MANAGING THE DEMAND

FOR

PUBLIC HOUSING

by

Edward Harris Kaplan

B.A. McGill University
(1977)

S.M. Massachusetts Institute of Technology
(1979)

M.C.P. Massachusetts Institute of Technology
(1979)

S.M. Massachusetts Institute of Technology
(1982)

Submitted to the Department of
Urban Studies and Planning
in Partial Fulfillment of the
Requirements of the
Degree of

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 1984

Signature of Author: ___
Department of Urban Studies and Planning
May 1, 1984

Certified by: ______________________
Thesis Supervisor

Accepted by: ______________________
Head, PhD. Committee

MANAGING THE DEMAND

FOR

PUBLIC HOUSING

by

Edward Harris Kaplan

Submitted to the Department of Urban Studies and Planning
on May 1, 1984 in partial fulfillment of the
requirements for the Degree of Doctor of Philosophy

ABSTRACT

This thesis describes the basic workings of public housing tenant assignment systems and presents the detailed assignment procedures utilized by several public housing authorities across the United States. Using these procedures as a guide, the theory of birth and death processes is used to develop realistic models for the prediction of applicant waiting times, tenant allocations, and project compositions. These models are applied to real data from the Boston Housing Authority to answer various policy questions.

A special case of tenant assignment occurs when large housing projects are redeveloped and tenants must be relocated. Scheduling models are derived for these redevelopment programs accounting for the fact that tenants must always be assigned to appropriate units. An application of the methods developed to a relocation problem in Boston is also presented.

The thesis concludes with a discussion of both the policy implications of the work reported, and areas deserving future research attention.

Thesis Supervisor: Dr. Richard C. Larson

Title: Professor of Electrical Engineering and Urban Studies; Co-Director of the Operations Research Center

ACKNOWLEDGEMENTS

During the two years spent producing this dissertation, I had the opportunity to work with several outstanding individuals, and enjoyed the warm support of many friends both inside and outside MIT. I would like to acknowledge all of these people.

I was introduced to public housing research by Gordon King at lunch one day. He was instrumental in arranging my initial relocation consulting work at the Boston Housing Authority (BHA), and remains an enthusiastic supporter of much of the work contained in this document. At the BHA, I worked closely with Michael Jacobs and David C. Gilmore on relocation problems, and Fran Price on tenant assignment issues. Peter Suffridini was responsible for providing data in a form readable by the MIT computers. The MIT Public Housing Research Group provided me with a forum for presenting my ideas on public housing operations as these ideas developed; I owe particular thanks to Christine Cousineau in this regard.

Faced with the task of abstracting more general research issues from a few specific problems, I received invaluable and virtually unlimited assistance from my research supervisor Dick Larson. Dick's concern for this work is characteristic of the attention he pays to all his student advisees. The approximation procedure presented in Chapter 7 is a direct result of his suggestions at the blackboard one day; it was also Dick's idea to survey housing authorities regarding their tenant assignment policies. Dick has served as my faculty advisor, research supervisor and thesis supervisor during my seven years at MIT. Such interaction led to the development of a highly valued personal friendship. This work strongly reflects his guidance over the years, and I thank him for that support.

The other members of my thesis committee, Joe Ferreira and Gary Hack, have contributed to both the development of this research and the creation of a synergistic environment within which to perform such work. Joe has been a teacher, co-teacher, and co-consultant of mine on a variety of occasions. I have learned a lot from the time spent with him. Gary gave me the freedom to pursue public housing work in the form of an experimental seminar entitled "Logistical Problems in Public Housing Management". Like Dick and Joe, Gary has advocated the introduction of quantitative methods to planning students. His support of this research (and other methodologically related issues) is gratefully acknowledged.

At the MIT Operations Research Center, my "home" for most of this work, I had the good fortune to share an office with Alan Minkoff and Jan Hammond, operations researchers *par excellence*. Alan was always an eager listener, and suggested the algorithm shown in Appendix 5.1 for solving the one strike refusal model with priorities and dropout. Jan made lots of interesting remarks at the strangest times; the best of these were recorded on our office blackboard. Most of the models in Chapter 4 were passed by fellow student Christian Schaack, who derived immense pleasure from obtaining some of the same results in completely different fashions for his own research.

Behind the scenes, I've enjoyed wonderful treatment from Marcia Ryder, Senior Administrator at the Operations Research Center. She made the Center a most hospitable place to work, while she also encouraged me to join in her quest to become a world champion body builder. Jackie LeBlanc of the Department of Urban Studies put up with every ridiculous request I made, and still managed not to lose her temper (at least not around me !).

I owe special thanks to Dean Linda Vaughan of the Office of Student Affairs. At various times during my stay at MIT, things looked bleak for a variety of reasons. Linda helped me place these discomforts in perspective; in so doing, I was able to regain my concentration and finish my work.

Cris Nelson was responsible for entering much of the data analyzed in Chaper 3 into the computer. The text of this thesis was masterfully typed by Julie Dean; I'm still amazed at how quickly and cheerfully she did it all! David Dean helped produce some of the figures.

On the home front, roommates Martha Matlaw and Ami Amir showed patience and understanding (perhaps too much!) in allowing me to escape the usual household chores to finish school. Many friends were key in picking me up when I was down; these include: Ami Amir, Carol Carter, Erica Crystal, Larry Denenberg, Matt Frohlich, George Kirby, Dick Larson, Martha Matlaw, Cris Nelson, Marcia Ryder, David Turner, and Ira Vishner. I would like to give special thanks to Karen Goldstein for her recent care and support in this regard. Of course, I have to acknowledge my friends and fellow dancers at MIT Israeli Folkdancing for providing me with a regular escape from the academic world.

Finally, I want to thank my parents, David and Harriett Kaplan, whose love for education is unsurpassed. They always wanted the best in life for their children, and it is to them that this thesis is dedicated.

Dedication

To my parents, David and Harriett Kaplan.

TABLE OF CONTENTS

TABLE OF CONTENTS (cont.)

TABLE OF CONTENTS (cont.)

LIST OF FIGURES

LIST OF FIGURES (cont.)

LIST OF TABLES

LIST OF EXHIBITS

Chapter I

Introduction

Public housing authorities exist in almost every major American city. The primary service of these authorities, the provision of affordable, decent housing for low income households, is clearly in heavy demand. One only has to examine applicant waiting lists to realize that the demand for public housing far exceeds supply. For example, there are currently about 10,000 households waiting for public housing assignments in Boston (Boston Housing Authority computer files). In Philadelphia, 15,000 applicants are on the waiting list (letter from Philadelphia Housing Authority dated Jan. 24/84), while over 40,000 applicants await public housing assignments in Baltimore (letter from Housing Authority of Baltimore City postmarked Jan 19/84).

With such burdens being placed on public housing programs, there is a clear need to develop means for effectively managing this demand. The consequences of poor demand management impact both the level of service provided by housing authorities and new applicants' perceptions of public housing. For example, in their management review of the Boston Housing Authority (BHA), Coopers and Lybrand report:

> "The demand for low-income housing, as evidenced by the number of applications received by the BHA, far exceeds the number of units that the BHA can make available in acceptable condition in its current situation ... it is estimated that the Occupancy Department spends 1000 hours or more per month of staff time in reviewing applications, interviewing applicants and determining eligibility. This time is provided at the expense of processing legitimate requests for transfer and assignment of existing

tenants to acceptable units. Just as importantly, acceptance and processing of applications probably creates a false sense of encouragement in applicants that their housing needs will be solved by the BHA and deters them from seeking other alternatives to their present situation."

(Coopers and Lybrand, 1980, p.III 16-17)

This dissertation is concerned with techniques for managing public housing demand. Our major contribution lies with identifying the impacts that alternative tenant assignment systems have on service quality (primarily expressed as waiting time for housing assignments) and housing authority objectives (for example, the racial integration of housing projects). Once housing authorities have the means to examine the implications of their adopted policies, it should prove easier for these authorities to develop policies which better achieve stated goals. As so much of our work will involve the tenant assignment process, we will briefly review the steps in this process as they might apply to a new applicant for public housing in a typical U.S. housing authority.

1.1 The Tenant Assignment Process

1.1.1 Arrival of New Applicant

The assignment process for a new tenant begins when that tenant applies for public housing. The application form for the Housing Authority of Baltimore City is shown in Figure 1.1; this form is typical of those used in American housing authorities. The tenant is asked to provide basic information necessary for determining the unit type required (e.g. household size and composition); eligibility (e.g.

Figure 1.1

For Office Use Only

Date ______________

App. No. ______________

Housing Authority of Baltimore City
Application for Public Housing and Section 8 Programs

Please print and fill out this form completely.

1. Name of Head of Household ______________ *Last Name* ______________ *First Name* ______________ *Middle Initial*
2. Present Address ______________ *Street Address* ______________ *Zip Code*
3. Telephone Number ______________ Social Security Number ______________
4. Is the head of the household or the spouse 62 years of age or older? Yes ____ No ____
5. Are all the members of your household over the age of 45? Yes ____ No ____
6. Is the head of the household or the spouse handicapped or disabled? Yes ____ No ____
7. Have you ever lived in public housing in Baltimore City before? Yes ____ No ____
   If so, where did you live? ______________ When? ________
8. How many people, including yourself, will be living in the household? ______________ *Number in family*
9. How many people who will be living in the household are under age 18? ______________ *Number of minors*
10. How many people who will be living in the household are female? ______________ *Number of females*
11. What is the total income coming into the household at this time? $ ______________ per ______________ *week, month, or year*
12. Do you or anyone living in the household receive income from any of the following sources?

Department of Social Services (DSS)	Yes ___ No ___	Supplemental Security Income	Yes ___ No ___
Social Security	Yes ___ No ___	Other/Miscellaneous	Yes ___ No ___
Employment (full time or part time)	Yes ___ No ___		

13. Please check the spaces below of the places where you would like to live.
    We will try to consider you for the developments of your choice.

____ Any family development
____ Any rehabilitated house
____ Any elderly development
____ Sheltered housing ____ Congregate Housing
____ Section 8 Existing Program
____ Section 8 Moderate Rehabilitation
____ Section 8 Regional Housing

Family developments	Number of Bedrooms
____ Anderson Village	1, 2, 3
____ The Broadway	2, 3, 4, 5
____ Brooklyn Homes	1, 2, 3
____ Cherry Hill Homes	1, 2, 3, 4, 5
____ Claremont Homes	1, 2, 3, 4, 5
____ Douglass Homes	1, 2, 3,
____ Fairfield Homes	1, 2, 3,
____ Flag House Courts	1, 2, 3, 4, 5
____ Gilmor Homes	1, 2, 3
____ Hollander Ridge	1, 2, 3, 4, 5, 6
____ Julian Gardens	3, 5
____ Lafayette Courts	1, 2, 3, 4
____ Latrobe Homes	1, 2, 3
____ Lexington Terrace	1, 2, 3, 4
____ McCulloh Homes	1, 2, 3, 4, 5
____ Mount Winans	2 and 4
____ Murphy Homes	1, 2, 3, 4, 5
____ O Donnell Heights	1, 2, 3, 4
____ Oswego Mall	3, 4
____ Perkins Homes	1, 2, 3
____ [illegible] Homes	[illegible]

Elderly and Handicapped Developments

(Most developments have efficiencies and one-bedroom apartments.)

____ Bel-Park Tower*
____ The Brentwood
____ The Broadway*
____ Chase House
____ Claremont Extension
____ Ellerslie Apartments
____ Govans Manor
____ Hollander Ridge*
____ Hollins House • one bedroom only
____ Lakeview Towers*
____ Bernard E. Mason Apts. • one bedroom only
____ McCulloh Extension 0, 1, & 2 bedroom**
____ Monument East Apartments
____ Primrose Place • one bedroom only**
____ Rosemont • one bedroom only
____ The West Twenty*
____ Wyman House

** Sheltered Housing available ** Congregate Housing available*

Please mail this application by folding it on the dotted lines so the address on the back faces outward or deliver it to:

Housing Authority of Baltimore City
Housing Application Office
American Building, 5th Floor
231 East Baltimore Street
Baltimore, Maryland 21202

income); assignment priority (e.g. handicapped or disabled status); and desired location for residence.

1.1.2 Determination of Eligibility

The application form is reviewed, and certain facts are verified by housing authority officials to determine whether or not the applicant is eligible for the public housing program. Eligibility is usually determined on the basis of income, though other attributes (e.g. past criminal record) affect eligibility as well. If a household is found ineligible, it is notified as such and dismissed from the system. Otherwise, the applicant is entered onto the waiting list for housing assignments.

1.1.3 Waiting List Processing

Essentially, eligible households wait until they are notified of an available unit. The particulars of waiting list management can be quite complicated, as housing priorities and tenant choices must be taken into account. In addition, households may choose to drop out while waiting for housing assignments, a wait that can take several years. During the wait for an assignment, households may be contacted periodically to reassess housing needs or to see if public housing is still desired. The details of waiting list management are discussed in Chapter 2.

1.1.4 Housing Assignments

Housing assignments are triggered by household moveouts from housing projects. As the waiting lists for public housing units are almost never empty, housing assignments can only occur when vacancies appear due to moveouts. When such moveouts arise, the managers of the relevant housing projects contact the central authority office to release those households next in queue for assignment from the waiting list. If the unit offered to the household in question is acceptable, a rental agreement is signed and the tenant occupies the unit shortly thereafter.

There are a number of questions associated with the tenant assignment process. First of all, it should be clear that the manner in which a housing authority organizes the waiting list for housing assignments greatly impacts the performance of the tenant assignment system. How do authorities manage waiting lists? What are the consequences of these management strategies for tenant waiting times? How will these rules effect ultimate project compositions? What is the role of tenant choice in a tenant assignment process? How long will households wait for assignments before dropping out of public housing waiting lists for a particular assignment scheme? These questions are addressed in detail in this dissertation.

1.2 Guide to the Thesis

In Chapter 2, the tenant assignment policies used in ten large U.S. housing authorities are analyzed in detail. The results of this analysis enable a characterization of tenant assignment schemes in terms of waiting list management, priority classes, methods for implementing priority assignments, and tenant choice. We will argue that the different assignment schemes reflect the different viewpoints held towards the function of public housing as a social service system, but that specific assignment systems may not be consistent with broader policy objectives.

The different policies reviewed in Chapter 2 are carefully modeled in Chapters 4 and 5. The idea is to develop a set of techniques which describe the consequences of a particular tenant assignment policy. The performance measures chosen include the waiting time from application to assignment for a new applicant, the demographic compositions of projects, and tenant allocations (numbers of tenants assigned to different projects; number of dropouts). These models are applied to real data from the Boston Housing Authority in Chapter 6. Issues considered include the effects of changing from the current assignment system in Boston to (i) a project based or (ii) a citywide system, and the time necessary to integrate a particular project following current policy. The models require various assumptions, and some of these are verified empirically in a study of household occupancy times presented in Chapter 3.

It is appropriate to mention the use of models at this point. The models developed throughout this thesis are somewhat novel in that they are designed to reflect the particulars of public housing operations. The models enable the policy maker to describe the implications of a particular policy <u>without actually implementing the policy</u>. This stands in stark contrast to other modes of scientific enquiry, such as social experimentation, which would require a tremendous effort in time and money to obtain results comparable to the ones reported throughout this thesis.

Thus far, we have focused on the problem of assigning new applicants to housing units. A very different form of tenant assignment occurs when housing projects are redeveloped. Here tenants must be relocated to temporary and new permanent units while large scale construction takes place. As these "relocation problems" are starting to occur more often due to the deterioration of public housing stock, methods for addressing these problems could prove quite valuable. A class of relocation models is developed in Chapter 7. The application of these models to an actual redevelopment project is described in Chapter 8.

The thesis concludes in Chapter 9 with a review of the implications of the work presented. Areas for future research are outlined, as are suggestions for implementing the research completed in this document.

1.3 Public Housing as an Urban Service System

Before closing this introductory chapter, I would like to place this work in perspective. Within the last fifteen years, operations researchers have begun to analyze the operations of urban service systems with an eye towards improving the quality of service these systems offer. In areas such as policing (Larson, 1972) and fire protection (Walker, Chaiken, Ignall; 1979), it is clear that this research has had an impact on the provision of the said services. Most of the major recommendations from these researchers were developed from mathematical models of the service system studied.

I would like to view this work on managing public housing demand as being in the same spirit as these earlier studies. Though I have chosen to focus on tenant assignment and relocation problems, public housing authorities have other logistical concerns such as the maintenance of housing stock; the design of rent collection and tenant accounting systems; and the provision of security to all public housing occupants. The work reported in the following pages represents only a sample of what could be learned from a detailed study of the operational problems of public housing management.

CHAPTER II

TENANT ASSIGNMENT POLICIES IN U.S. HOUSING AUTHORITIES

At the heart of public housing operations lies a fundamental resource allocation question: How are eligilble applicants to public housing assigned to public housing units? More precisely, what is the procedure used to determine which household is assigned to the next available apartment? The manner in which a housing authority answers these questions has far reaching consequences ranging from the determinaton of the waiting time until assignment for a newly arriving public housing applicant to the ultimate demographic compositions of housing projects and the happiness of the tenants living therein.

We will refer to the collection of procedures and decision rules used by housing authorities to assign households to housing units as *tenant assignment policies*. Tenant assignment policies, more than any other facet of public housing operations, reflect the true character of a public housing authority. These policies illustrate (and implement) the functions housing authorities perceive public housing programs to serve. Indeed, the public housing population within an authority's jurisdiction is a testimony to the tenant assignment practices (past and present) of that authority.

As mentioned in Chapter 1, public housing authorities are faced with demands for housing units that far exceed supply. This demand for public housing is essentially managed via tenant assignment policies. Tenant assignment policies dictate what form waiting lists will take, what choices prospective tenants receive in the assignment process, how prospective tenants are prioritized, and ultimately, how long prospective tenants are required to wait for a housing assignment. Thus, a careful study of tenant assignment policies is necessary if we are to understand the issues involved in managing public housing demand.

To develop an appreciation for tenant assignment policies and their attendant problems, it is useful to examine a range of assignment policies currently utilized by major U.S. housing authorities. Towards this end, I contacted sixteen large housing authoriites listed in the Council of Large Public Housing Authorities (CLPHA) directory requesting copies of their tenant assignment policies in whatever form they exist; the text of the request is shown in Exhibit 2.1. From these letters, I received detailed responses including stated tenant assignment policies from the following ten housing authorities: Baltimore, Boston, Cambridge, Chicago, Greensboro, Houston, Minneapolis, Omaha, Pittsburgh, and St. Paul. For the remainder of this Chapter, we will present an analysis of these policies, extracting important features for future consideration as we proceed. The issues raised in this chapter will form the basis for most our technical work in Chapters 4 and 5.

2.1 Objectives of Tenant Assignment Policies

That housing authorities share the broad social goal of providing decent, affordable housing for low-income households is evident from the stated objectives of these authorities. Here are four such statements:

"... provide decent, safe, sanitary, and uncongested rental

Exhibit 2.1

OPERATIONS RESEARCH CENTER
ROOM E40-164
MASSACHUSETTS INSTITUTE OF TECHNOLOGY
CAMBRIDGE, MASSACHUSETTS 02139
(617) 253-3601

November 25, 1983

The purpose of this letter is to request from you or a member of your staff some readily available information. As background, I am a doctoral candidate at the Massachusetts Institute of Technology undertaking research in the operational aspects of public housing occupancy policies. Much of my work, which to date has occurred almost exclusively in Boston, involves the modeling of tenant assignment systems. These models can be applied in day-to-day settings (e.g., to forecast the probable waiting times for new applicants), or as policy analytic tools (e.g., to study the effect of a tenant assignment scheme which prioritizes households on the basis of ethnicity, income, or some other criterion). I have also constructed models which aid in sequencing large redevelopment programs where tenant relocation is a major concern; these models have been used by the Boston Housing Authority.

It is my hope that this research will result in a flexible set of techniques specifically geared toward public housing occupancy planning and policy analysis. Toward this end, I would be most appreciative if you or a member of your staff would send me one or more of the following items:

1. A sample application form for public housing from your agency;
2. Any guidelines, directives, or procedures manuals pertaining to your agency's approach to tenant assignment (e.g., How are waiting lists managed--by development? citywide? When an apartment becomes available, how is the decision made regarding which household next occupies that apartment?).

In return, I would be delighted to provide you with a synopsis of my dissertation and details pertaining to the modeling effort as they become available.

Thank you very much for your cooperation, and I look forward to hearing from you.

Sincerely,

Edward Kaplan

housing for families with low incomes at rentals consistent with their incomes." (Section 7113, Occupancy Standards, Chicago Housing Authority)

"These policies are designed to meet the needs of limited-income families for decent, safe, sanitary low-rent housing which provides a suitable living environment and which fosters economic and social diversity and upward mobility." (Section 1.0, Occupancy Policy, Greensboro Housing Authority)

"The Tenant Selection and Assignment Policies have been designed by the Agency to take into consideration the needs of individual families for low income housing and the statutory purpose of developing and operating a socially and financially sound low-income housing program which provides a decent home and a suitable living environment, and fosters economic and social diversity in the tenant body as a whole." (Section 1, Statement of Policy, Minneapolis Community Development Agency)

"... the basic objective, within a reasonable period of time, of housing tenant families with a broad range of income, representative of the range of low-income families in this Authority's area of operation, as defined in state law, and with rent-paying ability sufficient to achieve financial stability of the project or projects." (Section 3.01 (D)), Resolution No. 27 of 1981, Housing Authority of the City of Pittsburgh)

The statements cited reveal other objectives besides the provision of low-income housing. Both the Minneapolis and Pittsburgh statements mention the necessity of achieving financial solvency in their authorities; such a goal necessarily requires a certain mix of incomes among project occupants, and could lead to explicit income mixing policies. The Pittsburgh statement makes reference to housing households "within a reasonable period of time "; as assignment policies have tremendous impacts upon waiting times, the Pittsburgh objective should lead to an efficient (in time) assignment scheme. The Omaha and Minneapolis statements mention "social diversity" as a policy objective; this refers to achieving demographically mixed project populations, and could lead to explicit racial mixing policies.

Towards this end, consider two of the stated objectives of the Boston Housing Authority:

> "... assure that no discrimination on the basis of race, creed, color, religion, national origin, marital status, sex, or handicap is practiced in the selection of applicants, assignment of tenants, or the granting of transfers... promote racial integration of public housing developments." Section IA and IB, Tenant Selection, Assignment, and Tranfer Plan, Boston Housing Authority)

To promote racial integration of projects, one would presumably implement differential assignment rates for different racial groups. Some may construe such differential assignment rates as a violation of the notion that "no discrimination on the basis of race... is practiced in the... assignment of tenants." The internal consistency of tenant assignment policies is perhaps questionable.

The preceding discussion has illustrated the more common stated objectives of public housing authorities to the extent that our ten responding authorities can be considered representative. We will now proceed to examine the specifics of tenant assignment policies, policies which presumably act to achieve the objectives mentioned above, beginning with the issues of waiting list management and tenant choice.

2.2 Waiting List Management and Tenant Choice

Much can be learned about an authority's tenant assignment process by examining the means by which waiting lists are managed. Typically, waiting lists are first differentiated by unit requirements. These requirements usually refer to apartment size (e.g. number of bedrooms), but may also include special features (e.g. apartments equipped with aids to the handicapped).

Waiting lists are also prioritized, with households in higher priorities receiving assignments before households in lower priorities. However, as authorities vary greatly in both the attributes considered to merit high priority status and the methods for implementing prioritized assignments, we will discuss priorities in detail later on.

Finally, waiting lists vary by geographic scale, in that any particular waiting list (already broken down by unit requirement and priority status) may be applied to a single project, a group of projects in a neighborhood or community, or all projects in the authority. Within the bounds of geographic scale, priority status and unit requirement, assignments are typically made in chronological order of tenant application.

The geographic scale covered by a waiting list has direct implications for tenant choice in the assignment process, and for the ultimate demographic compositions of projects. Consider the following two extremes. On the one hand, an authority could operate a system of project based waiting lists; each new applicant would select a single project, and wait until an appropriate unit becomes available. Such a system guarantees that all households are eventually offered units in their chosen projects. Such a system also causes the authority to abdicate control over the demographic design of projects, as tenants decide where to live; the authority cannot route tenants to projects to achieve some goal such as desegregation. Finally, a project based system will produce unbalanced waiting times, with households experiencing long waits at "popular" projects, and shorter waits elsewhere.

At the other extreme, an authority could operate a citywide first available unit system, where households are assigned to the first apartments vacated regardless of their locations. This system does not possess any guarantee that households will be offered units in desirable locations; rather, there is only a probability that a household will be assigned to a project viewed as desirable by that household. However, assuming for the moment that tenants don't quit the system, an assignment scheme of this form would integrate all projects in the same ratios as found on the waiting list. Also, the waiting times experienced by those on the waiting list would be much more balanced.

These two extremes in geographic scale illustrate a basic trade off that occurs in tenant assignment policies: one must balance tenant choice in the assignment process against the authority's ability to influence the demographic compositions of projects. In assessing this tradeoff, one must also remain cognizant of the waiting times implied by the assignment process chosen, as excess waiting times will cause tenants to quit the system.

The housing authorities in our survey practice assignment policies which cover the range between the two extreme examples presented. Recognizing that tenant choice is important for both continued participation in the housing program and ultimate tenant satisfaction, some of the authorities have devised mechanisms which grant prospective tenants some degree of choice via the right to refuse a certain number of offered units. Consider the following guidelines from the Housing Authority of Baltimore City:

> "Eligible applicants shall be offered suitable housing within the location wherein the highest number of vacancies exist. Rejection of three separate offers of suitable accomodations shall result in the placement of the applicant's name at the bottom of the eligible applicant list, unless the applicant shall prove undue hardship or handicap to the satisfaction of the Authority..." (Section IVG, Statement of Policies and Standards Governing Admission To and Occupancy of Low-Income Public Housing Operated by the Housing Authority of Baltimore City, Housing Authority of Baltimore City)

A similar policy is followed in Houston:

> "If there is a suitable vacant unit in more than one location, the applicant shall be offered the unit at the location that contains the largest number of vacancies. If the applicant rejects the first offer, he/she shall be offered a suitable unit at the

location containing the next highest number of vacancies. If the applicant rejects three (3) such offers he/she shall be placed at the bottom of the eligible list. The Authority shall make all such offers in sequence and there must be a rejection of a prior offer before the applicant may be offered another location. (Section IV, Admissions and Continued Occupancy Policy, Housing Authority of the City of Houston)

The assignment systems illustrated by the Baltimore and Houston statements will be refered to as refusal systems. We can formalize the notion of a refusal system through the following characterization: A k strike refusal system is a tenant assignment system where eligible applicants are offered up to k units, sequentially. Applicants may refuse any (or all) of the first k-1 units offered with no associated penalty. If an applicant refuses all k units, then the applicant must return to the bottom of the waiting list. In other words, in a k strike system, "k strikes and you're out." The Baltimore and Houston policies are both three strike systems (i.e. k=3).

It is interesting to note how refusal systems can cover the range from project based to citywide waiting lists. Suppose that an authority is operating a citywide system with one strike refusal; this situation gives tenants no choices other than accepting an offered unit or retreating to the bottom of the waiting list (or leaving the housing system altogether). Now, suppose that the authority offers an infinite number of strikes. This would afford applicants the luxury of refusing units without penalty until a desireable unit is offered, and effectively would represent a project based scheme. Applicants could decide a priori which projects to live in, and refuse offered units until an offer occurs in a desired project.

Eight of the authorities surveyed use refusal systems as a mechanism for implementing tenant choice; these authorities are listed in Table 2.1 along with the geographic scale of the waiting list managed, and the number of strikes in the refusal system. Note that city wide, one strike systems are in use. Consider the case of St. Paul:

> "Suitable vacancies arising at a given time at any location shall be offered to the eligible applicant first in sequence at such time. The eligible applicant must accept the vacancy offered or be moved to last place on the eligible applicant list." (Section C, Tenant Selection and Assignment Plan, Public Housing Agency of the City of St. Paul)

Thus, the degree of choice offered to new applicants via refusal systems is quite varied in U.S. Housing Authorities.

The Boston Housing Authority (BHA) has a tenant assignment system which is quite different from the refusal systems discussed above:

> "Applicants shall be asked to name up to three preferred locations for housing from among all BHA housing developments or leased housing on a community-wide basis. ... The interviewer shall explain to the applicant (1) that he/she will be offered only one of his/her preferred locations; (2) that the offer will be made in whichever requested development has the earliest appropriate vacancy; and (3) that if the applicant refuses to be housed at that location... his/her application will be treated as a refusal..." (Section III B, Tenant Selection, Assignment and Transfer Plan of the Boston Housing Authority, Boston Housing Authority)

Table 2.1

Eight Authorities with Refusal Systems

City	Geographic Scale of Waiting List	Number of Strikes (k)
Cambridge	Citywide	1
Greensboro	Citywide	3
Baltimore	Citywide	3
Minneapolis	Citywide	2
Houston	Citywide	3
Chicago	Project Based	1
Omaha	Project Based or Citywide at Applicant's Choice	1
St. Paul	Citywide	1

The policy towards refusals mimics that of one strike systems (with exceptions granted for various reasons).

Thus, the BHA system represents a different approach to tenant choice. Applicants pre-specify a collection of up to three projects, and the BHA guarantees that the unit offered to the applicant will fall within one of the projects specified. If an applicant is interested in only one project, the applicant can specify solely that project, so for some prospective tenants the BHA functions as a project based tenant assignment system. However, most new applicants specify two or three projects, and for these prospective tenants, the BHA functions as a multiqueue assignment scheme; households are on waiting lists at several projects simultaneously.

The BHA system appears to heavily favor the tenant choice side of the choice/project composition tradeoff discussed earlier, even more so than project based waiting lists. Yet, as mentioned before, one BHA objective is to promote the racial integration of projects. To achieve this goal in a heavily choice based assignment system is difficult. The way the BHA tries to integrate projects is through the use of priority structures. All housing authorities studied here also use priority structures, but for a variety of reasons. Let us now turn to examination of the types of priorities evidenced by the authorities in our survey; later we will consider the different methods used for implementing these priorities.

2.3 Priorities in Tenant Assignment

Within a given waiting list (broken down by unit requirements), all applicants are not treated equally. Some applicants are viewed as more needy, or more deserving of public housing than others. When one peruses the attributes which amount to different priority classes in different authorities, one is left with the sense that these priorities reflect the housing authority's view of its social mission. Consider the followilng statement regarding priorities:

> "1. Only those applicants who can pay a rent in the needed income range will be considered. In the event that there are no eligible applicants in this income range, the next highest range is used.
>
> 2. Within the applicants in this income range, displaced families will be given preference over nondisplaced families.
>
> 3. Within this group of displaced families, the family with the earliest date of applicaton will be selected.
>
> 4. If there are no displaced families, the nondisplaced family with the earliest date of application within the income range will be selected.
>
> GHA reserves the right to waive any provisions within these policies to meet emergency conditions; an emergency condition is defined as a situation in which failure to supply immediate relief would pose a serious threat to the health, life, or safety of the applicant." (Section 4.5, Occupancy Policy, Greensboro Housing Authority)

These statements clearly reflect the mission of public housing as perceived by the Greensboro Housing Authority. Emergencies, those with the greatest need, are housed as a top priority. After this, an income mix is enforced to ensure that the authority remains solvent. Finally, displaced families are prioritized over nondisplaced households, again reflecting relative need. Note that within priorities, assignments occur on a first come first housed basis.

The policies reviewed contain many different priority categories, and different orderings of these categories. Most authorities reserve their highest priority classifications for households exhibiting the greatest need; these households are typically referred to as emergencies or displaced households. Many authorities also attempt to house elderly applicants before assigning "regular" households.

However, not all authorities grant emergency or displaced households highest priority status. For example, the Chicago Housing Authority's highest priority status is defined as follows:

> "Both for initial occupancy and as vacancies occur in developments initially made available subsequent to November 24, 1969, dwelling units shall, depending upon bedroom size only, be offered first to eligible applicants residing at that time in the community area in which the development is located. This procedure is to be followed to the extent that such area residents shall have a priority to occupy 50% of the dwelling units in the development..." (Section 7142, Occupancy Standards, Chicago Housing Authority)

In fact, the application form for public housing administered by the Chicago Housing Authority explicitly states:

> "WE DO NOT HAVE EMERGENCY HOUSING, and you cannot be housed until we have housed all other families, of the same size as yours that are ahead of you on the waiting list." (Form CHA-315, Chicago Housing Authority Registration-Family Housing, Chicago Housing Authority)

Residency is also a factor in determining a household's priority in

St. Paul, where applicants receive a large number of "points" if they are either St. Paul residents, or are employed within the jurisdiction of the Public Housing Agency of the City of St. Paul (we will discuss the use of points in implementing priority schemes in the next section).

Other attributes taken into account when determining priority classifications include: household income (either for economic reasons of financial solvency, or social reasons of income diversity in project populations); transfers from other locatons in the public housing system; household ethnicity (for purposes of integrating projects); veteran or serviceman status; and relationship of rent at current private housing unit to household income. Table 2.2 presents the top four priority classes evidenced by the tenant assignment policies for eight of the authorities surveyed; the two other cities (Omaha and St. Paul) will be reviewed in the next section with scoring systems.

One thing is clear from Table 2.2; a given household with particular characteristics could receive greatly varying treatment from the different housing authorities owing to the different definitions of priorities across cities. This isn't entirely surprising, as the priority classes shown presumably represent the varied objectives of the housing authorities studied. What is not clear is whether or not the particular priority schemes used do in fact achieve the objectives set out by housing authorities; we will return to this issue at the end of this chapter.

2.4 Implementing Priorities

The last section described the different priorities housing

Table 2.2

Assignment Priorities in Eight U.S. Public Housing Authorities

Priority	Pittsburgh	Boston	Cambridge	Greensboro	Baltimore	Minneapolis	Houston	Chicago
1	Elderly displaced or sub-standard	Emergency, Emergency Transfer	Emergency	Emergency	Elderly or Displaced & Disabled	Elderly Displaced	Displaced by public action or catastrophe	Same neighborhood up to 50% occupancy
2	Elderly	Minority Preference	Displaced	Necessary income range	Other Displaced	Other Displaced by public action or natural disaster	Substandard living	Adjacent neighborhood
3	Veterans or servicemen displaced or sub-standard	Displaced by public action	Intra-project transfer	Displaced	Elderly or Disabled, not displaced	Income Range most under-represented	Elderly	Transfers from inside the authority
4	other displaced or sub-standard	Veterans and Servicemen	Veterans and Servicemen	-	-	Displaced Veteran and Servicemen	Veterans and Servicemen	Displaced
Source	Resolution No. 27 of 1981, Housing Authority of the City of Pittsburgh	Tenant Selection, Assignment and Transfer Plans, Boston Housing Authority	Applicant Selection and Transfer Plan, Cambridge Housing Authority	Occupancy Policy, Greensboro Housing Authority	Statement of Policies and Standards Governing Admission to and Occupancy of Low-Income Public Housing... Housing Authority of Baltimore City	Statement of Policy, Minneapolis Community Development Agency	Admission and Continued Occupancy Policy Housing Authority of the City of Houston	Occupancy Standards, Chicago Housing Authority

authorities have established in their tenant assignment policies. In this section, we will considr three different ways that authorities implement these priorities: Categorical priorities, blend priorities (or differential assignment rates), and score priorities.

2.4.1 Categorical Priorities

This method is the most common observed. Households are assigned to a priority category on the basis of their attributes. For example, a non-elderly, non-displaced household with an income in the range most underepresented would receive a priority of category 3 from the Minneapolis Community Development Agency according to Table 2.2. In a categorical priority system, no households in a priority category j can be assigned until all households in categories one through j-1 have been assigned. Within category j, assignment is in chronological order (i.e. first come first housed). Thus, our category 3 household in Minneapolis would not be housed until all households in categories 1 and 2 (elderly displaced, or others displaced by public action or natural disaster) are housed. In addition, newly arriving applicants in categories 1 through j-1 will be housed before applicants in priority category j initially present are housed. Completing our example, a newly arriving household displaced by public action in Minneapolis will be housed before a non-displaced, non-elderly household in priority category 3, regardless of how long the category 3 household has been waiting.

While the implementation of such categorical schemes is relatively straightfoward, these schemes do possess one problematic feature. If the rates at which high priority applicants arrive are sufficiently high to guarantee that such applicants are always present on the waiting list, then lower priority applicants will never be housed. We have not been able to study statistics for authorities across the country, but one could certainly conjecture that in some housing authorities, certain eligible applicants are effectively barred from receiving a public housing assignment due to the priority system in use.

2.4.2 Blend Priorities

One way to prioritize which does not have the drawback of the previously discussed categorical scheme is to assign different priority groups differential admission rates. For example, if one is attempting to integrate a predominantly non-white project, a means for doing this could be: assign k white applicants for every non-white applicant assigned to the project. If k is chosen to be very large, the effect of such a blend priority scheme would mimic that of a categorical scheme where white households are given highest priority, and non-white households are given lower priority. However, choosing k to be smaller (e.g. k=2 or 3) creates a situation where white applicants are being assigned at a faster pace than non-whites, but non-whites continue to be assigned. This form of prioritizing is being practiced in Boston with respect to household racial characteristics (white, non-white) and household incomes (above median income for family size, below median income for family size) to achieve various racial and income mixes in Boston Housing Authority projects (Price and Solomon, 1983).

2.4.3 Score Priorities

In two of the authorities studied, Omaha and St. Paul, applicants are actually assigned points on the basis of their housing need and other characteristics. Let:

w_i = points (or weight) assigned to attribute i, i = 1,..., I

$$x_{ij} = \begin{cases} 1 \text{ if household j possesses attribute i} \\ 0 \text{ if not} \end{cases}$$

Then the score for household j, s_j, is given by the sum

$$s_j = \sum_{i=1}^{I} w_i x_{ij} \qquad (2.1)$$

Households are assigned scores using equation (2.1); these scores are then rank ordered from highest to lowest. The households are then assigned in descending order of their scores. The attributes and attendant points awarded in Omaha and St. Paul are shown in Table 2.3.

It is very interesting to compare these two scoring systems. In Omaha, just under 50% of the total possible points is awarded to attributes demonstrating lack of housing. In St. Paul, just over 50% of the total possible points is awarded to residency/work location. Clearly, these two authorities have differing views of their missions as public housing agencies!

2.5 Impacts of Tenant Assignment Policies

The tenant assignment policies of a housing authority have direct impacts on the waiting times for prospective tenants, the demographic character of projects over time, and the ultimate allocations of tenants to projects (or the number of tenants who drop out). We raised the issue previously that tenant assignment policies are meant to reflect the objectives of housing authorities. Yet, it is not immediately clear that the policies reviewed here meet the objectives

Table 2.3

Score Priorities in Omaha and St. Paul

Omaha

Attribute	Points
Displaced, about to be displaced, no housing, or about to have no housing through no fault of applicant	100
Will move to a unit where race is a minority	45
Substandard housing	30-38
Rent above maximum percentage of income	20
Veteran/Serviceman or dependent	10

Source: Resident Selection and Assignment Plan, Omaha Housing Authority

St. Paul

Attribute	Points
St. Paul resident or employed within jurisdiction of authority	64
Displaced by government action	32
Without housing	16
Substandard housing	8
Rent above 30% of income	4
Elderly, disabled or handicapped	2
Veteran	1

Source: Memo to National Association of Housing and Redevelopment Officials, Public Housing Agency of City of St. Paul, 1983.

stated by the relevant authorities, nor is it immediately clear how one could check to see if these policies are consonant with the stated goals.

What is lacking is a set of well reasoned procedures which, when used thoughtfully, have the ability to predict the consequences of a given tenant assignment policy. Were such procedures available, housing officials could view the impacts of their policies on measures such as new applicant waiting times, project compositions and tenant allocations to see if in fact the policies are performing as intended. One could also assess the consequences of proposed changes to a tenant assignment policy on the performance measures mentioned. Finally, one could provide better information regarding waiting times to new applicants to aid them in their decisions regarding public housing.

The next several chapters embark on the development of procedures for addressing the issues raised here. Following an empirical analysis of occupancy times in Boston public housing in Chapter 3, the broad classes of tenant assignment policies reviewed in this chapter are translated into mathematical models. In Chapter 4, we construct detailed models for project based systems incorporating all three of the priority schemes presented here. Chapter 5 broadens the models to incorporate refusal systems, city wide first available unit systems, and multiqueue systems as used by the Boston Housing Authority. These models are applied to real data from the Boston Housing Authority in Chapter 6 to conclude our study of tenant assignment systems and models.

Chapter III

Analysis of Household Occupancy Times in the Boston Housing Authority

We have just completed a discussion of tenant assignment systems used in U.S. Housing Authorities. It was clear from our analysis that tenant assignment policies consist of rules for "front door" entrance and assignment to housing projects. We argued that these rules have long run impacts on the demographics of public housing projects among other things.

To gain a feeling for the time scale involved in serving public housing tenants, I conducted a study to examine the length of time households actually spend in public housing. The data compiled and analyzed in this study serve several purposes:

1) For the first time, basic estimates of occupancy time are available. These estimates can be used to determine the time necessary for projects to "turn over," and have implications for the demographics of projects over time.
2) The data can be used to verify certain assumptions made in models of the tenant assignment process; such models will be developed in Chapters 4 and 5.
3) The data can be used to assess the stability of public housing populations; are households spending more or less time in projects now compared to ten or twenty years ago?
4) Certain issues regarding tenant flow and intraproject transfers can be assessed.
5) The data should prove to be of interest in their own right to general housing researchers. For example, how do household

occupancy times in public housing compare to those for comparable households in private housing? We do not pursue such issues here, but these data could prove useful to the housing research community in answering various questions.

The remainder of this chapter is devoted to the description, presentation and analysis of the data collected in my study of household occupancy times in the Boston Housing Authority (BHA).

3.1 Data Collection and Goals of the Study

The data analyzed in this report were collected during June 1983. Six Boston housing projects were visited: Faneuil, Washington Beech, Mission Extension, Mission Hill, Mary Ellen McCormack, and Charlestown. These projects were chosen for two reasons. First, the necessary records for data extraction were available at these projects. Secondly, these projects are representative of the diverse physical and social conditions that pervade public housing in Boston. In addition, all of these projects are well established, the most recent of the group having housed tenants since 1950.

The information collected pertains to household occupancy times in project apartments. For every household that moved out of a project apartment in the years 1975 through June 1983 inclusive, the following data were recorded:

1) Identification of the apartment occupied
2) Bedroom size of the apartment occupied (i.e. number of bedrooms)
3) Move in date to the apartment occupied
4) Move out date from the apartment occupied

5) Transfer data (the bedroom size of the new apartment occupied as of the move out date for internal transfers, or a code indicating that the household left the project)

In addition, the move in dates and apartment bedroom sizes for all households currently living in the projects studied were recorded.

The major sources for these data are the Space Inventory Cards that are maintained at most developments (though some developments in the BHA have not maintained these files). Space Inventory Cards are meant to keep a history of the status of all apartments in a housing project. Thus, move in and move out dates, rental adjustments, major repairs, and rehabilitations are all examples of the data potentially retrievable from the Space Inventory Cards.

In some instances, however, these cards are not always accurate. Other data sources used include Tenant Status Review forms (TSR's), and development specific "Bibles" (log books that chronologically track move ins and move outs as they occur). When incomplete Space Inventory Cards were encountered, these secondary sources were utilized. In a few cases, however, it was not possible to reconstruct the required information; such cases were subsequently discarded from this study.

The major variable of interest to this study is household occupancy time, or length of stay (LOS) in public housing. For those households who moved out in the period January 1, 1975 through June 1, 1983 (henceforth referred to as the "complete" population), LOS is simply defined as the elapsed time between the move in and move out dates. For current occupants (henceforth referred to as the "current" population), LOS is defined as twice the elapsed time between the move in date and the date of data collection. The logic behind this is simple: on average, current tenants are halfway through their current LOS, thus an estimate of their ultimate LOS is given by doubling the observed amount of time they have spent in their apartments thus far. The properties of LOS for the complete and current populations are of course quite different; much shall be said about this later on.

There are several questions about LOS which we want to answer. First of all, we want to know how long households live in public housing and whether or not this duration varies by bedroom size and project. If LOS varies, how does it vary? This is a question of basic interest, for it defines the time frame within which housing authorities (like the BHA) serve their clients.

A technical question relates to the distribution of LOS. Models predicting waiting times for public housing assignments make assumptions regarding the LOS distribution, as do models of project mixing (e.g. differential assignments according to minority preference or income level). Are these assumptions warranted? This study will try to find out.

Thirdly, we wish to know if the public housing population is stable over time with respect to LOS. Are households spending the same amount of time in public housing now compared to clients ten, twenty or more years ago?

Finally, it is of interest (and practical utility for waiting time models) to determine transfer rates. For example, what fraction of households leaving two bedroom apartments transfer to three bedroom apartments? What fraction leaves the project altogether? Given that one policy under consideration by the BHA is the prioritization of transfers, this information is important.

3.2 Household Occupancy Times in Public Housing

The first step in analyzing LOS was to display the relevant data. Frequency distributions for LOS broken down by project and bedroom size may be found in Appendix 3.1 for both the complete and current populations. From these histograms, it is evident that LOS varies quite a bit.

The variation in LOS for the complete population is summarized in Table 3.1. The shortest average occupancy time observed is on the order of 3.5 years, while the longest observed mean occupancy times are on the order of 10.5 years. Overall, the mean LOS for the complete population (accounting for sampling variability) equals 5.2 years. To further summarize these data, the following questions were posed and answered:

1) How does mean LOS vary by bedroom size?

2) How does mean LOS vary by project?

3) Are these variations significant?

These questions were answered using weighted least squares. The results of the analysis are as follows:

3.2.1 Variation in LOS by Bedroom Size

LOS appears to increase with bedroom size from one to three bedroom apartments, then decrease from three to five bedroom apartments as shown in Table 3.2. However, the associated t-statistics indicate that there is no significant difference between the bedroom adjusted mean LOS and the overall mean occupancy time for any bedroom size (the computed t-statistics would have to exceed 2.101 in absolute value to reject the null hypothesis of no difference using a 5% level of significance). In fact, the hypothesis that mean LOS is equivalent for

Table 3.1

LOS for Complete Population: Summary Statistics

(LOS in years)

Project		Bedroom Size 1	2	3	4	5
Faneuil	Mean LOS	---	8.14	9.34	---	19.87
	St. Dev.		6.32	7.45		---
	Sample n		137	99		1
Wash. Beech	Mean LOS	8.27	5.54	5.17	6.10	8.09
	St. Dev.	8.54	5.87	4.70	6.02	4.55
	Sample n	67	178	85	16	14
Mission Extension	Mean LOS	3.66	5.34	6.67	7.13	---
	St. Dev.	4.07	4.66	5.32	7.48	
	Sample n	94	222	155	27	
Mission Hill	Mean LOS	3.36	4.39	5.70	5.13	3.41
	St. Dev.	2.60	2.98	3.46	3.03	1.88
	Sample n	210	403	180	70	5
Mary Ellen	Mean LOS	9.67	10.83	10.36	---	---
	St. Dev.	8.94	9.06	8.41		
	Sample n	395	266	63		
Charles-town	Mean LOS	5.59	5.49	5.89	5.70	---
	St. Dev.	5.20	4.79	4.70	4.11	
	Sample n	404	442	213	74	

Table 3.2

LOS by Bedroom Size
(Complete Population)

Bedroom Size	Mean LOS (years)	St. Error	t-statistic $\frac{(\text{Mean}-5.2)}{\text{St. Error}}$
1	4.6	.661	-.01
2	5.2	.538	0.00
3	6.1	.827	1.09
4	5.4	1.393	0.14
5	4.9	3.465	-.09

Model: LOS = $\beta_i + \varepsilon_i$

where β_i = LOS for bedroom size i; ε_i = error term all bedroom sizes cannot be rejected (F=.569 with 4 and 18 degrees of freedom; the associated significance level equals .688). Thus, one may conclude that mean LOS in the complete sample does not vary by bedroom size - however, this result will change modestly when different projects are identified and explicitly considered.

3.2.2 Variation in LOS by Project

LOS varies greatly with the project under consideration. This is clear from Table 3.3. Here, the t-statistics indicate that at a 5% significance level, three projects (Faneuil, Mission Hill and Mary Ellen McCormack) have mean occupancy times that differ from the overall population mean. The hypothesis that all projects have the same mean LOS is easily rejected (F=11.833 with 5 and 17 degrees of freedom; the likelihood of obtaining a result this extreme under the null hypothesis is essentially zero). Thus, we conclude that mean occupancy times are significantly above average in the Faneuil and Mary Ellen McCormack projects, significantly below average in the Mission Hill project, and about average in the Washington Beech, Mission Extension, and Charlestown projects.

3.2.3 Simultaneous Consideration of Bedroom Size and Housing Project

A more sophisticated model is presented in Table 3.4. Here, we may note that controlling for project, one can no longer accept the hypothesis that mean LOS is invariant over bedroom size (F=3.911 with 4 and 13 degrees of freedom; significance level is .027). However, the maximum difference in mean LOS attributable to bedroom size is on the order of 1.5 years; this variation is small compared to the differences in mean LOS attributable to the various housing projects studied. We

Table 3.3

LOS by Project
(Complete Population)

Project	Mean LOS (years)	St. Error	t-statistic $\frac{(\text{Mean}-5.2)}{\text{St. Error}}$
Faneuil	8.6	1.134	3.00
Washington Beech	5.8	.779	0.77
Mission Extension	5.3	.554	0.18
Mission Hill	4.3	.258	-3.49
Mary Ellen McCormack	10.2	.860	5.81
Charlestown	5.6	.373	1.07

Model: $\text{LOS} = \gamma_j + \varepsilon_j$

where γ_j = LOS for project j; ε_j = error term

Table 3.4

LOS by Bedroom Size and Project

MODEL: $LOS = \alpha + \beta_i + \gamma_j + \varepsilon_{ij}$

where α = common term

β_i = effect of bedroom size i (β_5=5BR effect $\equiv$ 0)

γ_j = effect of project j (γ_6 = Charlestown effect $\equiv$ 0)

ε_{ij} = error term associated with bedroom size i, project j

Coefficient	Value	St. Error	t-statistic (value/St. Error)
α	5.802	1.445	4.016
β_1 (1BR)	- .909	1.436	- .633
β_2 (2BR)	- .173	1.424	- .122
β_3 (3BR)	.664	1.445	.460
β_4 (4BR)	.390	1.517	.257
γ_1 (Faneuil)	2.635	.938	2.810
γ_2 (Washington Beech)	- .076	.684	- .112
γ_3 (Mission Extension)	- .384	.522	- .736
γ_4 (Mission Hill)	-1.262	.354	-3.562
γ_5 (Mary Ellen McCormack)	4.841	.735	6.586

thus conclude that the major variation in LOS is project specific; differences in LOS due to bedroom size can largely be ignored. This finding will greatly simplify future work, as it will not be necessary to differentiate between apartments on the basis of bedroom size when estimating occupancy times.

3.3 The Probability Distribution of LOS

When working with statistical data, it is often convenient to assume that the data come from a particular probability distribution (e.g. Normal, Poisson, exponential, gamma, etc.). Such an assumption, if warranted, greatly simplifies more detailed mathematical analysis, and can also provide an explanation of the process generating the data.

It is of particular interest to see if the household occupancy times correspond to the exponential probability distribution. One purpose for collecting LOS data, as previously mentioned, is for the estimation of waiting times for public housing assignments. The best understood models of this sort (known as queueing models) often assume that the service times (in our case, household LOS) are exponentially distributed. A second use of the exponential distribution will be demonstrated in the next section where we estimate mean cohort LOS assuming exponentiality. In this section, the appropriateness of the exponential assumption shall be examined.

