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Level 3 assets: Booking profits, concealing losses*

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Abstract

Fair value accounting forces institutions to revalue their inventory whenever a new transaction price is observed. An institution facing a balance sheet constraint can have incentives to suspend trading in Level 3 assets (traded on opaque over-the-counter markets) to avoid marking-to-market. This way the asset's book valuation can be kept artificially high, thereby relaxing the institution's balance sheet constraint. But, the institution loses direct control of its asset holdings, leading to possible excessive risk exposure. A regulator trying to reign in risk-taking faces ambiguous tools of increasing fines for mismarking and tightening capital requirements: although both make no-trading less like, conditional on no-trading they increase risk-taking. Random audits in general decrease risk-taking. Outside investors, who do not know at what price the asset would trade, reduce their valuation of the bank's balance sheet the longer the asset has not traded. Their expected discount from reported book value is convex in time since last trade.

Keywords: Mark-to-market; Mark-to-model; Level 3 assets; Balance sheet constraints; Toxic Assets; Optimal Stopping

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1 Introduction

Balance sheets matter, especially in times of financial crisis when credit channels tighten. The impact of illiquid, or so-called Level 3 assets, on banks' balance sheets plays an important role on their strategies. Level 3 assets include amongst others CDOs, ABSs and other – sometimes bespoke – structured credit products. Such hard-to-value assets made up nearly $600 billion of the balance sheets of the 8 largest US banks as of April 2008. More recently, many mortgage products became Level 3 assets as the ABX index ceased to serve as a reliable valuation basis.[1] Anecdotal evidence suggests that some assets are even actively kept off markets to obstruct price discovery to avoid adverse balance sheet impact.[2] Given the accounting flexibility that comes with the Level 3 category, institutions continue to list these assets at inflated values on their books. To regulators and the government, it is important to understand what drives a bank to suspend trading in a certain asset to effectively control risk-taking via capital requirements or other instruments.

I provide a model that derives when instutions suspend and restart trading of certain assets. Under fair value accounting, book valuations of securities generally have to be updated when new transaction prices are observed. However, over-the-counter markets can be so opaque that no continuously observable prices exist. An institution active in such markets has to take the accounting impact of its own trading decisions into consideration. If faced with regulatory capital requirements, it might be optimal to book gains immediately as fundamentals rise above accounting valuations, but suspend trading when fundamentals

[1] The ABX index, an index for asset backed securities, does not exist for its theoretically most recent vintage. The old vintages of the ABX have ceased to serve as a valuation basis for many OTC mortgage products (except for the specific names in the index). The website of Markit, the company that owns the Markit ABX.it index, states the following: 30 September 2008 - Per majority dealer vote, the roll date for the Markit ABX.it 05-2 index has been postponed due to the current market conditions. Markit will announce the new launch date in due course. `http://www.markit.com/information/products/category/indices/abx.html`

[2] As Norris (2008) writes in the New York Times: *"Did you ever hear of a broker who would not agree to earn a commission? Even if getting the money required absolutely no work at all? Apparently, some brokers think such a move could be wise. It's not that they don't like income, but they may fear that letting some securities trade at low prices could force them to report even larger losses than they are already posting."*

drop to obstruct price discovery, thereby delaying losses. As book value ceases to reflect current market prices, the capital requirement is relaxed. Asset holdings, however, become fixed, leading to potentially excessive risk exposure. The institution optimally balances the benefit of a relaxed balance sheet constraint against the cost of possible excessive exposure to determine when to stop and restart trading, and thus when to book gains and losses, and if to follow such a no-trading strategy at all. A regulator wanting to control leverage of a bank has as possible tools capital requirements, fines for mismarking and random audits. Increasing fines or capital requirements can lead to increased risk-taking by increasing the maximal leverage the institution is willing to accept. Random audits in general decrease the risk-taking of banks. These results are robust to an n bank extension – even if n banks each have to mark to each others prices, there can be no-trading equilibria. To an outside observer such as the regulator, the expected value of the balance sheet of a bank is decreasing and convex in time since last trade.

The model is set in a continuous-time stochastic market in which a risk-averse institution allocates its equity between a risk-free bond and a risky asset. Three assumptions drive the model. First, the institution is subject to capital requirements that take the form of a simple leverage constraint based on book values. Second, the risky asset has a stochastic underlying value or shadow price, i.e. the price at which the asset can be bought and sold in the market under current conditions. The shadow price only becomes the realized market price when a trade occurs. Furthermore, in an opaque market the shadow price – as part of the underlying market conditions – is only observed by inside investors. Third, in opaque markets where no continuously observable asset prices or pricing inputs exists, we model the option to mark-to-model by allowing marking to last observable trading price. If the market is OTC, an institution is able to ignore all but self-generated transaction prices. This assumption is later relaxed to a finite number of banks observing each other's trading prices.

The baseline model is a simple Merton setup – without any constraints, the optimal strategy is constant leverage which implies continuous portfolio rebalancing. A tight enough constraint on leverage prevents such a strategy from being implementable. If the market in question is widely traded, the institution cannot obstruct price discovery on its own, and

has to comply with the leverage constraint at each point in time. If the market is opaque, however, it is possible for the institution to obstruct price discovery by ceasing to trade. This introduces a trade-off: by continuing to trade, the balance sheet reflects market prices and the constraint applies, but exposure remains fully controlled. By suspending trade, the institution freezes its balance sheet value, thereby relaxing the constraint, but the now fixed asset position can lead to excessive risk exposure.

In opaque markets, once the institution stops trading, further losses on its position are concealed as the balance sheet is frozen. Yet actual leverage – based on the shadow price – continues to drift from reported leverage, thereby allowing a (passive) violation of the leverage constraint. There will be a shadow price at which the actual exposure has become too large for the institution's risk bearing capacity. At this point, it will *voluntarily* deleverage and reveal all previously concealed losses. If the shadow price reverts before hitting this *retrading boundary*, the institution will restart trading, having successfully avoided any constraint induced position adjustments. Thus, only when prices rise and the institution's actual leverage is at its constraint will continuous trading occur.

This trading behavior will generate a specific valuation profile on the institution's books. Price gains above previously reported valuations are booked immediately. When prices drop, however, the institution stops trading and losses to the balance sheet are concealed as valuations become stale. Figure 1 illustrates. The blue line traces the actual shadow price path whereas the red line traces out the reported asset values for accounting purposes. Shadow price innovations are not reflected on the balance sheet when the red line lies above the blue line. When the price deteriotes too much from its last reported value, the institution voluntary deleverages to avoid excessive risk exposure, leading to a large discrete jump in accounting valuations and balance sheet value. As an example, Merrill Lynch in the summer of 2008 decided to drastically deleverage out of mortgage related assets. This sale was done at prices significantly below reported valuations, thus leading to a large jump in book value.

A regulator cannot observe the fundamental process, but is nevertheless interested in regulating risk-taking of banks. The tools at its disposal are capital requirements (that translate to a leverage constraint), random audits to enforce the capital requirements and

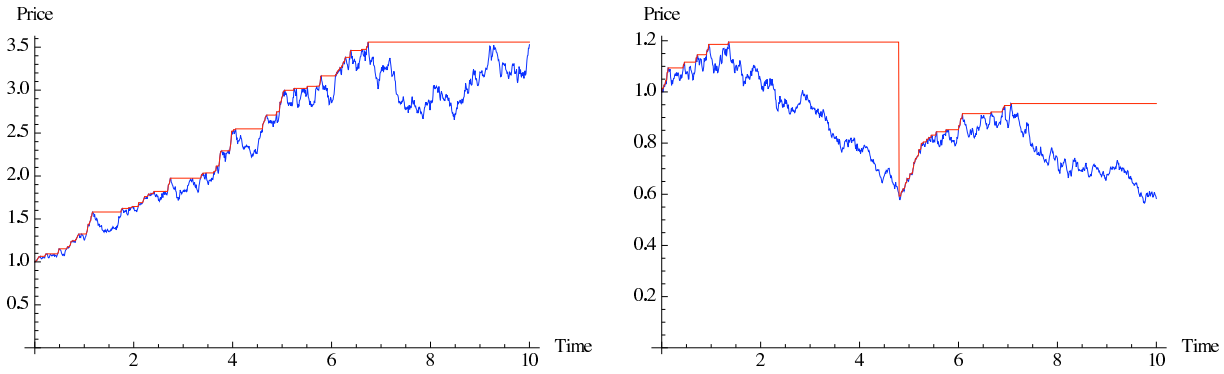

Figure 1: **Illustrative price paths:** Shadow price blue line, reported valuation red line

fines if mismarking of the books is detected. We show that tightening the leverage constraint or increasing fines has ambiguous effects: although both make the no-trading outcome less appealing, if a no-trading strategy is followed the maximal and thus average risk-taking of the firm increases.

A multiple firm extension adds strategic interaction via realized prices. A bank now has to acknowledge transaction prices of a set of finite other banks. We show that a bank's constraint defines it incentives to trade or not to trade. Thus, when leverage constraints are similar, incentives are aligned and a no-trading outcome can arise. However, when constraints are too far apart continuous trading ensues. Additionally, when each bank has an independent chance of a liquidity shock that forces it to exit the market, the more banks are in the market the less likely a no-trade outcome becomes.

Knowledge of the optimal no-trading region will allow an outside investor – who cannot observe the shadow price directly – to perform a valuation of the company's balance sheet. Monte carlos simulations show that the expected balance sheet value is decreasing in time since the last trade. The intuition is that the conditioning eliminates those paths that have strongly increases price paths.

The technical foundations of my model build on the literature in the area of risky arbitrage. Liu and Longstaff (2004) present a model in which an institution faces a leverage constraint in a fixed horizon model with a Brownian Bridge as the price process. Since the authors do not allow for possible accounting manipulation, the trade-off between no trading and stale valuations on the one hand and continuous trading on the other hand cannot arise in their

framework. Other models allowing for portfolio constraints are Grossman and Vila (1992), Cvitanic and Karatzas (1992), Pavlova and Rigobon (forthcoming) and Basak and Croitoru (2000).

Related papers focusing on the strategic interaction originating from balance sheet constraints are Brunnermeier and Pedersen (2005) and Attari, Mello, and Ruckes (2005). Here, competitors try to exploit the balance sheet constraint of an institution via market-based price manipulation. This effect propagates through an assumed temporary price-impact of trades. This motive is absent in this paper, and the interaction is through the balance sheet impact of imposing the shadow price on the books of other banks.

Two related papers with with respect to the fair value accounting component of the model are Plantin, Sapra, and Shin (forthcoming) and Heaton, Lucas, and McDonald (2009). Plantin, Sapra, and Shin (forthcoming) examine the systemic effects of marking-to-market in a static model that utilizes a game theoretic framework. The authors show that mark-to-market accounting can lead to a destabilization of the financial system in that it enhances feedback effects. In this paper we show that tightening capital constraints can lead to larger risk-exposure. Heaton, Lucas, and McDonald (2009) present an overview of the historic development of mark-to-market accounting and provide a model that shows how rigid accounting rules can be socially inefficient.

2 Model Setup

The model is set in continuous-time with $t \in [0, \infty)$ on a suitable defined probability space $(\Omega, \mathbb{P}, \mathcal{F})$. There is one type of agent, a financial institution such as a bank. The goal is to characterize the dynamic impact of accounting rules and balance sheet constraints on the institution's trading decisions and consequently on the path of reported valuations, and derive possible implication for estimating balance sheet value and for multi-bank behaviour. For tractability, I set the interest rate to zero, i.e. $r = 0$.

Shadow price process. Assets in thinly traded markets, such as certain OTC markets, can be subject to significant deviations of reported prices from fundamentals – that is there is a difference between what prices are reported on an institutions books and what price the asset could fetch if sold in the market.

In our model, there is only one risky asset the institution can invest in, simply termed the asset. The asset pays no dividends and has a fundamental value (or shadow price) P_t – this price should be understood as the (shadow) price at which the asset can be bought or sold at time t as the outcome of a matching process in the market. Once a trade occurs at t, this (shadow) price becomes the realized transaction price. An institution that is active in the market observes the fundamental value even when no transaction occurs, but outside investors – or the regulator / government – cannot observe the current shadow price. We assume the following (shadow) price dynamics

$$\frac{dP}{P} = \mu dt + \sigma dB_t \tag{1}$$

To get a frictionless market, there is another group of 'buy and hold' investors ready to buy and sell at the prevailing fundamental value. The possible trading impact of the institution is purely caused by the (accounting) balance sheet link.

By excluding asymmetric information considerations, we are able to isolate the mechanical effects fair value accounting and balance sheet constraints have on trading behavior. In the current crisis there is evidence of strong balance sheet driven effects influencing the trading behavior of institutions, as pointed out by Adrian and Shin (2008).

Wealth process. Financial institutions largely finance themselves through the standard instruments of debt and equity. At a bank, management's task is to maximize the value of existing equity.

