Problem #3 p. 64*

Recall the formula

\[ u(x,t) = \frac{1}{2} \left[ \phi(x+ct) + \phi(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(y) dy \]

Focus on the interval \( x > ct \)

\[ u(x,t) = \frac{1}{2} \left[ \phi(x+ct) - \phi(x-ct) \right] + \frac{1}{2} \int_{ct-x}^{ct+x} \psi(y) dy \]

In our case \( u(x,t) = \int f(x+ct) \) for \( x > 0 \)

\[ \frac{\partial u}{\partial t}(x,0) = \frac{\partial f}{\partial t}(x) \]

**t = 0** \( u(x,0) = f(x) \)

\[ \frac{\partial u}{\partial t}(x,0) = c \frac{\partial f}{\partial x}(x) \]

So immediately

\[ \frac{1}{2c} \int_{x-ct}^{x+ct} \frac{\partial f}{\partial t}(s) ds = \frac{1}{2} \left[ f(x+ct) - f(x-ct) \right] \]

and

\[ \frac{1}{2c} \int_{ct-x}^{ct+x} \frac{\partial f}{\partial t}(s) ds = \frac{1}{2} \left[ f(ct+x) - f(ct-x) \right] \]

\( x > ct \)

\( x < ct \)

You let $t > 0$ if $x > ct$

\[ V(x, t) = \frac{1}{2} \left[ f(x + ct) + f(x - ct) \right] + \frac{1}{2} \left[ f(x + ct) - f(x - ct) \right] \]

\[ = f(x + ct) \]

\[ \text{if} \]

\[ 0 < x < ct \]

\[ V(x, t) = \frac{1}{2} \left[ f(x + ct) - f(ct - x) \right] + \frac{1}{2} \left[ f(ct + x) - f(ct - x) \right] \]

\[ = f(x + ct) - f(ct - x) \]

To solve #4, first find a general formula for the Neuman problem and then predict the heat.

In homogeneous diffusion + non-equilibrium

Consider the diffusion equation problem

\[ \begin{cases} U_t - Ku_{xx} = f(x, t) \\ U(x, 0) = f(x) - \infty < x < \infty \quad t > 0 \end{cases} \]

Physically this means that as time progresses, there is a "sake of heat or cooling" that changes the temperature, the other wise constant of the temperature.
The solution of this problem is

\[ \mu(x, t) = \int_{-\infty}^{\infty} S(x-y, t) \phi(y) \, dy + \int_{-\infty}^{\infty} S(x-y, t-s) \phi(y) \, dy \]

where

\[ S(x, t) = \frac{1}{\sqrt{4\pi K t}} e^{-\frac{x^2}{4Kt}} \]

How do we find this formula?

The first approach is with the ODE

\[ \begin{cases} \frac{d}{dt} \theta(x, t) + K \theta(x, t) = f(t) \\ \theta(x, t) - \frac{\partial}{\partial x} \Theta(x, t) = \phi(x) \end{cases} \]

When \( A \) is a constant, then thanks to the factor at \( e \) we have

\[ g(t) = e^{\frac{-tA}{2}} \theta(x, t) + \int_{0}^{t} e^{\frac{-(s-t)A}{2}} f(s) \, ds \]

This is the solution of linear problem.

On the other hand this is not just an analogy.

In fact, if we take F.T. this becomes a proof.
\[ u(x,t) = \int S(x-y) f(y) \, dy + \int_0^t S(y) \, dt \]

so

\[ u(x,t) = e^{t} \Phi(x - t) + \int_0^t e^{-t} \Phi(x - t) \, dt = e^{t} \Phi(x) - \int_0^t e^{-s} \frac{\partial}{\partial x} \Phi(x) \, ds \]

You recall that

\[ v(0,0) = \Phi(x) \]

For the problem (3), becomes exactly (2)

Hint: I used the initial conditions.
Consider
\[\begin{align*}
&u_t - \kappa u_{xx} = f(x) \quad x > 0 \\
&u(x, 0) = \phi(x) \\
&u(0, t) = h(t)
\end{align*}\]

This problem has two sources of heat (cool): \(f(t)\) and \(h(t)\).