The probability density function for an exponential random variable x is given by

$$f(x) = \mu e^{-\mu x} \qquad x \geq 0,\ \mu > 0. \tag{3.1}$$

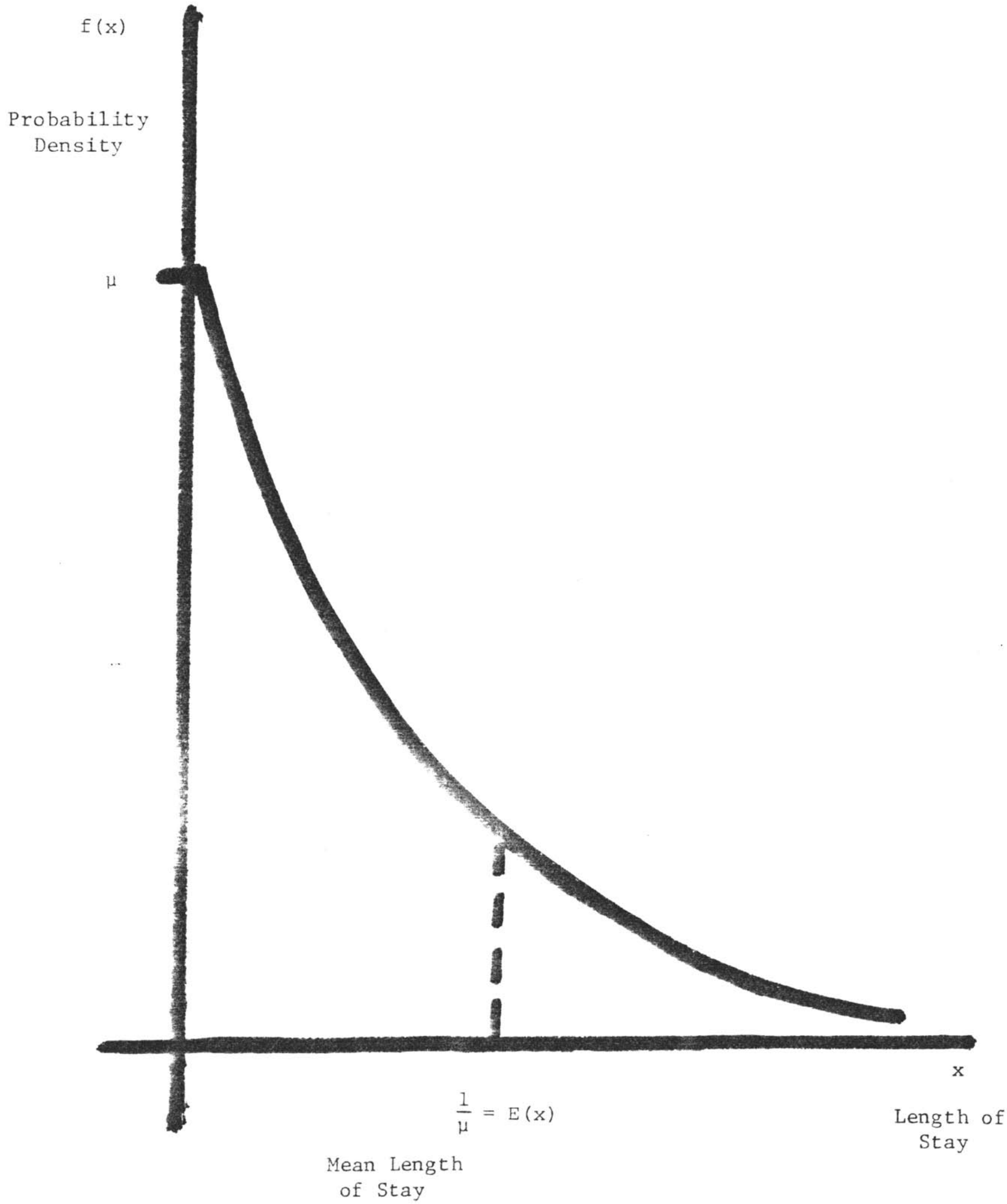
Here, x could represent household length of stay; the mean length of stay would then be given by $\frac{1}{\mu}$. A graph of the exponential density is shown in Figure 3.1.

Figure 3.1

An Exponential Density

for

Length of Stay



The LOS histograms for the complete population are shown in Appendix 3.1, and many of them do appear to have the shape of the exponential density. To see if these lengths of stay do in fact follow the exponential distribution, χ^2 tests were computed for each of the histograms shown. Testing the null hypothesis that the LOS data for the complete population came from exponential distributions yields mixed results. At a significance level of 5%, this hypothesis cannot be rejected for 9 out of 19 tests. In certain projects (notably Mary Ellen McCormack), the exponential distribution fits the LOS data remarkably well, while in other projects (notably Mission Hill), the exponential model does not work well at all. The results of these tests are shown in Table 3.5.

That the exponential distribution fails for Mission Hill is not entirely surprising; this project has undergone numerous physical transitions (including the closing of buildings) which would alter the "natural" move out dates of public housing tenants. The same is true to a degree at Mission Extension.

It would seem, then, that it would not be entirely misleading to treat LOS as an exponential variable. Practically, this is of great utility for future data collection: rather than painfully collecting move in and move out dates as in this study, the exponential model requires only the move out rate for estimation purposes. Thus, the number of move outs per apartment per year is all that needs to be computed to use the exponential model. A corresponding estimate of mean cohort LOS is then given by the reciprocal of the move out rate.

Table 3.5

Chisquare Tests for Exponentiality

Project	Bedroom Size	Sample Size	Degrees of Freedom	Chisquare	Significance
Faneuil	2	137	9	12.4	0.192
Faneuil	3	99	6	17.8	0.007
Wash. Beech	1	67	3	12.7	0.005
Wash. Beech	2	178	9	12.5	0.187
Wash. Beech	3	85	5	3.6	0.608
Mission Ext.	1	94	5	5.7	0.337
Mission Ext.	2	222	10	19.1	0.039
Mission Ext.	3	155	9	23.1	0.006
Mission Hill	1	210	7	43.6	0.000
Mission Hill	2	403	11	78.6	0.000
Mission Hill	3	180	9	67.0	0.000
Mission Hill	4	70	4	14.9	0.005
Mary Ellen	1	395	19	13.8	0.795
Mary Ellen	2	266	16	19.9	0.225
Mary Ellen	3	63	1	1.1	0.294
Charlestown	1	404	13	38.3	0.000
Charlestown	2	442	13	14.0	0.374
Charlestown	3	213	10	12.8	0.235
Charlestown	4	74	4	11.3	0.023

3.4 Stability of Household Occupancy Times

The analysis of the last two sections focused on the complete population: households who have completed move outs within the study period. To see if those currently living in public housing are following the same distributions of LOS evidenced in the complete population, occupancy times for the current population were compared to what would be expected based on the complete population using the following method.

Let f(x) refer to the probability density of LOS from the complete population, and h(x) be the probability density from the current population. If household occupancy times are equal, then arguments based on the theory of random incidence show that these two densities are related by (see Drake (1967, p. 157))

$$h(x) = \frac{xf(x)}{E(x)} \tag{3.2}$$

From (3.2), it is easy to show that the mean occupancy time for those in the current population, $E_h(x)$, is related to the first two moments of occupancy time in the complete population, $E_f(x)$ and $E_f(x^2)$, by the equation

$$E_h(x) = E_f(x^2)/E_f(x) \tag{3.3}$$

Also, the second moment of occupancy time for the current population is given by

$$E_h(x^2) = E_f(x^3)/E_f(x) \tag{3.4}$$

Thus, the variance of occupancy times for the current population $var_h(x)$ equals

$$var_h(x) = \frac{E_f(x^3)}{E_f(x)} - [\frac{E_f(x^2)}{E_f(x)}]^2 \tag{3.5}$$

To see if the current population is actually related to the complete population via equations (3.3) to (3.5), we compute the observed first three moments $E_f(x)$, $E_f(x^2)$ and $E_f(x^3)$ of the complete population, and treat these as known. We then estimate the mean occupancy time $\bar{x}_h$ from the current population, and construct the statistic

$$z = \frac{\bar{x}_h - E_f(x^2)/E_f(x)}{\sqrt{\left(\frac{E_f(x^3)}{E_f(x)} - \left[\frac{E_f(x^2)}{E_f(x)}\right]^2\right)/ \quad n_h}} \tag{3.6}$$

where n_h is the sample size taken from the current population. The z-statistic thus computed will roughly follow a Normal distribution with mean 0 and variance 1.

Table 3.6 summarizes the LOS data for the current population, while Table 3.7 shows expected information for the current population assuming that the trends of the complete population were followed. Finally, Table 3.8 presents the z-statistics which test whether or not the data shown in Table 3.6 match the expectations of Table 3.7.

The implications of Table 3.8 are clear. With few exceptions, the current and complete populations have significantly different mean lengths of stay (at a significance level of 5%, a z-statistic with absolute value greater than 1.96 is significant). Almost all of the z-statistics are positive. This indicates that the current population has longer mean occupancy times than would be expected according to the complete population.

A notable exception to this trend is found at Washington Beech. Here, current LOS's are less than would be expected for 1 and 2 bedroom apartments, more than would be expected for 3 bedroom apartments, and

Table 3.6

LOS for Current Population: Summary Statistics

(LOS in years)

Project		Bedroom Size 1	2	3	4	5
Faneuil	Mean LOS	---	16.17	18.45	---	36.45
	St. Dev.		17.03	18.30		17.53
	Sample n		124	119		6
Wash. Beech	Mean LOS	8.75	6.38	12.93	11.76	11.90
	St. Dev.	10.17	7.04	12.03	7.60	9.27
	Sample n	45	103	64	11	8
Mission Extension	Mean LOS	10.85	15.80	15.94	25.33	---
	St. Dev.	12.17	16.16	16.15	15.82	
	Sample n	31	38	31	13	
Mission Hill	Mean LOS	6.18	9.61	13.22	12.03	13.05
	St. Dev.	6.10	9.33	11.22	9.39	7.47
	Sample n	70	146	166	73	8
Mary Ellen	Mean LOS	15.79	24.71	26.18	---	---
	St. Dev.	16.30	21.72	18.69		
	Sample n	409	431	149		
Charles-town	Mean LOS	12.23	15.06	19.06	15.44	---
	St. Dev.	11.96	13.38	12.78	13.53	
	Sample n	273	283	156	51	

Table 3.7

Expected Summary Statistics for Current Population LOS

(LOS in years)

Project		Bedroom Size 1	2	3	4	5
Faneuil	Mean LOS	---	13.05	15.28	---	---
	St. Dev.		6.99	7.59		
Wash. Beech	Mean LOS	17.09	11.75	9.443	12.05	10.65
	St. Dev.	8.68	7.35	5.81	7.63	3.72
Mission Extension	Mean LOS	8.17	9.41	10.90	14.97	---
	St. Dev.	6.44	5.82	6.56	7.43	
Mission Hill	Mean LOS	5.37	6.41	7.81	6.92	4.45
	St. Dev.	3.42	3.32	3.48	3.03	1.80
Mary Ellen	Mean LOS	17.93	18.40	17.17	---	---
	St. Dev.	11.31	10.44	11.10		
Charles-town	Mean LOS	10.42	9.66	9.65	8.66	---
	St. Dev.	5.66	4.95	4.62	4.37	

Table 3.8

Z-Statistics for Stability Tests on Current Population LOS

Project	Bedroom Size 1	2	3	4	5
Faneuil	---	4.97	4.55	---	---
Washington Beech	-6.44	-7.42	4.81	-0.12	0.96
Mission Extension	2.31	6.78	4.27	5.01	---
Mission Hill	1.99	11.63	20.03	14.41	13.48
Mary Ellen McCormack	-3.83	12.54	9.89	---	---
Charlestown	5.26	18.35	25.48	11.07	---

about what would be expected for 4 and 5 bedroom apartments. It is unclear why this is so.

Now suppose that incoming public housing tenants have occupancy times consistent with the current population. A reasonable question to ask is: how long can an entering household be expected to stay in public housing? In other words, what would be the mean LOS for a cohort of households entering public housing now (as opposed to the mean LOS for those already living in public housing). In the last section, we supported the assumption that occupancy times are exponentially distributed. If we assume that an incoming cohort has exponential lengths of stay, then it is simple to estimate the implied mean cohort occupancy time based on the mean occupancy times for the current populaion.

For the exponential distribution, the first two moments of occupancy time are given by:

$$E_f(x) = 1/\mu \tag{3.7}$$

$$E_f(x^2) = 2/\mu^2 \tag{3.8}$$

Substituting these results into equation (3.3) yields

$$E_h(x) = \frac{2/\mu^2}{1/\mu} = 2/\mu \tag{3.9}$$

Thus, the mean cohort occupancy time equals $E_h(x)/2$, and we estimate this by

$$\text{Mean Cohort Occupancy Time} = \bar{x}_h/2 \tag{3.10}$$

These mean cohort occupancy times are presented in Table 3.9, along with the mean occupancy time from the complete population.

It appears that households are staying about two years longer in public housing when compared to the complete population, although there

Table 3.9

Observed Mean LOS for Complete Population and Estimated Mean Cohort LOS for Current Population

(LOS in years)

Project		Bedroom Size 1	2	3	4	5
Faneuil	Obs. Mean LOS	---	8.14	9.34	---	---
	Est. Mean LOS		8.09	9.23		
Wash. Beech	Obs. Mean LOS	8.27	5.54	5.17	6.10	8.09
	Est. Mean LOS	4.38	3.19	6.47	5.88	5.95
Mission Extension	Obs. Mean LOS	3.66	5.34	6.67	7.13	---
	Est. Mean LOS	5.43	7.90	7.97	12.67	
Mission Hill	Obs. Mean LOS	3.36	4.39	5.70	5.13	3.41
	Est. Mean LOS	3.09	4.81	6.61	6.02	6.53
Mary Ellen	Obs. Mean LOS	9.67	10.83	10.36	---	---
	Est. Mean LOS	7.85	12.36	13.09		
Charles-town	Obs. Mean LOS	5.59	5.49	5.89	5.70	---
	Est. Mean LOS	6.12	7.53	9.53	7.72	

are exceptions (notably Faneuil and Washington Beech). It should also be mentioned that Table 3.9 does not totally agree with Table 3.8; in some cases, the z-statistic indicates that mean LOS has increased where Table 3.9 suggests a decrease. These discrepancies can be explained by the fact that the exponential approximation is not always warranted, thus the results from Table 3.8 are more reliable.

3.5 Transfers and Termination of Occupancy

Finally, one of our stated goals was to investigate internal transfer rates. The observed transfer probabilities for the complete population are summarized in Table 3.10. The most noticeable feature of this table is that the vast majority of apartment occupancies terminate with the household leaving the project. However, in many projects and bedroom categories, over 10% of all occupancies end with a transfer to another on site apartment.

The two projects with the highest transfer probabilities (or equivalently the lowest exit probabilities) are Mission Hill and Mission Extension, where transfer rates are typically over 20% and often over 30%. However, these transfer likelihoods are artificially high due to the physical transitions at these projects; this is also consistent with the poor fit of the exponential model to household occupancy times at those projects. At the other projects, transfer probabilities are rarely higher than 15%.

One slightly disturbing feature is the regularity with which transfers occur between apartments of the same size. This is not supposed to occur (except in emergencies or for medical reasons), and

Table 3.10

Transfer Probabilities

To: Bedroom Size

Project	From (BR Size)	1	2	3	4	5	Exit	Sample n
	2	---	.081	.059	---	---	.860	136
Faneuil	3	---	---	.051	---	---	.949	98
	1	.015	.015	---	---	---	.970	67
	2	.039	.056	.028	---	---	.876	178
Washington	3	.012	.012	.035	---	.012	.929	85
Beech	4	---	.063	---	---	---	.937	16
	5	---	---	---	---	---	1.000	14
	1	.032	.160	.053	---	---	.755	94
Mission	2	.023	.081	.091	.009	---	.796	221
Extension	3	.026	.039	.117	.058	.013	.747	154
	4	.037	.037	---	.111	---	.815	27
	1	.091	.177	.034	---	---	.699	209
	2	.020	.112	.169	.020	.002	.677	403
Mission	3	.039	.039	.050	.106	---	.765	179
Hill	4	.057	.071	.014	.071	.043	.743	70
	5	---	---	---	.200	---	.800	5
Mary	1	.046	.046	.003	---	---	.906	395
Ellen	2	.030	.053	.026	---	---	.891	266
McCormack	3	.048	.191	.095	---	---	.667	63
	1	.064	.035	.005	.003	---	.894	404
	2	.027	.109	.048	.011	---	.805	442
Charlestown	3	.014	.033	.132	.033	---	.788	212
	4	---	.014	.068	.054	.014	.851	74

cannot be explained at present for all projects. The physical changes at Mission Hill and Mission Extension would have something to do with this phenomenon at those projects, however.

3.6 Summary of Findings

This chapter described the analysis of household occupancy times in six Boston Housing Authority projects (Faneuil, Washington Beech, Mission Extension, Mission Hill, Mary Ellen McCormack, and Charlestown). The results of this analysis are detailed in the body of the chapter along with the methodology employed. The main results of the study may be summarized as follows:

1) Mean occupancy times do not appear to vary by apartment size from the overall mean occupancy time of 5.2 years.
2) Mean occupancy times vary greatly by project, from a minimum of 4.3 years at Mission Hill to a maximum of 10.2 years at Mary Ellen McCormack.
3) Household occupancy times are often exponentially distributed. Thus, the properties of occupancy time distributions can be inferred from move out rates. Also, the assumption that occupancy times are exponential, used in models for assignment waiting times, can be justified in several instances.
4) Household occupancy times appear to have increased. In other words, households entering public housing today can be expected to remain longer in their apartments than households who entered several years ago, often by as much as two additional years.
5) Internal transfer probabilities are between 10% and 15% for most cases of interest. A non-negligible fraction of

transfers occur between apartments of the same size. It is unclear why this is so.

We have discussed the features of tenant assignment policies in Chapter 2, and the timing of household occupancies in this chapter. With this background material in mind, we are now prepared to develop detailed tenant assignment models. We will begin with single project models in the next chapter.

APPENDIX 3.1 EMPIRICAL DISTRIBUTIONS OF LOS

Part 1: Complete Population

LOS (yrs)	Two Bedroom Apartments at Faneuil	Frequency
1	xxxxxxxxxxxxxxxx	8
2	xxxxxxxxxxxxxxxxxxxx	10
3	xxxxxxxxxxxxxxxxxxxxxxxx	12
4	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	18
5	xxxxxxxxxxxxxxxx	8
6	xxxxxxxxxxxxxxxx	8
7	xxxxxxxxxxxxxxxxxxxxxx	11
8	xxxxxxxxxx	5
9	xxxxxxxxxxxx	6
10	xxxxxxxxxxxx	6
11	xxxxxxxxxxxxxxxx	8
12	xxxxxxxx	4
13	xxxxxxxxxxxx	6
14	xxxxxx	3
15	xxxxxx	3
16	xxxxxx	3
17	xxxxxx	3
18	xxxxxx	3
19	xxxxxx	3
20	xx	1
21	xx	1
22	xxxx	2
23	xx	1
24		0
25	xxxx	2
26		0
27	xx	1
28		0
29		0
30		0
31	xx	1

LOS (yrs)	Three Bedroom Apartments at Faneuil	Frequency
1	xxxxxxxx	4
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	16
3	xxxxxxxxxxxxxx	7
4	xxxxxxxxxxxxxxxxxx	9
5	xxxxxx	3
6	xxxx	2
7	xxxxxxxx	4
8	xxxxxxxx	4
9	xxxxxxxxxxxxxx	7
10	xxxxxx	3
11	xxxxxxxx	4
12	xx	1
13	xxxxxxxxxxxxxx	7
14	xx	1
15	xxxxxxxx	4
16	xxxxxxxxxx	5
17	xx	1
18	xx	1
19	xxxxxxxx	4
20	xxxxxxxxxx	5
21		0
22	xxxx	2
23	xx	1
24		0
25		0
26		0
27	xxxx	2
28		0
29		0
30		0
31	xx	1
32		0
33	xx	1

One Bedroom Apartments at Washington Beech

LOS (yrs)		Frequency
1	xxxxxxxxxxxxxxxxxxxxx	7
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	12
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	13
4	xxx	1
5	xxxxxx	2
6	xxxxxxxxx	3
7	xxxxxx	2
8	xxxxxx	2
9	xxxxxxxxx	3
10	xxx	1
11	xxx	1
12	xxxxxxxxx	3
13	xxxxxx	2
14	xxxxxx	2
15		0
16		0
17		0
18		0
19		0
20		0
21	xxxxxx	2
22	xxxxxx	2
23		0
24	xxxxxxxxx	3
25	xxxxxxxxx	3
26		0
27	xxxxxx	2
28		0
29	xxx	1

Two Bedroom Apartments at Washington Beech

1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxx	29
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	33
3	xxxxxxxxxxxxxxxxxxxxx	21
4	xxxxxxxxxxxxxxxxxxxxx	21
5	xxxxxxxxxxx	11
6	xxxx	4
7	xxxxxxxxxx	10
8	xxxx	4
9	xxxxxxx	7
10	xxxxxx	6
11	xxxx	4
12	xxx	3
13	xx	2
14	xxxxxxx	7
15	x	1
16	xx	2
17	x	1
18	x	1
19	xx	2
20	x	1
21	xx	2
22	x	1
23	x	1
24	x	1
25	xx	2
26		0
27		0
28	x	1

Three Bedroom Apartments at Washington Beech

LOS (yrs)		Frequency
1	xxxxxxxxxxxxxxxxxxxx	10
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	16
3	xxxxxxxxxxxxxxxxxxxxxx	11
4	xxxxxxxxxxxxxxxxxx	9
5	xxxxxxxxxxxx	6
6	xxxxxxxxxxxx	6
7	xxxxxxxxxxxx	6
8	xxxxxxxx	4
9	xxxx	2
10	xx	1
11	xxxxxx	3
12	xxxx	2
13	xx	1
14	xxxx	2
15	xx	1
16	xxxx	2
17	xxxx	2
18		0
19		0
20		0
21		0
22		0
23		0
24		0
25	xx	1

One Bedroom Apartments at Mission Extension

1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	19
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	19
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	17
4	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	15
5	xxxxxxxxxxxx	6
6	xxxxxxxxxx	5
7	xxxxxx	3
8		0
9	xxxx	2
10		0
11	xx	1
12	xx	1
13	xx	1
14		0
15	xx	1
16	xxxx	2
17		0
18		0
19		0
20		0
21	xx	1
22	xx	1

LOS (yrs)	Two Bedroom Apartments at Mission Extension	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxx	27
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	37
3	xxxxxxxxxxxxxxxx	16
4	xxxxxxxxxxxxxxxxxxxxxxxxxx	26
5	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	28
6	xxxxxxxxxxxxxxxxxx	18
7	xxxxxxxxxxxxxxx	15
8	xxxxxxxxx	9
9	xxxxxxxxxxx	11
10	xxxx	4
11	xxxxxxx	7
12	xxxx	4
13	xxx	3
14	x	1
15	xx	2
16	xxx	3
17	xx	2
18	x	1
19	x	1
20	xxxx	4
21	x	1
22	x	1
23	x	1

	Three Bedroom Apartments at Mission Extension	
1	xxxxxxxx	4
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	20
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	18
4	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	20
5	xxxxxxxxxxxxxxxxxxxxxxxx	12
6	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	15
7	xxxxxxxxxxxxxxxxxxxxxxxx	12
8	xxxxxxxxxxxxxxxxxx	9
9	xxxxxxxxxx	5
10	xxxxxxxxxxxxxx	7
11	xxxxxxxxxxxxxx	7
12	xxxxxxxxxx	5
13	xxxxxx	3
14	xxxxxxxx	4
15	xxxxxx	3
16		0
17	xx	1
18		0
19	xx	1
20	xxxx	2
21		0
22	xxxx	2
23	xxxx	2
24	xxxx	2
25	xx	1

One Bedroom Apartments at Mission Hill

LOS (yrs)	Frequency
1	22
2	54
3	40
4	28
5	25
6	19
7	3
8	4
9	3
10	5
11	2
12	3
13	0
14	1
15	0

Two Bedroom Apartments at Mission Hill

LOS (yrs)	Frequency
1	44
2	44
3	64
4	54
5	49
6	49
7	35
8	17
9	12
10	12
11	6
12	7
13	5
14	1
15	4

Three Bedroom Apartments at Mission Hill

LOS (yrs)	Frequency
1	8
2	15
3	23
4	22
5	21
6	21
7	10
8	13
9	5
10	20
11	4
12	7
13	5
14	3
15	1
16	1

Four Bedroom Apartments at Mission Hill

LOS (yrs)	Frequency
1	5
2	5
3	8
4	11
5	11
6	4
7	6
8	8
9	5
10	3
11	0
12	1
13	2
14	1

One Bedroom Apartments at Mary Ellen McCormack

LOS (yrs)	Frequency
1	29
2	39
3	24
4	31
5	28
6	22
7	23
8	21
9	16
10	20
11	12
12	15
13	9
14	11
15	13
16	12
17	9
18	7
19	5
20	3
21	1
22	4
23	5
24	3
25	7
26	2
27	2
28	2
29	1
30	2
31	0
32	1
33	2
34	0
35	1
36	1
37	2
38	1
39	1
40	3
41	2
42	2
43	0
44	1

LOS (yrs)	Two Bedroom Apartments at Mary Ellen McCormack	Frequency
1	xxxxxxxxxxxxxxxx	16
2	xxxxxxxxxxxxxx	14
3	xxxxxxxxxxxxxxxxxxxxxxxxx	25
4	xxxxxxxxxxxxxxxxxxxx	20
5	xxxxxxxxxxxxxxxx	16
6	xxxxxxxxxxxxx	13
7	xxxxxxxxxxxxx	13
8	xxxxx	5
9	xxxxxxxxxxxxx	13
10	xxxxxxxxxxxx	12
11	xxxxxxxxxxxxx	13
12	xxxxxxxxxxxx	12
13	xxxxxx	6
14	xxxxxxxxxxx	11
15	xxxxxxxxxx	10
16	xxxx	4
17	xxxx	4
18	xxxxxx	6
19	xxx	3
20	xxxxxxxxxxx	11
21	xxxxx	5
22	xxxx	4
23	xxxx	4
24	xxxx	4
25	xx	2
26	xx	2
27	xxxx	4
28	x	1
29		0
30	x	1
31		0
32	x	1
33		0
34	x	1
35	x	1
36		0
37		0
38	xxxx	4
39		0
40	xxx	3
41	xx	2

	Three Bedroom Apartments at Mary Ellen McCormack	
1	xxxxxxxxxxxxxx	2
2	xxxxxxxxxxxxxxxxxxxxx	3
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
4	xxxxxxxxxxxxxxxxxxxxx	3
5	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
6	xxxxxxxxxxxxxxxxxxxxx	3
7	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	[illegible]
8	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
9	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
10	xxxxxxxxxxxxxx	2
11	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
12	xxxxxxx	1
13	xxxxxxx	1
14	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
15	xxxxxxxxxxxxxxxxxxxxx	3
16	xxxxxxxxxxxxxx	2
17	xxxxxxx	1
18	xxxxxxx	1
19		0
20	xxxxxxx	1
21	xxxxxxxxxxxxxx	2
22	xxxxxxx	1
23	xxxxxxx	1
24		0
25	xxxxxxx	1

LOS (yrs)	One Bedroom Apartments at Charlestown	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	70
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	59
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	50
4	xxxxxxxxxxxxxxxxxxxxxxx	36
5	xxxxxxxxxxx	18
6	xxxxxxxxxxxxxxxxxxx	29
7	xxxxxxxxx	14
8	xxxxxxxxxxx	18
9	xxxxxxxxx	15
10	xxxxxxxxxxx	17
11	xxx	6
12	xxxxx	9
13	xxxxxxxxxxx	17
14	xxxxxxxxxx	16
15	x	3
16	x	2
17	xxx	5
18	xxx	6
19	xxx	6
20	xx	4
21		1
22		1
23		1
24		1

LOS (yrs)	Two Bedroom Apartments at Charlestown	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	69
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	65
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	49
4	xxxxxxxxxxxxxxxxxxxxxxxxxxx	41
5	xxxxxxxxxxxxxxxxxxxxxxxx	37
6	xxxxxxxxxxxxxxx	23
7	xxxxxxxxxxxx	19
8	xxxxxxxxxxxxxx	21
9	xxxxxxxx	12
10	xxxxxxxxx	14
11	xxxxxxxxxxxxx	20
12	xxxxxxxxxx	15
13	xxxxxxxx	13
14	xxxx	7
15	xxxxxxxx	12
16	xxxxx	8
17	xxxxxx	10
18	x	[illegible]
19	x	[illegible]
20	x	[illegible]
21		[illegible]
22		1

LOS (yrs)	Three Bedroom Apartments at Charlestown	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	32
2	xxxxxxxxxxxxxxxxxxxxxxxx	24
3	xxxxxxxxxxxxxxxxxxx	19
4	xxxxxxxxxxxxxxxx	16
5	xxxxxxxxxxxxxxxxxxxxx	21
6	xxxxxxxxxxx	11
7	xxxxxxxxxxxx	12
8	xxxxxxxxxxxx	12
9	xxxxxxxxxxxx	12
10	xxxx	4
11	xxxxxxxxxxxxx	13
12	xxxxxxxxxxxx	12
13	xxxx	4
14	xxxxxxx	7
15	xxxx	4
16	xxxxxx	6
17		0
18	x	1
19	x	1
20	x	1
21		0
22	x	1

LOS (yrs)	Four Bedroom Apartments at Charlestown	Frequency
1	xxxxxxxxxxxxxxx	5
2	xxxxxxxxxxxxxxxxxx	6
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	11
4	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	12
5	xxxxxxxxxxxxxxxxxxxxx	7
6	xxxxxxxxxxxxxxx	5
7	xxxxxxxxx	3
8	xxxxxxxxxxxxxxx	5
9	xxxxxxxxx	3
10	xxxxxxxxx	3
11	xxxxxxxxxxxx	4
12	xxxxxxxxxxxxxxx	5
13	xxxxxx	2
14		0
15		0
16	xxx	1
17	xxx	1
18		0
19	xxx	1

Part II: Current Population

LOS (yrs)	Two Bedroom Apartments at Faneuil	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	11
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	10
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	13
4	xxxxxxxxxxxxxxx	5
5	xxxxxxxxxxxxxxx	5
6	xxxxxxxxxxxxxxx	5
7	xxxxxxxxxxxxxxxxxx	6
8	xxxxxxxxxxxxxxxxxx	6
9	xxxxxxxxxxxxxxxxxxxxx	7
10	xxxxxxxxx	3
11	xxxxxxxxxxxxxxx	5
12	xxx	1
13	xxxxxx	2
14	xxx	1
15	xxx	1
16		0
17		0
18	xxx	1
19		0
20	xxxxxxxxx	3
21		0
22		0
23	xxx	1
24	xxx	1
25	xxx	1
26	xxxxxxxxxxxx	4
27		0
28	xxx	1
29		0
30	xxxxxx	2
31	xxx	1
32		0
33		0
34	xxxxxx	2
35		0
36	xxxxxxxxx	3
37	xxx	1
38		0
39	xxxxxxxxx	3
40	xxxxxxxxx	3
41	xxx	1
42	xxxxxx	2
43	xxxxxx	2
44	xxxxxx	2
45+	xxxxxxxxxxxxxxxxxxxxxxxxxxx	9

LOS (yrs)	Three Bedroom Apartments at Faneuil	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxx	9
2	xxxxxxxxxxxxxxxxxxxxx	7
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	10
4	xxxxxxxxxxxxxxxxxx	6
5	xxxxxx	2
6	xxxxxxxxxxxxxxxxxx	6
7	xxxxxxxxxxxxxxxxxxxxxxxx	8
8	xxx	1
9	xxxxxx	2
10	xxxxxxxxx	3
11	xxxxxxxxxxxxxxxxxx	6
12	xxxxxx	2
13	xxx	1
14	xxxxxx	2
15	xxx	1
16	xxxxxx	2
17	xxxxxx	2
18	xxxxxx	2
19	xxxxxx	2
20	xxx	1
21	xxxxxxxxx	3
22	xxx	1
23	xxxxxxxxx	3
24	xxxxxx	2
25		0
26	xxxxxx	2
27	xxx	1
28	xxxxxx	2
29	xxx	1
30	xxxxxxxxx	3
31	xxx	1
32		0
33	xxx	1
34	xxx	1
35		0
36	xxx	1
37	xxx	1
38	xxx	1
39	xxx	1
40	xxxxxx	2
41		0
42	xxxxxx	2
43	xxxxxx	2
44	xxx	1
45+	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	12

LOS (yrs)	One Bedroom Apartments at Washington Beech	Frequency
1	xxxxxxxxxx	2
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	9
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	8
4	xxxxx	1
5	xxxxx	1
6	xxxxxxxxxx	2
7		0
8		0
9	xxxxx	1
10	xxxxxxxxxxxxxxxxxxxxxxxxx	5
11	xxxxxxxxxxxxxxx	3
12	xxxxx	1
13	xxxxxxxxxxxxxxx	3
14	xxxxxxxxxx	2
15	xxxxx	1
16	xxxxx	1
17		0
18		0
19		0
20	xxxxxxxxxx	2
21		0
22	xxxxx	1
23		0
24		0
25		0
26		0
27		0
28	xxxxx	1

LOS (yrs)	Two Bedroom Apartments at Washington Beech	Frequency
1	xxxxxxxxxxxxxxxxxx	9
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	20
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	16
4	xxxxxxxxxxxxxx	7
5	xxxx	2
6	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	15
7	xxxx	2
8	xxxxxxxxxxxx	6
9	xxxxxxxx	4
10	xx	1
11	xxxxxxxx	4
12	xx	1
13	xxxx	2
14	xx	1
15	xx	1
16		0
17	xx	1
18	xxxx	2
19		0
20	xxxx	2
21	xxxx	2
22		0
23	xx	1
24		0
25		0
26	xx	1
27	xx	1
28		0
29		0
30		0
31		0
32		0
33	xx	1
34		0
35	xx	1

LOS (yrs)	Three Bedroom Apartments at Wasnington Beech	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	7
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	7
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
4	xxxxxxxxxxxxxxxxxx	3
5	xxxxxxxxxxxx	2
6	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
7	xxxxxxxxxxxxxxxxxxxxxxxx	4
8	xxxxxxxxxxxx	2
9	xxxxxxxxxxxx	2
10		0
11		0
12		0
13	xxxxxx	1
14		0
15		0
16	xxxxxxxxxxxx	2
17		0
18	xxxxxx	1
19	xxxxxxxxxxxxxxxxxx	3
20	xxxxxxxxxxxxxxxxxx	3
21		0
22		0
23	xxxxxx	1
24	xxxxxx	1
25	xxxxxxxxxxxx	2
26		0
27	xxxxxxxxxxxx	2
28		0
29	xxxxxxxxxxxxxxxxxx	3
30		0
31	xxxxxxxxxxxx	2
32		0
33	xxxxxx	1
34	xxxxxx	1
35		0
36	xxxxxx	1
37	xxxxxx	1
38		0
39		0
40		0
41		0
42		0
43		0
44	xxxxxxxxxxxx	2

LOS (yrs)	One Bedroom Apartments at Mission Extension	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
2	xxxxxxxxx	1
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
4	xxxxxxxxxxxxxxxxxx	2
5	xxxxxxxxxxxxxxxxxxxxxxxxxxx	3
6	xxxxxxxxx	1
7	xxxxxxxxx	1
8	xxxxxxxxxxxxxxxxxx	2
9		0
10		0
11	xxxxxxxxx	1
12		0
13	xxxxxxxxx	1
14		0
15	xxxxxxxxx	1
16	xxxxxxxxx	1
17	xxxxxxxxxxxxxxxxxx	2
18		0
19		0
20		0
21	xxxxxxxxx	1
22		0
23		0
24		0
25	xxxxxxxxxxxxxxxxxx	2
26		0
27	xxxxxxxxx	1
28		0
29		0
30		0
31		0
32		0
33		0
34		0
35		0
36		0
37		0
38		0
39		0
40		0
41		0
42		0
43		0
44	xxxxxxxxx	1
45+	xxxxxxxxx	1

LOS (yrs)	Two Bedroom Apartments at Mission Extension	Frequency
1	xxxxxxxxxxxxxx	2
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	6
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
4	xxxxxxx	1
5	xxxxxxx	1
6	xxxxxxx	1
7	xxxxxxx	1
8		0
9	xxxxxxx	1
10	xxxxxxx	1
11	xxxxxxx	1
12	xxxxxxxxxxxxxx	2
13	xxxxxxx	1
14		0
15		0
16	xxxxxxxxxxxxxx	2
17		0
18		0
19	xxxxxxxxxxxxxx	2
20	xxxxxxxxxxxxxx	2
21		0
22		0
23		0
24	xxxxxxx	1
25		0
26	xxxxxxx	1
27		0
28	xxxxxxx	1
29		0
30		0
31	xxxxxxx	1
32	xxxxxxx	1
33		0
34		0
35		0
36		0
37		0
38		0
39		0
40		0
41		0
42		0
43		0
44		0
45+	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5

LOS (yrs)	Three Bedroom Apartments at Mission Extension	Frequency
1	xxxxxxxxxxxxxxxxxxxxx	2
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	3
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	4
4	xxxxxxxxxxxxxxxxxxxxx	2
5	xxxxxxxxxxx	1
6		0
7	xxxxxxxxxxx	1
8	xxxxxxxxxxx	1
9	xxxxxxxxxxx	1
10		0
11	xxxxxxxxxxx	1
12	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	3
13		0
14		0
15		0
16		0
17	xxxxxxxxxxx	1
18	xxxxxxxxxxx	1
19		0
20		0
21		0
22		0
23		0
24		0
25	xxxxxxxxxxx	1
26		0
27		0
28	xxxxxxxxxxxxxxxxxxxxx	2
29	xxxxxxxxxxx	1
30		0
31		0
32		0
33	xxxxxxxxxxxxxxxxxxxxx	2
34	xxxxxxxxxxx	1
35		0
36		0
37		0
38		0
39		[illegible]
40		0
41		0
42		0
43	xxxxxxxxxxx	1
44		0
45+	xxxxxxxxxxxxxxxxxxxxx	2

LOS (yrs)	One Bedroom Apartments at Mission Hill	Frequency
1	xxxxxxxxxxxxxxx	5
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	13
3	xxxxxxxxxxxx	4
4	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	10
5	xxxxxxxxxxxxxxxxxxxxx	7
6	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	11
7	xxxxxxxxxxxxxxxxxxxxx	7
8	xxxxxx	2
9	xxx	1
10		0
11		0
12		0
13		0
14		0
15	xxxxxx	2
16	xxxxxx	2
17		0
18		0
19	xxx	1
20	xxx	1
21		0
22	xxx	1
23		0
24		0
25	xxx	1
26	xxx	1
27		0
28	xxx	1

	Two Bedroom Apartments at Mission Hill	
1	xxxxxxxxxxxxxxxxxxxxxx	11
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	19
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	15
4	xxxxxxxxxxxxxxxx	8
5	xxxxxxxxxxxxxxxxxxxxxxxxxxxx	14
6	xxxxxxxxxxxxxxxxxxxx	10
7	xxxxxxxxxxxxxxxxxxxxxxxx	12
8	xxxxxxxx	4
9	xxxxxx	3
10	xx	1
11	xxxx	2
12		0
13	xx	1
14	xxxxxx	3
15	xxxxxxxxxx	5
16	xxxxxx	3
17	xxxxxx	3
18	xxxxxx	3
19	xxxx	2
20	xxxx	2
21	xxxxxxxx	4
22	xxxx	2
23	xx	1
24		0
25	xxxx	2
26	xxxxxxxx	4
27		0
28	xxxx	2
29	xxxx	2
30	xx	1
31	xxxxxx	3
32		0
33	xxxx	2
34		0
35		0
36	xx	1
37		0
38		0
39	xx	1

Three Bedroom Apartments at Mission Hill

LOS (yrs)		Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	12
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	11
3	xxxxxxxxxxxxxxxxxxxxx	7
4	xxxxxxxxxxxxxxxxxxxxxxxxxxx	9
5	xxxxxxxxxxxxxxx	5
6	xxxxxxxxxxxxxxxxxxxxxxxx	8
7	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	14
8	xxxxxxxxxxxxxxxxxxxxxxxx	8
9	xxxxxx	2
10	xxxxxxxxxxxxxxx	5
11	xxxxxx	2
12	xxxxxx	2
13	xxxxxxxxxxxxxxxxxx	6
14	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	10
15	xxxxxxxxxxxxxxxxxx	6
16	xxx	1
17	xxxxxxxxxxxx	4
18	xxxxxxxxxxxxxxx	5
19	xxxxxxxxx	3
20	xxxxxxxxxxxx	4
21	xxxxxxxxxxxxxxx	5
22	xxxxxx	2
23	xxx	1
24	xxxxxxxxx	3
25	xxxxxx	2
26	xxxxxxxxxxxxxxx	5
27	xxxxxxxxx	3
28	xxxxxxxxx	3
29	xxxxxxxxxxxx	4
30	xxxxxxxxxxxx	4
31		0
32	xxxxxx	2
33	xxxxxxxxxxxx	4
34		0
35	xxxxxx	2
36		0
37		0
38	xxx	1
39		[illegible]
40		0
41		0
42		0
43		0
44		0
45+	xxx	1

LOS (yrs)	Four Bedroom Apartments at Mission Hill	Frequency
1	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
2	xxxxxxxxxxxx	2
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	7
4	xxxxxx	1
5	xxxxxxxxxxxxxxxxxx	3
6	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	7
7	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	7
8	xxxxxx	1
9	xxxxxxxxxxxxxxxxxx	3
10	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
11	xxxxxxxxxxxxxxxxxx	3
12	xxxxxx	1
13	xxxxxx	1
14	xxxxxxxxxxxxxxxxxx	3
15	xxxxxx	1
16		0
17	xxxxxxxxxxxx	2
18		0
19	xxxxxx	1
20	xxxxxxxxxxxxxxxxxx	3
21	xxxxxx	1
22		0
23		0
24	xxxxxx	1
25		0
26	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	5
27	xxxxxxxxxxxxxxxxxx	3
28	xxxxxxxxxxxx	2
29	xxxxxx	1
30	xxxxxxxxxxxx	2
31	xxxxxx	1
32		0
33	xxxxxx	1

LOS (yrs)	One Bedroom Apartments at Mary Ellen McCormack	Frequency
1	xxxxxxxxxxxxxxxxx	17
2	xxxxxxxxxxxxxxxxx	17
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	38
4	xxxxxxxxxxxxxxxxxxxxxxxxxxx	27
5	xxxxxxxxxxxxxxxxxxx	19
6	xxxxxxxxxxxx	12
7	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	31
8	xxxxxxxxxxxxxx	14
9	xxxxxxxxxxxxxxxxxxx	19
10	xxxxxxxx	8
11	xxxxxxxxxxxxxxxx	16
12	xxxxxxxxxxxxxxxxxx	18
13	xxxxxxxxxxxxx	13
14	xxxxxxxxxx	10
15	xxxxxxxx	8
16	xxxxx	5
17	xxxxx	5
18	xxxxx	5
19	xxxxxxxxx	9
20	xxxxxxx	7
21	xxxx	4
22	xxxxxxxx	8
23	xxxxx	5
24	xx	2
25	xxxxx	5
26	xxx	3
27	xxxxxx	6
28	xxxxxxx	7
29	x	1
30	xxxx	4
31	xxxxx	5
32	xxx	3
33	x	1
34	xx	2
35	x	1
36	xxxx	4
37		0
38	xx	2
39	xxx	3
40		0
41	xxx	3
42	xxxx	4
43		0
44	x	1
45+	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	37

LOS (yrs)	Two Bedroom Apartments at Mary Ellen McCormack Frequency
1	11
2	20
3	15
4	10
5	23
6	6
7	15
8	14
9	17
10	15
11	10
12	12
13	6
14	5
15	7
16	8
17	5
18	11
19	15
20	11
21	7
22	4
23	9
24	1
25	4
26	4
27	6
28	5
29	4
30	6
31	6
32	5
33	2
34	9
35	4
36	11
37	2
38	5
39	2
40	5
41	6
42	4
43	1
44	5
45+	79

LOS (yrs)	Three Bedroom Apartments at Mary Ellen McCormack	Frequency
1	xxx	3
2	xxx	3
3	xxx	3
4	x	1
5	xxx	3
6	xx	2
7	xxxxxxx	7
8	xxxxx	5
9	xxxx	4
10	xx	2
11	xxxxx	5
12	xxxx	4
13	xx	2
14	xxxx	4
15	xxx	3
16	xxxxxxxx	8
17	xx	2
18	xx	2
19	xxxxx	5
20	xxx	3
21	x	1
22	x	1
23	x	1
24	x	1
25		0
26	xxxxxx	6
27	xxxxx	5
28	xxxx	4
29	xx	2
30		0
31		0
32	xxxxxx	6
33	xxxx	4
34	xxx	3
35		0
36	xxx	3
37	xx	2
38	x	1
39	x	1
40	x	[illegible]
41	xx	2
42	xx	2
43		0
44	xx	2
45+	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	30

One Bedroom Apartments at Charlestown

LOS (yrs)		Frequency
1	xxxxxxxxxxxxxxxxxxx	19
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	35
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	40
4	xxxxxxxxxxxxxxxx	16
5	xxx	3
6	xxxxxxxx	8
7	xxxx	4
8	xxxxx	5
9	xxxxxxx	7
10	xxxxxx	6
11	xxxxxxxxxxxxx	13
12	xxxxxxxxxxx	11
13	xxxxxxx	7
14	xxxxxxx	7
15	xxx	3
16	xxxx	4
17	xxxxxxx	7
18	xxxxxxxx	8
19	xxxxx	5
20	x	1
21	xxxx	4
22	xxxxxxx	7
23	x	1
24	xxxxxx	6
25	xxx	3
26	x	1
27	xx	2
28	xx	2
29	x	1
30	x	1
31	xxxxxxx	7
32	x	1
33	xxxx	4
34	x	1
35	xxxx	4
36		0
37	xxx	3
38	xx	2
39	xxxxx	5
40		0
41	xx	2
42		0
43		0
44		0
45+	xxxxxx	6

LOS (yrs)	Two Bedroom Apartments at Charlestown	Frequency
1	xxxxxxxxxxxxxxxxxxx	19
2	xxxxxxxxxxxxxxxxxx	18
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	40
4	xxxxxxxx	8
5	xxxxxxxx	8
6	xxxxxxxxxxxxx	13
7	xxxxxx	6
8	xxxx	4
9	xxxxxx	6
10	xxxxxxxxx	9
11	xxxxxxxxxxxxxx	14
12	xxx	3
13	xxxxxxx	7
14	xxxxxx	6
15	xxxxxxxx	8
16	xxxxxxxxxx	10
17	xxxxxx	6
18	xxxxx	5
19	xx	2
20	xxx	3
21	xxx	3
22	xxxx	4
23		0
24	xx	2
25	xx	2
26	xxxxx	5
27	x	1
28	xxx	3
29	xxx	3
30	xxxx	4
31	xxxxxxxxxxxxxx	14
32	xx	2
33	xxxxxx	6
34	xxxx	4
35	xxxxx	5
36	x	1
37	xxxx	4
38	xx	2
39	xxxxxxxx	8
40	x	1
41	xxx	3
42	x	1
43	xx	2
44		0
45+	xxxxxxxx	8

LOS (yrs)	Three Bedroom Apartments at Charlestown	Frequency
1	xxxxx	1
2	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	9
3	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	9
4	xxxxxxxxxxxxxxx	3
5	xxxxxxxxxxxxxxx	3
6	xxxxxxxxxxxxxxx	3
7	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	6
8	xxxxxxxxxxxxxxx	3
9	xxxxxxxxxxxxxxxxxxxx	4
10	xxxxxxxxxxxxxxxxxxxxxxxxx	5
11	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	8
12	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	8
13	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	6
14	xxxxxxxxxx	2
15	xxxxxxxxxxxxxxx	3
16	xxxxxxxxxxxxxxx	3
17	xxxxxxxxxxxxxxx	3
18	xxxxxxxxxxxxxxx	3
19	xxxxxxxxxxxxxxx	3
20	xxxxxxxxxxxxxxxxxxxx	4
21	xxxxxxxxxx	2
22	xxxxx	1
23	xxxxxxxxxx	2
24	xxxxxxxxxxxxxxx	3
25	xxxxxxxxxxxxxxxxxxxxxxxxx	5
26	xxxxxxxxxx	2
27	xxxxx	1
28	xxxxx	1
29	xxxxxxxxxx	2
30	xxxxxxxxxxxxxxxxxxxxxxxxx	5
31	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	7
32	xxxxxxxxxxxxxxx	3
33	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	6
34	xxxxx	1
35	xxxxxxxxxx	2
36	xxxxxxxxxx	2
37	xxxxxxxxxxxxxxx	3
38		0
39	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	9
40		0
41	xxxxxxxxxxxxxxxxxxxxxxxxxxxxxx	6
42	xxxxx	1
43		0
44	xxxxx	1
45+	xxxxxxxxxx	2

LOS (yrs)	Four Bedroom Apartments at Charlestown	Frequency
1	XXXXXXXXXXXXXXX	3
2	XXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXXX	8
3	XXXXXXXXXXXXXXX	3
4	XXXXXXXXXXXXXXX	3
5	XXXXX	1
6	XXXXX	1
7	XXXXXXXXXXXXXXXXXXXX	4
8		0
9		0
10	XXXXX	1
11		0
12		0
13		0
14	XXXXX	1
15		0
16	XXXXXXXXXX	2
17	XXXXX	1
18	XXXXXXXXXXXXXXXXXXXX	4
19	XXXXX	1
20	XXXXX	1
21		0
22	XXXXX	1
23	XXXXX	1
24	XXXXX	1
25	XXXXXXXXXX	2
26		0
27		0
28		0
29	XXXXX	1
30	XXXXX	1
31	XXXXX	1
32		0
33	XXXXX	1
34		0
35		0
36	XXXXX	1
37	XXXXX	1
38		0
39	XXXXXXXXXXXXXXXXXXXX	4
40		0
41	XXXXX	1
42		0
43		0
44		0
45+	XXXXX	1

CHAPTER IV

TENANT ASSIGNMENT MODELS

In this chapter, we begin to develop models which describe major features of the tenant assignment systems discussed in Chapter II. We will begin with an overview of a generic tenant assignment system for a single project, and present some mathematical results which will prove useful in our later work. Following this, we will examine various aspects of single project tenant assignment systems; gradually we will incorporate dropout and prioritized assignment structures. By the end of this chapter we will have developed applicable models for single project assignment policies.

4.1 A General Assignment System

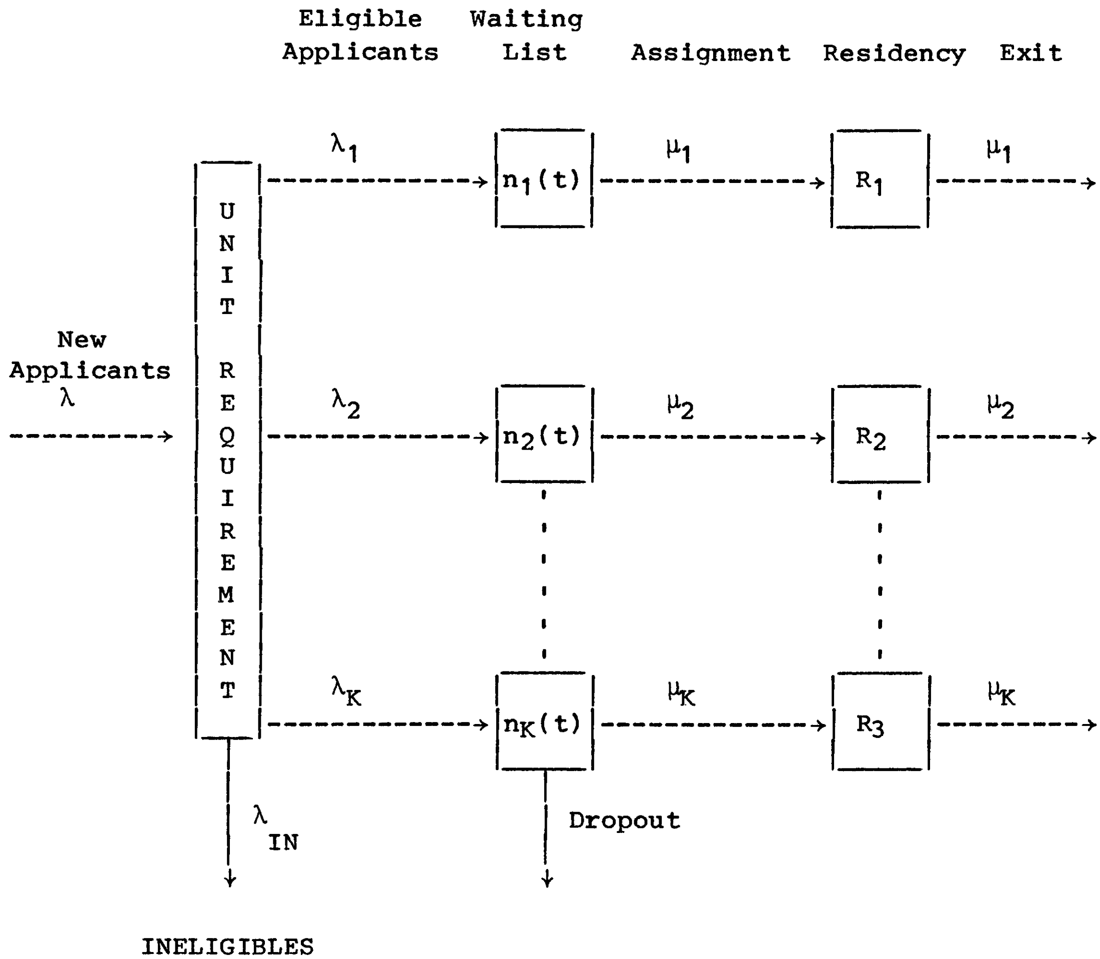
A simplified assignment procedure is diagramed below in Fig. 4.1. New applicants are assumed to arrive at a housing project in accordance with a Poisson process with rate λ. In any time period of length ℓ, the probability that exactly k new appliants arrive is assumed to equal

$$\Pr\{k \text{ applicants in period of length } \ell\} = \frac{(\lambda\ell)^k e^{-\lambda\ell}}{k!} \quad \ell > 0,\ \lambda > 0,\ k = 0,1,\ldots \qquad (4.1)$$

Upon application, households are assigned a unit requirement, or they are deemed ineligible. Unit requirments typically refer to apartment sizes, but they could also include special features such as aids to the handicapped or mobility impaired. All decisions regarding unit requirements and eligibility are assumed independent, thus the

Figure 4.1

A Tenant Assignment System



$$\lambda = \sum_{i=1}^{K} \lambda_i + \lambda_{IN}$$

effective arrival rate for new households with unit requirement i equals λ_i, and $\sum_{i=1}^{K} \lambda_i + \lambda_{IN} = \lambda$, where λ_{IN} equals the arrival rate for ineligible applicants, and K is the number of unit types.

Eligible households then join a waiting list for their unit requirement. At time t, the number of households waiting for type i units equals $n_i(t)$. Waiting lists may function as a simple accounting of those in queue for housing; they can also be prioritized in several ways. For example, "emergency" households may receive priority over standard applicants, while social goals such as racial integration may grant priorities to specific households. We will always assume that within unit requirements and priority structures, households are assigned to public housing on a first come, first served basis.