Let W_t denote the wealth or equity of the institution. The balance sheet in this simple world is made up of debt D_t (negative D_t denotes lending) with constant price 1 and N_t units

of the risky asset of price P_t each (negative N_t denotes shorting).

Assets	**Liabilities**
$N_t \times P_t$ Value of Assets	D_t Debt
	W_t Wealth / Equity

Naturally, W_t is restricted to be non-negative. Without loss of generality, we will be able to ignore this constraint due to our assumption of logarithmic utility below.

The bank's choice variable is $\phi \equiv \frac{NP}{W}$, its leverage or exposure. As they will feature subsequently, it is useful to derive the log wealth dynamics. Applying Ito's Lemma to $\log(W_t)$, imposing self-financing and recalling that $r = 0$, we have

$$d\log W = \underbrace{\left[\phi\mu - \frac{1}{2}\sigma^2\phi^2\right]}_{\equiv\mu_{\log W}} dt + \sigma\phi dZ \tag{2}$$

The drift of $\log W$, $\mu_{\log W}$, as a function of ϕ is independent of the level of W_t. The bank is endowed with a strictly positive and finite amount of initial wealth W_0.

I impose that ϕ has to be square integrable, i.e. $\phi \in \mathcal{L}^2$. Additionally, recapitalization of the bank is ruled out by assumption as it would allow adjustments of ϕ without trading. The main argument here is that outsiders who do not observe the underlying true balance sheet situation are reluctant to inject equity.

Preferences & Time horizons. We assume that the institution maximizes its utility from final wealth at a random time τ_ρ. Thus, the institution has no intermediate consumption. At time τ_ρ it realizes its current wealth W_{τ_ρ} and derives a utility $U\left(W_{\tau_\rho}\right) = \log\left(W_{\tau_\rho}\right)$.

The stopping time τ_ρ is an exponentially distributed random variable with intensity ρ that is independent of Z - this can be interpreted as a horizon time of the asset (it realizes and pays out P at this time) or a horizon time of the bank (it liquidates all assets at market

prices and exits the market). The value function can thus be written as

$$\sup_{\phi} \mathbb{E}_t \left[\log \left(W_{\tau_\rho} \right) \right] = \sup_{\phi} \mathbb{E}_t \left[\int_t^{\infty} \rho e^{-\rho(s-t)} \log \left(W_s \right) ds \right] \tag{3}$$

where $\mathbb{E}_t [\cdot]$ denotes the expectation operator w.r.t. to the filtration $\mathcal{F}_t$. Note that due to log utility, the institution will always keep equity positive, making debt risk-free.[3]

The choice of preference structure, although simplistic, allows us to capture the essential parts of market participants' behavior. First, decision making units at financial institutions have finite time horizons, though the exact horizon remains uncertain. Alternatively, the horizon can be understood as an extreme liquidity event that forces the realization of wealth, and thus prices, onto the bank. Second, investment banks display effective risk-aversion. To focus on the economic aspect of balance sheet constraints, we use a logarithmic utility definition, as it allows to abstract from any possible intertemporal hedging demands.

Leverage constraint. Financial institutions are constrained in their asset allocation decisions: financial market regulators require certain amounts of minimum capital and debt contracts often come with attached covenants. Any such constraint will curtail the institution's ability to exploit possible high drift assets by restricting the maximum position size. To make a statement about the impact of fair value accounting on an institution's portfolio decisions, we will focus on constraints that operate on the reported balance sheet of the institution.

Regulatory capital requirements force a bank to have sufficient capital base for their asset positions. Such capital requirements can be found in the Basel accords. We assume here that the regulator uses the reported balance sheet data for implementing capital requirements. A minimum capital requirement then gives a static leverage constraint in our model.

Additionally, debt often comes with attached covenants to limit the risk-taking ability of the firm. Such debt covenants fit into our story if they can be (a) summarized as a static

[3]Of course, we can easily incorporate final date management fees. Suppose that ξ is the porportion of assets under management that go to the bank. Then we have $\log(\xi W) = \log \xi + \log W$ and we simply have a level effect on the value function, but the choice between different strategies is not altered.

leverage constraint and (b) are enforceable in courts only based on the reported statements of the firm (i.e. the accounting statements).

In terms of our model, the balance sheet constraint takes the form of a leverage constraint

$$\phi \in \left[\underline{\phi}, \bar{\phi}\right] \tag{4}$$

where we assume $\underline{\phi} = 0$ (a no-shorting constraint) and $\bar{\phi} \geq 1$: the institution is allowed to invest all of its equity in the risky asset ($\phi = 1$) or in the bond ($\phi = 0$). Because of parameter restrictions we will make subsquently, only $\bar{\phi}$ will matter, and we will thus refer to $\bar{\phi}$ in the following simply as the constraint.

Accounting. Recent shifts away from historical cost accounting to fair value accounting have led to significant changes in how financial assets are valued. For accounting purposes, there are three asset categories, as specified in Financial Accounting Standards Board ("FASB") statement 157 that implemented fair value accounting: Level 1, Level 2 and Level 3.[4]

It is important to note the position of the SEC regarding the primacy of a market price in accounting valuations. Although FASB 157 allows for valuation exceptions (in case of distressed transactions), the interpretation of this rule by the regulator is strict. Applying a forward looking equilibrium view of valuations that ignores current illiquidity or imbalances in supply and demand is not permissible, as *"GAAP defines fair value as the amount at which an asset could be bought or sold in a current transaction*". However, with FASB 159 having been passed in February 2009, there has been a significant expansion of when it is allowed to use marking-to-model in illiquid markets.

To model marking-to-market and marking-to-model in a tractable way, I make the following simplifying assumptions: **(A1) The institution has the option to value Level 3 assets on its books at its last observable transaction price.** In other words, in a market with no observable prices, trading is intimately linked to own price realization.

[4]FASB 157 became effective November 15, 2007.

(A2) Once an institution trades a Level 3 asset at a price that reveals it to be in violation of the capital requirement, the institution is forced by the regulator to comply with the leverage constraint. The regulator may levy a fine on the institution at this point.

Allowing the institution to mark to the last observable transaction price permits us to remain agnostic about the specific model used by the institution to value the asset for its balance sheet. This affords us a certain degree of robustness while still maintaining the essential link between an institution's own trading decision and the value of its inventory. In real life, internal auditors are informed at an almost instantaneous basis of new transaction prices that are generated by the bank itself. Furthermore, the accounting balance sheet of the trading desk is reported daily to the regulator. We rule out fake trading (trades that are done between institutions at inflated prices) by noting that the possible punishment for such behaviour is much larger than for having slightly misleading marking-to-model valuations. In conversations with traders, such a strategy was always strongly denied because of possible jail-time punishment.

Let us briefly discuss the three accounting categories of assets.

Level 1 assets. Level 1 assets are liquid assets with publicly quoted prices, such as exchange traded assets. The unadjusted quoted prices have to be used for accounting purposes if the company has access to the market in question. Thus, the price of these assets is close to unambiguous and the institution has virtually no discretion in valuing its books. Examples of such assets would be common stock traded on the NYSE, such as a GM share

Level 2 assets. Although Level 2 assets are not exchange traded, the valuation of these assets is based on a model that requires the input of market observables. Such observables might be quoted prices of similar assets, interest rates, implied volatility or a related index. Level 2 assets, although market-to-model, are thus subject to continually updated inputs. An example of Level 2 assets are simple derivatives such as a plain vanilla option on a common stock.

As Level 2 assets have to be marked-to-model based on observable inputs, there are only few degrees of freedom available to the institution in how to influence the valuation. This

motivates the decision to treat Level 2 assets as having continuously updated valuations in this model.[5]

Level 3 assets. According to FASB 157, Level 3 assets are assets that have to be marked-to-model based on unobservable inputs. For the category Level 3 to apply, there have to be neither quoted prices of the asset itself nor any observable, i.e. Level 2, inputs available to the institution. This means that there are no actively traded similar or identical assets, nor are there any observable market inputs such as relevant indices or underlying assets. It is clear from the description that Level 3 assets are primarily found in OTC markets with high levels of opaqueness.

To model the valuation of Level 3 assets in a tractable way, we will make a third assumption: **(A3) the institution Level 3 trading happens in an opaque enough market to be able to ignore other institution's transaction prices of the same or similar assets for its own accounting treatment.** Financial accountants and auditors, even internal ones, do not have the expertise to be able to observe the shadow price process, nor do they have access to transaction prices of other institutions because of the decentralized nature of the market.[6] We relax the non-observability assumption in section 6.

3 Continuously updated valuations: Level 1 and 2 assets

In this section, we will look at the case of Level 1 and Level 2 assets for which accounting rules result in continuously updated balance sheet valuations.

[5]There is a possible exception in that accounts can be declared "hold to maturity". This option is often utilized by insurance companies, as many of their claims are not retraded. Declaring an account as "hold to maturity", however, restricts the ability to retrade in the future: trading non-trivial amounts of the asset will force the institution to re-declare the account as a trading or assets available for sale account, with the option to switch back to "hold to maturity" only after a fixed period of time.

[6]Current practice has internal auditors gathering quotes from other institutions if those provide them voluntarily. But even if such quotes are available, it is important to note that these are not trade quotes – they come from the other institution's risk management units, and not from the actual trading desks. Clearly, if there is no threat of trade on these quotes, and if institutions are in similar situations, there is little incentive to report the true actual quotes.

Denote the value function of the overall problem by $V(W)$. Then rewriting the definition of the value function, we get

$$\begin{aligned} V(W_t) &= \sup_{\phi\in\mathcal{A}} \mathbb{E}_t\left[\log(W_\tau)\right] = \sup_{\phi\in\mathcal{A}} \mathbb{E}_t\left[\int_0^\tau d\log W\right] \\ &= \sup_{\phi\in\mathcal{A}} \mathbb{E}_t\left[\int_0^\tau \phi_t\mu - \frac{\sigma^2\phi^2}{2}dt\right] + \mathbb{E}_t\left[\int_0^\tau \sigma\phi_t dB_t\right] \end{aligned} \tag{5}$$

where we already imposed $r = 0$. Maximizing the value function w.r.t. ϕ thus reduces to maximizing the drift of $\log W$, $\mu_{\log W}$. When $\mathcal{A}$ is unrestricted, a path-by-path maximization gives $\phi^* = \frac{\mu}{\sigma^2}$, the Merton outcome. Consider now a a leverage constraint as discussed in Section 2 with $\bar{\phi} \leq \phi^*$. Then since the quadratic function $\phi_t\mu - \frac{\sigma^2\phi^2}{2}$ is monotonically increasing in ϕ for $\phi < \phi^*$, the investor holds the highest possible ϕ he is allowed to hold, $\bar{\phi}$. We will call the strategy of keeping $\phi = \bar{\phi}$ under continuous trading the *compliance strategy*. To derive the value function, we use equation (3) to get the following HJB

$$\rho V = \sup_{\phi\in\mathcal{A}} \left\{\mu\phi W V_W + \frac{\sigma^2\phi^2}{2}W^2 V_{WW}\right\} + \rho\log W \tag{6}$$

We see that the value function will have the form $\bar{V} = \log W + \bar{g}$ with $\bar{g} = \frac{1}{\rho}\left(\mu\bar{\phi} - \frac{\sigma^2\bar{\phi}^2}{2}\right)$ and a maximum at $\bar{\phi} = \phi^*$ of $g^* = \frac{\mu^2}{2\rho\sigma^2}$. This value also applies for any $\bar{\phi} > \phi^*$, so that the unconstrained value function is simply $V(W) = \log W + \frac{\mu^2}{2\rho\sigma^2}$. In either case, keeping leverage constant when facing a stochastic price requires continuous trading and thus leads to continuously updated balance sheet valuations.

Let an upper bar $\bar{}$, i.e. $\bar{y}$, denote parameters and functions corresponding to the region that has continuous trading while the constraint is binding. Let us summarize:

Proposition 1 *The value function for the constrained institution without the option to mark-to-model is* $\bar{V}(W) = \log W + \bar{g}$ *with*

$$\bar{g} = \begin{cases} g^* = \frac{\mu}{2\rho\sigma^2} & \text{for } \phi^* < \bar{\phi} \\ \frac{1}{\rho}\left(\mu\bar{\phi} - \frac{\sigma^2\bar{\phi}^2}{2}\right) & \text{for } \phi^* \geq \bar{\phi} \end{cases} \tag{7}$$

and the optimal strategy is $\phi = \min\left\{\bar{\phi}, \phi^*\right\}$.

Proofs omitted from the text can be found in the appendix.

4 Possibly stale valuations: Level 3 assets

This section contains the main proposition of this paper that derives the optimal non-trading strategy and value function in the case of Level 3 assets, when the institution's trading decision can impact its financial reporting. The main insight is that although the institution cannot actively (via trading) violate the constraint, accounting rules allow it to passively do so by ceasing to trade.

