# Efficiency Loss in Resource Allocation Games

by

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## Abstract

The overarching goals of this thesis are to quantify the efficiency loss due to market participant strategic behavior, and to design proper pricing mechanisms that reduce the efficiency loss. The concept of efficiency loss is intimately related to the concept of "price of anarchy," which was advanced by Koutsoupias and Papadimitriou, and compares the maximum social welfare with that achieved at a worst Nash equilibrium.

This thesis focuses on the following two topics: (i) For a market with an arbitrary number of participants, how much is the Nash equilibrium close, in the sense of price of anarchy, to a social optimum? (ii) For a resource allocation/pricing mechanism, is the social welfare achieved at an economic equilibrium asymptotically optimal, as the number of market participants goes to infinity?

Regarding the first topic, we quantify the efficiency loss in classical Cournot oligopoly games, where multiple oligopolists compete by choosing quantities. We also compare the total profit earned at a Cournot equilibrium to the maximum possible total profit that would be obtained if the suppliers were to collude.

For the second topic, related to the efficiency in large economics, we analyze the efficiency of Kelly's proportional allocation mechanism in large-scale wireless communication systems. We study a corresponding Bayesian game in which each user has incomplete information on the state or type of the other users, and show that the social welfare achieved at a Bayes-Nash equilibrium is asymptotically optimal, as the number of users increases to infinity.

Finally, for electricity delivery systems, we propose a new dynamic pricing mechanism that explicitly encourages consumers to adapt their consumption so as to offset the variability of demand on conventional units. Through a dynamic game-theoretic formulation, we show that the proposed pricing mechanism achieves social optimality asymptotically, as the number of consumers increases to infinity.

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# Chapter 1

# Introduction

This thesis lies at the interface of operations research, economics, and engineering, with a focus on social welfare analysis and pricing mechanism design for a variety of static and dynamic market models motivated from communication networks to electricity markets. The overarching goals are to quantify the efficiency loss due to strategic behavior, and to design proper pricing mechanisms that reduce the efficiency loss. The concept of efficiency loss is intimately related to the concept of "price of anarchy," which was advanced by Koutsoupias and Papadimitriou in a seminal paper [54], and compares the maximum social welfare with that achieved at a worst Nash equilibrium.

In most of this thesis, game-theoretic approaches are employed to study the strategic interaction between multiple agents (e.g., customers in an electricity market, or oligopolists in an oligopolistic market). Particular attention is paid to the role that resource allocation/pricing mechanisms play in influencing the agents' strategic behavior and the social welfare (the sum of consumer surplus and supplier profit). In particular, the thesis focuses on the following two topics related to the analysis of social welfare in a number of settings motivated from communication networks to electricity markets:

1. For a market with an arbitrary number of participants, is the Nash equilibrium close, in the sense of price of anarchy, to a social optimum?

We quantify the efficiency loss in Cournot oligopoly games (Chapter 3), and in a two-sided market with an extension of Kelly's proportional allocation mechanism (Chapter 5).

The methodology we use can also be employed to quantify the profit loss in Cournot oligopoly games. In Chapter 4, we compare the total profit earned at a Cournot equilibrium to the maximum possible total profit, which would be obtained if the suppliers were to collude.

2. For a particular resource allocation mechanism, does asymptotic social optimality hold? That is, is the social welfare achieved at an economic equilibrium asymptotically optimal, as the number of market participants goes to infinity?

   For Kelly's proportional allocation mechanism, we show its asymptotic social optimality under some mild assumptions (Chapter 2).

   For future electricity delivery systems, we propose a new dynamic pricing mechanism that explicitly encourages consumers to adapt their consumption to offset the variation of renewable generation (Chapter 6). Through a dynamic game-theoretic formulation, we show that if all consumers use an equilibrium strategy, every consumer's expected payoff and the social welfare are asymptotically maximized, as the number of consumers increases to infinity.

In Sections 1.1-1.3, we overview some of the main contributions of this thesis.

## 1.1 Bayesian Proportional Resource Allocation Games

Communication networks are typically used by a large population of users of different types, who place different values on their perceived network performance. To capture each user's perceived utility from its allocated network resource, and to then efficiently allocate network resources to fulfill the needs of heterogeneous users, a variety of market mechanisms and associated congestion game models have been introduced.

In Chapter 2, we consider a resource allocation mechanism of the form proposed by Kelly [50]: every user submits a payment ("bid") and then a fraction of the resource is allocated to each user in proportion to its bid. We study a corresponding Bayesian game in which each user has incomplete information on the state or type of the other users. We are interested in the social welfare associated with the resulting allocation. More specifically, we focus on the following two issues: (i) price of anarchy, i.e., how much is the social welfare reduced if the users act strategically (in the game-theoretic sense), compared to a socially optimal solution? (ii) value of information, i.e., how is the social welfare affected by the amount of information available at each user (knowledge of all states versus knowledge of only its own state)?

Under some mild assumptions, we show that the social welfare achieved at a Bayes-Nash equilibrium is asymptotically optimal, as the number of users increases to infinity[1]. We also show that the worst case ratio between an ex ante optimal allocation and an ex post (based on all realized states) optimal allocation can be as bad as order $\Theta(N)$, where $N$ is the number of users.

## 1.2 Efficiency and Profit Loss in Cournot Oligopolies

In Chapter 3, we consider a Cournot oligopoly model where $N$ suppliers (oligopolists) compete by choosing quantities. Our work partially answers the classical question raised by Friedman in his book on oligopoly theory [31], "is the Cournot equilibrium close, in some reasonable sense, to the competitive equilibrium?" We compare the social welfare at a Cournot equilibrium to the maximum possible social welfare achieved at a competitive equilibrium. For the case where the inverse market demand function is convex, we establish a lower bound on the efficiency of Cournot equilibria in terms of a scalar parameter that captures key qualitative properties of the inverse demand function. To our knowledge, this is the first efficiency lower bound that holds for Cournot oligopoly models with general convex inverse demand functions.

The methodology used in our efficiency loss analysis can be also used to study the loss of aggregate profit due to competition. In Chapter 4, we derive a lower bound on the ratio of the aggregate profit earned by all suppliers at a Cournot equilibrium

[1] The most significant results of this chapter have been published in [90].

to the maximum possible aggregate profit, that is, the profit that would have been achieved if the suppliers were to collude.

## 1.3 Pricing of Fluctuations in Electricity Markets

In an electric power system, demand fluctuations may result in significant ancillary cost to suppliers. Furthermore, in the near future, deep penetration of volatile renewable electricity generation is expected to exacerbate the variability of demand on conventional thermal generating units. In Chapter 6, we address this issue by explicitly modeling the ancillary cost associated with demand variability. We show by example that a time-varying price that equals the suppliers' instantaneous marginal cost may not achieve social optimality.

We propose a new dynamic pricing mechanism that explicitly encourages consumers to adapt their consumption, with **negative** correlation to the variation of renewable generation: every consumer is charged an additional (possibly negative) price, equal to the marginal ancillary cost associated with the consumer's action taken at the previous stage. In a nonatomic dynamic game model featuring a continuum of consumers, we show that the proposed pricing mechanism achieves social optimality at a Rational Expectations Equilibrium (REE), under certain convexity assumptions. For a dynamic game with a large number of consumers, we show that if all consumers use a REE strategy of the nonatomic limit game, every consumer's expected payoff and the social welfare are approximately maximized. Numerical results demonstrate that compared with real-time marginal cost pricing, the proposed pricing mechanism reduces the peak load, and therefore has the potential to reduce the need for long-term investments in peaking plants.

# Chapter 2

# Bayesian Proportional Resource Allocation Games

## 2.1 Introduction

In this chapter, we consider proportional resource allocation mechanisms for allocating a certain resource among a set of heterogenous users, in the face of uncertainty. More precisely, we have $N$ users who share a divisible resource, which we refer to as **bandwidth**. We let $C > 0$ (the **capacity**) be the total available amount of the resource. Each user can be in one of several **states** $s$, chosen according to a known probabilistic mechanism. Each user is assigned a certain bandwidth $b$, according to some resource allocation mechanism, with the sum (over all users) of the allocated bandwidths bounded above by $C$. Each user derives a certain utility $U_s(b)$ that depends on its state and its allocated bandwidth.

The resource allocation mechanisms that we consider are of the form proposed by Kelly in the context of bandwidth allocation in communication networks [50]: every user submits a payment ("bid") and then a fraction of the resource is allocated to each user in proportion to its bid. We are interested in the total utility ("social welfare") derived from such an allocation. More specifically, our focus is on the following two issues:

(a) **Price of Anarchy:** By how much is the social welfare reduced if players act strategically (in the game-theoretic sense), compared to a socially optimal solution?

(b) **Value of Information:** How is the social welfare affected by the amount of information available at each user (i.e., knowledge of all states versus knowledge of only its own state)?

We note that in the case of limited information (every user only knows its own state) and strategic behavior, we obtain a Bayesian game; this game will be the subject of most of the results in this chapter.

Let us pause to mention some possible interpretations of the states and utility functions of the users.

(a) Consider a wireless communication network with frequency division multiplexing. Here $C$ represents the total bandwidth available and $b$ the bandwidth allocated to a user. The state $s$ of a user represents the quality of its channel, and determines the throughput that the user can obtain using the given bandwidth. Accordingly, the throughput and utility of a user is a function of both $b$ and $s$.

(b) Consider a wireline link with capacity $C$. We interpret $b$ as the rate allocated to a user, and interpret $s$ as the "type" of the user: users of different types make different uses of their allocated rate (e.g., voice versus video, or elastic versus time-critical communications), and accordingly derive different utilities from any given allocated rate.

We note that under either interpretation, it is plausible to assume that a user knows its own state, but not the state of the other users. On the other hand, it is reasonable to assume that the different states/types arise according to a probabilistic mechanism known to all users; this is our motivation for considering the Bayesian game setting. A Nash equilibrium of the deterministic games considered in earlier work [45] requires either an implicit assumption that every user knows the utility function (type) of every other user, or that there are repeated interactions and a learning process through which every user can obtain enough information on the behavior of the other users. In contrast, a Bayesian formulation can capture a one-shot game (no repeated interactions), without placing unrealistic assumptions on the available information.

### 2.1.1 Background and related research

Current communication networks are used by a large population of users of different types, who place different values on their perceived network performance. A fundamental difficulty in operating such large scale communication networks is how to allocate the network resources to maximize user satisfaction, instead of the total transmission rate. To efficiently allocate network resources to fulfill the needs of heterogeneous users, a variety of **congestion game** models, which use market mechanisms to allocate the scarce network resources, have emerged.

A possible approach to achieving efficient allocation of network resources to heterogeneous users is "**usage based pricing**", where users are charged based on their actual consumption of network resources. One of the prominent earlier proposals in congestion pricing is the "smart market" proposed by Mackie-Mason and Varian in [61], where an auction is employed to allocate each resource unit to the user who bids the highest. Following the classical results on Vickrey-Clarke-Groves (VCG) mechanisms (see [95], [20] and [37] for details), they show that each user has an incentive to reveal its true valuation of the network resource and thus the achieved allocation is efficient. However, due to the high information exchange requirement of VCG mechanisms [84], Mackie-Mason and Varian's approach is not scalable in a distributed communication network.

Rather than allocating network resources through an auction, an alternative approach views the resource allocation as a market-clearing process, and has received extensive attention in recent years. One of the earlier influential mechanisms is proposed in [50]. In Kelly's mechanism, each user submits a bid (the money it is willing to pay) to the network. Collecting the submitted bids of all users, the network sets a single price to clear the market, or to allocate the total available resources to the users; the rate received by each user is in proportion to its bid. For price-taking users, it is shown in [50] that such a mechanism maximizes aggregate utility.

If the users are price-anticipating, i.e, if they act strategically and take the influence of their bids on the price into account, Johari and Tsitsiklis show that the aggregate utility achieved at a Nash equilibrium of the Kelly mechanism cannot be less than 3/4 of its maximum value [45]. Furthermore, under certain assumptions, it is shown in in [63] and [48] that Kelly-mechanism has the lowest price of anarchy among all proportional allocation mechanisms. If price discrimination is allowed, mechanisms with one-dimensional strategy spaces that result in full efficient allocation at a Nash equilibrium have been designed in [98] and [48].

Different from the models studied in all these previous works, in this chapter we wish to study the performance of Kelly's mechanism in various environments with different types of available information. In our model, the utility received by each user not only depends on the allocated resources but also on its private state. The state of each user is generally regarded as "**private information**", which determines its utility function. For example, a wired internet user may have various types of applications which have different rate requirements. Also, the channel quality, e.g., the SINR (Signal to Interference plus Noise Ratio), may influence the utility received by a wireless user, even if the user receives a fixed amount of wireless bandwidth. We are interested in the value of centralizing this private information in the competition among multiple strategic users.

In this chapter, we assume that the prior probability distribution of the states of all users is a common knowledge. There are three different types of information for each user to make decisions, namely, i) **ex ante**, where users have not received any private information; ii) **interim**, where each user has received its private information, but does not know the other users' private information; iii) **ex post**, where the private information of all users is common knowledge. The price of anarchy result in [45] is for an ex post Nash equilibrium, where the utility function of each user is a common knowledge. In a communication network, to implement a protocol which requires the users to reveal and then exchange their true private information, e.g., the application type at the application layer or the SINR at the physical layer, may consume a considerable amount of network resources or even be impossible. Thus, for multiple users with "**asymmetric information**", it is necessary to study the efficiency of pricing mechanisms at the interim stage. When the state of other users is not available, each user could use the prior probability distribution to calculate the Bayesian Nash equilibrium separately, which makes the concept of Bayesian Nash equilibria more plausible than the Nash equilibrium. In this chapter, we will study the existence and the efficiency of Bayesian Nash equilibria at the interim stage. In particular, under a set of mild assumptions, we will prove the optimality of Bayesian Nash equilibria, as the number of users increases to infinity.

The Bayesian game in the interim stage, is sometimes referred to as a game with incomplete information or asymmetric information, and has been investigated in the field of economics. In particular, the auction mechanism design problem has received extensive attention. This line of literature studies the efficiency of auction games among multiple potential buyers for a limited number of indivisible goods, under the assumption that the seller(s) does (do) not know the valuation of each buyer. For a monopoly seller with limited supply, Myerson studies the efficiency of the second price auction with an appropriately chosen reserve price in [72]. In [68], the author considers the case of "large markets", where the number of sellers goes to infinity. He shows that it is an equilibrium for all sellers to employ a second price auction with no reserve price, regardless of the distribution of buyer valuations. More recently, the author of [78] studies a more complicated model where multiple sellers with limited supply strategically choose both the price formation rule and the allocation rule, and then buyers strategically choose to participate in one of the mechanisms set by sellers. In this two-stage game with incomplete information, it is shown in [78] that it is an equilibrium for all sellers to offer efficient mechanisms, as the number of sellers increases to infinity. More recently, Bodoh-Creed shows the asymptotic optimality of symmetric Bayesian Nash equilibria in a general economic model [13]; however, one of the key assumptions made in [13] does not hold in our model, because our payoff function is discontinuous at a feasible point of the action space (when all users bid zero).

For perfectly divisible goods (which is more similar to the model studied in this chapter), a Bayesian game for supply chains is studied in [15] to compare the total profit obtained by all retailers at a Bayesian Nash equilibrium with the maximum possible profit. In this model, the price is fixed and each retailer orders an amount of goods. If the total requirement exceeds the capacity, each retailer obtains a certain amount of goods in proportion to its order. The authors of [15] have shown that such a mechanism is inefficient because retailers will order more than they need to gain a more favorable allocation.

In the closer related area of communication networks, several works in the recent literature aim to explore the characteristics of Bayesian Nash equilibria in a game of multiple network users with asymmetric information. In [1], for multiple wireless devices with asymmetric information (the SINR of other transmitter-receiver pairs is not available), the spectrum sharing and power allocation problem is studied by calculating the Shannon capacity at a Bayesian Nash equilibrium. The authors of [85] study a hierarchical Stackelberg game where the service providers announce prices as leaders and the users respond with flows (usage) as followers. For communication networks with a large population of users, they show that the game with incomplete information behaves as if the type of each user were a common knowledge.

The most closely related result to ours is presented by Yu and Mannor in [99]. They model the interaction of users as a game with a heterogeneous population of players, characterized by random utility functions. They show that if the utility functions are bounded, then the (ex post) Nash equilibria are approximately as efficient as the social optimum, with high probability, when the number of users is large. In a large communication network, to implement an algorithm which allows all users to truthfully exchange their private information may consume a considerable amount of network resources. In such a network with many users, it is difficult for each user to gain the complete information about other users' utility functions. Hence, different from the model employed in [99], we model the strategic interaction among multiple users as a Bayesian game, i.e., where each player knows its own utility function, as well as the prior probability distribution of the utility function of other users. The model employed in this chapter has more reasonable informational requirements, but also raises some additional technical complications.

### 2.1.2 Summary and contributions of this chapter

We will study Kelly's resource allocation game with asymmetric information and explore the value of information. The main results of this chapter can be summed up as follows:

1. The worst case ratio between an ex ante optimal allocation and an ex post optimal allocation is $\Theta(N)$ (Theorem 2.1).

2. For linear utility functions, an interim optimal allocation is approximately optimal (Theorem 2.2).

3. For concave, strictly increasing and continuous utility functions, there always exists a Nash equilibrium for the Bayesian game; if the game is **symmetric**, then there exists a symmetric Nash equilibrium (Theorem 2.5).

4. Under a set of mild assumptions, the social welfare achieved at a Nash equilibrium of the Bayesian game is asymptotically optimal, as the number of users increases to infinity (Theorem 2.8).

5. For symmetric Bayesian games with two independent states and linear utility functions, we establish a lower bound on the efficiency of Bayesian Nash equilibria (Proposition 2.2).

The rest of the chapter is organized as follows. In the next section we will introduce the model employed in this chapter. In Section 2.3 we will compare the social welfare achieved in non-strategic formulations, where every user solves an optimization problem to maximize social welfare, based on different types of available information. In Section 2.4 we will show that, ex ante game with Kelly's mechanism is equivalent to the usual resource allocation game studied in [39] and [45], and thus the conclusions in previous works apply. In Section 2.5, we will prove the existence of Bayesian Nash equilibria. In Section 2.6, we will give some examples to illustrate some properties of Bayesian Nash equilibria. In Section 2.7, under a set of mild assumptions, we will provide the main result of this chapter, i.e., the asymptotic optimality of Bayesian Nash equilibria, as the number of users increases to infinity. For symmetric Bayesian games with two independent states and linear utility functions, in Section 2.8, we establish a lower bound on the efficiency of Bayesian Nash equilibria.

