \tag{163}$$

Then, retaining only the 2×2 block,

$$\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{c}{a - c\eta_1\eta_2}\begin{pmatrix} a(\eta_1+\eta_2) & a + d\eta_1\eta_2 \\ -(a + d\eta_1\eta_2) & -b\eta_1\eta_2(\eta_1+\eta_2) \end{pmatrix}. \tag{164}$$

To eliminate η_1, and η_2 we need the symmetric functions $\eta_1+\eta_2$ and $\eta_1\eta_2$. But these are easily obtained from the equation

$$Q(\eta) = 0 \tag{165}$$

satisfied by η_1, and η_2. Thus

$$\eta_1^2\eta_2^2 = \frac{a}{b}, \tag{166}$$

and so, because η_1 and η_2 have positive imaginary parts,

$$\eta_1\eta_2 = -\sqrt{\frac{a}{b}}. \tag{167}$$

Also

$$\eta_1^2 + \eta_2^2 = -\frac{ab-d^2}{bc} + \frac{2d}{b}. \tag{168}$$

Hence

$$\begin{aligned}(\eta_1+\eta_2)^2 &= \eta_1^2+\eta_2^2+2\eta_1\eta_2 \\ &= -\frac{ab-d^2}{bc} + \frac{2d}{b} - 2\sqrt{\frac{a}{b}}.\end{aligned} \tag{169}$$

Again, because of the disposition of η_1 and η_2 in the complex plane,

$$\eta_1 + \eta_2 = \frac{\mathrm{i}}{\sqrt{b}}\sqrt{\frac{ab-d^2}{c} + 2(\sqrt{ab}-d)}. \tag{170}$$

Thus, after some reduction,

$$\begin{aligned}&\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\Lambda\mathbf{U}^{-1} = \frac{1}{\sqrt{ab}+c} \times \\ &\begin{pmatrix} \mathrm{i}\sqrt{ac}\sqrt{\sqrt{ab}-d}\sqrt{\sqrt{ab}+2c+d} & c(\sqrt{ab}-d) \\ -c(\sqrt{ab}-d) & \mathrm{i}\sqrt{bc}\sqrt{\sqrt{ab}-d}\sqrt{\sqrt{ab}+2c+d} \end{pmatrix},\end{aligned} \tag{171}$$

and

$$\mathbf{L} = (\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\Lambda\mathbf{U}^{-1})^{-1} = \frac{1}{\sqrt{ab}+d}\times$$

$$\begin{pmatrix} -\mathrm{i}\sqrt{\frac{b}{c}}\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}} & 1 \\ -1 & -\mathrm{i}\sqrt{\frac{a}{c}}\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}} \end{pmatrix}. \tag{172}$$

The 1×1 Block

The calculation for the 1×1 block is trivial:

$$\mathbf{U}\Lambda\mathbf{U}^{-1} = \eta_3 = \mathrm{i}\sqrt{\frac{e}{f}}, \tag{173}$$

$$\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\Lambda\mathbf{U}^{-1} = f\eta_3 = \mathrm{i}\sqrt{ef}, \tag{174}$$

and

$$(\mathbf{K}_{21} + \mathbf{K}_{22}\mathbf{U}\Lambda\mathbf{U}^{-1})^{-1} = -\frac{\mathrm{i}}{\sqrt{ef}}. \tag{175}$$

Thus finally

$$\mathbf{L} = \begin{pmatrix} \frac{-\mathrm{i}}{\sqrt{ab}+d}\sqrt{\frac{b(\sqrt{ab}+2c+d)}{c(\sqrt{ab}-d)}} & \frac{1}{\sqrt{ab}+d} & 0 \\ \frac{-1}{\sqrt{ab}+d} & \frac{-\mathrm{i}}{\sqrt{ab}+d}\sqrt{\frac{a(\sqrt{ab}+2c+d)}{c(\sqrt{ab}-d)}} & 0 \\ 0 & 0 & \frac{-\mathrm{i}}{\sqrt{ef}} \end{pmatrix}. \tag{176}$$

The Body-Force Dipoles

Since we now have both $\mathbf{M}$ and $\mathbf{L}$ explicitly from (143) and (176) we may calculate $N(\mathrm{I})$ and N^*_{im}. From (148) and (150) we see that these depend on $\mathbf{L}$ and $\mathbf{M}$ only through the matrix product $\Re\{\mathrm{LM}\}$, which we easily find to be

$$\Re\{\mathrm{LM}\} = \frac{1}{\sqrt{ab}+d}\left[-\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \right.$$

$$\left. +\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}}\begin{pmatrix} \sqrt{\frac{b}{c}}\,\frac{r_1^2\sin^2\alpha + r_2^2\cos^2\alpha}{r_1 r_2} & -\sqrt{\frac{b}{c}}\,\frac{(r_1^2-r_2^2)\sin\alpha\cos\alpha}{r_1 r_2} \\ -\sqrt{\frac{a}{c}}\,\frac{(r_1^2-r_2^2)\sin\alpha\cos\alpha}{r_1 r_2} & \sqrt{\frac{a}{c}}\,\frac{r_1^2\cos^2\alpha + r_2^2\sin^2\alpha}{r_1 r_2} \end{pmatrix}\right], \tag{177}$$

where we have presented the leading 2×2 block only. Equation (148) gives $N(\mathrm{I})$ as the trace of this:

$$\begin{aligned} N(\mathrm{I}) \;=\; & \frac{1}{\sqrt{ab}+d}\left[-2+\right. \\ & \left. +\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}}\;\frac{(\sqrt{a}r_1^2+\sqrt{b}r_2^2)\cos^2\alpha + (\sqrt{a}r_2^2+\sqrt{b}r_1^2)\sin^2\alpha}{\sqrt{c}\,r_1 r_2}\right]. \end{aligned} \tag{178}$$

The expression for N^*_{im} in (150) involves contractions of tensors rather than multiplication of matrices and so, as a preliminary, we shall write the components of $\Re\{\mathrm{LM}\}$ as a 4-vector in the order 11, 12, 21, 22 and the components of $c_{imk\gamma}$ as a 6×4 matrix, and then form the matrix product of these. The order of the components in the matrix will be

$$\begin{pmatrix} c_{1111} & c_{1112} & c_{1121} & c_{1122} \\ c_{2211} & c_{2212} & c_{2221} & c_{2222} \\ c_{3311} & c_{3312} & c_{3321} & c_{3322} \\ c_{2311} & c_{2312} & c_{2321} & c_{2322} \\ c_{3111} & c_{3112} & c_{3121} & c_{3122} \\ c_{1211} & c_{1212} & c_{1221} & c_{1222} \end{pmatrix}. \tag{179}$$

Now we use (152) and the fact that the $c_{imk\gamma}$ vanish if any subscript appears an odd number of times to reduce the matrix to

$$\begin{pmatrix} a & 0 & 0 & d \\ d & 0 & 0 & b \\ c_{1133} & 0 & 0 & c_{2233} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & c & c & 0 \end{pmatrix}. \tag{180}$$

Using (176) and (180) in (150) we find the only non-zero components of N^*_{im} are

$$N^*_{11} = \frac{1}{\sqrt{ab}+d}\left[\sqrt{ab}-a+\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}}\times \frac{(d\sqrt{a}r_1^2+a\sqrt{b}r_2^2)\cos^2\alpha+(a\sqrt{b}r_1^2+d\sqrt{a}r_2^2)\sin^2\alpha}{\sqrt{c}\,r_1r_2}\right], \tag{181}$$

$$N^*_{22} = \frac{1}{\sqrt{ab}+d}\left[\sqrt{ab}-b+\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}}\times \frac{(b\sqrt{a}r_1^2+d\sqrt{b}r_2^2)\cos^2\alpha+(d\sqrt{b}r_1^2+b\sqrt{a}r_2^2)\sin^2\alpha}{\sqrt{c}\,r_1r_2}\right], \tag{182}$$

$$N^*_{33} = \frac{1}{\sqrt{ab}+d}\left[-(c_{1133}+c_{2233})+\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}}\times \frac{(c_{2233}\sqrt{a}r_1^2+c_{1133}\sqrt{b}r_2^2)\cos^2\alpha+(c_{1133}\sqrt{b}r_1^2+c_{2233}\sqrt{a}r_2^2)\sin^2\alpha}{\sqrt{c}\,r_1r_2}\right], \tag{183}$$

and

$$N^*_{12} = N^*_{21} = -\frac{\sqrt{c}(\sqrt{a}+\sqrt{b})}{\sqrt{ab}+d}\sqrt{\frac{\sqrt{ab}+2c+d}{\sqrt{ab}-d}}\,\frac{(r_1^2-r_2^2)\sin\alpha\cos\alpha}{r_1r_2}. \tag{184}$$

Transverse Isotropy, Axis in the 1-Direction

The only simplification is that

$$e=c, \quad c_{1133}=d, \quad \text{and} \quad c_{2233}=b-2f. \tag{185}$$

Transverse Isotropy, Axis Parallel to the 3-Direction

Here the only relationships between the constants are

$$a=b, \quad e=f, \quad d=a-2c, \quad \text{and} \quad c_{2233}=c_{1133}=g. \tag{186}$$

Isotropy

When the medium is isotropic with Lamé constants λ, μ

$$a=b=\lambda+2\mu, \quad c=e=f=\mu, \quad c_{2233}=c_{1133}=d=\lambda. \tag{187}$$

Then

$$\sqrt{ab} = \lambda + 2\mu, \qquad \sqrt{ab} - d = 2\mu,$$
$$\sqrt{ab} + 2c + d = 2(\lambda + 2\mu) \quad \sqrt{ab} + d = 2(\lambda + \mu), \tag{188}$$

leading to

$$N(\mathrm{I}) = \frac{1}{\lambda + \mu}\left[-1 + \frac{\lambda + 2\mu}{2\mu} \cdot \frac{r_1^2 + r_2^2}{r_1 r_2}\right], \tag{189}$$

and

$$N_{11}^* = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)} \cdot \frac{[\lambda r_1^2 + (\lambda + 2\mu) r_2^2] \cos^2 \alpha + [(\lambda + 2\mu) r_1^2 + \lambda r_2^2] \sin^2 \alpha}{r_1 r_2}, \tag{190}$$

$$N_{22}^* = \frac{\lambda + 2\mu}{2\mu(\lambda + \mu)} \cdot \frac{[(\lambda + 2\mu) r_1^2 + \lambda r_2^2] \cos^2 \alpha + [\lambda r_1^2 + (\lambda + 2\mu) r_2^2] \sin^2 \alpha}{r_1 r_2}, \tag{191}$$

$$N_{33}^* = \frac{\lambda}{\lambda + \mu}\left[-1 + \frac{\lambda + 2\mu}{2\mu} \frac{r_1^2 + r_2^2}{r_1 r_2}\right], \tag{192}$$

and

$$N_{12}^* = N_{21}^* = -\frac{\lambda + 2\mu}{\lambda + \mu} \cdot \frac{r_1^2 - r_2^2}{r_1 r_2} \sin \alpha \cos \alpha. \tag{193}$$