Given our assumptions from section 3, the institution holding Level 3 assets faces the following choice: If it trades at time t, the shadow price P_t is realized and accounting rules force an updated valuation of any inventory still held. The constraint is then enforced and possible fines levied. If no (own) trade occurs, however, the bank can keep level 3 assets at their old values on the books.

Denote by $\hat{\phi}$ the so called *retrading boundary*, the level of exposure at which the institution *voluntarily* reveals the shadow price (and thus its violation of the leverage constraint) and is forced (by the regulator) to immediately retrade to within $\left[0, \bar{\phi}\right]$. At this point the regulator levies a proportional fine of fW on the bank with $f \in [0, 1)$.[7] In the following, we will refer to the strategy that includes passively violating the leverage constraint the *no-trading strategy*.

Possibly stale valuations without leverage constraints. The option to keep valuations stale only generates value if it allows a relaxation of the balance sheet constraint. As this constraint is nonexistent for $\bar{\phi} = \infty$, the institution will find it optimal to continuously trade, marking-to-market its balance sheet at each point in time. Thus, the unconstrained solution from section 3 is the upper limit to any value function involving accounting manipulation.

[7]Note that the fine is not applied at the horizon time liquidation. However, a simple level shift would take place if the fine were applied, with the constant $\rho \log(1 - f) \leq 0$ added to the value function. As discussed in detail in the section 5, this would not affect the optimal strategy conditional on no-trading, but only if a no-trading strategy is followed or not.

Possibly stale valuations with leverage constraints. Once the institution faces a constraint on its leverage, the option to keep valuations stale may become valuable. The intuition for this is simple: with continuous trading and thus continually updated prices, the institution is forced to fulfill the leverage constraint and thus accept suboptimal leverage. With marking-to-model concealing losses via stale book valuations, the institution is able to keep its book value artificially inflated and thereby relax the leverage constraint.

Denote by $\tilde{\phi}_t$ the institution's actual exposure based on the current shadow price, as opposed to its accounting based exposure ϕ_t based on a possibly stale transaction price. In the previous section, as the institution was assumed to be trading continuously, these two variables coincided, i.e. $\tilde{\phi}_t = \phi_t$. When the institution stops trading at a time s its number of units of the risky asset, N_s, and the amount of debt, D_s, become fixed (recall $r = 0$). Thus, accounting equity on the balance sheet becomes fixed at $N_s P_s - D_s$.

We now conjecture the following strategy: the investor will let exposure drift on $\tilde{\phi} \in \left(\bar{\phi}, \hat{\phi}\right)$ by not trading, and will only readjust the portfolio at a point $\tilde{\phi} = \hat{\phi}$ back to $\bar{\phi}$. Any $\tilde{\phi} < \bar{\phi}$ is immediately (and costlessly) reset to $\bar{\phi}$. The following lemma will establish a helpful result.

Lemma 1 *Conditional on using a hiding strategy, the investor will never let exposure drift when $\tilde{\phi} \leq \bar{\phi}$.*

Proof. Consider a strategy that has $\tilde{\phi}$ drifting and has $\hat{\bar{\phi}} < \bar{\phi}$ as its lower end. Without loss of generality, consider a situation in which the agent has stopped trading for $\tilde{\phi}$ stricly inside $\left(0, \bar{\phi}\right)$ at $t = 0$. Then, for any path ω, let the stopping time s denote the first time $\tilde{\phi}$ crosses $\bar{\phi}$ from below. Thus, on $t \in (0, s)$ the agent has an exposure $\tilde{\phi}_t < \bar{\phi}$. But since he is not constrained on $(0, s)$, he can improve on this strategy by simply setting $\tilde{\phi} = \bar{\phi}$ on $t \in (0, s)$ by equation (5). Thus, $\hat{\bar{\phi}} < \bar{\phi}$ cannot be optimal. ■

Define the distance (in a geometric sense) of the current shadow price p_t from the last traded price p_s as $y_{t-s} \equiv \frac{p_t}{p_s}$. Thus, when the bank trades y is reset to 1. The next lemma will establish properties of the conjectured strategy to not trade on $\left(\bar{\phi}, \hat{\phi}\right)$.

Lemma 2 *Suppose the bank lets exposure drift on* $\tilde{\phi} \in \left(\bar{\phi}, \hat{\phi}\right)$*, and readjusts the portfolio once* $\tilde{\phi} = \hat{\phi}$ *back to* $\bar{\phi}$*. Any* $\tilde{\phi} < \bar{\phi}$ *is immediately reset to* $\bar{\phi}$*. Then*
(i) on the concealment or no-trading region $\tilde{\phi} \in \left(\bar{\phi}, \hat{\phi}\right)$*, the institution's actual leverage* $\tilde{\phi}$ *is only a function of* y_{t-s}*:*

$$\tilde{\phi}\left(y_{t-s}\right) \equiv \frac{\bar{\phi} y_{t-s}}{1 + \bar{\phi}\left(y_{t-s} - 1\right)} > \bar{\phi} \tag{8}$$

(ii) the wealth or equity of the institution becomes zero at the finite point $\bar{\bar{y}} = 1 - \frac{1}{\bar{\phi}}$ *, the bankruptcy point, and*
(iii) the optimal retrading boundary $\hat{y}$ *(where* $\hat{y}$ *solves* $\hat{\phi} = \tilde{\phi}\left(\hat{y}\right)$*) has to lie strictly within the finite interval* $\left(\bar{\bar{y}}, 1\right)$*.*

Proof. (i) By simple substitution, and recalling that $\bar{\phi} = \frac{N_s P_s}{W_s}$ and $r = 0$, we can derive

$$\begin{aligned} \tilde{\phi}_{t|s} &= \frac{N_s P_t}{W_t} = \frac{N_s P_t}{N_s P_t - D_t} = \frac{N_s P_t}{N_s P_s - D_s + N_s\left(P_t - P_s\right)} \\ &= \frac{N_s P_s \frac{P_t}{P_s}}{W_s + N_s P_s \frac{P_t - P_s}{P_s}} = \frac{\bar{\phi} y_{t-s}}{1 + \bar{\phi}\left(y_{t-s} - 1\right)} \end{aligned}$$

(ii) First, note that the numerator of $\tilde{\phi}$, $N_s P_t$, is everywhere positive and bounded. Thus, $\tilde{\phi}$ has a pole at the point at which the denominator becomes zero, $\bar{\bar{y}} = 1 - \frac{1}{\bar{\phi}}$. As $\bar{\phi} > 1$, we conclude $\bar{\bar{y}} < 1$.[8] Second, note that the actual exposure $\tilde{\phi}$ becomes negative for $y < \bar{\bar{y}}$. Thus wealth has to vanish at $\bar{\bar{y}}$.
(iii) By its definition $\bar{\bar{y}}$ is finite for any $\bar{\phi} > 1$. Thus, the set $\left\{t : y_t \leq \bar{\bar{y}}\right\}$ has non-zero probability mass if the bank does not retrade. We conclude that the value function $\tilde{V}$ of the institution for any retrading boundary $\hat{y} \leq \bar{\bar{y}}$ would result in a value of $-\infty$. This is clearly suboptimal, as any $\hat{y} > \bar{\bar{y}}$ yields finite value. We therefore conclude that $\hat{y} > \bar{\bar{y}}$. ■

Part (i) of the second lemma shows that stale prices allow the institution to expand its space of possible exposure $\tilde{\phi}$ beyond $\left[\underline{\phi}, \bar{\phi}\right]$, albeit in a passive way that is linked to the behavior of prices via $\tilde{\phi}\left(y\right)$. Thus, non-trading can lead to a closer approximation of ϕ^*. But

[8] We ruled out $\bar{\phi} \in [0, 1]$ by assumption. It should be clear to the reader that in this case the institution's wealth can never become negative as $\tilde{\phi} \in [0, 1]$.

as $y \to \overline{\overline{y}}$, the institution's actual exposure becomes unbounded, i.e. $\tilde{\phi} \to \infty$, and wealth becomes unboundedly risky. Consequently, the institution will retrade once the log shadow price reaches some point $\hat{y}$, as shown in part (iii) of the lemma, for any bounded fines levied by the regulator.

Suppose the institution stopped trading in the past and no $\tilde{\phi} > \bar{\phi}$. Then there are three possible outcomes: One, the exposure drops back to $\bar{\phi}$ via rising prices, at which point the institutions restarts trading to keep the exposure from dropping below $\bar{\phi}$. Two, the price detriorates so far that the exposure increases to the the retrading boundary $\hat{\phi}$ and the bank retrades back to $\bar{\phi}$, incurring regulatory fines but avoiding excessive exposure. Three, before $\tilde{\phi}$ reaches either $\hat{\phi}$ or $\bar{\phi}$ the horizon event realizes, the institution realizes its wealth (and exits the market).

The bank optimizes over $\hat{y}$, the point at which the mismarking is voluntarily revealed and a large balance sheet adjustment occurs. As a preliminary analysis, note that if there is zero cost to adjustment, i.e. $f = 0$, the agent is always better of compared to simply trading at the constraint $\tilde{\phi}_t = \bar{\phi}$ by following the (non-optimal) strategy of not trading on $y \in (y^*, 1)$ where $y^* \equiv \frac{(\bar{\phi}-1)\phi^*}{\bar{\phi}(\phi^*-1)} < 1$ so that $\tilde{\phi}(y^*) = \phi^*$. But we will see that the bank can do even better.

4.1 Solution

Let us now derive the value function. For this purpose, fix a $\bar{\phi}$. We now write the value function conditional on a no-trading strategy as $V(W, y) = \log W + g(y)$ where the deviation of the current shadow price from the last reported price, y, is now a state variable. We write the value function in terms of y and not in terms of $\tilde{\phi}$ because it will allow us closed form solutions up to a nonlinear equation for the retrading boundary $\hat{y}$. From simple optimality conditions (see Dixit (1993) or Dumas (1991)), the value function will satisfy value matching and smooth pasting at the boundary $\hat{y}$, as well as smooth pasting due to reflection at the boundary point $y = 1$ (a property implied by the value function being an expectation). We thus have 3 boundary conditions for a second order ODE with one free boundary $\hat{y}$, so the

function g is pinned down. Note that y follows a geometric Brownian motion on $(\hat{y}, 1)$ with reflection at $y = 1$ and resetting at $\hat{y}$. Thus, $\frac{dy}{y} = \mu dt + \sigma dB_t$ with $y_0 = 1$, so that the HJB will give the following linear second-order ODE on $[\hat{y}, 1]$ after plugging in $V(W, y) = \log W + g(y)$,

$$\frac{1}{2}\sigma^2 y^2 g''(y) + \mu y g'(y) - \rho g(y) + \mu_{\log W}(y) = 0$$

where $\mu_{\log W}(y)$ is the drift of the investor's log-wealth due to exposure $\tilde{\phi}$. The solution to the homogenous part of the ODE is

$$g_h(y) = c_+ y^{\eta_+} + c_- y^{\eta_-}$$

where $\eta_\pm = \frac{\frac{\sigma^2}{2} - \mu \pm \sqrt{2\rho\sigma^2 + \left(\frac{\sigma^2}{2} - \mu\right)^2}}{\sigma^2}$. Assume $\sigma^2 \geq \mu < \rho$. We then have $\eta_+ \geq 1 > -1 > \eta_-$ and $\phi^* \geq 1$. Define the Wronskian

$$Wr(y) = y^{\eta_+}\eta_- y^{\eta_- - 1} - \eta_+ y^{\eta_+ - 1} y^{\eta_-} = -(\eta_+ - \eta_-) y^{\eta_+ + \eta_- - 1}$$

We can now solve the particular part of the ODE, which we denote by g_p, by the method of variation of coefficients,

$$g_p(y|l) = \frac{2}{\sigma^2}\left[\int_y^l \frac{\mu_{\log W}(s)}{s^2} \frac{s^{\eta_+} y^{\eta_-} - y^{\eta_+} s^{\eta_-}}{Wr(s)} ds\right] \tag{9}$$

where l is an arbitrary limit of integration inside the range $[\hat{y}, 1]$ and $\mu_{\log W}(s) = \tilde{\phi}(s)\mu - \frac{1}{2}\sigma^2 \tilde{\phi}(s)^2$. As l can be freely chosen, we pick $l = 1$ for convenience. In the following, let $g_p(y) \equiv g_p(y|1)$.

