MATH 152: THE FOURIER TRANSFORM – THE INVERSION FORMULA

Recall that $S = S(\mathbb{R}^n)$ is the space of Schwartz functions, i.e. the functions $\phi \in C^\infty(\mathbb{R}^n)$ with the property that for any multiindices $\alpha, \beta \in \mathbb{N}^n$, $x^\alpha \partial^\beta \phi$ is bounded. Here we wrote $x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}$, and $\partial^\beta = \partial x_1^{\beta_1} \ldots \partial x_n^{\beta_n}$; with $\partial_{x_j} = \frac{\partial}{\partial x_j}$. (This notation with $\alpha$, $\beta$, is called the multiindex notation.)

We defined the Fourier transform on $S$ as

$$ (\mathcal{F}\phi)(\xi) = \hat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-ix \cdot \xi} \phi(x) \, dx, $$

and the inverse Fourier transform as

$$ (\mathcal{F}^{-1}\psi)(x) = (2\pi)^{-n} \int_{\mathbb{R}^n} e^{ix \cdot \xi} \psi(\xi) \, d\xi. $$

We showed by integration by parts that $\mathcal{F}, \mathcal{F}^{-1}$ satisfy

$$ \mathcal{F}D_{x_j} \phi = \xi_j \mathcal{F}\phi, \quad -D_{\xi_j} \mathcal{F}\phi = \mathcal{F}(x_j \phi), \quad D_{x_j} = i^{-1} \partial_j, $$

with similar formulae for the inverse Fourier transform:

$$ \mathcal{F}^{-1}D_{\xi_j} \psi = -x_j \mathcal{F}\psi, \quad D_{x_j} \mathcal{F}^{-1}\psi = \mathcal{F}^{-1}(\xi_j \psi). $$

We used this to show that $\mathcal{F} : S \to S$ and similarly for $\mathcal{F}^{-1}$; indeed, if $\phi \in S$, then $x^\alpha \partial^\beta \phi$ is bounded for all multiindices $\alpha, \beta$. But the Fourier transform of this a constant multiple of $\partial^\alpha \xi^\beta \hat{\phi}$. But we in fact have that $(1 + |x|^2)^{(n+1)/2}x^\alpha \partial^\beta \phi$ is also bounded (the first factor in effect simply increases $\alpha$), so $|x^\alpha \partial^\beta \phi| \leq C(1 + |x|^2)^{-(n+1)/2}$ for some $C > 0$. Thus,

$$ |\partial^\alpha \xi^\beta \hat{\phi}(\xi)| = |\int_{\mathbb{R}^n} e^{-ix \cdot \xi} (x^\alpha \partial^\beta \phi)(x) \, dx| $$

$$ \leq \int_{\mathbb{R}^n} |e^{-ix \cdot \xi} (x^\alpha \partial^\beta \phi)(x)| \, dx \leq \int_{\mathbb{R}^n} C(1 + |x|^2)^{-(n+1)/2} = M < +\infty, $$

so $\sup |\partial^\alpha \xi^\beta \hat{\phi}| < M$, i.e. $\partial^\alpha \xi^\beta \hat{\phi}$ is bounded indeed. Although the derivatives and the multiplications are in the opposite order as in the definition of $S$, using Leibniz’ rule (i.e. the product rule) for differentiation, we get other terms of the same form, so we conclude that $\hat{\phi} \in S$ indeed. The proof for the inverse Fourier transform is of course very similar.

We also calculated the Fourier transform of the Gaussian $\phi(x) = e^{-a|x|^2}$, $a > 0$, on $\mathbb{R}^n$ (note that $\phi \in S$) by writing it as

$$ \hat{\phi}(\xi) = \left( \int_{\mathbb{R}^n} e^{-a x_1^2} \, dx_1 \right) \ldots \left( \int_{\mathbb{R}^n} e^{-a x_n^2} \, dx_n \right), $$

hence reducing it to one-dimensional integrals which can be calculated by a change of variable and shift of contours. We can also proceed as follows. Write $x$ for the one-dimensional variable, $\xi$ for its Fourier transform variable for simplicity, and $\psi(x) = e^{-ax^2}$,

$$ \hat{\psi}(\xi) = \int_{\mathbb{R}} e^{-ix\xi} e^{-ax^2} \, dx = e^{-\xi^2/4a} \int_{\mathbb{R}} e^{-a(x+i\xi/(2a))^2} \, dx, $$