## 2.2 Model

In this section, we define more precisely the model outlined in Section 2.1, introduce our notation, and specify a variety of resource allocation mechanisms.

### 2.2.1 Formal description of the model

We now introduce formally the various elements of the model, and in the process also define some notation.

(a) **Users:** A set $\mathcal{N} = \{1, 2, \ldots, N\}$ of $N \geq 2$ users. We typically use the dummy variable $n$ to index the users.

(b) **States:** A common (for all users) finite set of states, $\mathcal{S} = \{1, 2, \ldots, S\}$. We use $s_n$ to denote the state of user $n$. A vector $\mathbf{s} = (s_1, s_2, \ldots, s_N) \in \mathcal{S}^N$ is called a **state vector**.

(c) **Resource:** An amount $C > 0$ of a divisible resource, to be allocated to the users. A nonnegative vector $\mathbf{b} = (b_1, \ldots, b_N)$ that satisfies $\sum_{n=1}^{N} b_n \leq C$ is a feasible **allocation**.

(d) **Utility functions:** We are given a collection of *concave* utility functions $U_s : [0, \infty) \to [0, \infty)$, $s \in \mathcal{S}$. Here, $U_s$ is the utility function of a user who happens to be at state $s$. We define the utility function vector by $\mathbf{U} = (U_1, \ldots, U_S)$.

(e) **State randomness:** The state $s_n$ of each user $n$ is a realization of an $\mathcal{S}$-valued random variable $S_n$. We define the random state vector $\mathbf{S}$ by $\mathbf{S} = (S_1, \ldots, S_N)$. Let $p_{n,s}$ be the probability that the state of user $n$ is $s$, i.e., $p_{n,s} = \mathbb{P}(S_n = s)$. The probability vector of user $n$ is $\mathbf{p}_n = (p_{n,1}, p_{n,2} \ldots, p_{n,S})$, and lies in the $S$-dimensional unit simplex, which we denote by $\mathcal{P}$. We define the (overall) probability vector $\mathbf{p}^N = (\mathbf{p}_1, \ldots, \mathbf{p}_N)$, which is an element of $\mathcal{P}^N$.

Throughout the chapter, we will use $\mathcal{M} = (\mathcal{N}, \mathcal{S}, C, \mathbf{U}, \mathbf{p}^N)$ to denote a particular fully specified model. We will say that a model is **symmetric** if the distribution of the random vector $(S_1, \ldots, S_N)$ is invariant under permutations of its elements; in particular, the probability vector $\mathbf{p}_n$ is the same for all $n$, and each user has the same probability of being at a particular state. Furthermore, all users at the same state form the same beliefs (i.e., conditional distribution) on the part of the state vector that they do not observe.

In general (that is, without the symmetry assumption), we allow some users to have zero probability of being at certain states. On the other hand, without loss of generality, we assume that for each state $s \in \mathcal{S}$, there exists some user $n$ for which $p_{n,s} > 0$. For any user $n$, we use $\mathbf{s}_{-n} = (s_1, \ldots, s_{n-1}, s_{n+1}, \ldots, s_N)$ and $\mathbf{S}_{-n} = (S_1, \ldots, S_{n-1}, S_{n+1}, \ldots, S_N)$ to indicate the states of the other users.

### 2.2.2 Resource allocation mechanisms

We are interested in the **social welfare** (the total expected utility derived by all the users) under different resource allocation mechanisms. The model we have introduced allows for a variety of such mechanisms, which can be classified along two different dimensions.

(a) **Information structure.** Different mechanisms are obtained depending on the amount of information used to make resource allocation decisions.

(i) In **ex ante** decision making, the bandwidth allocated to each user is chosen ahead of time, before the state vector $\mathbf{S}$ is realized.

(ii) In **interim** (or **Bayesian**) decision making, the bandwidth allocated to each user is determined by actions ("bids") chosen locally by each user $n$, on the basis of locally available state information (a realization of $S_n$).

(iii) In **ex post** decision making, bandwidth is allocated on the basis of the realization of the entire state vector $\mathbf{S}$. This corresponds to either immediate sharing of state information or to the presence of a central planner who can monitor the entire state vector.

(b) **Social optimization versus selfish behavior.** For any given information structure, we can assume either benevolent behavior whereby each user (or a central decision maker) strives to maximize social welfare, or selfish behavior whereby each user strives to maximize its own expected payoff, given the behavior of the other users. The first case leads to optimization problems; the second to game-theoretic variants.

For a game-theoretic version to be well-defined, a particular mechanism needs to be specified. We focus on proportional allocation mechanisms of the form introduced by Kelly [50]: each user submits a bid, interpreted as a payment, and bandwidth is allocated in proportion to the bids. We will assume quasilinear payoffs, i.e., that the payoff function of a user equals the expected utility minus the expected value of the bid. On the other hand, when calculating social welfare, we will only consider the resulting total expected utility; payments are viewed simply as transfers to other entities that do not affect social welfare. This way, we can focus on the efficiency of the resulting allocations, and ask the question of whether a certain mechanism results in an efficient utilization of the available resources.

The above classification leads to a total of six possible combinations, which we now proceed to define formally.

**(AO) Ex Ante Optimal Allocation.** Here a bandwidth vector $\mathbf{b} = (b_1, \ldots, b_N)$ is chosen before the state vector $\mathbf{S}$ is realized, to maximize the social welfare $\sum_{n=1}^{N} \sum_{s=1}^{S} p_{n,s} U_s(b_n)$ over all possible allocations, $\mathbf{b}$. For a given model $\mathcal{M} = (\mathcal{N}, \mathcal{S}, C, \mathbf{U}, \mathbf{p}^N)$, we use $W_{AO}(\mathcal{M})$ to denote the resulting optimal social welfare.

**(AG) Ex Ante Game**. Here, each user $n$ submits a bid $w_n \geq 0$, which is the same regardless of the user's state. We refer to $\mathbf{w} = (w_1, \ldots, w_N)$ as the **strategy profile**. If $w_n = 0$, then user $n$ obtains zero bandwidth, $b_n = 0$. If $w_n > 0$, user $n$ obtains bandwidth $b_n(\mathbf{w}) = Cw_n/(\sum_{i=1}^{N} w_i)$. For a given model $\mathcal{M} = (\mathcal{N}, \mathcal{S}, C, \mathbf{U}, \mathbf{p}^N)$, and a strategy profile $\mathbf{w}$, the (expected) payoff function of user $n$ is

$$Q_n(\mathbf{w}) = -w_n + \sum_{s=1}^{S} p_{n,s} U_s\big(b_n(\mathbf{w})\big). \tag{2.1}$$

These payoff functions define an $N$-person game. We are interested in the pure Nash equilibria of that game, that is, strategy profiles in which no user has a reason to modify its bid, given the bids of the others. The social welfare associated with a strategy profile $\mathbf{w}$ is $\sum_{n=1}^{N} \sum_{s=1}^{S} p_{n,s} U_s\big(b(\mathbf{w})\big)$. If the set of Nash equilibria is not empty, we are interested in the social welfare achieved at a Nash equilibrium, in the worst case over all Nash equilibria, which we denote by $W_{AG}(\mathcal{M})$.

**(BO) Interim (Bayesian) Optimal Allocation.** Here, each user knows its own state, but not the state of the other users, and is supposed to make a decision that is optimal in terms of overall social welfare. Such a decision cannot involve a direct choice of bandwidth, because distributed decisions have no way of enforcing the constraint $\sum_{n=1}^{N} b_n = C$. This leads us to consider again a proportional resource allocation mechanism.

Each user $n$, if found at state $s$, submits a bid $w_{n,s} \geq 0$. Without loss of generality, we assume that if $p_{n,s} = 0$, then $w_{n,s} = 0$. We use the notation $\mathbf{w}_n$ to denote the **strategy vector** of user $n$, i.e., $\mathbf{w}_n = (w_{n,1}, \ldots, w_{n,S})$, and $\mathbf{w}$ to denote the **strategy profile** $\mathbf{w} = (\mathbf{w}_1, \ldots, \mathbf{w}_N)$, and we let $\mathbf{w}_{-n}$ be the strategy profile of all users other than $n$, so that $\mathbf{w} = (\mathbf{w}_n, \mathbf{w}_{-n})$. The bandwidth allocated to user $n$, when the state vector is $\mathbf{s} = (s_1, \ldots, s_N)$, is

$$b_{n,\mathbf{s}}(\mathbf{w}) = \begin{cases} 0, & \text{if } w_{n,s_n} = 0 \\ \dfrac{w_{n,s_n}}{\sum_{i=1}^{N} w_{i,s_i}} C, & \text{otherwise.} \end{cases}$$

An optimal allocation is one obtained by choosing a nonnegative vector $\mathbf{w}$ that maximizes the social welfare

$$\sum_{n=1}^{N} \sum_{\mathbf{s} \in \mathcal{S}^N} U_{s_n}\big(b_{n,\mathbf{s}}(\mathbf{w})\big)\, \mathbb{P}(\mathbf{S} = \mathbf{s}). \tag{2.2}$$

For a given model $\mathcal{M} = (\mathcal{N}, \mathcal{S}, C, \mathbf{U}, \mathbf{p}^N)$, we use $W_{BO}(\mathcal{M})$ to denote the resulting optimal social welfare.

**(BG) Interim (Bayesian) Game.** Here, we use the same proportional allocation mechanism (and notation) as in the previous case of interim optimal allocation. The difference is that each user is now faced with an expected payoff at state $s$ of the form

$$Q_{n,s}(w_{n,s}, \mathbf{w}_{-n}) = -w_{n,s} + \sum_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_{s_n}\big(b_{n,s}(\mathbf{w})\big)\, \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n} \,\big|\, S_n = s). \tag{2.3}$$

For a given model $\mathcal{M}$ and strategy profile $\mathbf{w}$, the expected payoff of user $n$ is

$$Q_n(\mathbf{w}) = \sum_{s \in \mathcal{S}} p_{n,s} Q_{n,s}(w_{n,s}, \mathbf{w}_{-n}); \tag{2.4}$$

the resulting social welfare is given again by the expression (2.2). The (expected) payoff functions $Q_n(\mathbf{w})$ define an $N$-person game, which falls within the class of Bayesian games. We are interested in the pure Nash equilibria of this game, which we will be referring to as Bayes-Nash equilbria (or BNE, for short). Note that strategy profile $\mathbf{w}$ is a BNE if and only if there is no profitable deviation from $w_{n,s}$ whenever $p_{n,s} > 0$, or

$$p_{n,s} Q_{n,s}(w_{n,s}, \mathbf{w}_{-n}) \geq p_{n,s} Q_{n,s}(\overline{w}_{n,s}, \mathbf{w}_{-n}), \quad \forall\, \overline{w}_{n,s} \geq 0, \quad \forall\, n \in \mathcal{N}, \quad \forall\, s \in \mathcal{S}. \tag{2.5}$$

If the set of BNEs is not empty, we are interested in the social welfare achieved at the "worst" BNE (the infimum of the social welfare over all BNEs), which we denote by $W_{BG}(\mathcal{M})$.

**(PO) Ex Post Optimal Allocation.** An allocation based on knowledge of the entire state vector allocates a certain bandwidth $b_{n,\mathbf{s}}$ to agent $n$ when the state vector is $\mathbf{s} = (s_1, s_2, \ldots, s_N)$. An optimal allocation chooses, for each $\mathbf{s}$, a vector $(b_{1,\mathbf{s}}, \ldots, b_{N,\mathbf{s}})$ to maximize the social welfare

$$\sum_{n=1}^{N} \sum_{\mathbf{s} \in \mathcal{S}^{\mathbf{N}}} U_{s_n}(b_{n,\mathbf{s}})\, \mathbb{P}(\mathbf{S} = \mathbf{s}).$$

It is not hard to see that this is equivalent to maximizing $\sum_{n=1}^{N} U_{s_n}(b_{n,\mathbf{s}})$ separately for each $\mathbf{s}$. We let $W_{PO}(\mathcal{M})$ be the resulting optimal social welfare.

**(PG) Ex Post Game.** Here, we are essentially dealing with a collection of decoupled subgames, one for each possible value of $\mathbf{s}$. Under a given state vector $\mathbf{s}$, each user $n$ submits a bid $w_{n,\mathbf{s}}$, which yields a strategy profile $\mathbf{w}_{\mathbf{s}} = (w_{1,\mathbf{s}}, \ldots, w_{N,\mathbf{s}})$. User $n$ obtains a bandwidth $b_n(\mathbf{w}_{\mathbf{s}}) = C w_{n,\mathbf{s}} / \sum_i w_{i,\mathbf{s}}$, if $w_{n,\mathbf{s}} > 0$, and zero bandwidth if $w_{n,\mathbf{s}} = 0$. For that state vector, the payoff of user $n$ is

$$Q_n(\mathbf{w}_{\mathbf{s}}) = -w_{n,\mathbf{s}} + U_{s_n}(b_{n,\mathbf{s}}).$$

Thus, for any given state vector, we are dealing with a subgame identical to the game considered in [45]. The resulting social welfare is

$$\sum_{n=1}^{N} \sum_{\mathbf{s} \in \mathcal{S}^{\mathbf{N}}} U_{s_n}\big(b_n(\mathbf{w}_{\mathbf{s}})\big)\, \mathbb{P}(\mathbf{S} = \mathbf{s}).$$

Assuming existence of Nash equilibria for each one of the subgames, we let $W_{PG}(\mathcal{M})$ be the worst possible social welfare resulting from the Nash equilibria of the subgames.

Out of the six variants we have introduced, the most interesting, and the focus of this chapter, is the Bayesian game (BG), and especially its efficiency compared to the Bayesian Optimal Allocation (that is, the ratio $W_{BG}/W_{BO}$) or to the ex post optimal allocation (that is, the ratio $W_{BG}/W_{PO}$). Nevertheless, it is of interest to make some additional comparisons and observations, including developing some understanding on the social welfare losses that result from having access to less information (the "value of information"). This is the subject of the next section.

## 2.3 The Value of Information - Non-strategic Formulations

In this section, we set aside the game-theoretic aspects and focus on the value of information. In particular, we wish to compare the social welfare $W_{AO}$, $W_{BO}$, and $W_{PO}$ obtained under different informational assumptions. We will show that the gap between the social welfare achieved by ex ante optimal allocation and ex post optimal allocation is on the order of $N$. When utility functions are linear, interim optimal allocation can achieve approximately the same social welfare as ex post optimal allocation.

**Lemma 2.1.** *If all utility functions are concave, there exists an ex post optimal allocation, which allocates an equal amount of bandwidth to all users at the same state.*

**Proof.** The desired results follow the concavity of the utility functions. Indeed, suppose that in a best centralized allocation, user $i$ and $j$, both in state $s$, obtain different amounts of bandwidth, i.e.,

$$b_i \neq b_j$$

We can convert this allocation into another best centralized allocation which allocates equal amount of bandwidth to the users on the same state, because

$$U_s\left(b_i\right)+U_s\left(b_j\right)\leq 2U_s\left(\frac{b_i+b_j}{2}\right). \qquad \square$$

Actually, when all the utility functions are strictly concave, there exists a unique ex post optimal solution which allocates equal bandwidth to the users at the same state.

**Theorem 2.1.** *The worst case efficiency of an ex ante optimal allocation compared to an ex post optimal allocation is* $\Theta(1/N)$*, i.e.,*

1. *For any model* $\mathcal{M}^N$ *with* $N$ *users, we have*

$$\frac{W_{AO}(\mathcal{M}^N)}{W_{PO}(\mathcal{M}^N)}\geq\frac{1}{N}.$$

2. *There exists a positive constant* $c$ *such that for any* $N$*, there exists a model* $\mathcal{M}^N$ *that satisfies*

$$\frac{W_{AO}(\mathcal{M}^N)}{W_{PO}(\mathcal{M}^N)}\leq\frac{c}{N}.$$

*Furthermore, this is true if we restrict to symmetric models.*

**Proof.** 1) First we consider a "best" user $\overline{n}$ such that,

$$\overline{n}\in\arg\max_{n\in\mathcal{N}}\left\{\sum\nolimits_{s\in\mathcal{S}}p_{n,s}U_s(C)\right\}.$$

For any model $\mathcal{M}$, by allocating all the resources to the "best" user, an ex ante optimal allocation could at least achieve $1/N$ of the social welfare achieved at an ex post optimal allocation, because

$$W_{AO}(\mathcal{M}^N)\geq\sum\nolimits_{s\in\mathcal{S}}p_{\overline{n},s}U_s(C)\geq\frac{1}{N}\sum\nolimits_{n\in\mathcal{N}}\sum\nolimits_{s\in\mathcal{S}}p_{n,s}U_s(C)\geq\frac{1}{N}W_{PO}(\mathcal{M}^N).$$

2) Consider a symmetric model where $C=1$ and $S=2$. The utility functions

of all users at different states are $U_1(b) = b/N$ and $U_2(b) = b$. The expected utility achieved by any ex ante allocation is $p_2 + (1 - p_2)/N$. On the other hand, an ex post optimal allocation will allocate all available resources to the user at type 2, unless none exists. The resulting expected utility is

$$\left(1 - (1-p_2)^N\right) + \frac{1}{N}(1-p_2)^N.$$

Suppose that $p_2 = 1/N$. Then,

$$\lim_{N\to\infty} \frac{NW_{AO}(\mathcal{M}^N)}{W_{PO}(\mathcal{M}^N)} = \lim_{N\to\infty} \frac{\left(p_2 + \frac{1}{N}(1-p_2)\right) N}{1 - (1-p_2)^N + \frac{1}{N}(1-p_2)^N} = \lim_{N\to\infty} \frac{p_2 N}{1-(1-p_2)^N}$$
$$= \lim_{N\to\infty} \frac{1}{1-(1-\frac{1}{N})^N} = \frac{1}{1-e^{-1}},$$

which proves the second part of the theorem. □

The result that follows shows that in the absence of strategic behavior, and for the case of linear utility functions, an interim allocation can be arbitrarily close to an ex post optimal allocation. The idea is that the users can use their bids to encode their state, so that the resource allocation mechanism can then act with full information.