One more assumption is key regarding waiting lists. We will assume that waiting lists are never empty, that is, $n_i(t)>0 \ \forall i,t$. This assumption is almost always true empirically, and it has the following implication: the rate at which tenants are assigned equals the rate at which tenants leave the project; more succintly, the assignment rate equals the moveout rate. This result will greatly simplify our analysis.

Two things can happen to a household once it has been placed on a a waiting list; the household is eventually assigned to an apartment, or the household eventually drops out of the system. Dropout is an important feature of tenant assignment systems, as typical waiting times are sufficiently long to enable many of those waiting for public housing to find housing elsewhere. It is often the case that households are more likely to drop out than to receive an assignment. In our work, we will assume that the rate at which households drop out is proportional to the number of households waiting for assignments. Though this seems like a simple assumption, it will complicate our modeling effort. This complication is necessary, however, if we are to develop realistic tenant assignment models.

We earlier established that due to non-empty waiting lists, the rate at which tenants are assigned equals the rate at which tenants move out of the housing project. In Chapter 3, we presented evidence which suggested that the amount of time individual households live in public housing is approximately exponentially distributed. This being the case, the time between successive moveouts will also be exponentially distributed. If the project contains m units, and households live in public housing apartments for a mean of R time periods, then our assumptions imply that the length of time between succesive moveouts will be exponentially distributed with mean R/m. This in turn implies that the moveout process is Poisson with rate $\mu=m/R$. Finally, due to the equivalence of assignment and moveout processes, we see that the actual assignment process is Poisson with rate μ.

It should be mentioned that in order for the moveout process to be considered as Poisson, it is not necessary for individual household occupany times to be exponentially distributed. If the number m of apartments is sufficiently large, then the moveout process will approach a Poisson process, irrespective of the underlying distribution of household occupancy times. This is due to the fact that the pooled output from a large number of "renewal processes" approaches a Poisson process as the number of individual processes in the pool becomes large (see Cox (1970, p.77-79)). In our case, the individual processes are household specific moveouts, while the pooled process consists of the moveouts generated by the project as a whole.

We summarize our main assumptions as:

1. New applicants for type i units arrive in accordance with a Poisson process at rate λ_i.

2. The waiting list for type i units is never empty; that is, $n_i(t)>0 \ \forall \ i,t$. This implies that the assignment rate equals the moveout rate.

3. Households waiting for type i units drop out of the system at a rate proportional to the number of households in queue; if N such households are waiting, the dropout rate is assumed to equal $N\delta$, where δ is the household specific dropout rate.

4. Households in type i units reside in projects for exponentially distributed lengths of time. At 100% occupancy (which is always the case by assumption), a mean residency of length R in a project with m units implies that the lengths of time between successive moveouts are exponentially distributed with mean R/m. Equivalently, the moveout and tenant assignment processes are Poisson with rate $\mu=m/R$.

In studying tenant assignment systems, we will be interested in describing how the system looks to a newly arriving eligible applicant. In particular, we will try to answer the following questions:

1. Suppose a newly arriving eligible applicant finds N households waiting for housing assignments. How long will our household have to wait for a housing assignment?

2. While our household is waiting for an assignment, how many of the N households originally waiting will also be assigned? How many will drop out?

The answers to thse seemingly simple questions are usually quite difficult to derive, yet when we know this information, we can say a lot about tenant assignment. First of all, the estimation of waiting times for public housing should be basic to any housing authority . When potential public housing residents try to decide whether or not to remain in queue for housing, let alone choose which projects to live in, the amount of time required to wait could be a major factor impacting the decision. Thus, using models to be developed, new applicants can be informed of how long they can expect to wait for a housing assignment under the relevant tenant assignment policy. That the provision of this information will enable prospective tenants to make better decisions is sufficient to warrant our modeling effort!

Aside from this day to day application, our models will supply housing planners with important information. For example, one will be able to determine the length of time necessary to process all households waiting as of some given time (typically the end of a month), and the numbers of those waiting who will ultimately be housed or drop out. In addition, planners will be able to study the effects of alternative tenant assignment policies on the demographic compositions of projects, and determine how much time is necessary to achieve various social goals such as racial integration or income mixing. Another useful feature our models will provide is the ability to compare and contrast alternative tenant assignment policies such as those discussed in Chapter 2.

Throughout this chapter, we will develop models which predict waiting times and allocational quantities (such as the number of assignments and the number of dropouts) for various tenant assignment schemes. Most of these models will make use of some mathematical results associated with so called birth and death processs; these results will be summarized for later application. Following this, we will engage in the details of models for tenant assignment. We shall derive the necessary mathematical results, discuss statistical issues associated with using the models, and present numerical examples where relevant. Our work is reviewed at the end of the chapter.

4.2 Birth and Death Processes

Imagine a system characterized by a random variable which at any time can take on only non-negative integral values. One example of such a system is the number of households waiting for housing assignments at a given time. Denoting our random variable by $X(t)$, we say that the system occupies state n at time t if $X(t)=n$. Continuing with the tenant assignment example, the system would be in state n whenever n households are waiting for housing assignments.

Suppose we know that at some time t, the system is in state n, that is, $X(t)=n$. Our system corresponds to a birth and death process if the only possible states the system can next occupy are states n+1 (a birth), n-1 (a death), or n (a return). For tenant assignments, a birth corresponds to a new addition to the waiting list; a death corresponds to either a tenant assignment or a dropout, and a return corresponds to no change.

To make our process operational, we make the following two assumptions:

1. Occupancy Times are Exponentially Distributed

Given that the system enters state n at time t, the length of time the system will remain in state n, τ_n, is an exponentially distributed random variable with mean $\bar{\tau}_n$. Note that the length of time spent in state n, while dependent on n, is independent of time - τ_n does not depend on t.

2. <u>State Dependent Virtual Transitions</u>

Given the system enters state n at time t, the probability that the system next occupies state j (at time t + τ_n) is given by

$$\text{Prob}\{n \rightarrow j\} = \begin{cases} q_n & j = n - 1 \\ r_n & j = n \\ p_n & j = n + 1 \\ 0 & \text{all other values of } j \end{cases} \tag{4.2}$$

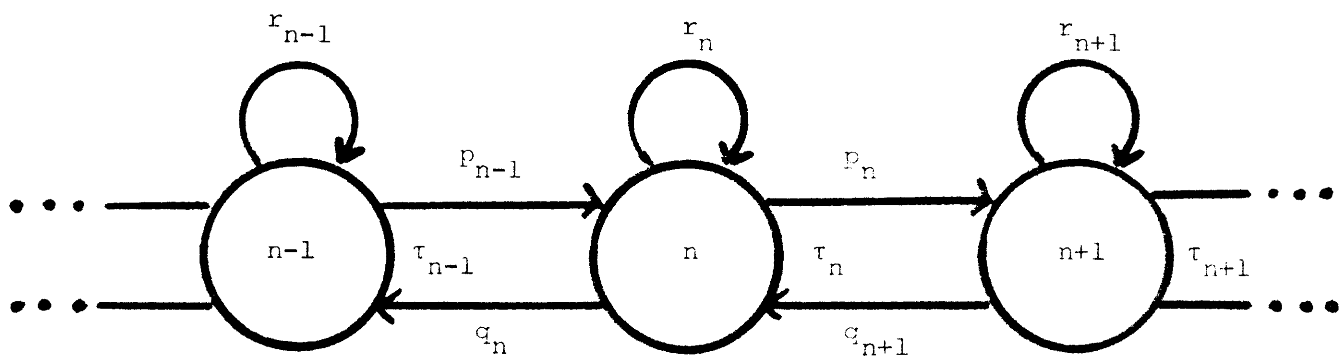
Also, we insist that $p_n + r_n + q_n = 1$ for all states n.

Equation (4.2) implies that the only state to state transitions allowed correspond to births, deaths, and returns. Note that the transition probabilities (p_n, r_n, q_n) are state dependent, but independent of time. Also note that by including a return probability r_n, we allow for virtual transitions - the process, upon leaving state n, returns to state n at time t + τ_n with probability r_n.

Assumptions 1 and 2 lead to the following description of our process: having entered state n at time t, the system remains in state n for τ_n time units, where τ_n is an exponentially distributed random variable with mean $\bar{\tau}_n$. At time t + τ_n, the system moves to state n-1 with probability q_n, returns to state n with probability r_n, or moves to state n+1 with probability p_n. This processed is summarized in the state transition diagram shown in Fig. 4.2.

Figure 4.2

State Transition Diagram for the Birth and Death Process



In our work, we will typically be interested in the random variable w_n, the amount of time necessary to first enter state 0 given that the system starts in state n. The variable w_n will describe the length of the time necessary to process n households on a waiting list, where a household is processed if it is assigned or if it drops out.

We will most often wish to compute the mean and variance of w_n. To do this, we make use of the fact that both the mean $E(w_n)$ and the second moment $E(w_n^2)$ can be found as the solution M_n of the difference equation.

$$M_n = p_n M_{n+1} + r_n M_n + q_n M_{n-1} + g_n \tag{4.3}$$

To obtain $E(w_n)$, one sets

$$g_n = \bar{\tau}_n \tag{4.4}$$

and solves (4.3) for M_n.

The interpretation of this is straightforward. Having entered state n, the process spends $\bar{\tau}_n$ units of time, on average, before changing states. With probabilities p_n, r_n and q_n, the process jumps to state n+1, n, or n-1. The expected times to reach state 0 from each of these states are $E(w_{n+1})$, $E(w_n)$ and $E(w_{n-1})$, respectively. Thus, the expected amount of time required to reach state 0 from state n equals the sum of the expected time spent in state n, plus the probabilistically weighted sum of the mean times to reach state 0 from each of states n+1, n, and n-1.

To obtain $E(w_n^2)$, one sets

$$g_n = E(\tau_n^2) + 2\,[p_n \bar{\tau}_{n+1} E(w_{n+1}) + r_n \bar{\tau}_n E(w_n) + q_n \bar{\tau}_{n-1} E(w_{n-1})] \tag{4.5}$$

and solves (4.3) for M_n. This result is not obvious; for its derivation see Howard (1971 b; p. 735). The approach is useful; in every case of interest to us, equation (4.3) can be solved.

To see this, rewrite the left hand side of (4.3) as $(p_n + r_n + q_n)M_n$, and re-express the equation as

$$p_n(M_{n+1} - M_n) - q_n(M_n - M_{n-1}) = -g_n \tag{4.6}$$

Next, define U_n to be the first difference of M_n, that is

$$U_n = M_n - M_{n-1} \tag{4.7}$$

Recall our definition for w_n; it follows that $w_0 = 0$. Thus, $M_0 = 0$ (since $E(w_0) = E(w_0{}^2) = 0$), and we have the relationship

$$M_n = \sum_{i=1}^{n} U_i \tag{4.8}$$

It will sometimes be the cause that $p_n = 0 \ \forall n$. When this is true, (4.6) may be written as

$$U_n = \frac{g_n}{q_n} \tag{4.9}$$

and we find using (4.8) that

$$M_n = \sum_{i=1}^{n} \frac{g_i}{q_i} \tag{4.10}$$

for this case.

When $p_n \neq 0 \ \forall n$ it will be true that $p_n > 0 \ \forall n$.

Dividing through (4.6) by p_n we obtain

$$U_{n+1} - \frac{q_n}{p_n} U_n = - \frac{g_n}{p_n} \tag{4.11}$$

This is a first order, linear difference equation with non-constant coefficients; it has the solution (Levy and Lessman, 1961; p. 153)

$$U_n = \prod_{i=1}^{n-1} \frac{q_i}{p_i} \left[U_1 - \sum_{k=1}^{n-1} \left(\frac{g_k}{p_k} \Big/ \prod_{j=1}^{k} \frac{q_j}{p_j}\right)\right] \tag{4.12}$$

For all of the models we will consider, the following conditions will always hold:

1) $\lim_{n\to\infty} \prod_{i=1}^{n} \frac{q_i}{p_i} = \infty$

2) $\lim_{n\to\infty} U_n = K \geq 0$

These two conditions enable the initial term U_1 to be expressed as

$$U_1 = \sum_{i=1}^{\infty} \left(\frac{g_i}{p_i} \Big/ \prod_{j=1}^{i} \frac{q_j}{p_j}\right) \tag{4.13}$$

Using this expression in (4.12), we find that

$$U_n = \sum_{i=n}^{\infty} \left(\frac{g_i}{p_i} \Big/ \prod_{j=n}^{i} \frac{q_j}{p_j}\right) \tag{4.14}$$

Finally, using (4.8) we have our general result

$$M_n = \sum_{k=1}^{n} \sum_{i=k}^{\infty} \left(\frac{g_i}{p_i} \Big/ \prod_{j=k}^{i} \frac{q_j}{p_j}\right) \tag{4.15}$$

Our approach will provide us with the first two moments of w_n irrespective of the complexity of the transition probabilities (p_n, r_n, q_n). This approach will always work provided the stated conditions

$$\left(\lim_{n\to\infty} \prod_{i=1}^{n} \frac{q_i}{p_i} = \infty \;,\; \lim_{n\to\infty} U_n = K \geq 0\right)$$

hold. Having presented the necessary results from birth and death processes, we can now return to the problems of modeling tenant assignment systems.

4.3 Tenant Assignment Models

All of the models we will develop will be variations on the following scenario: a new household (or "test applicant") applies for public housing and is found eligible. Upon joining the waiting list, the applicant finds N households already waiting for housing assignments. Each of these N households will either receive a unit or drop out of the system; once a household is assigned or drops out, the household is said to have been processed. All N households must be processed before our new applicant can be housed. The time necessary to process the N households found on the waiting list will be denoted by w_N.

The major purpose of our modeling effort is to predict the length of time households will have to wait until they receive public housing assignments. Thus, we will assume throughout that our new applicant will not drop out, but will wait whatever amount of time is necessary to receive an assignment. The length of time our applicant must wait from the time the N^{th} household originally present leaves the waiting list until the applicant is assigned an apartment will be denoted by w^*. The total amount of time our new applicant must wait from arrival until assignment, w_N^*, is thus given by

$$w_N^* = w_N + w^* \tag{4.16}$$

Most of our attention will focus on obtaining the mean and variance of w_N^* and related quantities.

4.4 Single Project, No Dropout, No Priorities

We begin our analysis with a simple case. Suppose that there is a single housing project, filled to capacity, consisting of m identical units. A new eligible applicant chooses this project, and finds $N > 0$ households already waiting for housing assignments. All households are willing to wait as long as necessary to receive assignments (i.e. there is no dropout), and no new applicants will be housed before our test applicant is assigned (i.e. there are no priorities).

Based on our empirical results from Chapter 3, we can reasonably assume that the length of time any household resides in the project (in the absence of household specific information) is exponentially distributed. If the mean length of project residency is given by R, then the lengths of time between successive household departures from the project will be independent and exponentially distributed with mean $\mu^{-1} = R/m$, as shown in Figure 4.3. This system and its attendant assumptions are summarized in Figure 4.4.

Under these assumptions, it is easy to show that w_N, the time to assign the N households found on the waiting list, follows the N^{th} order Erlang distribution with the density function

$$f_{w_N}(w) = \frac{\mu^N w^{N-1} e^{-\mu w}}{(N-1)!} \qquad \begin{array}{l} w > 0 \\ \mu > 0 \\ N=1,\ 2,\ \ldots \end{array} \qquad (4.17)$$

Also, the additional time our new applicant must wait, w^*, equals the time between two successive moveouts. The variable w^* is thus exponentially distributed as shown in Figure 4.3:

$$f_{w^*}(w) = \mu e^{-\mu w} \qquad \begin{array}{l} w > 0 \\ \mu > 0 \end{array} \qquad (4.18)$$

From (4.17) and (4.18) we can easily obtain the mean and variance of the waiting time for our test applicant:

$$E(w_N{}^*) = \frac{N + 1}{\mu} \qquad (4.19)$$

Figure 4.3

Distribution of Time Between Successive Moveouts

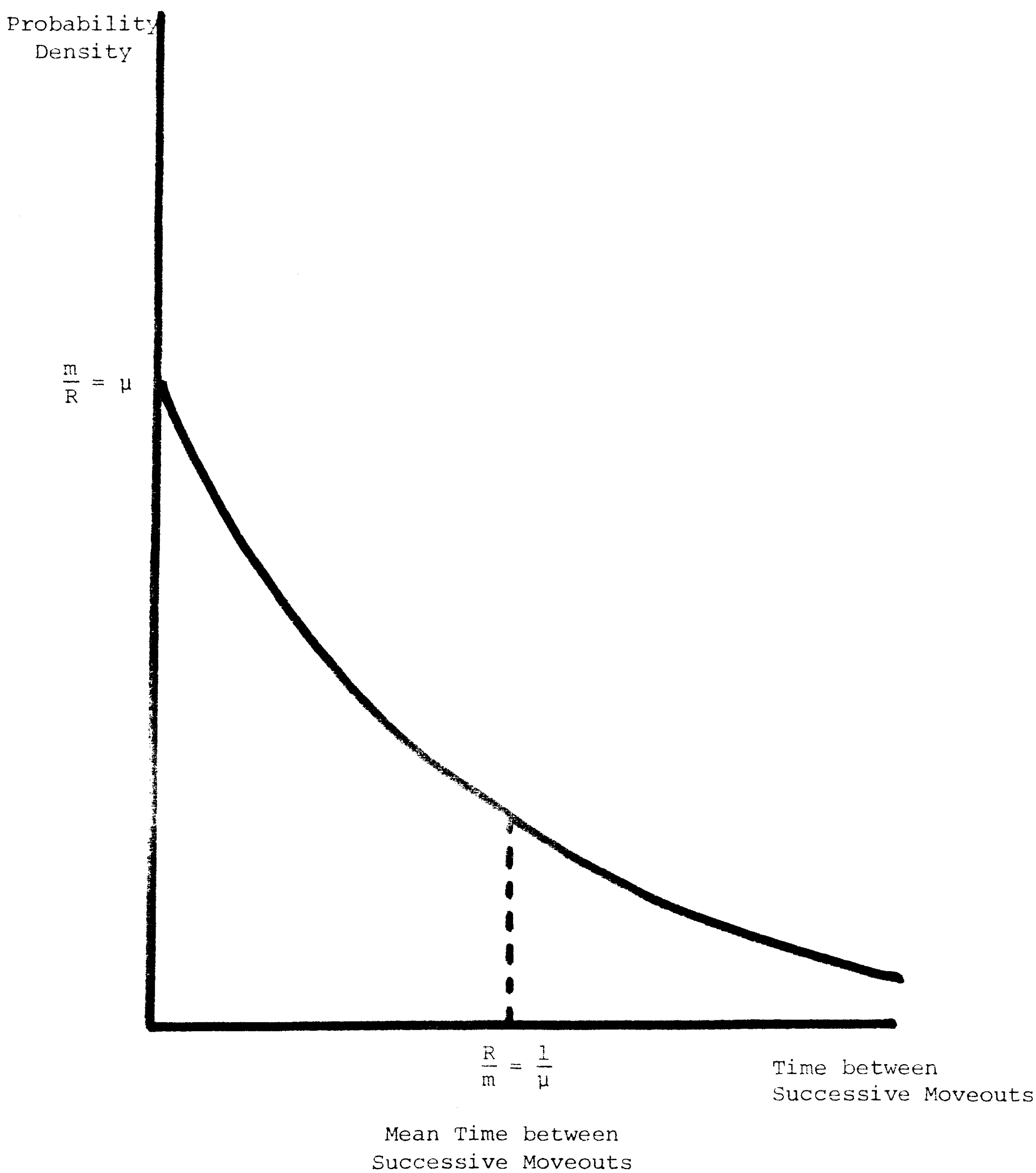
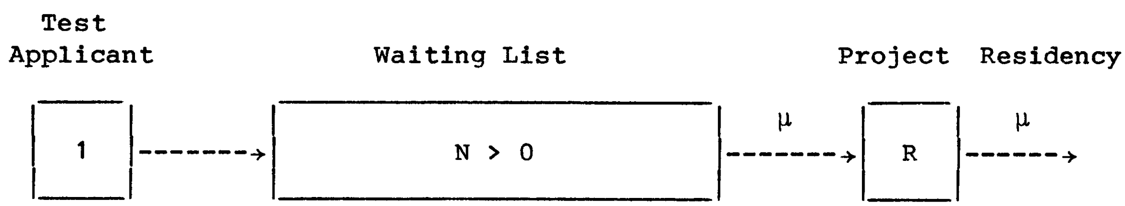


Figure 4.4

The Single Project, No Dropout, No Priorities System



Assumptions

1) N > 0 households are found waiting for housing assignments by a newly arriving test applicant.

2) Households are assigned in order of application.

3) No households drop out, and no new applicants are assigned prior to the test applicant.

4) The lengths of time between successive moveouts are exponentially distributed with mean $\mu^{-1}=R/m$. Equivalently, households are assigned to the project from the waiting list according to a Poisson process with rate μ.

$$\mathrm{var}(w_N^*) = \frac{N + 1}{\mu^2} \tag{4.20}$$

Thus, a newly arriving applicant could expect to wait $(N + 1)/\mu = (N + 1)R/m$ time units until assignment.

While these results are easily obtained directly, it is useful to derive them using the birth-and-death process described earlier. For this system, we have:

1) τ_n is exponentially distributed with mean μ^{-1}; $n=1,2,3, \ldots$

2) $p_n = 0$, $r_n = 0$, $q_n = 1$; $n=1,2,3, \ldots$

From the discussion following equation (4.9), we see that

$$M_n = \sum_{i=1}^{n} \frac{q_i}{g_i} \quad \text{(since } p_n = 0 \text{ for all n).}$$

To obtain $E(w_N)$, we set $g_n = \bar{\tau}_n = \frac{1}{\mu}$ and $q_i=1$ to obtain

$$E(w_N) = \sum_{i=1}^{N} \mu^{-1} = \frac{N}{\mu} \tag{4.21}$$

Similarly, to obtain $E(w_N^2)$, we set

$$g_n = \frac{2}{\mu^2} + 2\,\frac{1}{\mu}\,\frac{n-1}{\mu} = \frac{2n}{\mu^2} \tag{4.22}$$

and thus

$$E(w_N^2) = \sum_{i=1}^{N} \frac{2i}{\mu^2} / 1 = \frac{N(N+1)}{\mu^2} \tag{4.23}$$

yielding

$$\begin{aligned} \mathrm{var}(w_N) &= E(w_N^2) - [E(w_N)]^2 \\ &= \frac{N(N+1)}{\mu^2} - \left(\frac{N}{\mu}\right)^2 = \frac{N}{\mu^2} \end{aligned} \tag{4.24}$$

Since $E(w^*)=\mu^{-1}$, $\mathrm{var}(w^*)=\mu^{-2}$, and w_N and w^* are independent we finally have

$$E(w_N^*) = \frac{N}{\mu} + \frac{1}{\mu} = \frac{N+1}{\mu} \tag{4.25}$$

$$\text{var}(w_N^*) = \frac{N}{\mu^2} + \frac{1}{\mu^2} = \frac{N+1}{\mu^2} \tag{4.26}$$

These results agree with our earlier results obtained directly.

Formula (4.25) corresponds to a method often used by housing authorities to calculate waiting times. The parameter μ is set equal to the annual moveout rate from the apartment in question, and N is taken as the length of the current waiting list. Of course, this formula is simplistic in that:

(i) Dropout is not considered.

(ii) Priorities are ignored.

(iii) Tenant choice is dismissed - households must accept an offered unit.

We will address these shortcomings in subsequent models.

4.4.1 Statistical Issues

To use the model outlined in this section, one needs to estimate the unit turnover rate μ. Since we have reasonably assumed that the size of the waiting list is always positive, and the length of time between moveouts is exponentially distributed, it follows that the distribution of the number of moveouts (which equals the number of housing assignments) that occurs in a time period of length ℓ is Poisson with parameter $\mu\ell$; that is

$$\Pr(\text{number of moveouts in a period of length } \ell = k) = \frac{(\mu\ell)^k e^{-\mu\ell}}{k!} \tag{4.27}$$

$$\mu > 0$$
$$\ell > 0$$
$$k = 0, 1, 2, \ldots$$

The expected number of moveouts in a period of length ℓ simply equals $\mu\ell$. Thus, a simple (and good) estimate of μ is the empirical moveout rate. If M moveouts are observed during a period of length ℓ, then one estimates μ using

$$\hat{\mu} = M/\ell \tag{4.28}$$

The estimator $\hat{\mu}$ has other appealing properties in addition to its simplicity. First, the estimator is unbiased, that is, $E(\hat{\mu}) = \mu$. Secondly, the estimator is very stable for large time periods ℓ; this follows from the easily proven fact that $\text{var}(\hat{\mu}) = \mu/\ell$. Typically, one might set ℓ equal to one year, and update $\hat{\mu}$ on an annual basis; more frequent re-estimates are of course possible.

4.4.2 An Example

Suppose that the annual moveout rate at a project equals 20 households per year. Figures 4.5 and 4.6 report the values of $E(w_N^*)$ and $\text{var}(w_N^*)$ for this example as functions of N, the size of the waiting list encountered by our test applicant. We will continue to build upon this example as our model becomes increasingly complex.

4.5 Single Project, Dropout, No Priorities

4.5.1 The Incorporation of Dropout

Our first improvement on the pure assignment process discussed is the incorporation of dropout. We maintain all previous assumptions from before, and in addition we postulate that if n households are waiting for housing assignments at time t, then the probability that one of these households drops out of the system in the interval $(t, t+\Delta t)$ equals $n\delta\Delta t$. We refer to δ as the household dropout rate.

Figure 4.5

Mean Waiting Time in a

Single Project, No Dropout,

No Priority System

$\mu=20$

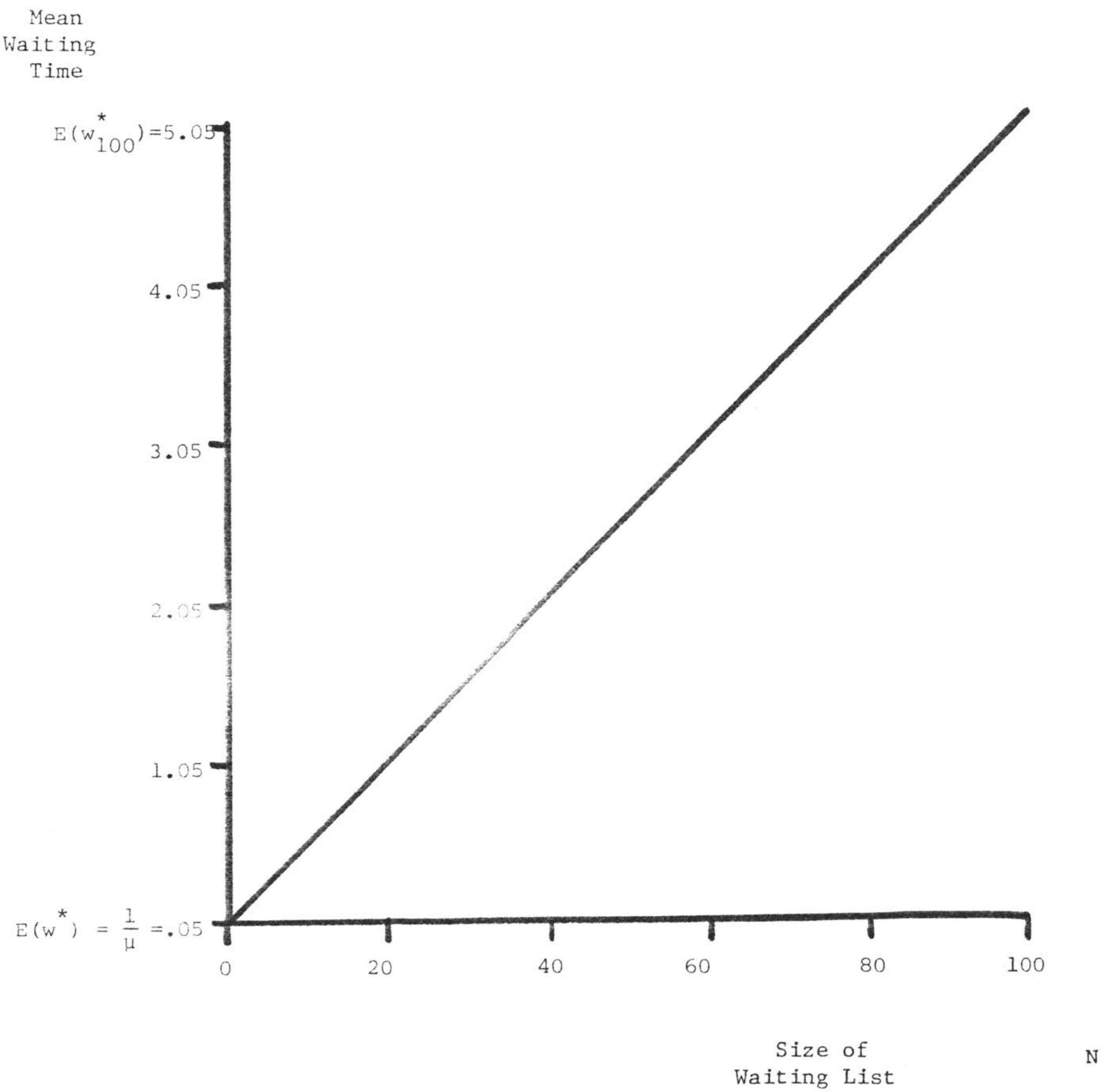
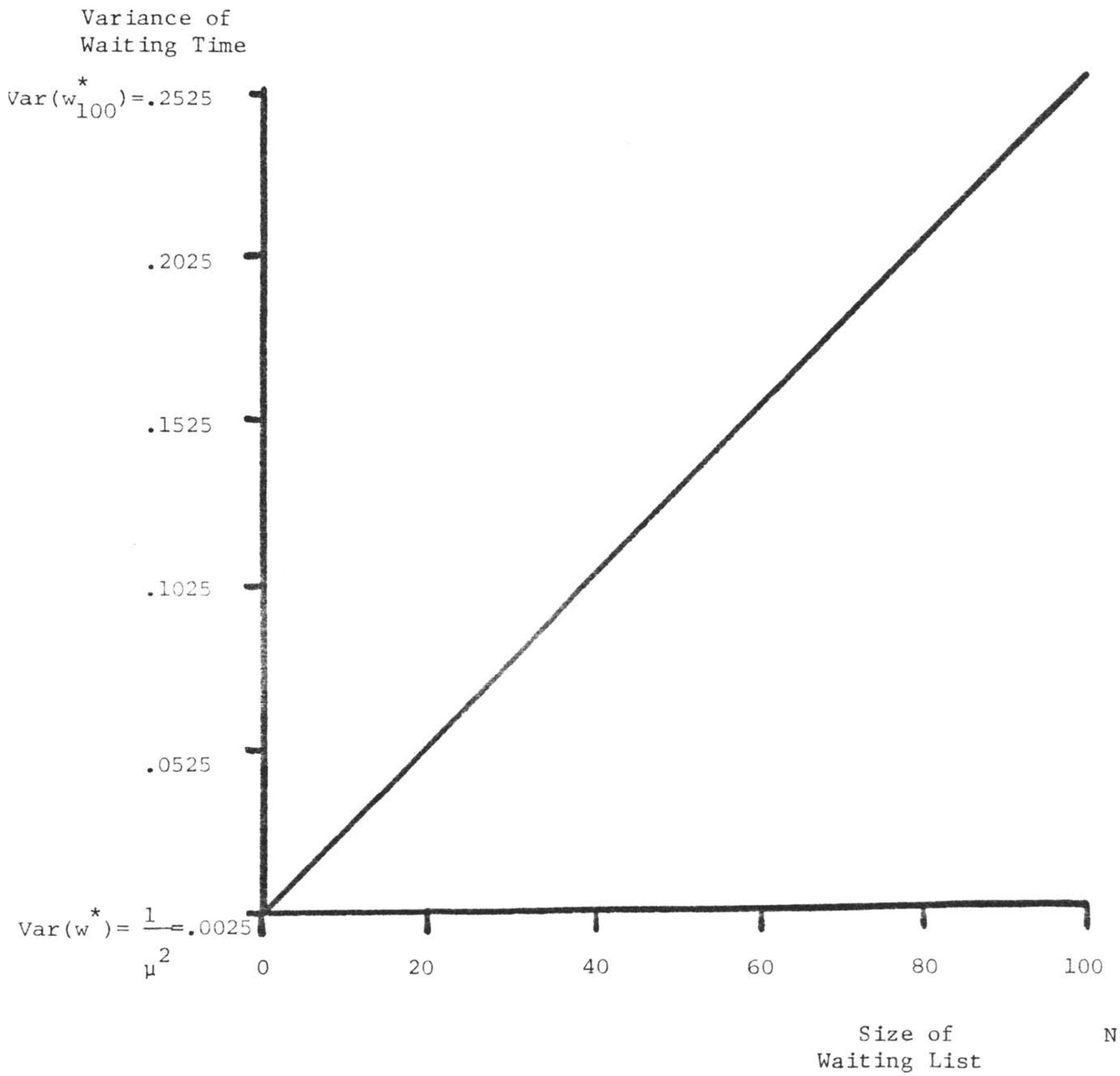


Figure 4.6

Variance of Waiting Time
in a Single Project, No
Dropout, No Priority System

μ=20



This system and its assumptions are summarized in Figure 4.7.

We will first derive the moments of w_N, the time necessary for the N households ahead of the test applicant to be housed or drop out. Given that n households are waiting, the expected time until a household either drops out or is assigned is given by $(n^{\delta}+\mu)^{-1}$. Thus, the expected time necessary for all N households to leave the waiting list is given by

$$E(w_N) = \sum_{n=1}^{N} \frac{1}{n^{\delta}+\mu} \tag{4.29}$$

Similarly, the variance of the time until a household either drops out or is housed is given by $(n^{\delta}+\mu)^{-2}$. Since all processing times are independent, we have for the variance of the time necessary to process all N households found waiting

$$\text{var}(w_N) = \sum_{n=1}^{N} \frac{1}{(n^{\delta}+\mu)^2} \tag{4.30}$$

Once all N households found waiting have been processed, our test applicant must wait for the next moveout to occur before assignment takes place. The amount of time necessary, w^*, is exponentially distributed with mean, μ^{-1}, thus

$$E(w^*) = \frac{1}{\mu} \tag{4.31}$$

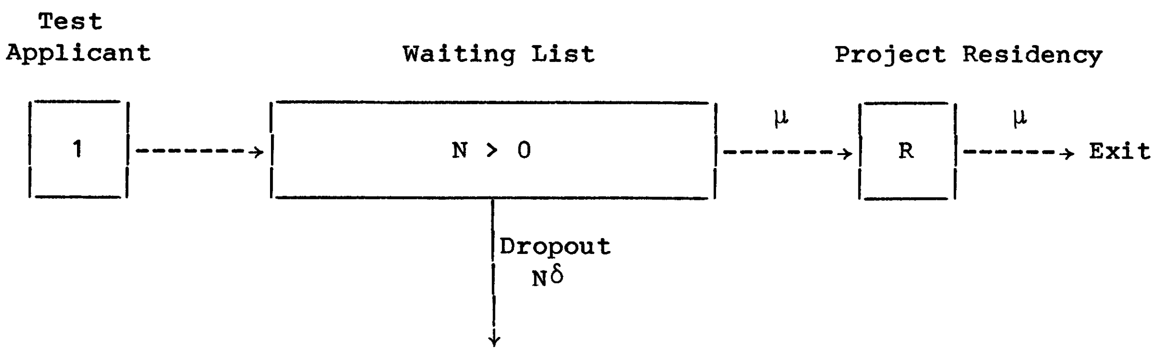
$$\text{var}(w^*) = \frac{1}{\mu^2} \tag{4.32}$$

Combining our results we obtain

$$E(w_N^*) = \frac{1}{\mu} + \sum_{n=1}^{N} \frac{1}{n^{\delta}+\mu} \tag{4.33}$$

<u>Figure 4.7</u>

<u>The Single Project, No Priorities System With Dropout</u>



<u>Assumptions</u>

1) $N > 0$ households are found waiting for housing assignments by a newly arriving test applicant.

2) Households are assigned in order of application.

3) Dropouts occur at rate $n\delta$ when n households are waiting for housing assignments; the test applicant will not drop out with certainty.

4) The lengths of time between successive moveouts are exponentially distributed with mean $\mu^{-1}=R/m$. Equivalently, households are assigned to the project from the waiting list according to a Poisson process with rate μ.

$$\mathrm{var}(w_N{*}) = \frac{1}{\mu^2} + \sum_{n=1}^{N} \frac{1}{(n\delta+\mu)^2} \tag{4.34}$$

as the first two moments for the amount of time the test applicant must wait to receive a housing assignment.

We can also derive these results using our general method based on the birth-and-death process. As was the case with the pure assignment process, we set $p_n=r_n=0$ and $q_n=1$. However, the state occupancy time τ_n is now exponentially distributed with mean $(n\delta+\mu)^{-1}$. Since $p_n=0$, we use the result $M_n = \sum_{i=1}^{n} g_i/q_i$.

To obtain $E(w_N)$, we set $g_n=\tau_n=(n\delta+\mu)^{-1}$ which yields

$$E(w_N) = \sum_{i=1}^{N} \frac{\frac{1}{i\delta+\mu}}{1} \tag{4.35}$$

Similarly, to obtain $E(w_N^2)$, we set

$$g_n = \frac{2}{(n\delta+\mu)^2} + 2\,\frac{1}{(n-1)\delta+\mu} \sum_{j=1}^{n-1} \frac{1}{j\delta+\mu} \tag{4.36}$$

yielding

$$E(w_N^{\,2}) = \sum_{n=1}^{N} \frac{2}{(n\delta+\mu)^2} + 2 \sum_{n=1}^{N} \frac{1}{(n-1)\delta+\mu} \sum_{j=1}^{n-1} \frac{1}{j\delta+\mu} \tag{4.37}$$

From (4.36) and (4.37), the formula $\mathrm{var}(w_N)=E(w_N^2)-E(w_N)^2$ yields equation (4.30) after some algebraic manipulation. Equations (4.33) and (4.34) then follow as has already been shown.

While the formulas for $E(w_N)$ and $\mathrm{var}(w_N)$ are not terribly complicated, we can obtain a simple approximation that eliminates the summations involved. Let $f(n)=(n\delta+\mu)^{-1}$. Since $f(n)$ is strictly decreasing, we know by the mean value theorum for integrals

(Purcell (1972), p.275) that there exists some number u such that:

i) $n < u < n+1$

ii) $f(n) > f(u) > f(n+1)$

iii) $\int_{n}^{n+1} f(x)dx = f(u)$

Conditions (ii) and (iii) together imply the inequality

$$f(n) > \int_{n}^{n+1} f(x)dx > f(n+1) \tag{4.38}$$

and thus

$$\sum_{n=1}^{N} f(n) > \sum_{n=1}^{N} \int_{n}^{n+1} f(x)dx = \int_{1}^{N+1} f(x)dx > \sum_{n=1}^{N} f(n+1) \tag{4.39}$$

Using the inequality (4.39) we establish that

$$\int_{1}^{N+1} f(x)dx < \sum_{n=1}^{N} f(n) < \int_{1}^{N} f(x)dx + f(1) \tag{4.40}$$

Now, the integrals involved are easily evaluated:

$$\int_{x=1}^{N} \frac{1}{x^{\delta}+\mu} dx = \frac{1}{\delta} \log\left[\frac{N^{\delta}+\mu}{\delta+\mu}\right] \tag{4.41}$$

Note that $\sum_{n=1}^{N} f(n) = E(w_N)$.

We thus have the bounds

$$\frac{1}{\delta} \log\left(\frac{(N+1)^{\delta}+\mu}{\delta+\mu}\right) < E(w_N) < \frac{1}{\delta} \log\left(\frac{N^{\delta}+\mu}{\delta+\mu}\right) + \frac{1}{\delta+\mu} \tag{4.42}$$

and can simply approximate $E(w_N)$ by averaging these bounds, that is,

$$E(w_N) \approx \frac{1}{2}\left[\frac{1}{\delta} \log\left\{\frac{[(N+1)^{\delta}+\mu][N^{\delta}+\mu]}{(\delta+\mu)^2}\right\} + \frac{1}{\delta+\mu}\right] \tag{4.43}$$

To approximate $\text{var}(w_N)$, we set $f(n)=(n^{\delta}+\mu)^{-2}$. Following exactly the same line of reasoning illustrated above, we obtain

$$\text{var}(w_N) \approx \frac{1}{2}\left[\frac{1}{\delta} \left\{\frac{2}{\delta+\mu} - \frac{1}{N^{\delta}+\mu} - \frac{1}{(N+1)^{\delta}+\mu}\right\} + \frac{1}{(\delta+\mu)^2}\right] \tag{4.44}$$

The approximations (4.43) and (4.44) are typically accurate to three decimal places for common values of δ and μ.

For planning purposes, it is useful to know the number of households that are actually assigned to the project. Since the assignment process is Poisson, we know that in a period of length ℓ, the mean and variance of the number of assignments equals $\mu\ell$. Now, suppose we want to estimate the number of households assigned to the project from the N households originally found on the waiting list by our test applicant. The expected length of time to process these households is $E(w_N)$, thus we obtain

$$E(\text{number housed from waiting list of size } N) = \mu E(w_N) \qquad (4.45)$$

An alternative derivation argues as follows. When a household is processed given n households waiting, the likelihood of an assignment equals $\mu(n\delta+\mu)^{-1}$. Thus, the expected number assigned equals the sum of the assignment likelihoods; this is the same as (4.45).

This second line of reasoning also yields the variance of the number assigned from those found waiting. Formally, let

$$x_n = \begin{cases} 1 & \text{next household processed is assigned given } n \text{ households waiting} \\ 0 & \text{next household processed drops out given } n \text{ households waiting} \end{cases} \qquad (4.46)$$

Probabilistically, we have the mass function

$$x_n = \begin{cases} 1 & \text{with probability } \mu(n\delta+\mu)^{-1} \\ 0 & \text{with probability } n\delta(n\delta+\mu)^{-1} \end{cases} \qquad (4.47)$$

Let N_A equal the number of households assigned from the initial group of N households found waiting. Clearly,

$$N_A = \sum_{n=1}^{N} x_n \qquad (4.48)$$

Thus, we have for the mean of N_A

$$E(N_A) = \sum_{n=1}^{N} E(x_n) = \sum_{n=1}^{N} \frac{\mu}{n\delta+\mu} = \mu E(w_N) \tag{4.49}$$

as already mentioned. Now, the variables x_n are mutually independent, each with variance $\mu n\delta(n\delta+\mu)^{-1}$. Thus,

$$\mathrm{var}(N_A) = \sum_{n=1}^{N} \mathrm{var}(x_n) = \sum_{n=1}^{N} \frac{\mu n\delta}{n\delta+\mu} \tag{4.50}$$

Finally, we note that having estimated the number of assignments, we can also estimate the number of households initially waiting that drop out. If N_D is the number of households from the N initially found waiting who drop out, then we must have,

$$N_D = N - N_A \tag{4.51}$$

Thus, the moments of the number of dropouts are given by

$$E(N_D) = N - E(N_A) \tag{4.52}$$

$$\mathrm{var}(N_D) = \mathrm{var}(N_A) \tag{4.53}$$

4.5.2 Statistical Issues

To use the dropout model we have discussed, one needs to estimate the household dropout rate δ. Depending upon the information at hand, one can obtain estimates with varying degrees of precision. Ideally, one would estimate δ by observing a cohort of applicants entering public housing at time t_o, following this cohort until some fixed time T, and then determining:

i) The time t_j at which household j is assigned to an apartment; $j \in A$ where A is the set of households assigned; $t_0 \leq t_j \leq T$.

ii) The time t_j at which household j drops out; $j \in D$ where D is the set of households that drops out; $t_0 \leq t_j \leq T$

iii) The number of "censored" households n_c who have yet to receive an assignment or drop out by time T, the end of the study period.

The likelihoods associated with (i) - (iii) are easily determined. If an assignment occurs at time t_j, we know that a dropout did not occur in the interval (t_0, t_j). The probability that a household does not drop out in (t_0, t_j) equals $e^{-\delta(t_j-t_o)}$, and hence the contribution to the likelihood of the abserved data from assigned households equals

$$L_{A_1} = \prod_{j\varepsilon A} e^{-\delta(t_j-t_o)} \tag{4.54}$$

If a dropout occurs at time t_j, its associated likelihood is $\delta e^{-\delta(t_j-t_o)}$. Thus, the contribution of dropouts to the likelihood is

$$L_{D_1} = \prod_{j\varepsilon D} \delta e^{-\delta(t_j-t_o)} \tag{4.55}$$

Finally, for those who have yet to drop out or receive assignments by time T, the contribution to the likelihood equals

$$L_{C_1} = e^{-n_c\delta(T-t_o)} \tag{4.56}$$

The overall likelihood function is given by the product of (4.54) through (4.56):

$$L_1 = \prod_{j\varepsilon A} e^{-\delta(t_j-t_0)} \prod_{j\varepsilon D} \delta e^{-\delta(t_j-t_0)} e^{-n_c\delta(T-t_0)} \tag{4.57}$$

Maximizing (4.57) with respect to δ yields the maximum likelihood estimate of the household dropout rate

$$\hat{\delta}_1 = \frac{\sum_{j\varepsilon D} 1}{\sum_{j\varepsilon A} (t_j-t_0) + \sum_{j\varepsilon D} (t_j-t_0) + n_c(T-t_0)}$$

$$= \frac{n_D}{T_{EX}} \tag{4.58}$$

where n_D equals the observed number of dropouts, and T_{EX} equals the total exposure time for all households in the cohort.

Unfortunately, the exact times of dropout are rarely known. One usually discovers dropouts when they are contacted for assignment. We will therefore consider some simpler estimates which do not require the precise times at which dropouts occur.

Suppose that at the end of the observation period, one knows the times $t_j, j \varepsilon A$ at which households were assigned, the number of dropouts n_D, and the number of households who have yet to drop out or be assigned n_C. The probabiltiy that n_D households drop out in the interval (t_O, T) is given by

$$L_{D_2} = [\ 1-e^{-\delta(T-t_o)}]^{n_D} \tag{4.59}$$

thus the total likelihood associated with the observed data is

$$L_2 = [1-e^{-\delta(T-t_o)}]^{n_D} \prod_{j\varepsilon A} e^{-\delta(t_j-t_o)} e^{-n_c\delta(T-t_o)} \tag{4.60}$$

Maximizing this with respect to δ yields

$$\hat{\delta}_2 = \frac{1}{T-t_o} \log \left[1 + \frac{n_D(T-t_o)}{n_c(T-t_o) + \sum_{j\varepsilon A}(t_j-t_o)}\right] \tag{4.61}$$

The estimate $\hat{\delta}_2$ is useful if one knows the times at which tenants are assigned, but only the <u>number</u> of dropouts in some interval (t_O, T).

Finally, if the only data available consists of the number of dropouts n_D in some interval (t_O, T), and the number of households N in the cohort at time t_O, we have for the likelihood function

$$L_3 = [1-e^{-\delta(T-t_o)}]^{n_D} e^{-(N-n_D)\delta(T-t_o)} \tag{4.62}$$

The maximum likehood estimate in this scarce data situation equals

$$\hat{\delta}_3 = \frac{1}{T-T_o} \log \left[\frac{1}{1-\frac{n_D}{N}} \right] \tag{4.63}$$

For small values of the ratio n_D/N, (4.63) can be approximated by

$$\hat{\delta}_3 \approx \frac{n_D}{N(T-t_o)} \tag{4.64}$$

By way of example, if $n_D/N = .25$, $\log [\frac{1}{1-.25}] = .29$, a small difference.

We have covered three approaches to estimating the household dropout rate δ. If one knows precise times of assignment and dropout from sample cohort data, the estimator $\hat{\delta}_1$ of (4.58) is appropriate,. If the times of dropout are unknown but assignment times are, the estimator $\hat{\delta}_2$ from (4.61) can be used. Finally, if only the number of dropouts are known, the estimator $\hat{\delta}_3$ from (4.63) or (4.64) is appropriate.

4.5.3 An Example

Continuing with the example from before, we assume an annual moveout rate of 20 households per year. Figures 4.8 and 4.9 plot $E(w^*_N)$ and $var(w^*_N)$, the mean and variance of the waiting time experienced by a test applicant, for various values of the household dropout rate δ. For example, at $\delta=.1$, households will wait, on average, 10 years before dropping out. With N=100, $E(w^*_N)$ has been reduced from just over 5 years to just over 4 years, almost a 20% reduction.

Figure 4.8

Mean Waiting Time in a
Single Project, No Priority System
with Dropout

μ=20

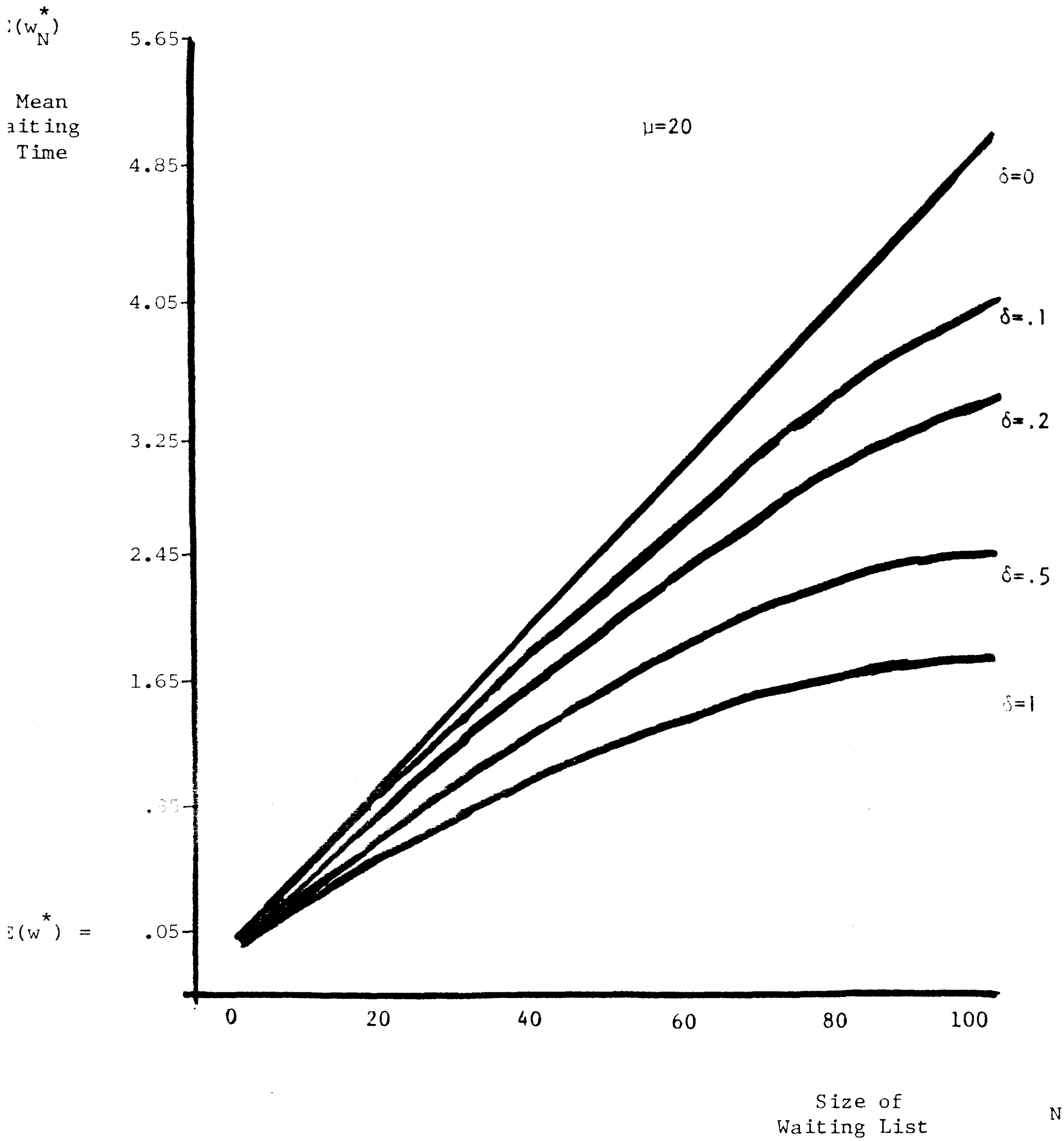
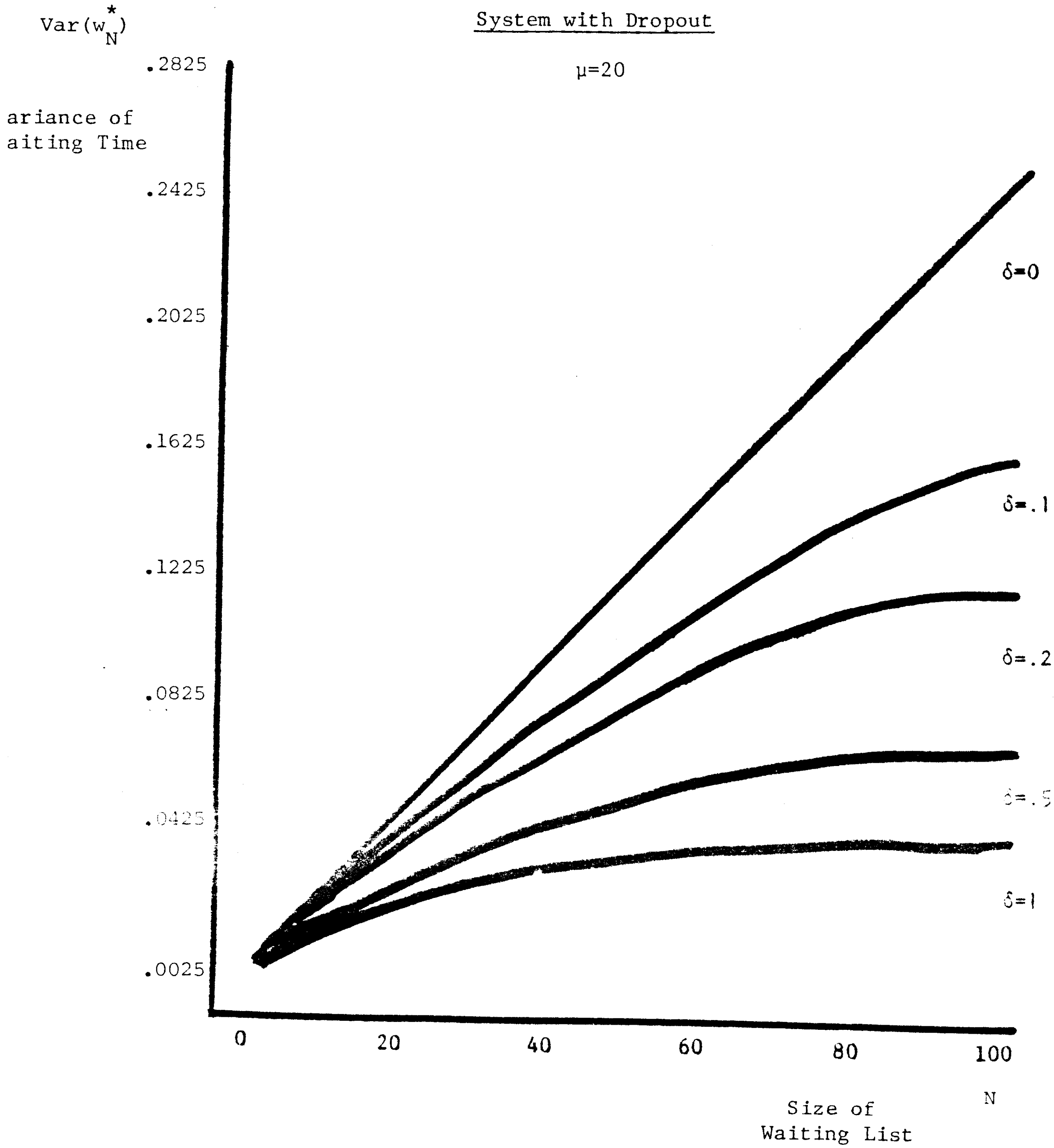


Figure 4.9

Variance of Waiting Time
in a Single Project, No Priority
System with Dropout



4.6 Blend Priorities

Our next improvement on the tenant assignment model reflects the often stated goals of cresting project communities with specific demographic characteristics. Racial integration is one case in point. As another example, the Boston Housing Authority intends to assign two "moderate" income households for every "low" income household at housing developments where the low income population comprises over 50% of the total project population; it is hoped that a more diverse range of incomes will help to stabilize public housing populations (Price and Solomon, 1983). In other cities, income mixing may be invoked to achieve financial solvency (e.g. Greensboro, North Carolina). Assignment policies such as those mentioned which attempt to "design" the demographic characteristics of housing developments often invoke a method we will refer to as a blend priority system.