Reduction to a known problem:

\[W(x, t) = u(x, t) - h(t)\]

Since \(h(t)\) is differentiable

\[W_t = u_t + h_t\]
\[W_t - \kappa W_{xx} = u_t - h_t - \kappa u_{xx} = f(t) - h_t(t)\]

This problem:
\[\begin{align*}
&W_t - \kappa W_{xx} = f(t) - h(t) \\
&W(x, 0) = \phi(x) - h(0) \\
&W(0, t) = 0
\end{align*}\]

Now, let \(\phi_{ext} = \text{odd extension of } \phi(x) - h(0)\)
and \(f_{ext} = - = = f(x, t) - h(t)\)
Our goal now is to solve
\[
\begin{aligned}
&\frac{\partial u}{\partial t} - ku \frac{\partial u}{\partial x} = f(t) - h(t), \\
&w(x, 0) = \phi(x)
\end{aligned}
\]

where
\[
W(x, t) = \int_{-\infty}^{\infty} S(x-y, t) \phi(y) \, dy + \int_{-\infty}^{t} S(x-y, t-s) [f(y) - h(s)] \, dy.
\]

and
\[
\begin{align*}
&u(x, t) = W(x, t) \bigg|_{t=0} \\
&u(x, t) = -\int_{-\infty}^{0} S(x-y) [\phi(y) - h(0)] \, dy \\
&\quad + \int_{0}^{\infty} S(x-y) [\phi(y) - h(0)] \, dy \\
&\quad + \int_{0}^{t} S(x-y, t-s) \left[ f(y, s) - h_{+}(s) \right] \, ds \\
&\quad + \int_{t}^{\infty} S(x-y, t-s) \left[ f(y, s) - h_{-}(s) \right] \, ds
\end{align*}
\]

This solves problem 4 in page 68.*

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Consider the problem:

\[ u_{tt} - c^2 u_{xx} = f(x, t) \]
\[ u(x, 0) = \phi(x) \]
\[ u_t(x, 0) = \psi(x) \quad -\infty < x < \infty, \ t > 0 \]

The solution for this problem is

\[ u(x, t) = \frac{1}{2} \left[ \phi(x+ct) + \phi(x-ct) \right] + \frac{1}{2c} \int_{x-ct}^{x+ct} \psi(s) ds + \frac{1}{2c} \int_{x-ct}^{x+ct} f(s, \tau) d\tau \]

where \( \Delta = [0, t] \times [x - c(t-s), x + c(t-s)] \)

Remark: It makes sense that the influence of the source function \( f(x, t) \) is only felt in the domain of influence.
Before we go to the actual proof of the formula for the solution, let's run it to obtain well-posedness.

**Well-posedness**

Recall that well-posedness has three components:

1. Existence of solutions
2. Uniqueness of solutions
3. Stability of solutions

1) We claim that \( u(x,t) \) is sol.

\[
\frac{\partial}{\partial t} u(x,t) = u_1(x,t) + u_2(x,t) + u_3(x,t)
\]

\[
\begin{align*}
\text{this solves} & \quad \text{this sol. ep.} \\
u_{tt} - c \frac{\partial^2 u}{\partial x^2} = 0 & \quad u_{tt} - c \frac{\partial^2 u}{\partial x^2} = 0
\end{align*}
\]

Clearly, that \( u_3(x,t) \) also solves the ep.

\[
\int_{x-c(t-s)}^{x+c(t-s)} f(s,y) \, dy \, ds
\]

and continue.

Hence, by linearity, \( u \) solves the ep.