Circular Borehole

When $r_1 = r_2$, (178) and (181) to (184) reduce to

$$N(\mathrm{I}) = \frac{1}{\sqrt{ab} + d}\left[-2 + \sqrt{\frac{\sqrt{ab} + 2c + d}{\sqrt{ab} - d}} \cdot \frac{\sqrt{a} + \sqrt{b}}{\sqrt{c}}\right], \tag{194}$$

and

$$N_{11}^* = \frac{1}{\sqrt{ab} + d}\left[\sqrt{ab} - a + \sqrt{\frac{\sqrt{ab} + 2c + d}{\sqrt{ab} - d}} \cdot \frac{d\sqrt{a} + a\sqrt{b}}{\sqrt{c}}\right], \tag{195}$$

$$N_{22}^* = \frac{1}{\sqrt{ab} + d}\left[\sqrt{ab} - a + \sqrt{\frac{\sqrt{ab} + 2c + d}{\sqrt{ab} - d}} \cdot \frac{b\sqrt{a} + d\sqrt{b}}{\sqrt{c}}\right], \tag{196}$$

$$N_{33}^* = \frac{1}{\sqrt{ab} + d}\left[-(c_{1133} + c_{2233}) + \sqrt{\frac{\sqrt{ab} + 2c + d}{\sqrt{ab} - d}} \cdot \frac{c_{2233}\sqrt{a} + c_{1133}\sqrt{b}}{\sqrt{c}}\right], \tag{197}$$

and N_{12}^* and N_{21}^* vanish.

The $N(\mathbf{I})$ and $\mathbf{N}^*$ for the Isotropic, Circular Case

Specializing (189) to (193) by setting $r_1 = r_2$ we get

$$N(\mathrm{I}) = \frac{1}{\mu}, \tag{198}$$

$$\mathbf{N}^* = \begin{pmatrix} \frac{\lambda+2\mu}{\mu} & 0 & 0 \\ 0 & \frac{\lambda+2\mu}{\mu} & 0 \\ 0 & 0 & \frac{\lambda}{\mu} \end{pmatrix}, \tag{199}$$

which may be written as

$$\mathbf{N}^* = \begin{pmatrix} \frac{\alpha^2}{\beta^2} & 0 & 0 \\ 0 & \frac{\alpha^2}{\beta^2} & 0 \\ 0 & 0 & \frac{\alpha^2}{\beta^2} - 2 \end{pmatrix}, \tag{200}$$

where α and β are the compressional and shear wave speeds in the isotropic medium. They are given by

$$\alpha^2 = \frac{\lambda+2\mu}{\rho}, \quad \beta^2 = \frac{\mu}{\rho}. \tag{201}$$

Equations (198)-(200) agree with the results of previous authors [White (1983), Lee and Balch (1982)].

THE FAR-FIELD RADIATION PATTERN

In this section we shall calculate the seismic far field generated by the pressure field calculated in Section 3.5 for an acoustic volume source in the borehole. We shall need N^*_{im} from (150) and the equivalent source distribution of body-force dipoles given in (103), we shall also need the far field Green's function in the anisotropic medium, and finally we shall integrate the Green's function against the source distribution to obtain the far field.

The Green's Function

We shall now calculate the seismic field radiated by the source distribution of equation (103). The far-field Green's function for a uniform anisotropic medium is given in Burridge (1967). In order to understand that result we shall need some background. We shall consider plane waves of the form

$$\mathbf{u} = \mathbf{P}(\boldsymbol{\xi}) f(t - \boldsymbol{\xi} \cdot \mathbf{x}), \tag{202}$$

where the constant vectors $\boldsymbol{\xi}$ and $\mathbf{P}(\boldsymbol{\xi})$ are the wave slowness and the polarization, respectively. For (202) to satisfy the elastic wave equation, $\boldsymbol{\xi}$ must be restricted to lie on a 3-sheeted surface S called the slowness surface. Each of the three sheets of S surrounds the origin. The vector $\mathbf{P}$ must be an eigenvector of a certain symmetric matrix whose entries are quadratic polynomials in $\boldsymbol{\xi}$ with elastic constants as coefficients. We shall normalize $\mathbf{P}$ to be a unit vector. Then in the far field (Burridge 1967, Section 6), the Green's function $\mathbf{G}(\mathbf{x}, t; \mathbf{x}', t')$ is given by

$$G_{ij}(\mathbf{x}, t; \mathbf{x}', t') = \sum_{\boldsymbol{\xi}} \frac{P_i(\boldsymbol{\xi}) P_j(\boldsymbol{\xi}) \boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')}{4\pi |C(\boldsymbol{\xi})|^{\frac{1}{2}} |\mathbf{x} - \mathbf{x}'|^2} \delta^*[t - t' - \boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')]. \tag{203}$$

For fixed $\mathbf{x}$, t', $\boldsymbol{\xi}$ and with t increasing from t', each of the planes

$$\boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}') = t - t' \tag{204}$$

in $\boldsymbol{\xi}$-space is orthogonal to $\mathbf{x} - \mathbf{x}'$ and its distance from the origin is

$$\frac{t - t'}{|\mathbf{x} - \mathbf{x}'|}. \tag{205}$$

When $t = t'$ the plane (204) passes through the origin and intersects all the sheets of S. For large $t - t'$ the plane is outside S and does not intersect it. For certain values of t the plane is tangent to S at certain points $\boldsymbol{\xi}$ (see Figure 2). These are the points $\boldsymbol{\xi}$ over which the sum is taken in (203). If these points $\boldsymbol{\xi}$ are points where S is smooth then they may be characterized as the points where the outward normal to S is parallel to $\mathbf{x} - \mathbf{x}'$. For each such point $\boldsymbol{\xi}$, (203) yields a singular wavefield arrival with time dependence

$$\delta^*[t - t' - \boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')], \tag{206}$$

where

$$\delta^* = \begin{cases} \delta, & \text{if } S \text{ is convex outward at } \boldsymbol{\xi}; \\ -\mathcal{H}\delta, & \text{if } S \text{ is saddle shaped at } \boldsymbol{\xi}; \\ -\delta, & \text{if } S \text{ is concave outward at } \boldsymbol{\xi}. \end{cases} \tag{207}$$

Here $\mathcal{H}$ is the Hilbert transform. The quantity $C(\boldsymbol{\xi})$ in (203) is the Gaussian curvature of S at $\boldsymbol{\xi}$. It is respectively positive, negative, and positive in the three cases of (207).

Formula (203) does not apply when $C(\boldsymbol{\xi}) = 0$ at any of the $\boldsymbol{\xi}$. Then a further more complicated expression for that particular term is required, which we shall not describe. It corresponds to a cuspidal edge on the wavefront and is akin to the expression for the wavefield near a caustic. Such points arise when the sheets of S fail to be convex.

Notice that each term of (203) has the form

$$G_{ij}(\mathbf{x}, t; \mathbf{x}', t') = F_{ij}(\mathbf{x}, \mathbf{x}')\delta[t - t' - \boldsymbol{\xi} \cdot (\mathbf{x} - \mathbf{x}')], \tag{208}$$

where F_{ij} is a smooth function of $\mathbf{x}, \mathbf{x}'$ away from $\mathbf{x} = \mathbf{x}'$, which never holds in the far field. If the medium is not homogeneous then G_{ij} has a similar structure in the ray theoretical approximation when no ray focussing occurs:

$$G_{ij}(\mathbf{x}, t; \mathbf{x}', t') = F_{ij}(\mathbf{x}, \mathbf{x}')\delta[t - t' - T(\mathbf{x}, \mathbf{x}')], \tag{209}$$

where F_{ij} and T are smooth functions, T being the characteristic travel time along the ray joining $\mathbf{x}$ and $\mathbf{x}'$. Since (103) is a distribution of dipoles we shall need the gradient of G with respect to $\mathbf{x}'$:

$$G_{ij,x'_k}(\mathbf{x}, t; \mathbf{x}', t') = \partial_{x'_k} G_{ij}(\mathbf{x}, t; \mathbf{x}', t') = -F_{ij}(\mathbf{x}, \mathbf{x}')T_{,x'_k}\delta'[t - t' - T(\mathbf{x}, \mathbf{x}')]. \tag{210}$$

But $-T_{,x'_k}$ is just the wave slowness for the ray at $\mathbf{x}'$. Let us write

$$\xi'_k = -T_{,x'_k}. \tag{211}$$

Then

$$G_{ij,x'_k} = F_{ij}\xi'_k\delta'[t - t' - T(\mathbf{x}, \mathbf{x}')]. \tag{212}$$

The Far Field

The far field radiated by the source of (103) may be written as a sum of terms like

$$\begin{aligned} u_i^{\boldsymbol{\xi}}(\mathbf{x}, t) &= \int\int f_j(\mathbf{x}', t')G_{ij,x'_k}(\mathbf{x}, t; \mathbf{x}', t')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t' \\ &= \int\int\int A(s)N^*_{jm}(s)p^{(0)}(s, t')\delta_{,m}[\mathbf{x}' - \mathbf{X}(s)]\,\mathrm{d}s\,G_{ij,x'_k}(\mathbf{x}, t; \mathbf{x}', t')\,\mathrm{d}\mathbf{x}'\,\mathrm{d}t' \\ &= \int\int A(s)N^*_{jm}(s)p^{(0)}(s, t')F_{ij}[\mathbf{x}, \mathbf{X}(s)]\xi'_m\delta'[t - t' - T(\mathbf{x}, \mathbf{X}(s))]\,\mathrm{d}s\,\mathrm{d}t', \end{aligned} \tag{213}$$

where we have performed the $\mathbf{x}'$ integration and used (212). Now the superscript $\boldsymbol{\xi}$ labels the different ray contributions. Performing the t' integral we obtain

$$u_i^{\boldsymbol{\xi}}(\mathbf{x}, t) = \int A(s)N^*_{jm}(s)p^{(0)}[s, t - T(\mathbf{x}, \mathbf{X}(s))]F_{ij}[\mathbf{x}, \mathbf{X}(s)]\xi'_m\,\mathrm{d}s, \tag{214}$$

and then using (84) we get

$$u_i^{\boldsymbol{\xi}}(\mathbf{x},t) = \int \frac{V_0 A(s)}{2A(0)} N^*_{jm}(s)\zeta^{\frac{1}{2}}(0)\zeta^{\frac{1}{2}}(s) \times g''[t - \operatorname{sgn}(s)\int_0^s \gamma_T(s')\,\mathrm{d}s' - T(\mathbf{x},\mathbf{X}(s))]F_{ij}[\mathbf{x},\mathbf{X}(s)]\xi'_m\,\mathrm{d}s. \quad (215)$$