Formally, a blend priority scheme assigns a probability b_j to the assignment of the next household in queue from group j on any tenant assignment. Groups may be defined in a variety of ways to reflect the particular policies of a given housing authority (e. g. low income whites, moderate income blacks). Within groups, households are processed in chronological order of application. The higher the value of b_j, the higher the priority given to group j. Also, as every household assigned is the member of some group, we must have $\sum_j b_j=1$.

A simple model demonstrates the consequences of employing blend priorities. Assume that a project is always filled to its capacity of H households; again all households are taken to require similar units. Let h_{jm} be the expected number of group j households in the project after the m^{th} tenant assignment from the time when blend priorities were implemented. We can model h_{jm} as:

$$h_{j,m+1} = h_{jm} + \Pr\{\text{household of type } j \text{ is assigned on move in}\} - \Pr\{\text{household of type } j \text{ leaves the project on move out}\} \quad (4.65)$$

The likelihood that a group j household is assigned on move m is the blend probability b_j. If we assume that the likelihood of a group j household leaving the project is proportional to the expected number of group j households present, then the probability of a group j departure equals h_{jm}/H. Equation (4.65) thus becomes:

$$h_{j,m+1} = h_{jm} + b_j - h_{jm}/H \quad (4.66)$$

This equation has the solution

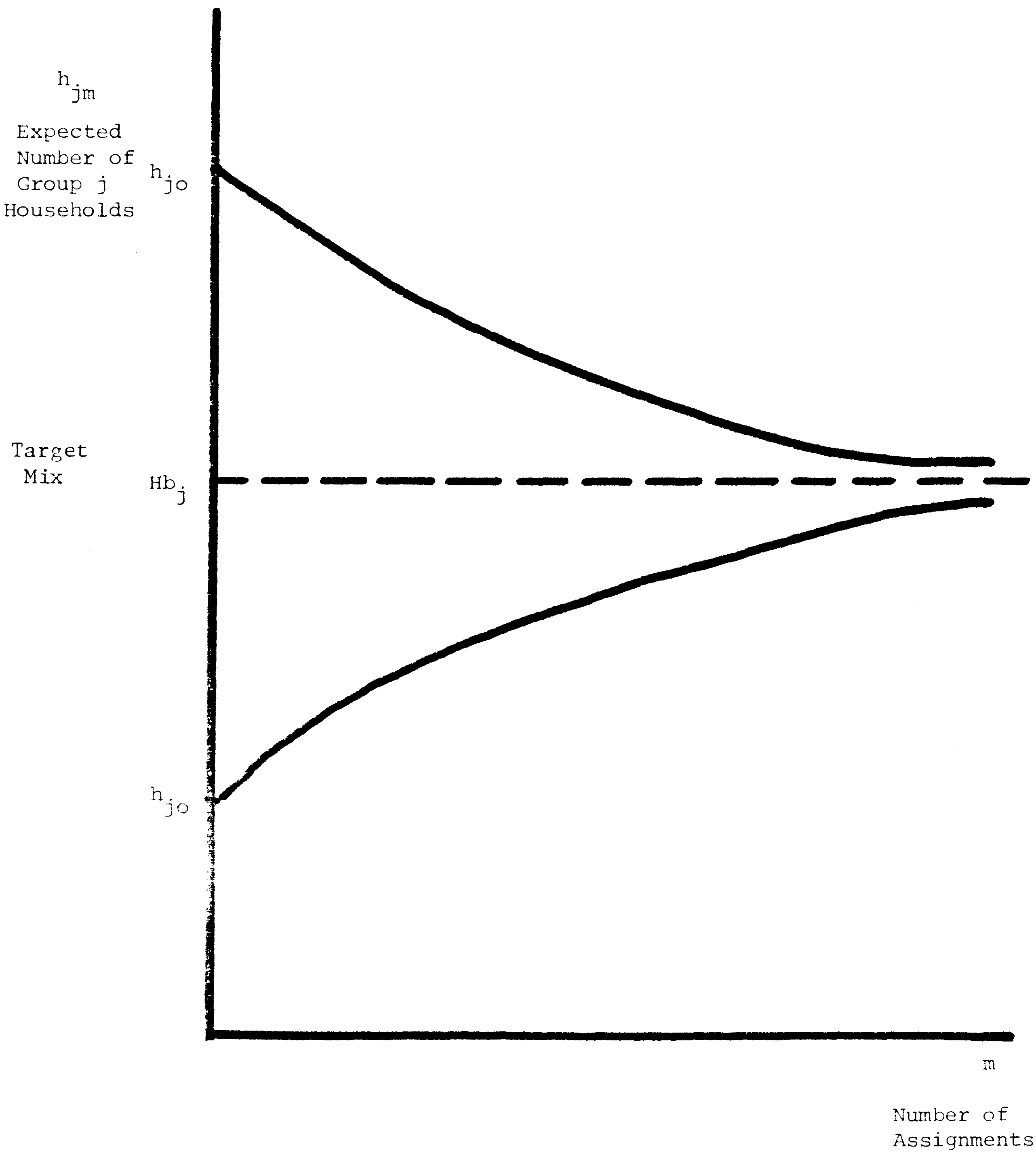
$$h_{jm} = Hb_j + [h_{j0} - Hb_j][1 - 1/H]^m \qquad m = 0, 1, 2, \ldots \quad (4.67)$$

where h_{j0} is the number of group j households present in the development when blend priorities are implemented. Equation (4.67) is shown graphically in Figure 4.10.

This model is quite useful in analyzing the consequences of various priority schemes. For example, if one wishes to racially integrate a project, the model can evaluate the time frame necessary to achieve the desired level of integration for alternative blend probabilities b_j. A practical scheme might take the form "admit k group 1 tenants for every group 2 tenant assigned"; the resulting blend probabilities for this example would set $b_1 = \frac{k}{k+1}$, $b_2 = \frac{1}{k+1}$.
Using these values for b_j, one can "follow" the changing demographics of the project using a graph similar to Figure 4.10. Although the model has been formulated in terms of the number of assignments

Figure 4.10

Blend Priorities



required to achieve a desired demographic composition, one can associate an expected length of time between tenant assignments with each moveout, resulting in a model where demographics change over chronological time. Finally, the model demonstrates the equivalence between specifying blend priorities probabilistically and designing the ultimate demographic composition of a housing project.

To forecast waiting times for new applicants in a blend priority setting, we argue as follows: tenant assignments occur according to a Poisson process with rate μ, the moveout rate from the project. When an assignment occurs, the probabiltiy that the household chosen is from group j equals b_j. Thus, tenant assignments from group j occur according to a Poisson process with rate $\mu_j = b_j \mu$, as long as successive assignments are assumed to be independent.

Having established this result, we may use the models already developed to forecast waiting times; a system diagram and attendant assumptions appear in Figure 4.11.

When an arriving group j test applicant finds N_j group j households waiting for housing assignments, equations (4.33) and (4.34) apply after substituting μ_j for μ and N_j for N yielding

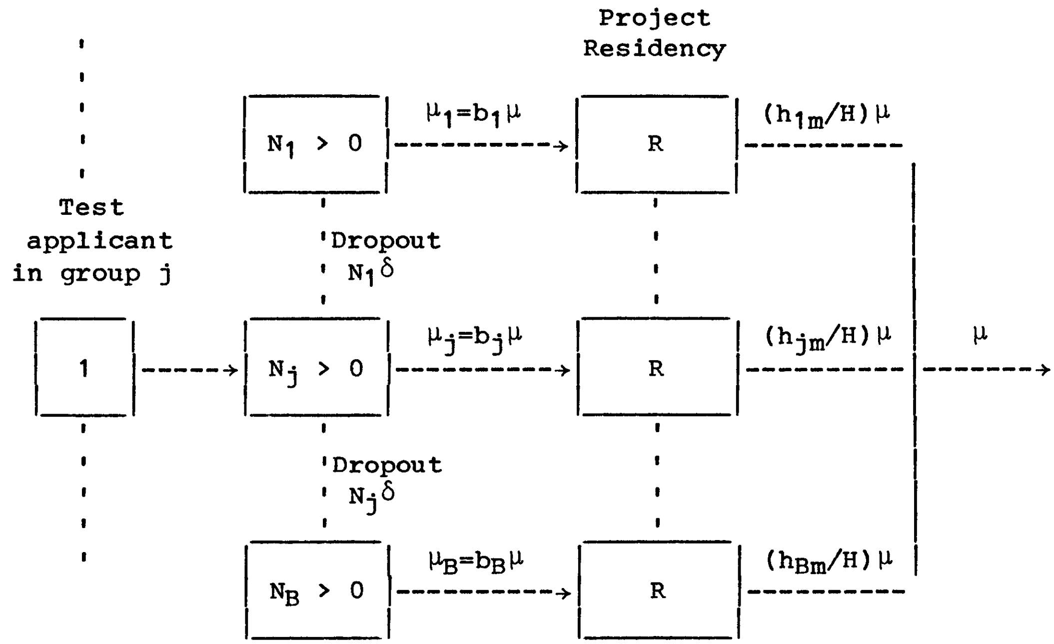
$$E(w^*_{N_j}) = \sum_{n=1}^{N_j} \frac{1}{n\delta+\mu_j} + \frac{1}{\mu_j} \tag{4.68}$$

$$\mathrm{var}(w^*_{N_j}) = \sum_{n=1}^{N_j} \frac{1}{(n\delta+\mu_j)^2} + \frac{1}{\mu_j^2} \tag{4.69}$$

as the mean and variance of the waiting time for our test applicant. One can also use the approximations found in equations (4.43) and (4.44) for quick estimates.

Figure 4.11

The Single Project, Blend Priority System With Dropout



Assumptions

1) $N_j > 0$ group j households are found waiting for housing assignments by a newly arriving group j test applicant

2) Within groups, households are assigned in order of application

3) Within groups, dropout occurs at rate n^δ when n households are waiting; test applicants will not drop out with certainty

4) Tenant assignments take place according to a Poisson process with rate μ. On any assignment, the probability that a group j tenant is assigned equals b_j. Successive assignments are independent, thus group j tenants are assigned acccording to a Poisson process with rate $\mu_j = b_j \mu$.

5) The probability that a departing household is from group j is proportional to the expected number of group j households in the project.

4.6.1 An Example

Again we consider a project with an annual moveout rate of μ=20 households per year. We will fix the household specific dropout rate to δ=.1, and consider two blend groups. Group 1 applicants receive assignments with probability b_1=.33, while group 2 applicants are assigned with probibility b_2=.67. The mean and variance of the waiting time for an arriving group i test applicant are shown in Figures 4.12 and 4.13.

4.7 Categorical Priorities

In every tenant assignment policy reviewed in Chapter 2, we discovered that housing authorities give absolute priorities on assignment to certain classes of households. Typical of this is is the policy in Minneapolis where "Individuals and Families displaced by public action or a natural disaster while residing within the jurisdiction of the Agency shall have preference over other individuals and Families." (Minneapolis Community Development Agency (1983, p.1)). Whenever households in such a priority class are present (i.e. waiting for an assignment), they are assigned before other households, regardless of the waiting times of these other households. We will refer to tenant assignment systems of this form as categorical priority systems, and will generalize our models to accomodate such schemes. Actual categorical priorities for several U. S. authorities were discussed in Chapter 2.

To model categorical priorities, we assume that there are J priority categories in the system, and a test applicant in category j arrives. The test applicant finds n_i households waiting in priority category i, i=1, 2, ..., j. All households have the same expected

Figure 4.12

Mean Waiting Time in a Single Project, Blend Priority System with Dropout

$\mu=20$
$\delta=.1$

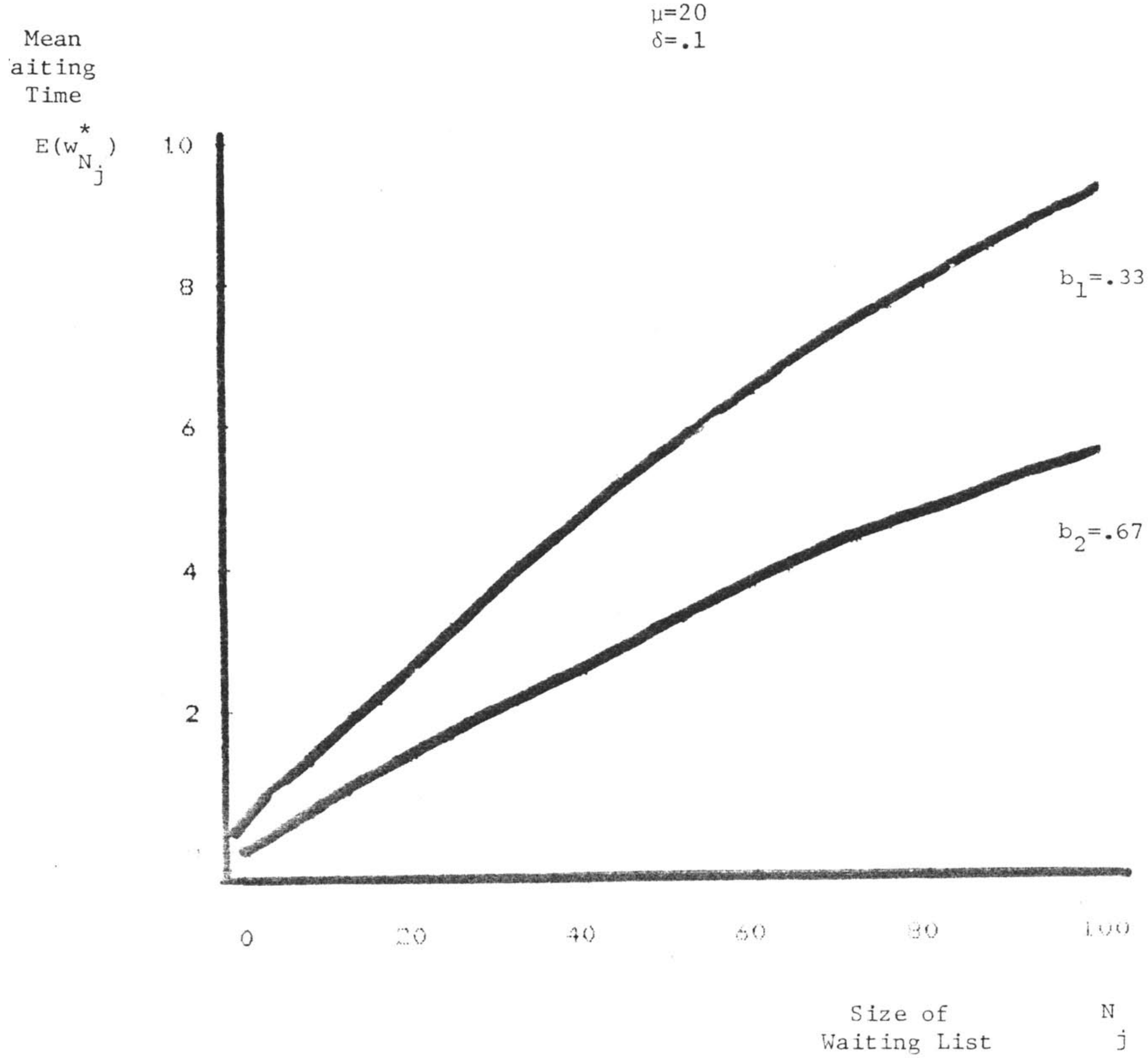
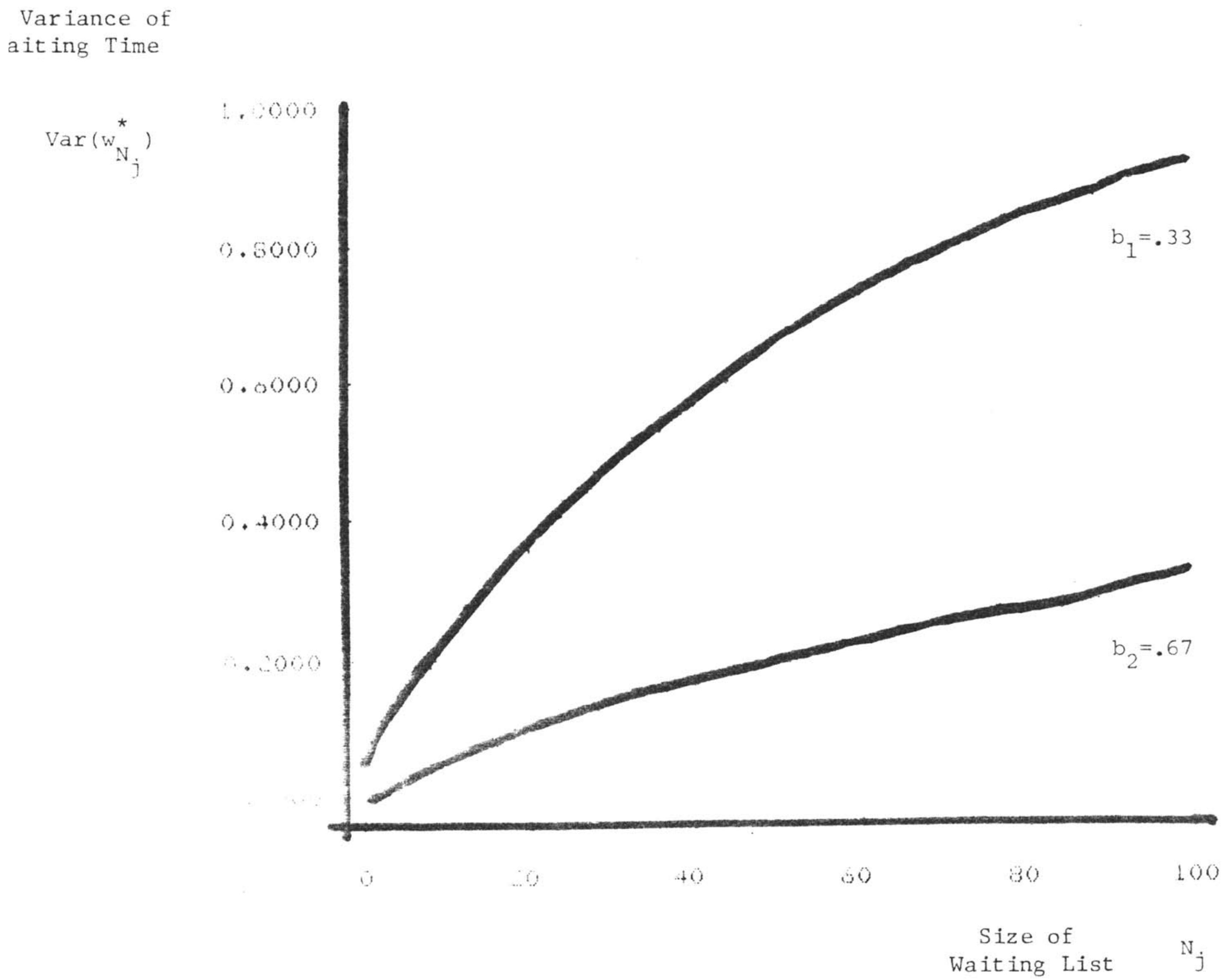


Figure 4.13

Variance of Waiting
Time in a Single
Project, Blend Priority
System with Dropout

$\mu=20$
$\delta=.1$



project residency R; as usual, the household assignment process is Poisson with parameter $\mu=m/R$ where m is the number of units in the project.

We further assume that households in priority category i arrive in a Poisson manner with rate λ_i, i=1, 2, ..., J. Initially, we will suppose that no households drop out; this assumption will be relaxed later. Of interest is the mean and variance of the time our test applicant will wait until assignment. The system diagram and a summary of our assumptions for this model are found in Figure 4.14.

Upon arrival, the total number of households found waiting by our test applicant in priorities 1 through j equals

$$N_j = \sum_{i=1}^{j} n_i > 0 \tag{4.70}$$

In addition to waiting for these N_j households to be assigned, our test applicant will be superseeded by newly arriving households in priorities 1 through j-1. These households arrive in Poisson fashion with rate

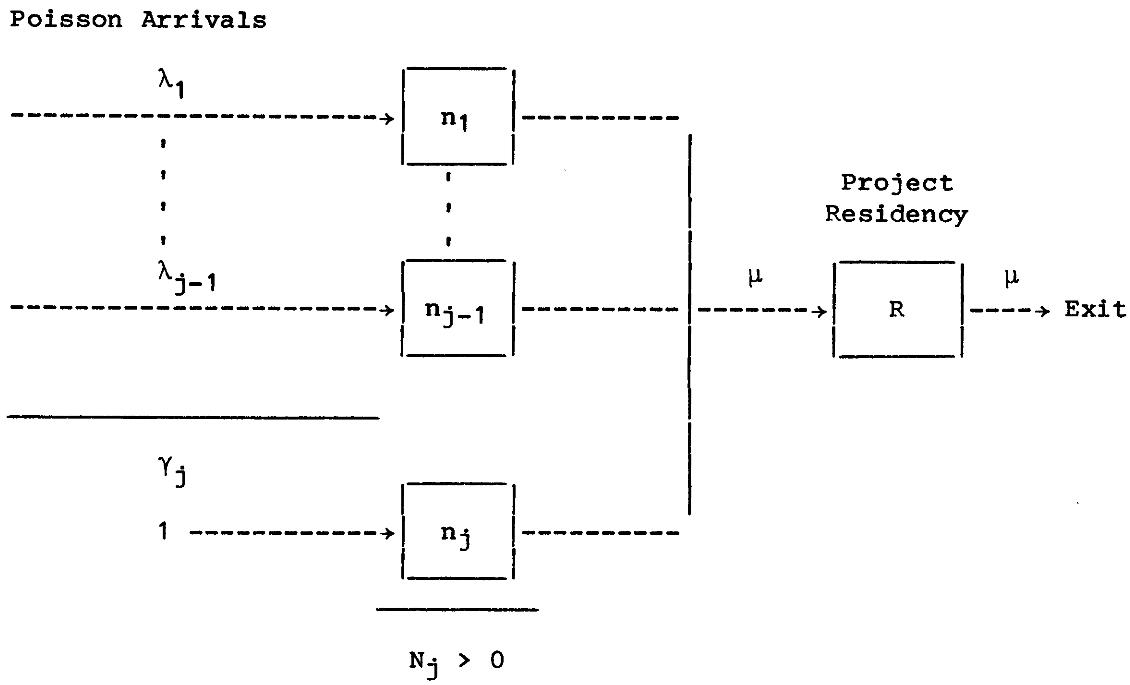
$$\gamma_j = \sum_{i=1}^{j-1} \lambda_i \tag{4.71}$$

We assume that $\gamma_j<\mu$.

Suppose that at some unspecified time, there are n households (<u>not</u> including the test applicant) waiting for housing assignments in priorities 1 through j. One of two events can next occur: a household will be assigned, reducing the number of households waiting to n-1, or a new household in one of priorities 1 through j-1 will arrive, increasing the number of households waiting to n+1. The probability that the former event will next occur equals $\mu/(\mu+\gamma_j)$, while the probability that the latter event will next occur equals $\gamma_j/(\mu+\gamma_j)$.

Figure 4.14

The Categorical Priorities, No Dropout System



Assumptions

(1) n_i priority i households are found waiting for housing assignments by a newly arriving priority j household, i=1,2, ..., j.

$$N_j \equiv \sum_{i=1}^{i} n_i > 0.$$

(2) Priority i households arrive according to a Poisson process with rate λ_i, i=1,2, ..., J. The arrival rate of class 1 through j-1 households, γ_j, is given by $\gamma_j \equiv \sum_{i=1}^{j-1} \lambda_i$.

(3) A priority i household is assigned only if no households in priorities 1, 2, ..., i-1 are present. Within priorities, assignment is in order of application.

(4) Tenant assignments take place in a Poisson fashion with rate μ; $\mu > \gamma_j$.

These probabilities arise from the Poisson nature of the assignment and priority arrival processes. In additiion, the length of time between successive events (assignments or arrivals) is exponentially distributed with parameter $\mu+\gamma_j$; this result also follows from the Poisson processes involved.

We are now ready to apply the results from birth-and-death processes derived earlier. Let:

$$p_n = \gamma_j/(\mu + \gamma_j) \tag{4.72}$$

$$r_n = 0 \tag{4.73}$$

$$q_n = \mu/(\mu + \gamma_j) \tag{4.74}$$

To obtain $E(w_{N_j})$, the expected time until the N_j households found by our test applicant in priorities 1 through j are assigned, we set

$$g_n = E(\tau_n) = 1/(\mu+\gamma_j) \tag{4.75}$$

Using these values for p_n, r_n, q_n, and g_n in equation (4.15) results in

$$M_{N_j} = E(w_{N_j}) = \frac{N_j}{\mu-\gamma_j} \qquad \gamma_j < \mu \tag{4.76}$$

Similarly, we obtain the second moment of w_{N_j} by setting

$$g_n = E(\tau_n^2) + 2[p_n\bar{\tau}_{n+1}E(w_{n+1}) + r_n\bar{\tau}_nE(w_n) + q_n\bar{\tau}_{n-1}E(w_{n-1})]$$

$$= \frac{2}{(\mu+\gamma_j)^2} + 2[\frac{\gamma_j}{\mu+\gamma_j}\,\frac{1}{\mu+\gamma_j}\,\frac{n+1}{\mu-\gamma_j} + 0 + \frac{\mu}{\mu+\gamma_j}\,\frac{1}{\mu+\gamma_j}\,\frac{n-1}{\mu-\gamma_j}]$$

$$= \frac{2}{(\mu+\gamma_j)^2} \cdot \frac{\mu+\gamma_j}{\mu-\gamma_j} \cdot n \qquad \gamma_j < \mu \tag{4.77}$$

Again, we use equation (4.15) to obtain

$$M_{N_j} = E(w_{N_j}^2) = \frac{N_j}{(\mu-\gamma_j)^2} \cdot \frac{\mu+\gamma_j}{\mu-\gamma_j} + \frac{N_j^2}{(\mu-\gamma_j)^2} \qquad \gamma_j < \mu \tag{4.78}$$

Combining (4.76) and (4.78) we obtain the variance of W_{N_j}

$$\mathrm{var}(w_{N_j}) = E(w_{N_j}^2) - E(w_{N_j})^2 = \frac{N_j}{(\mu-\gamma_j)^2} \frac{\mu+\gamma_j}{\mu-\gamma_j} \qquad \gamma_j < \mu \tag{4.79}$$

The additional time our test applicant must wait, w^*, is clearly the same as w_1, the length of time necessary to house a solitary household found waiting in priorities 1 through j. Thus we can use (4.76) and (4.79) to obtain

$$E(w^*) = \frac{1}{\mu-\gamma_j} \tag{4.80}$$

$$\mathrm{var}(w^*) = \frac{1}{(\mu-\gamma_j)^2} \cdot \frac{\mu+\gamma_j}{\mu-\gamma_j} \tag{4.81}$$

Finally, we can combine our results to obtain the mean and variance of the waiting time for a newly arriving priority j test applicant given that N_j households are found waiting in priorities 1 through j:

$$E(w_{N_j}^*) = \frac{N_j+1}{\mu-\gamma_j} \tag{4.82}$$

$$\mathrm{var}(w_{N_j}^*) = \frac{N_j+1}{(\mu-\gamma_j)^2} \cdot \frac{\mu+\gamma_j}{\mu-\gamma_j} \tag{4.83}$$

A simplistic (but not entirely correct) interpretation of (4.82) is that the assignment rate μ has been reduced by the amount γ_j - of the μ apartments available per unit time for assignment, γ_j must be allocated to newly arriving households in priorities 1 through j-1. Thus, the "effective" assignment rate is $\mu-\gamma_j$, and equation (4.82) is the same as equation (4.25) for the pure service model with μ replaced by $\mu-\gamma_j$.

However, we see from (4.83) that the effect of priorities is more complicated than a simple adjustment to the assignment rate. If the only effect of categorical priorities was to reduce μ to $\mu-\gamma_j$, then the variance of the waiting time for our test applicant would equal $(N_j+1)/(\mu-\gamma_j)^2$. The actual variance, given in equation (4.83), inflates this amount by the factor $(\mu+\gamma_j)/(\mu-\gamma_j)$. As the arrival rate of priority applicants approaches the assignment rate, this inflation factor becomes quite large. Thus, a major impact of categorical priority schemes on waiting times rests with the increase in the variability of the time until assignment.

4.7.1 Statistical Issues

The model of this section has introduced a new group of quantities which require estimation - the arrival rates of applicants in various priority categories. By assumption, these arrival processes are Poisson, thus a reasonable estimate of λ_i, the arrival rate of priority i applicants, is given by the observed number of priority i applicants arriving in some time period divided by the length of the time period:

$$\hat{\lambda}_i = A_i/\ell \qquad (4.84)$$

where A_i is the number of priority i arrivals in a time period of length ℓ. This estimator has the same properties as the assignment rate estimator $\hat{\mu}$ discussed earlier in equation (4.28).

To estimate the arrival rate of new applicants in priority categories 1 through j-1, γ_j, we merely sum the individual category arrival rate estimates; that is

$$\hat{\gamma}_j = \sum_{i=1}^{j-1} \hat{\lambda}_i \tag{4.85}$$

As γ_j is also the parameter of a Poisson proces, $\hat{\gamma}_j$ possesses the same properties is $\hat{\mu}$ and $\hat{\lambda}_i$. The estimation of γ_j should not pose any special problem for housing administrators.

4.7.2 An Example

As with our previous examples, we assume a project with an annual moveout rate of μ=20 households per year. However, we now consider a situation where applicants in priorities 1 through j-1 arrive at a rate of γ_j households per year for various values of γ_j. The corresponding mean waiting times appear in Figure 4.15. Figure 4.16 plots the variance. In the absence of dropout, both of these quantities grow linearly with N_j, the number of househoulds in priorities 1 through j found waiting by our test applicant. The effect of increasing γ_j on waiting time is clearly seen from these plots.

4.7.3. Applications of Categorical Priorities: Multiple Unit Types and Transfers

The analysis performed thus far has assumed that every household requires the same unit type, and all project units are identical. In fact, our analysis is valid for multiple unit types if we treat unit

Figure 4.15

Mean Waiting Time
in a Categorical Priority
System without Dropout

$\mu=20$

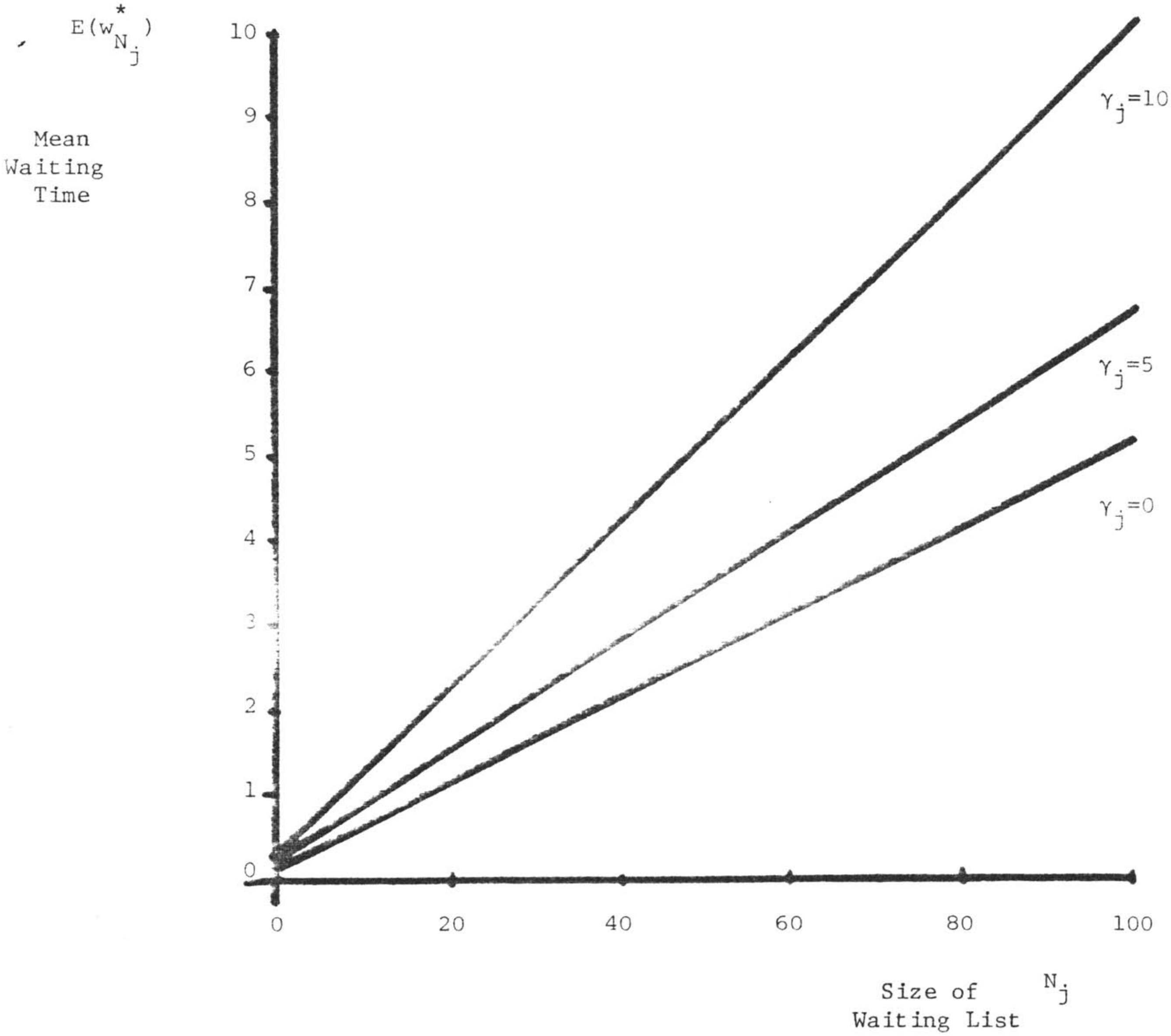
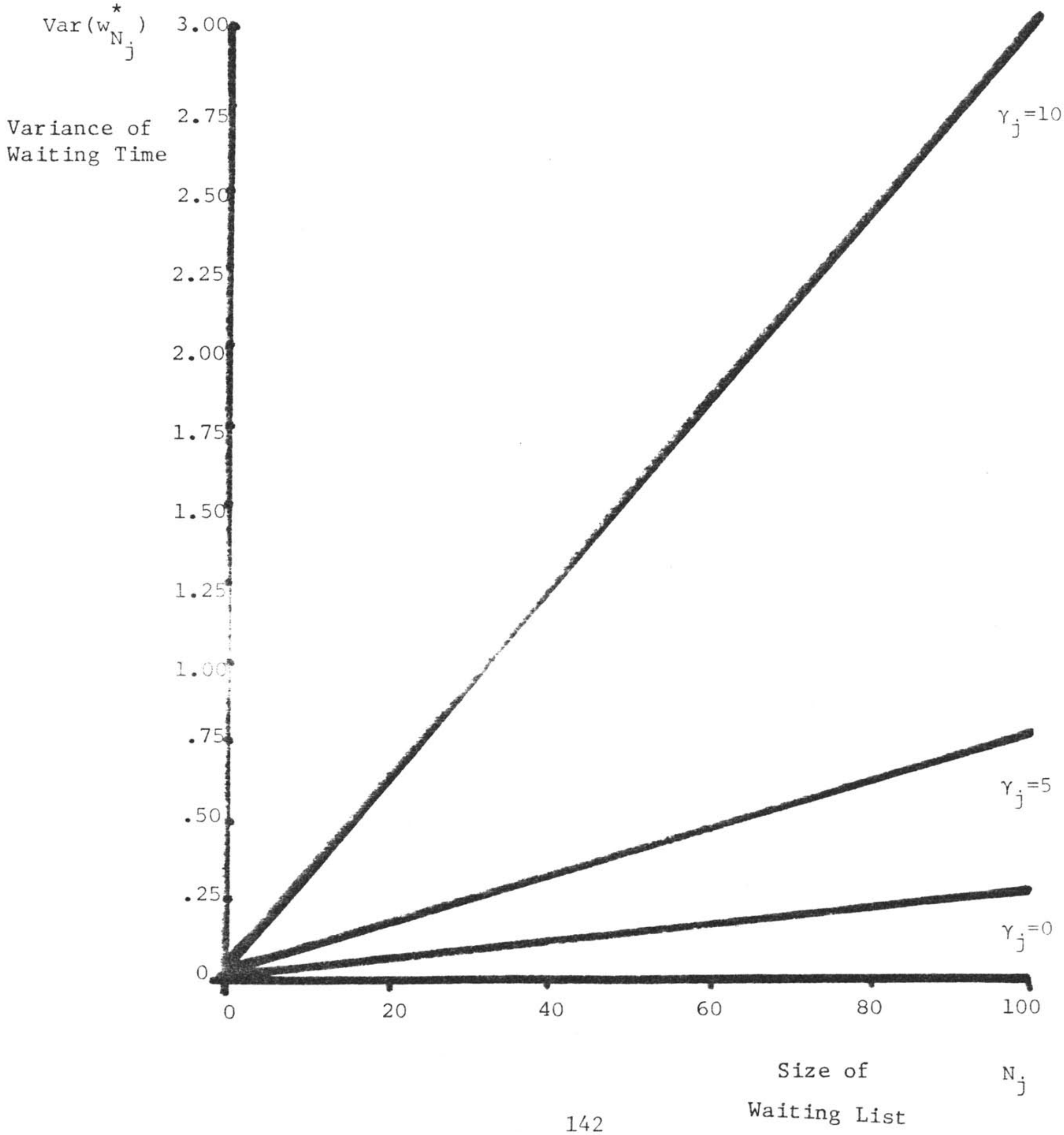


Figure 4.16

Variance of Waiting
Time in a Categorical
Priority System
Without Dropout

$\mu=20$



types independently of each other. If $\mu^{(k)}$ is the moveout rate for type k units, $\gamma_j^{(k)}$ the arrival rate for applicants in priorities 1 through j-1 requiring type k units, and $N_j^{(k)}$ the number of households in priorities 1 through j requiring type k units found by a newly arriving test applicant in priority j who also requires a type k unit, all of our previous results hold; the analysis is simply interpreted to be conditional on households who require type k units. This approach assumes no interaction between different household unit requirements. For example, a household which applies for a type k unit, cannot change its unit requirement to some other type ℓ. This is not a major problem in as much as such unit type changes are relatively infrequent. Also, if the model is frequently used, a household requiring a type k unit in one month, but a type ℓ unit in some subsequent month (perhaps due to a change in family size) will appear on the type ℓ waiting list (and hence in the data base for a "type ℓ model") in that subsequent month; as our analysis is always conditional, this change in unit requirements can be incorporated into a new waiting time forecast.

In certain tenant assignment systems, <u>intra project transfers</u> from type k to type ℓ units receive categorical priorities over new applicants. Suppose that when a household terminates a period of residency in a type k unit, the household transfers to a type ℓ unit in the same project with probability $q_{k\ell}$. We let q_{ko} represent the likelihood that a household leaves the project after residency in a type k unit (q_{ko} is referred to as the "exist" probability), and since all households leaving type k units must go somewhere, we require

$\sum_{\ell=0} q_{k\ell}=1$.

Suppose that there are two priority categories: intra-project transfers and new applicants. To obtain the "application rate" for intra-project transfers <u>into</u> type ℓ units, we set

$$\gamma(\ell) = \sum_{k=1} \mu(k)\, q_{k\ell} \tag{4.86}$$

and proceed as before. Extension to the case where intra-project transfers represent the j^{th} priority category is straightforward.

Though equation (4.86) is correct if we focus our attention on type ℓ units, there is a problem of dependence between moves -a transfer from a type k unit to a type ℓ unit necessitates a new assignment (or perhaps a new transfer) into the type k unit being vacated. In fact, that new type ℓ vacancies triggering transfers from type k units cause new assignments of type k to be made destroys the Poisson assignment process to type k units. However, we choose to ignore this difficulty for the sake of modeling simplicity. As long as the intra-project transfer probabilities $q_{k\ell}, \ell>0$ are low relative to the exit probabilities q_{ko}, such dependence should not cause any major changes to occur in our results. Table 3.10 presents empirical intra-project transfer probabilities for six housing projects in Boston. Note that the exit probabilities are typically over 70%, and are often over 80%.

4.7.4 Applications of Category Priorities: Score Priorities

In Chapter 2, we noted that two of the cities in our survey of tenant assignment practices, St. Paul and Omaha, assign points to new applicants; these points reflect the priorities assigned by the housing authorities to the households. Applicants are processed in the order of their scores, form highest to lowest, with ties being broken by date of application (for a discussion of the specifics of the St. Paul and Omaha systems, we Chapter 2).

We will now show that the assignment of scores to applicants is just a special case of the category priority model we have been discussing. Let S be the random variable representing the score obtained by a randomly chosen applicant. We assume that the probability law of S is characterized by the (known) density function $f_S(s)$. Let s^* be the score assigned to a newly arriving test applicant, and let $N(s^*)$ be the number of households found on the waiting list by the test applicant with scores greater than or equal to s^*.

Suppose now that new applicants arrive according to a Poisson process with parameter λ. Some of these households will receive scores that are greater than s^*; the fraction of new applicants in this situation equals

$$\Pr\{S > s^*\} = \int_{s^*}^{\infty} f_S(s)ds \tag{4.87}$$

Thus, the arrival rate of new applicants <u>with scores greater than s^*</u> is given by

$$\gamma(s^*) = \lambda \Pr\{S > s^*\} \tag{4.88}$$

The score priority system is thus a categorical priority system, but with continuous categories. The mean and variance of the waiting time for our test applicant is given by equations (4.82) and (4.83) substituting $N(s^*)$ for N_j and $\gamma(s^*)$ for γ_j. It should be noted that in practice, the integral in (4.87) would be replaced by the sum

$$\Pr\{S > s^*\} = \sum_{s=s^*+1}^{s_{max}} f_s \tag{4.89}$$

where f_s is the relative frequency at which a score with value s occurs, and s_{max} is the maximum possible score.

4.8 Merging Dropout and Categorical Priorities

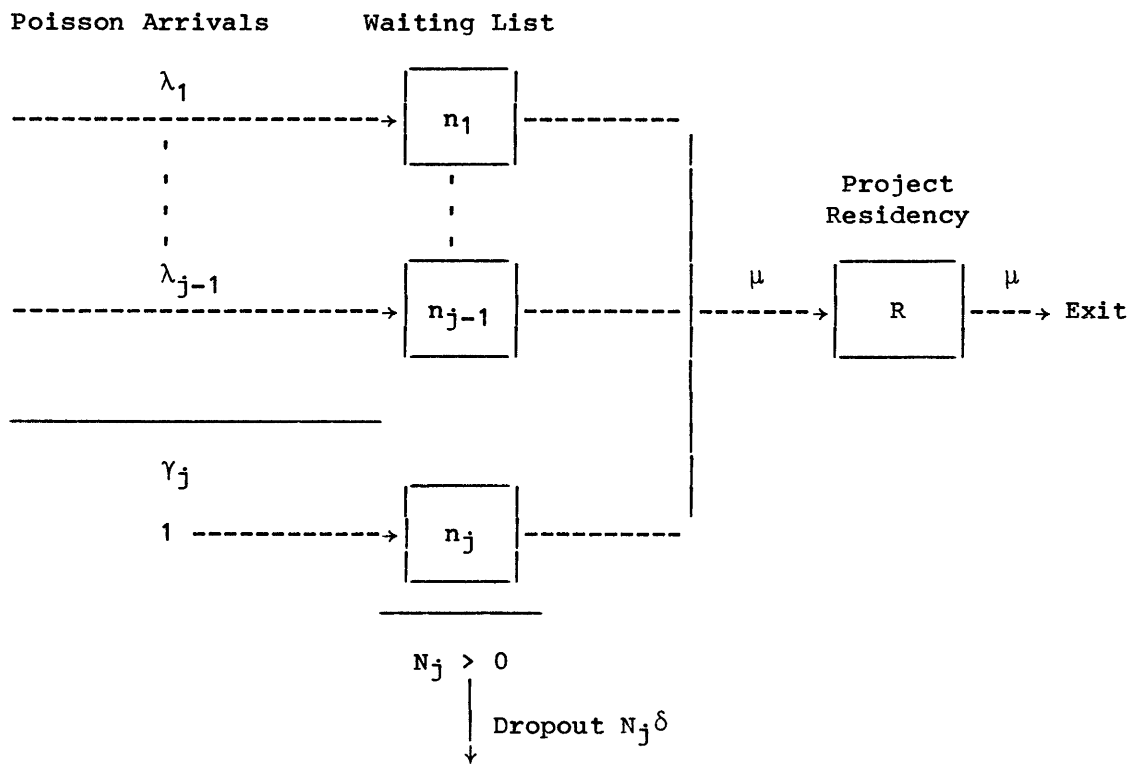
Thus far, we have presented models which exhibit specific features of tenant assignment systems; the analysis of this section will begin to consolidate these features into realistic tenant assignment models suitable for use by public housing authorities. Our first step is to merge our earlier model of dropout with the categorical priority systems just discussed. Figure 4.17 provides a graphic depiction of the system and a listing of its attendant assumptions.

The situation is as follows: a newly arriving test applicant in priority j finds $N_j>0$ households in priorities 1 through j waiting for housing assignments. New applicants in priorities 1 through j-1 arrive in Poisson fashion with rate γ_j. As usual, the tenant assignment process is Poisson with rate μ. Finally, in a manner similar to our earlier dropout model, we assume that when n households in priorities 1 through j are waiting for assignments, the dropout rate for the system equals $n\delta$; these n households are <u>ahead</u> of our test applicant, and the test applicant by assumption will not drop out.

Suppose that at some unspecified time, n households await housing assignments in priorities 1 through j. A household could be assigned, or a household could drop out, reducing the number of households waiting to n-1. Alternatively, a new household in one of priorities 1 through j-1 could arrive, increasing the number of households waiting to n+1. The probability that the next event to occur is a tenant assignment or a dropout equals $(n\delta+\mu)/(n\delta+\mu+\gamma_j)$, while the probability that the next event is the arrival of a new

Figure 4.17

The Categorical Priorities System With Dropout



Assumptions

1) n_i priority i households are found waiting for housing assignments by a newly arriving priority j household, i=1,2,...,j. $N_j \equiv \sum_{i=1}^{j} n_i > 0$.

2) Priority i households arrive according to a Poisson process with rate λ_i, i=1,2,...,J. $\gamma_j \equiv \sum_{i=1}^{j-1} \lambda_i$ = the arrival rate of households in priorities 1 through j-1.

3) A priority i household is assigned only if no households in priorities 1,2,...,j-1 are present. Within priorities, assignment is in order of application.

4) Dropouts occur at rate $n\delta$ when n households are waiting for housing assignments in priorities 1 through j. The test applicant will not drop out with certainty.

5) Tenant assignments take place in a Poisson fashion with rate μ.

applicant in one of priorities 1 through j-1 equals $\gamma_j/(n\delta+\mu+\gamma_j)$. These probabilities follow from the Poisson arrival, assignment, and dropout processes involved. In addition, the length of time between successive events (assignments, dropouts or arrivals) given n households waiting is exponentially distributed with parameter $n\delta+\mu+\gamma_j$.

Returning to our results from birth and death processes, we set

$$p_n = \gamma_j/(n\delta+\mu+\gamma_j) \tag{4.90}$$

$$r_n = 0 \tag{4.91}$$

$$q_n = (n\delta+\mu)/(n\delta+\mu+\gamma_j) \tag{4.92}$$

To obtain $E(w_{N_j})$, the expected time until all N_j households found by our test applicant in priorities 1 through j are assigned or drop out, we set

$$g_n = E(\tau_n) = 1/(n\delta+\mu+\gamma_j) \tag{4.93}$$

Using these results in equation (4.15) we obtain

$$M_{N_j} = E(w_{N_j}) = \sum_{k=1}^{N_j} \sum_{i=k}^{\infty} \left(\gamma_j^{i-k} / \prod_{\ell=k}^{i}(\ell\delta+\mu)\right) \tag{4.94}$$

Once one computes the values of $E(w_{N_j})$ using equation (4.94), the second moment $E(w_{N_j}^2)$ is found by setting

$$g_n = \frac{2}{(n\delta+\mu+\gamma_j)^2} + 2\Big[\frac{\gamma_j}{n\delta+\mu+\gamma_j}\,\frac{1}{(n+1)\delta+\mu+\gamma_j}\,E(w_{n+1})$$

$$+ \frac{n\delta+\mu}{n\delta+\mu+\gamma_j}\,\frac{1}{(n-1)\delta+\mu+\gamma_j}\,E(w_{n-1})\Big] \tag{4.95}$$

and using this result in equation (4.15). As one can see, the analytic results become messy; computationally, there is no problem in obtaining

$E(w_{N_j}^2)$. The variance of w_{N_j} is then found by subtraction.

The additional time w^* our test applicant must wait is not equal to w_1, the length of time necessary to house a solitary household found waiting in priorities 1 through j. This is due to our assumption that the test applicant will not drop out. To handle this scenario, we modify the dropout rate to $(n-1)\delta$ when n households, including the test household, are waiting for assignments. In this manner, when n equals 1 (i.e. when only the test applicant is waiting), the dropout rate equals zero (i.e. the test applicant cannot dropout). Utilizing our previous arguments, we obtain for the mean additional waiting time

$$E(w^*) = \sum_{i=1}^{\infty} \left(\gamma_j^{i-1} / \prod_{\ell=1}^{i} \{(\ell-1)\delta+\mu\}\right) \qquad (4.96)$$

To find $E(w^{*2})$, we set g_n as in equation (4.95), but we substitute a dropout rate of $(n-1)\delta$ for $n\delta$ throughout. This also involves a recalculation of the equivalent "mean" waiting times $E(w_n)$; equation (4.94) may be used, but again the dropout rate $\ell\delta$ must be modified to $(\ell-1)\delta$.

An entirely equivalent (and perhaps less confusing!) procedure is to compute $E(w_{N_j}^*)$ and $E(w_{N_j}^{*2})$ directly by initially adjusting the dropout rate to $(n-1)\delta$ and re-setting N_j to N_j+1. This approach directly yields

$$E(w_{N_j}^*) = \sum_{k=1}^{N_j+1} \sum_{i=k}^{\infty} \left(\gamma_j^{i-k} / \prod_{\ell=k}^{i} \{(\ell-1)\delta+\mu\}\right) \qquad (4.97)$$

as the mean waiting time for our test applicant. The second moment is obtained by setting g_n as in equation (4.95), correcting $n\delta$ to $(n-1)\delta$, and setting $E(w_{n+1})$ to $E(w_n^*)$, and $E(w_{n-1})$ to $E(w_{n-2}^*)$ where $E(w_n^*)$ is conputed using equation (4.97).