**Theorem 2.2.** *For any model $\mathcal{M}$ such that all utility functions are linear, we have*

$$W_{BO}(\mathcal{M}) = W_{PO}(\mathcal{M}).$$

**Proof.** Suppose not. Then there exists a model $\mathcal{M}$ with linear utility functions and a constant $\varepsilon > 0$ such that for any $\mathbf{w} \geq 0$, we have

$$W_{PO}(\mathcal{M}) - \sum_{n=1}^{N} \sum_{\mathbf{s}\in\mathcal{S}^N} U_{s_n}\big(b_{n,\mathbf{s}}(\mathbf{w})\big)\, \mathbb{P}(\mathbf{S} = \mathbf{s}) \geq \varepsilon, \tag{2.6}$$

In this model $\mathcal{M}$, we let $\alpha_s$ be the slope of the utility function $U_s$. Without loss of generality, we assume that $\alpha_1 \leq \alpha_2 \leq \ldots \leq \alpha_S$. Now we construct a strategy profile

**w** such that

$$w_{n,s} = \left( \frac{\alpha_S C N}{\varepsilon} \right)^{s-1}, \quad s = 1, 2, \ldots S, \quad , \forall n. \tag{2.7}$$

For a state vector **s**, we let

$$\overline{s} = \max \{s_1, s_2 \ldots, s_N\}.$$

At the state vector **s**, the social welfare achieved by an ex post optimal allocation is $U_{\overline{s}}(C) = \alpha_{\overline{s}} C$. On the other hand, with the strategy profile **w** defined in (2.7), at the state vector **s**, the social welfare achieved by an interim optimal allocation cannot be less than

$$U_{\overline{s}} \left( \frac{C w_{n,\overline{s}}}{w_{n,\overline{s}} + (N-1) w_{n,\overline{s}-1}} \right) = \frac{\alpha_{\overline{s}} C w_{n,\overline{s}}}{w_{n,\overline{s}} + (N-1) w_{n,\overline{s}-1}},$$

which is a lower bound for the aggregate utility obtained by users at state $\overline{s}$. We further have

$$\begin{aligned} U_{\overline{s}} \left( \frac{C w_{n,\overline{s}}}{w_{n,\overline{s}} + (N-1) w_{n,\overline{s}-1}} \right) &= \frac{\alpha_{\overline{s}} C w_{n,\overline{s}}}{w_{n,\overline{s}} + (N-1) w_{n,\overline{s}-1}} \\ &= \alpha_{\overline{s}} C \left( 1 - \frac{(N-1) w_{n,\overline{s}-1}}{w_{n,\overline{s}} + (N-1) w_{n,\overline{s}-1}} \right) \\ &= \alpha_{\overline{s}} C \left( 1 - \frac{N-1}{\frac{\alpha_S C N}{\varepsilon} + N - 1} \right) \\ &\geq \alpha_{\overline{s}} C \left( 1 - \frac{(N-1)\,\varepsilon}{\alpha_S C N} \right) \geq \alpha_{\overline{s}} C - \varepsilon \end{aligned}$$

where the first and second equalities are trivial, the third follows (2.7), and the last inequality is true because $\alpha_{\overline{s}} \leq \alpha_S$. The above inequality contradicts our hypothesis in (2.6). The desired result follows. □

For the case of nonlinear utility functions, the proof of Theorem 2.2 does not go through: while $w_{n,s}$ can still be used to encode the state of user $n$, the proportional allocation mechanism is not flexible enough to exploit the available information. This suggests that the proportional allocation mechanism is not an appropriate one for this setting: when users are non-strategic (and therefore truth-telling), and when we allow communication to a central decision maker, we should just have the users transmit their states, and let the central decision maker construct an ex post optimal allocation. For this reason, we do not consider interim optimal allocations any further.

## 2.4 Ex Ante and Ex Post Games with Kelly Mechanism

In this section we will apply the results in previous works to ex ante and ex post games involving the Kelly mechanism. We will show that the worst case efficiency of a Nash equilibrium in an ex ante or in an ex post game is 3/4, compared with an ex ante optimal allocation and an ex post optimal allocation, respectively.

### 2.4.1 Ex Ante Game

The ex ante game is essentially the resource allocation game studied in Hajek and Gopalakrishnan (2002) and Johari and Tsitsiklis (2004). Given a strategy profile $\mathbf{w}$, the utility function of user $n$ can be given by

$$\overline{U}_n(w_1, \ldots, w_N) = \sum\nolimits_{\mathbf{s} \in \mathcal{S}^N} U_{s_n}\Big(\frac{Cw_n}{\sum_{i=1}^N w_i}\Big)\mathbb{P}(\mathbf{S} = \mathbf{s}) \tag{2.8}$$

For user $n$, if all the utility functions under different states are concave, strictly increasing, and continuously differentiable, it is not hard to see that the expected utility function $\overline{U}_n(\mathbf{w})$ is also concave, strictly increasing, and continuously differentiable. Following [39], when all users are price-anticipating, there exists a unique Nash equilibrium and according to [45], the efficiency loss at a Nash equilibrium of the ex ante game cannot be more than 25%.

**Theorem 2.3.** *Suppose that for every state $s$, the utility function $U_s(\cdot)$ is concave, strictly increasing, and continuously differentiable. The worst case ratio between the social welfare achieved at a Nash equilibrium of an ex ante game and an optimal ex* *ante allocation is* 3/4*, i.e.,*

$$\inf_{\mathcal{M}} \frac{W_{PG}(\mathcal{M})}{W_{PO}(\mathcal{M})} = \frac{3}{4}.$$

**Proof.** If we regard $\overline{U}_n(\mathbf{w})$ in (2.8) as the utility function of each user $n$, the ex ante game defined in Section 2.2 is the resource allocation game studied in [45]. If the conditions required in this theorem are fulfilled, for each user $n$, it can be shown that $\overline{U}_n(\mathbf{w})$ must be concave, strictly increasing, and continuously differentiable. Thus, the desired result then follows Theorem 3 in [45]. □

### 2.4.2 Ex post Game

According to the main results in [45], the efficiency achieved at any Nash equilibrium cannot be less than 3/4.

**Theorem 2.4.** *Suppose that for every state $s$, the utility function $U_s(\cdot)$ is concave, strictly increasing, and continuously differentiable. The worst case ratio between the social welfare achieved at a Nash equilibrium of an ex post game and by an optimal ex post allocation allocation is* 3/4*, i.e.,*

$$\inf_{\mathcal{M}} \frac{W_{PG}(\mathcal{M})}{W_{PO}(\mathcal{M})} = \frac{3}{4}.$$

**Proof.** Given a state vector $\mathbf{s}$, if we regard $U_{s_n}(w_{1,\mathbf{s}}, \ldots, w_{N,\mathbf{s}})$ as the utility function of each user $n$, the ex post game is equivalent to the resource allocation game studied in [45]. For each user $n$, it can be shown that $U_{s_n}(w_{1,\mathbf{s}}, \ldots, w_{N,\mathbf{s}})$ must be concave, strictly increasing, and continuously differentiable. Based on Theorem 3 in [45], in any model $\mathcal{M}$, at every state vector $\mathbf{s} \in \mathcal{S}^N$, the social welfare achieved at a Nash equilibrium of the ex post game cannot be less than three fourths of that achieved by optimal ex post allocation. Hence, we have

$$\inf_{\mathcal{M}} \frac{W_{PG}(\mathcal{M})}{W_{PO}^{N}(\mathcal{M})} \geq \frac{3}{4}.$$

On the other hand, the bound 3/4 has been shown in [45] to be tight even if $|\mathcal{S}| = 1$. Hence, the bound 3/4 is also tight here. □

## 2.5 Bayesian Game: Existence of Equilibria

In the remainder of the chapter, we focus exclusively on the Bayesian game (BG) associated with the proportional allocation mechanism. In this section, we prove under minimal assumptions the existence of a BNE, and the existence of a symmetric BNE for symmetric games. We continue with some examples (Section 2.6), and with some general results on efficiency losses (Section 2.8). In Section 2.7, we show that under certain assumptions, and as the number of users increases, the social welfare associated with a BNE is asymptotically the same as that associated to an ex post optimal allocation.

Recall from Section 2.2 that a model is symmetric if the distribution of the state vector is invariant under permutations. When the model is symmetric, we also say that the corresponding Bayesian game is symmetric. A strategy profile $\mathbf{w}$ is called *symmetric* if there exist a set of real numbers $\{\hat{w}_s \mid s \in \mathcal{S}\}$ such that

$$w_{n,s} = \hat{w}_s, \qquad \forall\, n \in \mathcal{N}, \quad \forall\, s \in \mathcal{S}.$$

A symmetric BNE is a symmetric strategy profile which is a BNE. The main result of this section follows.

**Theorem 2.5.** *Consider the Bayesian game (BG) defined in Section 2.2. Suppose that for each state $s \in \mathcal{S}$, the utility function $U_s$ is concave, strictly increasing, and continuous on $[0, \infty)$. Then, a BNE is guaranteed to exist. Furthermore, if the game is symmetric, there exists a symmetric BNE.*

Theorem 2.5 can be viewed as an extension to a Bayesian setting of the Nash equilibrium existence result proved by Hajek and Gopalakrishnan [39] for the non-Bayesian setting. In that work, existence was proved by showing that the Nash equilibrium conditions are equivalent to optimality conditions for a related convex optimization problem. Assuming that the utility functions of all users are concave, strictly increasing, and continuous, then the convex optimization problem has a unique optimal solution, which is also a unique Nash equilibrium of the game (see [45] for details). Furthermore, the game is symmetric if and only if all users have the same utility function. In that case, the optimal solution of the convex optimization problem (and therefore, the Nash equilibrium as well) is symmetric. However, for the Bayesian games studied in this chapter, a reformulation in terms of a convex optimization problem is not apparent (and is unlikely to be possible), and the proof is more complicated.

The proof of Theorem 2.5 is given in Appendix A.1. We provide here a high-level summary of the main steps. The main technical difficulty to be overcome is that the payoff is a discontinuous function of the strategy profile.

We first consider a "bounded game" in which we fix some $B > 0$ and restrict the bids $w_{n,s}$ to lie in the compact set $[0, B]$. We then construct a perturbed game in which we fix some $\varepsilon > 0$ and let the bandwidth allocated to a user be $Cw_{n,s_n}/(\varepsilon + \sum_i w_{i,s_i})$; this is equivalent to introducing an additional virtual user who always submits a bid of $\varepsilon > 0$. Using Rosen's existence theorem [80], we obtain that a BNE $\mathbf{w}^\varepsilon$ exists for this perturbed bounded game. Taking the limit as $\varepsilon \to 0$, we consider a limit point of $\mathbf{w}^\varepsilon$, and show that it is a BNE of the unperturbed bounded game. Finally we show that a BNE of the unperturbed bounded game is also a BNE of the original Bayesian game, as long as $B$ is chosen large enough.

For the case of symmetric games, the proof follows the argument given by Nash in [73] to show that $\mathbf{w}^\varepsilon$ can be taken to be symmetric, which by taking the limit as $\varepsilon \to 0$ leads to a symmetric BNE of the original game.

We note that Theorem 2.5 is not true under the weaker assumption that the utility functions are only non-decreasing, even for the deterministic case (single state). While the perturbed game always has a Nash equilibrium, the original game need not have one. We demonstrate this with the example that follows.

**Example 2.1.** Consider a game with $N = 2, C = 3$, and $S = 1$. Let the utility functions of each user be

$$U(b) = \begin{cases} b, & \text{if } 0 \le b \le 1, \\ 1, & \text{if } 1 < b. \end{cases}$$

Fix a small positive constant $\varepsilon$, and consider the perturbed game. It is not hard to verify that having each user submit a bid equal to $\varepsilon$ is a Nash equilibrium. At that equilibrium, each user obtains one unit of bandwidth.

We now argue that the original game does not have a Nash equilibrium. To see this, first note that $(0, 0)$ is not a Nash equilibrium because each user could benefit by giving a small positive bid to obtain the whole bandwidth. Second, a strategy profile such as $(w_{1,1}, 0)$ with $w_{1,1} > 0$ is not a Nash equilibrium because the user who submits the positive bid can reduce her bid without reducing the amount of bandwidth received. Finally, let us consider strategy profiles $(w_{1,1}, w_{2,1})$ with $w_{1,1} > 0$ and $w_{2,1} > 0$. Without loss of generality, suppose that $w_{1,1} \geq w_{2,1} > 0$. Then, user 1 could still obtain one unit of bandwidth (and hence the largest possible utility) by reducing her bid to $w_{2,1}/2$. Thus, such a strategy profile is not a Nash equilibrium either.

We note that the strategy profile $(0, 0)$ is a limit point of the sequence of Nash equilibria of the perturbed games, as $\varepsilon \to 0$. However, for the present example, the constant $d$ in the proof of Theorem 2.5 (cf. Eq. (A.9) in Appendix A.1) is zero, and the proof breaks down. □

## 2.6 Bayesian Game Examples

In this section, we illustrate the nature of the BNE by considering an example in which one of the users can be in either a "good" or a "bad" state. We observe that the bid is a monotonic function of the state's quality, and then proceed to generalize by proving a result to this effect.

**Example 2.2.** We consider a Bayesian game with two users, $N = 2$, and $C = 1$. The state of second user is deterministic, and its utility function is linear, $U_2(b) = b$, with $b > 0$. With probability $p$, user 1 will be at a "good" state $s = 1$, with utility function $U_{1,1}(b) = a_1 b$, where $a_1 > 1$. With probability $1-p$, user 1 will be at a "bad" state $s = 2$, with utility function, $U_{1,2}(b) = a_2 b$, where $0 < a_2 < a_1$. The following conditions are necessary and sufficient conditions for a strategy profile, $(w_{1,1}, w_{1,2}, w_2)$, to be a BNE:

$$\begin{cases} a_1 \dfrac{w_2}{(w_{1,1}+w_2)^2} = 1, & \text{if } w_{1,1} > 0 \\ \dfrac{a_1}{w_2} \leq 1, & \text{if } w_{1,1} = 0 \\ a_2 \dfrac{w_2}{(w_{1,2}+w_2)^2} = 1, & \text{if } w_{1,2} > 0 \\ \dfrac{a_2}{w_2} \leq 1, & \text{if } w_{1,2} = 0 \end{cases},$$

and

$$\begin{cases} p\dfrac{w_{1,1}}{(w_{1,1}+w_2)^2} + (1-p)\dfrac{w_{1,2}}{(w_{1,2}+w_2)^2} = 1, & \text{if } \ w_2 > 0, \\ \dfrac{p}{w_{1,1}} + \dfrac{1-p}{w_{1,2}} \leq 1, & \text{if } \ w_2 = 0. \end{cases}$$

Since $U_2(C) = 1$, we conclude that at a BNE we must have $w_2 < 1$. Since $a_1 > 0$, the condition $a_1/w_2 \leq 1$ cannot hold, and therefore we must have $w_{1,1} > 0$. We also argue that at a BNE, $w_2$ must be positive. Indeed, if $w_2 = 0$, user 1 can always improve its payoff by reducing the value of $w_{1,1}$ to a smaller positive number (while still obtaining the entire bandwidth). This would preclude the existence of an equilibrium with $w_{1,1} > 0$, a contradiction.

The above discussion implies that we can restrict attention to two possible types of BNEs. Either (i) all three bids are positive, or (ii) $w_{1,1} > 0$, $w_{1,2} = 0$, and $w_2 > 0$. Suppose first that $(w_{1,1,}, 0, w_2)$ is a BNE. The conditions give earlier imply that

$$w_{1,1} = \frac{a_1^2 p}{p + a_1}, \qquad w_2 = \frac{a_1 p^2}{(p+a_1)^2}.$$

Let us now consider the case where all three components of a BNE $(w_{1,1,}, w_{1,2}, w_2)$ are positive. The conditions give earlier imply that

$$x_2 \triangleq w_2 + w_{1,2} = \frac{p\sqrt{\dfrac{a_2}{a_1}} + (1-p)}{1 + \dfrac{1-p}{a_2} + \dfrac{p}{a_1}},$$

and

$$w_2 = \frac{(x_2)^2}{a_2}, \quad w_{1,2} = x_2 - w_2, \quad w_{1,1} = \sqrt{\frac{a_1}{a_2}} x_2 - w_2.$$

For either of the two cases, the above conditions show that a BNE of a certain type is uniquely determined. We will now prove uniqueness of a BNE by showing that it is impossible to have two BNEs, one of each type.

Suppose that there exists a BNE in which user 1 bids zero at state 2, i.e., $w_{1,2} = 0$. Then,

$$a_2 \leq w_2 \leq \frac{a_1 p^2}{(p + a_1)^2}.$$

Suppose, to derive a contradiction, that there also exists a BNE $(w_{1,1}, w_{1,2}, w_2)$ in which user 1 submits a positive bid at state 2, i.e., $w_{1,2} > 0$. Since

$$x_2 \geq a_2 \Leftrightarrow \frac{p\sqrt{\frac{a_2}{a_1}} + (1-p)}{1 + \frac{1-p}{a_2} + \frac{p}{a_1}} \Leftrightarrow \frac{p\sqrt{\frac{a_2}{a_1}}}{1 + \frac{p}{a_1}} \geq a_2 \Leftrightarrow \frac{a_1 p^2}{(p + a_1)^2} \geq a_2,$$

we conclude that the existence of a Nash equilibrium where user 1 bids zero at state 2 implies that $x_2 \geq a_2$. Hence, a Nash equilibrium where user 1 submits positive bid at state 2 cannot exist because

$$w_2 = \frac{(x_2)^2}{a_2} \geq x_2 = w_2 + w_{1,2}.$$

We observe that at the unique Nash equilibrium of this game, user 1 always gives a higher bid at the good state 1 than at the bad state 2. Indeed, this is immediate if $w_{1,2} = 0$. If on the other hand $w_{1,2} > 0$, we use the inequality $a_1 > a_2$ to obtain

$$w_{1,2} = x_2 - w_2 < \sqrt{\frac{a_1}{a_2}} x_2 - w_2 = w_{1,1} \qquad \square.$$

We now provide a definition of a state being "better" than another and prove that equilibrium bids are higher at better states. The definition is in terms of marginal utility.

