Lecture # 7

- Mid-term next Thu.
- Final exam Dec. 16, 9-12
- Next homework due next Tuesday.

Remember from last time:

$u$ solve $u_t = ku_{xx}$ on $R = [0, T] \times [0, 1]$

We want to prove

$$\max_{\Omega} u = \max_{\partial \Omega} u$$

We introduced

$$v(x,t) = u(x,t) + \epsilon x^2$$

We are left to prove that if

$$M = \max_{\partial \Omega} u(x,t)$$

then

$$v(x,t) \leq M + \epsilon x^2$$

for all $(x,t) \in R$

Proof of this fact

Observe that

$$|v|_{x=e} = |u|_{x=e} + \epsilon e^2 \leq M + \epsilon e^2$$

$$|v|_{x=0} = |u|_{x=0} \leq M$$

$$|v|_{t=0} = |u|_{t=0} + \epsilon x^2 \leq M + \epsilon x^2$$

So

$$\max_{\partial \Omega} v(x,t) \leq M + \epsilon x^2$$
We want to show that
\[ \max_{V_R} V = \max_{V_R} V \]
\[ (x_0, t_0) \]
Assume \( (x_0, t_0) \) is a point s.t. \( \max_{V_R} V = V(x_0, t_0) \)
if \( (x_0, t_0) \) is in \( R \) \( \Rightarrow \) done
So far we assume:

**Case 1:** \( (x_0, t_0) \) is in \( R \) (interior of \( R \))

**Case 2:** \( t_0 = T \) and \( 0 < x, \epsilon \)

In the first case
\[ U_x(x_0, t_0) = U_t(x_0, t_0) = 0 \]
\[ \forall x \]
\[ U_{xx}(x_0, t_0) \leq 0 \]
\[ U_t = U_x, \quad U_x = \ell U_x + 2 \epsilon \quad U_{xx} = U_{xx} + 2 \epsilon \]
\[ U_t = \ell U_{xx} = \ell (U_{xx} - 2 \epsilon) < 0 \]
\[ \Rightarrow \]
\[ U_t \]
\[ 0 \]
\[ \Rightarrow \] contradiction
In the second case:

\[ f(x) = u(x, T) \quad \text{on } \mathcal{L} \]

\[
\lim_{\delta \to 0^+} \frac{u(x_0, T) - u(x_0 - \delta, T)}{\delta} = u_T(x_0, T) > 0
\]

and again for \( \mathcal{G} \) we get a contra-diction.

**Uniqueness:** Consider the boundary initial value problem

\[
\begin{cases}
    u_t - ku_{xx} = f(x, t) \\
    u(x, 0) = \phi \\
    u(0, t) = g(t) \\
    u(l, t) = h(t)
\end{cases}
\]

Minimun of the solution for (x) means that for given \( \phi, g, h \) there is only one solution for (x), \( \phi, g, h \) determines it completely.

Suppose there were 2 solutions for (x), call them \( u_1 \) and \( u_2 \). Then \( W = u_1 - u_2 \) solves

\[
\begin{cases}
    W_t - kW_{xx} = 0 \\
    W(x, 0) = 0 \\
    W(0, t) = 0 \\
    W(l, t) = 0
\end{cases}
\]

By the max (min) principle \( \min W = \min W_{\mathcal{R}} = 0 \)

\[
\max W = \max W_{\mathcal{R}} = 0
\]
\[ E = \frac{1}{2} \int_0^L u(x,t)^2 \, dx \] 
Assume \( u(l,t) = u(0,t) = 0 \)

\[ \frac{\partial E}{\partial t} = \frac{1}{2} \int_0^L \left[ \mathcal{L} u + \mathcal{F} u \right] \, dx = 0 \]

\[ = \frac{1}{2} \int_0^L \left[ -k \frac{\partial u}{\partial x} \right] \, dx \]

\[ = -k \int_0^L \frac{\partial u}{\partial x} \, dx = -k \int_0^L \frac{\partial u}{\partial x} \, dx \]

\[ = ku_x \bigg|_0^L = 0 - k \int_0^L u_x^2 \, dx \leq 0 \]

So \( E(t) \downarrow \).

We can also prove uniqueness by using this fact:

\[ 0 \leq E(t) \leq E(0) = \frac{1}{2} \int_0^L \mathcal{W}(x,0)^2 \, dx \geq 0 \]

So \( E(t) = 0 \) \( \forall t \in [0, T] \)

\[ \int_0^L \mathcal{W}(x,t)^2 \, dx = 0 \Rightarrow \mathcal{W} \equiv 0 ! \]

[Definition of Stability]: In general we say that a system is stable if "close" initial data generates "close" solutions.
Consider the diffusion equation $u_{tt} = ku_{xx}$.

We now give an example of stability.

Suppose $u_1$ and $u_2$ are two solutions with $u_1(x, 0) = \phi_1$ and $u_2(x, 0) = \phi_2$.

Let $W = u_1 - u_2$. Then the equation and initial conditions give

\[
E(W) = \int (u_1 - u_2)^2 \, dx \leq \int (\phi_1 - \phi_2)^2 \, dx,
\]

so if $\phi_1$ and $\phi_2$ are "close" in the sense of the distance above, then $u_1$ and $u_2$ are also "close" uniformly in time.

We now suppose we define the distance as

\[
\text{dist}(f, g) = \max_{x_0, x_1} |f(x) - g(x)|.
\]
Assume that $u_1$ and $u_2$ solve the two problems

\[
\begin{cases}
    u_t = ku_{xx} \\
    u(x,0) = \phi_1 \\
    u(0,t) = g_1 \\
    u(l,t) = f_1
\end{cases}
\quad \begin{cases}
    v_t = kv_{xx} \\
    v(x,0) = \phi_2 \\
    v(0,t) = g_2 \\
    v(l,t) = f_2
\end{cases}
\]

Then $u_1 - u_2 = w$ solves

\[
\begin{cases}
    w_t = kw_{xx} \\
    w(x,0) = \phi_1 - \phi_2 \\
    w(0,t) = w(l,t) = 0
\end{cases}
\]

Then by the maximum principle

\[
\max_{\partial \Omega} w = \max_{\partial \Omega} (\max_{[0,T]} (\phi_1 - \phi_2), 0)
\]

\[
\max_{\Omega} w = \max_{\Omega} (\max_{[0,T]} (\phi_1 - \phi_2), 0)
\]

So

\[
\max_{[0,T]} |u_1 - u_2| \leq \max_{[0,T]} |\phi_1 - \phi_2|
\]

for all $t$.

Finding solutions for a diffusion equation.

We have considered only the initial value problem

\[
\begin{cases}
    u_t = ku_{xx} \\
    u|_{t=0} = \phi(x)
\end{cases}
\]
The idea here is to find a source function $S(x,t)$ such that any solution for $(\star)$ can be written in terms of $S$ and the initial data $u(x,0)$ for the wave equation.

For this proof it is better to list four a-priori properties for the solution of a diffusion equation $(\star) \quad u_t = k u_{xx}$

a) **Translation invariance**: If $u(x,t)$ solves $(\star)$ then for any fixed $y$

   $u(x-y,t)$ also solves $(\star)$

b) **If $u$ is a solution for $(\star)$ then all derivatives of any order of $u$ also solves $(\star)$

c) **A linear combination of solutions for $(\star)$ is also a solution**

d) **If $u(x,t)$ is a solution then for any $g$ smooth function $v(x,t) = \int u(x-y,t) g(y) \, dy$ is also a solution for $(\star)$

e) **Solving**; if $u(x,t)$ is a solution then $u(x,x^2 t) = \alpha \exp \gamma$.