This model appears complicated, but it is the first model with sufficient realism to be of actual use. We have already seen that categorical priority systems can be construed to represent a range of different assignment policies (including transfers and point scoring systems). The addition of dropout to this model, while increasing the difficulty of the analysis involved, provides us with a reasonable approach to forecasting waiting times. Implementation of this model on a digital computer poses no special difficulties, as the following example demonstrates.

4.8.1 An Example

Again we assume a project with an annual moveout rate of $\mu=20$ households per year. We fix the arrival rate of new applicants in priorities 1 through j-1 to $\gamma_j=10$ households per year. The mean and variance of the waiting times for a test applicant are shown for various values of δ, the household specific dropout rate, in Figures 4.18 and 4.19. Note the dramatic effect of increasing δ on the waiting time. It appears that mean waiting time grows logarithmically with the size of the waiting list for positive values of δ; the variance of waiting time appears to approach a limit as the waiting list becomes large. In housing authorities where categorical priorities more or less define the entire tenant assignment system, graphs such as Figures 4.18 and 4.19 could be used to conveniently forecast waiting times for new applicants.

Figure 4.18

Mean Waiting Time in
a Categorical Priority
System with Dropout

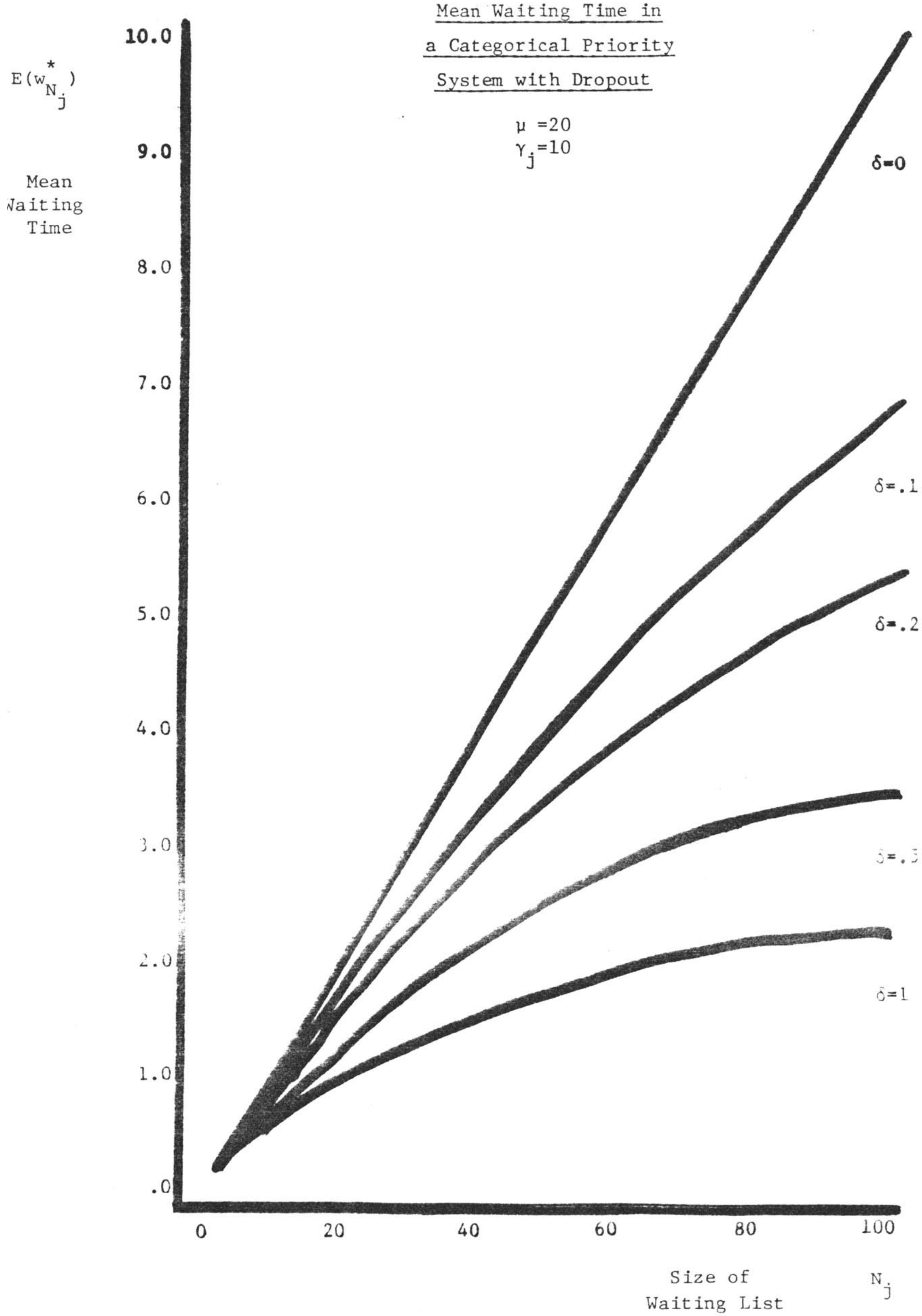
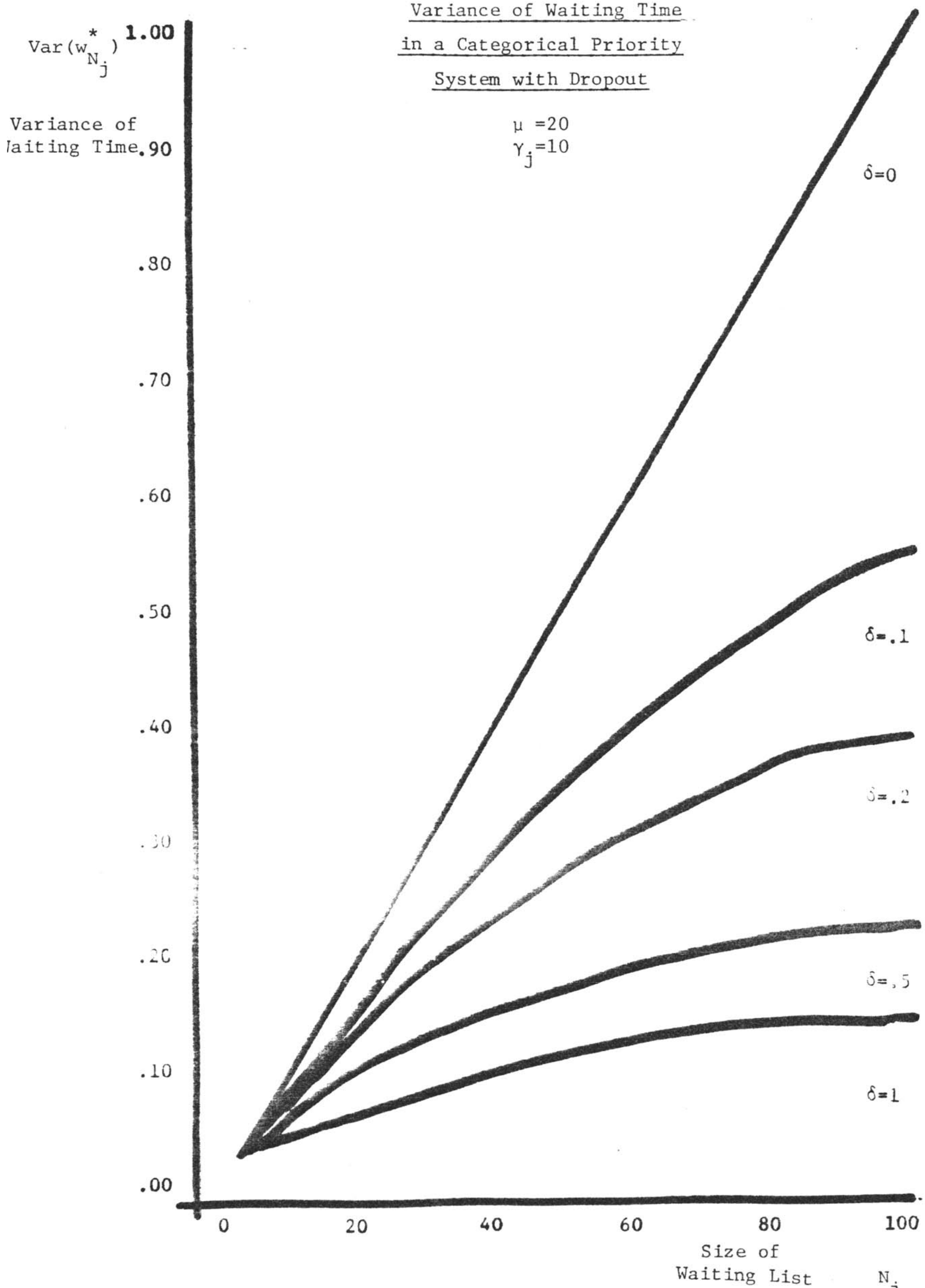


Figure 4.19

Variance of Waiting Time in a Categorical Priority System with Dropout

$\mu = 20$
$\gamma_j = 10$



4.8.2 A Balance Equation

In accounting for tenant flow, housing officials are sometimes interested in the number of tenants housed, the number of dropouts, and the number of new applicants in a given time period. If we define

$N_{A_j} \equiv$ number of households assigned in priorities 1 through j

$N_{D_j} \equiv$ number of households who drop out in priorities 1 through j

$N_{\gamma_j} \equiv$ number of new applicants in priorities 1 through j-1

$N_j \equiv$ number of households found waiting by a newly arriving priority j test applicant

we see that over the period of time until our test applicant is housed, we must have

$$N_{A_j} = 1 + N_j + N_{\gamma_j} - N_{D_j} \tag{4.98}$$

where the "1" refers to the certainty of our test applicant being assigned. As this equation must hold in expected value, we obtain

$$\mu E(w^*_{N_j}) = 1 + N_j + \gamma_j E(w^*_{N_j}) - E(N_{D_j}) \tag{4.99}$$

where we have taken advantage of the Poisson assignment and priority arrival processes, and the known quantity $1+N_j$.

We have argued that

$$E(N_{A_j}) = \mu E(w^*_{N_j}) \tag{4.100}$$

$$E(N_{\gamma_j}) = \gamma_j E(w^*_{N_j}) \tag{4.101}$$

and using equation (4.99) we find that

$$E(N_{D_j}) = 1 + N_j + (\gamma_j - \mu) E(w^*_{N_j}) \tag{4.102}$$

Note that if $\gamma_j > \mu$ (a possibility for this model), the expected number of dropouts will be larger than the number of households originally waiting; if $\gamma_j < \mu$, the expected number of dropouts will be less than the size of the initial waiting list. This result holds even for very small dropout rates; in fact, the result does not depend on the specific dropout model we have assumed - any dropout process will yield equation (4.102).

4.9 Categorical Priorities, Blend Priorities and Dropout

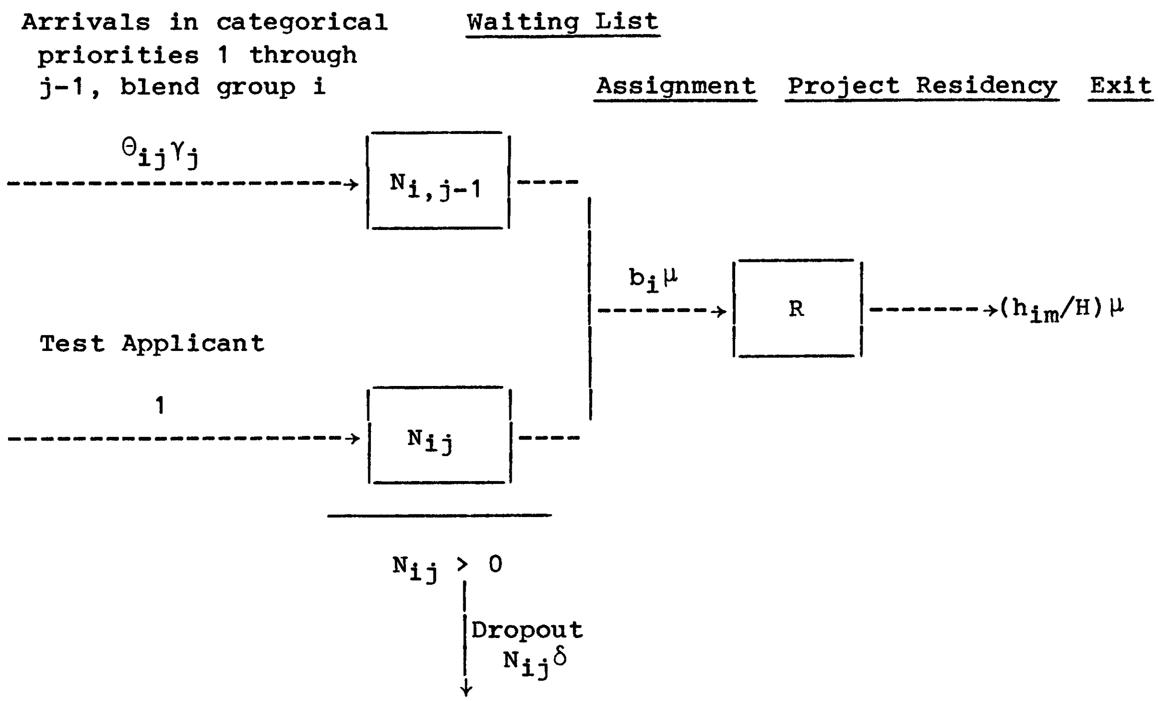
We conclude our discussion of single project tenant assignment models by considering a realistic system with categorical priorities, blend priorities and dropout. The system is depicted graphically in Figure 4.20; a list of assumptions is also shown. Though this system appears complex, it follows quite naturally from our earlier work.

As usual, we are interested in the waiting time faced by a test applicant; in this case, our test household is a member of blend group i and categorical priority j. Our test applicant finds N_{ij} households waiting in categorical priorities 1 through j who are also members of blend group i. Households in categorical priorities 1 through j-1 arrive according to a Poisson process with parameter γ_j. Of these new arrivals, a fraction Θ_{ij} will be members of blend group i, $\sum_i \Theta_{ij} = 1$. Thus, the arrival rate for new blend group i households in categorical priorities 1 through j-1 equals $\Theta_{ij}\gamma_j$.

Tenant assignments take place at rate μ. However, on each assignment, the probability of assigning a blend group i tenant is set equal to b_i, the desired fraction of group i tenants in the project. Therefore, blend group i tenants are assigned according to a Poisson process with rate $b_i\mu$.

Figure 4.20

Categorical Priorities, Blend Priorities And Dropout



Assumptions

1) n_{ij} categorical priority j households in blend group i are found waiting for housing assignments by a newly arriving categorical priority j blend group i household.

$$N_{ij} = \sum_{k=1}^{j} n_{ik}.$$

2) Categorical priority k households arrive according to a Poisson process with rate λ_k. $\gamma_j = \sum_{k=1}^{j-1} \lambda_k$ = arrival rate of all households in categorical priorities 1 through j-1.

3) The probability that a newly arriving household in categorical priorities 1 through j-1 is also in blend group i equals θ_{ij}; $\sum_{i=1}^{B} \theta_{ij} = 1$ where B equals the number of blend groups.

4) A categorical priority j household is assigned only if no households in categorical priorities 1,...,j-1 are present.

5) Within categorical priorities, a household from blend group i is assigned with probability b_i, $\Sigma b_i=1$. Within categorical and blend groups, assignment is in order of application.

6) Dropouts occur at rate $n\delta$ when n households (not including the test applicant) are waiting for housing assignments. The test applicant will not drop out with certainty.

7) Tenant assignments take place in a Poisson fashion with rate μ; tenants from blend group i are assigned at rate $b_i\mu$.

Finally, dropout is assumed to occur at a rate proportional to the number of households waiting. The test applicant does not drop out by assumption.

For our test applicant, the relevant facts are:

1) N_{ij} households are already waiting.

2) Assignments take place at rate $b_i\mu$.

3) Higher priority applicants arrive at rate $\Theta_{ij}\gamma_j$.

4) Dropout is proportional to $n\delta$ when n households in blend group i, categorical priorities 1 through j are waiting ahead of the test applicant.

We can therefore use our earlier results to obtain $E(w^*_{N_{ij}})$, the expected waiting time for our test applicant from arrival until assignment, by making the following adjustments in equation (4.97):

1) Substitute N_{ij} for N_j. (initial waiting list)

2) Substitute $b_i\mu$ for μ. (assignment rate)

3) Substitute $\Theta_{ij}\gamma_j$ for γ_j. (categorical priority arrival rate)

These changes yield

$$E(w^*_{N_{ij}}) = \sum_{k=1}^{N_{ij}+1} \sum_{m=k}^{\infty} \left([\Theta_{ij}\gamma_j]^{m-k} / \prod_{\ell=k}^{m} [(\ell-1)\delta + b_i\mu]\right) \tag{4.103}$$

The second moment of $w^*_{N_{ij}}$ is obtained from the arguments following equation (4.97) making the substitutions indicated above. As long as the probabilities Θ_{ij} can be estimated, the implementation of this model should pose no special problems. As presented, the model is quite rich in the variety of situations that can be studied.

4.9.1 Statistical Issues

This section has introduced the new quantity Θ_{ij}, the likelihood that a new arrival in categorical priority 1, 2,..., j-1 is a member of blend group i. Rather than estimating Θ_{ij} directly, we will choose an indirect approach that is easily understood. Let β_{ij} represent the probability that a newly arriving household in categorical priority j is in blend group i, $\sum_i \beta_{ij}=1$. Then the likelihood that an arrival in one of categorical priorities 1 through j-1 is in blend group i equals

$$\Pr\{\text{arrival in blend group } i \mid \text{arrival in one of categorical priorities 1 through } j-1$$
$$= \sum_{k=1}^{j-1} \beta_{ik} \Pr\{\text{arrival in categorical priority } k \mid \text{arrival in one of categories } i \text{ through } j-1 \qquad (4.104)$$

by the laws of conditional probability. However, since categorical priority j applicants arrive according to a Poisson process with parameter λ_j, we have

$$\Pr\{\text{arrival in categorical priority } k \mid \text{arrival in one of priorities 1 through } j-1\}$$
$$= \lambda_k / \sum_{\ell=1}^{j-1} \lambda_\ell \qquad (4.105)$$

Thus, we can express θ_{ij} as

$$\theta_{ij} = \sum_{k=1}^{j-1} \beta_{ik}\lambda_k / \sum_{\ell=1}^{j-1} \lambda_\ell$$
$$= \frac{1}{\gamma_j} \sum_{k=1}^{j-1} \beta_{ik}\lambda_k \qquad (4.106)$$

Now, the quantities λ_k and γ_j are easily estimated as has already been discussed in equations (4.84) and (4.85). Also, to estimate β_{ik}, one merely observes the fraction of newly arriving categorical priority k households who are also members of blend group i. These estimates may be denoted by $\hat{\beta}_{ik}$, and we can thus estimate θ_{ij} by

$$\hat{\theta}_{ij} = \frac{1}{\hat{\gamma}_j} \sum_{k=1}^{j-1} \hat{\beta}_{ik} \hat{\lambda}_k \tag{4.107}$$

4.9.2 An Example

As before, we consider a project with an annual moveout rate of μ=20 households per year. New households in categorical priorities 1 through j-1 arrive at a rate of γ_j=10 per year. We assume a household specific dropout rate of δ=.1. In addition, we consider two blend groups; group 1 tenants receive 33% of all assignments, while group 2 tenants receive 67% of all assignments. Of those households arriving in categorical priorities 1 through j-1, 50% are from blend group 1, and 50% are from blend group 2.

The expected waiting time faced by an arriving group i test applicant who finds N_{ij} group i households waiting for assignments is plotted in Figure 4.21; the corresponding variances are shown in Figure 4.22. Note the long waits for group 1 applicants; this is due to the fact that the effective arrival rate $\theta_{1j}\gamma_j$ is close to the effective service rate $b_1\mu$. In addition to longer mean waits, the uncertainty associated with waiting time is relatively higher for group 1 than group 2 applicants, as is clearly seen from Figure 4.22.

4.10 Summary

This chapter has developed a variety of models for studying tenant assignment policies in single project housing systems. The models are

Figure 4.21

Mean Waiting Time in a Categorical Priority, Blend Priority System with Dropout

$\mu = 20$
$\gamma_j = 10$
$\delta = .1$

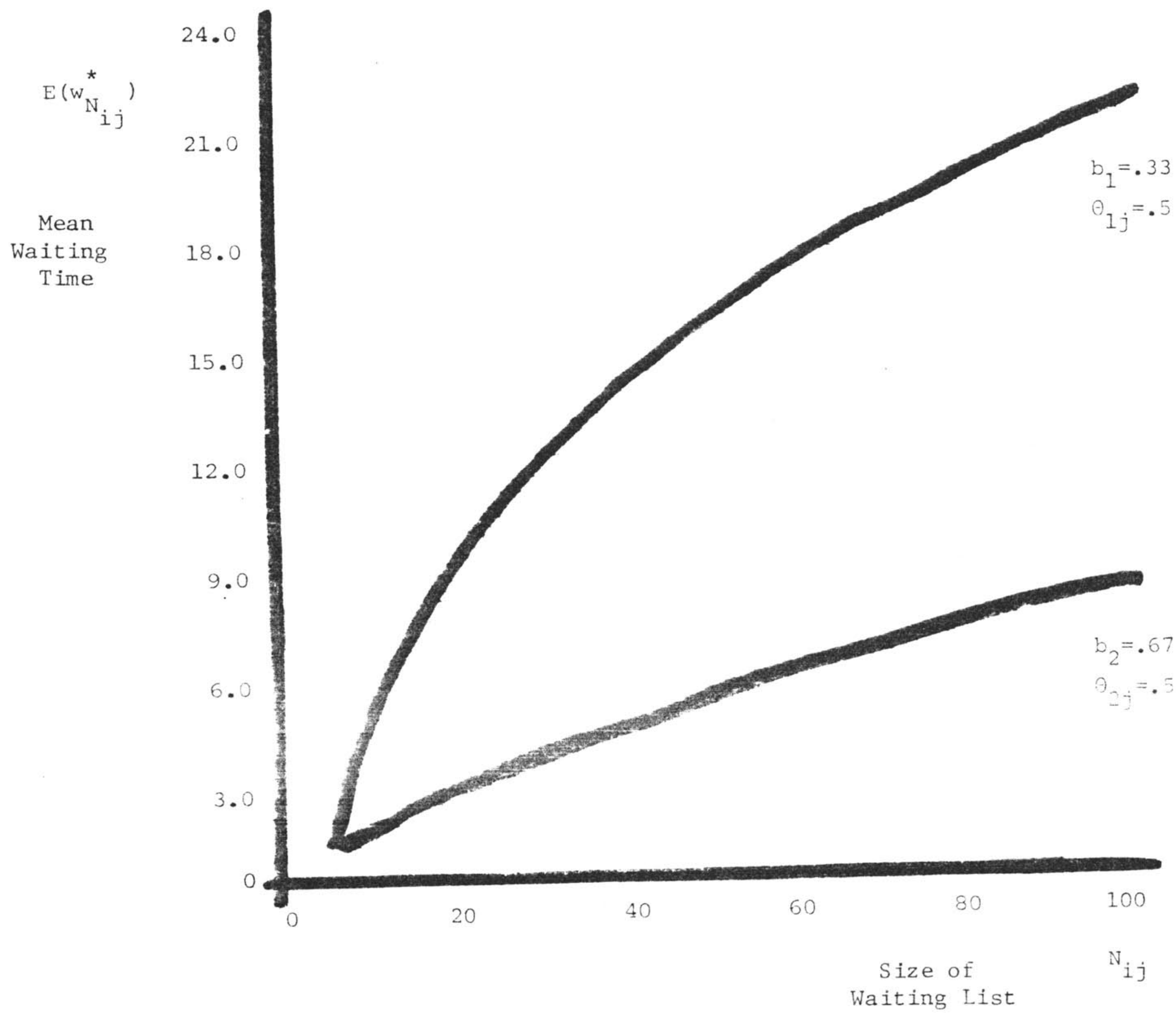
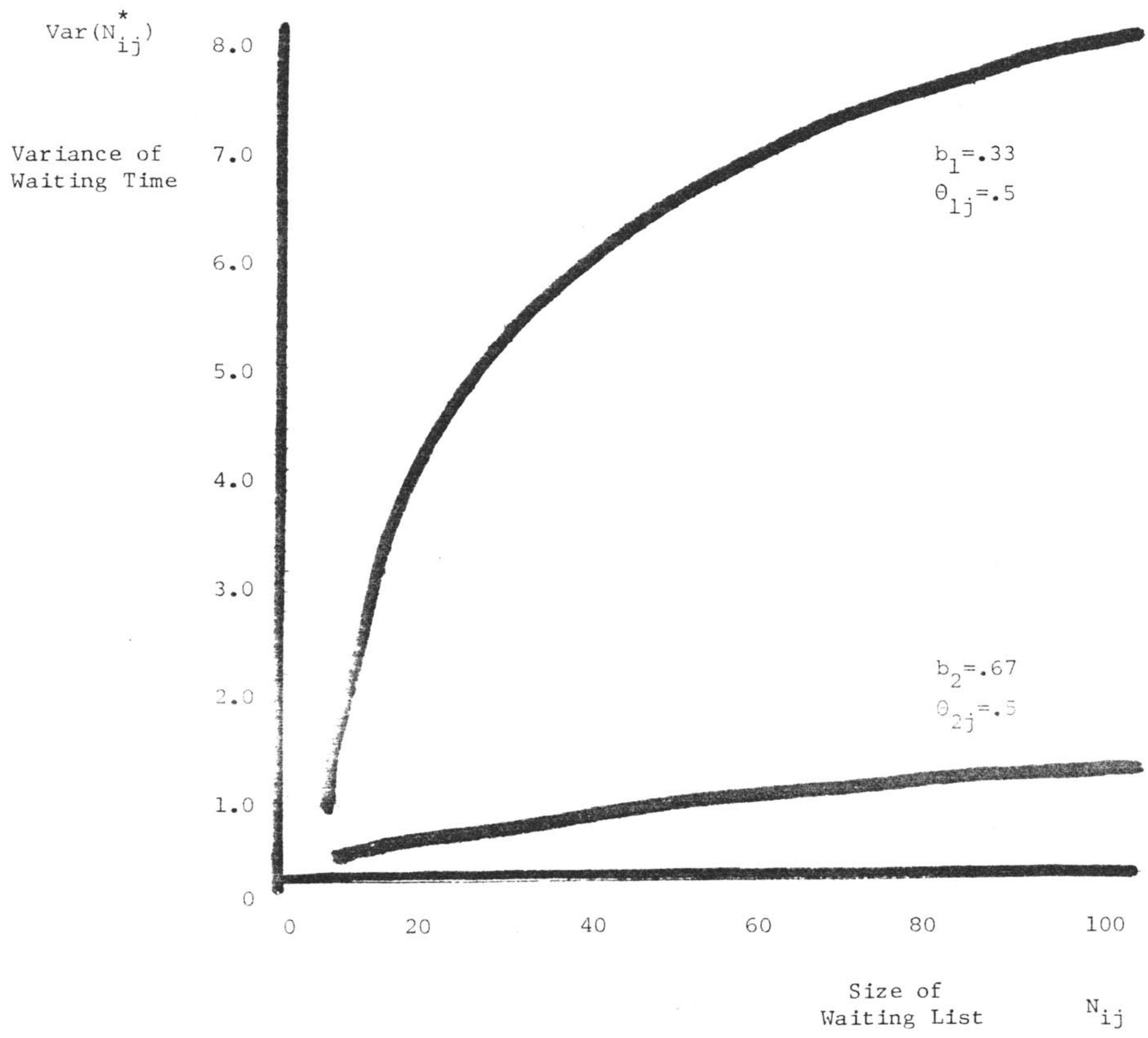


Figure 4.22

Variance of Waiting Time in a Categorical Priority, Blend Priority System with Dropout

$\mu = 20$
$\gamma_j = 10$
$\delta = .1$



rich in that they incorporate:

-- project moveout rates

-- dropout

-- blend priorities (e.g. racial integration)

-- categorical priorities (e.g. emergencies)

-- the number of households waiting for assignments

The major focus has been on modeling the time from when a new applicant arrives until that applicant is housed. We also examined allocational quantities such as the expected number of households assigned and the expected number of dropouts.

The models assume that there is no interaction among unit types; the models can account for transfer policies within the context of categorical priorities. More importantly, the models reflect single project, no choice assignment systems. They are not immediately applicable to multi-project schemes or systems with an appreciable degree of tenant choice. In the next chapter, we will build on the work just completed to design models for multi-project systems with various degrees of tenant choice.

Chapter V

Assignment Models with Tenant Choice

One characteristic that distinguishes many tenant assignment schemes from other queueing systems is the choice given to the tenants (customers) regarding which project (server) they are assigned to (receive service from). From our review of tenant assignment policies in Chapter 2, we can identify two systems for addressing the issue of tenant choice: refusal systems and multi-queue systems. We will briefly discuss each of these before modeling waiting times for these classes of tenant assignment systems.

5.1 Refusal Systems

In many cities, when a household is offered an apartment, the household can refuse to accept the apartment, for whatever reasons, without penalty - to a point. For example, a household might be allowed three offers; if the household rejects all three offers, then it must retreat to the back of the waiting list. If an apartment is rejected by a certain household, it is immediately offered to the next household on the waiting list.

Refusal systems allow households some degree of choice, albeit by after-the-fact rejection as opposed to before-the-fact selection. The higher the number of refusals allowed, the more flexible the system is from the prospective tenant's standpoint. Of course, the ability of the authority to maintain control over the composition of housing projects decreases as the number of allowed refusals increases.

5.2 Multiqueue Systems

Multiqueue systems, while difficult to analyze, are in many ways the fairest tenant assignment processes from the prospective tenants' viewpoint. In a multiqueue system, tenants specify a number of projects (up to some maximum) in which they are willing to reside. The household's name is placed on a waiting list at every project in the relevant choice set - the set of projects under consideration for residency. The household is assigned to that project in the choice set where a unit becomes first available. Thus, households are guaranteed to be offered a unit in a project of their own choosing. If a household is very choosy, they need only specify a single choice - guaranteeing an assignment to that choice.

The drawbacks to such a system are twofold. First of all, since some projects will be more "popular" than others, waiting times will become unbalanced. Secondly, the housing authority loses a good deal of control over project composition in such systems. Even though priorities may be instituted to promote social goals such as racial integration and income mixing, these priorities may never gain the chance to be enacted if households choose projects strictly along ethnic or other socioeconomic lines.

5.3 Models for Refusal Systems

We will now proceed to modify some of our results from Chapter 4 to incorporate the possibility that households may refuse offered apartments. The essential addition to our models is the probability that a prospective tenant will accept an offer. As noted in Chapter 2, refusal systems differ according to the number of apartments tenants are allowed to refuse without penalty. It is very difficult to model an assignment scheme with an arbitrary number of "strikes" but two special cases which bound all possibilities are tractable and of considerable interest:

(i) Infinite Refusal - tenants may refuse offers indefinitely until accepting an apartment

(ii) One Strike Refusal - tenants are required to either accept the first apartment offered or face dismissal from the system.

As refusal models become complicated, we will focus our attention solely on mean waiting times. To clarify the notion of choice by refusal, we first consider models which ignore dropout and priority structures; these features will be reinstated later on in the chapter.

5.3.1 Infinite Refusal: No Dropout, No Priorities

The situation developed in this model is similar to the model in equations (4.19) and (4.20), the only difference being the incorporation of refusal. A newly arriving test applicant finds $N>0$ households waiting for housing assignments at a given project. Households leave the project in Poisson fashion at rate μ. Whenever a household leaves the project, the apartment vacated is immediately offered to the first household on the waiting list. This apartment is accepted with probability α, $0 \leq \alpha \leq 1$. If the apartment is refused, it is offered to the next household or the waiting list; this next household also accepts with probability α.

Thus, we have a situation where all households are assumed to accept offered apartments with the same acceptance probability α, independently of the actions of other households. We make one exception to this rule: our test applicant will <u>not</u> accept any apartment that has been offered to other households; when our test applicant is offered a "fresh" unit, the offer is accepted with probabiltiy . The justification for this assumption lies in the resulting mathematical simplifications introduced; the actual impact of this assumption on numerical results is negligible.

We make one additional assumption. If at some time there are n households waiting for assignments ahead of our test applicant, and a unit is offered and refused, by all n+1 households of interest on the waiting list, the unit is then automatically filled by a household from a backup list of infinite size. This assumption preserves the identity between moveouts and tenant assignments, and is made for that purpose. A diagram of this system and a list of its attendant assumptions are found in Figure 5.1.

As in the models of Chapter 4, we let w_N denote the time necessary to process the N households found waiting by our test applicant. The assumptions of our process lead to the following equation for the mean of w_N:

$$E(w_N) = \frac{1}{\mu} + (1- \ )^N E(w_N) + [1-(1- \ )^N]\ E(w_{N-1}) \qquad (5.1)$$

This equation is of the form described by (4.3) with:

$p_n=0,\ r_n=(1-\alpha)^n,\ q_n=1-(1-\alpha)^n,\ g_n=\mu^{-1}$

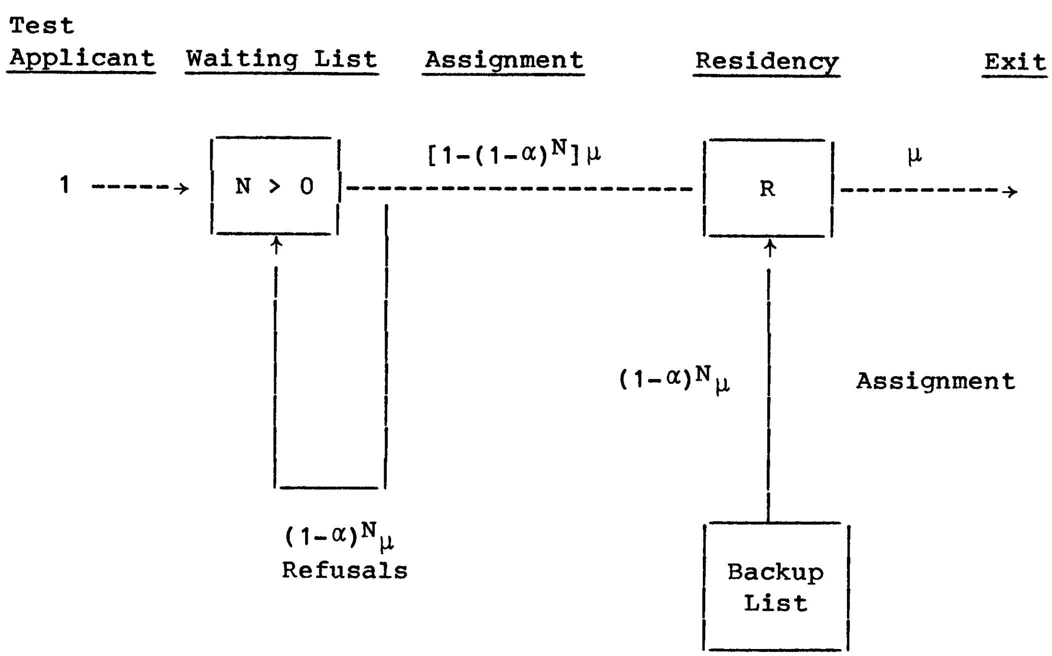
Since $p_n=0\ \forall n$, we see from (4.10) that the solution of (5.1) is given by

$$E(w_N) = \frac{1}{\mu} \sum_{i=1}^{N} \frac{1}{1-(1-\alpha)^i} \qquad (5.2)$$

Now, the additional mean time $E(w^*)$ our test applicant must wait clearly equals $E(w_1)$, the time necessary to assign a single household found waiting. From (5.2) we have

Figure 5.1

Infinite Refusal: No Dropout, No Priorities



Assumptions

1) N>0 households are found waiting for housing assignments by a newly arriving test applicant.

2) Households are offered units in order of application. Households accept units with probability α, independently of the decisions of other households. The test applicant refuses any unit previously offered to another household, and accepts "fresh" units with probability α.

3) No households drop out. If all households (including the test applicant) refuse an offer, the unit is assigned to a household from a backup list.

4) Households leave the project according to a Poisson process with rate μ.

$$E(w^*) = \frac{1}{\mu}\frac{1}{\alpha} \tag{5.3}$$

Combining (5.2) and (5.3), we find for the mean time to house our test applicant from arrival given that N households are found waiting:

$$E(w_N^*) = \frac{1}{\mu}\left[\frac{1}{\alpha} + \sum_{i=1}^{N} \frac{1}{1-(1-\alpha)^i}\right] \tag{5.4}$$

Note that if α, the acceptance probability, equals one, (5.4) reduces to $(N+1)/\mu$, the same result obtained for our simplest model of Chapter 4 in equation (4.19).

Using the arguments based on the mean value theorem developed in equations (4.38) through (4.40), we can approximate (5.4) by a closed form expression. Noting that

$$\int \frac{1}{1-(1-\alpha)^x}\,dx = x - \frac{\log[1-(1-\alpha)^x]}{\log[1-\alpha]} + \text{constant} \tag{5.5}$$

we have the approximation

$$\sum_{i=1}^{n} \frac{1}{1-(1-\alpha)^i} \approx \frac{1}{2}\left[2n-1+\frac{1}{\alpha} + \frac{1}{\log(1-\alpha)}\log\left\{\frac{\alpha^2}{[1-(1-\alpha)^{n+1}][1-(1-\alpha)^n]}\right\}\right] \tag{5.6}$$

Using this result in (5.4) yields

$$E(w_N^*) \approx \frac{1}{\mu}\left[\frac{1}{\alpha} + \frac{1}{2}\left[2N-1+\frac{1}{\alpha} + \frac{1}{\log(1-\alpha)}\log\left\{\frac{\alpha^2}{[1-(1-\alpha)^{N+1}][1-(1-\alpha)^N]}\right\}\right]\right] \tag{5.7}$$

As N becomes large, (5.7) tends towards

$$E(w_N^*) \approx \frac{1}{\mu}\left[\frac{1}{\alpha} + N - \frac{1}{2} + \frac{1}{2\alpha} + \frac{\log\alpha}{\log(1-\alpha)}\right]$$

$$= \frac{N}{\mu} + C \tag{5.8}$$

where

$$C = \frac{1}{\mu}\left[\frac{3}{2\alpha} - \frac{1}{2} + \frac{\log \alpha}{\log(1-\alpha)}\right] \qquad (5.9)$$

Thus, the expected waiting time for our test applicant grows linearly with the number of households found waiting for assignments. Note that as α, the acceptance probability, approaches 1, the constant C approaches μ^{-1}, yielding $E(w_N^*) = (N+1)/\mu$, as expected.

Although the analysis of this section has assumed that prospective tenants have the right to refuse an infinite number of offers, the results may serve as approximations to systems offering small numbers of units for consideration, providing the acceptance probability α is relatively high. For example, if a system allows for 3 strikes, and α=.8, the likelihood that a tenant is dismissed from the system equals $(1-.8)^3$=.008. This is rather close to zero, the corresponding dismissal probability in an infinite refusal system. Therefore, we feel that infinite refusal models offer a reasonable approximation to finite strike systems as long as the acceptance probability is relatively high. Unfortunately, we currently have no estimates of α; methods for estimating α will follow our discussions of one strike systems.

5.3.2 One Strike Refusal: No Dropout, No Priorities

In a one strike system, a household is offered exactly one unit. If this unit is refused, the household is immediately dismissed from the system. To analyze one strike schemes, we assume that households accept offers with acceptance probability α, and that an available unit is offered until it is accepted. We will now consider the case where a newly arriving test applicant finds N households.

The process realized by one strike refusal is not of the birth and death type described in Chapter 4, so our usual methods no longer apply. The state transition diagram associated with one-strike assignment is shown in Figure 5.2 for the case where N=2, and our test applicant is thus the 3rd household on the waiting list. The "states" of the system indicate the number of households waiting for housing assignments. Once a state is entered, the system remains in that state for a period of length τ, where τ is exponentially distributed with mean μ^{-1}. Households are dismised from the system whenever an offer is refused; units are offered until they are accepted.

Defining w_n to be the time necessary to process the first n households on the waiting list, we obtain the following equation for the mean wait:

$$E(w_n) = \frac{1}{\mu} + \sum_{k=1}^{n-1} \alpha(1-\alpha)^{k-1} E(w_{n-k}) \qquad (5.10)$$

Given that n households are waiting, the likelihood that the next unit offered is accepted by the kth household in queue equals $\alpha(1-\alpha)^{k-1}$. When this event occurs, (k-1) households are dismissed for refusing an offer, while an additional household (the kth) is assigned. Thus, there are n-k households remaining to be processed. Of course, the time to process zero households is zero, while the mean time between successive unit offers equals μ^{-1}. These arguments taken together produce equation (5.10).

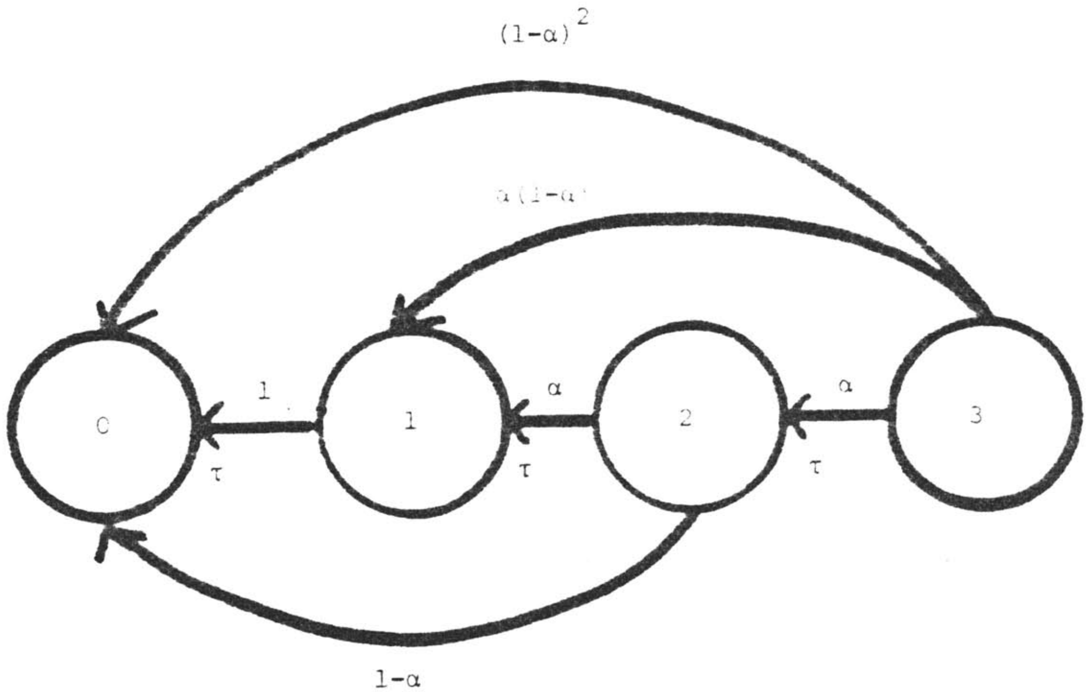
This equation can be solved by induction to yield

$$E(w_n) = \frac{1}{\mu} [1+(n-1)\alpha] \qquad (5.11)$$

as the expected time necessary to process n households. Now, when our test applicant discovers N households waiting, our applicant becomes

Figure 5.2

State Transition Diagram for the
One Strike Refusal Model



Assumptions

1) Households accept offers independently with probability α. Units are offered until they are accepted. If a household refuses an offer, it is dismissed from the system.

2) The length of time between successive moveouts, τ, is exponentially distributed with mean μ^{-1}.

the (N+1)st household in queue. Using (5.11), the expected waiting time for our test applicant is seen to equal

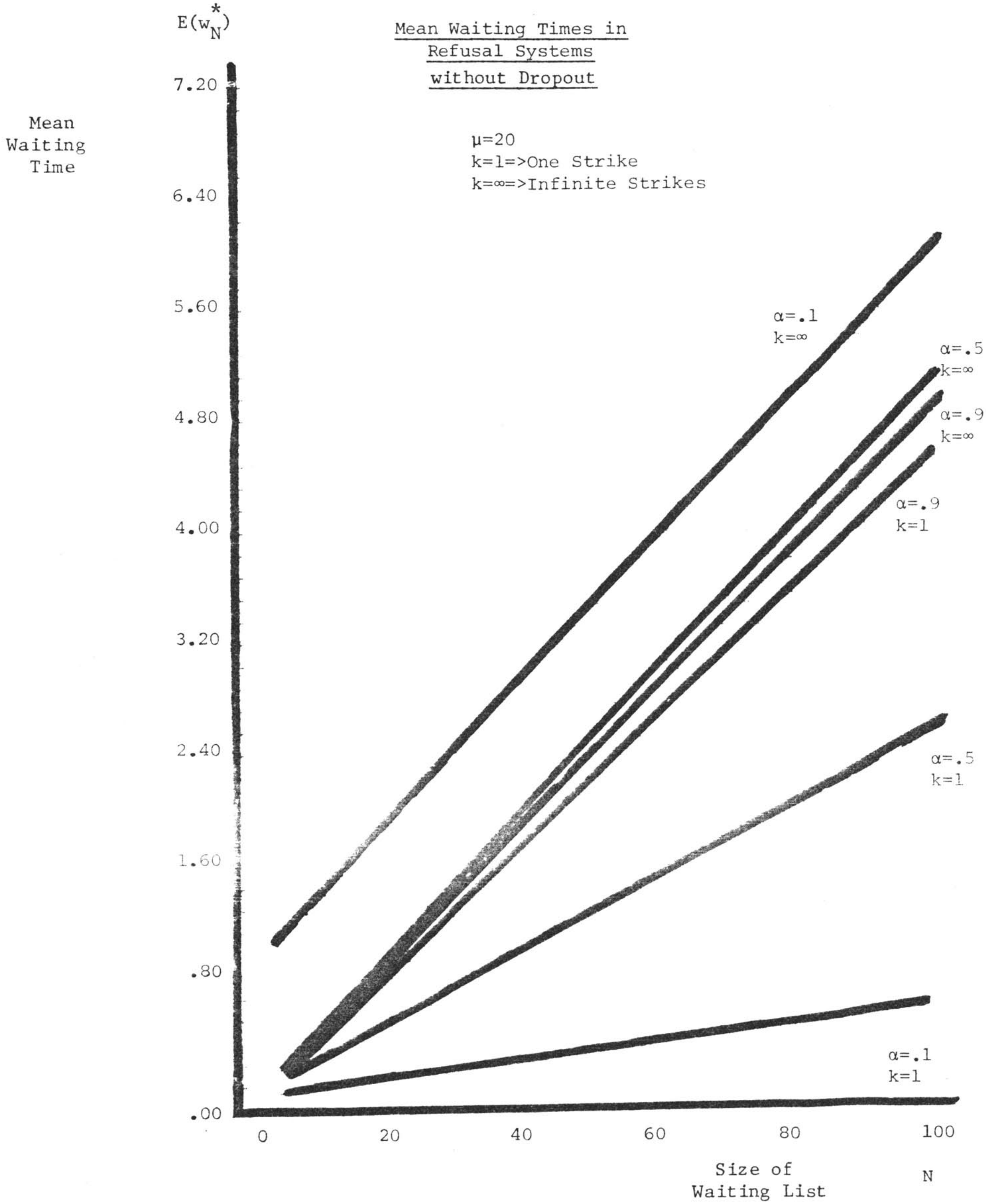
$$E(w_N^*) = \frac{1}{\mu}[1+N\alpha] \tag{5.12}$$

This result has a simple interpretation. Since α is the acceptance likelihood, $N\alpha$ is the expected number of units accepted by those households found on the waiting list. The quantity $N\alpha$ is also the expected number of units offered to the N households found by our test applicant. Our test applicant is guaranteed to receive an offer, thus the total number of units offered equals $1+N\alpha$ in expectation. As the mean time between offered units equals μ^{-1}, the time necessary to process our test applicant equals $(1+N\alpha)\mu^{-1}$. Note that this result requires no assumption about the acceptance or rejection of a unit by the test applicant based on decisions made by other households.

It is interesting to compare (5.12) to the equivalent result for infinite strike systems developed in (5.4) through (5.8). As expected, the two schemes are equivalent when $\alpha=1$; both yield a mean wait of $\frac{N+1}{\mu}$ time units. As α decreases, the mean wait increases for infinite strike systems, and decreases for one strike systems. For any particular finite strike refusal system, the mean waiting time can be bounded from above by the infinite refusal model, and from below by the one strike model. If α is relatively high, these bounds are fairly tight. the situation is illustrated graphically in Fig. 5.3 where equations (5.8) and (5.12) are graphed as functions of N for various values of α with $\mu=20$.

Figure 5.3

Mean Waiting Times in Refusal Systems without Dropout



5.3.3 Estimating the Acceptance Probability

To use the models described, one needs to estimate the acceptance probability α. Perhaps the easiest approach to this problem is to count the number of offers required for each newly available unit to be assigned. Regardless of the number of strikes in a refusal policy, the probability that a newly available unit will be assigned on the k^{th} offer is given by

$$p_k = \alpha(1-\alpha)^{k-1} \qquad k=1,2... \tag{5.13}$$

Thus, the likelihood of observing a sample of n units where unit i required k_i offers until acceptance, i=1, ..., n is given by

$$L = \prod_{i=1}^{n} \alpha(1-\alpha)^{k_i-1} = \alpha^n(1-\alpha)^{\Sigma k_i-n} \tag{5.14}$$

Maximizing this expression with respect to α yields the maximum likelihood estimate

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} k_i} \tag{5.15}$$

Of course, $\hat{\alpha}$ is the well known maximum likelihood estimate for the "success" probability when sampling from the geometric distribution.

To estimate α then, one merely counts the total number of offers required to fill a pre-selected set of newly available apartments, and divides this into the number of units in question. As mentioned, this procedure will work for refusal schemes of any number of strikes, as long as the assumption of a constant acceptance probability is met. In fact, one could test the validity of this assumption by seeing if the observed numbers of offers required to assign units follows a geometric distribution, but we have not yet obtained sufficient data to do so.

5.4 Refusal Models with Dropout and Categorical Priorities

The models of the previous section have served to illustrate the effect of introducing choice by refusal into tenant assignment systems. However, to return to a more realistic modeling scenario, we will reintroduce dropout and categorical priorities as introduced in Chapter 4. The incorporation of blend priorities is relatively straightforward and will therefore be omitted from the ensuing discussion.

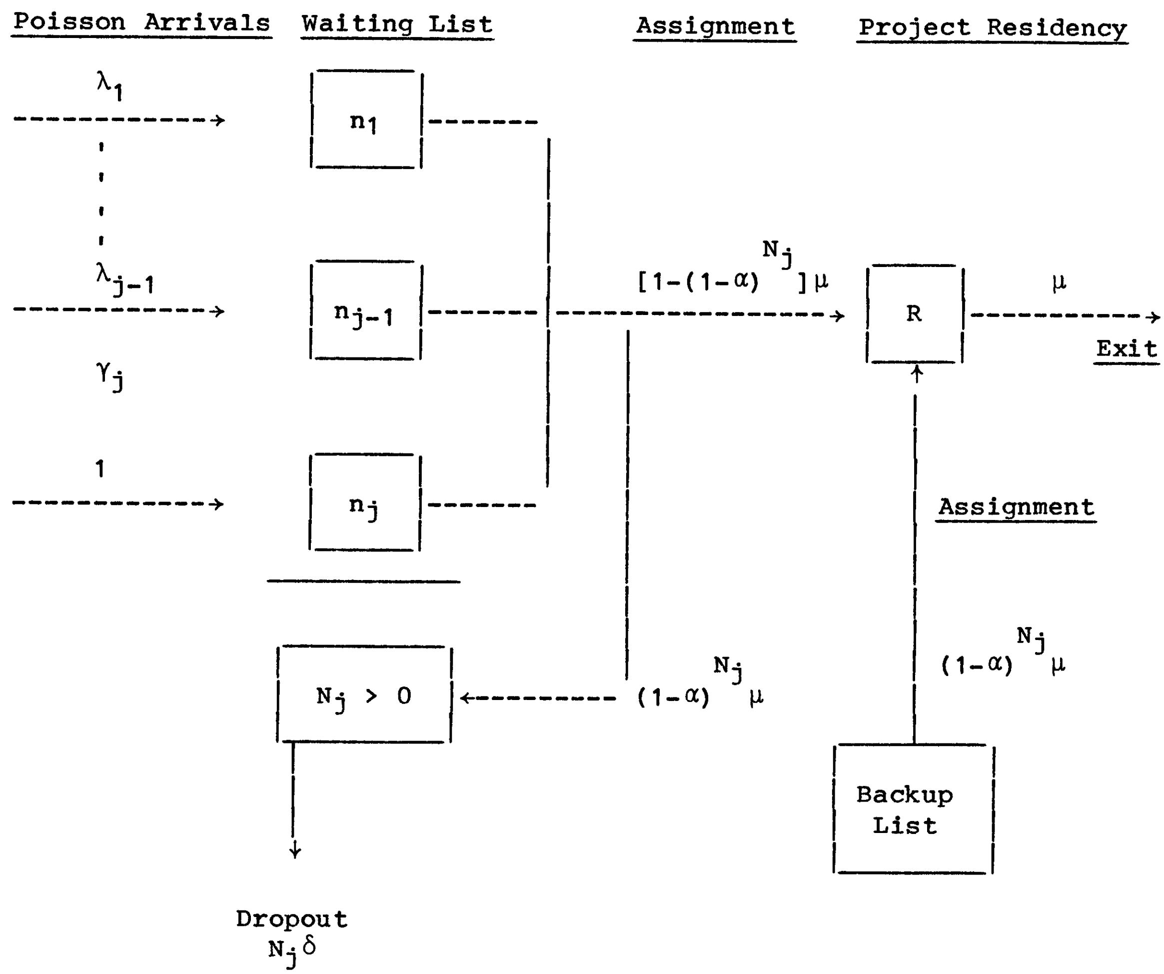
5.4.1 Infinite Refusal with Dropout and Categorical Priorities

We assume that new applicants in categorical priorities 1 through j-1 arrive according to a Poisson process with parameter γ_j. Our newly arriving test applicant finds $N_j>0$ households waiting in priorities 1 through j. Households drop out of the system at rate $n\delta$ when n households (not including the test applicant) are waiting for assignments; by assumption, the test applicant will not drop out. As usual the assignment process is Poisson with rate μ. In addition, we assume that households independently accept offered units with acceptance probability α, and that households may indefinitely refuse units before accepting an offer. Finally, we assume that our test applicant will not accept an apartment that has been previously refused by another household. This model is diagrammed in Figure 5.4.