**Definition 2.1.** *Suppose that the utility functions are differentiable, and let $U'$ denote* *the derivative of a function $U$. We say a state $s$ is "better" than $t$, if*

$$U_s'(b) \geq U_t'(b), \qquad \forall\, b \geq 0.$$

**Theorem 2.6.** *Suppose that the states of users (the random variables $S_1, \ldots, S_N$) are independent. Suppose furthermore that the utility functions $U_s$, for $s \in \mathcal{S}$, are concave, strictly increasing, and continuously differentiable. If state $s$ is better than state $t$, and if $p_{n,s}$, $p_{n.t}$ are both positive, then*

$$w_{n,s} \geq w_{n,t}, \qquad \forall\, n \in \mathcal{N},$$

*at every BNE.*

**Proof.** By setting the derivative (with respect to $w_{n,s}$) of the payoff function $Q_{n,s}$ to zero, we note that if $p_{n,s} > 0$ a BNE must satisfy the condition

$$\sum_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s'\left(\frac{w_{n,s}C}{w_{n,s} + \sum_{i \neq n} w_{i,s_i}}\right) \frac{C \sum_{i \neq n} w_{i,s_i}}{\left(w_{n,s} + \sum_{i \neq n} w_{i,s_i}\right)^2} \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n}) = 1, \quad \text{if } w_{n,s} > 0.$$

If $w_{n,t} = 0$ at a BNE, the desired conclusion is immediate. We can therefore assume that $w_{n,t} > 0$. Suppose, to derive a contradiction, that $w_{n,s} < w_{n,t}$, $p_{n,s} > 0$, and $p_{n,t} > 0$. Let us fix the strategy profile $\mathbf{w}_{-n}$ of all users other than $n$. The function

$$\sum_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_t'\left(\frac{w_{n,t}C}{w_{n,t} + \sum_{i \neq n} w_{i,s_i}}\right) \frac{C \sum_{i \neq n} w_{i,s_i}}{\left(w_{n,t} + \sum_{i \neq n} w_{i,s_i}\right)^2} \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n})$$

can be seen to be strictly decreasing in $w_{n,t}$. Since $w_{n,s} < w_{n,t}$, the above necessary conditions imply that

$$\sum_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_t'\left(\frac{w_{n,s}C}{w_{n,s} + \sum_{i \neq n} w_{i,s_i}}\right) \frac{C \sum_{i \neq n} w_{i,s_i}}{\left(w_{n,s} + \sum_{i \neq n} w_{i,s_i}\right)^2} \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n}) > 1.$$

Since state $s$ is better than $t$, we further have

$$\sum_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s' \left( \frac{w_{n,s} C}{w_{n,s} + \sum_{i \neq n} w_{i,s_i}} \right) \frac{C \sum_{i \neq n} w_{i,s_i}}{(w_{n,s} + \sum_{i \neq n} w_{i,s_i})^2} \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n}) > 1,$$

which contradicts the necessary conditions for $w_{n,s}$ being the bid at a BNE. □

The assumption that the states of different users are independent is essential to the proof of Theorem 2.6. If this assumption is relaxed, the result does not hold, as illustrated by our next example.

**Example 2.3.** Consider a game with two users and three states. The utility functions are,

$$U_1(b) = b, \qquad U_2(b) = 1.2b, \qquad U_3(b) = 6b,$$

so that states with a higher index are better. User 1 can be at state 1 or 2; user 2 can be at state 2 or 3. The states of the two users are correlated, as in Table 1.1. Given

Table 2.1:

| Probability | User 2 at state 1 | User 2 at state 3 |
|---|---|---|
| User 1 at state 1 | 0.3 | 0.05 |
| User 1 at state 2 | 0.2 | 0.45 |

the state of user 1, the conditional probability distribution of the state of user 2 is,

$$\mathbb{P}(S_2 = 1 \mid S_1 = 1) = \frac{6}{7}, \ \mathbb{P}(S_2 = 3 \mid S_1 = 1) = \frac{1}{7}, \ \mathbb{P}(S_2 = 1 \mid S_1 = 2) = \frac{4}{13},$$

and

$$\mathbb{P}(S_2 = 3 \mid S_1 = 2) = \frac{9}{13}.$$

Thus, if user 1 is at the "bad" state $s = 1$, there is high probability (6/7) that the other user will also be at the bad state. On the other hand, if user 1 is at the "good" state $s = 2$, there is high probability that user 2 will be at the even better state $s = 3$. This correlation leads to a BNE where user 1 submits a smaller bid at the good state 2 than at the bad state 1. A numerical calculation verifies that the following strategy profile is a BNE:

$$w_{1,1} = 0.2373, \quad w_{1,2} = 0.2204, \quad w_{2,1} = 0.2495, \quad w_{2,2} = 0.9320 \qquad \Box.$$

## 2.7 Bayesian Games with a Large Number of Users

In this section, we carry out an asymptotic analysis, as the number $N$ of users grows, and the capacity also grows proportionally; we will assume, for concreteness, that $C = N$. On the other hand, we will fix the utility functions, $\{U_s(\cdot)\}_{s=1}^{S}$, associated to the different states, independently of $N$. (This corresponds, for example, to a system that serves a few predetermined types of users, as the user population grows.) We will make a number of assumptions that guarantee that no single user can have a sizable impact on the equilibrium unit price of the divisible resource. Thus, the users effectively become price takers, and we obtain a situation similar to the competitive equilibria in the economics literature. It is then natural to expect that in the limit, a BNE is efficient when compared to the Bayesian optimal or even the ex post optimal allocation. Indeed, under rather weak assumptions on the distribution of the state vector, we show that for any two of several variants of the problem, the ratio of their corresponding social welfare converges to 1, as $N$ increases. The four variants under consideration are:

(a) The welfare at a BNE of the Bayesian game.

(b) The welfare at a Bayesian optimal allocation.

(c) The ex post optimal welfare.

(d) The optimal welfare in a new formulation where the actual frequencies of the different states are set deterministically equal to the expected values associated with the original stochastic model.

On the technical side, a key step is provided by the fact that, at a BNE, the bids of different users that happen to be at the same state must be very close (cf. Theorem 2.7); loosely speaking, we can say that a BNE must be close to symmetric, and this is true even without assuming that the original game is symmetric.

Asymptotic efficiency results of this type, in the limit of a large number of users, are not uncommon in the economics literature; see, e.g., [67]. Closer to our subject, an asymptotic efficiency result was established in in Section 2.2 of [44] (Corollary 2.8).

### 2.7.1 Assumptions, preliminaries, and approximate symmetry of BNEs

In this subsection, we introduce certain assumptions, use them to establish an upper and a lower bound (independent of $N$) on the users' bids, and show that a BNE must be approximately symmetric.

**Assumption 2.1.** *There is a constant $D$ (independent of $N$) such that $U_s(b) \leq D$, for all $b \geq 0$, and all $s \in \mathcal{S}$.*

Assumption 2.1 states that a user cannot derive an unbounded amount of utility by increasing its bandwidth allocation, and is often reasonable. As a consequence of this assumption, a user has no incentive to place a very large bid.

**Lemma 2.2.** *If a Bayesian game satisfies Assumption 2.1, if $\mathbf{w}$ is an associated BNE, and if $p_{n,s} > 0$, then $w_{n,s}^N \leq D$.*

**Proof.** If $w_{n,s} > D$, then user $n$ can achieve a higher expected payoff (zero) by submitting a zero bid at state $s$, which contradicts the equilibrium property. □

For any single user to not have a large impact on the resulting price of the resource, we need the bid of that user to not be very large, and we also need the bids of the remaining users to not be very small. The first objective was met by Assumption 2.1 and Lemma 2.2. For the second objective, we could impose an artificial requirement that each user bid at least $\varepsilon$, for some positive constant $\varepsilon$, independent of $N$. An alternative and perhaps more elegant assumption is to assume that the derivative of the utility function at zero is infinite.

**Assumption 2.2.** *For every $s \in \mathcal{S}$, the right derivative of $U_s(\cdot)$ at zero, denoted by $U_s'(0)$, is infinite.*

Note that $U_s'(0)$ is guaranteed to be well defined because of our standing assumption that $U_s(\cdot)$ is concave. Under Assumption 2.2, a user will never submit a bid of zero at a BNE. Furthermore, the lowest conceivable bid corresponds to a situation where all the other users submit the largest possible bid, which is the constant $D$ in Lemma 2.2. By considering such an extreme situation, we obtain a lower bound on the equilibrium bids. This intuition is made precise in the proof of the following lemma, which is given in Appendix A.2.

**Lemma 2.3.** *Suppose that Assumptions 2.1 and 2.2 hold, and that for every $s \in \mathcal{S}$, the utility function $U_s(\cdot)$ is strictly increasing. Then, there exists a positive constant $\overline{a}$, completely determined by the utility functions, such that for any BNE $\mathbf{w}$, any pair of $n$ and $s$ that satisfies $p_{n,s} > 0$, we have $w_{n,s} \geq \overline{a}$.*

By Lemmas 2.2 and 2.3, the bid of any individual user is too small relative to the total to make a difference. Thus, the situation faced by two users that happen to be at the same state is essentially the same. Accordingly, their bids (at a BNE) will be approximately the same. This is the intuition behind the next result, proved in Appendix A.3. In Theorem 2.7 and in subsequent results in this section, we make an additional assumption that the states of different users are independent. This assumption is mostly for convenience and can be relaxed to various forms of weak dependence. We also make the assumption that the utility functions are *strictly* concave. It is unclear whether this assumption can be relaxed.

**Theorem 2.7.** *Suppose that the states of different users are independent. Suppose, furthermore, that for every $s \in \mathcal{S}$, the utility function $U_s(\cdot)$ is strictly increasing, strictly concave, continuously differentiable on $(0, \infty)$, and satisfy Assumptions 2.1 and 2.2. Fix some $\varepsilon > 0$. Then, there exists an integer $N_0$ such that if the number $N$*

*of users is at least $N_0$, and if* $\mathbf{w}$ *is a BNE, then*

$$|w_{n,s} - w_{m,s}| \leq \varepsilon,$$

*for any $s \in \mathcal{S}$ and any pair of users $n$ and $m$ for which $p_{n,s} > 0$ and $p_{m,s} > 0$.*

### 2.7.2 Asymptotic efficiency

We are now ready to proceed to our asymptotic efficiency results. Any discussion of asymptotics needs to refer to a sequence of games, with an increasing number of users; this corresponds to a sequence of models/games, which we will denote by $\mathcal{M}^N$. In the particular asymptotics that we consider, we increase the number of users (and proportionally scale the capacity); we require the user states to be independent, but we allow the distribution of each user's state to be fairly arbitrary; and, crucially, we fix the space of possible states and the utility function associated with each state. When necessary, we will use a superscript $N$ to indicate quantities associated with the $N$-user model $\mathcal{M}^N$. For example, $p_{n,s}^N$ stands for the probability that $S_n = s$ in the $N$-user model.

We will also make one additional assumption, which guarantees that (with high probability) there are enough users present at each state. This assumption introduces a minimal amount of statistical regularity, which we exploit in our proof. We will use the notation

$$p_s^N = \frac{1}{N}\sum_{n=1}^{N} p_{n,s}^N.$$

**Assumption 2.3.** *There exists some $N_0$ and a positive absolute constant $h$ such that*

$$p_s^N \geq h, \qquad \forall\ s \in \mathcal{S}, \quad \forall\ N \geq N_0.$$

Our result compares $W_{BG}(\mathcal{M}^N)/N$, the per user welfare at a BNE, with $W_{PO}(\mathcal{M}^N)/N$, and the ex post optimal (centralized) welfare; see Section 2.2 for the precise definitions. We will show that they are both equal, asymptotically, to $W_{CE}(\mathcal{M}^N)/N$, defined as the optimal value of the objective function in the following certainty equivalent optimization problem:

$$\begin{aligned} \underset{b_1^N,\ldots,b_S^N}{\text{maximize}} \quad & \sum\nolimits_{s=1}^{S} p_s^N U_s(b_s^N) \\ \text{subject to} \quad & \sum\nolimits_{s=1}^{S} p_s^N b_s^N \leq 1, \qquad b_s^N \geq 0. \end{aligned} \tag{2.9}$$

Notice that this is the ex post optimization problem obtained if it were known that the number of users at state $N$ was exactly equal to its expected number, $p_s^N \cdot N$.

**Theorem 2.8.** *Consider a sequence of models/games $\mathcal{M}^N$ in which, for all $N$:*

*(a) $C = N$, for all $N$;*

*(b) the set $\mathcal{S}$ of states is the same for $N$;*

*(b) the user states are independent random variables;*

*(c) the state probabilities $p_{n,s}$ satisfy Assumption 2.3;*

*(d) the utility function $U_s(\cdot)$ associated with each state $s$ does not change with $N$, is strictly increasing, strictly concave, continuously differentiable on $(0,\infty)$, and satisfies Assumptions 2.1-2.2.*

*Then,*

$$\lim_{N\to\infty} \frac{W_{BG}(\mathcal{M}^N)}{N} - \frac{W_{CE}(\mathcal{M}^N)}{N} = \lim_{N\to\infty} \frac{W_{PO}(\mathcal{M}^N)}{N} - \frac{W_{CE}(\mathcal{M}^N)}{N} = 0$$

The proof of Theorem 2.8 is given in Appendix A.4.

As an immediate corollary of Theorem 2.8, we obtain that the asymptotic efficiency loss at a BNE of the Bayesian game, compared to an ex post optimal allocation, is zero. This result incorporates two effects: first, there is no efficiency loss due to the incomplete information pattern imposed by decentralization; second, there is no efficiency loss due to the strategic behavior of the users.

**Corollary 2.1.** *Under the assumptions of Theorem 2.8,*

$$\lim_{N\to\infty} \frac{W_{BG}(\mathcal{M}^N)}{W_{PO}(\mathcal{M}^N)} = 1.$$

**Proof.** Note that

$$\frac{W_{BG}(\mathcal{M}^N)}{W_{PO}(\mathcal{M}^N)} = 1 + \frac{W_{BG}(\mathcal{M}^N) - W_{PO}(\mathcal{M}^N)}{N} \cdot \frac{N}{W_{PO}(\mathcal{M}^N)}. \tag{2.10}$$

From Theorem 2.8, the term $(W_{BG}(\mathcal{M}^N) - W_{PO}(\mathcal{M}^N))/N$ converges to zero. By considering an allocation where each user is given unit bandwidth, we see that $W_{PO}(\mathcal{M}^N) \geq N \cdot \min_{s\in\mathcal{S}} U_s(1)$, or $W_{PO}(\mathcal{M}^N)/N \geq \min_{s\in\mathcal{S}} U_s(1) > 0$, so that the ratio $N/W_{PO}(\mathcal{M}^N)$ is bounded above. It follows that the right-hand side in Eq. (2.10) converges to one. □

The basic idea behind our results is that due to independence and the large number of users, statistical fluctuations are washed out. Thus, each user is faced with an approximately deterministic situation. Furthermore, because of the boundedness of the utility functions, no user has any reason to place a large bid, and therefore no single user can have a significant effect on the overall allocation. Therefore, the Bayesian game becomes approximately the same as a model with perfect information and a large number of small users, a setting for which efficiency is naturally expected.

Let us now discuss the scope of the asymptotic efficiency result. For the deterministic setting considered in [45] an efficiency loss of 25% is possible. However, the examples that demonstrate such an efficiency loss violate several of our assumptions: they involve utility functions that are linear (hence violate our Assumptions 2.1 and 2.2) and also change with $N$; furthermore they involve a single user with "large market power" and $N-1$ "small" users. Such a situation violates our Assumption 2.3, which requires that each user type is a positive fraction of the user population.

It would be interesting to determine the exact extent to which our assumptions are necessary for asymptotic efficiency to hold. For example, we may consider the following cases, and ask whether, in the limit of increasing $N$, the ratios $\lim_{N\to\infty} W_{BG}(\mathcal{M}^N)/W_{BO}(\mathcal{M}^N)$ and $\lim_{N\to\infty} W_{BG}(\mathcal{M}^N)/W_{PO}(\mathcal{M}^N)$ converge to one, are bounded below by a positive number, or can be made arbitrarily small. Some questions of interest are the following:

(a) What happens if we remove Assumptions 2.1 and/or 2.2?

(b) What happens if we allow the utility functions to depend on $N$ while keeping all other assumptions?

(c) What happens if we remove Assumption 2.3, while keeping all other assumptions?

At present, we are only able to assert that if we fix a set of utility functions, which does not depend on $N$, and remove Assumptions 2.1, 2.2, and 2.3, we can essentially replicate the example from [45] and show that $\lim_{N\to\infty} W_{BG}(\mathcal{M}^N)/W_{BO}(\mathcal{M}^N)$ and $\lim_{N\to\infty} W_{BG}(\mathcal{M}^N)/W_{PO}(\mathcal{M}^N)$ can be made as small as 3/4. But several open problems remain.

## 2.8 Efficiency Loss in Bayesian Games

In this section, we will prove some bounds on the price of anarchy for the Bayesian game with arbitrary number of users. First we derive a lower bound that depends on the utility functions. Then for a special case where there are two independent states with linear utility functions, we establish a lower bound on the efficiency of BNEs over arbitrary number of users, utility functions and probability vectors.

**Proposition 2.1.** *Suppose that Assumptions 2.1 and 2.2 hold, and all the utility functions are strictly increasing. Then the efficiency achieved at any Bayesian Nash equilibrium is lower bounded by a utility function based parameter, i.e.,*

$$\frac{W_{BG}(\mathcal{M})}{W_{PO}(\mathcal{M})} \geq \frac{\min_s \left\{ U_s\left(\frac{\overline{a}(\boldsymbol{U})}{D}\right)\right\}}{D}, \quad \forall \mathcal{M},$$

*where $\overline{a}(\boldsymbol{U})$ is a constant only depending on utility functions.*

**Proof.** At any Bayesian Nash equilibrium, in Lemmas 2.2 and 2.3 we have proved that for every pair of user $n$ and state $s$ such that $p_{n,s} > 0$,

$$D \geq w_{n,s}^N \geq \overline{a}(\mathbf{U}), \ \ \forall N \geq 2.$$

Hence, at every state vector $\mathbf{s}$, the utility received by every user $n$ cannot be less than $U_{s_n}(\overline{a}(\mathbf{U})/D)$. On the other hand, at every state vector $\mathbf{s}$, the optimal social welfare cannot be more than $DN$ due to Assumption 2.1. Hence, at any state vector $\mathbf{s} \in \mathfrak{S}$, the efficiency achieved at a Bayesian Nash equilibrium is at least

$$\frac{\min_s \left\{ U_s\left(\frac{\overline{a}(\mathbf{U})}{D}\right)\right\}}{D} W_{PO}(\mathcal{M}),$$

which implies the desired result. $\square$

**Proposition 2.2.** *For a Bayesian game $\mathcal{M}$ with two states ($|\mathcal{S}| = 2$), suppose that the two utility functions are linear with positive slopes, and that the states of users are independent. A symmetric Nash equilibrium, if exists, must achieve a social welfare that is no less than*

$$\left(1 - \frac{1}{\sqrt{3}} + \frac{1}{3\sqrt{3}}\right) W_{PO}(\mathcal{M}) > 0.6 W_{PO}(\mathcal{M}).$$

We put the proof for the proposition on Appendix A.5. Note that the results in the previous proposition is quite conservative. The price of anarchy of Kelly mechanism remains an open question, if some restrictions are removed, for example,

1. the states of users are correlated;

2. some utility functions are not linear;

3. there are more than two available states in the set $\mathcal{S}$.