where we simply completed the square. We wish to show that

$$ f(\xi) = \int_{\mathbb{R}} e^{-a(x+i\xi/(2a))^2} \, dx $$
is a constant, i.e. is independent of \( \xi \), and in fact it is equal to \( \sqrt{\pi/\alpha} \). But that is easy: differentiating \( f \), we obtain \( f'(\xi) = -i \int_\mathbb{R} (x + i\xi/(2\alpha))e^{-a(x+i\xi/(2\alpha))^2} dx \). The integrand is the derivative of \((-1/(2\alpha))e^{-a(x+i\xi/(2\alpha))^2}\) with respect to \( x \), so by the fundamental theorem of calculus, \( f'(\xi) = (i/(2\alpha))e^{-a(x+i\xi/(2\alpha))^2}\) \( \left[ \frac{-\infty}{x=-\infty} \right. \) = 0, due to the rapid decay of the Gaussian at infinity. This says that \( f \) is a constant, so for all \( \xi \), \( f(\xi) = f(0) = \int_\mathbb{R} e^{-x^2} dx \) which can be evaluated by the usual polar coordinate trick, giving \( \sqrt{\pi/\alpha} \). Returning to \( \mathbb{R}^n \), the final result is thus that

\[
\hat{f}(\xi) = (\pi/\alpha)^n/2 e^{-|\xi|^2/4\alpha},
\]

which is hence another Gaussian. A similar calculation shows that for such Gaussians \( \mathcal{F}^{-1} \hat{\phi} = \phi \), i.e. for such Gaussians \( T = \mathcal{F}^{-1} \mathcal{F} \) is the identity map.

Now we can show that \( T \) is the identity map on all Schwartz functions using the following lemma.

**Lemma 0.1.** Suppose \( T : \mathcal{S} \to \mathcal{S} \) is linear, and commutes with \( x_j \) and \( D_{x_j} \). Then \( T \) is a scalar multiple of the identity map, i.e. there exists \( c \in \mathbb{C} \) such that \( Tf = cf \) for all \( f \in \mathcal{S} \).

**Proof.** Let \( y \in \mathbb{R}^n \). We show first that if \( \phi(y) = 0 \) and \( \phi \in \mathcal{S} \) then \( (T \phi)(y) = 0 \). Indeed, we can write, essentially by Taylor’s theorem, \( \phi(x) = \sum_{j=1}^n (x_j - y_j) \phi_j(x_j) \), with \( \phi_j \in \mathcal{S} \) for all \( j \). In one dimension this is just a statement that if \( \phi \) is Schwartz and \( \phi(y) = 0 \), then \( \phi_1(x) = \phi(x)/(x-y) = \phi(x) - \phi(y)/(x-y) \) is Schwartz: smoothness near \( y \) follows from Taylor’s theorem, while the rapid decay with all derivatives from \( \phi_1(x) = \phi(x)/(x-y) \). For the multi-dimensional version, one can take \( \phi_j(x) = (x_j - y_j) \phi(x)/(x-y)^2 \) for \( |x-y| \geq 2 \), say, suitably modified inside this ball. Thus,

\[
T \phi = \sum_{j=1}^n (x_j - y_j) (T \phi_j),
\]

where we used that \( T \) is linear and commutes with multiplication by \( x_j \) for all \( j \). Substituting in \( x = y \) yields \( (T \phi)(y) = 0 \) indeed.

Thus, fix \( y \in \mathbb{R}^n \), and some \( g \in \mathcal{S} \) such that \( g(y) = 1 \). Let \( c(y) = (Tg)(y) \). We claim that for \( f \in \mathcal{S} \), \( (Tf)(y) = c(y)f(y) \). Indeed, let \( \phi(x) = f(x) - f(y)g(x) \), so \( \phi(y) = f(y) - f(y)g(y) = 0 \). Thus, \( 0 = (T \phi)(y) = (Tf)(y) - f(y)(Tg)(y) = (Tf)(y) - c(y)f(y) \), proving our claim.

We have thus shown that there exists \( c : \mathbb{R}^n \to \mathbb{C} \) such that for all \( f \in \mathcal{S} \), \( y \in \mathbb{R}^n \), \( (Tf)(y) = c(y)f(y) \), i.e. \( Tf = cf \). Taking \( f \in \mathcal{S} \) such that \( f \) never vanishes, e.g. a Gaussian as above, shows that \( c = Tf/f \) is \( C^\infty \), since \( Tf \) and \( f \) are such.

We have not used that \( T \) commutes with \( D_{x_j} \) so far. But

\[
c(y)(D_{x_j} f)(y) = T(D_{x_j} f)(y) = D_{x_j}(T f)|_{x=y} = D_{x_j}(c(y) f(x))|_{x=y} = (D_{x_j} c)(y) f(y) + c(y)(D_{x_j} f)(y).
\]

Comparing the two sides, and taking \( f \) such that \( f \) never vanishes, yields \( (D_{x_j} c)(y) = 0 \) for all \( y \) and for all \( j \). Since all partial derivatives of \( c \) vanish, \( c \) is a constant, proving the lemma. \( \square \)

The actual value of \( c \) can be calculated by applying \( T \) to a single Schwartz function, e.g. a Gaussian, and then the explicit calculation from above shows that \( c = 1 \), so \( \mathcal{F}^{-1} \mathcal{F} = \text{Id} \) indeed.