These assumptions define a birth and death process, thus the methods of equations (4.2) through (4.15) apply. It is easiest to solve first for the expected value of w_{N_j}, the time necessary to process the N_j households found waiting. We will then find $E(w^*)$, the additional mean time until our test applicant is housed. Consistent with our general formulation for birth and death processes, we set:

Figure 5.4

Infinite Refusal with Dropout and Categorical Priorities



$$p_n = \frac{\gamma_j}{n\delta+\mu+\gamma_j} \tag{5.16}$$

$$r_n = \frac{(1-\alpha)^n\mu}{n\delta+\mu+\gamma_j} \tag{5.17}$$

$$q_n = \frac{n\delta + \left(1-(1-\alpha)^n\right)\mu}{n\delta+\mu+\gamma_j} \tag{5.18}$$

To obtain $E(w_{N_j})$, we assign

$$g_n = \frac{1}{n\delta+\mu+\gamma_j} \tag{5.19}$$

and obtain from equation (4.15)

$$E(w_{N_j}) = \sum_{k=1}^{N_j} \sum_{i=k}^{\infty} \left(\gamma_j^{i-k} / \prod_{\ell=k}^{i} \{\ell\delta + (1-(1-\alpha)^{\ell})\mu\}\right) \tag{5.20}$$

Now, to obtain $E(w^*)$, we correct the drop out rate to $(n-1)\delta$ to reflect the assumption that our test applicant doesn't drop out. This yields

$$E(w^*) = \sum_{i=1}^{\infty} \left(\gamma_j^{i-1} / \prod_{\ell=1}^{i} \{(\ell-1)\delta + (1-(1-\alpha)^{\ell})\mu\}\right) \tag{5.21}$$

Combining (5.20) and (5.21) we obtain

$$E(w^*_{N_j}) = E(w_{N_j}) + E(w^*) \tag{5.22}$$

as the expected time until our test applicant is assigned.

Though (5.20) and (5.21) represent the most complicated instance of our general results from equation (4.15) this model is easily seen to reduce to some special cases we have already studied. For example, setting $\alpha=1$ in (5.20) and (5.21) yields the same results as equations (4.94) and (4.96). If we assign zero to both δ and γ_j in (5.20) and (5.21) these equations reproduce (5.2) and (5.3) if we interpret 0° to equal 1. Finally, if we set α equal to zero, (5.20) produces the mean time until all N_j households found waiting drop out (as $\alpha=0$ implies no assignments are made), while (5.21) equals infinity. If we assume that our test applicant does not drop out then the waiting time for the applicant is infinite when $\alpha=0$.

5.4.2 An Example

For a numerical example, we assume a project with a turnover rate of $\mu=20$ apartments per year. Higher priorities arrive at rate $\gamma=10$ households per year, while the household specific dropout rate is fixed at $\delta=.2$. We consider three cases for infinite refusal: $\alpha=.1$, $\alpha=.5$ and $\alpha=1$. The resulting expected waiting times as calculated from (5.22) are shown in Figure 5.5. Note the long waits associated with $\alpha=.1$; this is expected as most offered units are refused. Also, note how similar the waiting times are for $\alpha=.5$ and $\alpha=1$. The small difference in mean waits for these cases is owed to the low likelihood of all households present rejecting an offered unit. Thus, for the infinite refusal model, expected waiting times are not terribly sensitive to the acceptance probability α once this probability reaches an appreciable magnitude (e.g. $\alpha=.5$).

5.4.3 One Strike Refusal with Dropout and Priorities

We will now generalize our earlier work on one-strike models to include the effect of dropout and categorical priorities. Summarizing our assumptions, we postulate:

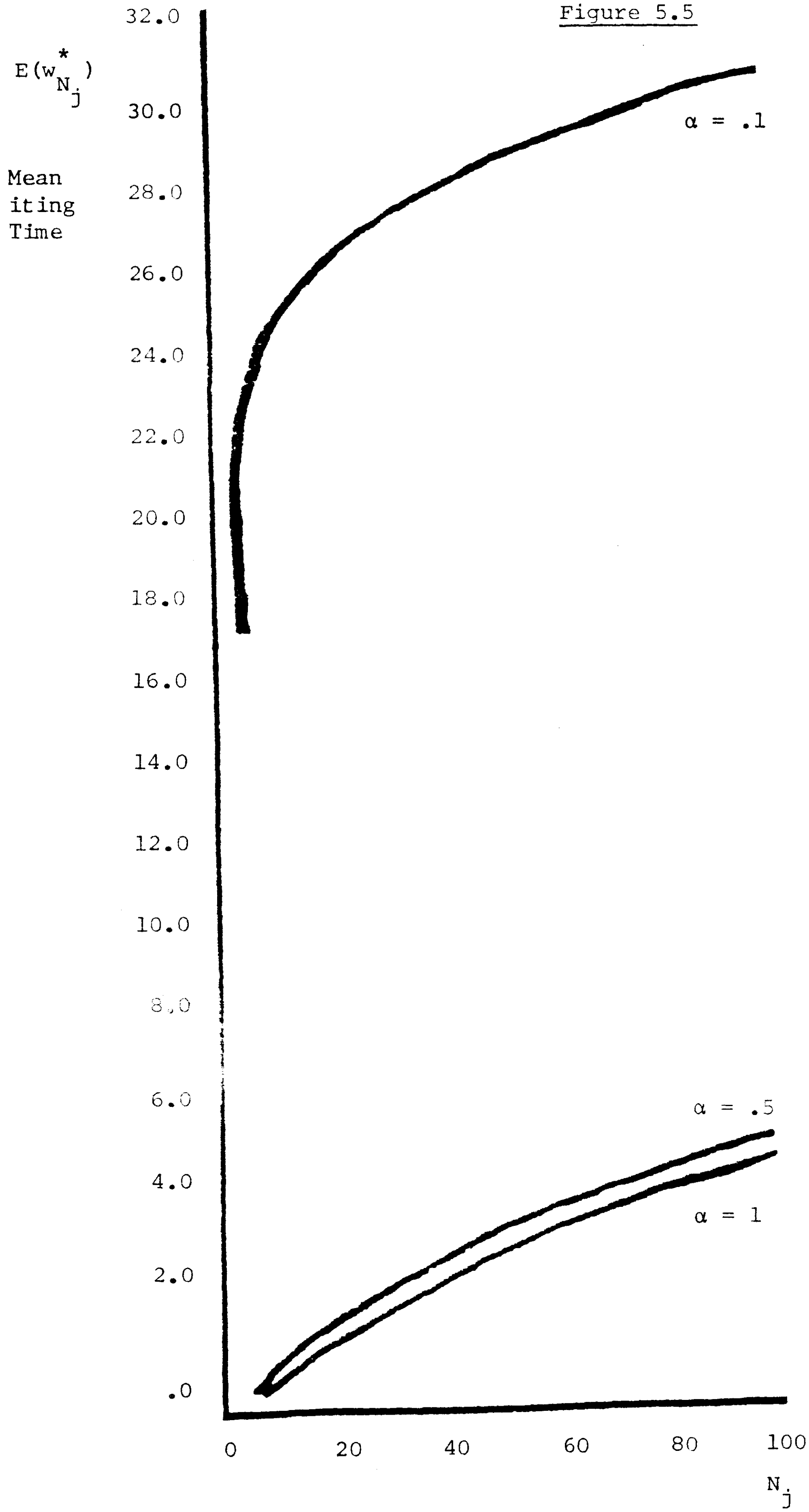
(i) New applicants in categorical priorities 1 through j-1 arrive according to a Poisson process with parameter γ_j

(ii) A newly arriving test applicant finds $N_j>0$ households waiting in priorities 1 through j

Figure 5.5

Mean Waiting Time in an Infinite Refusal System with Dropout

μ=20
γ_j=10
δ=.2



(iii) The system dropout rate equals $n\delta$ whenever n households (not including the test applicant) are waiting for assignments; the test applicant does not drop out by assumption

(iv) Households are assigned from the waiting list in Poisson fashion with rate μ

(v) Households independently accept offered units with acceptance probability α

(vi) Households receive only one offer; if this offer is rejected, they are dismissed from the system

These assumptions define a rather complicated stochastic process; the associated state transition diagram is shown in Figure 5.6. We can write down an equation for the mean of w, the time necessary to process n households found on the waiting list as

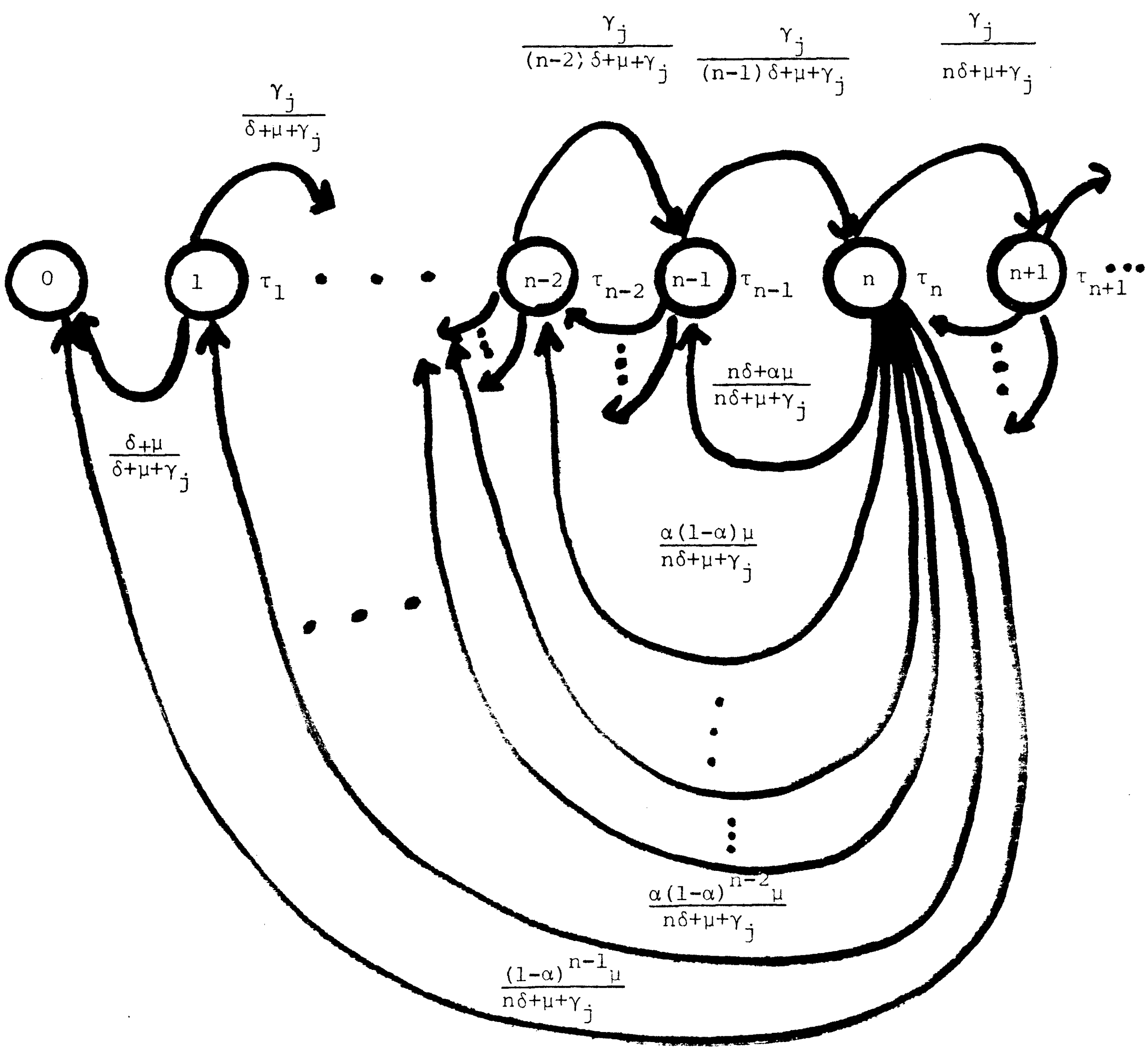
$$\begin{aligned} E(w_n) &= \frac{1}{n\delta+\mu+\gamma_j} + \frac{n\delta}{n\delta+\mu+\gamma_j} E(w_{n-1}) \\ &+ \frac{\mu}{n\delta+\mu+\gamma_j} \sum_{k=1}^{n-1} \alpha(1-\alpha)^{k-1} E(w_{n-k}) \qquad (5.23) \\ &+ \frac{\gamma_j}{n\delta+\mu+\gamma_j} E(w_{n+1}) \end{aligned}$$

Unfortunately, this infinite set of equations does not possess a closed form solution comparable to equation (4.15). However, we can rewrite (5.23) as

$$\sum_{i=1}^{n+1} a_{ni} E(w_i) = \frac{1}{n\delta+\mu+\gamma_j} \qquad (5.24)$$

Figure 5.6

State Transition Diagram for the One Strike Refusal Model with Dropout and Categorical Priorities



$$
\text{where } a_{ni} = \begin{cases} -\dfrac{\mu\alpha(1-\alpha)^{n-i-1}}{n\delta+\mu+\gamma_j} & i=1,2,\ldots,\ n-2 \\ -\dfrac{(n\delta+\mu\alpha)}{n\delta+\mu+\gamma_j} & i=n-1 \\ 1 & i=n \\ -\dfrac{\gamma_j}{n\delta+\mu+\gamma_j} & i=n+1 \end{cases} \tag{5.25}
$$

As n, the number of households on the waiting list becomes large, the probability that the next event to occur is a high priority arrival approaches 0. This suggests that one may presume a maximum size for the waiting list, and assume that once the waiting list reaches this size, the next event to occur must be a dropout or a tenant assignment. Formally, we assume that when n equals a maximum size η, we have

$$
a_{\eta i} = \begin{cases} -\dfrac{\mu\alpha(1-\alpha)^{\eta-i-1}}{\eta\delta+\mu} & i=1,2,\ldots,\eta-2 \\ -\dfrac{(\eta\delta+\mu\alpha)}{\eta\delta+\mu} & i=\eta-1 \\ 1 & i=\eta \end{cases} \tag{5.26}
$$

and $\sum_{i=1}^{\eta} a_{\eta i}\, E(w_i) = (\eta\delta+\mu)^{-1}$.

Using (5.25) and (5.26), we can define a finite set of linear simultaneous equations for $E(w_1)$ through $E(w_\eta)$ by allowing n to run from 1 through η.

This set of equations can be efficiently solved, as the coefficient matrix obtained from (5.25) and (5.26) is nearly in echelon form; an algorithm based on the LU factorization of the coefficient matrix is described in Appendix 5.1.

To obtain the mean additional time required to process our test applicant, $E(w^*)$, we condition on whether or not the applicant immediately preceding our test applicant has accepted an offered unit. If the preceding applicant has accepted an apartment, then we must take into account the probability of a higher priority applicant arriving during our test applicant's wait for a new unit. Otherwise, the test applicant is either housed or leaves the waiting list. More formally, we have

$$E(w^* \mid \text{acceptance by preceeding applicant}) = \frac{1}{\mu+\gamma_j} + \frac{\gamma_j}{\mu+\gamma_j} E(w_1+w^*) \qquad (5.27)$$

$$E(w^* \mid \text{rejection by preceeding applicant}) = 0 \qquad (5.28)$$

By assumption, all applicants accept units with probability α. Thus, the unconditional mean of w^* is given by

$$E(w^*) = \alpha\left\{\frac{1}{\mu+\gamma_j} + \frac{\gamma_j}{\mu+\gamma_j} E(w_1+w^*)\right\} \qquad (5.29)$$

and upon solving for $E(w^*)$, we find that

$$E(w^*) = \frac{\alpha\left(1+\gamma_j E(w_1)\right)}{\mu+(1-\alpha)\gamma_j} \qquad (5.30)$$

Using the system of equations developed in (5.24) through (5.26) with equation (5.30), we can set the expected waiting time until our test applicant is either housed or dismissed from the system equal to the sum of $E(w_{N_j})$ and $E(w^*)$.

5.4.4 An Example

We again examine a project with a moveout rate of μ=20, priority arrival rate of γ=10, and a household specific dropout rate of δ=.2. The expected waiting times (solved using the finite state approximation) for the one strike model are presented for the cases $\alpha=.1$, $\alpha=.5$, and $\alpha=1$ in Figure 5.7. Unlike infinite refusal models, the waiting times for one strike models are heavily dependent on α, the acceptance probability. It is evident that most applicants are dismissed from the system as α decreases from 1.

5.5 Multiproject Models

The modeling effort thus far has focused exclusively on single project tenant assignment systems. As noted from our review of currently used tenant assignment schemes, several cities operate city-wide waiting lists where assignments are made on a first available unit basis, regardless of the particular project involved. In this section, we will show how single project models are easily adapted to multiproject systems of the form described.

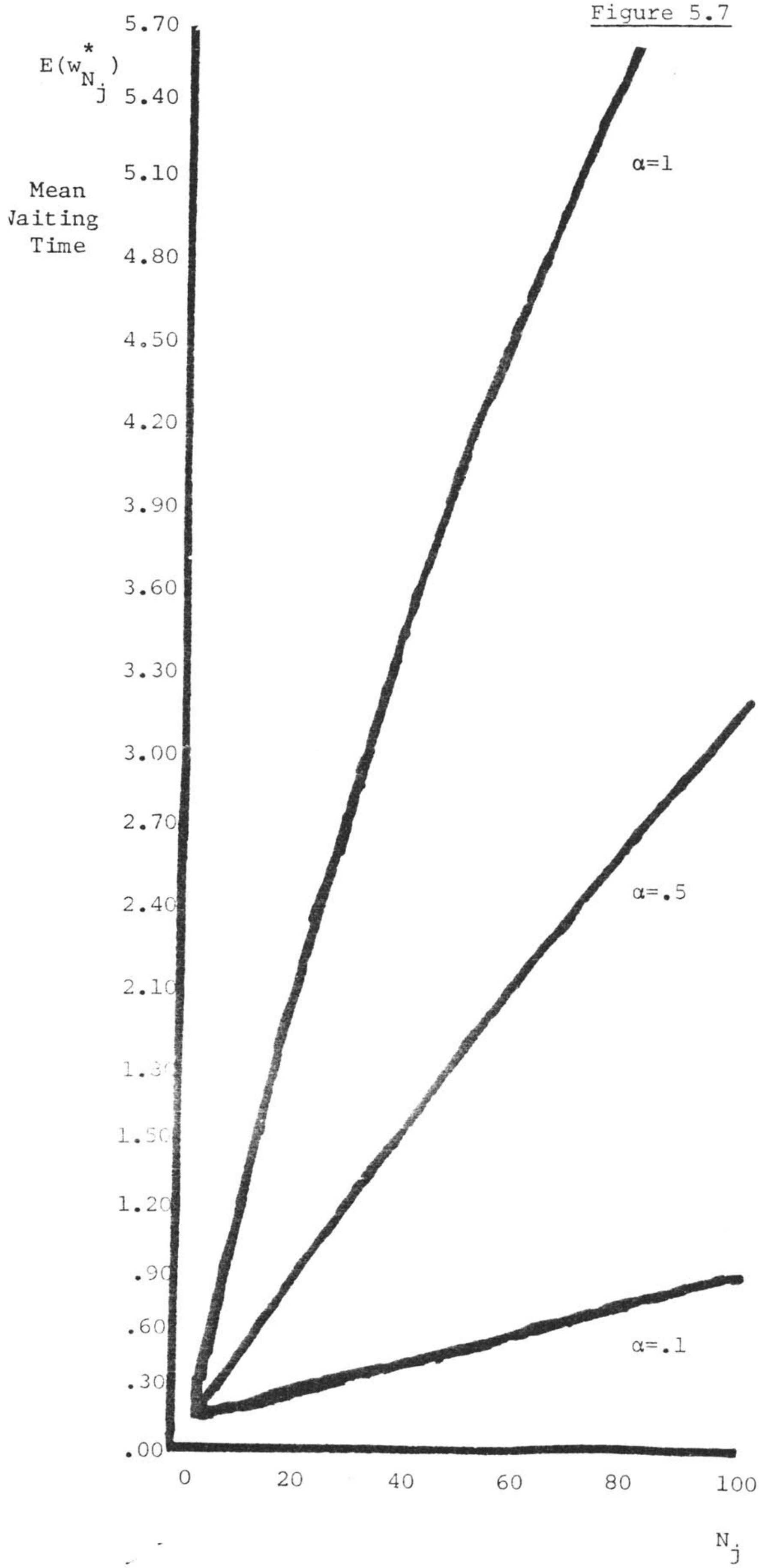
Suppose that for a particular unit type, the assignment process at project i is Poisson with rate μ_i, i=1,...,I. the overall assignment process for the entire housing authority will also be Poisson with rate

$$\mu_{sys} = \sum_{i=1}^{I} \mu_i \qquad (5.31)$$

since in first available unit systems, assignments at projects are mutually independent. Thus, one can consider the entire authority to function as a "mega-project" with respect to housing assignments. One such situation is demonstrated schematically in Figure 5.8.

The prediction of waiting times in multiproject first available unit systems is therefore rather straightforward. One only needs to substitute μ_{sys} for μ in the relevant single project tenant assignment model to obtain estimation of the mean waiting time (or the variance of waiting time) for a newly arriving test applicant. The features of

Figure 5.7

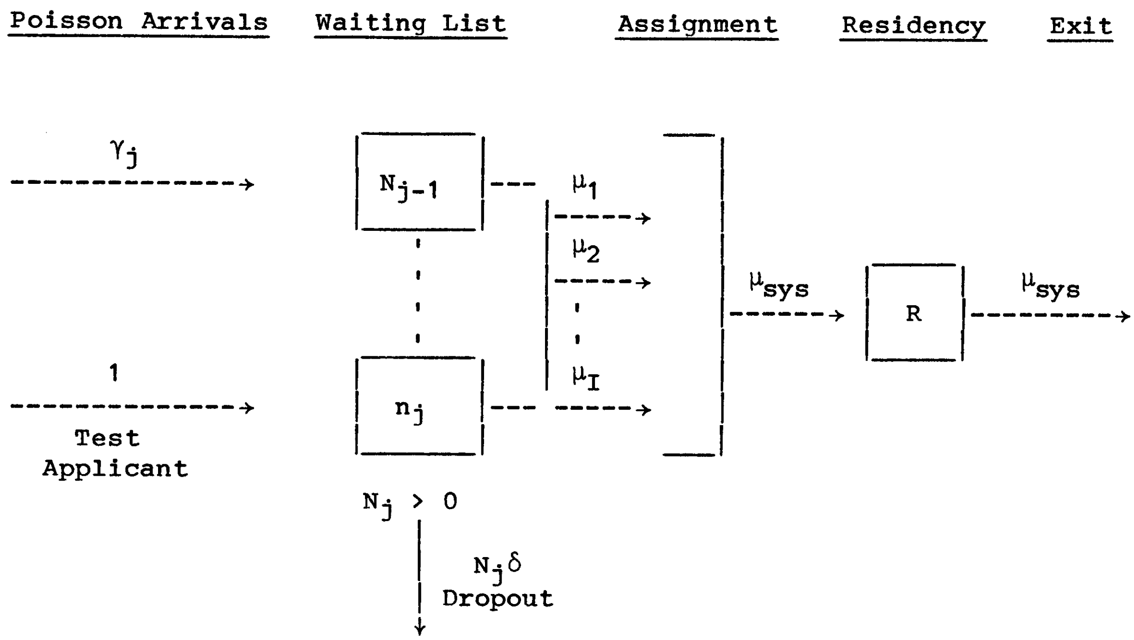


Mean Waiting Time in a One Strike Refusal System with Dropout

$\mu = 20$
$\gamma_j = 10$
$\delta = .2$

Figure 5.8

A Multiproject First Available Unit System with Categorical Priorities and Dropout



Assumptions

1) N_j households in categorical priorities 1 through j are found waiting by a newly arriving priority j household.

2) Households in categorical priorities 1 through j-1 arrive in a Poisson process with rate γ_j.

3) Tenant assignments at project i occur in Poisson fashion with rate μ_i. The overall system assignment rate is given by

$$\mu_{sys}=\sum_{i=1}^{I} \mu_i .$$

4) Dropout occurs at a rate proportional to the number of households waiting; the test applicant does not drop out by assumption.

blend priorities, categorical priorities, and tenant refusal can all be incorporated in this framework.

Of additional interest is the number of households who will be assigned to specific projects. These quantities are easily obtained from the product of the project specific assignment rates with the mean waiting time:

$$E(\text{number housed at project } i) = \mu_i E(w_N^*) \tag{5.32}$$

where μ_i is the assignment rate at project i, and w_N^* is the time necessary to process a test applicant who finds N households waiting for assignments (note that the waiting list may be prioritized).

One rather unrealistic feature of our multiproject analysis surfaces when refusal systems are considered. It is difficult to imagine an authority where the likelihood of a household accepting a unit is independent of the project in which the unit is situated. A more realistic model would assign a project specific acceptance probability α_i to the acceptance of an offered unit located in project i. Unfortunately, such a model is too complicated to consider at the present time; the development of refusal models with project specific acceptance probabilities could prove to be an interesting topic for future research.

5.6 Multiqueue Systems

Refusal systems offer one means for allowing tenants some degree of choice in an assignment procedure. An entirely different approach would be to guarantee tenants assignment to a member of a pre-chosen group of projects. A system of this form is currently used by the Boston Housing Authority. The advantage of such a system is that tenants can indeed specify where they are willing to live. The disadvantage is that the authority using this system relinquishes control over the demographic composition of housing projects.

To model the waiting times for such systems, we must take into account the detailed choices and positions of all tenants on the waiting list. We can summarize tenant choices via the notion of a choice set. Formally, the choice set C_j consists of all projects chosen by tenant j as acceptable for assignment. Thus, tenant j can only be assigned to project i if $i \in C_j$.

The positioning of tenants on the waiting list is arranged in order of application date. However, as tenants' choice sets overlap, one may think of the waiting list as a number of queues, one for each project, with a tenant appearing in all queues for projects in his or her choice set (hence the term "multiqueue"). An example is shown in Figure 5.9 for a simple system with three projects and two choices per choice set.

The tenant assignment rule for a multiqueue system (ignoring dropout and priorities) states that a newly available unit at project i is assigned to the household at the front of the waiting list for project i. The household assigned is then removed from all other project waiting lists. Thus, in a multiqueue system, tenants are assigned to that project in their choice set in which the first available unit appears. An example of an assignment sequence using the waiting list in Figure 5.9 is shown in Figure 5.10. Note that the households are not assigned in strict chronological order; this is due to the restriction of assignments to choice sets. Also, note the dependence of the waiting list at one project on assignments that occur at other projects. In the example of Figure 5.10 household 4 is

Figure 5.9

A Multiqueue Waiting List

Tenant ID (j)	Choice Set (C_j)
1	1,3
2	2,3
3	1,2
4	1,3
5	2,3
6	1,2
7	2,3

Project (i)	Waiting List (By Tenant ID)
1	1,3,4,6
2	2,3,5,6,7
3	1,2,4,5,7

Figure 5.10

A Sample Assignment Sequence for the Multiqueue Waiting List of Figure 5.9

Project with New Available Unit	Household Assigned
2	2
3	1
2	3
1	4
1	6
3	5
2	7

assigned to project 1 on the first assignment made to project 1, even though household 4 is initially third in queue at project 1. Of course, this is due to earlier assignments made at other projects.

Before continuing, it is worth noting that we have already considered two special cases of multiqueue assignment systems. If the choice set for every tenant consists of a single project, then the entire assignment system decomposes into a series of independent single project first come first serve waiting lists. This situation has been discussed in detail in Chapter 4 and the early part of this chapter. Alternatively, if the choice set for every tenant consists of <u>all</u> projcts in the system, then the situation is the same as the multiproject first come first serve assignment scheme considered in the previous section.

The last concept we need to introduce before formulating a model for multiqueue systems is that of a "state." Previously, it was sufficient to know the number of households on the waiting list in order to make probabilistic statements about future events (e.g. an assignment, a dropout). Now, it is necessary to know the configuration of all households in the system on the multiqueue waiting list after each assignment is made. Each distinct, possible configuration will be referred to as a state. The actual state transitions for the assignment sequence in Figure 5.10 are illustrated in Figure 5.11. Note that these states represent the particular sequence that was observed; they do not represent all possible states.

We are now ready to model the multiqueue system. As usual, we assume that the moveout (and hence the assignment) rate at project i equals μ_i; and that the assignment process is Poisson. If we set

Figure 5.11

State Transitions for the Assignment Sequence of Figure 5.10

State 0:	Project	1	1,3,4,6
		2	2,3,5,6,7
		3	1,2,4,5,7
State 1:	Project	1	1,3,4,6
		2	3,5,6,7
		3	1,4,5,7
State 2:	Project	1	3,4,6
		2	3,5,6,7
		3	4,5,7
State 3:	Project	1	4,6
		2	5,6,7
		3	4,5,7
State 4:	Project	1	6
		2	5,6,7
		2	5,7
State 5:	Project	1	-
		2	5,7
		3	5,7
State 6:	Project	1	-
		2	7
		3	7
State 7:	Project	1	-
		2	-
		3	-

$\mu_{sys}=\sum_i \mu_i$, then the probability that an assignment is made to project i on any transition is given by

$$\Pr\{\text{Assignment made to project } i\} = \frac{\mu_i}{\mu_{sys}} \qquad (5.33)$$

Now, suppose the system currently occupies some state k, where all possible states have been arbitrarily numbered from 0 (the initial state, or starting configuration) to D_o (the first state in which all households present in state 0 have been assigned). We define

$$q_{kl} = \Pr\{\text{System next occupies state } \ell \mid \text{System currently occupies state } k\} \quad \begin{matrix} k,\ell=0\ldots,D_o \\ k\neq\ell \end{matrix} \qquad (5.34)$$

If we let $G_{k\ell}$ be the group of projects which would carry the system from state k to state ℓ if an assignment is made at any project in the set $G_{k\ell}$, then we see that

$$q_{k\ell} = \frac{\sum_{i\varepsilon G_{k\ell}} \mu_i}{\mu_{sys}} \qquad (5.35)$$

Finally, we define $p_{ij}(n|S_k)$ to be the n-step transition probability for the assignment of household j to project i given that the system is currently in state k. This probability gives the likelihood that household j will be assigned to project i in exactly n assignments, given that the system is currently described by state k, and is computed recursively as

$$p_{ij}(n|S_k) = \begin{cases} \sum_{\ell=0}^{D_o} q_{k\ell}p_{ij}\ (n-1|S_\ell) & i \in C_j \\ 0 & i \notin C_j \end{cases} \qquad (5.36)$$

for each project i and household j on the waiting list.

Once these n-step probabilities have been computed, we can easily compute the mean length of time necessary to house all those initially present on the waiting list, and the expected number of assignments that will occur at each project. First, we compute the ultimate probability of assigning household j to project i, given that the system is initially in state 0 as

$$p_{ij}(S_o) = \begin{cases} \sum_{n=1}^{\infty} p_{ij}(n|S_o) & i \varepsilon C_j \\ 0 & i \not\varepsilon C_j \end{cases} \tag{5.37}$$

The expected number of households assigned to project i of those originally present initially is then given by

$$E\,(\text{Assigned to project } i|S_o) = \sum_{j=1}^{N} p_{ij}(S_o) \tag{5.38}$$

Finally, the expected time necessary to assign those who ultimately are assigned to project i is given by

$$E(w_N^i|S_o) = \frac{1}{\mu_i} \sum_{j=1}^{N} p_{ij}(S_o) \tag{5.39}$$

If a test applicant choosing only project i were to enter the housing system in state 0, the mean waiting time for our test applicant would equal

$$E(w_N^{i^*}|S_o) = \frac{1}{\mu_i} + E(w_N^i|S_o) \tag{5.40}$$

Once one is familiar with the notion of state in the multiqueue context, the formulation just presented is rather straightforward. Unfortunately, the concept of state also serves to make the preceding analysis unworkable for all but the most trivial problems. The number of possible states explodes geometrically with the number of projects and the size of the waiting list, and combinatorically with the number of projects allowed in a choice set. For example in the simple system of Figure 5.9, enumeration yields a total of 24 possible states. In a system such as the Boston Housing Authority with up to three projects per choice set, over fifty projects, and waiting lists of over 2000 households for most bedroom sizes the number of states attainable is in the billions. These realities render our approach inoperable.

However, it is not al all difficult to simulate the operations of a multiqueue tenant assignment scheme. A simulation would generate events in a manner consistent with the process as it has been described, and calculate statistics such as waiting times and the number of project specific assignments based on several "realizations" of the system. We will now describe an algorithm for simulating a multiqueue assignment scheme (ignoring dropout and priorities), and present some examples.

5.6.1 Simulating Multiqueue Systems

A simple method for simulating multiqueue systems involves generating exponential inter-moveout times, and assigning households to projects in a manner consistent with the given choice sets for the households on the waiting list at these simulated moveout times. We will describe an algorithm to do this using the following notation:

μ_i = moveout rate at project i

s_{ik} = time elapsed between the (k-1)st and the k^{th} moveout at project i

w_{ik} = time at which the k^{th} moveout (and hence the k^{th} assignment) occurs at project i

t_ℓ = time at which the ℓ^{th} tenant assignment occurs (system wide)

$i(\ell)$ = ID of the project where the ℓ^{th} tenant assignment occurs

C_j = set of projects chosen by household j (i.e., the choice set of household j). Households are ordered by application date (i.e., in order of arrival).

Ψ_j = 1 if household j has been assigned

0 if not

N_i = number of households assigned to project i

T_i = time necessary to assign all households ultimately assigned to project i

The simulation proceeds in three major blocks:

Block 1: Generate Inter-moveout Times, Compute Assignment Times

1) For every project in the system generate K random inter-moveout times S_{ik}, k=1,...,K using an exponential distribution with parameter μ_i for project i

2) For every project in the system, compute k assignment times using the recursions

$$w_{ik}=w_{i,k-1} + S_{ik}; \quad w_{io} = 0; \quad k=1,...K$$

At the end of Block 1, the simulation has generated K assignment times for each project in the system.

Block 2: Sort the Assignment Times to Obtain a System Assignment Sequence

In this block, all of the project assignment times w_{ik} are sorted in ascending order, and stored in the variable t_ℓ. Thus, the time at which the ℓ^{th} system wide assignment occurs is given by t_ℓ. While sorting, the project ID's associated with each assignment time are saved. At the end of Block 2, the variable $i(\ell)$ identifies the project at which the ℓ^{th} system wide assignment occurs.

Block 3: Allocate Households to Projects and Compute Project Specific Waiting Times

(1) Set $T_i=0$ and $N_i=0$ for every project i.
Set $\Psi_j=0$ for every household j.

(2) Set $\ell=0$.

(3) Set $\ell \leftarrow \ell+1$

(4) Find min j such that $i(\ell) \varepsilon C_j$ and $\Psi_j=0$.
If no such j, go to (6).

(5) Set $\Psi_j \leftarrow 1$; $T_{i(\ell)} \leftarrow t_\ell$; $N_{i(\ell)} \leftarrow N_{i(\ell)}+1$.

(6) Is $\Psi_j=1 \; \forall j$? If yes, go to (7).
If no, go to (3).

(7) STOP.

Block 3 performs the allocation of households to projects. After initializing the variables T_i, N_i and Ψ_j, the algorithm locates the household that is first in line for each new unit that occurs; assigns the household to the relevant project; removes the household from consideration at other projects; updates the number of households assigned and the waiting time for the relevant project; and checks to see if all households have been assigned. When all households have been assigned, the routine is complete. To compute the expected number of households assigned and the average waiting times at each project, the entire algorithm (Blocks 1 through 3) is repeated several times, and the resultant values of T_i and N_i from the various runs are averaged to compute $\overline{T}_i$ and $\overline{N}_i$; the mean waiting time and mean number of households assigned at project i. As an example, a simulation allocating 100 households to five projects in a 2 project per choice set multiqueue system was performed for various values of μ_i, the moveout rates, and various combinations of project choices. A computer coding of the model may be found in Appendix 5.2.

Project choice sets were generated by sampling without replacement from the pool of five projects according to predetermined probabilities. Each household was given an initial probability p_i of selecting project i, i=1,...,5. Having selected project i, the conditional likelihood of selecting project i' equals

$$\Pr\{\text{choose } i' \mid \text{chosen } i\} = p_{i'}/(1-p_i) \quad \begin{matrix} i,i'=1,\ldots,5 \\ i'\neq i \end{matrix} \tag{5.41}$$

Thus, the probability that the choice set for any household consists of projects i and i' equals

$$\Pr\{\text{Choice set contains } i \text{ and } i'\} = p_i p_{i'} \left[\frac{1}{1-p_i} + \frac{1}{1-p_{i'}}\right] \tag{5.42}$$

$$i,i'=1,\ldots,5 \quad i\neq i'$$

A total of 18 experiments were performed. In each experiment, choice sets generated by a particular set of p_i's were combined with moveout rates given by a particular set of μ_i's to compute the mean time to house all assigned households by project, and the mean number of households assigned to each project. For each experiment, the simulation model was run 100 times; choice sets were <u>not</u> regenerated with each run for a given experiment. Table 5.1 shows the parameters used in generating the experimental runs, while Table 5.2 presents the resultant mean numbers of assignments and average waiting times by project.

In the first experiment, where assignment rates and project desirability are homogeneous across projects, the resulting tenant allocation and waiting times are also invariant. Both the assignment rates and the choice structures have been systematically varied in the successive experiments. It is interesting to note the interaction

Table 5.1

Parameters for the Simulation Experiments

Project Choice Probabilities

Case	P_1	P_2	P_3	P_4	P_5
A	1/5	1/5	1/5	1/5	1/5
B	1/8	1/8	1/8	1/8	1/2
C	1/15	2/15	3/15	4/15	5/15

Yearly Moveout Rates

Case	μ_1	μ_2	μ_3	μ_4	μ_5
1	10	10	10	10	10
2	10	10	10	10	50
3	50	10	10	10	10
4	10	20	20	50	50
5	50	50	20	20	10
6	10	20	30	40	50
7	50	40	30	20	10

Table 5.2

Experimental Results

Project Choice Case	Moveout Rate Case	Mean Number of Households Assigned $\bar{N}_1$	$\bar{N}_2$	$\bar{N}_3$	$\bar{N}_4$	$\bar{N}_5$	Mean Time to Assign all Households $\bar{T}_1$	$\bar{T}_2$	$\bar{T}_3$	$\bar{T}_4$	$\bar{T}_5$
A	1	19.5	20.2	20.7	19.8	19.8	1.94	1.97	1.94	1.96	1.91
B	1	20.6	21.4	17.4	19.0	21.7	2.07	2.05	1.80	1.86	2.25
C	1	15.1	20.4	21.7	21.0	21.8	1.52	2.03	1.98	2.09	2.09
A	2	14.4	14.8	15.5	14.8	40.5	1.44	1.45	1.42	1.45	0.81
B	2	12.2	10.9	10.3	11.0	55.6	1.22	0.97	1.01	1.04	1.13
C	2	11.0	12.4	12.7	11.8	52.1	1.04	1.23	1.10	1.14	1.05
B	3	34.7	17.9	14.1	15.1	18.3	0.70	1.70	1.43	1.48	1.83
C	3	19.5	19.2	20.6	20.1	20.6	0.40	1.91	1.89	1.96	1.99
A	4	7.1	14.2	16.0	30.9	31.8	0.67	0.69	0.74	0.62	0.63
B	4	8.5	15.1	13.5	24.2	38.7	0.87	0.70	0.68	0.48	0.79
C	4	6.6	13.6	14.2	32.0	33.6	0.57	0.66	0.62	0.64	0.67
B	5	29.3	31.7	15.0	15.0	9.0	0.59	0.61	0.76	0.74	0.90
C	5	17.4	28.2	21.1	21.9	11.3	0.35	0.57	0.98	1.09	1.05
A	6	6.8	14.0	21.6	26.5	31.1	0.62	0.68	0.69	0.67	0.62
B	6	8.2	14.5	17.6	22.8	36.9	0.84	0.67	0.61	0.57	0.76
C	6	6.6	13.6	20.5	25.8	33.5	0.56	0.67	0.63	0.65	0.66
B	7	29.8	28.7	18.4	14.5	8.5	0.60	0.69	0.64	0.72	0.86
C	7	17.6	26.7	25.2	20.3	10.2	0.35	0.67	0.79	1.00	0.96

between choice structure and assignment rates. For example, in the fourth experiment, using a homogeneous choice structure, an average of 40.5 households were assigned to project 5; in the fifth experiment using the same assignment rates of 10 for projects 1 through 4 and 50 for project 5, an average of 55.6 households went to project 5 with the choice structure biased in favor of project 5. Now, the same choice structures used in experiments 4 and 5 were also used in experiments 1 and 2, but with homogeneous assignment rates. Here, the number assigned to project 5 increased from 19.8 to 21.7 on average - hardly the same as the increase from 40.5 to 55.6 noted earlier.

An informal analysis of these results leads one to conclude that the interaction between choice structures and assignment rates is complex. Certainly, projects with higher assignment rates will house more tenants; projects which are more likely to be chosen are more likely to house tenants as well. Beyond these almost trivial observations, it is difficult to characterize the behavior of multiqueue systems.

That the choice aspect of multiqueue systems combined with the exact ordering of tenants on the waiting list is critical to the outcomes of the assignment process is illustrated by the following example. Imagine a system with two projects, both with equal assignment rates. Also, imagine 100 tenants to be housed; 50 will accept either project 1 or 2, the other 50 will accept only project 1. If the waiting list is such that the fifty households willing to go to either project are the first fifty tenants on the list, then one would expect 25 of these tenants to be assigned to the first project, and 25 to the second. The remaining 50 tenants, having chosen only project 1, would be assigned to project 1, yielding a final tally of 75 households assigned to project 1 and 25 assigned to project 2.

Alternatively, if the 50 households choosing only project 1 were processed first, then any new units opening up at the second project will be assigned to households who are indifferent to living in either project (since the 50 households in the first half of the list will only accept project 1). Thus, by reversing the order of the waiting list, one would expect a final tally of 50 households assigned to project 1 and 50 households assigned to project 2. Clearly, choice and order are important in the outcome of a multiqueue assignment process.

While we have developed a simulation model for calculating the mean times to "clear" waiting lists and the expected number of project assignments for multiqueue systems, we have not really succeeded in understanding the qualitative features of these assignment schemes. The simulation model will prove useful for actually forcasting waiting times in a multiqueue system (once modified to incorporate dropout and priorities, for example). However, the development of analytic models for the multiqueue model poses a difficult research problem.

5.7 Conclusions

This chapter extended the analysis of Chapter 4 to incorporate certain aspects of tenant choice found in the tenant assignment schemes of U.S. Housing Authorities. We examined the notion of refusal systems, and developed models for the cases of one-strike and infinite-strike refusal. The applicability of our models to multiproject first come first serve assignment schemes was demonstrated. Following this, we developed a simulation model for application to multiqueue systems, and presented the results of some preliminary experiments with this model.

We have devoted a great deal of time to the development of tenant assignment models. In the next chapter, we will use these ideas to analyze some real data from the Boston Housing Authority in an attempt to demonstrate the utility of tenant assignment modeling in a policy analysis setting.

Appendix 5.1

An Algorithm for Obtaining $E(w_N)$ in the One Strike Refusal Model with Dropout and Categorical Priorities

Define:

(i)
$$a_{ij} = \begin{cases} -\mu\alpha(1-\alpha)^{i-j-1}/(i\delta+\mu+\gamma_j) & j=1,2,\ldots,i-2 \\ -(\mu\alpha+i\delta)/(i\delta+\mu+\gamma_j) & j=i-1 \\ 1 & j=i \\ -\gamma_j/(i\delta+\mu+\gamma_j) & j=i+1 \\ 0 & j>i+1 \end{cases}$$
$$i=1,2,\ldots,\eta-1$$

(ii)
$$a_{\eta j} = \begin{cases} -\mu\alpha(1-\alpha)^{\eta-j-1}/(\eta\delta+\mu) & j=1,2,\ldots,\eta-2 \\ -(\mu\alpha+\eta\delta)/(\eta\delta+\mu) & j=\eta-1 \\ 1 & j=\eta \end{cases}$$

(iii)
$$b_i = 1/(i\delta+\mu+\gamma_j) \qquad i=1,\ldots,\eta-1$$

(iv)
$$b_\eta = 1/(\eta\delta+\mu)$$

The expected waiting times $E(w_n)$ are given by the solution to the set of η simultaneous linear equations

$$\sum_{j=1}^{\eta} a_{ij}\, E(w_j) = b_i \qquad i=1,2,\ldots,\eta$$

Owing to the special structure of the coefficients a_{ij}, these equations are easily solved by the following algorithm:

1) $b_1 \leftarrow b_1/a_{11}$

2) $v_1 \leftarrow a_{12}/a_{11}$

3) FOR i=2 to η

4) $\ell_o \leftarrow a_{i1}$

5) $b_i \leftarrow b_i - \ell_o b_1$

6) IF i=2 THEN GO TO 12)

7) FOR j=2 TO i-1

8) $\ell_1 \leftarrow a_{ij} - \ell_o v_{j-1}$

9) $b_i \leftarrow b_i - \ell_1 b_j$

10) $\ell_o \leftarrow \ell_1$

11) END j

12) $\ell_1 \leftarrow a_{ii} - \ell_o v_{i-1}$

13) $b_i \leftarrow b_i/\ell_1$

14) $v_i \leftarrow a_{i,i+1}/\ell_1$

15) END i

16) $E(w_\eta) \leftarrow b_\eta$

17) FOR i=η-1 TO 1

18) $E(w_i)=b_i-v_i b_{i+1}$

19) END i

```
#CONTROL USLINIT,NOSOURCE
C
C     SIMULATION OF EFFECTIVE QUEUE LENGTHS FOR COOPERATIVE QUEUES
C     FIVE QUEUES, TWO CHOICES PER TEAM, FIXED CHOICE STRUCTURE
C
      DIMENSION X(5,300),T(5,300),ST(3000),IFAC(3000),IQ(5,300),L(5),
     +LQ(5),W(5),AVGL(5),AVGL2(5),AVGW(5),AVGW2(5),ISERVE(300),
     +LAMBDA(5),LOLD(5),N(5,300),NSTAR(5,300),OLDSTR(5,300),
     +L1(5),L2(5)
      REAL LAMBDA,L,NSTAR,LOLD,L1,L2,LL
C
C     INPUT PARAMETERS
C
      DISPLAY "ENTER # PROJECTS, # HOUSEHOLDS, # SLOTS"
      ACCEPT NP,NH,NT
      DISPLAY "ENTER SERVICE RATES"
      ACCEPT (LAMBDA(I),I=1,NP)
      DISPLAY "ENTER BREAKPOINTS FOR CHOICE STRUCTURE"
      ACCEPT B1,B2,B3,B4
C
C     INITIALIZE FOR CHOICE STRUCTURE GENERATION
C
      DO 5 I=1,NP
      DO 5 J=1,NH
      X(I,J)=0
      IQ(I,J)=0
    5 CONTINUE
      DISPLAY "ENTER SEED"
      ACCEPT SEED
C
C     GENERATE CHOICE STRUCTURE
C
      DO 10 J=1,NH
   11 CALL CHOICE(I1,B1,B2,B3,B4,SEED)
      CALL CHOICE(I2,B1,B2,B3,B4,SEED)
      IF(I1.EQ.I2) GO TO 11
      X(I1,J)=1
      X(I2,J)=1
   10 CONTINUE
C
C     BUILD ORIGINAL QUEUES
C
      DO 15 I=1,NP
      IS=0
      DO 15 J=1,NH
      IF(X(I,J).EQ.0) GO TO 15
      IS=IS+1
      IQ(I,IS)=J
   15 CONTINUE
C
C     WRITE ORIGINAL QUEUES TO FILE 8
C
      DO 20 IS=1,NH
      DO 25 I=1,NP
      IF(IQ(I,IS).GT.0) GO TO 21
   25 CONTINUE
      GO TO 22
   21 WRITE(8,23) (IQ(I,IS),I=1,NP)
   23 FORMAT(5(I3,1X))
```

```
   20  CONTINUE
   22  CONTINUE
C
C      INITIALIZE SIMULATION LOOP
C
       DISPLAY "ENTER DESIRED NUMBER OF SIMULATIONS"
       ACCEPT NSIM
       DO 30 I=1,NP
       AVGL(I)=0
       AVGL2(I)=0
       AVGW(I)=0
       AVGW2(I)=0
   30  CONTINUE
       DO 1000 KSIM=1,NSIM
C
C      INITIALIZE SIMULATION VARIABLES
C
       DO 35 I=1,NP
       L(I)=0
       LQ(I)=1
       W(I)=0
       DO 40 K=1,NT
       T(I,K)=0
   40  CONTINUE
   35  CONTINUE
       DO 45 J=1,NH
       ISERVE(J)=0
   45  CONTINUE
C
C      GENERATE SERVICE TIMES
C
C
       DO 50 I=1,NP
       T(I,1)=-ALOG(RAND(SEED))/LAMBDA(I)
       DO 50 K=2,NT
       T(I,K)=T(I,K-1)-ALOG(RAND(SEED))/(LAMBDA(I))
   50  CONTINUE
C
C      SORT THE SERVICE TIMES
C
       KKK=0
       DO 55 K=1,NT
       DO 55 I=1,NP
       KKK=KKK+1
       ST(KKK)=T(I,K)
       IFAC(KKK)=I
   55  CONTINUE
       NNTT=NP*NT
   60  ICHECK=0
       DO 65 KKK=2,NNTT
       IF(ST(KKK-1).LE.ST(KKK)) GO TO 65
       ICHECK=1
       SWAP=ST(KKK)
       KSWAP=IFAC(KKK)
       ST(KKK)=ST(KKK-1)
       IFAC(KKK)=IFAC(KKK-1)
       ST(KKK-1)=SWAP
       IFAC(KKK-1)=KSWAP
   65  CONTINUE
       IF(ICHECK.EQ.1) GO TO 60
```

```
C
C     COUNT THE TRUE EFFECTIVE QUEUE LENGTHS
C
      DO 100 K=1,NNTT
      I=IFAC(K)
      IF(LQ(I).EQ.NH) GO TO 100
      DO 200 J=LQ(I),NH
      IF(X(I,J).EQ.0.OR.ISERVE(J).GT.0) GO TO 200
      L(I)=L(I)+1
      ISERVE(J)=I
      W(I)=ST(K)
      LQ(I)=J+1
      GO TO 100
  200 CONTINUE
  100 CONTINUE
C
C     RECORD QUEUE LENGTHS AND TOTAL WAITS ON FILE 9
C
      WRITE(9,260) (L(I),W(I),I=1,NP)
  260 FORMAT(5(F4.0,2X,F7.2,2X))
C
C     UPDATE MEAN AND MEAN SQUARE STATISTICS
C
      DO 270 I=1,NP
      AVGL(I)=AVGL(I)+L(I)
      AVGW(I)=AVGW(I)+W(I)
      AVGL2(I)=AVGL2(I)+L(I)*L(I)
      AVGW2(I)=AVGW2(I)+W(I)*W(I)
  270 CONTINUE
C
C     END SIMULATION LOOP
C
 1000 CONTINUE
C
C     COMPUTE SUMMARY STATISTICS
C
      DO 280 I=1,NP
      AVGL(I)=AVGL(I)/FLOAT(NSIM)
      AVGW(I)=AVGW(I)/FLOAT(NSIM)
      AVGL2(I)=AVGL2(I)/FLOAT(NSIM)
      AVGW2(I)=AVGW2(I)/FLOAT(NSIM)
  280 CONTINUE
C
C     DISPLAY AVERAGE RESULTS
C
      WRITE(10,284)
  284 FORMAT(" SIMULATION RESULTS")
      WRITE(10,286)
  286 FORMAT(" QUEUE    L     L**2     W      W**2")
      DISPLAY "AVERAGE RESULTS FOR EACH QUEUE"
      DISPLAY "QUEUE","L","L**2","W","W**2"
      DO 290 I=1,NP
      DISPLAY I,AVGL(I),AVGL2(I),AVGW(I),AVGW2(I)
      WRITE(10,*) I,AVGL(I),AVGL2(I),AVGW(I),AVGW2(I)
  290 CONTINUE
C
C     END OF SIMULATION
C
      STOP
      END
```

```
C
C     SUBROUTINE SETS UP ASSIGNMENTS FOR CHOICE STRUCTURE
C
      SUBROUTINE CHOICE(I,B1,B2,B3,B4,SEED)
      FLIP=RAND(SEED)
      IF(FLIP.LE.B1) I=1
      IF(B1.LT.FLIP.AND.FLIP.LE.B2) I=2
      IF(B2.LT.FLIP.AND.FLIP.LE.B3) I=3
      IF(B3.LT.FLIP.AND.FLIP.LE.B4) I=4
      IF(B4.LT.FLIP) I=5
      RETURN
      END
```

Chapter VI

Using Tenant Assignment Models: Examples from Boston

The Boston Housing Authority (BHA) administers over 14,000 public housing units in 69 family and elderly developments (including leased housing). As of November 1983, about 8,000 households were waiting for project assignments in the one through five bedroom apartment range. The BHA assigns tenants using the multiqueue assignment scheme discussed in Chapter 5. However, the authority is considering changes in its tenant assignment policies.