By the end of the section, we give two examples on the efficiency of symmetric Bayesian Nash equilibria.

**Example 2.4.** *Consider a symmetric Bayesian game with two states, where the states of users are independent. The utility functions are*

$$U_1(b) = b, \;\; U_2(b) = 3b.$$

*For any user n, let* $p_{n,2} = 1 - p_{n,1} = 3/(2N)$*, where* $N$ *is the number of users. According to Theorem 2.5, there must exist a symmetric Nash equilibrium for this symmetric Bayesian game. Let* $(w_1, w_2)$ *be a symmetric Nash equilibrium. From the first-order equilibrium conditions we have*

$$\begin{aligned} &\alpha_1 \sum_{v=0}^{N-1} P(M_1 = v+1 \mid S_n = 1) \frac{vl + (N-1-v)h}{(l + vl + (N-1-v)h)^2} \\ &= \alpha_2 \sum_{v=0}^{N-1} P(M_1 = v \mid S_n = 2) \frac{vl + (N-1-v)h}{(vl + (N-v)h)^2} \\ &= 1, \end{aligned} \tag{2.11}$$

*where* $M_1$ *is a random variable denoting the number of users at state 1. Based on the previous Nash equilibrium condition, through a simple calculation we have*

$$\lim_{N\to\infty} \frac{w_2}{N w_1} = 0.9311,$$

*and*

$$\lim_{N\to\infty} \frac{\widetilde{W}_{BG}(\mathcal{M}^N)}{W_{PO}(\mathcal{M}^N)} = 0.7721,$$

*where* $\widetilde{W}_{BG}(\mathcal{M}^N)$ *is the social welfare achieved at the symmetric Nash equilibrium in the* $N$*-user Bayesian game.*

**Example 2.5.** *Consider a symmetric Bayesian game with two states, where the states of users are correlated. The utility functions are*

$$U_1(b) = b, \;\; U_2(b) = 3b.$$

*The probability that all users are at the bad state is* $1/4$*, i.e.,*

$$P(\boldsymbol{S} = (1, \ldots, 1)) = \frac{1}{4}.$$

*For each user n, the probability that* $S_n = 2$ *and* $S_i = 1$ *for all* $i \neq n$ *is* $3/(4N)$*.*

*Hence, for any user n, we have*

$$P\left(\boldsymbol{S}_{-n} = (1,\ldots,1) \mid S_n = 2\right) = 1,$$

*and*

$$P\left(\boldsymbol{S}_{-n} = (1,\ldots,1) \mid S_n = 1\right) = \frac{\frac{1}{4}}{\frac{1}{4} + \frac{3(N-1)}{4N}} = \frac{N}{N+3(N-1)},$$

*and*

$$P\left(M_1 = N-1 \mid S_n = 1\right) = \frac{3(N-1)}{N+3(N-1)}.$$

*As the number of users increases to infinity, we have*

$$\lim_{N\to\infty} P\left(\boldsymbol{S}_{-n} = (1,\ldots,1)\middle|S_n = 1\right) = \frac{1}{4}, \quad \lim_{N\to\infty} P\left(M_1 = N-1\middle|S_n = 1\right) = \frac{3}{4}.$$

*According to Theorem 2.5, there must exist a symmetric Nash equilibrium for this symmetric Bayesian game. Based on the Nash equilibrium condition in (2.11), through a simple calculation we have*

$$\lim_{N\to\infty} \frac{w_2}{Nw_1} = 1.2751,$$

*and*

$$\lim_{N\to\infty} \frac{\widetilde{W}_{BG}(\mathcal{M}^N)}{W_{PO}^N(\mathcal{M}^N)} = \frac{\frac{1}{4} + \frac{3}{4} \cdot \frac{3 \cdot 1.2751 + 1 \cdot 1}{1.2751 + 1}}{\frac{1}{4} + 3 \cdot \frac{3}{4}} = 0.7363.$$

# Chapter 3

# Efficiency Loss in a Cournot Oligopoly with Convex Market Demand

## 3.1 Introduction

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities, with a focus on the case where the inverse market demand function is convex. Our objectives are to compare the optimal social welfare to: (i) the social welfare at a Cournot equilibrium and (ii) the social welfare achieved when the suppliers collude to maximize the total profit, or, equivalently, when there is a single supplier.

### 3.1.1 Background

In a book on oligopoly theory (see Chapter 2.4 of [31]), Friedman raises two questions on the relation between Cournot equilibria and competitive equilibria. First, "is the Cournot equilibrium close, in some reasonable sense, to the competitive equilibrium?" Furthermore, "will the two equilibria coincide as the number of firms goes to infinity?" The answer to the second question seems to be positive, in general. Indeed, the efficiency properties of Cournot equilibria in economies and markets with a large number of suppliers and/or consumers have been much explored. For the case of a large number of suppliers, it is known that every Cournot equilibrium is approximately a socially optimal competitive equilibrium [34, 77, 75]. Furthermore, the author of [94] derives necessary and/or sufficient conditions on the relative numbers of consumers and suppliers for the efficiency loss associated with every Cournot equilibrium to approach zero, as the number of suppliers increases to infinity.

In more recent work, attention has turned to the efficiency of Cournot equilibria in settings that involve an arbitrary (possibly small) number of suppliers or consumers. The authors of [6] quantify the efficiency loss in Cournot oligopoly models with concave demand functions. However, most of their results focus on the relation between consumer surplus, producer surplus, and the aggregate social welfare achieved at a Cournot equilibrium, rather than on the relation between the social welfare achieved at a Cournot equilibrium and the optimal social welfare.

The concept of efficiency loss is intimately related to the concept of "price of anarchy," advanced by Koutsoupias and Papadimitriou in a seminal paper [54]; it provides a natural measure of the difference between a Cournot equilibrium and a socially optimal competitive equilibrium. In the spirit of [54], we define the efficiency of a Cournot equilibrium as the ratio of its aggregate social welfare to the optimal social welfare. Recent works have reported various efficiency bounds for Cournot oligopoly with affine demand functions. The authors of [52] compare the social welfare and the aggregate profit earned by the suppliers under Cournot competition to the corresponding maximum possible, for the case where suppliers produce multiple differentiated products and demand is an affine function of the price. The authors of [46] establish a 2/3 lower bound on the efficiency of a Cournot equilibrium, when the inverse demand function is affine. They also show that the 2/3 lower bound applies to a monopoly model with general concave demand.

The efficiency loss in a Cournot oligopoly with some specific forms of convex inverse demand functions has received some recent attention. The authors of [22] study the special case of convex inverse demand functions of the form $p(q) = \alpha - \beta q^{\gamma}$, analyze the efficiency loss at a Cournot equilibrium and show that when $\gamma > 0$, the worst case efficiency loss occurs when an efficient supplier has to share the market with infinitely many inefficient suppliers. A class of inverse demand functions that solve a certain differential equation (for example, constant elasticity inverse demand functions belong to this class), is considered in [38], where efficiency lower bounds that depend on equilibrium market shares, the market demand, and the number of suppliers are established.

In this chapter, we study the efficiency loss in a Cournot oligopoly model with general convex demand functions[1]. Convex demand functions, such as the negative exponential and the constant elasticity demand curves, have been widely used in oligopoly analysis and marketing research [11, 27, 92]. In general, a Cournot equilibrium need not exist when the inverse demand function is convex. However, it is well known that a Cournot equilibrium will exist if the inverse demand function is "not too convex" (e.g., if the inverse demand function is concave), in which case the quantities supplied by different suppliers are strategic substitutes [10, 9]. Existence results for Cournot oligopolies for the case of strategic substitutes can be found in [76, 35, 88], and [58]. Note however, that the strategic substitutes condition is not necessary for the existence of Cournot equilibria. For example, using the theory of ordinally supermodular games, it is shown in [5] that the log-concavity of inverse demand functions guarantees the existence of a Cournot equilibrium. In this chapter, we will not address the case of concave inverse demand functions, which appears to be qualitatively different, as will be illustrated by an example in Section 3.3.5.

### 3.1.2 Our contribution

For Cournot oligopolies with convex and nonincreasing demand functions, we establish a lower bound of the form $f(c/d)$ on the efficiency achieved at a Cournot equilibrium. Here, $f$ is a function given in closed form; $c$ is the absolute value of the slope of the inverse demand function at the Cournot equilibrium; and $d$ is the abso-

[1] Since a demand function is generally nonincreasing, the convexity of a demand function implies that the corresponding inverse demand function is also convex.

lute value of the slope of the line that agrees with the inverse demand function at the Cournot equilibrium and at a socially optimal point. For convex and nonincreasing inverse demand functions, we have $c \geq d$; for affine inverse demand functions, we have $c/d = 1$. In the latter case, our efficiency bound is $f(1) = 2/3$, which is consistent with the bound derived in [46]. More generally, the ratio $c/d$ can be viewed as a measure of nonlinearity of the inverse demand function. As the ratio $c/d$ goes to infinity, the lower bound converges to zero and arbitrarily high efficiency losses are possible. The usefulness of this result lies in that it allows us to lower bound the efficiency of Cournot equilibria for a large class of Cournot oligopoly models in terms of qualitative properties of the inverse demand function, without having to restrict to the special case of affine demand functions, and without having to calculate the equilibrium and the social optimum.

An interesting special case of our model arises when $N = 1$, in which case we are dealing with a single, monopolistic, supplier. The previous lower bounds for Cournot equilibria continue to hold. However, by using the additional assumption that $N = 1$, we can hope to obtain sharper (i.e., larger) lower bounds in terms of the same scalar parameter $c/d$. Let us also note that the case $N = 1$ also covers a setting where there are multiple suppliers who choose to collude and coordinate production so as to maximize their total profit.

### 3.1.3 Outline of the chapter

The rest of the chapter is organized as follows. In the next section, we formulate the model and review available results on the existence of Cournot equilibria. In Section 3.3, we provide some mathematical preliminaries on Cournot equilibria that will be useful later, including the fact that efficiency lower bounds can be obtained by restricting to linear cost functions. In Section 3.4, we consider affine inverse demand functions and derive a refined lower bound on the efficiency of Cournot equilibria that depends on a small amount of ex post information. We also show this bound to be tight. In Section 3.5, we consider a more general model, involving convex inverse demand functions. We show that for convex inverse demand functions, and for the purpose of studying the worst case efficiency loss, it suffices to restrict to a special class of piecewise linear inverse demand functions. This leads to the main result of this chapter, a lower bound on the efficiency of Cournot equilibria (Theorem 3.2). Based on this theorem, in Section 3.6 we derive a number of corollaries that provide efficiency lower bounds that can be calculated without detailed information on these equilibria, and apply these results to various commonly encountered convex inverse demand functions. In Section 3.7, we establish a lower bound on the efficiency of monopoly outputs (Theorem 3.3), and show by example that the social welfare at a monopoly output can be higher than that achieved at a Cournot equilibrium (Example 3.8). Finally, in Section 6.9, we make some brief concluding remarks.

## 3.2 Formulation and Background

In this section, we define the Cournot competition model that we study in this chapter. We also review available results on the existence of Cournot equilibria.

We consider a market for a single homogeneous good with inverse demand function $p : [0, \infty) \to [0, \infty)$ and $N$ suppliers. Supplier $n \in \{1, 2, \ldots, N\}$ has a cost function $C_n : [0, \infty) \to [0, \infty)$. Each supplier $n$ chooses a nonnegative real number $x_n$, which is the amount of the good to be supplied by her. The **strategy profile** $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ results in a total supply denoted by $X = \sum_{n=1}^{N} x_n$, and a corresponding market price $p(X)$. The payoff to supplier $n$ is

$$\pi_n(x_n, \mathbf{x}_{-n}) = x_n p(X) - C_n(x_n),$$

where we have used the standard notation $\mathbf{x}_{-n}$ to indicate the vector $\mathbf{x}$ with the component $x_n$ omitted. A strategy profile $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ is a Cournot (or Nash) equilibrium if

$$\pi_n(x_n, \mathbf{x}_{-n}) \geq \pi_n(x, \mathbf{x}_{-n}), \qquad \forall\, x \geq 0, \quad \forall\, n \in \{1, 2, \ldots, N\}.$$

In the sequel, we denote by $f'$ and $f''$ the first and second, respectively, derivatives of a scalar function $f$, if they exist. For a function defined on a domain $[0, Q]$, the derivatives at the endpoints 0 and $Q$ are defined as left and right derivatives, respectively. For points in the interior of the domain, and if the derivative is not guaranteed to exist, we use the notation $\partial_+ f$ and $\partial_- f$ to denote the right and left, respectively, derivatives of $f$; these are guaranteed to exist for convex or concave functions $f$.

### 3.2.1 Existence results

Some results on the existence of Cournot equilibrium are provided in [89], but require the concavity of the inverse demand function. The author of [69] provides an existence result under minimal assumptions on the inverse demand function $p(\cdot)$, but only for the special case where all suppliers have the same cost function $C(\cdot)$ — a rather restrictive assumption. The most relevant result for our purposes is provided in [76], which does not require the suppliers to be identical or the inverse demand functions to be concave.

**Proposition 3.1.** *[76] Suppose that the following conditions hold:*

*(a) The inverse demand function $p(\cdot)$ is continuous.*

*(b) There exists a real number $Q > 0$ such that $p(q) = 0$ for $q \geq Q$. Furthermore, $p(\cdot)$ is twice continuously differentiable and strictly decreasing on $[0, Q)$.*

*(c) For every $q \in [0, Q)$, we have $p'(q) + qp''(q) \leq 0$.*

*(d) The cost functions $C_n(\cdot)$, $n = 1, 2, \ldots, N$, are nondecreasing and lower-semi-continuous.*

*Then, there exists a Cournot equilibrium.*

If the inverse demand function $p(\cdot)$ is convex, the condition (c) in the preceding proposition implies that

$$\frac{\partial^2 \pi_n}{\partial x_n \partial x_m}(X) \leq 0, \qquad \forall\ m \neq n, \quad \forall\ X \in (0, Q),$$

i.e., that the quantities supplied by different suppliers are strategic substitutes. We finally note that the existence of a Cournot equilibrium is proved in [5] in a setting where the strategic substitutes condition does not hold. Instead, this reference assumes that the inverse demand function $p(\cdot)$ is strictly decreasing and log-concave.

## 3.3 Preliminaries on Cournot Equilibria

In this section, we introduce several main assumptions that we will be working with, and some definitions. In Section 3.3.1, we present conditions for a nonnegative vector to be a social optimum or a Cournot equilibrium. Then, in Section 3.3.2, we define the efficiency of a Cournot equilibrium. In Sections 3.3.3 and 3.3.4, we derive some properties of Cournot equilibria that will be useful later, but which may also be of some independent interest. For example, we show that the worst case efficiency occurs when the cost functions are linear.

**Assumption 3.1.** *For any $n$, the cost function $C_n : [0, \infty) \to [0, \infty)$ is convex, continuous, and nondecreasing on $[0, \infty)$, and continuously differentiable on $(0, \infty)$. Furthermore, $C_n(0) = 0$.*

**Assumption 3.2.** *The inverse demand function $p : [0, \infty) \to [0, \infty)$ is continuous, nonnegative, and nonincreasing, with $p(0) > 0$. Its right derivative at $0$ exists and at every $q > 0$, its left and right derivatives also exist.*

Note that we do not yet assume that the inverse demand function is convex. The reason is that some of the results to be derived in this section are valid even in the absence of such a convexity assumption. Note also that some parts of our assumptions are redundant, but are included for easy reference. For example, if $C_n(\cdot)$ is convex and nonnegative, with $C_n(0) = 0$, then it is automatically continuous and nondecreasing.

**Definition 3.1.** *The **optimal social welfare** is the optimal objective value in the* *following optimization problem,*

$$\begin{aligned} &\text{maximize} \quad \int_0^X p(q)\, dq - \sum_{n=1}^{N} C_n(x_n) \\ &\text{subject to} \quad x_n \geq 0, \qquad n = 1, 2, \ldots, N, \end{aligned} \tag{3.1}$$

*where* $X = \sum_{n=1}^{N} x_n$.

In the above definition, $\int_0^X p(q)\, dq$ is the aggregate consumer surplus and $\sum_{n=1}^{N} C_n(x_n)$ is the total cost of the suppliers. The objective function in (3.1) is a measure of the social welfare across the entire economy of consumers and suppliers, the same measure as the one used in [94] and [6].

For a model with a nonincreasing continuous inverse demand function and continuous convex cost functions, the following assumption guarantees the existence of an optimal solution to (3.1), because it essentially restricts the optimization to the compact set of vectors $\mathbf{x}$ for which $x_n \leq R$, for all $n$.

**Assumption 3.3.** *There exists some* $R > 0$ *such that* $p(R) \leq \min_n\{C_n'(0)\}$.

### 3.3.1 Optimality and equilibrium conditions

We observe that under Assumption 3.1 and 3.2, the objective function in (3.1) is concave. Hence, we have the following *necessary and sufficient* conditions for a vector $\mathbf{x}^S$ to achieve the optimal social welfare:

$$\begin{cases} C_n'(x_n^S) = p\left(X^S\right), & \text{if } x_n^S > 0, \\ C_n'(0) \geq p\left(X^S\right), & \text{if } x_n^S = 0, \end{cases} \tag{3.2}$$

where $X^S = \sum_{n=1}^{N} x_n^S$.