It is convenient to write this integral in the form

$$\begin{aligned}\int_{-\infty}^{+\infty} G(s)g''[t-\phi(s)]\,\mathrm{d}s &= \int_{-\infty}^{+\infty} G(s)\delta[t-\phi(s)]\,\mathrm{d}s *_t g''(t) \\ &= I(t) *_t g''(t), \end{aligned} \quad (216)$$

say. We shall now investigate $I(t)$, which appears to represent the response to $g''(t) = \delta(t)$, i.e., $g(t) = tH(t)$, where H is the Heaviside step function. However, because of the restriction to slow time variation, (216) is only relevant when g varies slowly and smoothly in time. Because of the occurrence of δ in I we may evaluate the integral

$$\begin{aligned} I(t) &= \int G(s)\delta[t-\phi(s)]\,\mathrm{d}s \\ &= \sum_{\{s|\phi(s)=t\}} \frac{G(s)}{|\phi'(s)|}, \end{aligned} \quad (217)$$

where the sum extends over all values of s for which $\phi(s) = t$. We shall assume for simplicity that $\phi(s)$ has only one local minimum s_0. We shall distinguish two cases: $s_0 = 0$, and $s_0 \neq 0$, in which case we shall assume for definiteness that $s_0 > 0$. In fast formations, for which $\gamma_T > \boldsymbol{\xi}' \cdot \mathbf{t}$, $s_0 = 0$. In slow formations this inequality may fail for some positions of the receiver, and at $s = 0$ we may have $\gamma_T < \boldsymbol{\xi}' \cdot \mathbf{t}$. We shall consider these two cases separately. We shall see that when $s_0 = 0$ the only geometrical wave arrival emanates from the source point $s = 0$. When $s_0 > 0$ there are two arrivals, one of which emanates from the source, but there is an earlier "conical" wave arrival corresponding to the point s_0.

Let us now examine the function ϕ

$$\begin{aligned}\phi(s) &= \operatorname{sgn}(s)\int_0^s \gamma_T(s')\,\mathrm{d}s' + T(\mathbf{x},\mathbf{X}(s)) \\ &= \phi_1(s) + \phi_2(s), \end{aligned} \quad (218)$$

say. Thus

$$\phi_1'(s) = \operatorname{sgn}(s)\gamma_T(s), \quad \phi_2'(s) = -\boldsymbol{\xi}' \cdot \mathbf{t}. \quad (219)$$

For a fast formation $\gamma_T > \xi' \cdot \mathbf{t}$ at $s = 0$, so that $\phi = \phi_1 + \phi_2$ has the form shown in Figure 3. In this case $\phi(s)$ has a unique minimum at $s = 0$ with

$$\begin{aligned}\phi'(-0) &= -[\gamma_T + \boldsymbol{\xi}' \cdot \mathbf{t}]_{s=0} < 0, \\ \phi'(+0) &= [\gamma_T - \boldsymbol{\xi}' \cdot \mathbf{t}]_{s=0} > 0. \end{aligned} \quad (220)$$

Thus for $t < \phi(0)$ there are no terms in the sum (217) and $I(t) = 0$. When $t > \phi(0)$ there are two values, $s_1 < 0$ and $s_2 > 0$, of s for which $\phi(s) = t$, and from (217) we may write

$$I(t) = \frac{G(s_1)}{\gamma_T(s_1) + \boldsymbol{\xi}(s_1)' \cdot \mathbf{t}(s_1)} + \frac{G(s_2)}{\gamma_T(s_2) - \boldsymbol{\xi}(s_2)' \cdot \mathbf{t}(s_2)}. \tag{221}$$

At $t = \phi(0) + 0$

$$\begin{aligned} I(t) &= G(0)\left(\frac{1}{\gamma_T(0) + \boldsymbol{\xi}(0)' \cdot \mathbf{t}(0)} + \frac{1}{\gamma_T(0) - \boldsymbol{\xi}(0)' \cdot \mathbf{t}(0)}\right), \\ &= \frac{2G(0)\gamma_T(0)}{\gamma_T(0)^2 - (\xi(0)' \cdot \mathbf{t}(0))^2}, \end{aligned} \tag{222}$$

and so $I(t)$ jumps by this amount at $t = \phi(0)$ and is smooth elsewhere. $I(t)$ therefore has a graph as shown in Figure 4. In a slow formation, let us suppose that for the field point under consideration $s_0 > 0$. The graph of ϕ is as in Figure 5. Then $\phi(s_0) < \phi(0)$ and for $t < \phi(s_0)$ there are no s for which $\phi(s) = t$, and so the sum (217) is vacuous. For $t > \phi(s_0)$ there are two such s, $s_1 < s_0$ and $s_2 > s_0$. When $t - \phi(s_0) > 0$ is small then $|s_i - s_0|$, $i = 1, 2$, are both small and we have approximately

$$\begin{aligned} \phi(s_i) - \phi(s_0) &= \tfrac{1}{2}\phi''(s_0)(s_i - s_0)^2, \\ \phi'(s_i) &= \phi''(s_0)(s_i - s_0). \end{aligned} \tag{223}$$

Then approximately for $i = 1, 2$

$$\phi'(s_i) = \operatorname{sgn}(s_i - s_0)\sqrt{2\phi''(s_0)[\phi(s_i) - \phi(s_0)]}. \tag{224}$$

Thus for $t > \phi(s_0)$, $I(t)$ has a reciprocal square-root singularity

$$I(t) = \sqrt{\frac{2}{\phi''(s_0)[t - \phi(s_0)]}}\,G(s_0) + O([t - \phi(s_0)]^{-\frac{1}{2}}). \tag{225}$$

At $t = \phi(0) - 0$, $s = +0$ and

$$|\phi'(+0)| = \xi(0)' \cdot \mathbf{t}(0) - \gamma_T(0). \tag{226}$$

At $t = \phi(0) + 0$, $s = -0$ and

$$|\phi'(-0)| = \boldsymbol{\xi}(0)' \cdot \mathbf{t}(0) + \gamma_T(0). \tag{227}$$

Thus at $t = \phi(0)$, $I(t)$ jumps by

$$\begin{aligned} I(\phi(0) + 0) - I(\phi(0) - 0) &= G(0)\left(\frac{1}{\gamma_T + \boldsymbol{\xi}(0)' \cdot \mathbf{t}(0)} + \frac{1}{\gamma_T - \boldsymbol{\xi}(0)' \cdot \mathbf{t}(0)}\right) \\ &= \frac{2G(0)\gamma_T}{\gamma_T^2 - (\xi(0)' \cdot \mathbf{t}(0))^2} < 0. \end{aligned} \tag{228}$$

The graph of $I(t)$ is now as shown in Figure 6.

We notice that formulae (222) and (228) are identical. Thus in all cases there is an arrival corresponding to the jump discontinuity, which seems to emanate from the source point. On referring back to (215) we identify $G(0)$ and then, interpreting convolution with a step function as integration, we obtain

$$u_i(\mathbf{x},t) = \sum_{\boldsymbol{\xi}} u_i^{\boldsymbol{\xi}}(\mathbf{x},t) = \sum_{\boldsymbol{\xi}} \frac{\rho_f V_0 N_{jm}^*(0) F_{ij}(\mathbf{x},\mathbf{0})\xi_m'}{\gamma_T^2 - (\boldsymbol{\xi}'\cdot\mathbf{t})^2} g'[t - T(\mathbf{x},\mathbf{0})]. \tag{229}$$

For a uniform medium we may identify $F_{ij}\xi_m'$ by comparing (208) and (203). Thus

$$F_{ij}(\mathbf{x},0) = \frac{P_i(\boldsymbol{\xi})P_j(\boldsymbol{\xi})\boldsymbol{\xi}\cdot\mathbf{x}}{4\pi\rho|C(\boldsymbol{\xi})|^{\frac{1}{2}}|\mathbf{x}|^2}, \quad \xi_m' = \xi_m, \tag{230}$$

and so the wave arrival which emanates from the source is

$$u_i(\mathbf{x},t) = \sum_{\boldsymbol{\xi}} \frac{\rho_f V_0 N_{jm}^*(0) P_i(\boldsymbol{\xi})P_j(\boldsymbol{\xi})\boldsymbol{\xi}\cdot\mathbf{x}\xi_m}{4\pi\rho|C(\boldsymbol{\xi})|^{\frac{1}{2}}|\mathbf{x}|^2[\gamma_T^2 - (\boldsymbol{\xi}\cdot\mathbf{t})^2]} g'[t - T(\mathbf{x},\mathbf{0})]. \tag{231}$$

In addition, in slow formations and at certain positions of the source, there will be an arrival corresponding to the stationary point s_0 with a pulse shape which is the fractional derivative of order $\frac{1}{2}$ of the g' appearing in (231). This pulse shape arises by convolution of g'' with the reciprocal square root. In the isotropic case this arrival has a circular conical wave front. Explicitly we may write this contribution in the form

$$\begin{aligned} u_i(\mathbf{x},t) &= \sum_{\boldsymbol{\xi}} \frac{V_0 A(s_0) N_{jm}^*(s_0)\zeta^{\frac{1}{2}}(0)\zeta^{\frac{1}{2}}(s_0)P_i(\boldsymbol{\xi})P_j(\boldsymbol{\xi})\boldsymbol{\xi}\cdot\mathbf{x}\xi_m}{8\pi\rho A(0)|C(\boldsymbol{\xi})|^{\frac{1}{2}}|\mathbf{x}|^2}\sqrt{\frac{2}{\phi''(s_0)}} \\ &\quad \times \int \frac{1}{[t - t' - \phi(s_0)]^{\frac{1}{2}}} g'(t')\,\mathrm{d}t'. \end{aligned} \tag{232}$$

The slowness $\boldsymbol{\xi}$ must be evaluated at the point $s = s_0$ and the quantity $\phi''(s_0)$ may be evaluated explicitly for isotropic media and will then be proportional to $|\mathbf{x}|^{-1}$. The sum in (232) is over only those $\boldsymbol{\xi}$ for which $\boldsymbol{\xi}(0)'\cdot\mathbf{t}(0) - \gamma_T(0) > 0$. Typically this will only happen for the (quasi-)S waves.