We are now able to write the overall solution including boundary conditions as

$$\begin{aligned} g(y) &= g_h(y) + g_p(y) = c_+ y^{\eta_+} + c_- y^{\eta_-} + g_p(y) \\ &\textit{subject to} \\ g'(1) &= \eta_+ c_+ + \eta_- c_- = 0 \\ g(\hat{y}) &= g(1) + \log(1-f) \\ g'(\hat{y}) &= 0 \end{aligned}$$

with $\hat{y} \in (\bar{\bar{y}}, 1)$ and where we used the fact that $g_p'(y)$ vanishes at $y = l = 1$. We see that we are left with a non-linear system of equations. Solving for the value function given an (arbitrary) retrading boundary $\hat{y}$, we get

$$g(y|\hat{y}) = \frac{\log(1-f) - g_p(\hat{y})}{\eta_+(\hat{y}^{\eta_-} - 1) - \eta_-(\hat{y}^{\eta_+} - 1)}(\eta_+ y^{\eta_-} - \eta_- y^{\eta_+}) + g_p(y) \tag{10}$$

so that the final equation determining $\hat{y}$ is

$$0 = g_y(y|\hat{y})|_{y=\hat{y}} = \frac{\log(1-f) - g_p(\hat{y})}{\eta_+(\hat{y}^{\eta_-} - 1) - \eta_-(\hat{y}^{\eta_+} - 1)} \eta_+ \eta_- \left(\hat{y}^{\eta_- - 1} - \hat{y}^{\eta_+ - 1}\right) + g_p'(\hat{y}) \tag{11}$$

Finally, denote the optimal value function (that is $g(y|\hat{y})$ with the optimal $\hat{y}$) conditional on a no-trading strategy by $g(y)$. Let us summarize:

Proposition 2 *Suppose the bank employs a no-trading strategy on $(\hat{y}, 1)$ with $\hat{y}$ bounded away from 1. Then the optimal retrading boundary $\hat{y}$ is given by equation (11) and the bank's value function given $\hat{y}$ is given by equation (10). The bank will follow a no-trading strategy if and only if $g(1) > \bar{g}$.*

For closed form solutions up to the determination of the boundary $\hat{y}$ that do not contain integrals, we need $\eta_\pm \in \mathbb{Z}$. For arbitrary $N \in \mathbb{N}$ we have mixed log-polynomial solutions

$$I_N(s) \equiv \int \frac{s^N}{1 + \bar{\phi}(s-1)} ds = \sum_{n=1}^{N} s^n \cdot cst_n + \log\left[1 + \bar{\phi}(s-1)\right] cst_0$$

and

$$I_N^2(s) \equiv \int \frac{s^N}{\left(1+\bar{\phi}(s-1)\right)^2} ds = \sum_{n=1}^{N} s^n \cdot \tilde{cst}_n + \log\left[1+\bar{\phi}(s-1)\right] \tilde{cst}_0 + \left(1+\bar{\phi}(s-1)\right)^{-1} \tilde{cst}_{-1}$$

where $cst_n, \tilde{cst}_n$ are some constants. I_N and I_N^2 are the solutions to the two integrals that appear in $g_p(y)$ and $g_p'(y)$. For $\eta_\pm \notin \mathbb{Z}$, we can numerically integrate via very robust methods to get $g_p(y)$ and $g_p'(y)$.

4.2 The impact of fines on risk-taking

We can prove the following comparative static statement with respect to f:[9]

Corollary 1 *Suppose that $\hat{y}$ is optimally chosen so that $g''(\hat{y}) > 0$. Then*
(i) the initial value of the no-trading option goes down as fines increase, $\frac{dg(1)}{df} < 0$, but
(ii) conditional on following a no-trading strategy, the retrading boundary $\hat{y}$ decreases and thus the accepted maximum leverage $\hat{\phi}$ increases as fines increase, $\frac{d\hat{y}}{df} < 0$.

The intuition for part (i) of the corollary is straighforward - as fines increase, the value derived from a non-trading strategy declines. Part (ii) is the more interesting part of the corollary, and the intution hinges on the fact that it is conditional on a no-trading strategy. As fines increase, given that the bank is following a no-trading stragegy, resetting from y to 1 is now costlier. Consequently, the bank will be more willing to wait a little longer to see if the process y drifts upward again. Essentially, higher fines will make the bank more willing to gamble for low values of y.

An important implication of this result is that increasing fines for the mismarking of the balance sheet has an *ambiguous* effect on the risk-taking of a financial institution. Suppose the regulator increases fines for misrepresenting one's balance sheet. Part (i) of the corollary states that the overall attractivness of the no-trading strategy shrinks - $g(1)$ decreases whereas

[9]No such clean analytical results can be obtained for the other parameters ρ, μ, σ as they enter the characteristic roots of the differential equation and are thus much more entwined in the solution. Numerical comparative statics are readily available, but dropped for brevity.

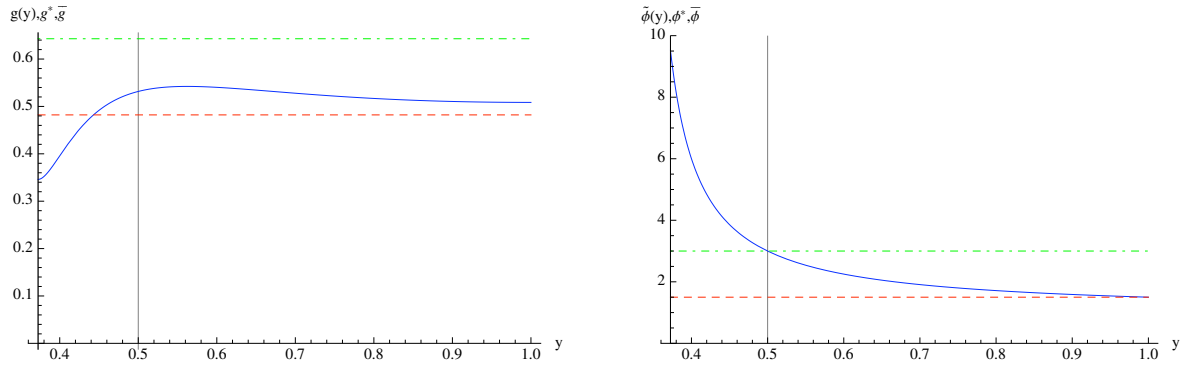

Figure 2: **Value function & exposure:** Left panel: Benchmark value functions: optimal Level 3 behaviour blue line ($g(y)$), continuous trading given constrained dashed red ($\bar{g}$), unconstrained dash-dot green (g^*); Right panel: exposure/leverage: optimal Level 3 behaviour leverage blue line ($\tilde{\phi}(y)$), constraint dashed red ($\bar{\phi}$), unconstrained optimal leverage dash-dot green (ϕ^*)

$\bar{g}$ is unaffected by possible fines. As the set of institutions following a no-trading strategy shrinks, this effect decreases risk-taking. But part (ii) states that if a no-trading strategy remains optimal, risk-taking actually increases. Thus, fines have a dual effect - they make following the no-trading strategy less attractive, but conditional on an instution following a no-trading strategy increase the maximal and thus also the average risk-taking.

4.3 Benchmark scenario

As the benchmark scenario, assume $\mu = .06$, $\sigma^2 = .02$ (and thus $\sigma = \frac{\sqrt{2}}{10} \approx .141$), so that $\phi^* = 3$. Also, pick $\rho = .14$. The resulting roots are $\eta_\pm = \{-7, 2\}$, and we can solve in closed form up to $\hat{y}$.

Figure 2 shows the value function $g(y)$ in our benchmark case for $\bar{\phi} = 1.5$ and $f = .15$ as the blue line. The green line depicts the unconstrained value function g^* whereas $\bar{g}$ the value function resulting from the compliance strategy. The value function $g(y)$ increases at first when prices, i.e. y, drop as leverage $\tilde{\phi}(y)$ gets closer to ϕ^*. But there will be a point beyond which the value function $g(y)$ starts decreasing as prices drop. As the institution's exposure $\tilde{\phi}$ starts increasing beyond ϕ^*, the value function $g(y)$ can even drop below $\bar{g}$.[10] In

[10]This can only occur for $f > 0$.

other words, there is regret: $g(1) > \bar{g}$ and $g(\hat{y}) < \bar{g}$ imply that when the agent made the decision to follow a no-trading strategy at $y = 1$, it dominated the compliance strategy. But as prices deteriorate, there will be a point at which the bank regrets having stopped trading in the past instead of having followed a compliance strategy.

Figure 3 shows the behaviour of the optimal retrading boundary $\hat{y}$ when we vary f and $\bar{\phi}$ (odd rows), and the corresponding value the initial point $y_0 = 1$ (even rows). We see that $\hat{y}$ is close to the bankruptcy boundary $\bar{\bar{y}}$ for low values of $\bar{\phi}$. For our benchmark scenario of $f = .15$, the no-trading strategy is dominated by the compliance strategy for $\bar{\phi} > 2.1$ - this is the upward jump in the solid blue line describing the optimal $\hat{y}$ (with $\hat{y} = 1$ implying following the compliance strategy). We can also see this in the even rows as the function $g(1)$, which is the red dashed line, dips below the function $\bar{g}$, which is the green dot-dashed line. As fines increase, the set of $\bar{\phi}$ on which the no-trading strategy is optimal shrinks slowly as the function $g(1)$ slopes downard more rapidly. We can see that the boundary $\hat{y}$ gets closer to bankruptcy boundary $\bar{\bar{y}}$ as corollary 1 showed analytically. The comparative statics w.r.t. ρ are not shown here, but the conditional on no-trading boundary $\hat{y}$ only moves very little for reasonable changes in ρ ($\pm 50\%$). The impact of ρ on the attractiveness of the no-trading option is discussed in section 4.4.

Note that for all parameter values we have $\hat{y}$ increasing in $\bar{\phi}$. From a regulators perspective, increasing capital requirements (and thus decreasing $\bar{\phi}$) leads to more conditional risk-taking (as $\hat{y}$ decreases) and possibly switching away from a compliance strategy to a no-trading strategy that increases risk-taking.

An optimal no-trading strategy will thus result in a specific reported valuation profile. Figure 1 illustrates. When $\tilde{\phi}$ is at $\bar{\phi}$, and the price rises, the bank immediately trades to keep exposure from dropping below $\bar{\phi}$. Thus, the rising price is reported and the gains to the balance sheet are immediately realized. The realized price path traced out by this trading behaviour resembles a maximum process of the price, as the asset's valuation remains stale on $\left(\bar{\phi}, \hat{\phi}\right)$. However, as opposed to the maximum process, there are downward jumps to the reported price path in case $\tilde{\phi}$ reaches $\hat{\phi}$. At this point, the reported price drops (proportionally) by $(1 - \hat{y})$. This leads to a large and abrupt change in the reported balance

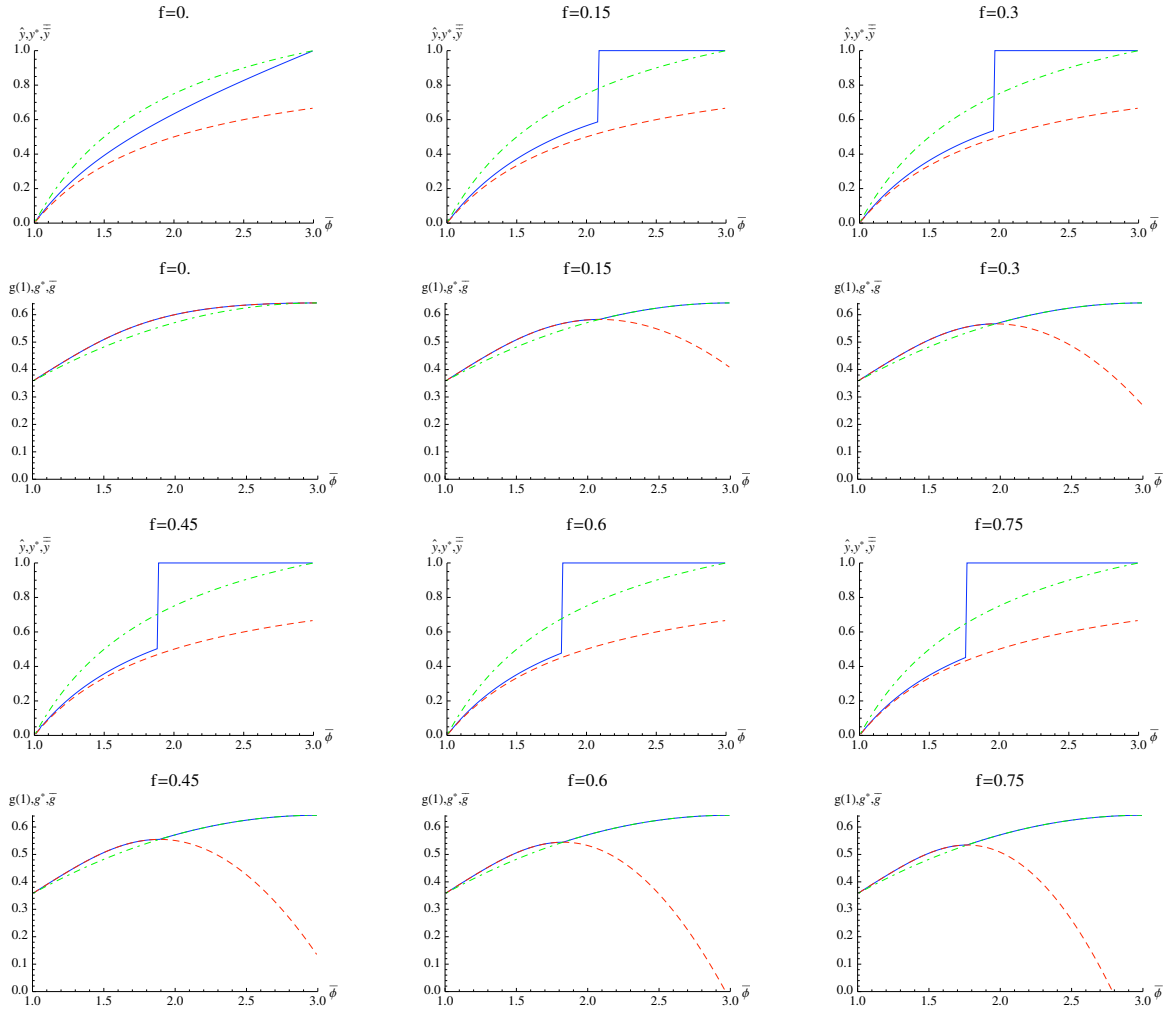

Figure 3: **Optimal trading boundary and corresponding initial value:** Odd rows: Optimal retrading boundary blue line ($\hat{y}$), bankruptcy boundary dashed red ($\bar{\bar{y}}$), unconstrained optimal exposure boundary dash-dot green (y^*); Even rows: Optimal value at $y = 1$ blue line ($\max\{g(1), \bar{g}\}$), value of no-hiding dashed red ($g(1)$), value of continuous updating dash-dot green ($\bar{g}$)

