Lecture 13
Eigenvalue Problems

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Introduction to Numerical Methods

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The Eigenvalue Decomposition

- Eigenvalue problem for $m \times m$ matrix A:

$$Ax = \lambda x$$

with *eigenvalues* λ and *eigenvectors* x (nonzero)

- *Eigenvalue decomposition* of A:

$$A = X \Lambda X^{-1} \quad \text{or} \quad AX = X\Lambda$$

with eigenvectors as columns of X and eigenvalues on diagonal of Λ

- In “eigenvector coordinates”, A is diagonal:

$$Ax = b \;\; \longrightarrow \;\; (X^{-1}b) = \Lambda(X^{-1}x)$$

Multiplicity

- The eigenvectors corresponding to a single eigenvalue λ (plus the zero vector) form an *eigenspace*
- Dimension of E_λ = $\dim(\text{null}(A - \lambda I))$ = *geometric multiplicity* of λ
- The *characteristic polynomial* of A is

$$p_A(z) = \det(zI - A) = (z - \lambda_1)(z - \lambda_2) \cdots (z - \lambda_m)$$

- λ is eigenvalue of $A \Longleftrightarrow p_A(\lambda) = 0$
  - Since if λ is eigenvalue, $\lambda x - Ax = 0$. Then $\lambda I - A$ is singular, so $\det(\lambda I - A) = 0$
- Multiplicity of a root λ to p_A = *algebraic multiplicity* of λ
- Any matrix A has m eigenvalues, counted with algebraic multiplicity

Similarity Transformations

- The map $A \mapsto X^{-1}AX$ is a *similarity transformation* of A
- A and B are *similar* if there is a similarity transformation $B = X^{-1}AX$
- A and $X^{-1}AX$ have the same characteristic polynomials, eigenvalues, and multiplicities:
  - The characteristic polynomials are the same:

$$\begin{aligned} p_{X^{-1}AX}(z) &= \det(zI - X^{-1}AX) = \det(X^{-1}(zI - A)X) \\ &= \det(X^{-1})\det(zI - A)\det(X) = \det(zI - A) = p_A(z) \end{aligned}$$

  - Therefore, the algebraic multiplicities are the same
  - If E_λ is eigenspace for A, then $X^{-1}E_\lambda$ is eigenspace for $X^{-1}AX$, so geometric multiplicities are the same

Algebraic Multiplicity $\geq$ Geometric Multiplicity

- Let n first columns of $\hat{V}$ be orthonormal basis of the eigenspace for λ
- Extend $\hat{V}$ to square unitary V, and form

$$B = V^* A V = \begin{bmatrix} \lambda I & C \\ 0 & D \end{bmatrix}$$

- Since

$$\det(zI - B) = \det(zI - \lambda I)\det(zI - D) = (z - \lambda)^n \det(zI - D)$$

the algebraic multiplicity of λ (as eigenvalue of B) is $\geq n$

- A and B are similar; so the same is true for λ of A

Defective and Diagonalizable Matrices

- If the algebraic multiplicity for an eigenvalue $>$ its geometric multiplicity, it is a *defective eigenvalue*
- If a matrix has any defective eigenvalues, it is a *defective matrix*
- A *nondefective* or *diagonalizable* matrix has equal algebraic and geometric multiplicities for all eigenvalues
- The matrix A is nondefective $\Longleftrightarrow A = X\Lambda X^{-1}$
  - ($\Longleftarrow$) If $A = X\Lambda X^{-1}$, A is similar to Λ and has the same eigenvalues and multiplicities. But Λ is diagonal and thus nondefective.
  - ($\Longrightarrow$) Nondefective A has m linearly independent eigenvectors. Take these as the columns of X, then $A = X\Lambda X^{-1}$.