The Two-Borehole Problem

We consider now the problem of a source in one well and a receiver in another well. The source is of a volume injection type, supplying accumulated volume $V_0 g(t)$ up to time t, and the receiver is sensitive to pressure. For simplicity, we will assume that the medium surrounding the boreholes is homogeneous but anisotropic. The boreholes need not be straight, and the distance between them is $O(1/\epsilon^2)$. The one-dimensional acoustic system in the receiver borehole follows from (62), (64), and (65), with the source term G set to zero. This leads to

$$(\sigma^R + N^R(\mathrm{I}))p^{(0)}_{,t} + v^{(0)}_{3,s} = -[\delta_{ij} - t_i t_j + (\delta_{mn} - t_m t_n)N^R_{mnpq}c_{pqij}]u^{I(0)}_{i,jT}, \tag{233}$$

$$\rho^R_f v^{(0)}_{3,t} + p^{(0)}_{,s} = 0, \tag{234}$$

where s is the arclength along the receiver borehole, and the superscript R indicates quantities related to the receiver as opposed to the source. We will use the superscript S in relation to the source. The incident field $u^{I(0)}_i(\mathbf{x}(s),t)$ in the case of a fast formation (with respect to the source borehole) is as given by (231), which we repeat here

$$u^{I(0)}_i(\mathbf{x}(s),t) = \sum_{\boldsymbol{\xi}(s)} \frac{\rho^S_f V_0 N^{*S}_{jm}(0) P_i(\boldsymbol{\xi}(s)) P_j(\boldsymbol{\xi}(s)) \boldsymbol{\xi}(s)\cdot\mathbf{x}(s)\xi_m(s)}{4\pi\rho|C(\boldsymbol{\xi}(s))|^{\frac{1}{2}}|\mathbf{x}(\mathrm{s})|^2[(\gamma^S_T)^2 - (\boldsymbol{\xi}(s)\cdot\mathbf{t}^S)^2]} g'[t - T_1(\mathbf{x}(s),\mathbf{0})], \tag{235}$$

where $\boldsymbol{\xi}(s)$ is such that $\boldsymbol{\xi}(s)\cdot\mathbf{x}(s) = t$, and $T_1(\mathbf{x}(s),\mathbf{0})$ is the travel time from the source to the point $\mathbf{x}(s)$ which is along the receiver borehole.

Following (101) and the reciprocity relation (A.10) from the Appendix, equation (233) can be rewritten as

$$(\sigma^R + N^R(\mathrm{I}))p^{(0)}_{,t} + v^{(0)}_{3,s} = -N^{*R}_{ij}(s)u^{I(0)}_{i,jT}(\mathbf{x}(s),t). \tag{236}$$

The solution of (234) and (236) follows from the results of Section 3. The pressure is thus given by

$$p^{(0)}(s,t) =$$

$$-\int_{-\infty}^{+\infty}\sum_{\boldsymbol{\xi}} \frac{\rho^S_f V_0 N^{*S}_{km}(0) N^{*R}_{ij}(s') P_i(\boldsymbol{\xi}(s')) P_k(\boldsymbol{\xi}(s'))\xi_m(s')\xi_j(s')\boldsymbol{\xi}(s)\cdot\mathbf{x}(s)[\zeta^R(s')\zeta^R(s)]^{\frac{1}{2}}}{8\pi\rho|C(\boldsymbol{\xi}(s'))|^{\frac{1}{2}}|\mathbf{x}(s')|^2[(\gamma^S_T)^2 - (\boldsymbol{\xi}(s)\cdot\mathbf{t}^S)^2]}$$

$$\times\, \delta(t - T_1(\mathbf{x}(s'),0) - T_2(s,s')) *_t g'''(t)\,\mathrm{d}s', \tag{237}$$

where $T_2(s,s')$ is the tube wave travel time from s' to s in the receiver borehole. Using the result obtained in (216) and (217) for the integral over s', we finally obtain

$$\begin{aligned} p^{(0)}(s,t) \quad &= \quad \sum_{\boldsymbol{\xi}} \frac{\rho^S_f \rho^R_f V_0 N^{*S}_{km}(0) N^{*R}_{ij}(s) P_i(\boldsymbol{\xi}(s)) P_k(\boldsymbol{\xi}(s))\xi_m(s)\xi_j(s)\boldsymbol{\xi}(s)\cdot\mathbf{x}(s)}{4\pi\rho|C(\boldsymbol{\xi}(s))|^{\frac{1}{2}}|\mathbf{x}(s)|^2[(\gamma^S_T)^2 - (\boldsymbol{\xi}(s)\cdot\mathbf{t}^S)^2][(\gamma^R_T)^2 - (\boldsymbol{\xi}(s)\cdot\mathbf{t}^R)^2]} \\ &\quad \times g''(t - T_1(\mathbf{x}(s),0)). \end{aligned} \tag{238}$$

If the tube wave in either borehole is faster than the body wave under consideration, then there will be additional "conical" wave arrivals which we shall not treat here. If the slowness surface is not convex at $\boldsymbol{\xi}$, however, we may replace g'' by $g^{*\prime\prime}$ where, in keeping with (207),

$$g^* = \begin{cases} g, & \text{if } S \text{ is convex outward at } \boldsymbol{\xi}; \\ -\mathcal{H}g, & \text{if } S \text{ is saddle shaped at } \boldsymbol{\xi}; \\ -g, & \text{if } S \text{ is concave outward at } \boldsymbol{\xi}. \end{cases} \tag{239}$$

SOME EXAMPLES

In this section we shall illustrate the results obtained in the previous sections with examples of boreholes in isotropic and transversely isotropic homogeneous formations.

Circular Borehole in a Homogeneous Isotropic Medium

We consider a straight borehole of circular cross section of radius r_0 penetrating a homogeneous isotropic formation characterized by a volume density of mass ρ and Lamé parameters λ and μ. The compressional and shear wave speeds in the solid are α and β, as given in (201). The density of the fluid in the borehole is ρ_f and its wave speed α_f is given by

$$\frac{1}{\alpha_f^2} = \rho_f \sigma, \tag{240}$$

[see (10) and (11)]. Under these conditions, the expression for the equivalent body force system [equation (103)] reduces to

$$f_i(\mathbf{x}, t) = -AN^*_{im} \int p^{(0)}(x'_3, t)\delta_{,m}(\mathbf{x} - \mathbf{X}(x'_3))\mathrm{d}x'_3, \tag{241}$$

where $A = \pi r_0^2$, the borehole axis is taken as $\mathbf{X}(x_3) = (0, 0, x_3)$, and the tensor N^* is given in (200), which we repeat here:

$$\mathrm{N}^* = \begin{pmatrix} \alpha^2/\beta^2 & 0 & 0 \\ 0 & \alpha^2/\beta^2 & 0 \\ 0 & 0 & \alpha^2/\beta^2 - 2 \end{pmatrix}. \tag{242}$$

The far-field displacements can be computed from (231). We first notice that due to the homogeneity of the medium we have $\boldsymbol{\xi} = \boldsymbol{\xi}'$. Furthermore, due to isotropy there are only two $\boldsymbol{\xi}$ vectors, namely $\boldsymbol{\xi}_\alpha = \mathbf{x}/\alpha|\mathbf{x}|$ and $\boldsymbol{\xi}_\beta = \mathbf{x}/\beta|\mathbf{x}|$. The Gaussian curvatures associated with these vectors are $C(\boldsymbol{\xi}_\alpha) = 1/\alpha^2$ and $C(\boldsymbol{\xi}_\beta) = 1/\beta^2$, and the polarization vectors are $\mathbf{P}(\xi_\alpha) = \mathbf{x}/|\mathbf{x}|$ and $\mathbf{P}(\xi_\beta) = \mathbf{y}/|\mathbf{y}|$ where $\mathbf{y} = \mathbf{x} \times (\mathrm{e}_3 \times \mathbf{x})$.

The third polarization corresponding to SH waves is not excited due to the circular symmetry of the borehole. Finally from (73), (198), and (240) we have $\gamma_T = 1/\alpha_T = \sqrt{1/\alpha_f^2 + \rho_f/\rho\beta^2}$. The far-field displacements are then given by

$$
\begin{aligned}
u_i(\mathbf{x},t) \quad = \quad & -\frac{\rho_f V_0 \gamma_i}{4\pi\rho}\left[\frac{1}{\alpha^3}\,\frac{(\alpha^2/\beta^2 - 2\cos^2\varphi)}{(\cos^2\varphi/\alpha^2 - 1/\alpha_T^2)}\,\frac{g'(t-|\mathbf{x}|/\alpha)}{|\mathbf{x}|}\right. \\
& \left. + \frac{1}{\beta^3}\,\frac{(N^*_{\underline{ii}} - \alpha^2/\beta^2 + 2\cos^2\varphi)}{(\cos^2\varphi/\beta^2 - 1/\alpha_T^2)}\,\frac{g'(t-|\mathbf{x}|/\beta)}{|\mathbf{x}|}\right], \qquad (243)
\end{aligned}
$$

where the summation convention is suppressed for underlined indices, and $\gamma_i = \mathbf{x}\cdot\mathbf{e}_i$ ($i = 1,2,3$) with $\gamma_3 = \cos\varphi$. The far-field displacements in terms of spherical coordinates (R,φ,θ) can be computed from

$$
\begin{aligned}
u_R &= \gamma_i u_i, && (244)\\
u_\varphi &= \sqrt{u_1^2 + u_2^2}\,\gamma_3 - u_3\sqrt{1-\gamma_3^2}, && (245)
\end{aligned}
$$

resulting in the following expressions

$$
\begin{aligned}
u_R(R,\varphi,\theta,t) &= \frac{\rho_f \alpha_T^2 V_0}{4\pi\rho\alpha\beta^2}\,\frac{(1-2\beta^2\cos^2\varphi/\alpha^2)}{(1-\alpha_T^2\cos^2\varphi/\alpha^2)}\,\frac{g'(t-R/\alpha)}{R}, && (246)\\
u_\varphi(R,\varphi,\theta,t) &= \frac{\rho_f \alpha_T^2 V_0}{2\pi\rho\beta^3}\,\frac{\sin\varphi\cos\varphi}{(1-\alpha_T^2\cos^2\varphi/\beta^2)}\,\frac{g'(t-R/\beta)}{R}, && (247)
\end{aligned}
$$

where $R = (\mathbf{x}\cdot\mathbf{x})^{1/2}$. These expressions agree with the ones given by Lee and Balch (1982), who derived them by using a stationary phase approximation to the exact integral expression for the displacements induced by a point source inside a fluid-filled borehole.