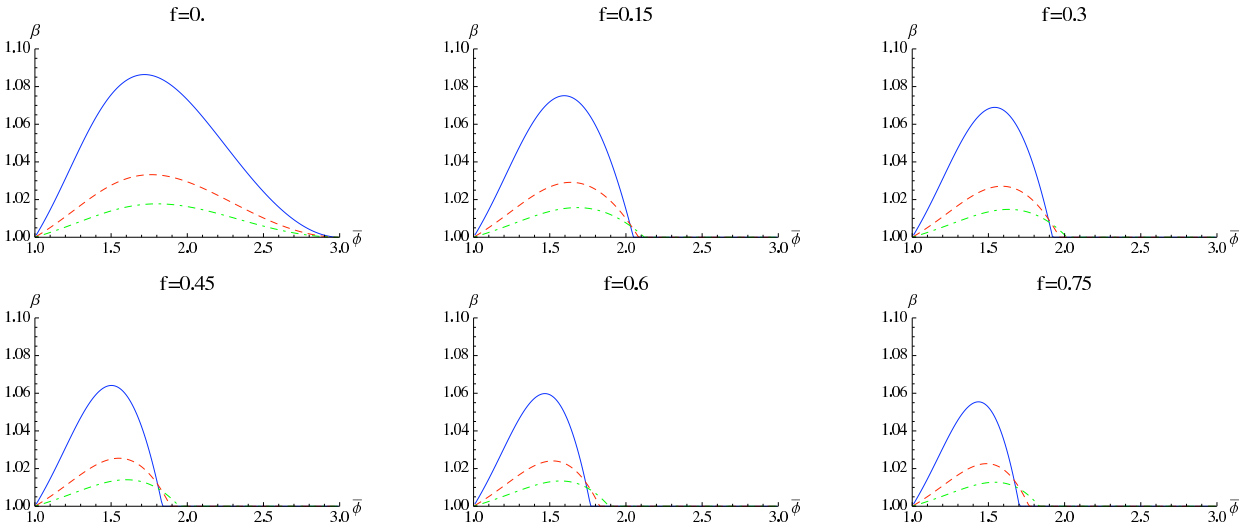

Figure 4: **Value of Level 3 accounting option (β):** Blue line $\rho = .06$, red line $\rho = .14$, green line $\rho = .24$

sheet value of the firm, as is shown in the right panel of figure 1.

4.4 Value of option to mark-to-model

We can now derive how big the wealth gain is, for a given constraint $\bar{\phi}$, of having accounting flexibility that allows for more leverage. Let $\beta \geq 1$ be the multiple that has to be applied to wealth W so the agent is willing to forego the accounting flexibility that comes with Level 3 assets at time 0. Then

$$\beta \quad = \quad \max\left\{\exp\left[g\left(1\right) - \frac{1}{\rho}\left(\mu\bar{\phi} - \frac{\sigma^2\bar{\phi}^2}{2}\right)\right], 1\right\}$$

Figure 4 illustrates β for different hoirzon intensities.

First, we see that the β curve is bell-shaped. As $\bar{\phi} \to \phi^*$, clearly the accounting option becomes less valuable: the constrained value function $\bar{g}$ will asymptote towards the unconstrained value function g as $\bar{\phi} \to \infty$. Therefore, the additional value to be gained from discretionary accounting has to vanish as well, as $\tilde{g}$ lies between g and $\bar{g}$ at $y = 1$.

On the other hand, at $\bar{\phi} = 1$, the discretionary accounting option does not enlarge the strategy space: if the institution stops trading at $\bar{\phi} = 1$, its actual and reported leverage will

always be constant at 1, as debt is zero. By continuity, the possible value gain is small for $\bar{\phi}$ close to 1: for a given $\hat{y}$, $\tilde{\phi}$ remains flat for a large range of y, resulting only in marginal improvements in closer approximating ϕ^*. As $\bar{\phi}$ increases, $\tilde{\phi}$ becomes steeper as a function of y, allowing the institution to closer approximate ϕ^* and leading to more value to be gained from stale prices.

4.5 Discussion of applicability of leverage constraint

The reader might be concerned that the leverage rules imposed by the regulator are static: given that the bank is not reporting any continuous trades, the regulator knows that its leverage is higher than allowed. There are two answers to why the model still makes sense.

One, there is the enforcability issue. The regulator might be well aware that mismarking is going on, but this is not enough to intervene. An intervention has to be sanctioned by a court, and thus evidence has to be produced as to what the real underlying price is. The argument is that the counter-positive "the bank is not trading and thus has to be mismarking" does not hold up in court. A thorough audit has to be performed that will produce the evidence necessary to intervene. Such audits can be handled by extensions discussed in section 5.

Two, the model can still go through even if there is enforcability via the counter-positive by inserting some small transaction costs into the model. The reason is the following: with proportional transaction costs, the agent may not use $\bar{\phi}$ has his lower exposure limit. Rather, with sufficient transaction costs, the bank might choose a no-trading region with lower limit $\hat{\bar{\phi}} < \bar{\phi}$. If sufficient probability mass lies on the set $\left[\hat{\bar{\phi}}, \bar{\phi}\right]$, the expectation of the exposure of the bank can lie below $\bar{\phi}$, ruling out intervention by the regulator. Although the transaction costs extension can be incorporated into the model, the level of technical detail involved is beyond the scope of the paper.

5 Auditing, random detection & liquidity shocks

Consider now the situation in which the regulator audits randomly and with intensity ξ finds enforceable[11] evidence of mismarking of the balance sheet. In this case, the regulator levies a fine $\hat{f}W$ and forces immediate compliance with the leverage constraint. If the agent himself decides to retrade, a fine fW is imposed if revealed to be in violation of the constraint, with potentially different $f \leq \hat{f}$. An observationally equivalent interpretation of this setup would be that the bank faces liquidity shocks that force it to realize prices at random times that are exponentially distributed with intensity ξ. This interpreation forces $f = \hat{f}$ for consistency. For the rest of this subsection we will follow the audit interpretation.

A random audit thus punishes in two ways: (1) it imposes a cost on the firm for non-compliance in the form of a fine, and (2) it imposes an additional cost on the firm by forcing a trade down to $\tilde{\phi} = \bar{\phi}$. The first part of the punishment is essentially a constant flow cost (in value function terms) of $\xi \log\left(1-\hat{f}\right)$. The second part of the punishment, however, is not constant and will depend on how far y is away from y^*. Can this second effect outweigh the first effect? Clearly, this will depend on y, for a y very close to 1 the first effect will dominate, whereas for y around y^*, the second effect may dominate. Writing out the HJB/ODE for this specification, we have

$$\begin{aligned}
\rho g(y) &= \mathcal{L}^y g(y) + \mu_{\log W}(y) + \xi\left[\left(\log\left[\left(1-\hat{f}\right)W\right] + g(1)\right) - (\log W + g(y))\right] \\
&= \mathcal{L}^y g(1) + \mu_{\log W}(y) + \xi\left[\Delta g(y) + \log\left(1-\hat{f}\right)\right]
\end{aligned}$$

where this is a very simple version of a differential-difference equation (here, the difference being $\Delta g(y) \equiv [g(1) - g(y)]$) and $\mathcal{L}^y g(y) \equiv \mu_y y g'(y) + \frac{\sigma^2 y^2}{2} g''(y)$ is the linear generator of y. The punishment can now be written as the sum of (1) $\xi \log\left(1-\hat{f}\right)$, which is the constant flow punishment cost, and (2) $\xi \Delta g(y)$, which is the state dependent punishment cost.

The ODE is easy to solve due to its underlying linear nature. There are now different roots of the homogenous equation, $\eta_{\pm} = \frac{\frac{\sigma^2}{2} - \mu \pm \sqrt{2(\rho+\xi)\sigma^2 + \left(\frac{\sigma^2}{2}-\mu\right)^2}}{\sigma^2}$, but the solution to the

[11] See discussion in 4.5.

homogenous equation stays in the same form

$$g_h(y) = c_+ y^{\eta_+} + c_- y^{\eta_-}$$

Secondly, the particular part now becomes

$$g_p(y|l) = \frac{2}{\sigma^2}\int_y^l \frac{\mu_{\log W}(s)}{s^2}\frac{s^{\eta_+}y^{\eta_-} - y^{\eta_+}s^{\eta_-}}{Wr(s)}ds + C$$

where $C = \frac{\xi}{\rho+\xi}\left[g(1) + \log\left(1-\hat{f}\right)\right]$ is a constant. As C includes the unknown value $g(1)$, we need to evaluate $g(y)$ at $y = 1$ to get

$$C = \frac{\xi}{\rho}\left[c_+ + c_- + \log\left(1-\hat{f}\right)\right]$$

The boundary conditions are the same as in section 4

$$\begin{aligned} g'(1) &= g_h'(1) = \eta_+ c_+ + \eta_- c_- = 0 \\ g(\hat{y}) &= g(1) + \log(1-f) \\ \iff g_h(\hat{y}) + \frac{2}{\sigma^2}\int_{\hat{y}}^1 ...ds &= g_h(1) + \log(1-f) \\ g'(\hat{y}) &= g_h'(y) + \frac{2}{\sigma^2}\left(\int_{\hat{y}}^1 ...ds\right)' = 0 \end{aligned}$$

where $\frac{2}{\sigma^2}\int_{\hat{y}}^1 ...ds$ ($\frac{2}{\sigma^2}\left(\int_{\hat{y}}^1 ...ds\right)'$ resp.) took the place of $g_p(\hat{y})$ ($g_p'(\hat{y})$ resp.) in comparison to the solution of section 4 as the particular solution g_p now contains both an integral and the constant C.

Note that C does not enter the first and third boundary condition, as it drops out when taking the derivative, and drops out of the second boundary condition as it enters both sides of the equation additively. Consequently, the only difference to the model in section 2 with a horizon intensity $\rho + \xi$ will be in the level of the value function as represented by C. Of course, the level matters for the choice of the no-trading versus the compliance strategy. But conditional on the no-trading strategy, the solution to the optimal retrading boundary

is equivalent to the no-auditing solution with $\rho + \xi$. Thus, we can write the value function (conditional on no-trading) as

$$g^{\rho,\xi}(y) = g^{\rho+\xi}(y) + \frac{\xi}{\rho}\left[g^{\rho+\xi}(1) + \log\left(1-\hat{f}\right)\right] \tag{12}$$

where we used the notation $g^{\rho,\xi}(y)$ for the model with horizon intensity ρ and auditing intensity ξ and $g^{\rho+\xi}$ for the model without auditing but overall horizon intensity $\rho + \xi$.[12] This equality holds due to the fact that the constants c_+, c_- are the same for $g^{\rho,\xi}$ and $g^{\rho+\xi}$ (as they are determined by the same roots and the same boundary conditions), so we can make use of $g^{\rho+\xi}(1) = c_+ + c_-$. We summarize:

Corollary 2 *Suppose the bank employs a no-trading strategy on $(\hat{y}, 1)$ with $\hat{y}$ bounded away from 1. Suppose further that it faces random auditing with intensity ξ that forces it conform to the balance sheet constraint and brings fines of level $\hat{f}W$ with it. Then the optimal stratgey is the same as in Proposition 2 as if facing a horizon intensity of $\rho+\xi$ with no auditing, and the overall value function is given by equation (12). The bank will employ a hiding strategy if and only if $g^{\rho,\xi}(1) > \bar{g}^{\rho}$.*

What now is the effect of stochastic auditing or liquidity shocks on the trading decision of the firm?[13] Figure 5 supplies the optimal retrading boundaries $\hat{y}$ in the case of our benchmark parameters with $\xi = .1$. The solid blue line depicts the retrading boundaries under no-auditing $\hat{y}^{\rho}$, whereas the dashed red line depicts the retrading boundaries under auditing $\hat{y}^{\rho+\xi}$. We see that conditional on no-trading, the trading boundaries are almost identical - as mentioned in section 4.3, reasonable variations in ρ do not impact the optimal retrading boundary much. Thus, the main impact of random auditing will come from the impact it has on the attractiveness of the no-trading versus compliance strategy.