The social optimization problem (3.1) may admit multiple optimal solutions. However, as we now show, they must all result in the same price. We note that the differentiability of the cost functions is crucial for this result to hold.

**Proposition 3.2.** *Suppose that Assumptions 3.1 and 3.2 hold. All optimal solutions to (3.1) result in the same price.*

**Proof.** Suppose not, in which case there exist two optimal solutions, $\mathbf{x}^S$ and $\overline{\mathbf{x}}^S$, such that $p(X^S) \neq p(\overline{X}^S)$. Without loss of generality, we assume that $p(X^S) > p(\overline{X}^S)$. Since $p(\cdot)$ is nonincreasing, we must have $X^S < \overline{X}^S$. For all $n$ such that $\overline{x}_n^S > 0$, the optimality conditions (3.2) yield

$$C_n'(\overline{x}_n^S) = p(\overline{X}^S) < p(X^S) \leq C_n'(x_n^S).$$

Using the convexity of the cost functions, we obtain

$$\overline{x}_n^S < x_n^S, \qquad \text{if } \overline{x}_n^S > 0,$$

This contradicts the assumption that $X^S < \overline{X}^S$, and the desired result follows. □

There are similar equilibrium conditions for a strategy profile $\mathbf{x}$. In particular, under Assumptions 3.1 and 3.2, if $\mathbf{x}$ is a Cournot equilibrium, then

$$C_n'(x_n) \leq p\left(X\right) + x_n \cdot \partial_- p\left(X\right), \quad \text{if } x_n > 0, \tag{3.3}$$

$$C_n'(x_n) \geq p\left(X\right) + x_n \cdot \partial_+ p\left(X\right), \tag{3.4}$$

where again $X = \sum_{n=1}^{N} x_n$. Note, however, that in the absence of further assumptions, the payoff of supplier $n$ need not be a concave function of $x_n$ and these conditions are, in general, not sufficient.

We will say that a nonnegative vector $\mathbf{x}$ is a **Cournot candidate** if it satisfies the necessary conditions (3.3)-(3.4). Note that for a given model, the set of Cournot equilibria is a subset of the set of Cournot candidates. Most of the results obtained in this section, including the efficiency lower bound in Proposition 3.6, apply to all Cournot candidates.

For convex inverse demand functions, the necessary conditions (3.3)-(3.4) can be further refined.

**Proposition 3.3.** *Suppose that Assumptions 3.1 and 3.2 hold, and that the inverse demand function* $p(\cdot)$ *is convex. If* $\mathbf{x}$ *is a Cournot candidate with* $X = \sum_{n=1}^{N} x_n > 0$*, then* $p(\cdot)$ *must be differentiable at* $X$*, i.e.,*

$$\partial_{-} p\left(X\right) = \partial_{+} p\left(X\right).$$

**Proof.** Let $\mathbf{x}$ be a Cournot candidate with $X > 0$. The conditions (3.3)-(3.4) applied to some $n$ with $x_n > 0$, imply that

$$p\left(X\right) + x_n \cdot \partial_{-} p\left(X\right) \geq p\left(X\right) + x_n \cdot \partial_{+} p\left(X\right).$$

On the other hand, since $p(\cdot)$ is convex, we have $\partial_{-} p\left(X\right) \leq \partial_{+} p\left(X\right)$. Hence, $\partial_{-} p\left(X\right) = \partial_{+} p\left(X\right)$, as claimed. □

Because of the above proposition, when Assumptions 3.1 and 3.2 hold and the inverse demand function is convex, we have the following necessary (and, by definition, sufficient) conditions for a nonzero vector $\mathbf{x}$ to be a Cournot candidate:

$$\begin{cases} C_n'(x_n) = p\left(X\right) + x_n p'(X), & \text{if } x_n > 0, \\ C_n'(0) \geq p\left(X\right) + x_n p'(X), & \text{if } x_n = 0. \end{cases} \tag{3.5}$$

### 3.3.2 Efficiency of Cournot equilibria

As shown in [30], if $p(0) > \min_n\{C_n'(0)\}$, then the aggregate supply at a Cournot equilibrium is positive; see Proposition 3.4 below for a slight generalization. If on the other hand $p(0) \leq \min_n\{C_n'(0)\}$, then the model is uninteresting, because no supplier has an incentive to produce and the optimal social welfare is zero. This motivates the assumption that follows.

**Assumption 3.4.** *The price at zero supply is larger than the minimum marginal cost of the suppliers, i.e.,*

$$p(0) > \min_n\{C_n'(0)\}.$$

**Proposition 3.4.** *Suppose that Assumptions 3.1, 3.2, and 3.4 hold. If* $\mathbf{x}$ *is a Cournot candidate, then* $X > 0$.

**Proof.** Suppose that $p(0) > \min_n\{C_n'(0)\}$. Then, the vector $\mathbf{x} = (0, \dots, 0)$ violates condition (3.4), and cannot be a Cournot candidate. $\square$

Under Assumption 3.4, at least one supplier has an incentive to choose a positive quantity, which leads us to the next result.

**Proposition 3.5.** *Suppose that Assumptions 3.1-3.4 hold. Then, the social welfare achieved at a Cournot candidate, as well as the optimal social welfare [cf. (3.1)], are positive.*

**Proof.** Using Assumption 3.4, we can choose some $k$ such that $p(0) > C_k'(0)$. The right derivative with respect to $x_k$ of the obejctive function in (3.1), evaluated at $\mathbf{x} = (0, 0, \dots, 0)$, is $p(0) - C_k(0) > 0$. Hence the optimal value of the objective is strictly larger than the zero value obtained when $\mathbf{x} = (0, 0, \dots, 0)$. Thus, the optimal social welfare is positive.

Now consider the social welfare achieved at a Cournot candidate $\mathbf{x} = (x_1, \dots, x_N)$. Because of Assumption 3.4, Proposition 3.4 applies, and we have $X > 0$. For any supplier $n$ such that $x_n > 0$, the necessary conditions (3.3) and the fact that $p(\cdot)$ is nonincreasing imply that $C_n'(x_n) \le p\,(X)$. Suppose that $C_n'(x_n) = p\,(X)$ for every supplier $n$ with $x_n > 0$. Then, the necessary conditions (3.3)-(3.4) imply that $\mathbf{x}$ satisfies the sufficient optimality conditions in (3.2). Thus, $\mathbf{x}$ is socially optimal and the desired result follows.

Suppose now that there exists some supplier $n$ with $x_n > 0$ and $C_n'(x_n) < p\,(X)$. Then,

$$\sum_{n=1}^{N} C_n'(x_n)x_n < Xp\,(X) \le \int_0^X p(q)\,dq, \tag{3.6}$$

where the last inequality holds because the function $p(\cdot)$ is nonincreasing. Since for each $n$, $C_n(\cdot)$ is convex and nondecreasing, with $C_n(0) = 0$, we have

$$\sum_{n=1}^{N} C_n(x_n) \le \sum_{n=1}^{N} C_n'(x_n)x_n < Xp\,(X) \le \int_0^X p(q)\,dq. \tag{3.7}$$

Hence, the social welfare at the Cournot candidate, $\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)$, is positive. $\square$

We now define the efficiency of a Cournot equilibrium as the ratio of the social welfare that it achieves to the optimal social welfare. It is actually convenient to define the efficiency of a general vector $\mathbf{x}$, not necessarily a Cournot equilibrium.

**Definition 3.2.** *Suppose that Assumptions 3.1-3.4 hold. The* ***efficiency*** *of a non-negative vector* $\boldsymbol{x} = (x_1, \ldots, x_N)$ *is defined as*

$$\gamma(\boldsymbol{x}) = \frac{\displaystyle\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\displaystyle\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N C_n(x_n^S)}, \tag{3.8}$$

*where* $\mathbf{x}^S = (x_1^S, \ldots, x_N^S)$ *is an optimal solution of the optimization problem in (3.1) and* $X^S = \sum_{n=1}^N x_n^S$.

We note that $\gamma(\mathbf{x})$ is well defined: because of Assumption 3.4 and Proposition 3.5, the denominator on the right-hand side of (3.8) is guaranteed to be positive. Furthermore, even if there are multiple socially optimal solutions $\mathbf{x}^S$, the value of the denominator is the same for all such $\mathbf{x}^S$. Note that $\gamma(\mathbf{x}) \leq 1$ for every nonnegative vector $\mathbf{x}$. Furthermore, if $\mathbf{x}$ is a Cournot candidate, then $\gamma(\mathbf{x}) > 0$, by Proposition 3.5.

### 3.3.3 Restricting to linear cost functions

In this section, we show that in order to study the worst-case efficiency of Cournot equilibria, it suffices to consider linear cost functions. We first provide a lower bound on $\gamma(\mathbf{x})$ and then proceed to interpret it.

**Proposition 3.6.** *Suppose that Assumptions 3.1-3.4 hold and that* $p(\cdot)$ *is convex. Let* $\mathbf{x}$ *be a Cournot candidate which is not socially optimal, and let* $\alpha_n = C_n'(x_n)$. *Consider a modified model in which we replace the cost function of each supplier* $n$

*by a new function $\overline{C}_n(\cdot)$, defined by*

$$\overline{C}_n(x) = \alpha_n x, \qquad \forall\ x \geq 0.$$

*Then, for the modified model, Assumptions 3.1-3.4 still hold, the vector* $\mathbf{x}$ *is a Cournot candidate, and its efficiency, denoted by* $\overline{\gamma}(\mathbf{x})$*, satisfies* $0 < \overline{\gamma}(\mathbf{x}) \leq \gamma(\mathbf{x})$.

**Proof.** We first observe that the vector $\mathbf{x}$ satisfies the necessary conditions (3.3)-(3.4) for the modified model. Hence, the vector $\mathbf{x}$ is a Cournot candidate for the modified model. It is also not hard to see that Assumptions 3.1 and 3.2 are satisfied by the modified model. Since $\alpha_n \geq C_n'(0)$ for every $n$, Assumption 3.3 also holds in the modified model.

We now show that Assumption 3.4 holds in the modified model, i.e., that $p(0) > \min_n\{\alpha_n\}$. Since the vector $\mathbf{x}$ is a Cournot candidate in the original model, Proposition 3.4 implies that $X > 0$, so that there exists some $n$ for which $x_n > 0$. From the necessary condition (3.3) we have that $\alpha_n \leq p(X)$. Furthermore, if $\alpha_n = p(X)$, then $\partial_- p(X) = 0$, and the convexity of $p(\cdot)$ implies that $\partial_+ p(X) = 0$. Hence, the vector $\mathbf{x}$ satisfies the optimality condition (3.2), and thus is socially optimal in the original model. Under our assumption that $\mathbf{x}$ is not socially optimal in the original model, we conclude that $\alpha_n < p(X)$, which implies that Assumption 3.4 holds in the modified model.

Let $\mathbf{x}^S$ be an optimal solution to (3.1) in the original model. Since $\mathbf{x}^S$ satisfies the optimality conditions in (3.2) for the modified model, it remains a social optimum in the modified model. In the modified model, since Assumptions 3.1-3.4 hold, the efficiency of the vector $\mathbf{x}$ is well defined and given by

$$\overline{\gamma}(\mathbf{x}) = \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N \alpha_n x_n}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \alpha_n x_n^S}. \tag{3.9}$$

Note that the denominator on the right-hand side of (3.9) is the optimal social welfare and the numerator is the social welfare achieved at the Cournot candidate $\mathbf{x}$, in the modified model. Proposition 3.5 implies that both the denominator and the numerator on the right-hand side of (3.9) are positive. In particular, $\overline{\gamma}(\mathbf{x}) > 0$.

Since $C_n(\cdot)$ is convex, we have

$$C_n(x_n^S) - C_n(x_n) - \alpha_n(x_n^S - x_n) \geq 0, \qquad n = 1, \ldots, N.$$

Adding a nonnegative quantity to the denominator cannot increase the ratio and, therefore,

$$\gamma(\mathbf{x}) = \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N C_n(x_n^S)} \geq \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right)} > 0. \tag{3.10}$$

Since $C_n(\cdot)$ is convex and nondecreasing, with $C_n(0) = 0$, we also have

$$\sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n \leq 0. \tag{3.11}$$

Since the right-hand side of (3.10) is in the interval $(0, 1]$, adding the left-hand side of Eq. (3.11) (a nonpositive quantity) to both the numerator and the denominator cannot increase the ratio, as long as the numerator remains nonnegative. The numerator remains indeed nonnegative because it becomes the same as the numerator in the expression (3.9) for $\overline{\gamma}(\mathbf{x})$. We obtain

$$\begin{aligned}
\gamma(\mathbf{x}) &\geq \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n)}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right)} \\
&\geq \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N C_n(x_n) + \left(\sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n\right)}{\int_0^{X^S} p(q)dq - \sum_{n=1}^N \left(\alpha_n(x_n^S - x_n) + C_n(x_n)\right) + \left(\sum_{n=1}^N C_n(x_n) - \sum_{n=1}^N \alpha_n x_n\right)} \\
&= \frac{\int_0^X p(q)\,dq - \sum_{n=1}^N \alpha_n x_n}{\int_0^{X^S} p(q)\,dq - \sum_{n=1}^N \alpha_n x_n^S} \\
&= \overline{\gamma}(\mathbf{x}).
\end{aligned}$$

The desired result follows. $\square$

If $\mathbf{x}$ is a Cournot equilibrium, then it satisfies Eqs. (3.3)-(3.4), and therefore is a Cournot candidate. Hence, Proposition 3.6 applies to all Cournot equilibria that are not socially optimal. We note that if a Cournot candidate $\mathbf{x}$ is socially optimal for the original model, then the optimal social welfare in the modified model could be zero, in which case $\gamma(\mathbf{x}) = 1$, but $\overline{\gamma}(\mathbf{x})$ is undefined; see the example that follows.

**Example 3.1.** Consider a model involving two suppliers ($N = 2$). The cost function of supplier $n$ is $C_n(x) = x^2$, for $n = 1, 2$. The inverse demand function is constant, with $p(q) = 1$ for any $q \geq 0$. It is not hard to see that the vector $(1/2, 1/2)$ is a Cournot candidate, which is also socially optimal. In the modified model, we have $\overline{C}_n(x) = x$, for $n = 1, 2$. The optimal social welfare achieved in the modified model is zero. $\square$

Note that even if $\mathbf{x}$ is a Cournot equilibrium in the original model, it need not be a Cournot equilibrium in the modified model with linear cost functions, as illustrated by our next example. On the other hand, Proposition 3.6 asserts that a Cournot candidate in the original model remains a Cournot candidate in the modified model. Hence, to lower bound the efficiency of a Cournot equilibrium in the original model, it suffices to lower bound the efficiency achieved at a worst Cournot candidate for a modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) restrict to the case of linear cost functions, and study the worst case efficiency over all Cournot candidates.

**Example 3.2.** Consider a model involving only one supplier ($N = 1$). The cost function of the supplier is $C_1(x) = x^2$. The inverse demand function is given by

$$p(q) = \begin{cases} -q + 4, & \text{if } 0 \leq q \leq 4/3, \\ \max\{0, -\frac{1}{5}(q - 4/3) + 8/3\}, & \text{if } 4/3 < q, \end{cases}$$

which is convex and satisfies Assumption 3.2. It can be verified that $x_1 = 1$ maximizes the supplier's profit and thus is a Cournot equilibrium in the original model. In the modified model, $\overline{C}_1(\cdot)$ is linear with a slope of 2; the supplier can maximize its profit at $x_1 = 7/3$. Therefore, in the modified model, $x_1 = 1$ remains a Cournot candidate, but not a Cournot equilibrium. $\square$

### 3.3.4 Other properties of Cournot candidates

In this subsection, we collect a few useful and intuitive properties of Cournot candidates. We show that at a Cournot candidate there are two possibilities: either $p(X) > p(X^S)$ and $X < X^S$, or $p(X) = p(X^S)$ (Proposition 3.7); in the latter case, under the additional assumption that $p(\cdot)$ is convex, a Cournot candidate is socially optimal (Proposition 3.8). In either case, imperfect competition can never result in a price that is less than the socially optimal price.

**Proposition 3.7.** *Suppose that Assumptions 3.1-3.4 hold. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot candidate and an optimal solution to (3.1), respectively. If* $p(X) \neq p(X^S)$*, then* $p(X) > p(X^S)$ *and* $X < X^S$*.*

**Proof.** By Assumption 3.4, $p(0) > \min_n\{C'_n(0)\}$. According to Proposition 3.4, we have $X > 0$. Since $p(\cdot)$ is nonincreasing, the conditions in (3.3) imply that

$$C'_n(x_n) \leq p\,(X)\,, \qquad \text{if } x_n > 0.$$

Suppose that $X \geq X^S$. Since the inverse demand function is nonincreasing and $p(X) \neq p(X^S)$, we have $p(X) < p(X^S)$ and $X > X^S$. For every supplier $n$ with $x_n > 0$, we have

$$C'_n(x_n) \leq p\,(X) < p(X^S) \leq C'_n(x_n^S),$$

where the last inequality follows from (3.2). Since, $C_n(\cdot)$ is convex, the above inequality implies that

$$x_n < x_n^S, \qquad \text{if } x_n > 0,$$

from which we obtain $X < X^S$. Since we had assumed that $X \geq X^S$, we have a contradiction.

The preceding argument establishes that $X < X^S$. Since $p(\cdot)$ is noincreasing and $p(X) \neq p(X^S)$, we must have $p(X) > p(X^S)$. □

For the case where $p(X) = p(X^S)$, Proposition 3.7 does not provide any comparison between $X$ and $X^S$. While one usually has $X < X^S$ (imperfect competition results in lower quantities), it is also possible that $X > X^S$, as in the following example.

**Example 3.3.** Consider a model involving two suppliers ($N = 2$). The cost function of each supplier is linear, with slope equal to 1. The inverse demand function is convex, of the form

$$p(q) = \begin{cases} 2 - q, & \text{if } \ 0 \leq q \leq 1, \\ 1, & \text{if } \ 1 < q. \end{cases}$$

It is not hard to see that any nonnegative vector $\mathbf{x}^S$ that satisfies $x_1^S + x_2^S \geq 1$ is socially optimal; $x_1^S = x_2^S = 1/2$ is one such vector. On the other hand, it can be verified that $x_1 = x_2 = 1$ is a Cournot equilibrium. Hence, in this example, $2 = X > X^S = 1$. □

**Proposition 3.8.** *Suppose that Assumptions 3.1-3.4 hold and that the inverse demand function is convex. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot candidate and an optimal solution to (3.1), respectively. If* $p(X) = p(X^S)$*, then* $p'(X) = 0$ *and* $\gamma(\mathbf{x}) = 1$.