Distributed Versus Localized Sources Along the Borehole

The equivalent force system given by (241) consists of three mutually orthogonal dipoles moving along the borehole axis at the tube wave speed. Carrying out the integration indicated in (241) we obtain

$$
\mathbf{f}(\mathbf{x},t) = -\pi r_0^2 \begin{pmatrix} N^*_{11}\, p^{(0)}(x_3,t)\delta'(x_1)\delta(x_2) \\ N^*_{22}\, p^{(0)}(x_3,t)\delta(x_1)\delta'(x_2) \\ N^*_{33}\, p^{(0)}_{,3}(x_3,t)\delta(x_1)\delta(x_2) \end{pmatrix}. \qquad (248)
$$

The pressure field $p^{(0)}(x_3,t)$ can be further expanded in terms of multipoles by testing it with a scalar function $\phi(x_3)$ and using the distributional definition of the derivatives

$$
I = \int_{-\infty}^{+\infty} p^{(0)}(x_3,t)\phi(x_3)\,\mathrm{d}x_3
$$

$$= \int_{-\infty}^{+\infty} p^{(0)}(x_3,t)\left(\sum_{n=0}^{\infty}\phi^{(n)}(0)\frac{x_3^n}{n!}\right) \mathrm{d}x_3. \tag{249}$$

From (84) we obtain

$$p^0(x_3,t) = \frac{\rho_f \alpha_T V_0}{2A} g'(t - \frac{|x_3|}{\alpha_T}), \tag{250}$$

which substituted into (249) gives

$$\begin{aligned} I &= \frac{\rho_f \alpha_T V_0}{2A}\sum_{n=0}^{\infty}\phi^{(n)}(0)\int_{-\infty}^{+\infty}\frac{x_3^n}{n!}\; g'(t-\frac{|x_3|}{\alpha_T})\; \mathrm{d}x_3 \\ &= \frac{\rho_f \alpha_T V_0}{A}\sum_{m=0}^{\infty}\phi^{(2m)}(0)\int_0^{\infty}\frac{x_3^{2m}}{(2m)!}\; g'(t-\frac{x_3}{\alpha_T})\; \mathrm{d}x_3 \\ &= \frac{\rho_f \alpha_T V_0}{A}\sum_{m=0}^{\infty}\alpha_T^{2m+1}\phi^{(2m)}(0)\int_0^{\infty}\frac{y^{2m}}{(2m)!}\; g'(t-y)\; \mathrm{d}y. \end{aligned} \tag{251}$$

The first term in the above expansion corresponding to $m = 0$ is simply given by

$$\begin{aligned} I^{(0)} &= \frac{\rho_f \alpha_T^2 V_0}{A}\phi(0)g(t) \\ &= \int_{-\infty}^{+\infty}\frac{\rho_f \alpha_T^2 V_0}{A} g(t)\delta(x_3)\phi(x_3)\; \mathrm{d}x_3, \end{aligned} \tag{252}$$

which when compared to (249) gives the following approximation for $p^{(0)}(x_3,t)$

$$p^{(0)}(x_3,t) \approx \frac{\rho_f \alpha_T^2 V_0}{A} g(t)\delta(x_3). \tag{253}$$

Similarly, a multipole expansion of $p^{(0)}_{,3}(x_3,t)$ can be carried out resulting in the following approximation

$$p^{(0)}_{,3}(x_3,t) \approx \frac{\rho_f \alpha_T^2 V_0}{A} g(t)\delta'(x_3). \tag{254}$$

Under these approximations, the force system in (241) has been localized at the origin. It consists of three mutually orthogonal dipoles, or equivalently, a monopole with moment $M_0 = \rho_f \alpha_T^2 V_0 \alpha^2/\beta^2$ and a dipole in the x_3-direction with moment $M = -2(\beta^2/\alpha^2)M_0$. This same result was obtained by Ben-Menahem and Kostek (1991), whose analysis assumed from the outset that a point monopole source in a fluid-filled borehole could be replaced by a localized mechanism in an infinite homogeneous medium.

To investigate the consequences of this approximation we compute, as before, the far-field displacements induced by such a localized force system. The far-field displacements can be computed as in (231) giving the following displacements

$$u_i(\mathbf{x},t) = \frac{\rho_f \alpha_T^2 V_0 \gamma_i}{4\pi\rho}\left[\frac{(\alpha^2/\beta^2 - 2\cos^2\varphi)}{\alpha^3}\,\frac{g'(t-|\mathbf{x}|/\alpha)}{|\mathbf{x}|} + \frac{(N^*_{\underline{ii}} - \alpha^2/\beta^2 + 2\cos^2\varphi)}{\beta^3}\,\frac{g'(t-|\mathbf{x}|/\beta)}{|\mathbf{x}|}\right]. \tag{255}$$

Using expressions (244) and (245) we can express the far-field displacements in terms of spherical coordinates as

$$u_R(R,\varphi,\theta,t) = \frac{\rho_f \alpha_T^2 V_0}{4\pi\rho\alpha}\,\frac{(1-2\beta^2\cos^2\varphi/\alpha^2)}{\beta^2}\,\frac{g'(t-R/\alpha)}{R}, \tag{256}$$

$$u_\varphi(R,\varphi,\theta,t) = \frac{\rho_f \alpha_T^2 V_0}{2\pi\rho\beta}\,\frac{\sin\varphi\cos\varphi}{\beta^2}\,\frac{g'(t-R/\beta)}{R}. \tag{257}$$

Comparing these expressions with the ones derived earlier we notice that the bracketed term in the denominator of (246) and (247) is missing in both of these expressions. That term in the radiation pattern originates from the motion of the three mutually orthogonal dipole sources, as pointed out by Kurkjian et al. (1992). The numerical results in Ben-Menahem and Kostek (1991), for both the point source in a fluid-filled borehole and the above localized mechanism in an infinite solid medium, are in good agreement because their particular choice of parameters was such that $\pi/4 \le \varphi \le 3\pi/4$ and $\alpha_T < \beta$.

Borehole in a Transversely Isotropic Medium

We consider now a circular borehole in a transversely isotropic medium with symmetry axis along the x_3-direction, i.e., the borehole axis. The particular medium has the elastic properties of Cotton-Valley shale (Thomsen, 1986), and if we take a local reference system with the x_3-direction along the symmetry axis of the medium, the elastic constants are: $c_{11} = 74.73$; $c_{12} = 14.75$; $c_{13} = 25.29$; $c_{33} = 58.84$; $c_{44} = 22.05$; $c_{66} = 29.99$, in units of GPa. The volume density of mass of this material is $\rho = 2640.0$ kg/m^3. The borehole fluid density is $\rho_f = 1000.0$ kg/m^3, its wave speed is $\alpha_f = 1500.0$ m/s, and the borehole radius is taken as $r_0 = 0.1$ m.

Using (228) we computed the radiation patterns of quasi-P (qP) and quasi-SV (qSV) waves generated by the tube wave in the borehole. These are shown in Figures 7 and 11, while their corresponding wavefront surfaces are shown in Figures 8 and 12. For comparison we also show the radiation patterns of P and SV waves in an isotropic medium (Figures 9 and 13, respectively), with the same density as the transversely isotropic medium defined above but with compressional and shear wave speeds given by $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$, respectively. The corresponding wavefront surfaces are shown in Figures 10 and 14. These radiation patterns could have also been obtained by the same technique as employed by Lee and Balch (1982), since the problem is amenable to analytical calculations due to the symmetry.

A case which cannot be analyzed by their technique is, for instance, that of a transversely isotropic medium with symmetry axis perpendicular to the borehole axis. We take the symmetry axis along the x_1-direction. In Figures 15 and 19 we show the qP and qSV wave radiation patterns. The corresponding wavefront surfaces are shown in Figures 16 and 20. Here, the qSV polarization is such that the particle motion is in planes containing the x_1-axis. For comparison, we also show the P and SV radiation patterns for the isotropic medium defined above (Figures 17 and 21, respectively). The P-wave radiation patterns shown in Figures 13 and 17 are the same, but the SV radiation patterns in Figures 13 and 21 are different because of the different definitions of the SV polarization in both cases. The wavefront surfaces for the P and SV waves in the isotropic medium are shown in Figures 18 and 22, respectively. Finally, we show the SH radiation pattern in Figure 23 with the corresponding wavefront surface shown in Figure 24. We also show in Figures 25 and 26 the radiation pattern and wavefront surface for the isotropic medium, with the definition that the polarization vector is perpendicular to planes containing the x_1-axis.

CONCLUSIONS

We have set up a formalism for calculating the seismic radiation from a borehole asymptotically in the limit as the ratio of the borehole diameter to wavelength goes to zero. In this limit an acoustic source in the borehole acts indirectly as a seismic source. The source first generates a tube wave, which is an acoustic wave in the fluid filling the borehole, and the pressure field of the tube wave, by distorting the borehole wall, in turn generates the seismic wave. The action of the tube wave is equivalent in the narrow borehole approximation to a line distribution of body force along the borehole centerline and acting in the intact elastic solid, i.e., the solid with no hole bored in it. We have found expressions for this source distribution, which turns out to be a distribution of dipoles.

In previous work the asymptotic limit of a narrow hole was calculated by taking the low frequency approximation to an exact solution for a circular cylindrical hole in an isotropic medium. By directly calculating the asymptotic limit one is able to find these body-force equivalents and other aspects of the solution in quite general circumstances, without the need for exact solutions. In fact we have found the asymptotic solution for a curved hole with elliptic cross-section in an arbitrary anisotropic medium. When these results are specialized to right circularly cylindrical holes in an isotropic medium they agree with previously appearing work in the literature.

We have also considered the radiation problem from the equivalent source distribution in both fast and slow formations, and have provided far-field expressions for the displacements. In slow (anisotropic) formations there can be either one or two "conical" waves arriving earlier than the direct wave from source to receiver. We have illustrated our results by plotting radiation patterns for quasi-P, quasi-SV, and SH waves in isotropic and transversely isotropic media and the corresponding wavefront surfaces. We illustrate the cases where the borehole axis and the TI axis are parallel, and also when they are perpendicular. These two cases can be completely solved analytically. More general situations may require the numerical solution of sextic equations. It is interesting that the expression for the body-force distribution in the equivalent source is obtained as a product of two matrices, one a function of the material properties of the (anisotropic) medium, including its orientation relative to the borehole, and the other a function of the parameters of the elliptical cross-section of the borehole.

Finally, we considered the problem of computing the pressure field in one borehole induced by a volume injection source in another borehole. The far-field solution is obtained in closed form, and in particular it clearly shows the reciprocal nature of the problem.