As a preliminary application of our tenant assignment models, the BHA was interested in forecasting waiting times and tenant allocations under the current system and making these predictions known to newly arriving applicants. Additional policy questions address the addition of newly rehabilitated units to the system; the institution of income and racial mixing priorities; and the implementation of an alternative tenant assignment scheme (e.g. single project or citywide first available unit systems). In this chapter, we will utilize many of the techniques developed in Chapter 4 and 5 to study some of these issues.

6.1 Simulating BHA Tenant Assignments

In order to analyze BHA waiting lists, the simulation model described in Chapter 5 was modified to incorporate the effect of dropout. An initial decision to treat emergency applicants as regular applicants was made due to the relatively few numbers of emergency applicants in the system. To include dropout, we modify Block 1 of the simulation by including the following step:

For every household in the system, generate a random time until dropout using an exponential distribution with parameter δ. In Block 2 of the simulation, these times until dropout are sorted, saving household identifiers, along with the assignment times generated in Block 1. Finally, Block 3 is modified such that possible tenant assignments include dropouts; if the "project" identified in step 4 of Block 3 corresponds to a dropout, and the household has yet to be assigned, then the household drops out, and the number of dropouts is incremented. A complete computer listing of the simulation model used for the BHA analysis is found in Appendix 6.1.

The data analyzed in this chapter all stem from BHA computer files. The November 1983 waiting list, replete with project choices for each household listed in order of application date, was obtained. This list reflects all households waiting for housing assignments in BHA projects as of November 1983. Table 6.2 shows the breakdown of households on the waiting list by bedroom size and project choices; as households can choose up to three projects in the BHA assignment system, the project figures in Table 6.1 overstate the true number of households waiting. The true size of the waiting list for each bedroom size is indicated at the bottom of Table 6.1.

Two sets of parameters need to be estimated in order to implement the model; the moveout rates for each project by bedroom category, and the household specific dropout rates, also estimated by bedroom size. The moveout rates were estimated by observing the actual one year moveout rates from January 1 through December 31, 1983 for each project with available data by bedroom size; moveout rates were unavailable for

Table 6.1

Waiting Lists by Bedroom Size

Project	ID	1BR	2BR	3BR	4BR	5BR
Charlestown	101	58	111	115	28	9
Mission Hill	103	138	342	300	111	27
Lenox Street	104	175	454	322	-	-
Orchard Park	105	56	167	150	52	10
Cathedral	106	145	330	317	91	-
Maverick	108	57	180	140	27	7
Franklin Hill	109	119	301	267	80	-
Whittier St.	111	108	323	228	71	-
Beech St.	113	89	267	190	61	14
Mission Extension	114	118	192	129	38	16
Columbia Point	120	3	14	6	10	2
Mary Ellen McCormack	123	283	316	219	-	-
Old Colony	124	223	268	176	21	5
West Newton St.	158	242	532	379	105	19
Rutland	174	125	271	186	-	-
Collins	226	21	7	-	-	-
Annapolis	227	4	2	-	-	-
Ashmont	228	22	8	-	-	-
Holgate	229	9	1	-	-	-
Foley Apts	230	156	-	-	-	-
Groveland	232	8	4	-	-	-
Davison	234	17	-	-	-	-
Washington	235	155	8	-	-	-
West 9th St.	236	132	7	-	-	-
Carrol Apts.	237	127	7	-	-	-
Meade Apts.	238	7	8	-	-	-
Warren Tower	240	21	-	-	-	-
Eva W. White	241	21	1	-	-	-
Walnut Park	242	16	7	-	-	-
Tremont St.	244	32	-	-	-	-
Amory St.	245	24	2	-	-	-
Warren Apts .	247	27	5	-	-	-
Torre Unidad	249	43	-	-	-	-
Rockland	250	24	-	-	-	-
Codman Apts.	251	21	8	-	-	-
Heritage	252	84	8	24	14	-
St. Botolph	253	38	-	-	-	-
Pasciucco	254	12	-	-	-	-
Lower Mills	257	45	8	-	-	-
Ausonia Homes	261	115	5	-	-	-
Hassan	262	23	4	-	-	-
West Roxbury	270	61	13	-	-	-
Washington Cory	271	176	16	-	-	-
Cliffmont Roslindale	272	9	2	-	-	-
Bellflower	277	54	3	-	-	-
Peabody Square	283	52	1	-	-	-
Northampton	298	23	-	-	-	-

Table 6.1 (con't)

Waiting Lists by Bedroom Size

Project	ID	1BR	2BR	3BR	4BR	5BR
1701 Washington St.	299	21	-	-	-	-
Broadway	501	8	15	9	1	-
Camden St.	502	70	243	162	-	-
Commonwealth	503	11	9	19	7	-
Faneuil	504	-	246	172	-	16
Fairmont	505	1	199	162	-	-
Archdale	507	79	249	222	59	6
Orient Heights	508	41	222	145	43	6
Gallivan Blvd	510	-	485	455	111	-
Franklin Field	511	4	13	24	15	-
South St.	512	100	266	224	72	-
Franklin Elderly I	601	-	-	2	-	-
Franklin Elderly II	602	1	-	-	-	-
L St.	603	185	11	-	-	-
Summer St. Hyde Park	605	37	-	-	-	-
ACTUAL TOTAL (True number of Households Waiting)		2092	2783	2150	493	74

four projects. These rates are shown in Table 6.2. For projects with waiting lists but no observed moveouts, the moveout rate was arbitrarily set to .1.

The estimation of dropout utilized the simple estimator $\hat{\delta}_3$ presented in equation (4.64). All households who applied for public housing in August 1982 were examined at the end of a one year period. Those households who withdrew or were found to be ineligible during the one year period were considered to have dropped out. The data used to estimate household specific dropout rates, along with the computed estimates of δ and the estimated mean time until dropout for each bedroom category appear in Table 6.3.

Expected tenant allocations and depletion times were estimated from 100 runs of the simulation model. Households choosing one of the projects with missing data were treated as though they has not chosen the project involved, and households choosing only projects with missing data were deleted from the analysis. In addition to computing the expected number of tentants assigned to projects and the associated mean time to assign these tenants, the model computed the standard deviations of the number of tenants assigned and the time to deplete the initial waiting lists. The model also computed the mean and standard deviation of the number of dropouts that occurred over the 100 simulation runs. These results are presented in Tables 6.4 through 6.8.

In reviewing these results, a number of features are evident. First, there is a tremendous variability in the number of households assigned to the different housing projects. This reflects both the differences in project popularity (as evidenced by the figures in Table

Table 6.2

Moveout Rates by Bedroom Size (1982)

Project	ID	1BR	2BR	3BR	4BR	5BR
Charlestown	101	75	56	6	7	.1
Mission Hill	103	18	35	12	6	1
Lenox Street	104	23	9	.1	-	-
Orchard Park	105	25	23	17	1	.1
Cathedral	106	39	25	13	.1	-
Maverick	108	11	18	13	3	1
Franklin Hill	109	7	26	7	1	-
Whittier St.	111	.1	13	3	1	-
Beech St.	113	16	36	13	4	1
Mission Extension	114	25	19	4	3	.1
Columbia Point	120	1	13	8	2	.1
Mary Ellen McCormack	123	56	27	5	-	-
Old Colony	124	33	22	9	5	.1
West Newton St.	158	8	.1	.1	.1	.1
Rutland	174	2	5	4	-	-
Collins	226	5	.1	-	-	-
Annapolis	227	3	1	-	-	-
Ashmont	228	4	1	-	-	-
Holgate	229	6	.1	-	-	-
Foley Apts	230	5	-	-	-	-
Groveland	232	.1	.1	-	-	-
Davison	234	7	-	-	-	-
Washington	235	11	2	-	-	-
West 9th St.	236	10	.1	-	-	-
Carrol Apts.	237	9	1	-	-	-
Meade Apts.	238	1	.1	-	-	-
Warren Tower	240	8	-	-	-	-
Eva W. White	241	8	.1	-	-	-
Walnut Park	242	4	2	-	-	-
Tremont St.	244	3	-	-	-	-
Amory St.	245	8	3	-	-	-
Warren Apts .	247	4	.1	-	-	-
Torre Unidad	249	2	-	-	-	-
Rockland	250	.1	-	-	-	-
Codman Apts.	251	4	.1	-	-	-
Heritage	252	14	3	.1	.1	-
St. Botolph	253	5	-	-	-	-
Pasciucco	254	4	-	-	-	-
Lower Mills	257	3	1	-	-	-
Ausonia Homes	261	3	1	-	-	-
Hassan	262	11	.1	-	-	-
West Roxbury	270	13	.1	-	-	-
Washington Cory	271	13	.1	-	-	-
Cliffmont Roslindale	272	8	1	-	-	-
Bellflower	277	10	1	-	-	-
Peabody Square	283	8	4	-	-	-
Northampton	298	5	-	-	-	-

Table 6.2 (con't)

Moveout Rates by Bedroom Size (1982)

Project	ID	1BR	2BR	3BR	4BR	5BR
1701 Washington St.	299	2	-	-	-	-
Broadway	501	.1	1	1	2	-
Camden St.	502	7	3	2	-	-
Commonwealth	503	.1	2	1	.1	-
Faneuil	504	-	25	14	-	.1
Fairmont	505	.1	3	1	-	-
Archdale	507	5	30	8	1	.1
Orient Heights	508	6	31	19	1	.1
Gallivan Blvd	510	-	3	1	1	-
Franklin Field	511	2	7	2	.1	-
South St.	512	3	2	5	.1	-
Franklin Elderly I	601	1	3	.1	-	-
Franklin Elderly II	602	3	.1	-	-	-
L St.	603	2	.1	-	-	-
Summer St. Hyde Park	605	7	1	-	-	-
SYSTEM ASSIGNMENT RATES		576.6	470.4	168.4	38.6	3.9

Table 6.3

Estimation of Household
Dropout Rates (Aug. 82 - July 83)

BEDROOM SIZE	NUMBER DROPOUTS	INITIAL SAMPLE SIZE	$\hat{\delta}$	ESTIMATED MEAN TIME UNTIL DROPOUT (YEARS)
1	64	183	.3497	2.86
2	40	232	.1724	5.80
3	28	183	.1530	6.54
4	9	40	.2250	4.44
5	1	6	.1667	6.00

Table 6.4

AGGREGATE SYSTEM RESULTS - UNIT TYPE 1   2092 HOUSEHOLDS PRESENT AT START   100 SIMULATION ROUNDS

PROJECT ID	MEAN NUMBER SERVED	Std. Deviation	MEAN DEPLETION TIME	Std. Deviation
101	51.52	2.52	0.70	0.09
103	43.59	5.59	2.41	0.17
104	55.53	6.00	2.44	0.17
105	40.03	3.21	1.61	0.23
106	82.30	5.79	2.10	0.13
108	22.66	3.08	2.18	0.30
109	21.68	3.23	3.04	0.41
111	0.71	0.73	2.47	2.99
113	39.70	5.07	2.43	0.21
114	56.39	6.69	2.22	0.15
120	1.67	0.96	1.23	0.89
123	124.28	8.50	2.22	0.12
124	74.34	7.66	2.24	0.13
158	32.86	4.06	4.14	0.41
174	8.12	2.75	3.84	0.68
226	9.08	2.11	1.84	0.39
227	2.58	0.98	0.69	0.28
228	9.28	2.58	2.28	0.39
229	6.39	1.44	1.06	0.31
230	22.14	3.38	4.42	0.57
232	0.32	0.55	0.56	1.47
234	8.99	2.06	1.27	0.30
235	35.66	5.06	3.22	0.27
236	39.92	4.95	3.93	0.37
237	27.25	4.04	3.08	0.26
238	2.46	0.98	2.03	0.38
240	10.29	2.57	1.27	0.23
241	11.83	2.30	1.46	0.29
242	5.73	1.91	1.32	0.34
244	6.96	2.26	2.22	0.54
245	13.27	2.51	1.61	0.26
247	12.29	2.09	3.05	0.57
249	7.64	2.09	3.68	0.83
250	0.45	0.59	1.15	2.08
251	7.72	1.93	1.85	0.40
252	42.58	4.29	3.06	0.34
253	11.99	2.82	2.29	0.41
254	6.38	1.77	1.45	0.36
257	9.42	2.41	3.25	0.61
261	19.90	3.49	6.87	0.97
262	15.26	2.01	1.38	0.29
270	27.87	3.34	2.21	0.33
271	41.74	5.59	3.20	0.27
272	7.24	1.25	0.90	0.31
277	23.58	2.98	2.34	0.27
283	21.66	3.15	2.63	0.40
298	9.81	2.61	1.88	0.28
299	5.44	1.81	2.31	0.65
501	0.60	0.65	1.81	2.54
502	18.02	3.77	2.57	0.30
503	0.56	0.75	1.90	2.89
505	0.25	0.43	0.63	1.45
507	14.41	3.14	2.82	0.49
508	13.16	2.72	2.14	0.29
511	1.20	0.62	0.48	0.47
512	12.97	2.83	4.25	0.75
602	0.89	0.31	0.28	0.26
603	10.63	2.30	5.11	0.88
605	14.64	2.60	2.12	0.36

EXPECTED NUMBER OF DROPOUTS= 866.17   Std. Deviation= 20.26

Table 6.5

AGGREGATE SYSTEM RESULTS - UNIT TYPE 2   2783 HOUSEHOLDS PRESENT AT START   100 SIMULATION ROUNDS

PROJECT ID	MEAN NUMBER SERVED	Std. Deviation	MEAN DEPLETION TIME	Std. Deviation
101	94.09	3.77	1.71	0.18
103	145.01	9.70	4.12	0.17
104	61.04	6.37	6.83	0.65
105	92.40	7.88	3.95	0.20
106	108.10	8.31	4.25	0.19
108	63.93	7.23	3.53	0.18
109	112.46	9.90	4.36	0.19
111	73.43	5.74	5.66	0.47
113	133.18	10.54	3.70	0.16
114	78.20	8.32	4.07	0.20
120	10.17	1.54	0.78	0.22
123	139.12	7.44	3.81	0.19
124	85.37	8.26	3.84	0.18
158	1.69	1.17	11.53	7.70
174	45.61	5.24	9.19	0.83
226	0.94	0.83	4.56	4.64
227	1.39	0.56	1.23	0.91
228	3.53	1.30	3.12	1.40
229	0.30	0.46	0.54	1.16
232	0.66	0.70	2.21	3.27
235	5.29	1.36	2.60	0.90
236	0.53	0.71	1.92	3.26
237	2.85	1.13	2.52	1.12
238	0.90	0.75	3.65	4.29
241	0.22	0.41	0.42	1.10
242	4.29	1.24	2.30	0.36
245	1.67	0.55	0.50	0.41
247	0.94	0.86	4.32	5.21
251	0.95	0.96	3.92	4.50
252	5.96	1.18	1.85	0.57
257	2.79	1.09	2.43	1.23
261	2.67	0.86	2.52	1.28
262	0.83	0.81	2.96	3.83
270	1.27	1.05	7.25	6.21
271	1.13	0.83	6.24	5.25
272	0.97	0.67	0.48	0.51
277	1.94	0.80	1.80	1.09
283	0.98	0.14	0.25	0.23
501	3.50	1.11	3.29	1.52
502	25.34	4.82	8.49	0.82
503	3.70	1.20	1.83	0.68
504	102.20	8.40	4.09	0.20
505	30.60	3.73	9.84	1.11
507	110.30	8.91	3.68	0.17
508	109.39	7.06	3.53	0.19
510	35.46	4.89	11.82	1.18
511	8.85	1.40	1.29	0.31
512	23.23	3.84	11.73	1.48
603	1.14	1.07	5.51	5.68

EXPECTED NUMBER OF DROPOUTS=1042.50   Std. Deviation= 21.02

Table 6.6

AGGREGATE SYSTEM RESULTS - UNIT TYPE 3 2150 HOUSEHOLDS PRESENT AT START 100 SIMULATION ROUNDS

PROJECT ID	MEAN NUMBER SERVED	Std. Deviation	MEAN DEPLETION TIME	Std. Deviation
101	37.46	5.04	6.28	0.57
103	94.35	7.98	7.93	0.44
104	2.04	1.33	13.10	8.45
105	92.16	5.24	5.39	0.44
106	101.55	8.65	7.78	0.44
108	58.81	6.62	4.61	0.33
109	61.01	7.13	8.74	0.66
111	31.44	4.81	10.51	1.14
113	85.04	7.37	6.60	0.46
114	30.83	4.33	7.59	0.56
120	5.63	0.54	0.76	0.29
123	41.15	5.98	8.08	0.68
124	68.01	6.53	7.64	0.61
158	1.92	1.23	13.52	7.95
174	40.51	4.55	9.96	1.01
252	1.20	0.92	6.63	6.16
501	3.88	1.53	3.52	1.45
502	20.85	3.93	10.32	1.17
503	5.75	1.61	5.32	1.50
504	90.09	4.86	5.79	0.48
505	14.63	3.09	15.02	2.22
507	56.16	7.42	7.08	0.44
508	84.94	6.94	4.49	0.28
510	17.99	3.14	18.31	2.12
511	10.47	2.39	5.27	1.03
512	43.71	4.56	8.66	0.70
601	0.65	0.73	2.46	3.65

EXPECTED NUMBER OF DROPOUTS=1057.71 Std. Deviation= 18.49

Table 6.7

AGGREGATE SYSTEM RESULTS - UNIT TYPE 4 493 HOUSEHOLDS PRESENT AT START 100 SIMULATION ROUNDS

PROJECT ID	MEAN NUMBER SERVED	Std. Deviation	MEAN DEPLETION TIME	Std. Deviation
101	16.55	2.52	2.45	0.41
103	38.25	4.26	6.51	0.69
105	8.43	2.45	7.56	1.45
106	1.51	1.18	8.06	5.92
108	10.53	2.35	3.50	0.76
109	9.73	2.35	9.21	1.36
111	9.09	2.47	8.23	1.37
113	23.72	3.89	5.91	0.74
114	16.41	2.71	5.28	0.83
120	6.36	1.45	3.12	0.99
124	13.35	2.36	2.69	0.54
158	1.51	1.18	9.12	7.21
252	0.55	0.62	1.53	2.37
501	0.83	0.38	0.33	0.37
503	0.50	0.53	1.63	2.56
507	8.43	2.73	7.90	1.27
508	6.72	1.99	6.83	1.92
510	10.56	2.73	10.21	1.92
511	0.88	0.90	3.41	4.07
512	1.45	1.20	7.65	5.84

EXPECTED NUMBER OF DROPOUTS= 307.52 Std. Deviation= 9.60

Table 6.8

AGGREGATE SYSTEM RESULTS - UNIT TYPE 5   74 HOUSEHOLDS PRESENT AT START   100 SIMULATION ROUNDS

PROJECT ID	MEAN NUMBER SERVED	Std. Deviation	MEAN DEPLETION TIME	Std. Deviation
101	0.74	0.86	3.92	4.94
103	9.30	2.57	9.19	2.19
105	0.86	0.88	3.12	3.71
108	4.22	1.23	3.79	1.28
113	6.47	1.76	6.27	1.77
114	1.10	0.96	5.27	5.16
120	0.58	0.68	2.23	3.71
124	0.72	0.76	2.99	4.07
158	1.43	1.10	7.10	5.82
504	1.40	1.15	6.88	6.07
507	0.58	0.70	2.07	2.96
508	0.53	0.64	2.23	3.21

EXPECTED NUMBER OF DROPOUTS= 46.07   Std. Deviation= 4.11

6.1), and the differences in project assignment rates (as evidenced by the figures in Table 6.2).

For the same reasons, there is a large variability in mean depletion times. These numbers have the following meaning: if a test applicant choosing only project i applied for public housing, and this applicant is guaranteed not to drop out, then the expected time until the test applicant is assigned would equal

$$E(w_N^{i*}) = E(\text{depletion time at project i}) + \frac{1}{\mu_i} \qquad (6.1)$$

As the mean depletion times vary from well under six months to well over six years, one would expect that making this information available to new applicants would influence their decisions as to which projects to choose.

Another interesting feature of these results rests with the large number of households who are predicted to drop out. By bedroom category, we have:

1 Bedroom - 41.40% expected to drop out

2 Bedroom - 37.46% expected to drop out

3 Bedroom - 49.20% expected to drop out

4 Bedroom - 62.38% expected to drop out

5 Bedroom - 62.26% expected to drop out

These dropout percentages seem high, considering the magnitudes of the household specific dropout rates in Table 6.3. However, when one considers the product of the household dropout rates with the number of households waiting for assignments, it becomes clear that for many projects, the aggregate dropout rate for all households waiting is higher than the project assignment rate. This yields the large numbers of dropouts observed.

We have shown the simulation model to be useful in predicting the implications of the BHA's multiqueue tenant assignment system for tenant allocations and waiting list depletion times. To place these consequences in some sort of comparative context, we will reanalyze the BHA data under the following two schemes:

1) Suppose each household is allowed only 1 choice (arbitrarily chosen to be the first one listed on the application form). This represents a series of single project assignment schemes.

2) Suppose all households are assigned on a first available unit basis citywide, and that households are indifferent among projects. This represents a system-wide multiproject assignment scheme.

As the analysis is the same for all bedroom sizes, we will focus our attention on those households requiring 3 bedroom units.

6.2 Single Project Assignment Scheme

To model the implications of a single project scheme, we recall that for a system with dropout but no priorities, the expected amount of time to process a waiting list of size N is given by

$$E(w_N) = \sum_{n=1}^{N} \frac{1}{n\delta+\mu} \tag{6.2}$$

The mean number of households assigned then equals $\mu E(w_N)$. The expected number of dropouts, $E(N_D)$, is then computed as

$$E(N_D) = N - \mu E(w_N) \tag{6.3}$$

following the reasoning of equation (4.51).

Table 6.9 shows the three bedroom unit moveout rate for each project, along with the number of households who listed that project first on their application form. The mean number of assignments, expected depletion times and numbers of dropouts were computed using the equations above. As some 159 households listed a project for which no moveout information was available as a first choice, the total number of households considered reduces from 2150 to 1991.

Compared to the multiqueue system, the single project scheme creates even more variability in tenant allocations and waiting list depletion times. This is to be expected, as unlike the multiqueue system where assignments at one project effect assignments elsewhere, tenant allocations in the single project scheme are independent across projects. Thus, the single project scheme creates maximum variability in tenant allocations and waiting times across the entire system.

In considering dropout, it is clear that those projects with lower assignment rates will induce relatively larger numbers of dropouts. Systemwide, the expected number of dropouts equals 1123.51, or 56.43% of those initially waiting for assignments. This represents a noticeable increase from the 49.2% of all three bedroom households expected to drop out under the multiqueue system.

6.3 Citywide First Available Unit System

To model the implications of a citywide assignment scheme, we use the multiproject approach of Chapter 5. First, we set

$$\mu_{sys} = \sum_i \mu_i = 168.4 \tag{6.4}$$

as the system wide annual assignement rate for three bedroom units. From

Table 6.9

Results from Single Project Assignment Scheme

PROJECT	μ	N	Mean Number Assigned	$E(w_N)$ (years)	$E(N_D)$
Charlestown	6	69	39.48	6.58	29.52
Mission Hill	12	179	92.88	7.74	86.12
Lenox Street	.1	128	3.07	30.65	124.94
Orchard Park	17	76	57.63	3.39	18.37
Cathedral	13	127	77.35	5.95	49.65
Maverick	13	73	52.52	4.04	20.48
Franklin Hill	7	104	53.90	7.70	50.10
Whittier St.	3	74	30.27	10.09	43.73
Beech St.	13	68	49.79	3.83	18.21
Mission Extension	4	45	25.84	6.46	19.16
Columbia Point	8	3	2.88	0.36	.12
Mary Ellen McCormack	5	144	54.75	10.95	89.25
Old Colony	9	65	43.56	4.84	21.44
West Newton St.	.1	127	3.06	30.60	123.94
Rutland	4	26	17.80	4.45	8.20
Heritage	.1	7	1.26	12.60	5.74
Broadway	1	2	1.63	1.63	.37
Camden	2	35	16.66	8.33	18.34
Commonwealth	1	11	6.15	6.15	4.85
Faneuil	14	79	56.70	4.05	22.30
Fairmont	1	67	15.38	15.38	51.62
Archdale	8	71	44.56	5.57	26.44
Orient Heights	19	53	44.08	2.32	8.92
Gallivan Blvd	1	253	23.59	23.59	229.41
Franklin Field	2	15	9.72	4.86	5.28
South St.	5	88	42.35	8.47	45.65
Franklin Elderly	.1	2	.64	6.42	1.36

equation (6.2) we obtain the expected time to process all 2150 households awaiting assignments as:

$$E(w_{2150}) = \sum_{n=1}^{2150} \frac{1}{(.153n + 168.4)} = 7.08 \qquad (6.5)$$

This result is somewhat astonishing in its own right. Ignoring the possibility of priority assignments, and assuming that households would accept whatever units are offered whenever they are located, it would take about 7 years to house all three bedroom households waiting for assignments as of November 1983. This represents the shortest amount of time in which these assignments could occur! The result suggests that waiting lists should be closed, at least for some projects, owing to excessive waiting times.

In a first available unit system, mean depletion times are equal at all projects. Recall that the expected number of households assigned to project i simply equals $\mu_i E(w_N)$. These figures are presented in Table 6.10. All variability in tenant allocations can now be attributed to the different assignment rates at the different projects. As such, the citywide first available unit system demonstrates the smallest variability in housing assignments among tenant assignment systems for our data.

The expected number of dropouts in the citywide system is given by equation (6.3) using μ_{sys} and the total number of households waiting citywide for three bedroom units; the actual figure equals $2150 - 168.4 \times 7.08 = 957.73$. Thus, one would expect 44.55% of all households to drop out. This represents a decrease from the 49.2% of all households that are expected to drop out in the multiqueue system.

Table 6.10

Results from Citywide First Available Unit System

Project	Moveout Rate	Mean Number Assigned
Charlestown	6	42.48
Mission Hill	12	84.96
Lenox St.	.1	.71
Orchard Park	17	120.36
Cathedral	13	92.04
Maverick	13	92.04
Franklin Hill	7	49.56
Whittier St.	3	21.24
Beech St.	13	92.04
Mission Extension	4	28.32
Columbia Point	8	56.64
Mary Ellen McCormack	5	35.40
Old Colony	9	63.72
West Newton St.	.1	.71
Rutland	4	28.32
Heritage	.1	.71
Broadway	1	7.08
Camden	2	14.16
Commonwealth	1	7.08
Faneuil	14	99.12
Fairmont	1	7.08
Archdale	8	56.64
Orient Heights	19	134.52
Gallivan Blvd	1	7.08
Franklin Field	2	14.16
South St.	5	35.40
Franklin Elderly	.1	.71

An interesting question to consider relates to the number of households in a citywide first available unit system who would have received an assignment to a project in their choice set. If this figure is relatively high, one would not expect to encounter major objections from tenants if a switch to a citywide system was proposed. If this figure is low, then a change to a citywide system could have the effect of causing many would be tenants to drop out of public housing rather than accept a unit in an undesirable location.

To calculate the likelihood that a tenant is assigned to a project in their choice set under a citywide assignment scheme, we note that

$$\Pr\left\{\begin{array}{l}\text{assignment to a unit}\\ \text{in choice set}\end{array}\right\} = \Pr\left\{\begin{array}{l}\text{assigned unit is in}\\ \text{a choice set}\end{array} \middle| \begin{array}{l}\text{assignment}\\ \text{occurs}\end{array}\right\} \times \Pr\{\text{assignment occurs}\} \qquad (6.6)$$

The probability that an assignment occurs is simply the dropout probability subtracted from 1. For our data,

$$\Pr\{\text{assignment occurs}\} = 1 - .4455 = .5545 \qquad (6.7)$$

Now, the probability that an assigned unit is in a choice set given that an assignment occurs can be estimated by

$$\Pr\left\{\begin{array}{l}\text{assigned unit is in}\\ \text{a choice set}\end{array} \middle| \begin{array}{l}\text{assignment}\\ \text{occurs}\end{array}\right\} =$$

$$\frac{1}{N}\sum_{j=1}^{N} \Pr\left\{\begin{array}{l}\text{household j is assigned a}\\ \text{unit in the choice}\\ \text{set } C_j\end{array} \middle| \begin{array}{l}\text{household j}\\ \text{is assigned}\end{array}\right\} \qquad (6.8)$$

In words, we will estimate the conditional likelihood of an assignment occuring in a choice set by averaging the household specific likelihoods of this same event. These household specific probabilities are easy to obtain:

$$\Pr\left\{\begin{array}{l}\text{household j is assigned a}\\ \text{unit in the choice set } C_j\end{array} \middle| \begin{array}{l}\text{household j}\\ \text{is assigned}\end{array}\right\}$$

$$= \frac{\sum_{i \varepsilon C_j} \mu_i}{\sum_i \mu_i} \tag{6.9}$$

where the μ_i's represent the assignment rates at the various projects. Thus, we estimate the conditional probability of an assignment belonging to some choice set given an assignment occurs as

$$\Pr\left\{ \begin{array}{l} \text{assigned unit is in} \\ \text{a choice set} \end{array} \middle| \begin{array}{l} \text{assignment} \\ \text{occurs} \end{array} \right\} = \frac{1}{N} \sum_{j=1}^{N} \frac{\sum_{i \varepsilon C_j} \mu_i}{\sum_i \mu_i} \tag{6.10}$$

For our data, application of equation (6.10) yields a conditional probability of assignment to a choice set given that an assignment occurs of .0855. Combining our results, we obtain

$$\Pr\left\{ \begin{array}{l} \text{assignment to a} \\ \text{unit in a choice set} \end{array} \right\} = .0855 \times .5545 \tag{6.11}$$

$$= .0474$$

A switch to a citywide first available unit system would result in less than 5% of the 2150 households waiting as of November 1983 for 3 bedroom apartments receiving assignments in desirable projects. Given this result, any attempt to change from the current multiqueue system to a citywide first available unit system must be viewed as unwise.

6.4 Other Issues

We have focused on the impact of changing the current multiqueue system to either a single project assignment scheme or a citywide first available unit system. There are a number of other issues one could address using the methods we have developed; two of them shall be briefly mentioned.

6.4.1 Categorical Priorities

Our models ignored the effect of prioritized arrivals as these represent a small percentage of all applicants for BHA housing units. However, given the relatively long waiting times for housing assignments, the effect of priorities (such as emergencies) could have a major impact upon the waiting times for standard applicants, and hence upon the number of dropouts from the system. We already know how to incorporate the affect of categorical priorities into single project and citywide first available unit systems. The simulation model could also be modified to incorporate the effect of priorities on tenant allocations and waiting times.

6.4.2 Blend Priorities

Two major policy reforms are currently being reviewed by the BHA. The first involves income mixing, while the second involves racial integration. In both cases, the BHA has specified the desired project compositions, in terms of racial and income mixes, for most projects in the authority. Models of blend priorities could be used to determine the time necessary to achieve these goals, and the impact of these policies on tenant waiting times.

As an example, the Gallivan Blvd. housing project currently possesses a racial mix of 46.4% white households and 53.6% non-white households (from BHA records). The new racial mixing plan calls for two white households to be admitted for every non-white household, implying a target mix of 67% white households, and 33% non-white households. The project is comprised of 250 households with a total moveout rate of 5 households per year across all apartment types. From equation (4.67) of Chapter 4, we see that if current trends continue, the expected number of white households in Gallivan Blvd. on the m^{th} move after implementing the stated racial blend priority equals

$$h_{white,m} = 250 \times .67 + (250 \times .464 - 250 \times .67)(1 - 1/250)^m \qquad m=0,1,2,...$$

$$= 167.5 - 51.5\ (.996)^m \qquad (6.12)$$

At a pace of roughly five moves per year, we see that after 10 years (or 50 moves), the expected number of white households in Gallivan Blvd would equal (using m=50 in equation (6.12)) 125, or 50% of the total project population. After 20 years (or 100 moves), the mean number of white households would equal 133, or about 53% of the project population. The process could only move faster if the moveout rate increased, or if the blend probabilities were changed to favor white households more heavily. While these results are hardly exact, they should serve to convince the reader that the integration of the Gallivan Blvd project will take a long time to achieve.

6.5 Summary

This chapter has illustrated a number of useful points. Fist, we showed that one can predict waiting times and tenant allocations in a complex system like the BHA using our models. Secondly, we showed how one can perform various policy analyses using the models developed in Chapters 4 and 5. We examined the impacts of single project and citywide first available unit systems on housing assignments and waiting times, and also considered the choice effects associated with a citywide scheme. Finally, we mentioned some other issues that could be studied, such as the impact of categorical priorities, and the impact of racial and income mixing on tenant allocations, waiting times, and project compositions. While we have not tried to be exhaustive, this chapter has hopefully served the purpose of demonstrating the potential usefulness of tenant assignment modeling in a policy context.

Appendix 6.1

```
c
c     This program simulates a tenant assignment system where tenants are
c     assigned on a first come first serve basis within choice sets.  Dropout
c     is incorporated into the model.  The model computes household specific
c     measures, as well as system wide statistics.  This program is the
c     property of Ed Kaplan, Dept. of Urban Studies and Planning, MIT.
c
c


      double precision dseed
      common proj(3,3000),x(3,3000),w(3,3000),drop(3000),tdrop(3000),iserve(3000),avgl(70),avglsq(70),sigl(70),avgwq(70),sigw(70),p(
3),avgw(3)

      common rl(70),rlsq(70),rlsum(70),wtot(70),wtotsq(70),rwsum(70),rate(50),avgwsq(70),
     &ihshld(70,1000),rmu(70),nq(70),id(70),tt(9000),ifac(9000),lserve(70),ip(9000)

      common t,tally,dmu,dropn,dropn2,drpsum,np,nn,nc,nsim,ksim,c,ntypes,itype,dseed

      integer c,proj,tally



c
c     here is the main calling sequence
c

      call input
      call init
      do 10 ksim=1,nsim
      if(ksim/10*10.eq.ksim) print,"round",ksim
      call service
      call update
   10 continue
      call report
      stop
      end

      subroutine input

      double precision dseed
      common proj(3,3000),x(3,3000),w(3,3000),drop(3000),tdrop(3000),iserve(3000),avgl(70),avglsq(70),sigl(70),avgwq(70),sigw(70),p(
3),avgw(3)

      common rl(70),rlsq(70),rlsum(70),wtot(70),wtotsq(70),rwsum(70),rate(50),avgwsq(70),
     &ihshld(70,1000),rmu(70),nq(70),id(70),tt(9000),ifac(9000),lserve(70),ip(9000)

      common t,tally,dmu,dropn,dropn2,drpsum,np,nn,nc,nsim,ksim,c,ntypes,itype,dseed

      integer c,proj,tally



c
c     This routine inputs data from the terminal and from files 7 and 8.
c

      rewind 7
```

```
      rewind 8

c     Input parameters from terminal

      print,"Input number of unit types, type desired for this run,"
      print,"seed for random number generator."
      print,"number of choices allowed, and number of simulations."
      read, ntypes,itype,dseed,nc,nsim

c     read in project information

      i=0
 10   i=i+1
      read(7,15,end=30) id(i),(rate(k),k=1,ntypes)
 15   format(v)
      if(id(i).eq.0) go to 20
      rmu(i)=rate(itype)
      go to 10

c     set up dropout rate

 20   np=i-1
      dmu=rate(itype)
      go to 40

 30   print,"Error on input - project file (unit 07)"
      stop

c     read in individual information

 40   j=0
 50   j=j+1
      read(8,15,end=60) (proj(c,j),c=1,nc)
c     check to see that at least one choice is a real project

      do 42 i=1,np
      do 42 c=1,nc
      if(proj(c,j).eq.id(i)) go to 50
 42   continue
      j=j-1
      go to 50
 60   nh=j-1

      print,"nh=",nh
      return
      end


      subroutine init

      double precision dseed
      common proj(3,3000),x(3,3000),w(3,3000),drop(3000),tdrop(3000),iserve(3000),avgl(70),avglsq(70),sigl(70),avgwq(70),sigw(70),p(
 3),avgw(3)

      common rl(70),rlsq(70),rlsum(70),wtot(70),wtotsq(70),rwsum(70),rate(50),avgwsq(70),
     &ihshld(70,1000),rmu(70),nq(70),id(70),tt(9000),ifac(9000),lserve(70),ip(9000)

      common t,tally,dmu,dropn,dropn2,drpsum,np,nh,nc,nsim,ksim,c,ntypes,itype,dseed

      integer c,proj,tally
```

```
c       this routine initializes simulation variables

c       initialize aggregate variables

        do 10 i=1,np
        rl(i)=0
        rlsq(i)=0
        rlsum(i)=0
        wtot(i)=0
        wtotsq(i)=0
        rwsum(i)=0
        nq(i)=0
        lserve(i)=1
   10   continue


c       initialize dropout measures

        dropn=0
        dropn2=0
        drpsum=0

c       initialize counter

        tally=0


c       initialize household variables

        do 20 j=1,nh
        drop(j)=0
        tdrop(j)=0
        iserve(j)=0
        do 20 c=1,nc
        x(c,j)=0
        w(c,j)=0
   20   continue

c       form initial queues at projects

        do 30 j=1,nh
        do 40 c=1,nc
        do 50 i=1,np
        if(proj(c,j).ne.id(i)) go to 50
        nq(i)=nq(i)+1
        inshld(i,nq(i))=j
        go to 40
   50   continue
   40   continue
   30   continue

        return
        end

        subroutine service
```

```
      double precision dseed
      common proj(3,3000),x(3,3000),w(3,3000),drop(3000),tdrop(3000),iserve(3000),avgl(70),avglsq(70),sigl(70),avgwq(70),sigw(70),p(
3),avgw(3)

      common rl(70),rlsq(70),rlsum(70),wtot(70),wtotsq(70),rwsum(70),rate(50),avgwsq(70),
     &ihshld(70,1000),rmu(70),nq(70),id(70),tt(9000),ifac(9000),lserve(70),ip(9000)

      common t,tally,dmu,dropn,dropn2,drpsum,np,nh,nc,nsim,ksim,c,ntypes,itype,dseed

      integer c,proj,tally

c     This is a service/allocate routine that generates, sorts and
c     locates service/dropout times

c     set up the vector of transition times

      kkk=0
      do 10 i=1,np
      if(nq(i).eq.0) go to 10
      t=0
      do 20 k=1,nq(i)
      t=t-alog(ggubfs(dseed))/rmu(i)
      kkk=kkk+1
      tt(kkk)=t
      ifac(kkk)=i
   20 continue
   10 continue


c     generate dropout times

      do 50 j=1,nh
      kkk=kkk+1
      tt(kkk)=-alog(ggubfs(dseed))/dmu
      ifac(kkk)=np+j
   50 continue

c     sort the transition times using shell sort

      igap=kkk
   51 igap=igap/2
      nmg=kkk-igap
      do 53 j1=1,nmg
      i1=j1+igap
      jj=j1
   52 if(tt(jj).le.tt(i1)) go to 53
      swap=tt(i1)
      tt(i1)=tt(jj)
      tt(jj)=swap
      iswap=ifac(i1)
      ifac(i1)=ifac(jj)
      ifac(jj)=iswap

      i1=jj
      jj=jj-igap
      if(jj.ge.1) go to 52
   53 continue
      if(igap.gt.1) go to 51
```

```
c       allocate households

        do 70 kk=1,kkk
        if(tally.eq.nh) return
        i=ifac(kk)

c       check for dropout

        if(i.gt.np) go to 80
        if(lserve(i).gt.nq(i)) go to 70
        do 66 k=lserve(i),nq(i)
        j=ihshld(i,k)
        if(j.gt.nh) print,"ERROR - j>nh",kk,ifac(kk),j,ksim
        if(j.gt.nh) stop
        if(iserve(j).eq.1) go to 66
        lserve(i)=k+1
        iserve(j)=1
        t=tt(kk)
        tally=tally+1
c       assign to project, update measures

        rlsum(i)=rlsum(i)+1
        rwsum(i)=t
        do 90 c=1,nc
        if(proj(c,j).ne.id(i)) go to 90
        x(c,j)=x(c,j)+1
        w(c,j)=w(c,j)+t
        go to 70
   90   continue

        print, "ERROR - HSHLD ASSIGNED TO ILLEGAL PROJECT!!"
        stop

   66   continue

c       set project out of process
        lserve(i)=nq(i)+1
        go to 70
c       process dropout

   80   j=i-np
        if(iserve(j).eq.1) go to 70
        tally=tally+1
        t=tt(kk)
        iserve(j)=1
        drpsum=drpsum+1
        drop(j)=drop(j)+1
        tdrop(j)=tdrop(j)+t
   70   continue

        return
        end


        subroutine update

        double precision dseed
        common proj(3,3000),x(3,3000),w(3,3000),drop(3000),tdrop(3000),iserve(3000),avgl(70),avglsq(70),sigl(70),avgwq(70),sigw(70),p(
3),avgw(3)
```

```
      common rl(70),rlsq(70),rlsum(70),wtot(70),wtotsq(70),rwsum(70),rate(50),avgwsq(70),
     &ihshld(70,1000),rmu(70),nq(70),id(70),tt(9000),ifac(9000),iserve(70),ip(9000)

      common t,tally,dmu,dropn,dropn2,drpsum,np,nh,nc,nsim,ksim,c,ntypes,itype,dseed

      integer c,proj,tally


c     this routine updates all counters after each simulation run

c     update dropout counters

      dropn=dropn+drpsum
      dropn2=dropn2+drpsum**2
      drpsum=0

c     update project info

      do 10 i=1,np
      rl(i)=rl(i)+rlsum(i)
      rlsq(i)=rlsq(i)+rlsum(i)**2
      rlsum(i)=0
      wtot(i)=wtot(i)+rwsum(i)
      wtotsq(i)=wtotsq(i)+rwsum(i)**2
      rwsum(i)=0
      iserve(i)=1
   10 continue


c     update tally

      tally=0

c     update iserve

      do 20 j=1,nn
      iserve(j)=0
   20 continue

      return
      end

      subroutine report
      double precision dseed
      common proj(3,3000),x(3,3000),w(3,3000),drop(3000),tdrop(3000),iserve(3000),avgl(70),avglsq(70),sigl(70),avgwq(70),sigw(70),p(
3),avgw(3)

      common rl(70),rlsq(70),rlsum(70),wtot(70),wtotsq(70),rwsum(70),rate(50),avgwsq(70),
     &ihshld(70,1000),rmu(70),nq(70),id(70),tt(9000),ifac(9000),iserve(70),ip(9000)

      common t,tally,dmu,dropn,dropn2,drpsum,np,nh,nc,nsim,ksim,c,ntypes,itype,dseed

      integer c,proj,tally
```

```
c     this routine computes the final system statistics and produces reports

c     process aggregate measures - first dropouts

      rn=float(nsim)
      avgdrp=dropn/rn
      avgdrp2=dropn2/rn
      sigdrp=sqrt(avgdrp2-avgdrp**2)

c     process project information

      do 10 i=1,np
      avgl(i)=rl(i)/rn
      avglsq(i)=rlsq(i)/rn
      sigl(i)=sqrt(avglsq(i)-avgl(i)**2)
      avgwq(i)=wtot(i)/rn
      avgwsq(i)=wtotsq(i)/rn
      sigw(i)=sqrt(avgwsq(i)-avgwq(i)**2)
   10 continue

c     process household information


c     print out individual reports

      write(11,205)
  205 format("INDIVIDUAL RESULTS")
      write(11,206)
  206 format(//"HOUSEHOLD",t15,"WBAR",t20,"PREF1",t27,"Prob.",t34,"Wait",
     &t40,"PREF2",t47,"Prob.",t54,"Wait",t60,"PREF3",t67,"Prob.",t74,"Wait",
     &t80,"Prob. Dropout",t97,"Avg. Time to Dropout"//)
  207 format(3x,i4,t15,f5.1,t20,2x,i3,t27,f5.3,t34,f5.1,t40,2x,i3,t47,
     &f5.3,t54,f5.1,t60,2x,i3,t67,f5.3,t74,f5.1,t80,4x,f5.3,t97,5x,f5.1)

      do 20 j=1,nh
      pdrop=drop(j)/rn
      atdrop=0
      if(drop(j).gt.0) atdrop=tdrop(j)/drop(j)
      wbar=0
      do 30 c=1,nc
      p(c)=x(c,j)/rn
      avgw(c)=0
      if(x(c,j).eq.0) go to 35
      avgw(c)=w(c,j)/x(c,j)
   35 wbar=wbar+p(c)*avgw(c)
   30 continue
      if(pdrop.eq.1) go to 25
      wbar=wbar/(1.-pdrop)
   25 write(11,207) j,wbar,proj(1,j),p(1),avgw(1),proj(2,j),p(2),avgw(2),
     &proj(3,j),p(3),avgw(3),pdrop,atdrop
   20 continue

c     produce output - first report - aggregate results to file10

      write(10,201) itype,nh,nsim
  201 format("AGGREGATE SYSTEM RESULTS - UNIT TYPE",1x,i3,5x,i4," HOUSEHOLDS PRESENT AT START",3x,i4," SIMULATION ROUNDS")
      write(10,202)
  202 format(//"PROJECT",t10,"MEAN NUMBER SERVED",t30,"Std. Deviation",t50,
     &"MEAN DEPLETION TIME",t70,"Std. Deviation"//)
```

```
      do 50 i=1,np
      if(nq(i).eq.0) go to 50
      write(10,203) id(i),avgl(i),sigl(i),avgwq(i),sigw(i)
  203 format(2x,i3,t10,5x,f6.2,t30,4x,f6.2,t50,5x,f6.2,t70,4x,f6.2)
  50  continue

c     write out dropout info
      write(10,204) avgdrp,sigdrp
  204 format(//"EXPECTED NUMBER OF DROPOUTS=",f7.2,4x,"Std. Deviation=",f7.2)

      return
      end
```

Chapter VII

Relocation Models for Public Housing Redevelopment Programs

A very different type of tenant assignment issue arises in the context of project redevelopment programs. Given that many public housing projects were constructed in the period following World War II; these projects have literally come of age. In the Boston area alone, four major housing projects are undergoing physical redevelopment at tremendous expense; some relevant data are shown in Table 7.1. Given these expenses, public housing authorities must be able to determine how large developments can be rehabilitated within cost, time, occupancy and other constraints.

Clearly, there are many complex issues involved in any redevelopment process, including physical design problems, financing techniques, tenant participation in program planning, construction management, and overall program control. One issue central to all public housing redevelopment efforts is the relocation problem - in order to redevelop public housing stock current tenants must be relocated to temporary quarters, and assigned to upgraded housing units once these units are available. Relocation problems involve two broad classes of issues. First, it should be clear that when relocating tenants in a large public housing project, one is dealing with a large population and its associated set of heterogeneous social concerns. Some tenants may have occupied their current units for several years, and could be understandably reluctant to move. Other groups of tenants might insist on being moved together to neighboring units (e.g. an elderly parent and his/her extended family). In implementing relocation programs, these idiosyncracies cannot be ignored.

Table 7.1

Boston Area Redevelopment Projects

Project	Total Redevelopment Costs	Number of Units	Cost per Unit
Commonwealth	$31,566,275	392	$80,526
Jefferson Park	$12,500,000	175	$71,429
West Broadway	$29,176,000	341	$85,560
Franklin Field	$32,780,000	346	$94,740

Source: New Lives for Old Projects: Revitalizing Public Housing, Public Housing Research Group, MIT, 1983.

On the other hand, there are the technical issues relating to project feasibility. For example, it must be possible to relocate all households from a building before that building can be constructed; this must be true for all buildings. From a technical perspective, there are three major components to any redevelopment project: a design which dictates the distribution of completed apartments (by type) across the housing project; a sequence which dictates the order in which buildings are to be redeveloped; and a relocation plan which dictates where tenants will move temporarily (permanently), when these temporary (permanent) moves will occur, and the "rules" which govern these moves.

Methods for surfacing relocation strategies should be of major interest to public housing officials, yet to date, no proposals for systematically attacking relocation problems are evident. In this chapter, I discuss some technical aspects of relocation planning. We will begin with the formulation of a scheduling model for a redevelopment program at a project with a homogeneous tenant population. The properties of this model are examined via a numerical example. As the procedure for solving the model is computationally complex, we propose some approximations. We close by considering a number of improvements to the model formulation aimed at incorporating more realistic aspects of relocation problems.

7.1 The Basic Relocation Problem

The notation used in this section is summarized in Table 7.2. We begin by considering a homogeneous project consisting of B buildings. Each building b in the project initially contains n_b households, $b=1,2,...B$.

Table 7.2

Notation for the Basic Relocation Model

n_b = number of households initially living in building b

a_b = number of apartments to be contained in building b as a result of the redevelopment program

L_b = length of time necessary to redevelop building b

B = number of buildings in the project

$x_{bt} = \begin{cases} 1 & \text{if building b undergoes redevelopment in week t} \\ 0 & \text{if not} \end{cases}$

N_t = number of households relocated from their initial homes in week t

A_t = number of new apartments completed and available for occupancy project wide in week t

V_t = number of vacant units available for occupancy in week t

D = project duration

M = maximum feasible project duration

All households in the project require the same unit type, and all apartments in the project are of that required type.

The proposed redevelopment design for the project calls for a_b apartments to be present in building b after building b has been redeveloped. Note that if building b represents a new building not previously present, then $n_b=0$. Also note that if building b is to be demolished, then $a_b=0$. In general, a_b can be greater than, equal to, or less than n_b. Of course, a necessary (but not sufficient) condition for the feasibility of the relocation design is that

$$\sum_{b=1}^{B} a_b \geq \sum_{b=1}^{B} n_b \quad . \tag{7.1}$$

We assume that the time necessary to redevelop building b is known and equal to L_b. Building b cannot undergo redevelopment until all occupants of the building have been relocated. If building b undergoes redevelopment in week t, then the work is completed in week $t+L_b-1$, and the building may be reoccupied in week $t+L_b$.

Suppose that the building populations n_b and new apartment allocations a_b are such that it is feasible to redevelop the project; all buildings can be rehabilitated while all households are guaranteed to be housed throughout the entirety of the project. The <u>maximum amount</u> of time necessary to redevelop a feasible project equals

$$M = \sum_{b=1}^{B} L_b + 1 \tag{7.2}$$

This follows from the fact that the longest feasible schedule is given by redeveloping buildings one at a time with no overlap in the redevelopment process.

We now define the indicator variable x_{bt} as

$$x_{bt} = \begin{cases} 1 \text{ if building b undergoes redevelopment in week t} \\ 0 \text{ if not} \end{cases} \qquad (7.3)$$

$$b=1,2,\ldots,B$$
$$t=1,2,\ldots,M-L_b$$

The variable x_{bt} will inform us of when building b begins to be redeveloped. Clearly, we only need to consider starting weeks up to $M-L_b$, as the project, if feasible must be completed by week M, and a start date for building b beyond week $M-L_b$ would imply a project completion date beyond week M, a contradiction. We also note that the variables x_{bt} are constrained by

$$\sum_{t=1}^{M-L_b} x_{bt} = 1 \quad . \qquad b=1,2,\ldots,B \qquad (7.4)$$

This result simply ensures that all buildings undergo rehabilitation exactly once.

Once the decision variables x_{bt} are determined (we have not yet stated how), several other interesting quantities may be defined. First of all, we may define the number of households relocated from their initial homes in week t, N_t, as

$$N_t = \sum_{b=1}^{B} n_b x_{bt} \qquad t=1,2,\ldots M-\min_b L_b \qquad (7.5)$$

since when x_{bt} equals 1, n_b households must be relocated from building b. Similarly, we define the number of new apartments available for occupancy in week t, A_t, as

$$A_t = \sum_{b=1}^{B} n_b \, x_{b,t-L_b} \qquad t=1,2,\ldots,M \qquad (7.6)$$

since when $x_{b,t-L_b}$ equals 1, building b undergoes redevelopment in week $t-L_b$, and hence is available for reoccupancy in week t. Upon availability for reoccupancy, building b contributes a_b new apartments for assignment.

Finally, we define V_t to be the number of vacant units in the project in week t. This quantity is clearly given by the balance equation

$$V_t = V_{t-1} + A_t - N_t \qquad t=1,2,\ldots,M \qquad (7.7)$$

The number of vacancies in week t equals the number of vacancies in week t-1 plus the number of new apartments available for occupancy in week t, minus the number of households relocated in week t. The redevelopment program is assumed to start with an initial endowment of vacancies V_o. Also, to guarantee that all households always are housed in every week of the redevelopment program, we require that

$$V_t \geq 0 \quad . \qquad t=1,2,\ldots,M \qquad (7.8)$$

The number of vacancies is never allowed to become negative.