**Proof.** Since Assumption 3.4 holds, Proposition 3.4 implies that $X > 0$. Since $p(\cdot)$ is convex, Proposition 3.3 shows that $p(\cdot)$ is differentiable at $X$ and the necessary conditions in (C.1) are satisfied.

We will now prove that $p'(X) = 0$. Suppose not, in which case we have $p'(X) < 0$. For every $n$ such that $x_n > 0$, from the convexity of $C_n(\cdot)$ and the conditions in (C.1), we have

$$C_n'(0) \leq C_n'(x_n) < p(X) = p(X^S).$$

Then, the social optimality conditions (3.2) imply that $x_n^S > 0$. It follows that

$$C_n'(x_n) < p(X^S) = C_n'(x_n^S),$$

where the last equality follows from the optimality conditions in (3.2). Since $C_n(\cdot)$ is convex, we conclude that $x_n < x_n^S$. Since this is true for every $n$ such that $x_n > 0$, we obtain $X < X^S$. Since the function $p(\cdot)$ is nonincreasing, and we have $p'(X) < 0$

and $X < X^S$, we obtain $p(X) > p(X^S)$, which contradicts the assumption that $p(X) = p(X^S)$. The contradiction shows that $p'(X) = 0$.

Since $p'(X) = 0$ and the Cournot candidate $\mathbf{x}$ satisfies the necessary conditions in (C.1), it also satisfies the optimality conditions in (3.2). Hence, $\mathbf{x}$ is socially optimal and the desired result follows. $\square$

Proposition 3.2 shows that all social optima lead to a unique "socially optimal" price. Combining with Proposition 3.8, we conclude that if $p(\cdot)$ is convex, a Cournot candidate is socially optimal if and only if it results in the socially optimal price.

### 3.3.5 Concave inverse demand functions

In this section, we argue that the case of concave inverse demand functions is fundamentally different. For this reason, the study of the concave case would require a very different line of analysis, and is not considered further in this chapter.

According to Proposition 3.8, if the inverse demand function is convex and if the price at a Cournot equilibrium equals the price at a socially optimal point, then the Cournot equilibrium is socially optimal. For nonconvex inverse demand functions, this is not necessarily true: a socially optimal price can be associated with a socially suboptimal Cournot equilibrium, as demonstrated by the following example.

**Example 3.4.** Consider a model involving two suppliers ($N = 2$), with $C_1(x) = x$ and $C_2(x) = x^2$. The inverse demand function is concave on the interval where it is positive, of the form

$$p(q) = \begin{cases} 1, & \text{if } 0 \le q \le 1, \\ \max\{0, -M(q-1)+1\}, & \text{if } 1 < q, \end{cases}$$

where $M > 2$. It is not hard to see that the vector $(0.5, 0.5)$ satisfies the optimality conditions in (3.2), and is therefore socially optimal. We now argue that $(1/M, 1 - 1/M)$ is a Cournot equilibrium. Given the action $x_2 = 1/M$ of supplier 2, any action on the interval $[0, 1 - 1/M]$ is a best response for supplier 1. Given the action

$x_1 = 1 - (1/M)$ of supplier 1, a simple calculation shows that

$$\arg \max_{x \in [0,\infty)} \left\{ x \cdot p(x + 1 - 1/M) - x^2 \right\} = 1/M.$$

Hence, $(1/M, 1 - 1/M)$ is a Cournot equilibrium. Note that $X = X^S = 1$. However, the optimal social welfare is 0.25, while the social welfare achieved at the Cournot equilibrium is $1/M - 1/M^2$. By considering arbitrarily large $M$, the corresponding efficiency can be made arbitrarily small. □

The preceding example shows that arbitrarily high efficiency losses are possible, even if $X = X^S$. The possibility of inefficient allocations even when the price is the correct one opens up the possibility of substantial inefficiencies that are hard to bound.

## 3.4 Affine Inverse Demand Functions

We now turn our attention to the special case of affine inverse demand functions. It is already known from [46] that 2/3 is a tight lower bound on the efficiency of Cournot equilibria. In this section, we refine this result by providing a tighter lower bound, based on a small amount of ex post information about a Cournot equilibrium.

Throughout this section, we assume an inverse demand function of the form

$$p(q) = \begin{cases} b - aq, & \text{if } 0 \leq q \leq b/a, \\ 0, & \text{if } b/a < q, \end{cases} \tag{3.12}$$

where $a$ and $b$ are positive constants.[2] Under the assumption of convex costs (Assumption 3.1), a Cournot equilibrium is guaranteed to exist, by Proposition 3.1.

The main result of this section follows.

**Theorem 3.1.** *Suppose that Assumption 3.1 holds (convex cost functions), and that*

[2]Note that the model considered here is slightly different from that in [46]. In that work, the inverse demand function is literally affine and approaches minus infinity as the total supply increases to infinity. However, as remarked in that paper (p. 20), this difference does not affect the results.

*the inverse demand function is affine, of the form* (3.12).[3] *Suppose also that* $b > \min_n\{C_n'(0)\}$ *(Assumption 3.4). Let* $\mathbf{x}$ *be a Cournot equilibrium, and let* $\alpha_n = C_n'(x_n)$*. Let also*

$$\beta = \frac{aX}{b - \min_n\{\alpha_n\}},$$

*If* $X > b/a$*, then* $\mathbf{x}$ *is socially optimal. Otherwise:*

*(a) We have* $1/2 \le \beta < 1$.

*(b) The efficiency of* $\mathbf{x}$ *satisfies,*

$$\gamma(\mathbf{x}) \ge g(\beta) = 3\beta^2 - 4\beta + 2.$$

*(c) The bound in part (b) is tight. That is, for every* $\beta \in [1/2, 1)$ *and every* $\epsilon > 0$*, there exists a model with a Cournot equilibrium whose efficiency is no more than* $g(\beta)+\epsilon$*.*

*(d) The function* $g(\beta)$ *is minimized at* $\beta = 2/3$ *and the worst case efficiency is* 2/3*.*

The proof of Theorem 3.1 is given in Appendix B.1. The lower bound $g(\beta)$ is illustrated in Fig. 3-1. Consider a Cournot equilibrium such that $X \le b/a$. For the special case where all the cost functions are linear, of the form $C_n(x_n) = \alpha_n$, Theorem 3.1 has an interesting interpretation. We first note that a socially optimal solution is obtained when the price $b - aq$ equals the marginal cost of a "best" supplier, namely $\min_n \alpha_n$. In particular, $X^S = (b - \min_n\{\alpha_n\})/a$, and $\beta = X/X^S$. Since $p'(X) = -a < 0$, Proposition 3.8 implies that $p(X) \ne p(X^S)$, and Proposition 3.7 implies that $\beta < 1$. Theorem 3.1 further states that $\beta \ge 1/2$. i.e., that the total supply at a Cournot equilibrium is at least half of the socially optimal supply. Clearly, if $\beta$ is close to 1 we expect the efficiency loss due to the difference $X^S - X$ to be small. However, efficiency losses may also arise if the total supply at a Cournot equilibrium is not provided by the most efficient suppliers. (As shown in Example 3.4, in the nonconvex case this effect can be substantial.) Our result shows that, for the convex

[3] Note that Assumptions 3.2 and 3.3 hold automatically.

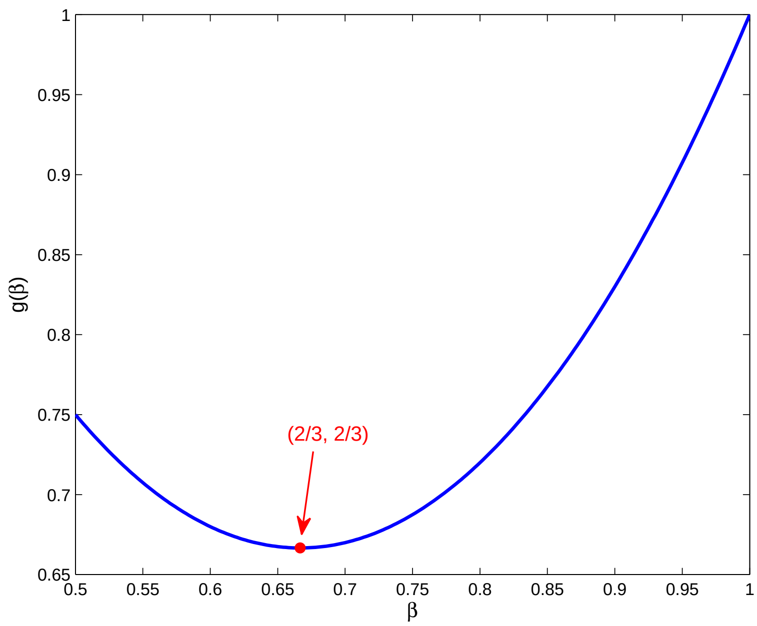


Figure 3-1: A tight lower bound on the efficiency of Cournot equilibria for the case of affine inverse demand functions.

case, $\beta$ can be used to lower bound the total efficiency loss due to this second factor as well; when $\beta$ is close to 1, the efficiency indeed remains close to 1. (This is in sharp contrast to the nonconvex case where we can have $X = X^S$ but large efficiency losses.) Somewhat surprisingly, the worst case efficiency also tends to be somewhat better for low $\beta$, that is, when $\beta$ approaches $1/2$, as compared to intermediate values ($\beta \approx 2/3$).

## 3.5 Convex Inverse Demand Functions

In this section we study the efficiency of Cournot equilibria under more general assumptions. Instead of restricting the inverse demand function to be affine, we will only assume that it is convex. A Cournot equilibrium need not exist in general, but it does exist under some conditions (cf. Section 3.2.1). Our results apply whenever a Cournot equilibrium happens to exist.

We first show that a lower bound on the efficiency of a Cournot equilibrium can be established by calculating its efficiency in another model with a piecewise linear inverse demand function. Then, in Theorem 3.2, we establish a lower bound on the efficiency of Cournot equilibria, as a function of the ratio of the slope of the inverse demand function at the Cournot equilibrium to the average slope of the inverse demand function between the Cournot equilibrium and a socially optimal point. Then, in Section 3.6, we will apply Theorem 3.2 to specific convex inverse demand functions. Recall our definition of a Cournot candidate as a vector $\mathbf{x}$ that satisfies the necessary conditions (3.3)-(3.4).

**Proposition 3.9.** *Suppose that Assumptions 3.1-3.4 hold, and that the inverse demand function is convex. Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and an optimal solution to (3.1), respectively. Assume that $p(X) \neq p(X^S)$ and let[4] $c = |p'(X)|$. Consider a modified model in which we replace the inverse demand function by a new function $p^0(\cdot)$, defined by*

$$p^0(q) = \begin{cases} -c(q-X) + p(X), & \text{if} \quad 0 \leq q \leq X, \\ \max\left\{0, \dfrac{p(X^S) - p(X)}{X^S - X}(q - X) + p(X)\right\}, & \text{if} \quad X < q. \end{cases} \tag{3.13}$$

*Then, for the modified model, with inverse demand function $p^0(\cdot)$, the vector $\mathbf{x}^S$ remains socially optimal, and the efficiency of $\mathbf{x}$, denoted by $\gamma^0(\mathbf{x})$, satisfies*

$$\gamma^0(\mathbf{x}) \leq \gamma(\mathbf{x}).$$

**Proof.** Since $p(X) \neq p(X^S)$, Proposition 3.7 implies that $X < X^S$, so that $p^0(\cdot)$ is well defined. Since the necessary and sufficient optimality conditions in (3.2) only involve the value of the inverse demand function at $X^S$, which has been unchanged, the vector $\mathbf{x}^S$ remains socially optimal for the modified model. Let

$$A = \int_0^X p^0(q)\, dq, \; B = \int_X^{X^S} p(q)\, dq,$$

[4]According to Proposition 3.3, $p'(X)$ must exist.

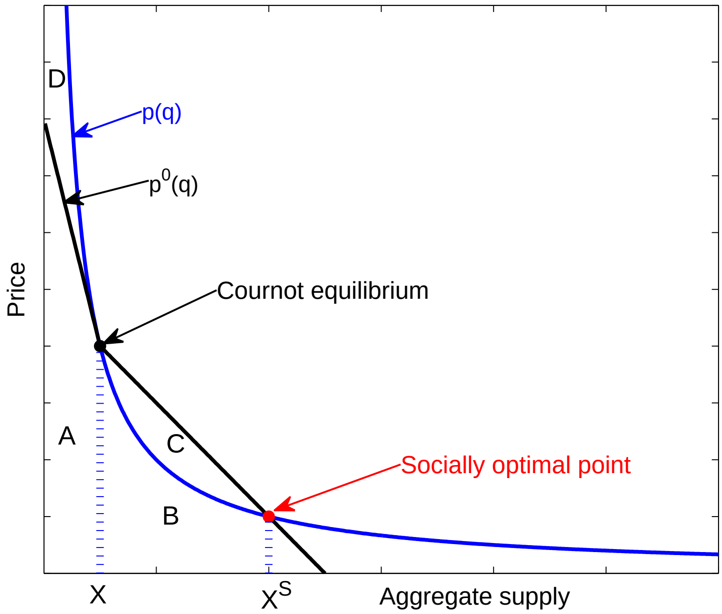


Figure 3-2: The efficiency of a Cournot equilibrium cannot increase if we replace the inverse demand function by the piecewise linear function $p^0(\cdot)$. The function $p^0(\cdot)$ is tangent to the inverse demand function $p(\cdot)$ at the equilibrium point, and connects the Cournot equilibrium point with the socially optimal point.

and

$$C = \int_X^{X^S} (p^0(q) - p(q))\, dq, \;\; D = \int_0^X (p(q) - p^0(q))\, dq.$$

See Fig. 3-2 for an illustration of $p(\cdot)$ and a graphical interpretation of $A$, $B$, $C$, $D$. Note that since $p(\cdot)$ is convex, we have $C \geq 0$ and $D \geq 0$. The efficiency of $\mathbf{x}$ in the original model with inverse demand function $p(\cdot)$, is

$$0 < \gamma(\mathbf{x}) = \frac{A + D - \sum_{n=1}^N C_n(x_n)}{A + B + D - \sum_{n=1}^N C_n(x_n^S)} \leq 1,$$

where the first inequality is true because the social welfare achieved at any Cournot candidate is positive (Proposition 3.5). The efficiency of $\mathbf{x}$ in the modified model is

$$\gamma^0(\mathbf{x}) = \frac{A - \sum_{n=1}^N C_n(x_n)}{A + B + C - \sum_{n=1}^N C_n(x_n^S)}.$$

Note that the denominators in the above formulas for $\gamma(\mathbf{x})$ and $\gamma^0(\mathbf{x})$ are all positive, by Proposition 3.5. If $A - \sum_{n=1}^N C_n(x_n) \leq 0$, then $\gamma^0(\mathbf{x}) \leq 0$ and the result is clearly true. We can therefore assume that $A - \sum_{n=1}^N C_n(x_n) > 0$. We then have

$$\begin{aligned} 0 \;\; < \;\; \gamma^0(\mathbf{x}) = \frac{A - \sum\limits_{n=1}^N C_n(x_n)}{A + B + C - \sum\limits_{n=1}^N C_n(x_n^S)} &\leq \frac{A + D - \sum\limits_{n=1}^N C_n(x_n)}{A + B + C + D - \sum\limits_{n=1}^N C_n(x_n^S)} \\ \leq \;\; \frac{A + D - \sum\limits_{n=1}^N C_n(x_n)}{A + B + D - \sum\limits_{n=1}^N C_n(x_n^S)} = \gamma(\mathbf{x}) \leq 1, & \end{aligned}$$

which proves the desired result. □

Note that unless $p(\cdot)$ happens to be linear on the interval $[X, X^S]$, the function $p^0(\cdot)$ is not differentiable at $X$ and, according to Proposition 3.3, $\mathbf{x}$ cannot be a Cournot candidate for the modified model. Nevertheless, $p^0(\cdot)$ can still be used to derive a lower bound on the efficiency of Cournot candidates in the original model, as will be seen in the proof of Theorem 3.2, which is our main result.

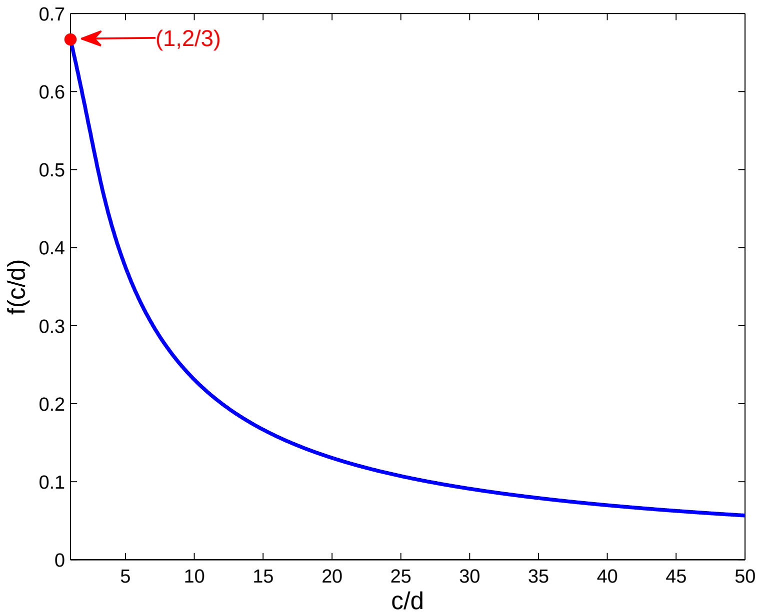


Figure 3-3: Plot of the lower bound on the efficiency of a Cournot equilibrium in a Cournot oligopoly with convex inverse demand functions, as a function of the ratio $c/d$ (cf. Theorem 3.2).