We see that the monotonicity of the cut-off between no-trading and compliance that we saw in 4.3 disappears - the bank will become less willing to use a no-trading strategy for very

[12] A similar notation applies for $\bar{g}$: $\bar{g}^{\rho}$ denotes the outside option w.r.t. an intensity ρ. Note however that $\bar{g}^{\rho} = \bar{g}^{\rho,\xi} \neq \bar{g}^{\rho+\xi}$.

[13] Note that this discussion also applies approximately to pure level effects as discussed in footnote 7 because equation (12) contains a shift down via the constant $\frac{\xi}{\rho}\log\left(1-\hat{f}\right)$.

low and very high levels of $\bar{\phi}$. This is cause by the level shift of the value function due to the constant. For very low $\bar{\phi}$, and without auditing, following a no-trading strategy instead of a compliance strategy will be approximately the same as the occurence of fines is a remote event with $\hat{y}$ that far out. But with auditing, there is a constant probability of detection with a fine that is not proportional to the deviation of $\tilde{\phi}$ from $\bar{\phi}$. Thus, for low values of $\bar{\phi}$ the bank will follow a compliance strategy. From a regulators perspective, very tight capital requirements combined with moderate fines can now ensure a compliance strategy.

Additionally, we see that fines now have a much stronger impact, as they also influence the shift down of the value function. Although we still have the fact that fines shift the no-trading boundary down, we note that a much lower level of fines is required to enforce compliance across the whole range of $\bar{\phi} \in [1, \phi^*]$.

6 Multiple banks

The previous case described the optimal strategy of a single bank. Consider instead a change to our assumption (3) to: **(A3') there are n institutions that each have to acknowledge each other's transaction prices in their Level 3 asset trading but can ignore all other transaction prices.** These banks can have possibly different leverage constraints $\bar{\phi}$. If there is continuous price updating, there is no strategic interaction between the banks, as we take the price process as exogenous. With price impact and continuous updating, we would be in the predatory trading case treated by Brunnermeier and Pedersen (2005). In their model, the driving mechanism for predatory trading is the round-trip benefit in which a predatory seller sells for more on the way down and then buys back for less on the way up as one less player is bidding. This mechanism is absent here by design as the underlying price process is exogenous – there is no round-trip benefit. The only impact of a trade is the endogenous realization of the underlying (shadow) price. Strategic interaction arises in markets for level 3 assets, as prices are not continuously updated and a bank's trade imposes transaction prices not only on itself but also on its competitors.

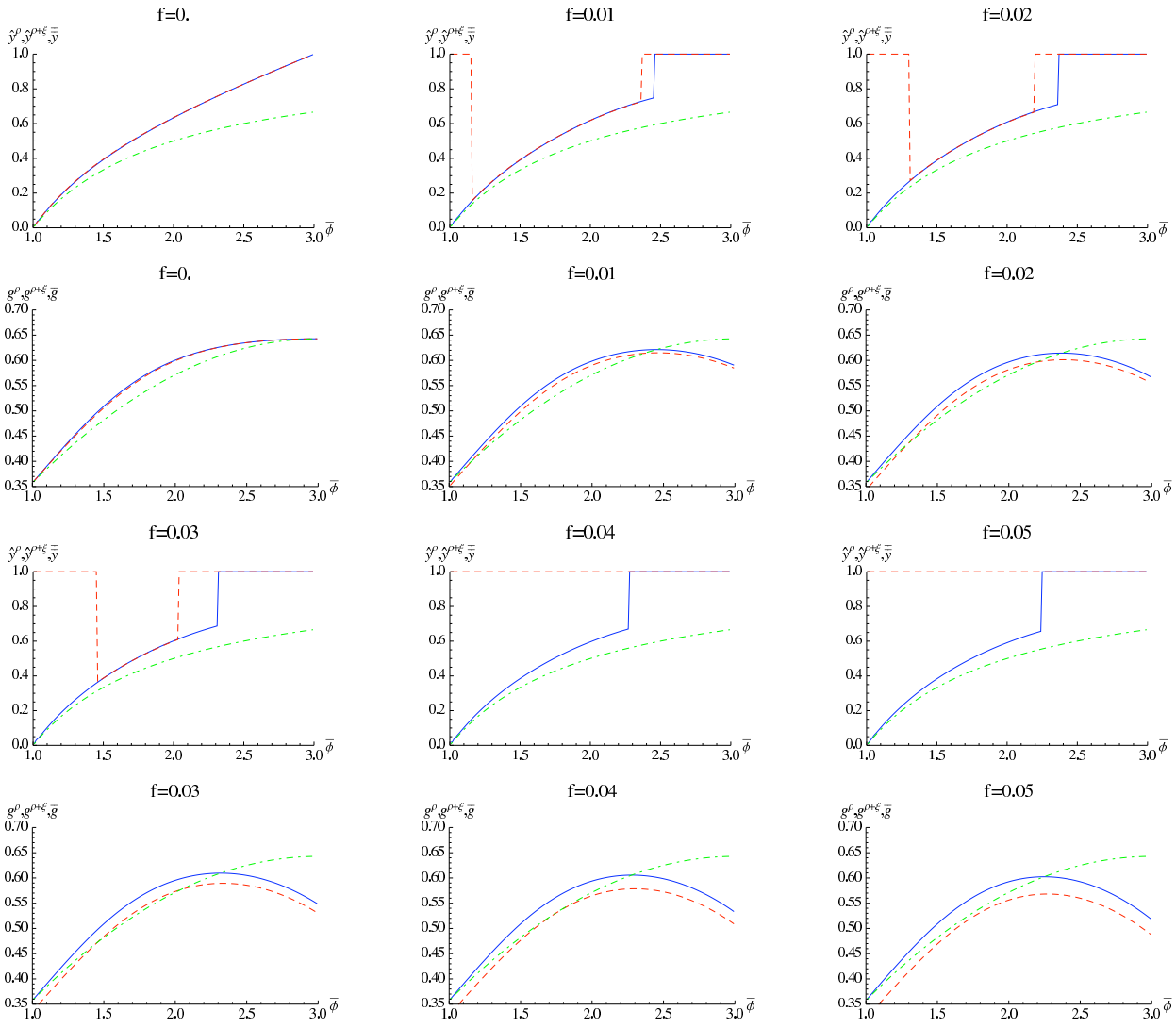

Figure 5: **Optimal trading boundary and corresponding initial value under auditing:** Odd rows: Optimal retrading boundary without auditing blue line ($\hat{y}^{\rho}$), optimal retrading boundary with auditing dashed red line ($\hat{y}^{\rho+\xi}$), bankruptcy boundary dash-dot green (y^{*}); Even rows: Value of no-trading without auditing blue line ($g^{\rho}(1)$), value of no-trading with auditing blue line ($g^{\rho+\xi}(1)$), value of continuous updating dash-dot green ($\bar{g}$); $\hat{f} = f$ throughout

6.1 Different horizon times

Consider the situation in which all banks have the same leverage constraint $\bar{\phi}$. Assume that the banks horizon times (or liquidity shocks, see section 5) are independent. When i has to liquidiate (and is replaced by a new idential institution, so stationary is preserved) or has to trade due to a liquidity shock at time τ_i , the price that is realized is also imposed on the books of institution j. Essentially, we can frame this situation in terms of the setup in 5 with $\xi = \sum_{j \neq i}^{n} \rho_j = (n-1) \times \rho$ and $\hat{f} = f$ where we assumed that $\rho_j = \rho, \forall j$ so that we have a symmetric problem.

6.2 Different constraints

Let us now consider the possibility that the leverage constraints $\bar{\phi}$ amongst the banks are not the same. This might be due to unmodeled lines of businesses of the banks. Consider the two bank case with banks A and B – this is without loss of generality as only the least constrained institution and most constrained institution amongst n are pivotal. Let us assume for ease of exposition that the banks share the same horizon time and there is no auditing, but the banks are facing different leverage constraints. Let bank A be less constrained than bank B, i.e. $\bar{\phi}_A > \bar{\phi}_B$. What then are possible outcomes of the trading game of the agents?

First, there is an always-trading equilibrium in which all banks trade continuously. The optimal response to an always-trading player is to always trade as well – stopping trading at the constraint will simply lead to an immediate violation of the constraint that brings substantial (and very frequent) fines with it.

Second, there can be an equilibrium in which trading stops. Consider the situation in which bank A, the least constraint bank, decides to follow its single player optimal strategy and picks $\hat{y}_A$ as its retrading boundary. As the curve $\hat{y}\left(\bar{\phi}\right)$ is upward sloping (see figure 3), a more lax leverage constraint (higher $\bar{\phi}$) leads to an individually optimal retrading boundary $\hat{y}$.

Bank B now has two options: it can either (1) always-trade and ensure itself a value $\bar{g}$ or it can (2) mimic bank A and get some value $g_B\left(1|\hat{y}_A\right)$ (that is not equal to $g_B\left(1\right)$ as the

retrading point $\hat{y}_A$ is not individually optimal for bank B). If bank B is much more constrained than bank A, it will have significantly lower leverage going into the no-trading region. B will then have to weigh if it is worth to stop trading, given that it will be forced to liquidate at $\hat{y}_A$ – it would be willing to not trade until $\hat{y}_B < \hat{y}_A$, but A's trading reveals the violation of the leverage constraint on B's books. Clearly, if bank B individually is unwilling to stop trading given an optimal $\hat{y}_B$, it will not accept a suboptimal boundary $\hat{y}_A$. But if bank B individually would be willing to stop trading, it might accept bank A's boundary.

Alternatively put, banks' constraints and their positions define their incentives to stop trading. When positions (influenced by their constraints) lie closely together, incentives to stop trading are aligned and there exists an equilibrium that yields no-trading. The no-trading interval is determined by the least constrained bank, but the most constrained bank will determine if this equilibrium actually exists. When the no-trading equilibrium exists, it is the Pareto optimal one from the view of the firms. Let us summarize:

Proposition 3 *Let there be n banks with different leverage constraints $\bar{\phi}$ but with possibly different horizon times and independent liquidity shocks. These banks have to acknowledge each other's transaction prices for accounting purposes. Further, assume that all $\bar{\phi}_i$'s are such that banks would individually stop trading. Then*
(i) There always exists an always-trading Nash equilibrium in which all banks trade continuously.
(ii) There can exists a second, no-trading Nash equilibrium when the leverage constraints of the least and most constrained bank lie close enough together. The no-trading interval will be determined by the least constrained bank, and the least constrained bank will still find it optimal to follow a no-trading strategy, i.e.

$$g_B\left(1|\hat{y}_A\right) > \bar{g}_B$$

. This second equilibrium Pareto dominates the always-trading equilibrium from the banks' perspective.