Determinant and Trace

- The *trace* of A is $\operatorname{tr}(A) = \sum_{j=1}^{m} a_{jj}$
- The determinant and the trace are given by the eigenvalues:

$$\det(A) = \prod_{j=1}^{m} \lambda_j, \qquad \operatorname{tr}(A) = \sum_{j=1}^{m} \lambda_j$$

since $\det(A) = (-1)^m \det(-A) = (-1)^m p_A(0) = \prod_{j=1}^{m} \lambda_j$ and

$$p_A(z) = \det(zI - A) = z^m - \sum_{j=1}^{m} a_{jj} z^{m-1} + \cdots$$

$$p_A(z) = (z - \lambda_1) \cdots (z - \lambda_m) = z^m - \sum_{j=1}^{m} \lambda_j z^{m-1} + \cdots$$

Unitary Diagonalization and Schur Factorization

- A matrix A is *unitary diagonalizable* if, for a unitary matrix Q, $A = Q\Lambda Q^*$
- A hermitian matrix is unitarily diagonalizable, with real eigenvalues (because of the Schur factorization, see below)
- A is unitarily diagonalizable $\Longleftrightarrow$ A is normal ($A^*A = AA^*$)
- Every square matrix A has a Schur factorization $A = QTQ^*$ with unitary Q and upper-triangular T
- Summary, Eigenvalue-Revealing Factorizations
  - Diagonalization $A = X\Lambda X^{-1}$ (nondefective A)
  - Unitary diagonalization $A = Q\Lambda Q^*$ (normal A)
  - Unitary triangularization (Schur factorization) $A = QTQ^*$ (any A)

Eigenvalue Algorithms

- The most obvious method is ill-conditioned: Find roots of $p_A(\lambda)$
- Instead, compute Schur factorization $A = QTQ^*$ by introducing zeros
- However, this can not be done in a finite number of steps:

Any eigenvalue solver must be iterative

- To see this, consider a general polynomial of degree m

$$p(z) = z^m + a_{m-1}z^{m-1} + \cdots + a_1 z + a_0$$

- There is no closed-form expression for the roots of p: (Abel, 1842)

In general, the roots of polynomial equations higher than fourth degree cannot be written in terms of a finite number of operations

Eigenvalue Algorithms

- (continued) However, the roots of p are the eigenvalues of the *companion matrix*

$$A = \begin{bmatrix} 0 & & & & & -a_0 \\ 1 & 0 & & & & -a_1 \\ & 1 & 0 & & & -a_2 \\ & & 1 & \ddots & & \vdots \\ & & & \ddots & 0 & -a_{m-2} \\ & & & & 1 & -a_{m-1} \end{bmatrix}$$

- Therefore, in general we cannot find the eigenvalues of a matrix in a finite number of steps (even in exact arithmetic)

- In practice, algorithms available converge in just a few iterations

Schur Factorization and Diagonalization

- Compute Schur factorization $A = QTQ^*$ by transforming A with similarity transformations

$$\underbrace{Q_j^* \cdots Q_2^* Q_1^*}_{Q^*} \, A \, \underbrace{Q_1 Q_2 \cdots Q_j}_{Q}$$

which converge to a T as $j \to \infty$

- Note: Real matrices might need complex Schur forms and eigenvalues (or a *real Schur factorization* with 2×2 blocks on diagonal)

- For hermitian A, the sequence converges to a diagonal matrix

Two Phases of Eigenvalues Computations

- General A: First to *upper-Hessenberg* form, then to upper-triangular

$$\underset{A \neq A^*}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{\text{Phase 1}} \underset{H}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}} \xrightarrow{\text{Phase 2}} \underset{T}{\begin{bmatrix} \times & \times & \times & \times & \times \\ & \times & \times & \times & \times \\ & & \times & \times & \times \\ & & & \times & \times \\ & & & & \times \end{bmatrix}}$$

- Hermitian A: First to *tridiagonal* form, then to diagonal

$$\underset{A \neq A^*}{\begin{bmatrix} \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times \end{bmatrix}} \xrightarrow{\text{Phase 1}} \underset{T}{\begin{bmatrix} \times & \times & & & \\ \times & \times & \times & & \\ & \times & \times & \times & \\ & & \times & \times & \times \\ & & & \times & \times \end{bmatrix}} \xrightarrow{\text{Phase 2}} \underset{D}{\begin{bmatrix} \times & & & & \\ & \times & & & \\ & & \times & & \\ & & & \times & \\ & & & & \times \end{bmatrix}}$$