APPENDIX: A RECIPROCITY RELATION

Consider a borehole of elliptic cross-section with axis in the direction $\mathbf{t}$, $|\mathbf{t}| = 1$. Suppose that the surface of the borehole is acted on by a traction $\boldsymbol{\tau} \cdot \mathbf{n}$ derivable from a constant stress $\boldsymbol{\tau}$. Suppose, moreover, that there is no extension in the direction $\mathbf{t}$. (This is the anisotropic equivalent of plane strain.) Then it is known that the displacement $\mathbf{u}$ on the borehole wall is, up to a rigid body motion, a linear function of position:

$$u_i = \sigma_{ik} x_k. \tag{A.1}$$

Here the matrix σ may be taken to be such that

$$\sigma_{ik} t_k = 0. \tag{A.2}$$

We shall consider reciprocity between two such stress states $\{\mathbf{u}^{(1)}, \boldsymbol{\sigma}^{(1)}, \boldsymbol{\tau}^{(1)}\}$ and $\{\mathbf{u}^{(2)}, \boldsymbol{\sigma}^{(2)}, \boldsymbol{\tau}^{(2)}\}$ for the same borehole in the same anisotropic elastic medium. Betti's reciprocity theorem (Love, 1927, p. 173) implies that, in the notation of Section 4,

$$\int_{\partial\Sigma} u_i^{(1)} \tau_{ij}^{(2)} n_j \, \mathrm{d}s' = \int_{\partial\Sigma} u_i^{(2)} \tau_{ij}^{(1)} n_j \, \mathrm{d}s', \tag{A.3}$$

where $\partial\Sigma$ is the perimeter of a right cross-section Σ of the borehole, $\mathbf{n}$ is the unit normal to the borehole wall, and $\mathrm{d}s'$ is the element of arclength along $\partial\Sigma$. Let the constant stress fields $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$ be extended as constant functions of position into the interior of the borehole, and the displacement fields $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$ as the linear functions of (A.1). Using equation (96):

$$n_j = \epsilon_{jpq} t'_p t_q \tag{A.4}$$

in the left member of (A.3), and then applying Stokes's theorem we get

$$\begin{aligned}
\int_{\partial\Sigma} u_i^{(1)} \tau_{ij}^{(2)} n_j \, \mathrm{d}s' &= \int_{\partial\Sigma} u_i^{(1)} \tau_{ij}^{(2)} \epsilon_{jpq} t'_p t_q \, \mathrm{d}s' \\
&= \int_{\partial\Sigma} \epsilon_{rnp} u_{i,n}^{(1)} \tau_{ij}^{(2)} \epsilon_{jpq} t_q t_r \, \mathrm{d}A \\
&= \int_{\partial\Sigma} (\delta_{qr}\delta_{nj} - \delta_{qj}\delta_{nr}) \sigma_{in}^{(1)} \tau_{ij}^{(2)} t_q t_r \, \mathrm{d}A \\
&= A\sigma_{in}^{(1)} (\delta_{nj} - t_n t_j) \tau_{ij}^{(2)} \\
&= A\sigma_{ij}^{(1)} \tau_{ij}^{(2)},
\end{aligned} \tag{A.5}$$

by (A.2). Applying a similar calculation to the right member of (A.3) and equating the two we obtain

$$\sigma_{ij}^{(1)} \tau_{ij}^{(2)} = \sigma_{ij}^{(2)} \tau_{ij}^{(1)}. \tag{A.6}$$

But $\boldsymbol{\sigma}$ is a linear function of $\boldsymbol{\tau}$, say

$$\sigma_{ij} = N_{ijpq} \tau_{pq}. \tag{A.7}$$

Then using (A.7) in (A.6) we obtain

$$N_{ijpq} \tau_{pq}^{(1)} \tau_{ij}^{(2)} = N_{ijpq} \tau_{pq}^{(2)} \tau_{ij}^{(1)}. \tag{A.8}$$

The tensors $\boldsymbol{\tau}^{(1)}$ and $\boldsymbol{\tau}^{(2)}$ are symmetric but otherwise arbitrary, and so

$$N_{ijpq} = N_{pqij}, \tag{A.9}$$

and we may assume $\mathbf{N}$ has the symmetries

$$N_{ijpq} = N_{jipq} = N_{ijqp}. \tag{A.10}$$

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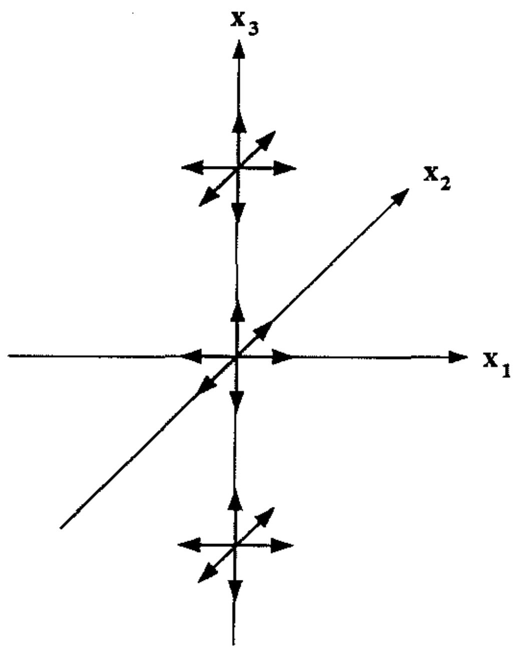

Figure 1: Symmetrical distribution of dipoles for a straight borehole, corresponding to the integrand in equation (103).

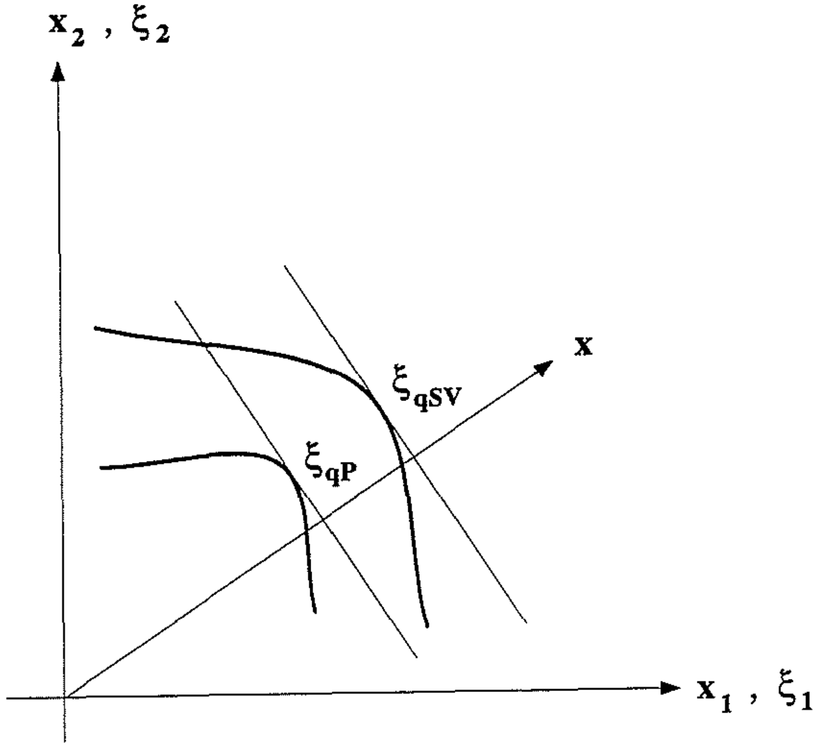

Figure 2: qP and qSV slowness surfaces for a TI medium with symmetry axis along the x_3-direction. The indicated points on the two slowness sheets (touched by the planes perpendicular to $\mathbf{x}$) give the most singular contribution at $\mathbf{x}$ in the far field.

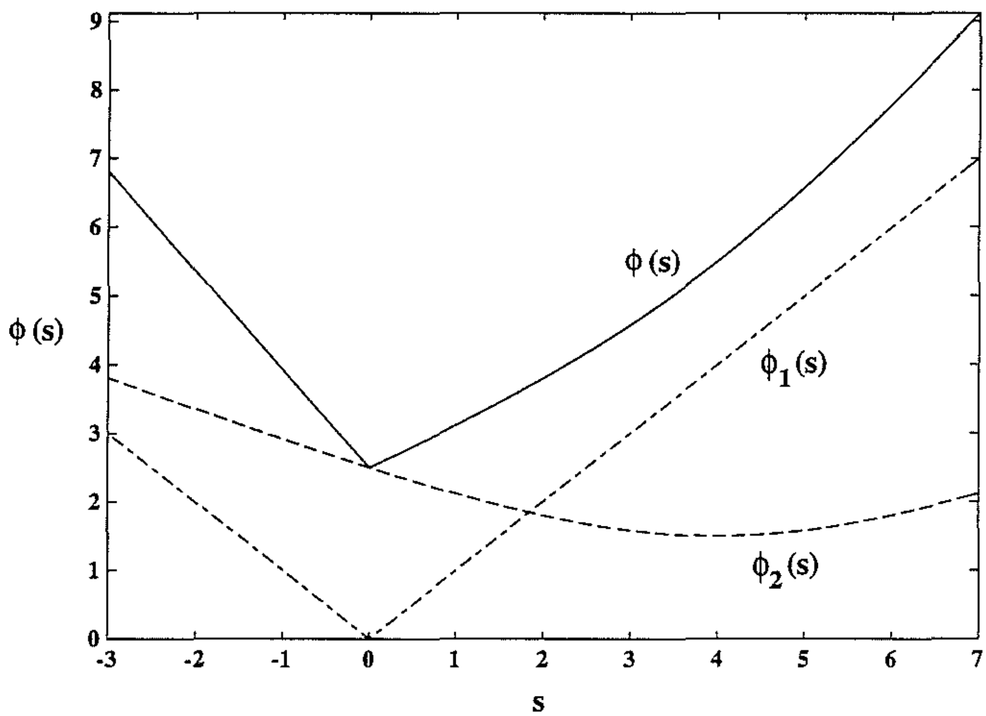

Figure 3: Phase function $\phi(s) = \phi_1(s) + \phi_2(s)$ for a fast formation. At $s = 0$, $\phi(s)$ has a finite jump discontinuity in its first derivative.