We are now able to formulate our basic relocation model. The objective will be to find a sequence of construction which minimizes the total time necessary to redevelop the project, subject to the constraints that all households are always housed throughout the redevelopment program. Let D be the duration of the redevelopment project. We formulate the model as:

$$\text{minimize } D \qquad (7.9)$$

subject to:

$$\text{(i)} \quad \sum_{t=1}^{M-L_b} t x_{bt} + L_b \leq D \qquad b=1,2,\ldots,B$$

(ii) $\sum_{t=1}^{M-L_b} x_{bt} = 1$ $\qquad b=1,2,\ldots,B$

(iii) $N_t = \sum_{b=1}^{B} n_b x_{bt}$ $\qquad t=1,2,\ldots,M-\min_b L_b$

(iv) $A_t = \sum_{b=1}^{B} a_b x_{b,t-L_b}$ $\qquad t=1,2,\ldots,M$

(v) $V_t = V_{t-1} + A_t - N_t$ $\qquad t=1,2,\ldots,M$

(vi) $V_t \geq 0$ $\qquad t=1,2,\ldots,M$

(vii) $x_{bt} = 0 \text{ or } 1$ $\qquad t=1,2,\ldots,M-L_b$; $b=1,2,\ldots,B$

(viii) V_o is given

The model minimizes project duration subject to a set of constraints. Constraint (i) states that all buildings are completed within the project duration, and in fact defines the project duration. Constraint (ii) ensures that all buildings are constructed exactly once. Constraints (iii) through (v) define the number of households relocated in week t, the number of new apartments available for occupancy in week t, and the number of vacancies in week t. Constraint (vi) insists that the number of vacancies remains non-negative throughout the life of the redevelopment program; this guarantees that all households are always housed. Constraint (vii) merely enforces our coding device for identifying start times for buildings, while constraint (viii) identifies the initial endowment of vacancies.

As formulated, the model is an integer program. Various integer programming codes could be used to implement this model; the program I used is a zero-one code from the University of Illinois at Urbana-Champaign named ILLIP-2 (Young, Liu, Baugh, and Muroga (1977)).

To demonstrate the properties of this model, we will consider a numerical example.

7.2 A Numerical Example

As a numerical example, we will consider a project consisting of five buildings with building populations n_b, apartment distributions a_b, and redevelopment times L_b as shown in Table 7.3. It appears that the project is conceivably workable, depending upon the initial number of vacancies V_o, as there are enough new apartments being created (57) to house the initial project population (54). Using the integer program developed, we determined optimal construction sequences for values of V_o ranging from 8 to 54. For V_o less than 8, it would be impossible to evacuate any building, while for V_o greater than or equal to 54, all buildings may be emptied instantaneously.

In Figure 7.1, the optimal project duration is plotted as a function of the initial number of vacancies V_o. The first fact noticed is that the minimum project duration monotonically decreases with the initial vacancy endowment. However, the relationship between D and V_o is not continuous; rather, there are many ranges of V_o within which D remains constant. For example, any value of V_o in the range 42 to 53 inclusive yields a minimum project duration of 11 weeks. If these data represented a real project where initial vacancies could cost $10,000 per unit to provide, the analysis in Figure 7.1 could conceivably save over $100,000 of needless expenditures by noting the ineffectiveness of providing additional vacancies in the cited range.

Secondly, it is interesting to note that the minimum number of vacancies necessary to guarantee project feasibility is 10 even though building 2 only contains 8 households at the start, and building 5

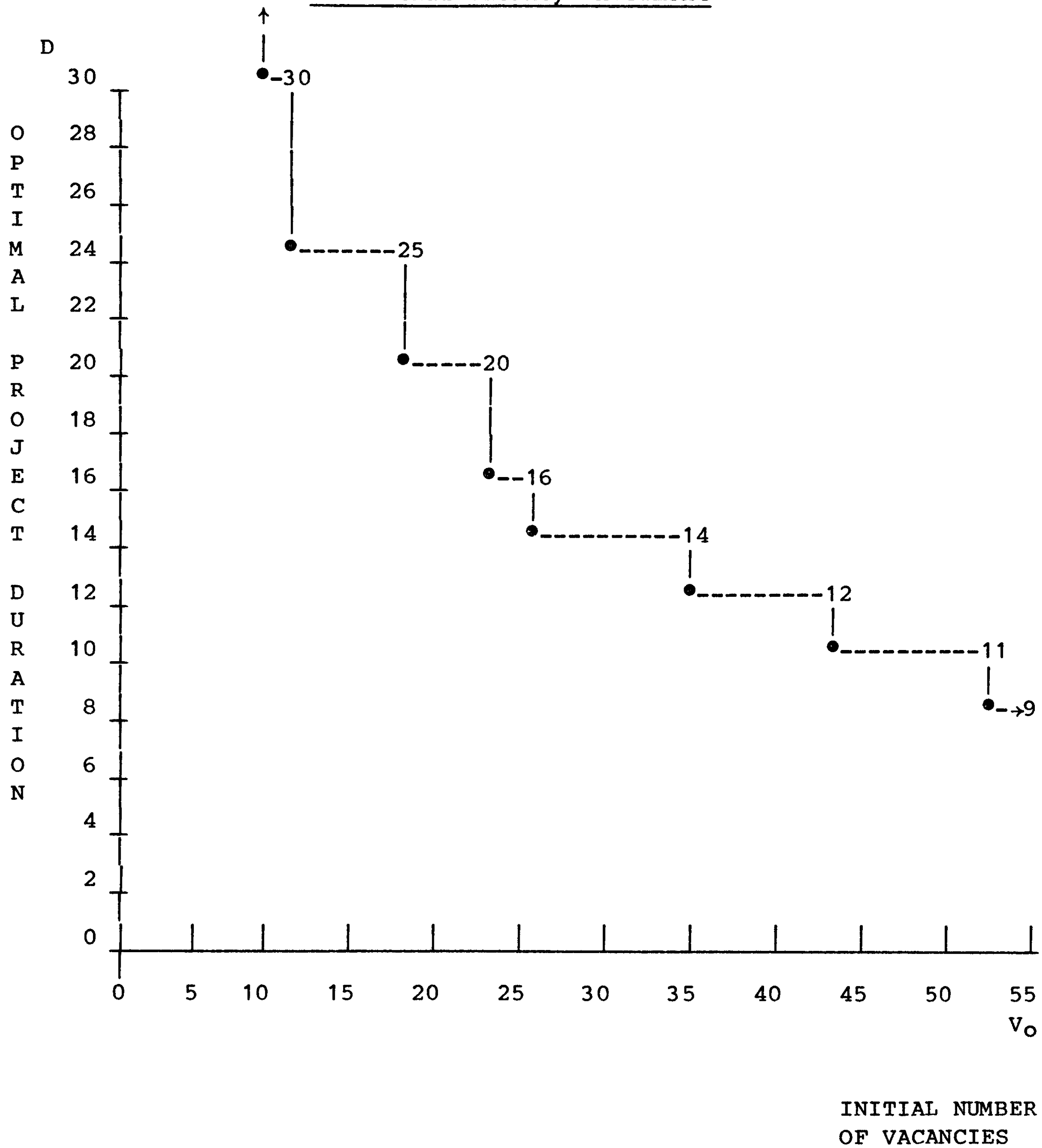
Table 7.3

Data for the Relocation Example

b	n_b	a_b	L_b
1	10	8	5
2	8	12	6
3	12	10	5
4	15	18	8
5	9	9	5
	54	57	29

Figure 7.1

Optimal Project Duration as a Function of Initial Vacancy Endowment



contains only 9 initial households. Let us see why this is true. Suppose V_o equals 8. We could begin by clearing building 2; upon completion of building 2 we would now possess 12 vacancies (8 initial vacancies minus 8 households cleared plus 12 new units). With 12 vacancies, we could clear either building 1, building 3 or building 5. Building 5 does not effect the vacancy pool, as $n_5=a_5=9$. However, clearing either building 1 or building 3 will dictate a net loss of vacancies. Suppose we clear building 3 first. After finishing 3, we are left with 10 vacancies, sufficient for clearing building 1. Unfortunately, after finishing building 1, there are only 8 vacancies left, an insufficient number to clear building 4. Thus, the sequence is infeasible. Had we cleared building 1 before building 3, we would have become stuck even earlier. We have just shown that no sequence is feasible for $V_o=8$. Similar reasoning shows that $V_o=9$ also dictates an infeasible project.

With respect to the actual sequences resulting from the minimum project time criterion, there are often multiple optima for given values of V_o; this is especially true as V_o increases. This has important implications in practice; some construction sequences may be preferred to others for reasons of geographic proximity, ease of movement or access, etc. The changes in sequence (and project duration) occur when new possibilities for emptying buildings arise. For example, at $V_o=14$, the building sequence 2-3-4-1-5 is optimal and yields a project duration of 25 weeks. When V_o increases to 15, the building sequence 4-2-5-1-3 is optimal and yields a project duration of 20 weeks. This shift is attributable to the fact that at $V_o=15$, it is possible to clear building 4, a feat not possible if $V_o<15$.

A feature of good sequences is that they efficiently utilize the initial vacancies provided, and efficiently allocate new units as they become available. To see this, one can examine V_t, the number of vacancies available in week t (i.e. the number of unoccupied units). Figure 7.2 plots V_t as a function of time for the building sequence 4-2-5-1-3 with V_o=15. Note how efficiently units are utilized. For the first nine weeks of the project, no units are unoccupied. For the next five weeks, a single unit is left vacant; there are no vacancies for the ensuing five weeks. All buildings have begun redevelopment by week 15. Vacancies only accumulate at the end of the program as buildings become complete.

This example also demonstrates the amount of overlap possible in an efficient sequence; this is summarized in Figure 7.3. From weeks 9 through 19, there are always two buildings simultaneously being redeveloped. Note how the completion of some buildings triggers the beginning of redevelopment for others; the completion of building 4 enables the start of buildings 2 and 5, the completion of 5 enables the start of building 1, and the completion of building 2 enables the start of building 3.

As this example has demonstrated, our model is quite useful in determining redevelopment sequences and analyzing the consequences of a particular sequence. However, the integer programming solution is complicated; most housing authorities do not have the capability to routinely solve large mathematical programs. Therefore, it is useful to consider some approximations; these are the subject of the next section.

Figure 7.2

Unoccupied Units for the Sequence
4-2-5-1-3 with V_o=15

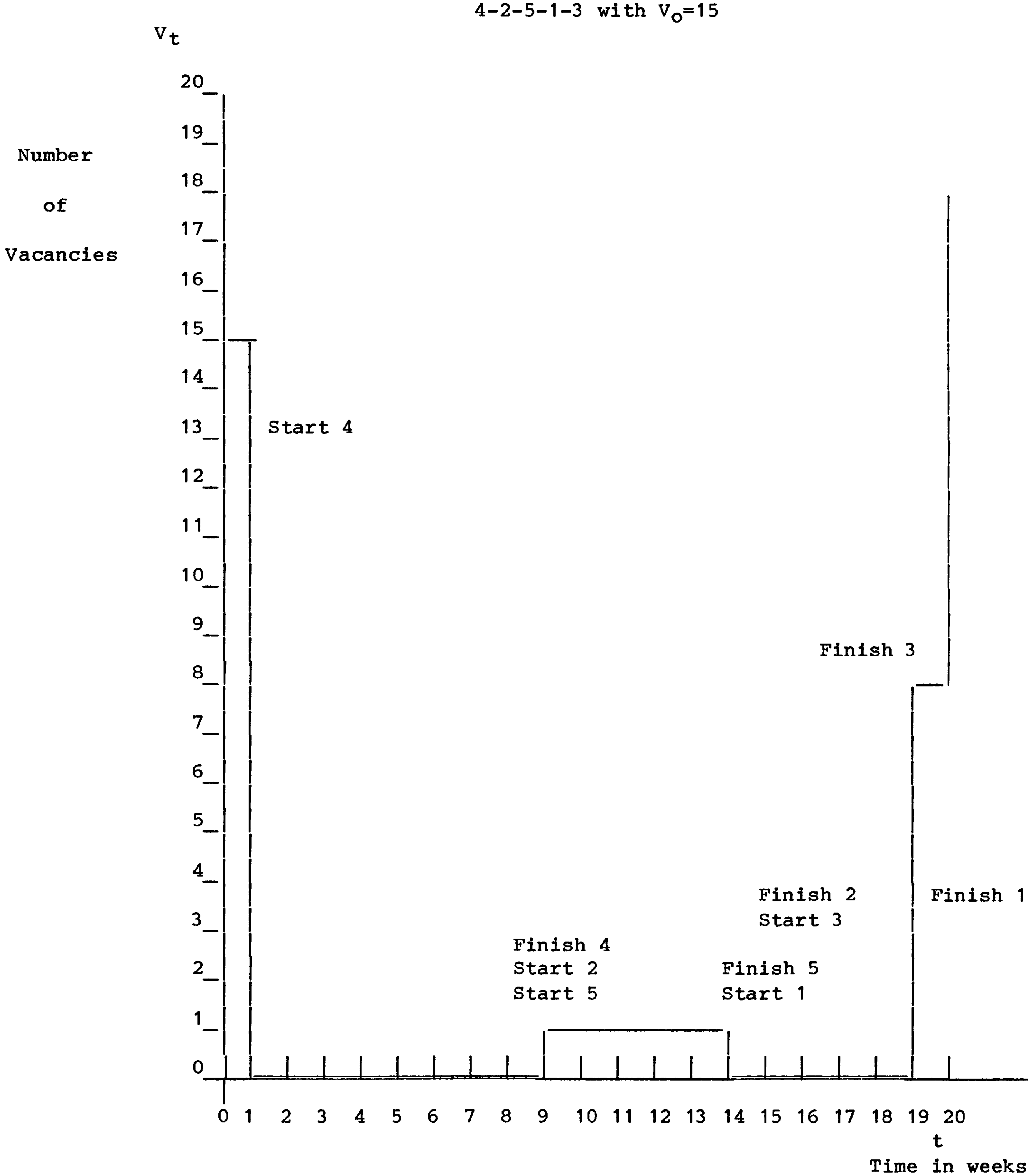
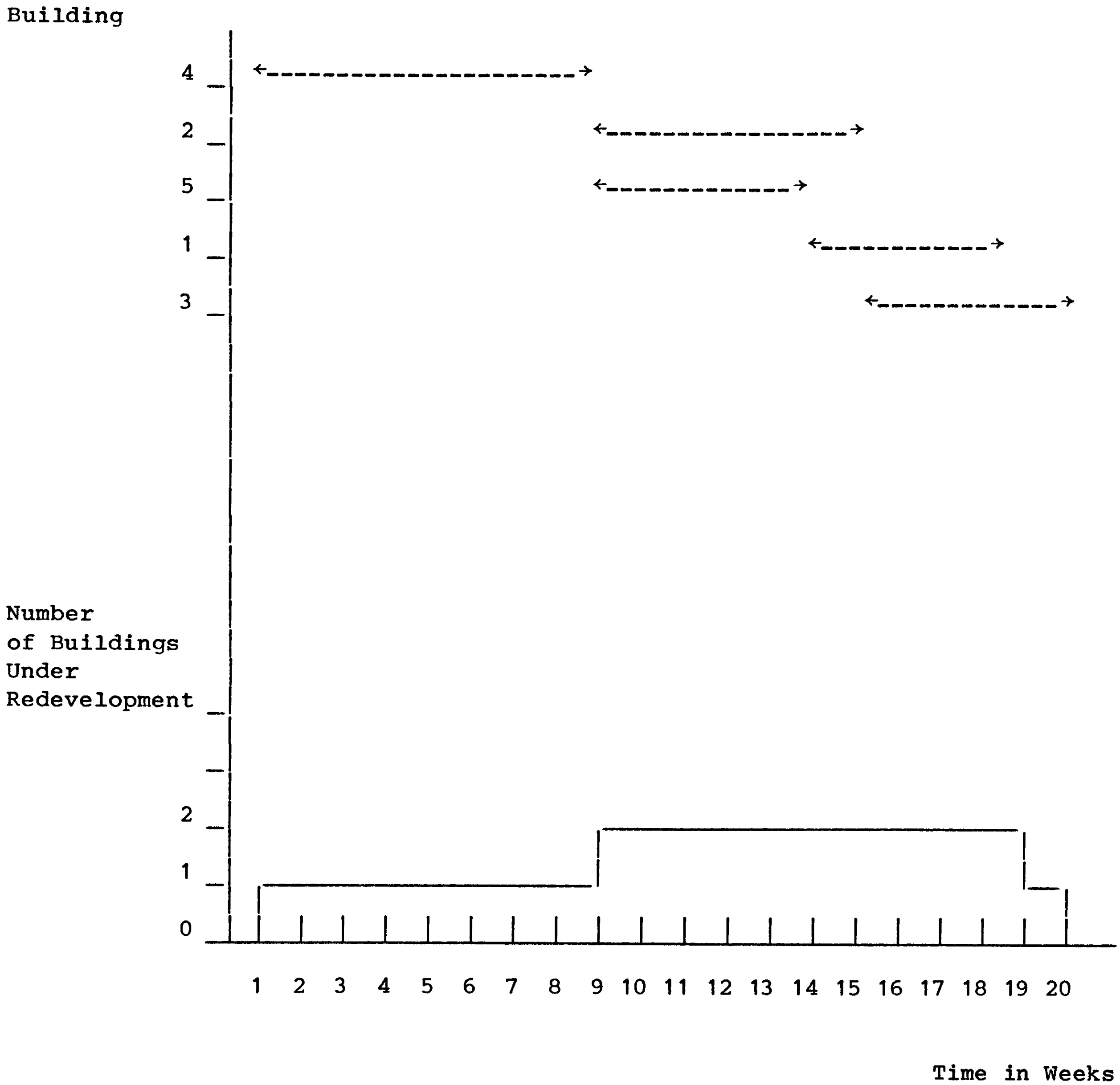


Figure 7.3

Redevelopment Sequence
4-2-5-1-3 with V_o=15



7.3 Approximations to the Relocation Model

7.3.1 Linear Programming Relaxation

Perhaps the most obvious modification to our procedure involves relaxing the integrality constraint (vii) on x_{bt} in the formulation (7.9). Such a relaxation would allow our model to be solved as a linear program rather than as an integer program, and linear programs are much easier to solve. If the values for x_{bt} produced by a linear program were almost always 0 or 1, then one could still contruct useful schedules from the approximate results. However, if the resulting values of x_{bt} are heavily fractional, the linear programming approach would not prove useful.

As an experiment, I ran our model as a linear program using the package available through the Consistent System (Klensin and Dawson, 1981) for the data presented in Table 7.3 using values of 8, 12 and 20 for V_o. The results were not at all encouraging. The values for x_{bt} were not only fractional; they were not even closely linked chronologically.

As an example, consider the case where V_o equals 8. From our previous work, we know that this case represents an infeasible project (i.e. $D=\infty$). Yet the linear program computes a project duration of 21.16 weeks. As for the values of x_{bt}, consider the "fractional starting dates" for building 1:

$x_{1,14} = .0347$

$x_{1,15} = .5698$

$x_{1,16} = 0$

$x_{1,17} = .1928$

$x_{1,18} = 0$

$x_{1,19} = .2027$

The results say that about 3.5% of building 1 starts in week 14, 57% starts in week 15, 19% starts in week 17, etc. The results for other buildings and the other values of V_o mentioned are no better. Thus, despite the simplicity involved in removing the integrality constraint, linear programming does not represent a useful approach to approximating our model.

7.3.2 Myopic Algorithm

A very different approximation can be developed based on physical reasoning. One would suspect that to efficiently complete a relocation schedule, one wants to "produce" new apartments as quickly as possible. For building b, the rate at which apartments are produced is given by a_b/L_b, the ratio of the number of apartments created to the redevelopment time for the building. Thus, a reasonable criterion to use in determining which buildings to initially redevelop is

$$\text{maximize} \quad \sum_{b=1}^{B} \left(\frac{a_b}{L_b}\right) x_{b1} \quad . \tag{7.10}$$

This objective function attempts to maximize the rate of production of new apartments project wide.

Of course, not all buildings can be selected. We still have to satisfy an occupancy constraint of the form

$$\sum_{b=1}^{B} n_b \, x_{b1} \leq V_o \quad . \tag{7.11}$$

Constraint (7.11) states that the number of households relocated in the first week cannot exceed the available number of vacancies available.

Combining (7.10) and (7.11) with the integrality constraint $x_{b1}=0$ or 1, we have a procedure for deciding which buildings to begin in the first week, namely:

$$\text{maximize} \quad \sum_{b=1}^{B} \left(\frac{a_b}{L_b}\right) x_{b1} \tag{7.12}$$

$$\text{subject to} \quad \sum_{b=1}^{B} n_b x_{b1} \leq V_o$$

$$x_{b1} = 0 \text{ or } 1 \qquad b=1,\ldots,B$$

The program (7.12) is a knapsack problem (for example, see Shapiro (1979, p.116)), and can be easily solved using dynamic programming.

Let $C_\beta(v)$ be the solution to the partial problem

$$\text{maximize} \quad \sum_{b=1}^{\beta} \left(\frac{a_b}{L_b}\right) x_{b1} \tag{7.13}$$

$$\text{subject to} \quad \sum_{b=1}^{\beta} n_b x_{b1} \leq V_o$$

$$x_{b1} = 0 \text{ or } 1$$

Then a recursion relating $C_\beta(\cdot)$ to $C_{\beta+1}(\cdot)$ can be defined as

$$C_{\beta+1}(v) = \underset{\{x_{\beta+1,1}\}}{\text{maximum}} \left(C_\beta(v-n_{\beta+1}x_{\beta+1,1}) + \frac{a_{\beta+1}}{L_{\beta+1}} x_{\beta+1,1}\right) \tag{7.14}$$

As $x_{\beta+1,1}$ can only take on the values 0 or 1, the recursion simplifies to

$$C_{\beta+1}(v) = \text{maximum} \left(C_\beta(v-n_{\beta+1}) + \frac{a_{\beta+1}}{L_{\beta+1}}\ ,\ C_\beta(v)\right) \tag{7.15}$$

If we define $C_o(v) \equiv 0$, recursions (7.14) or (7.15) can be iterated until $C_\beta(v)$ has been tabulated. The optimal value of the objective function $C^*_B(V_o)$ yields the maximum apartment production rate attainable, while the optimal mix of buildings to begin in the first week is given by those values of b for which $x_{b1}=1$. This procedure could easily be programmed on a microcomputer, or performed by hand for small problems.

We have outlined an approach for making an initial decision. To schedule an entire project, we can use our approach sequentially. After a set of buildings has been chosen for redevelopment, we use the knapsack model to decide which buildings should next be redeveloped. To do this, we determine the earliest date at which the pool of vacancies will change, and update the vacancy pool at that time. We can then reapply the knapsack model to the buildings not chosen in the first round, using the updated vacancy pool as a constraint. This process is repeated until all buildings have been scheduled for redevelopment, or until a particular knapsack attempt proves infeasible. This "myopic algorithm" is formally described below:

Myopic Algorithm

1) Initialize $v = V_o$, $t=1$, all x_{bj}'s$=0$ $\quad b=1,\ldots,B$; $j=1,\ldots,M-L_b$

2) Define the set $F_t = \{b \mid \sum_{j=1}^{t-1} x_{bj}^* = 0\}$

(Note that F_1 contains all buildings)

3) Solve maximum $\sum_{b \varepsilon F_t} \frac{a_b}{L_b} x_{bt}$

Subject to $\sum_{b \varepsilon F_t} n_b x_{bt} \leqslant v$, $\quad x_{bt} = 0$ or $1 \quad b \varepsilon F_t$

$x_{bt} = 0 \quad b \varepsilon F_t$

Call optimal solution x_{bt}^* , or STOP if infeasible.

4) $v \leftarrow v - \sum_{b \varepsilon F_t} n_b x_{bt}^*$

5) For each building b, set $\ell_b = \sum_{j=1}^{t} (j+L_b)\, x_{bj}^*$

6) Set $\ell \leftarrow \min_b \{\ell_b \mid \ell_b > t\}$

7) Set $t \leftarrow \ell$

8) $v \leftarrow v + \sum_{b=1}^{B} a_b x_{b,t-L_b}^*$

9) Are there any b such that $\sum_{j=1}^{t} x_{bj}^* = 0$?

If yes, go to (2)

If no, STOP. x_{bj}^*'s give optimal schedule, D = most recent value of t.

The steps of this algorithm will now be briefly summarized. Step 1 initializes the number of vacancies to V_o, sets the time counter to 1, and enables all buildings available for redevelopment. Step 2 identifies all buildings that have yet to undergo construction, while Step 3 chooses those buildings to redevelop next on the basis of apartment production rates or determines that the sequence being proposed is infeasible and halts the process. Step 4 updates the vacancy pool by accounting for newly relocated households. Steps 5 and 6 determine when the next building completion occurs, and Step 7 sets the time counter to that event in time. In Step 8, the vacancy pool is updated to reflect new apartments just completed. Step 9 checks to see if all buildings have undergone redevelopment; if not, the process returns to Step 2. If all buildings have been assigned starting dates, the algorithm halts with the "optimal" schedule and project duration.

This myopic algorithm was applied by hand to the data from Table 7.3 for the cases $V_o=8$, $V_o=12$ and $V_o=19$. For $V_o=8$, the algorithm terminated with an infeasible sequence, as expected. At $V_o=12$, the algorithm produced the building sequence 2-4-1-5-3 for a project duration of 25 weeks; this sequence is in fact optimal. At $V_o=19$, the algorithm produced the sequence 2-5-1-3-4 for a project duration of 20 weeks. This sequence is also optimal. It cannot be concluded that the algorithm will always produce optimal results, but these examples are certainly encouraging and warrant further study of the myopic approach.

7.4 Generalization of the Relocation Model

We have invested a good deal of effort in analyzing the properties of a basic relocation model. We will now consider a few modifications of the model which should render it more realistic at least in its formulation.

7.4.1 Labor Force Considerations

As our model is currently formulated, we could achieve some paradoxical results. Suppose there are two buildings, 1 and 2, with $n_1=a_1=n_2=a_2=5$, and $L_1=L_2=10$. Also, suppose that building 3 has $n_3=a_3=10$ and $L_3=20$. If $V_o=20$, the three buildings would be completed 20 weeks after starting, as all buildings could be cleared.

Suppose $V_o=10$. Either building 3 could be emptied, taking 20 weeks to produce 10 units, or both buildings 1 and 2 could be cleared, also producing 10 units, but in only 10 weeks. Something is wrong - the hidden factor is that twice the effort is required to actually work on the two small buildings compared to the one large building. Our formulations have, in effect, assumed an "infinite labor force" which can be used at will.

If we insist that some maximum number of apartments under construction, say α_{max}, cannot be exceeded in any week, then a reasonable constraint is given by

$$\sum_{b=1}^{B} \sum_{t=j-L_b+1}^{j} \left(\frac{a_b}{L_b}\right) x_{bt} \leq \alpha_{max} \qquad j=1,2,\ldots,M \qquad (7.16)$$

This constraint says that the total number of apartments under construction per week cannot exceed α_{max}. Note that the limits of summation run from $j-L_b+1$ to j. This is consistent with our postulate that buildings take L_b weeks to be redeveloped. Also note that the number of apartments being redeveloped in a given week is estimated by a_b/L_b, the same production rate measure used in our myopic model.

7.4.2 Sequence Constraints

It may be in a particular application that certain constraints are posed on feasible sequences. For example, it may be that two buildings b and b' must follow in order, that is, building b must precede building b'. This is easily coded as

$$\sum_{t=1}^{M-L_b} t x_{bt} < \sum_{t=1}^{M-L_{b'}} t x_{b't} \tag{7.17}$$

7.4.3 Multiple Unit Types

Perhaps the most obvious deficiency to our model is that it fails to distinguish different unit types. Suppose that households of type k are now matched to units of type k, k=1,...,K. Types could refer to size, special features for the handicapped, or other attributes. The initial building populations would now be denoted by n_{bk}, the number of type k households in building b. Similarly, the number of created apartments of type k in building b would be denoted by a_{bk}. Finally, the initial number of type k vacancies could be denoted by V_{ok}.

Using an obvious notation, we could define the number of type k households relocated, type k apartments made available, and type k vacancies in week t as

$$N_{tk} = \sum_{b=1}^{B} n_{bk} x_{bt} \qquad k=1,\ldots,K \tag{7.18}$$

$$A_{tk} = \sum_{b=1}^{B} a_{bk} x_{b,t-L_b} \qquad k=1,\ldots,K \tag{7.19}$$

$$V_{tk} = V_{t-1,k} + A_{tk} - N_{tk} \qquad k=1,\ldots,K \tag{7.20}$$

and insist that $V_{tk} \geq 0$ for every week t and unit/household type k.

If the unit type classifications strictly refer to size, then it is acceptable to place type k households into type k, k+1, ..., K units; households must be housed in units that are of sufficient size. This formulation requires us to define N_{tk}, A_{tk} and V_{tk} in terms of the number of households relocated, apartments created, and vacancies available in week t for households requiring units of <u>type k or larger</u>. Our constraints would become

$$N_{tk} = \sum_{\ell=k}^{K} \sum_{b=1}^{B} n_{b\ell}\, x_{bt} \tag{7.21}$$

$$A_{tk} = \sum_{\ell=k}^{K} \sum_{b=1}^{B} a_{b\ell}\, x_{b,t-L_b} \tag{7.22}$$

V_{tk} would remain defined as before. This particular formulation will be demonstrated within the context of an actual redevelopment effort in Chapter 8.

7.5 Summary

This chapter has developed some theory for determining schedules for public housing redevelopment programs. The key to our approach has been to recognize that all tenants must always be assigned to appropriate housing units. A basic model was developed in detail, and a promising approximation was also presented. Modifications to our basic model aimed at incorporating more of the realities of relocation problems were also considered.

While more work remains to be done with models for relocation programs (particularly with the approximation procedure), the models are sufficiently developed for application to real problems. Such an application is illustrated in the next chapter.

Chapter VIII

Applying Relocation Models: The Franklin Field Project

8.1 Background

Franklin Field is a housing project operated by the Boston Housing Authority. The project, initially occupied in 1954, was designed to house 504 households in nineteen buildings. The original unit mix consisted of 150 one bedroom, 179 two bedroom, 100 three bedroom and 75 four bedroom apartments.

For a variety of reasons, the physical condition of the Franklin Field project has become problematic. According to the MIT Public Housing Research Group,

> "The buildings are, in general, deteriorated. The shared entries and halls, for example, receive excessive use/traffic by families with many children. A single entry commonly serves 12 units. Security cannot be maintained since apartments are too small and stairway access is uncontrolled. Thus common areas become play areas."
>
> "The apartments are small and poorly laid out and do not adequately house the activities of family living. The livingroom in a typical unit is laid out to work as a corridor, kitchens and bathrooms are small, dining rooms are lacking, storage is inadequate, bedrooms are too small for double occupancy, and there is only one bedroom in large units."
>
> (Public Housing Research Group, 1983, p.66)

Issues such as these led to the current drive to redevelop the Franklin Field project. Beginning in 1978 with a $3.5 million modernization grant from the Massachusetts Executive Office of

Communities and Development (Franklin Field is owned by the Commonwealth of Massachusetts) to renovate 48 apartments, the redevelopment effort now involves some $32,780,000 in both state and federal monies. The original unit count of 504 is being reduced to 346 redesigned apartments; thus the redevelopment program is costing about $94,740 per unit.

In reducing the number of units from 504 to 346, the Public Housing Research Group notes that

> "The primary goal of the redevelopment program is the adequate housing of those families who currently live in Franklin Field. Of the 346 proposed redesigned units, 26% are either duplex or triplex to provide larger families with greater privacy and separate entries and to thus reduce the use of shared entries and enhance security for both shared and private entries.
> ... The proposal calls for all units to increase in size and to be designed to better facilitate their use by family members."
>
> (Public Housing Research Group, 1983, p.66)

To implement the redevelopment project, it was necessary to determine a sequence of construction that allowed for the feasible relocation of those households already living in the project. The redevelopment planning team (consisting of Boston Housing Authority planners, representatives of Carr, Lynch Associates and Wallace, Floyd Associates, redevelopment architects, and the Franklin Field Tenant Task Force) initially moved the occupants of nine buildings (some 93 households) off-site leaving 198 households on-site. It was at this point that I was contacted to aid in determining a feasible construction sequence and relocation strategy for the Franklin Field redevelopment program.

In the next few sections, I will describe how this problem was approached. The data involved will be presented, and the particular version of our relocation model from Chapter 7 used for this project will be discussed. Finally, a quick method used for allocating households to units will be described.

8.2 Data for the Franklin Field Project

The first issue faced was the identification of the scale of construction to take place. As buildings in the Franklin Field project fall into rather natural groupings, it was decided that with the exception of a single building, all construction and reoccupancy would take place according to building pairs. As the project buildings are roughly the same size and require comparable amounts of work, it was also decided to assign equal redevelopment times to all pairs of buildings. It was felt that each building pair would require roughly six months of work. However, given the assumption that all building pairs require equal redevelopment times, the actual time involved becomes immaterial to the determination of an optimal construction sequence. Thus, without loss of generality, the times for all building pairs were assigned the value 1.

The Franklin Field redevelopment plan calls for the creation of some 31 distinct unit types. For occupancy purposes, the design distinctions between many units become unimportant. The project staff from Wallace, Floyd Associates therefore reduced the number of unit types necessary to consider from 31 to 11; the characteristics of these unit types are shown in Table 8.1.

Table 8.1

Unit Types for the Franklin Field
Redevelopment Program

Unit Type	Bedrooms	Number of Occupants
1	1	2
2	1 1/2	3
3	1 2/2	4
4	2	4
5	2 1/2	5
6	3	6
7	2 2/2	6
8	3 1/2	7
9	4	8
10	3 2/2	8
11	5	10

Source: Memo from Wallace, Floyd Associates, February 11, 1983.

Table 8.2

Distribution and Demand for Units by Type for the Franklin Field Redevelopment Program

Building Pair (b)

Unit Type (k)	1		2		3		4		5		6		7		8		9		10		11 (off-site)	
	n_{1k}	a_{1k}	n_{2k}	a_{2k}	n_{3k}	a_{3k}	n_{4k}	a_{4k}	n_{5k}	a_{5k}	n_{6k}	a_{6k}	n_{7k}	a_{7k}	n_{8k}	a_{8k}	n_{9k}	a_{9k}	n_{10k}	a_{10k}	n_{11k}	a_{11k}
1	0	0	0	6	0	4	0	2	0	2	12	14	4	8	3	4	4	4	2	2	10	0
2	0	0	0	0	0	4	0	2	0	2	10	0	9	2	7	4	13	4	25	26	27	0
3	0	0	0	0	0	0	0	0	0	0	5	6	10	0	9	0	13	0	14	0	25	0
4	0	7	0	10	0	8	0	11	0	11	0	2	0	6	0	8	0	8	0	4	3	0
5	0	4	0	6	0	4	0	5	0	6	5	10	5	5	3	4	4	4	4	2	9	0
6	0	2	0	11	0	4	0	4	0	4	0	12	0	11	2	4	0	4	0	2	5	0
7	0	1	0	1	0	0	0	1	0	1	7	0	4	0	1	0	4	0	3	0	12	0
8	0	1	0	1	0	4	0	2	0	3	4	0	2	2	1	4	0	4	0	2	2	0
9	0	0	0	3	0	2	0	1	0	1	0	4	0	4	0	2	0	2	0	1	0	0
10	0	1	0	1	0	0	0	1	0	1	3	0	1	0	1	0	1	0	1	0	0	0
11	0	0	0	0	0	2	0	1	0	1	1	0	1	1	0	2	0	2	0	1	0	0

Table 8.3

Vacancies by Type Resulting from the Completion of Building Pairs 1 through 5

Unit Type (k)	Number of Vacancies	(V_{ok})
1	14	149
2	8	135
3	0	127
4	47	127
5	25	80
6	25	55
7	4	30
8	11	26
9	7	15
10	4	8
11	4	4

Source: Table 8.2

Having identified these 11 unit types, the next step was to determine the distribution of created units by type across building pairs and to determine the number of households in each building pair (and off-site) requiring at a minimum particular unit types. This information was also compiled by the Wallace, Floyd Associates team and is present in Table 8.2 using the notation of Chapter 7.

As mentioned previously, several buildings were initially emptied, and their occupants were temporarily relocated off-site in other available public housing units. The buildings vacated (represented by building pairs 1 through 5 in Table 8.2) were initially redeveloped. Upon completion, these buildings provided a pool of vacancies for the relocation of the remaining residents of the Franklin Field project. Table 8.3 reports the number of vacancies available by type resulting from the completion of building pairs 1 through 5.

8.3 Model Formulation

As the unit types house monotonically increasing household sizes, it was decided that households determined to require a type k unit could be legally assigned to units of type k, k+1, k+2, ..., 11 for k=1,2, ..., 11. Thus, the model formulation presented at the end of Chapter 7 is applicable. The initial numbers of vacant units of type k through 11, V_{ok}, have been tabulated in Table 8.3. We can now formulate the model used in Franklin Field using the notation of Chapter 7:

minimize D (8.1)

subject to

$$\sum_{t=1}^{6} t x_{bt} + 1 \leq D \qquad b=1, \ldots, 6$$

$$\sum_{t=1}^{6} x_{bt} = 1 \qquad b=1, \ldots, 6$$

$$N_{tk} = \sum_{\ell=k}^{11} \sum_{b=1}^{6} n_{b\ell} x_{bt} \qquad t=1, \ldots, 6; \; k=1, \ldots, 11$$

$$A_{tk} = \sum_{\ell=k}^{11} \sum_{b=1}^{6} a_{b\ell} x_{b,t-1} \qquad t=1, \ldots, 7; \; k=1, \ldots, 11$$

$$V_{tk} = V_{t-1,k} + A_{tk} - N_{tk} \qquad t=1, \ldots, 7; \; k=1, \ldots, 11$$

$$V_{tk} \geq 0 \qquad t=1, \ldots, 7; \; k=1, \ldots, 11$$

$$x_{bt} = 0 \text{ or } 1 \qquad b=1, \ldots, 6; \; t=1, \ldots, 6$$

V_{ok} are given in Table 8.3.

Note that in formulating this model, we have:

(i) set B=6, representing building pairs 6 through 10 and the households initially moved off-site

(ii) set L_b=1 for b=1, ..., 6

(iii) set $M=7=\sum_{b=1}^{6} L_b + 1$

(iv) set K=11

This model was solved using the data in Table 8.2 for n_{bk} and a_{bk} using the ILLIP-2 program mentioned in Chapter 7 (see Young, Liu, Baugh and Muroga, 1977).

8.4 Model Results

The ILLIP-2 program located 10 optimal solutions (there could well be more). However, one of these solutions corresponded to a sequence that the redevelopment planning team had hoped to implement a priori. This solution was:

Period 1: Redevelop pairs 1 through 5

Period 2: Redevelop pairs 6 through 8

Period 3: Redevelop pairs 9 and 10

Period 4: Return households from off-site (i.e. Redevelop pair 11)

The rationale behind this sequence was that the building pairs are geographically contiguous making it easy to move heavy construction equipment from site to site.

Having determined a sequence, the issue of how to relocate households arose. We wanted to arrive at a tenant assignment scheme that did not require households to move more than once. To do this, we took advantage of our decision that households of type k could occupy units of type k or larger, and developed the following scheme.

For notation, let:

h_i = number of type i households to assign from a particular building; $i=1, \ldots, K$

u_j = number of type j units available for occupancy in this assignment round; $j=1, \ldots, K$

α_{ij} = number of type i households assigned to type j units

An allocation scheme α_{ij} for a given group of households h_i, $i=1, \ldots, K$ and a set of units u_j, $j=1, \ldots, K$ is feasible if the following four conditions hold:

(i) $\alpha_{ij} = 0$ if $i>j$ (8.2)

(ii) $\alpha_{ij} \geq 0$ if $i \leq j$ (8.3)

(iii) $\sum_{j=1}^{K} \alpha_{ij} = h_i \quad i=1, \ldots, K$ (8.4)

(iv) $\sum_{i=1}^{K} \alpha_{ij} \leq u_j \quad j=1, \ldots, K$ (8.5)

Conditions (i) and (ii) state that only feasible assignments can be made. Condition (iii) states that all households are assigned to apartments, while condition (iv) guarantees that the number of apartments assigned does not exceed the number of apartments available.

Thus, to find a feasible allocation scheme, one solves the system of inequalities (8.2) through (8.5) for α_{ij}. There are many ways to do this, but one simple approach utilizes a technique known as the <u>northwest corner method</u>. An algorithm for this approach is:

(1) set $\alpha_{ij} \leftarrow 0$ for all i and j

(2) set $i \leftarrow 0$

(3) set $i \leftarrow i+1$

(4) set $j \leftarrow i-1$

(5) set $j \leftarrow j+1$

(6) is $u_j = 0$? If yes, go to (11)

(7) set $\alpha_{ij} = \min(h_i, u_j)$

(8) set $h_i \leftarrow h_i - \alpha_{ij}$

(9) set $u_j \leftarrow u_j - \alpha_{ij}$

(10) Is $h_i = 0$? If yes, go to (13)

(11) Is $j < K$? If yes, go to (5)

(12) STOP - INFEASIBLE ALLOCATION

(13) Is $i < K$? If yes, go to (3)

(14) STOP - FEASIBLE ALLOCATION

The algorithm just described is simple enough to perform by hand. Using the algorithm, we were able to find an allocation scheme that assigned all households in single moves - no households were required to move twice (with the exception of the households initially relocated off site). This result was quite pleasing especially to the tenant task force.

8.5 Summary

The application of our relocation model in this instance verified a sequence that had been previously selected by redevelopment planners. Of course, it could be that in other applications, sequences selected a priori could prove to be sub-optimal, or even infeasible! That the model identified the desired sequence as an optimal schedule was a great relief and confidence booster to the redevelopment team. As of the date of this writing, the Boston Housing Authority's Redevelopment Director for the Franklin Field project, David C. Gilmore, has stated that construction is proceeding according to the sequence determined and relocation is proceeding without difficulty (personal communication, April 19, 1984).

Chapter IX

Conclusions and Areas for Future Research

This thesis has addressed some problems associated with managing public housing demand. From a review of tenant assignment policies utilized by U.S. housing authorities, we were able to develop models describing the impacts of these policies on household waiting times, project composition, and tenant allocations. We also addressed the problem of determining construction sequences in a redevelopment project which guarantee that tenants are always assigned to appropriate units. An explicit procedure for identifying sequences resulting in minimum project time was developed and applied to an actual project.

In this concluding chapter, we will discuss some of the policy implications of our work. Areas for future research will be identified where appropriate.

Many of our results relate to models which predict the waiting times for new applicants to public housing; indeed, the detailed analysis of Chapters 4 and 5 is devoted to this problem. A major reason for modeling waiting times relates to the attempt to provide new housing applicants with the best information available regarding housing options. One would think that waiting times would play a major role in a prospective tenant's decision regarding which housing projects to choose for potential assignment, or whether to remain interested in public housing at all. New applicants should receive waiting time estimates along with the other information typically presented (e.g. location of projects; unit mix; project populations and demographics; age of project etc.). Indeed, waiting time estimates would reduce the level of uncertainty involved in the new applicant's decision; certain projects may be instantly disregarded due to long waiting times, for example.

For our work to be truly useful in this regard, it is necessary to encode the models we have developed into user friendly computer programs that could be operated by people with no computer knowledge. The situation would be akin to an automatic teller machine at a bank where customers make requests by answering simple questions presented on a screen display. In the public housing context, new applicants would enter information regarding their development choices and household status by providing the answers to simple questions. For example, questions regarding household size and composition would provide the necessary information to determine a unit requirement. Questions regarding income and ethnicity could extract information pertaining to priority status. Applicants could then receive estimated waiting times by development; these waiting times would be computed using a model reflecting the particular rules of the housing authority involved. The data for the models could be obtained directly from housing authority computer files and would thus provide waiting time estimates using the most recent data available. By using our models in this fashion, new applicants could make a decision regarding development choice taking waiting time into account. Applicants could also formulate realistic expectations of waiting time for the particular decision made.

The idea presented here is indeed feasible for most of the waiting time models considered. Only the multiqueue system used by the Boston Housing Authority requires a model too complex for implementation on a microcomputer. Yet, it should be possible to develop good approximations to the multiqueue simulation model which could be efficiently programmed on small computers. The development of a menu driven, user friendly software package for implementing the waiting time models developed in this thesis is one pragmatic area for future work.

At the research level, it is unknown just how important a role waiting times play in the decision to request assignments at specific projects. Conceptually, one may imagine a utility function for housing. The attributes of this utility function include site amenities (such as location, demographic characteristics of the local population), cost (i.e., rent) and waiting time. This utility function could apply to both public and private housing. If new applicants to public housing were given waiting time information, it would then be possible, by observing the decisions made by these applicants, to determine the extent to which waiting times actually effect housing decisions. The results of such an experiment could have direct policy implications. For example, if waiting time is not viewed as important, then the policy of using reduced waiting times as an incentive for achieving social goals such as project integration will undoubtedly prove to be ineffective. Should waiting times prove important in housing decisions, then it would become possible to "market" projects on the basis of their waiting times.

Another pragmatic use of waiting time data relates to waiting list management. If the waiting times at certain projects are sufficiently long, say several years, then perhaps applications for residence at those projects should be refused until expected waits subside. Given the amount of authority effort necessary in the processing of application forms (e.g. reference checking, applicant interviews) such a policy could free up staff time for other tasks. The closing of certain projects to new applicants would also erase any possibility of tenants feeling incorrectly secure about future housing assignments at the projects in question.

As another application, tenants with projected waiting times longer than a certain threshold could be referred to housing elsewhere as a matter of policy. Thus, housing authorities could maintain a standard of housing applicants within a given time period, or not accepting applications for assignments. Again, this sort of policy would more realistically reflect the ability of housing authorities to respond to the excessive demands currently experienced.

Our waiting time models could also be used to question and perhaps revamp the priority structures currently existing in some authorities. For example, many housing authorities claim to house emergencies as a top priority. However, imagine the following situation: an emergency household applies for housing, but an appropriate unit only becomes available after a six month wait. Several questions arise. How much of an "emergency" still exists after six months? Should this "emergency" be granted an assignment instead of a regular applicant, who may have waited several years for an assignment? If our models predicted that the mean wait involved for a high priority applicant is so long that by the time a unit could be offered, the applicant's priority status could be questioned, then perhaps it is unwise to assign a high priority to the new applicant initially. On the other hand, households with very low priorities may well face waiting times that exceed their expected lifetimes! Surely such households should not be encouraged to apply for public housing assignments.

The past several arguments all point to a basic use of our waiting time models: households with excessively long expected waiting times are essentially not able to be housed by a public housing authority. These long waiting times could be due to tenant choices (e.g. insistence on assignment to a very popular project), or due to the particular priority structure of a tenant assignment policy (e.g. a household in a low priority class may receive an infinite expected wait). As every application to public housing requires authority staff time to process, every application has a cost. Only applications resulting in housing assignments may be thought to have some benefit. It seems clear that one would only wish to incur the cost of processing an application if some benefit is accrued.

Our waiting time models provide a method for identifying applicants with little to no chance of receiving an assignment. It should be possible to dismiss such applicants from the system before processing their applications on the grounds that following current authority policy, such households could never be housed. This use of our models would preclude the cost involved in processing applications of zero benefit and could actually serve to redirect hours of potentially wasted authority staff time to more useful tasks.

Our models of tenant assignment have been descriptive, but they could be developed for prescriptive use if the characteristics of "good" tenant assignment policies could be made explicit. For example, if one wishes to integrate a project within a certain time frame, an objective could explicitly state: assign tenants such that X% of the project population consists of Group G tenants with T time periods.

Using the models of blend priorities from Chapter 4, one could determine the differential admission rates required to achieve this policy or conclude that the policy is in effect infeasible.

As another example, one could formulate models which assign tenants to projects in order to achieve certain objectives. Within the context of a multiqueue system, one could guarantee assignment to one of the projects in an applicant's choice set, but prescribe assignment probabilities to achieve some objective (such as the minimization of waiting time, or the equalization of racial balance). I have formulated a small number of models which begin to address these issues; their further development is a topic for future work.

More work remains to be done with our relocation model from Chapter 7. The most practical starting point is to see if the approximation procedure suggested works well in a wide variety of test cases. If so, this procedure can be coded and distributed to housing authorities for use on a microcomputer.

The relocation model is primarily useful for generating cost effective construction sequences. The results from analyses using our model can be used as standards to which proposed schedules received from bidding construction firms can be compared. Indeed, the model can be used to dictate sequences as it stands now.

The model could be made more realistic, however. The actual costs involved in a redevelopment project could be incorporated. For example, the cost incurred due to project duration is not the only cost involved; initial vacancies also cost money. In addition, one could envision a cost incurred due to the presence of unoccupied units throughout the redevelopment process. A better model would determine a sequence which minimizes total project cost. The results of this model could include the optimal number of initial vacancies to provide, in addition to the optimal construction sequence.

Finally, the impacts of a redevelopment project on tenants could be quantified and included in a relocation model. For example, one could consider the length of stay in temporary housing as a measure of discomfort to project residents; this measure could have a cost (albeit psychological) attached to it. In extreme cases, certain tenants felt to be disadvantaged could be offered monetary compensation for each day spent in temporary housing beyond some threshold. This form of cost could be incorporated. Another example relates to the relative composition of project buildings at the completion of a project. It is unlikely that a relocation plan which results in the break up of social groupings within the project (e.g. neighborhoods) will be well received. This sort of outcome should be considered.

These then are some ideas which remain to be explored within a larger research context. The particular issues raised in this thesis, and the extensions to these issues mentioned in the past few pages, have not by any means exhausted the research agenda relating to tenant allocation issues in public housing. However, this work could result in some reassessments of public housing policy. At a minimum, I hope that the contents of this thesis generate some interest by others in the issues involved with managing the demand for public housing.

Bibliography

1. Admissions and Continued Occupancy Policy (1983); Housing Authority of the City of Houston; Houston, Texas.

2. Applicant Selection and Transfer Plan (1980); Cambridge Housing Authority; Cambridge, Massachusetts.

3. Coopers and Lybrand (1980); Management Review of Boston Housing Authority; National Housing Group; Coopers and Lybrand, Inc.; Washington, D.C.

4. Cox, D.R. (1970); Renewal Theory; Methuen & Company, Ltd.; London.

5. Drake, A.W. (1967); Fundamentals of Applied Probability Theory; McGraw-Hill, Inc.; New York.

6. Howard, R.A. (1971a); Dynamic Probabilistic Systems - Volume I: Markov Models; John Wiley and Sons, Inc.; New York.

7. Howard, R.A. (1971b); Dynamic Probabilistic Systems - Volume II: Semi-Markov and Decision Processes; John Wiley and Sons, Inc.; New York.

8. Klensin, J. and Dawson, R. (1981); Consistent System: Program Summaries; Laboratory of Architecture and Planning; MIT.

9. Kolodny, R. (1979); Exploring New Strategies for Improving Public Housing Management; U.S. Department of Housing and Urban Development; Washington, D.C.

10. Larson, R.C. (1972); Urban Police Patrol Analysis; MIT Press; Cambridge, Massachusetts.

11. Larson, R.C. and Odoni, A.R. (1981); Urban Operations Research; Prentice-Hall, Inc.; Englewood Cliffs, New Jersey.

12. Levy, H. and Lessman, H. (1961); Finite Difference Equations; MacMillan Company; New York.

13. Occupancy Policy (1983); Greensboro Housing Authority; Greensboro, North Carolina.

14. Occupancy Standards ( ); Chicago Housing Authority; Chicago, Illinois.

15. Public Housing Research Group (1983); Revitalizing Public Housing Projects; Department of Urban Studies and Planning; MIT; Cambridge, Massachusetts.

16. Purcell, E.J. (1972); Calculus with Analytic Geometry; Prentice-Hall, Inc.; Englewood Cliffs, New Jersey.

17. Price, F. and Solomon, R. (1983); "A Final Proposal for Income Mixing"; memo to Lewis H. Spence, Receiver/Administrator of the Boston Housing Authority.

18. Resident Selection and Assignment Plan (1982); Omaha Housing Authority; Omaha, Nebraska.

19. Resolution No. 27 of 1981 (1981); Housing Authority of the City of Pittsburgh; Pittsburgh, Pennsylvania.

20. Shapiro, J.F. (1979); Mathematical Programming: Structures and Algorithms; John Wiley and Sons, Inc.; New York.

21. Statement of Policies and Standards Governing Admission to the Occupancy of Low-Income Public Housing Operated by the Housing Authority of Baltimore City (1979); Housing Authority of Baltimore City; Baltimore, Maryland.

22. Statement of Policy (1980); Minneapolis Community Development Agency; Minneapolis, Minnesota.

23. Tenant Selection and Assignment Plan (1983); Public Housing Agency of St. Paul; St. Paul, Minnesota.

24. Tenant Selection, Assignment and Transfer Plan of the Boston Housing Authority (1978); Boston Housing Authority; Boston, Massachusetts.

25. Walker, W.E.; Chaiken, J.M.; and Ignall, E.J. (eds.)(1979); Fire Department Deployment Analysis; North Holland; New York.

26. Young, M.H.; Liu, T.K.; Baugh, C.R.; and Muroga, S. (1977); A Code for Zero-One Integer Programming ILLIP-2; Department of Computer Science; University of Illinois at Urbana-Champaign; Urbana, Illinois.