Figure 6 shows the possible multiple bank equilibria for the n bank case. The blue line

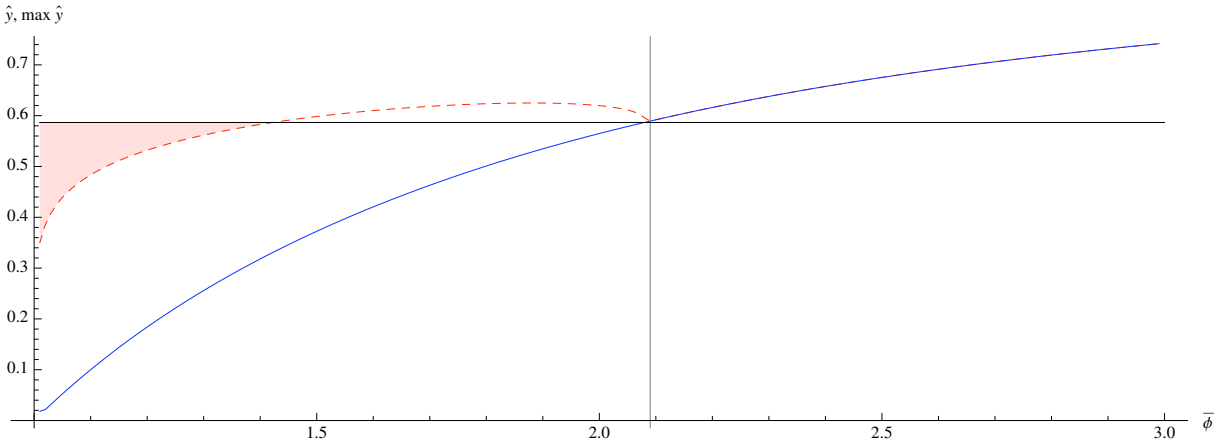

Figure 6: **Multiple bank equilibrium:** The blue line shows the optimal $\hat{y}$ for each $\bar{\phi}$. The dashed-red line is the maximal $\hat{y}$ a bank with constraint $\bar{\phi}$ is willing to accept to still use a no-trading strategy. The shaded red area is the set of $\bar{\phi}$'s for which only an always trading equilibrium exists. Beyond the vertical line at $\bar{\phi}_B = 2.09$ only an always trading equilibrium exists.

shows the optimal $\hat{y}$ for each $\bar{\phi}$, whereas the dashed red line shows the maximal acceptable $\hat{y}_A$ for the most constrained player $\bar{\phi}_B$. To use the graph, first figure out the maximal $\hat{y}_B^{max}$ bank $\bar{\phi}_B$ is willing to accept a no-trading equilibrium. Then via the blue line we translate it back to the maximal leverage constraint that A can have for the no-trading equilibrum to exits, $\bar{\phi}^{-1}\left(\hat{y}_B^{max}\right)$. For example, for $\bar{\phi}_B = 1.1$, the maximal leverage of the least constraint bank that still allows a no-trading equilibrium is around $\bar{\phi}_A^{max} = 1.8$. For any $\bar{\phi}_B \geq 1.4$, the maximal bank A leverage constraint is $\bar{\phi}_A = 2.09$. Thus, there can be sizable differences in the constraints faced by each bank that can still sustain a no-trading equilibrium. Most major banks carried sizable positions in related level 3 assets into the current crisis, their incentives to stop trading were aligned. Thus, the model provides some explanation of why banks might refuse to trade. Additionally, the current situation shows that these balance sheet incentives can outweigh possible predatory trading incentives that are abstracted from in this current model.

7 Estimating true balance sheet value

Takeovers, forced mergers or refinancing at short notice involving companies heavily invested in Level 3 assets pose a formidable challenge to any acquirer or outside investor. Although the target company will open its books, outside investors often lack the necessary expertise to accurately value some of the assets held by the target. As our model showed, it is likely that book valuations based on stale prices are out of line with current shadow prices. As an example, the Korean Development Bank was in such a situation during its talks over a controlling stake at Lehman Brothers shortly before the latter's demise. Given the rapidly closing window of opportunity and the consequently very tight time frame, the Koreans ultimately passed on investing in Lehman Brothers, partially because of their difficulty appraising some of Lehman Brothers assets.

Our model can give some guidance in how to estimate the market price of such assets to an outsider with only access to the internal transaction prices and book valuations. Recall that we assumed that only institutions active in the market observe the shadow price process. With the derived no-trading boundaries, we are now in a position to examine the expected shadow price of non-traded assets to an outsider.

Our aim is to answer the following question: After prices have become stale, what is the expected shadow price given t units of time have passed since the last trade? Suppose the asset last traded at time s at a price P_s. If the asset has not been traded on $[s, s+t]$, the expected value of the asset at time $s+t$ is

$$\mathbb{E}^o_{s+t}\left[P_{s+t}\right] = P_s \cdot \mathbb{E}_s\left[y_{s+t} | y_r \in (\hat{y}, 1) \text{ for all } r \in [s, s+t], y_s = 1\right]$$

where $\mathbb{E}^o_t$ denotes an expectation w.r.t. the outsider information set and accounting information provided by the insider.

The last price at which significant amounts of the asset traded, P_s, is observable from the company's books, as is the time since last trade t. We rely on Monte Carlo methods to derive expected price paths as a full-fledged examination of the behavior of $\mathbb{E}^o_{s+t}\left[P_{s+t}\right]$ is beyond

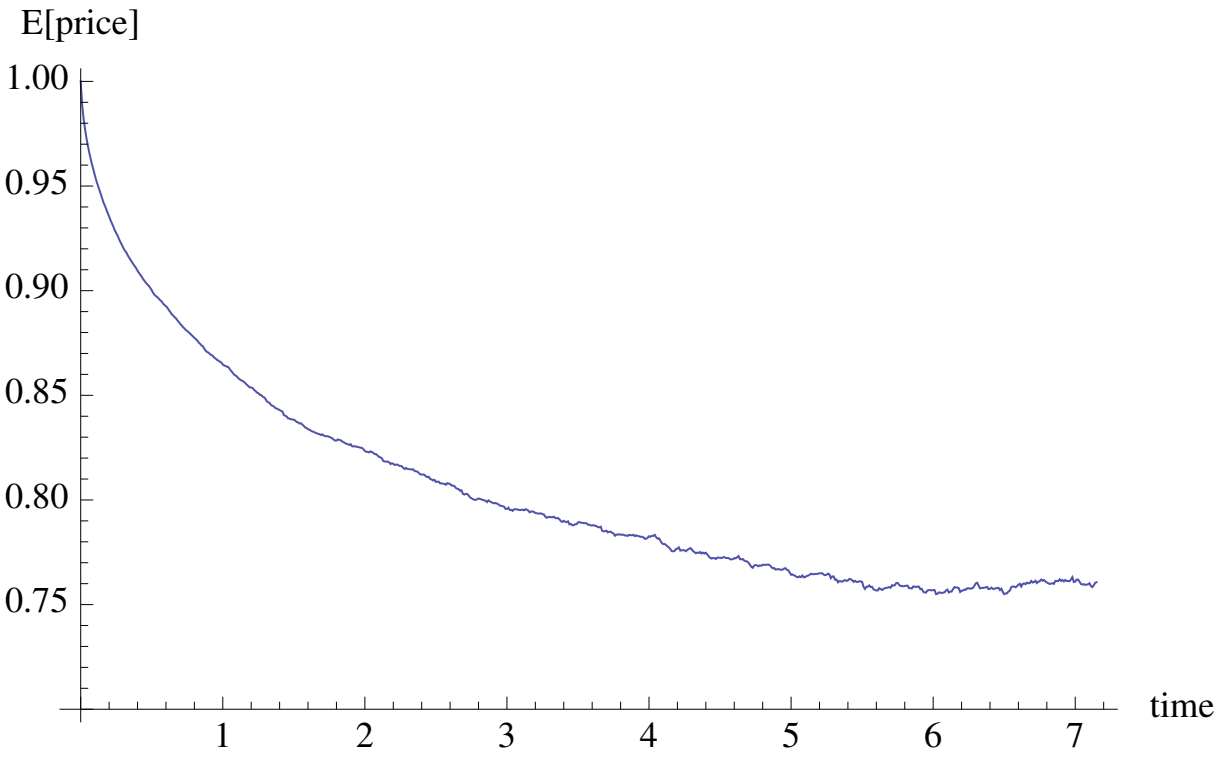

Figure 7: **Expected price path for stale asset:** $P_s = 1$, benchmark parameters, $\hat{y} = .37$

the scope of this paper.

Figure 7 shows one such expected price path. The path is monotonically decreasing in time: the longer an asset has been non-traded, the lower on average its price will be. This outcome is counter to the underlying shadow price process being increasing in its unconditional expectation. It is the property of the conditioning set $[\hat{y}, 1]$ that leads to strong enough 'bad news' conditioning to reverse the drift of the price process. Intuitively, 'good' paths that revert back to 1 before time $s+t$ will not be included in the conditioning set, as they result in the institution trading on $(s, s+t)$ – we are dropping the strongest upward drift paths. The expected price path is further convex in t, i.e. it decreases very fast at the beginning, but the rate of the decrease diminishes. Essentially, the information content of the asset staying stale for another Δt periods vanishes for assets that have not been traded for some large t.

It is important to note that the model is driven by the static balance sheet constraints that relies on self-reported transaction prices – the bank games the accounting rules to optimally breach this constraint. If the constraint could take the above derived decreasing price

path when no trade occurs into account, then the no-trade outcome could unravel unless we introduce transaction prices as discussed in 4.5. However, as we argued before, enforceability in a juridical sense requires some positive proof and such proof cannot be acquired instantenously.

8 Conclusion

This paper examined the dynamic effects of fair value accounting on the trading behavior of financial institutions and its implications for the valuation of non-traded assets on the institution's books. Certain OTC markets are so opaque that an institution under marking-to-market accounting only has to acknowledge self-generated transaction prices. Consequently, there can be incentives for balance sheet constrained institutions active in such markets to obstruct price discovery by keeping assets off the market.

I provide a model that derives when instutions suspend and restart trading of certain assets. Under fair value accounting, book valuations of securities generally have to be updated when new transaction prices are observed. However, OTC markets can be so opaque that no continuously observable prices exist. An institution active in such markets has to take the accounting impact of its own trading decisions into consideration. If faced with regulatory capital requirements, it might be optimal to book gains immediately as fundamentals rise above accounting valuations, but suspend trading when fundamentals drop to obstruct price discovery, thereby delaying losses. As book value ceases to reflect current market prices, the capital requirement is relaxed. Asset holdings, however, become fixed, leading to potentially excessive risk exposure. The institution optimally balances the benefit of a relaxed balance sheet constraint against the cost of possible excessive exposure to determine when to stop and restart trading, and thus when to book gains and losses, and if to follow such a no-trading strategy at all. A regulator wanting to control leverage of a bank has as possible tools capital requirements, fines for mismarking and random audits. Increasing fines or capital requirements can lead to increased risk-taking by increasing the maximal leverage the institution is willing to accept. Random audits in general decrease the risk-taking of banks.

These results are robust to an n bank extension – even if n banks each have to mark to each others prices, there can be no-trading equilibria. To an outside observer such as the regulator, the expected value of the balance sheet of a bank is decreasing and convex in time since last trade.

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A Appendix

A.1 Solutions to the ODE: Method of variation of coefficients

By the method of variation of coefficients, we can write the particular solution of an ODE in integral form involving its linearly independent solutions (see for example Coddington (1961), Ch.3, Sec. 10, Thm. 11). In our case, the second order ODE has linearly independent solutions y^{η_+} and y^{η_-}, and the solution will be of the stated form. Note that $l \in [\hat{y}, 1]$ is completely free, and does not influence the solution.

Lemma 3 *For the variation of coefficients solution*

$$g_p(y|l) = \frac{2}{\sigma^2}\left[\int_y^l \frac{\mu_{\log W}(s)}{s^2}\frac{s^{\eta_+}y^{\eta_-} - y^{\eta_+}s^{\eta_-}}{Wr(s)}ds\right]$$

the function and its first derivative w.r.t. y vanish at the limit of integration l

$$g_p(l|l) = \left.\frac{\partial g_p(y|l)}{\partial y}\right|_{y=l} = 0 \tag{A.1}$$

but the second derivative w.r.t. y does not

$$\left.\frac{\partial^2 g_p(y|l)}{\partial y^2}\right|_{y=l} = \frac{2}{\sigma^2}\frac{\mu_{\log W}(l)}{l^2}$$

Proof. Taking derivatives, we see that

$$\begin{aligned}
\frac{\partial g_p(y|l)}{\partial y} &= \frac{2}{\sigma^2}\left[\begin{array}{c}\int_y^l \frac{\mu_{\log W}(s)}{s^2}\frac{\eta_- s^{\eta_+}y^{\eta_-} - \eta_+ y^{\eta_+}s^{\eta_-}}{y\cdot Wr(s)}ds \\ -\left(\frac{\mu_{\log W}(y)}{y^2}\frac{y^{\eta_+}y^{\eta_-} - y^{\eta_+}y^{\eta_-}}{Wr(y)}\right)\end{array}\right] \\
&= \frac{2}{\sigma^2}\int_y^l \frac{\mu_{\log W}(s)}{s^2}\frac{\eta_- s^{\eta_+}y^{\eta_-} - \eta_+ y^{\eta_+}s^{\eta_-}}{y\cdot Wr(s)}ds \\
\frac{\partial^2 g_p(y|l)}{\partial y^2} &= \frac{2}{\sigma^2}\left[\begin{array}{c}\int_y^l \frac{\mu_{\log W}(s)}{s^2}\frac{\eta_-(\eta_- -1)s^{\eta_+}y^{\eta_-} - \eta_+(\eta_+ -1)y^{\eta_+}s^{\eta_-}}{y^2\cdot Wr(s)}ds \\ -\left(\frac{\mu_{\log W}(y)}{y^2}\frac{\eta_- y^{\eta_+ + \eta_- -1} - \eta_+ y^{\eta_+ + \eta_- -1}}{Wr(y)}\right)\end{array}\right] \\
&= \frac{2}{\sigma^2}\left[\int_y^l \frac{\mu_{\log W}(s)}{s^2}\frac{\eta_-(\eta_- -1)s^{\eta_+}y^{\eta_-} - \eta_+(\eta_+ -1)y^{\eta_+}s^{\eta_-}}{y^2\cdot Wr(s)}ds - \frac{\mu_{\log W}(y)}{y^2}\right]
\end{aligned}$$

Plugging in $y = l$, the integrals vanish as the integrands are all bounded, and we are left with the result. ■

A.2 Optimality proofs

Proof of Proposition 1. Verification argument:
We will verify the optimal strategy ϕ^* directly. Once the optimality of ϕ^* is established, what remains is essentially computing an expectation for a fixed ϕ^*. Note that τ is independent of Z, so that we can make use of conditional expectations. Also note that τ is almost surely finite. We can then write out the expectation as

$$\max_\phi \mathbb{E}\left[\log W_\tau\right] = \max_\phi \mathbb{E}^\tau\left[\mathbb{E}^Z\left[\int_0^\tau \mu\phi - \frac{1}{2}\sigma^2\phi^2 ds\right] + \mathbb{E}^Z\left[\int_0^\tau \sigma\phi dZ\right]\right] \tag{A.2}$$

where $\mathbb{E}^\tau$ and $\mathbb{E}^Z$ are expectations w.r.t. τ and Z respectively. By our assumption that ϕ is in $\mathcal{L}^2$, and by τ a.s. finite, we know that the stochastic integral $\int_0^\tau \sigma\phi dZ$ is a martingale and its expectation is therefore zero.