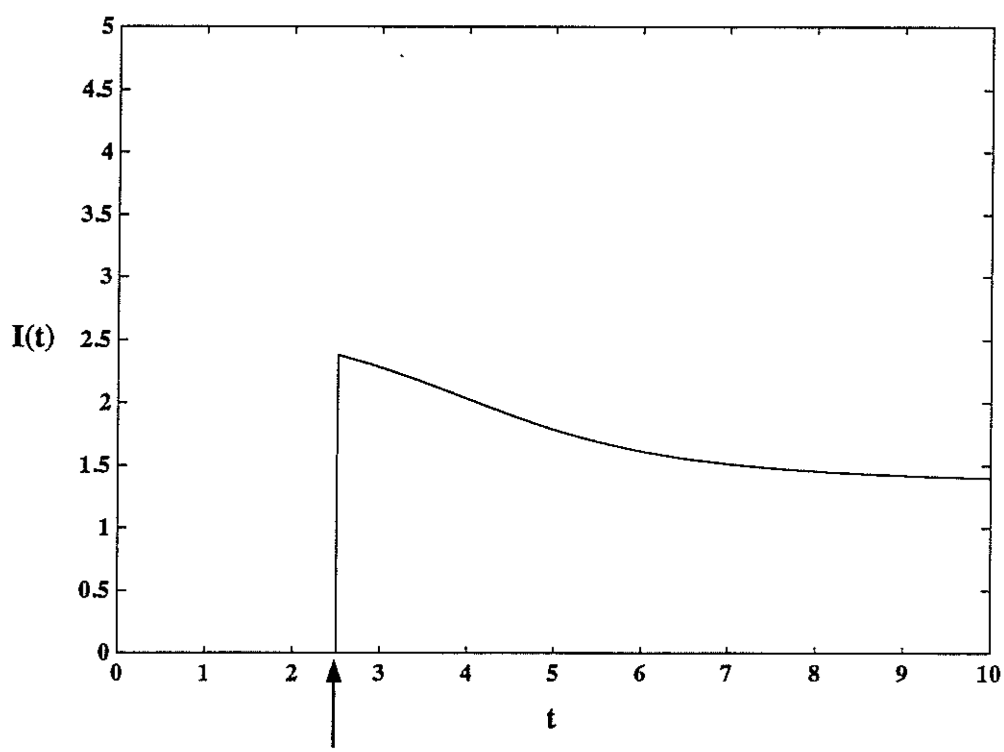

Figure 4: Graph of $I(t)$ for a fast formation. The arrow indicates the arrival time of a ray emanating directly from the source.

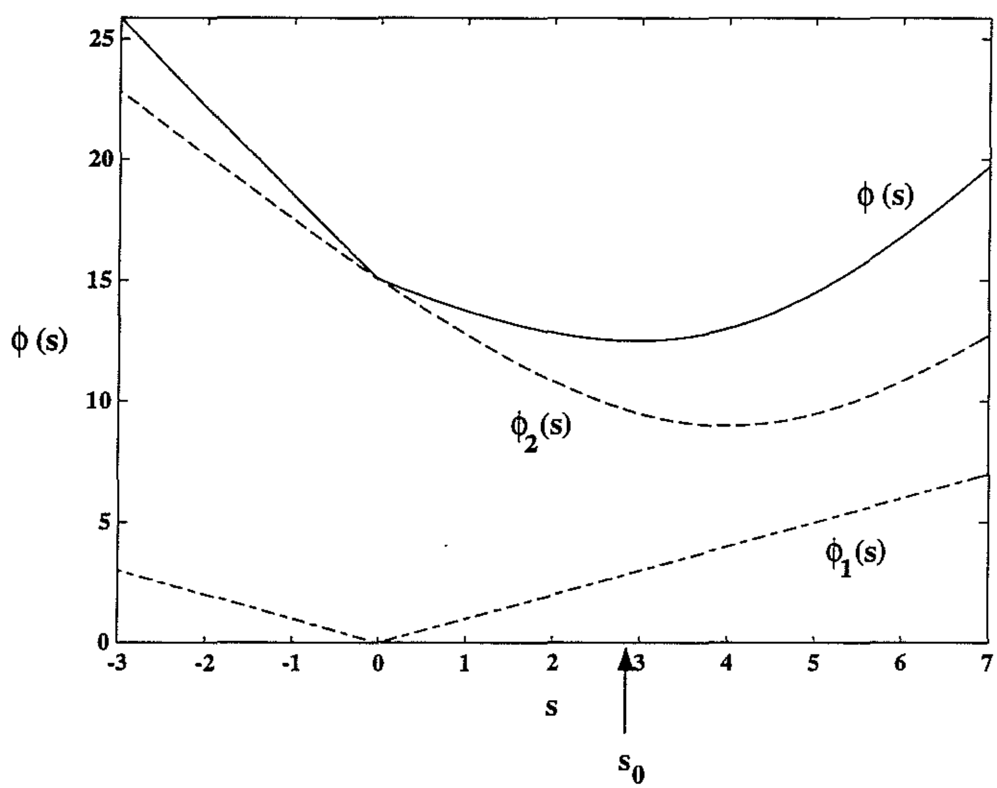

Figure 5: Phase function $\phi(s) = \phi_1(s) + \phi_2(s)$ for a slow formation. At $s = 0$, $\phi(s)$ has a finite jump discontinuity in its first derivative. The stationary point (a minimum) is shown at $s = s_0$.

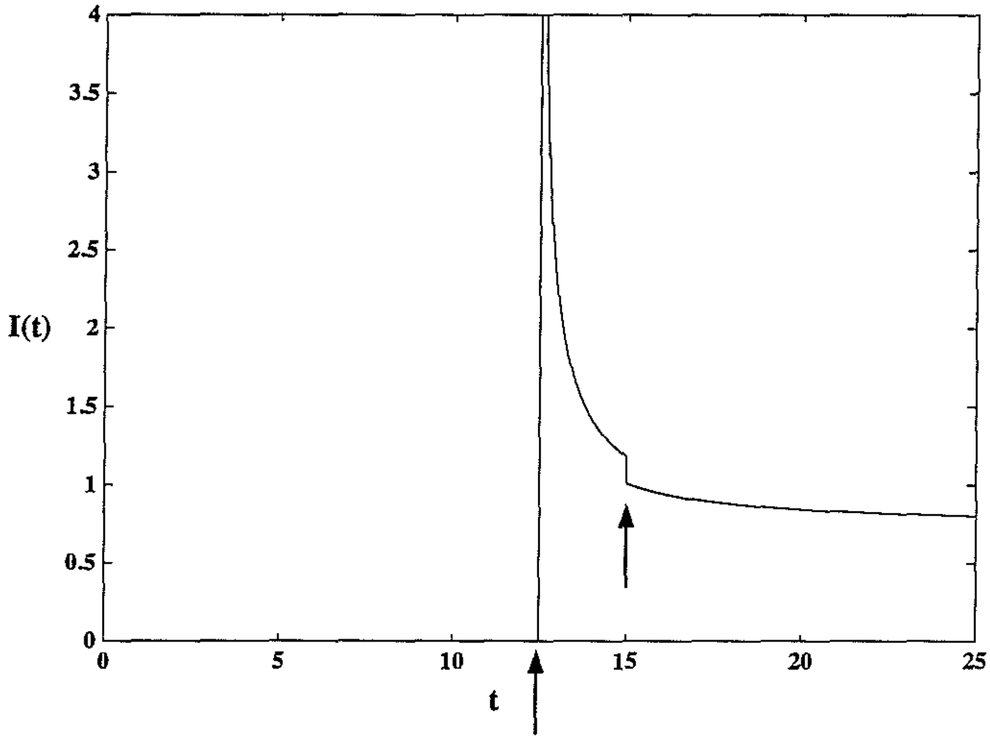

Figure 6: Graph of $I(t)$ for a slow formation. The first arrow indicates the arrival time of the "conical" wave. The second arrow indicates the arrival time of a ray emanating directly from the source.

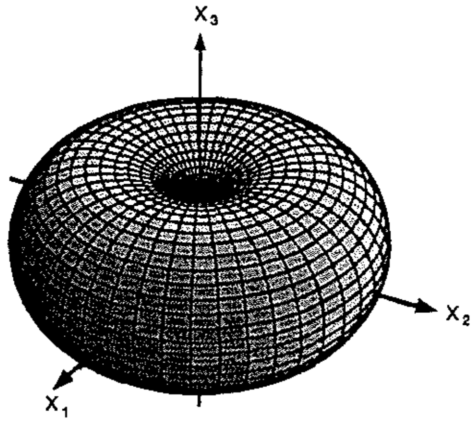

Figure 7: Radiation pattern of the qP-wave for symmetry axis along the x_3-direction (parallel to borehole axis).

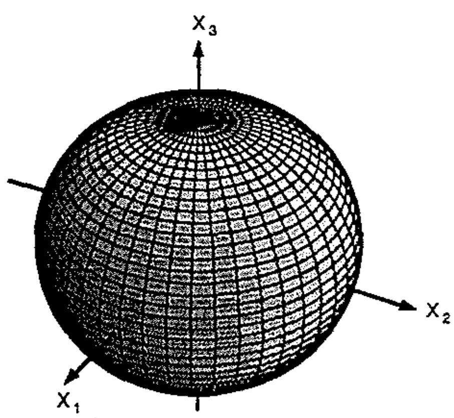

Figure 8: Wavefront surface of the qP-wave for symmetry axis along the x_3-direction (parallel to borehole axis).

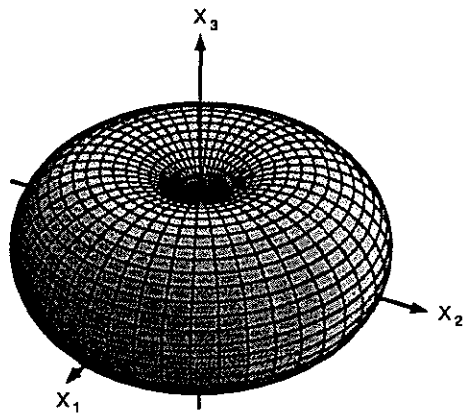

Figure 9: Radiation pattern of the P-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

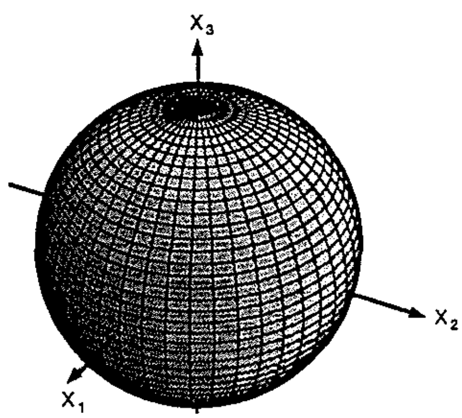

Figure 10: Wavefront surface of the P-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

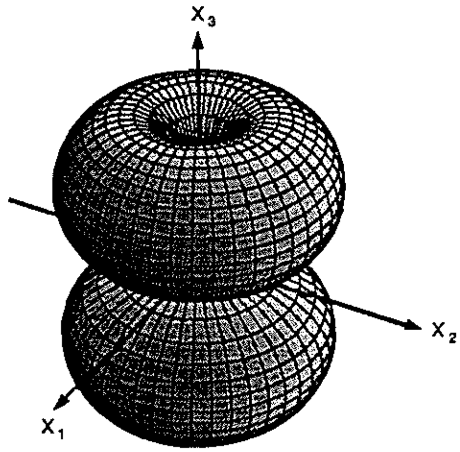

Figure 11: Radiation pattern of the qSV-wave for symmetry axis along the x_3-direction (parallel to borehole axis).