For initial data:

\[
\begin{align*}
t = 0 & \quad u(x,0) = \phi \\
t = 0 & \quad u_t(x,0) = \gamma
\end{align*}
\]
2) Uniqueness: Let \( u_1, u_2 \) be two solutions, then let
\[
W = u_1 - u_2 \quad \text{and} \quad W = u_1 - u_2
\]
\[
W_t + f(x) W_x = f(t) - f(t) = 0
\]
\[
W(x, 0) = 0 \quad \Rightarrow \quad \text{Uniqueness: } W(t) = 0 \quad \text{since } W(x, 0) = 0
\]
3) Stability: We define the "distance of two functions" as
\[
\| f_1 - f_2 \| = \max_{-\infty < x < \infty}, 0 \leq t \leq T | f_1(x, t) - f_2(x, t) |
\]
Also if \( \phi \) is no dependency on time then
\[
\| \phi_1 - \phi_2 \| = \max_{-\infty < x < \infty} | \phi_1(x) - \phi_2(x) |
\]
Assume for the initial data that
\[
\| \psi_1 - \psi_2 \|, \| \phi_1 - \phi_2 \| < \delta \quad \Rightarrow \quad \| \phi_1 - \phi_2 \| < \delta
\]
Then
\[
| U_1(x, t) - U_2(x, t) | = ...
\]
\[ \frac{1}{2} \left[ \phi_1(x+c) - \phi_2(x+c) \right] + \frac{1}{2} \left[ \phi_1(x-c) - \phi_2(x-c) \right] \]

\[ + \frac{1}{2c} \int_{x-c}^{x+c} \left[ \psi_1(s) - \psi_2(s) \right] ds + \frac{1}{2c} \int_{\Delta} \left[ g_1(\cdot) - g_2(\cdot) \right] d\gamma(\cdot) \]

\[ \leq \frac{1}{2} \| \phi_1 - \phi_2 \| + \frac{1}{2} \| \phi_1 - \phi_2 \| + \frac{1}{2c} \| \psi_1 - \psi_2 \| + \frac{1}{2c} \| \psi_1 - \psi_2 \| |

\[ + \frac{1}{2c} \| \phi_1 - \phi_2 \| + \text{Area}(\Delta) \]

\[ \text{Area}(\Delta) = 2ct \cdot t = 2ct^2 \]

\[ \leq 5 + 5T + 5T^2 < \varepsilon \]

So for any \( \varepsilon > 0 \) we can find \( \delta = \delta(T) \) s.t.

\[ \| \mu_1 - \mu_2 \| \leq \varepsilon. \]

**How do we find the functional?**

Again, we are going to use F.T. to the simpler problem

\[ \begin{cases} U_{tt} - c^2 U_{xx} = f(x,t) \\ U(t,0) = 0 \\ U(t,\pi) = 0 \end{cases} \quad \begin{cases} \tilde{U}_{tt} - c^2 \tilde{U}(x,t) = \tilde{f}(x,t) \\ \tilde{U}(0,\xi) = 0 \\ \tilde{U}(\pi,\xi) = 0 \end{cases} \]
This is not much of a restriction because we know
the effect of the initial data e we have
linearity. So if we put \( g \) in yet this ODE
\[
\begin{aligned}
\frac{\partial^2 g}{\partial t^2} + c^2 \frac{\partial^2 g}{\partial x^2} &= f(x,t) \\
g(0) = 0 & \quad g_t(0) = 0
\end{aligned}
\]
again the solution of this is easy:
\[
U(\xi, t) = u(t) = \int_0^t \frac{1}{|\xi|} \sin(|\xi| \tau) \chi(E) \frac{\partial}{\partial \xi} f(\xi, s) \, ds
\]

After one takes the inverse F.T.
\[
U(x,t) = \int_0^t \mathcal{F}^{-1} \left( \frac{1}{|\xi|} \sin(|\xi| (t-s)) \chi(E) \right) \chi(x) \, ds
\]

Using the fact that
\[
H(a \cdot 1_{[0,1]}) (\xi) = \frac{\pi}{\xi} \sin \pi a \xi
\]
and some arithmetic manipulations, one
based on properties of \( \sin \) one
gets the results.

\[ Q \]