Maximizing the expectation thus reduces to maximizing the expectation of the time integral $\mathbb{E}\left[\int ...ds\right]$. As τ is independent of Z, we can maximize path-by-path. Applying simple calculus of variations will give ϕ^*. The value function of the main part can thus be interpreted as the solution to the expectation of $\mathbb{E}\left[\log W_\tau\right]$ with fixed strategy ϕ^*. This verifies that our value functions is indeed optimal. This proof is similarly applicable to the constraint $\bar{\phi} < \phi^*$ with only minor modifications.

■

Proof of Proposition 2. First, note that

$$c_+ = -\frac{\eta_-}{\eta_+} c_-$$

from the first boundary condition. Plugging this back into the second boundary condition gives

$$\begin{aligned} -\frac{\eta_-}{\eta_+} c_- \hat{y}^{\eta_+} + c_- \hat{y}^{\eta_-} + g_p(\hat{y}) &= -\frac{\eta_-}{\eta_+} c_- + c_- + \log(1-f) \\ \iff c_-(\hat{y}) &= \frac{\log(1-f) - g_p(\hat{y})}{\frac{\eta_-}{\eta_+}(1-\hat{y}^{\eta_+}) + (\hat{y}^{\eta_-} - 1)} \end{aligned}$$

so plugging this into the definition of $g(y|\hat{y})$ will give

$$\begin{aligned} g(y|\hat{y}) &= c_+ y^{\eta_+} + c_- y^{\eta_-} + g_p(y) \\ &= c_- \left(y^{\eta_-} - \frac{\eta_-}{\eta_+} y^{\eta_+} \right) + g_p(y) \\ &= \frac{\log(1-f) - g_p(\hat{y})}{\eta_+(\hat{y}^{\eta_-} - 1) - \eta_-(\hat{y}^{\eta_+} - 1)} (\eta_+ y^{\eta_-} - \eta_- y^{\eta_+}) + g_p(y) \end{aligned} \tag{A.3}$$

We then take the derivative w.r.t. y and evaluate at $y = \hat{y}$ to get equation (11).[14]

Verification:

Note that $\tilde{\phi}(y)$ is potentially unbounded and thus may not fulfill our assumption $\tilde{\phi} \in \mathcal{L}^2$. From Lemma 2, we know that there must exist an $\varepsilon > 0$ such that $\tilde{y} > \overline{\overline{y}} + \varepsilon$. We pick an arbitrary but small $\varepsilon = 10^{-10}$, from which boundedness of $\tilde{\phi}$ follows. Although this ε might theoretically not be small enough, we found it sufficient for all our numerical solutions.

Pick an arbitrary stopping time $\tilde{\tau}$ at which the agent retrades. We can then write, by Ito's formula

$$\begin{aligned} e^{-\rho\tilde{\tau}} V(W_{t+\tilde{\tau}}, y_{t+\tilde{\tau}}) &= V(W_t, y_t) \\ &\quad + \int_t^{t+\tilde{\tau}} \left[\mu y V_y + W\tilde{\phi}\left(\mu + \frac{1}{2}\sigma^2\right) V_W + \frac{1}{2} V_{yy}\sigma^2 y^2 + \frac{1}{2} V_{WW}\sigma^2\tilde{\phi}^2 + V_{yW}\sigma^2\tilde{\phi} y \right] ds \\ &\quad + \int_t^{t+\tilde{\tau}} \left[\sigma y V_y + \sigma\tilde{\phi} V_W W \right] dZ \end{aligned}$$

Conjecture the value function to be $V = \log W + g(y)$ and plug in to get

$$e^{-\rho\tilde{\tau}} V(W_{t+\tilde{\tau}}, y_{t+\tilde{\tau}}) = V(W_t, y_t) + \int_t^{t+\tilde{\tau}} (\text{ODE})\, ds + \int_t^{t+\tilde{\tau}} \left[\sigma y g'(y) + \sigma\phi\right] dZ$$

We now need to show that $\int \left[\sigma g'(y) + \sigma\tilde{\phi}(y)\right] dZ$ is a martingale. Any alternative stopping strategy can be described by a retrading point $y' \in \left[\overline{\overline{y}} + \varepsilon, 1\right]$. We then know by our closed form solution from Proposition 10 that for this alternative strategy x' we have

[14] Optimality dictates, by Dumas (1991) and Dixit (1993), a smooth pasting condition at $\hat{y}$. We can also derive this smooth pasting condition from the optimization $\max_{\hat{y}} g(y|\hat{y})$ for which we conveniently can choose $y = 1$. See equation (A.5).

(i) $g'(y)$ bounded for all $y \in \left[\overline{\overline{y}} + \varepsilon, 1\right]$ for $\varepsilon > 0$.
(ii) $\tilde{\phi}(y)$ bounded for all $y \in \left[\overline{\overline{y}} + \varepsilon, 1\right]$.
Therefore, the stochastic integral is a martingale on $[y', 1]$.
By construction, the value function is C^1 at the optimal $\hat{y}$. Also, on $y \in [\hat{y}, 1]$, (ODE) $= 0$ holds by construction. On $y \in (\bar{\bar{y}} + \varepsilon, \hat{y})$ we have $g(y) = \log(1-f) + g(1)$, a constant that only depends on $\hat{y}$. At $y = \hat{y}$, by the ODE we also have

$$0 = \frac{\sigma^2}{2}\hat{y}g''(\hat{y}) - \rho\left[\log(1-f) + g(1)\right] + \mu_{\log W}(\hat{y}) \tag{A.4}$$

as matching g and g' fixes g'' via the ODE.

Numerically, we can show for all parameters in the paper that equation (11) has a unique solution $\hat{y} < y^*$ and that at this optimal point $g''(\hat{y}) > 0$. Thus, we know that $\mu_{\log W}(y)$ is increasing in y on $(\bar{\bar{y}} + \varepsilon, y^*)$. We conclude that (ODE) $= -\rho\left[\log(1-f) + g(1)\right] + \mu_{\log W}(y) < 0$ on $(\bar{\bar{y}} + \varepsilon, \hat{y})$, and thus our initial strategy is optimal.

■

Lemma 4 *Let $h(\hat{y}) \equiv \eta_+(\hat{y}^{\eta_-} - 1) - \eta_-(\hat{y}^{\eta_+} - 1)$. On $\hat{y} \in (0,1)$, $h(\hat{y}) > 0$, $h'(\hat{y}) < 0$ and $h''(\hat{y}) > 0$.*

Proof.

The first derivative of $h(\hat{y})$ is $h'(\hat{y}) = \eta_+\eta_-\left(\hat{y}^{\eta_- - 1} - \hat{y}^{\eta_+ - 1}\right)$. Note that $\left(\hat{y}^{\eta_- - 1} - \hat{y}^{\eta_+ - 1}\right) > 0 \iff (\eta_- - 1)\log\hat{y} > (\eta_+ - 1)\log\hat{y}$ which holds as $\hat{y} \in (0,1)$ implies $\log\hat{y} < 0$. We thus have $h'(\hat{y}) < 0$. In conjuctino with $h(1) = 0$ this implies $h(\hat{y}) > 0$ on $\hat{y} \in (0,1)$. Further note that

$$h''(\hat{y}) = \eta_+\eta_-\left[(\eta_- - 1)\hat{y}^{\eta_- - 2} - (\eta_+ - 1)\hat{y}^{\eta_+ - 2}\right] > 0$$

as we made parameter restrictions implying $\eta_+ > 1$.

■

Proof of Corollary 1.

(i) Let us first take the derivative $\frac{dg(1|\hat{y})}{df} = \frac{\partial g(1|\hat{y})}{\partial f} + \frac{\partial g(1|\hat{y})}{\partial \hat{y}}\frac{d\hat{y}}{df}$. Evaluate equation (10) at $y = 1$ to get $g(1|\hat{y}) = \frac{\log(1-f) - g_p(\hat{y})}{h(\hat{y})}\Delta\eta$ where $\Delta\eta = \eta_+ - \eta_-$, we substituted in for $h(\hat{y})$ and we used the fact that $g_p'(y)$ vanishes at $y = \bar{l} = 1$. Taking the derivative gives

$$\begin{aligned}\frac{\partial g(1|\hat{y})}{\partial \hat{y}} &= -\Delta\eta\frac{g_p'(\hat{y})}{h(\hat{y})} - \Delta\eta\frac{\log(1-f) - g_p(\hat{y})}{h(\hat{y})^2}h'(\hat{y}) \\ &= -\frac{\Delta\eta}{h(\hat{y})}\underbrace{\left[\frac{\log(1-f) - g_p(\hat{y})}{h(\hat{y})}h'(\hat{y}) + g_p'(\hat{y})\right]}_{= g_y(y|\hat{y})|_{y=\hat{y}}} = 0\end{aligned} \tag{A.5}$$

Clearly, this is just the envelope theorem, and we thus have $\frac{dg(1|\hat{y})}{df} = \frac{\partial g(1|\hat{y})}{\partial f} = -\Delta\eta\frac{1}{1-f}\frac{1}{h(\hat{y})} < 0$. This of course is straight-forward: when the fines increase, the value function has to go down.

(ii) How does the retrading boundary $\hat{y}$ move as f changes? First, multiply equation (11) through by $h(\hat{y}) > 0$ to get

$$Q(f, \hat{y}) \equiv \left[\log(1-f) - g_p(\hat{y})\right]h'(\hat{y}) + g_p'(\hat{y})\,h(\hat{y}) = 0$$

Now, by the implicit function theorem, we know that $\frac{d\hat{y}}{df} = -\frac{\partial Q}{\partial f}/\frac{\partial Q}{\partial \hat{y}}$. The partial derivatives are

$$\frac{\partial Q}{\partial f} = -\frac{1}{1-f}h'(\hat{y}) > 0$$

and

$$\begin{aligned}\frac{\partial Q}{\partial \hat{y}} &= \left[\log\left(1-f\right)-g_p\left(\hat{y}\right)\right]h''\left(\hat{y}\right)-g_p'\left(\hat{y}\right)h'\left(\hat{y}\right)+g_p'\left(\hat{y}\right)h'\left(\hat{y}\right)+g_p''\left(\hat{y}\right)h\left(\hat{y}\right)\\ &= \left[\log\left(1-f\right)-g_p\left(\hat{y}\right)\right]h''\left(\hat{y}\right)+g_p''\left(\hat{y}\right)h\left(\hat{y}\right)=g''\left(\hat{y}\right)h\left(\hat{y}\right)\end{aligned}$$

where we took the second derivative of equation (10), $g''\left(y\right)=\frac{\log(1-f)-g_p(\hat{y})}{h(\hat{y})}h''\left(y\right)+g_p''\left(y\right)$, and evaluated it at $y=\hat{y}$ for the last step. As we have $g''\left(\hat{y}\right)>0$ (this is given numerically from our optimization) and $h\left(\hat{y}\right)>0$ from the lemma above, we are left with $\frac{\partial Q}{\partial \hat{y}}>0$. We conclude that $\frac{d\hat{y}}{df}<0$.

■