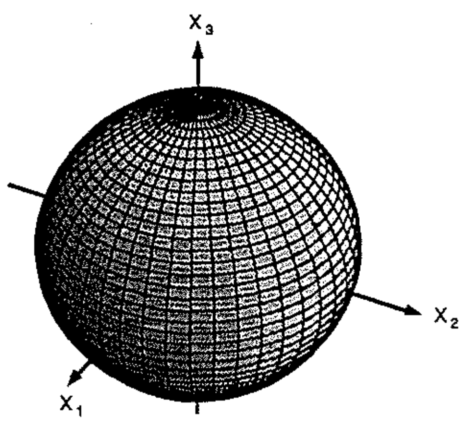

Figure 12: Wavefront surface of the qSV-wave for symmetry axis along the x_3-direction (parallel to borehole axis).

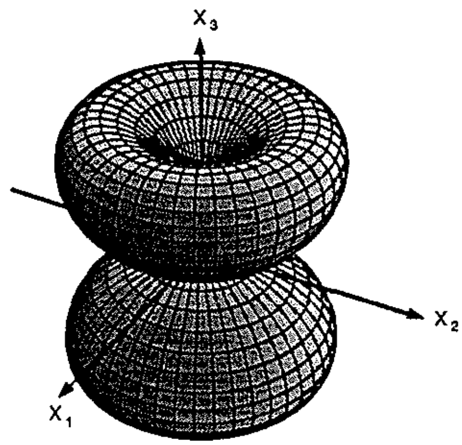

Figure 13: Radiation pattern of the SV-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

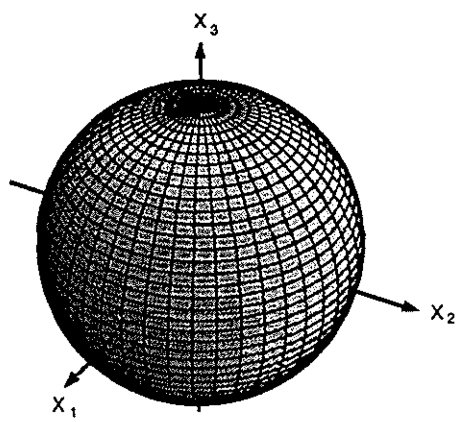

Figure 14: Wavefront surface of the P-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

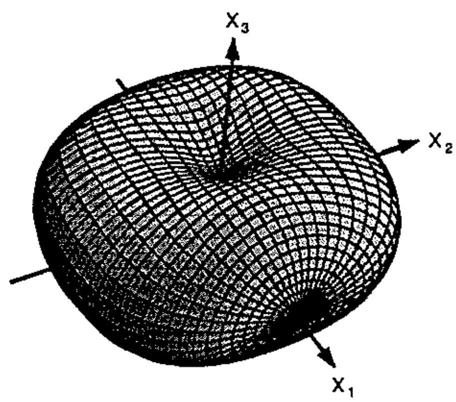

Figure 15: Radiation pattern of the qP-wave for symmetry axis along the x_1-direction (perpendicular to borehole axis).

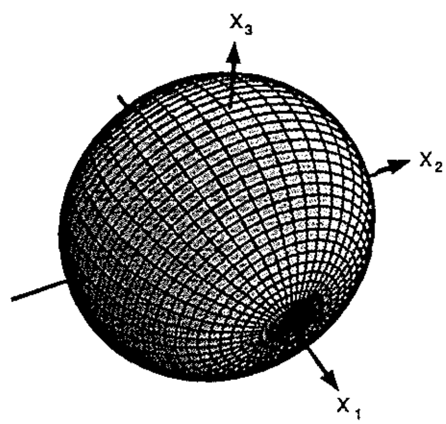

Figure 16: Wavefront surface of the qP-wave for symmetry axis along the x_1-direction (perpendicular to borehole axis).

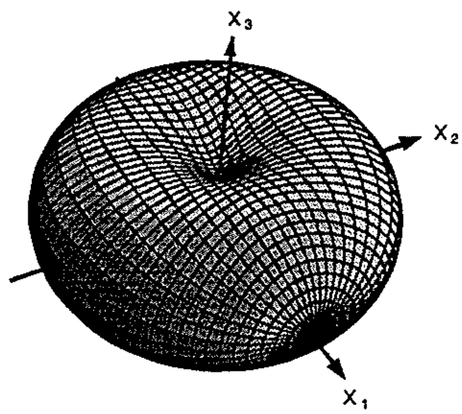

Figure 17: Radiation pattern of the P-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

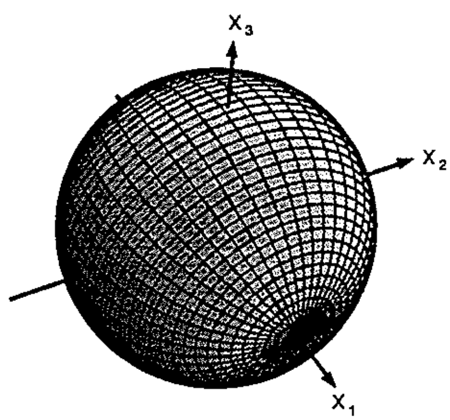

Figure 18: Wavefront surface of the P-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

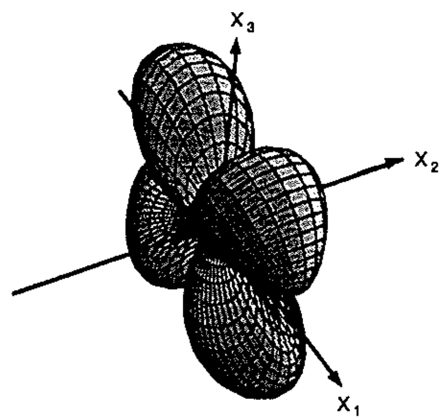

Figure 19: Radiation pattern of the qSV-wave for symmetry axis along the x_1-direction (perpendicular to borehole axis).

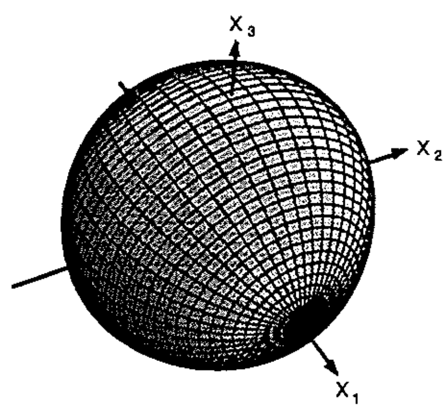

Figure 20: Wavefront surface of the qSV-wave for symmetry axis along the x_1-direction (perpendicular to borehole axis).

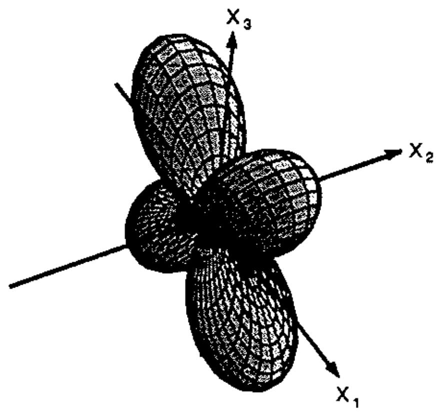

Figure 21: Radiation pattern of the SV-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

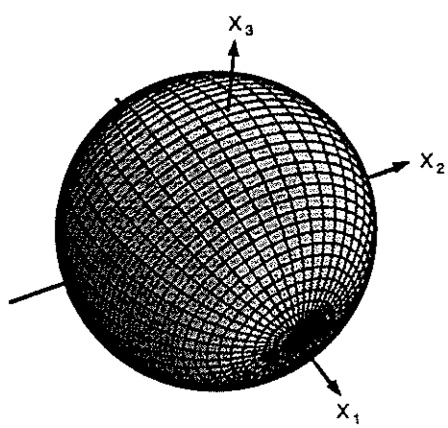

Figure 22: Wavefront surface of the SV-wave for isotropic medium with $\alpha = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

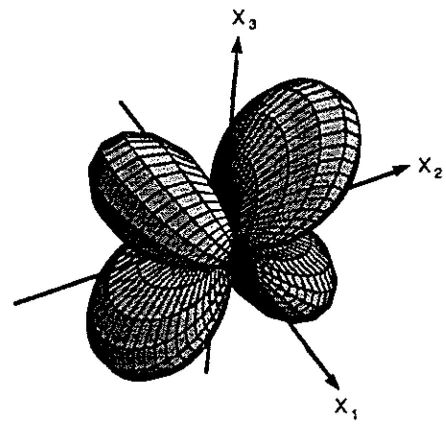

Figure 23: Radiation pattern of the SH-wave for symmetry axis along the x_1-direction (perpendicular to borehole axis).

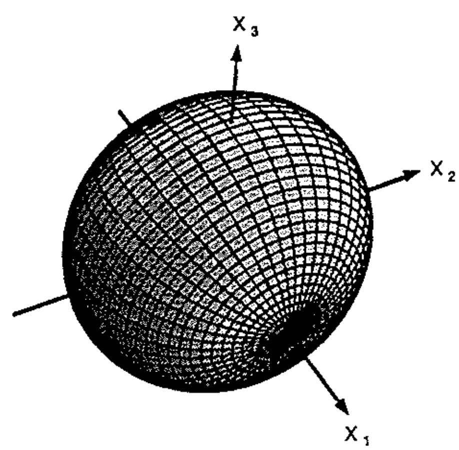

Figure 24: Wavefront surface of the SH-wave for symmetry axis along the x_1-direction (perpendicular to borehole axis).

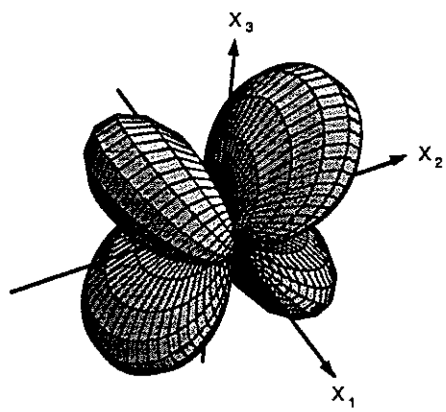

Figure 25: Radiation pattern of the SH-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.

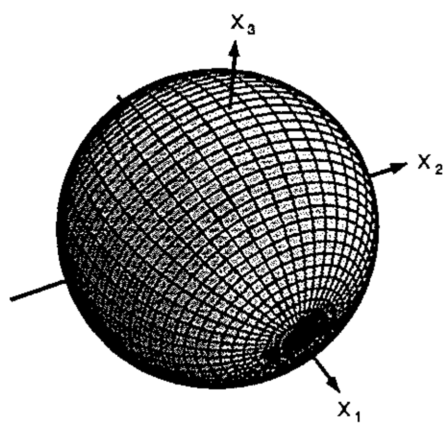

Figure 26: Wavefront surface of the SH-wave for isotropic medium with $\alpha^2 = c_{33}/\rho$ and $\beta^2 = c_{44}/\rho$.