**Theorem 3.2.** *Suppose that Assumptions 3.1-3.4 hold, and that the inverse demand function is convex. Let* $\mathbf{x}$ *and* $\mathbf{x}^S$ *be a Cournot candidate and a solution to (3.1), respectively. Then, the following hold.*

*(a) If* $p(X) = p(X^S)$*, then* $\gamma(\mathbf{x}) = 1$*.*

*(b) If* $p(X) \neq p(X^S)$*, let* $c = |p'(X)|$*,* $d = |(p(X^S) - p(X))/(X^S - X)|$*, and* $\overline{c} = c/d$*. We have* $\overline{c} \geq 1$ *and*

$$1 > \gamma(\mathbf{x}) \geq f(\overline{c}) = \frac{\phi^2 + 2}{\phi^2 + 2\phi + \overline{c}}, \tag{3.14}$$

*where*

$$\phi = \max\left\{ \frac{2 - \overline{c} + \sqrt{\overline{c}^2 - 4\overline{c} + 12}}{2}, 1 \right\}.$$

We do not know whether the lower bound in Theorem 3.2 is tight. The difficulty in proving tightness is due to the fact that the vector $\mathbf{x}$ need not be a Cournot equilibrium in the modified model.

We give the proof of Theorem 3.2 in Appendix B.2. Fig. 3-3 shows a plot of the lower bound $f(\overline{c})$ on the efficiency of Cournot equilibria, as a function of $\overline{c} = c/d$. If $p(\cdot)$ is affine, then $\overline{c} = c/d = 1$. From (3.14), it can be verified that $f(1) = 2/3$, which agrees with the lower bound in [46] for the affine case. We note that the lower bound $f(\overline{c})$ is monotonically decreasing in $\overline{c}$, over the domain $[1, \infty)$. When $\overline{c} \in [1, 3)$, $\phi$ is at least 1, and monotonically decreasing in $\overline{c}$. When $\overline{c} \geq 3$, $\phi = 1$.

## 3.6 Applications

For a given inverse demand function $p(\cdot)$, the lower bound derived in Theorem 3.2 requires some knowledge on the Cournot candidate and the social optimum, namely, the aggregate supplies $X$ and $X^S$. Even so, for a large class of inverse demand functions, we can apply Theorem 3.2 to establish lower bounds on the efficiency of Cournot equilibria that do not require knowledge of $X$ and $X^S$. With additional information on the suppliers' cost functions, the lower bounds can be further refined. At the end of this section, we apply our results to calculate nontrivial quantitative efficiency bounds for various convex inverse demand functions that have been considered in the economics literature.

**Corollary 3.1.** *Suppose that Assumptions 3.1-3.4 hold and that the inverse demand function is convex. Suppose also that $p(Q) = 0$ for some $Q > 0$, and that the ratio, $\mu = \partial_+ p(0)/\partial_- p(Q)$, is finite. Then, the efficiency of a Cournot candidate is at least $f(\mu)$.*

**Proof.** Let $\mathbf{x}$ be a Cournot candidate. Since the inverse demand function is convex, we have that $\mu \geq 1$. If $X > Q$, then $p(X) = p'(X) = 0$. The necessary conditions (3.3)-(3.4) imply the optimality condition in (3.2), and thus $\gamma(\mathbf{x}) = 1 > f(\mu)$.

Now consider the case $X \leq Q$. If $p(X) = p(X^S)$ for some social optimum $\mathbf{x}^S$, then Proposition 3.8 implies that $\gamma(\mathbf{x}) = 1 > f(\mu)$. Otherwise, for any social optimum $\mathbf{x}^S$, we have that $\overline{c} = c/d \leq \mu$. Theorem 3.2 shows that the efficiency of every Cournot candidate cannot be less than $f(\overline{c})$. The desired result then follows from the fact that $f(\overline{c})$ is decreasing in $\overline{c}$. □

For convex inverse demand functions, e.g., for negative exponential demand, with

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad 0 < \alpha, \qquad 0 < \beta, \qquad 0 \leq q,$$

Corollary 3.1 does not apply, because the left derivative of $p(\cdot)$ at 0 is infinite. This motivates us to refine the lower bound in Corollary 3.1. By using a small amount of additional information on the cost functions, we can derive an upper bound on the total supply at a social optimum, as well as a lower bound on the total supply at a Cournot equilibrium, to strengthen Corollary 3.1.

**Corollary 3.2.** *Suppose that Assumptions 3.1-3.4 hold and that $p(\cdot)$ is convex. Let*[5]

$$s = \inf\{q \mid p(q) = \min_n C_n'(0)\}, \quad t = \inf\left\{q \;\Big|\; \min_n C_n'(q) \geq p(q) + q\partial_+ p(q)\right\}. \quad (3.15)$$

*If $\partial_- p(s) < 0$, then the efficiency of a Cournot candidate is at least $f\left(\partial_+ p(t)/\partial_- p(s)\right)$.*

**Proof.** If there exists a "best" supplier $n$ such that $C_n'(x) \leq C_m'(x)$, for any other supplier $m$ and any $x > 0$, then the parameters $s$ and $t$ depend only on $p(\cdot)$ and $C_n'(\cdot)$.

Let $\mathbf{x}$ and $\mathbf{x}^S$ be a Cournot candidate and a social optimum, respectively. If $p(X) = p(X^S)$, for some social optimum $\mathbf{x}^S$, then $\gamma(\mathbf{x}) = 1$ and the desired result holds trivially. Now suppose that $p(X) \neq p(X^S)$. We first derive an upper bound on the aggregate supply at a social optimum, and then establish a lower bound on the aggregate supply at a Cournot candidate. The desired results will follow from the fact that the function $f(\cdot)$ is strictly decreasing.

*Step 1: There exists a social optimum with an aggregate supply no more than $s$.*

According to Proposition 3.5 we have $X^S > 0$ and there exists a supplier $n$ such that $x_n^S > 0$. From the optimality conditions (3.2) we have $p(X^S) = C_n'(x_n^S)$, which implies that $p(X^S) \geq C_n'(0)$, due to the convexity of the cost functions. We conclude that

$$p(X^S) = C_n'(x_n^S) \geq C_n'(0) \geq \min_n C_n'(0). \quad (3.16)$$

[5]Under Assumption 3.3, the existence of the real numbers defined in (3.15) is guaranteed.

If $p(X^S) > \min_n C_n'(0)$, then from the definition of $s$ in (3.15), and the assumption that $p(\cdot)$ is nonincreasing, we have that $X^S < s$.

If $p(X^S) = \min_n C_n'(0)$, by (3.16) we know that for any $n$ such that $x_n^S > 0$, we must have $C_n'(x_n^S) = C_n'(0) = p(X^S)$. Since $C_n(\cdot)$ is convex, we conclude that $C_n(\cdot)$ is actually linear on the interval $[0, x_n^S]$. We now argue that there exists a social optimum $\mathbf{x}^S$ such that $X^S \leq s$. If $X^S \leq s$, then we are done. Otherwise, we have $X^S > s$. Let $\mathcal{N}$ be the set of the indices of suppliers who produce a positive quantity at $\mathbf{x}^S$. Since $p(\cdot)$ is nonincreasing, and $p(s) = \min_n C_n'(0) = p(X^S)$, we know that for any $q \in [s, X^S]$, $p(q) = C_n'(0)$ for every $n \in \mathcal{N}$. Combing with the fact that for each supplier $n$ in the set $\mathcal{N}$, $C_n(\cdot)$ is linear on the interval $[0, x_n^S]$, we have

$$\int_s^{X^S} p(q)\, dq = (X^S - s) C_n'(x), \qquad \forall n \in \mathcal{N}, \quad \forall x \in [0, x_n^S),$$

from which we conclude that the vector, $(s/X^S) \cdot \mathbf{x}^S$, yields the same social welfare as $\mathbf{x}^S$, and thus is socially optimal. Note that the aggregate supply at $(s/X^S) \cdot \mathbf{x}^S$ is $s$.

If $p(X) = p(X^S)$, then $\gamma(\mathbf{x}) = 1$ and the desired result holds trivially. Otherwise, since $p(\cdot)$ is nonincreasing and convex, we have

$$|\partial_- p(s)| \leq \left| (p(X^S) - p(X))/(X^S - X) \right| = d. \tag{3.17}$$

*Step 2: The aggregate supply at a Cournot candidate* $\mathbf{x}$ *is at least* $t$.

Since $p(\cdot)$ is convex, $\mathbf{x}$ satisfies the necessary conditions in (C.1). Therefore,

$$C_n'(x_n) \geq p(X) + x_n p'(X), \qquad \forall n. \tag{3.18}$$

Since $X \geq x_n$, we have

$$C_n'(x_n) \leq C_n'(X), \qquad X p'(X) \leq x_n p'(X), \tag{3.19}$$

where the first inequality follows from the convexity of the cost functions, and the second one is true because $p'(X) \leq 0$. Combing (4.16) and (3.19), we have

$$C_n'(X) \geq p(X) + Xp'(X), \qquad \forall n,$$

which implies that $X \geq t$. Since $p(\cdot)$ is nonincreasing and convex, we have

$$c = |p'(X)| \leq |\partial_+ p(t)| . \tag{3.20}$$

Since $\partial_- p(s) < 0$, Eqs. (3.17) and (3.20) yield

$$\bar{c} = c/d \leq \partial_+ p(t)/\partial_- p(s).$$

The desired result follows from Theorem 3.2, and the fact that $f(\cdot)$ is strictly decreasing. $\square$

We now apply Corollary 3.2 to three examples.

**Example 3.5.** Suppose that Assumptions 3.1, 3.3, and 3.4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. Eq. (6) in [11])

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad 0 < q, \tag{3.21}$$

where $\alpha$ and $\beta$ are positive constants. Note that Corollary 3.1 does not apply, because the left derivative of $p(\cdot)$ at 0 is infinite.[6] Since

$$p'(q) + qp''(q) = \frac{-\beta}{q} + \frac{q\beta}{q^2} = 0, \qquad \forall q \in (0, \exp(\alpha/\beta)),$$

Proposition 3.1 implies that there exists a Cournot equilibrium. Through a simple

[6] In fact, $p(0)$ is undefined. This turns out to not be an issue: for a small enough $\epsilon > 0$, we can guarantee that no supplier chooses a quantity below $\epsilon$. Furthermore, $\lim_{\epsilon \downarrow 0} \int_0^{\epsilon} p(q)\, dq = 0$. For this reason, the details of the inverse demand function in the vicinity of zero are immaterial as far as the chosen quantities or the resulting social welfare are concerned.

calculation we obtain

$$s = \exp\left(\frac{\alpha - c}{\beta}\right), \qquad t = \exp\left(\frac{\alpha - \beta - c}{\beta}\right).$$

From Corollary 3.2 we obtain that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{\exp\left((\alpha - c)/\beta\right)}{\exp\left((\alpha - \beta - c)/\beta\right)}\right) = f\left(\exp\left(1\right)\right) \geq 0.5237. \tag{3.22}$$

Now we argue that the efficiency lower bound (3.22) holds even without the assumption that there is a best supplier associated with a linear cost function. From Proposition 3.6, the efficiency of any Cournot equilibrium $\mathbf{x}$ will not increase if the cost function of each supplier $n$ is replaced by

$$\overline{C}_n(x) = C_n'(x_n)x, \qquad \forall x \geq 0.$$

Let $c = \min_n\{C_n'(x_n)\}$. Since the efficiency lower bound in (3.22) holds for the modified model with linear cost functions, it applies whenever the inverse demand function is of the form (3.21). $\square$

**Example 3.6.** Suppose that Assumptions 3.1, 3.3, and 3.4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. Eq. (5) in [11])

$$p(q) = \max\{\alpha - \beta q^{\delta}, 0\}, \qquad 0 < \delta \leq 1, \tag{3.23}$$

where $\alpha$ and $\beta$ are positive constants. Note that if $\delta = 1$, then $p(\cdot)$ is affine; if $0 < \delta \leq 1$, then $p(\cdot)$ is convex. Assumption 3.4 implies that $\alpha > \chi$. Since

$$p'(q) + qp''(q) = -\beta\delta q^{\delta-1} - \beta\delta(\delta - 1)q^{\delta-1} = -\beta\delta^2 q^{\delta-1} \leq 0, \quad 0 \leq q \leq \left(\frac{\alpha}{\beta}\right)^{1/\delta},$$

Proposition 3.1 implies that there exists a Cournot equilibrium. Through a simple calculation we have

$$s = \left(\frac{\alpha - c}{\beta}\right)^{1/\delta}, \quad t = \left(\frac{\alpha - c}{\beta(\delta+1)}\right)^{1/\delta}.$$

From Corollary 3.2 we know that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{-\beta\delta t^{\delta-1}}{-\beta\delta s^{\delta-1}}\right) = f\left((\delta+1)^{\frac{1-\delta}{\delta}}\right).$$

Using the argument in Example 3.5, we conclude that this lower bound also applies to the case of general convex cost functions. □

As we will see in the following example, it is sometimes hard to find a closed form expression for the real number $t$. In such cases, since $s$ is an upper bound for the aggregate supply at a social optimum (cf. the proof of Corollary 3.2), Corollary 3.2 implies that the efficiency of a Cournot candidate is at least $f\left(\partial_+ p(0)/\partial_- p(s)\right)$. Furthermore, in terms of the aggregate supply at a Cournot equilibrium $X$, we know that $\gamma(\mathbf{x}) \geq f\left(p'(X)/\partial_- p(s)\right)$.

**Example 3.7.** Suppose that Assumptions 3.1, 3.3, and 3.4 hold, and that there is a best supplier, whose cost function is linear with a slope $c \geq 0$. Consider inverse demand functions of the form (cf. p. 8 in [27])

$$p(q) = \begin{cases} \alpha\left(Q - q\right)^{\beta}, & 0 < q \leq Q, \\ 0, & Q < q, \end{cases} \tag{3.24}$$

where $Q > 0$, $\alpha > 0$ and $\beta \geq 1$. Assumption 3.4 implies that $c < \alpha Q^{\beta}$. Note that Corollary 3.1 does not apply, because the right derivative of $p(\cdot)$ at $Q$ is zero. Through a simple calculation we obtain

$$s = Q - \left(\frac{c}{\alpha}\right)^{1/\beta},$$

and

$$p'(s) = \alpha\beta\left(\frac{c}{\alpha}\right)^{(\beta-1)/\beta}, \qquad \partial_+ p(0) = \alpha\beta Q^{\beta-1}.$$

Corollary 3.2 implies that for every Cournot equilibrium $\mathbf{x}$,

$$\gamma(\mathbf{x}) \geq f\left(\frac{\partial_+ p(0)}{p'(s)}\right) = f\left(\left(\frac{\alpha Q^\beta}{c}\right)^{(\beta-1)/\beta}\right) = f\left(\left(\frac{p(0)}{c}\right)^{(\beta-1)/\beta}\right).$$

Using information on the aggregate demand at the equilibrium, the efficiency bound can be refined. Since

$$p'(X) = \alpha\beta(Q - X)^{\beta-1},$$

we have

$$\gamma(\mathbf{x}) \geq f\left(\frac{p'(X)}{p'(s)}\right) = f\left(\left(\frac{\alpha(Q-X)^\beta}{c}\right)^{(\beta-1)/\beta}\right) = f\left(\left(\frac{p(X)}{c}\right)^{(\beta-1)/\beta}\right), \quad (3.25)$$

so that the efficiency bound depends only on the ratio of the equilibrium price to the marginal cost of the best supplier, and the parameter $\beta$. For affine inverse demand functions, we have $\beta = 1$ and the bound in (3.25) equals $f(1) = 2/3$, which agrees with Theorem 3.1. $\square$

## 3.7 Monopoly and Social Welfare

In this section we study the special case where $N = 1$, so that we are dealing with a single, monopolistic, supplier. As we explain, this case also covers a setting where multiple suppliers collude to maximize their total profit. By using the additional assumption that $N = 1$, we obtain a sharper (i.e., larger) lower bound, in Theorem 3.3. We then establish lower bounds on the efficiency of monopoly outputs that do not require knowledge of $X$ and $X^S$.

In a Cournot oligopoly, the maximum possible profit earned by all suppliers (if they collude) is an optimal solution to the following optimization problem,

$$\begin{aligned} &\text{maximize} \;\; p\left(\sum_{n=1}^N x_n\right) \cdot \sum_{n=1}^N x_n - \sum_{n=1}^N C_n(x_n) \\ &\text{subject to} \;\; x_n \geq 0, \quad n = 1, \ldots, N. \end{aligned} \quad (3.26)$$

We use $\mathbf{x}^P = (x_1^P, \ldots, x_N^P)$ to denote an optimal solution to (3.26) (a *monopoly output*), and let $X^P = \sum_{n=1}^{N} x_n^P$.

It is not hard to see that the aggregate supply at a monopoly output, $X^P$, is also a Cournot equilibrium in a modified model with a single supplier ($N$=1) and a cost function given by

$$\begin{aligned} \overline{C}(X) &= \inf \sum_{n=1}^{N} C_n(x_n), \\ \text{subject to } & x_n \geq 0, \quad n = 1, \ldots, N; \quad \sum_{n=1}^{N} x_n = X. \end{aligned} \tag{3.27}$$

Note that $\overline{C}(\cdot)$ is convex (linear) when the $C_n(\cdot)$ are convex (respectively, linear). Furthermore, the social welfare at the monopoly output $\mathbf{x}^P$, is the same as that achieved at the Cournot equilibrium, $x_1 = X^P$, in the modified model. Also, the socially optimal value of $X$, as well as the resulting social welfare is the same for the $N$-supplier model and the above defined modified model with $N = 1$. Therefore, the efficiency of the monopoly output equals the efficiency of the Cournot equilibrium of the modified model. To lower bound the efficiency of monopoly outputs resulting from multiple colluding suppliers, we can (and will) restrict to the case with $N = 1$.

**Theorem 3.3.** *Suppose that Assumptions 3.1-3.4 hold, and the inverse demand function is convex. Let $\mathbf{x}^S$ and $\mathbf{x}^P$ be a social optimum and a monopoly output, respectively. Then, the following hold.*

*(a) If $p(X^P) = p(X^S)$, then $\gamma(\mathbf{x}^P) = 1$.*

*(b) If $p(X^P) \neq p(X^S)$, let $c = |p'(X^P)|$, $d = |(p(X^S) - p(X^P))/(X^S - X^P)|$, and $\overline{c} = c/d$. We have $\overline{c} \geq 1$ and*

$$\gamma(\mathbf{x}^P) \geq \frac{3}{3 + \overline{c}}. \tag{3.28}$$

*(c) The bound is tight at $\overline{c} = 1$, i.e., there exists a model with $\overline{c} = 1$ and a monopoly output whose efficiency is 3/4.*

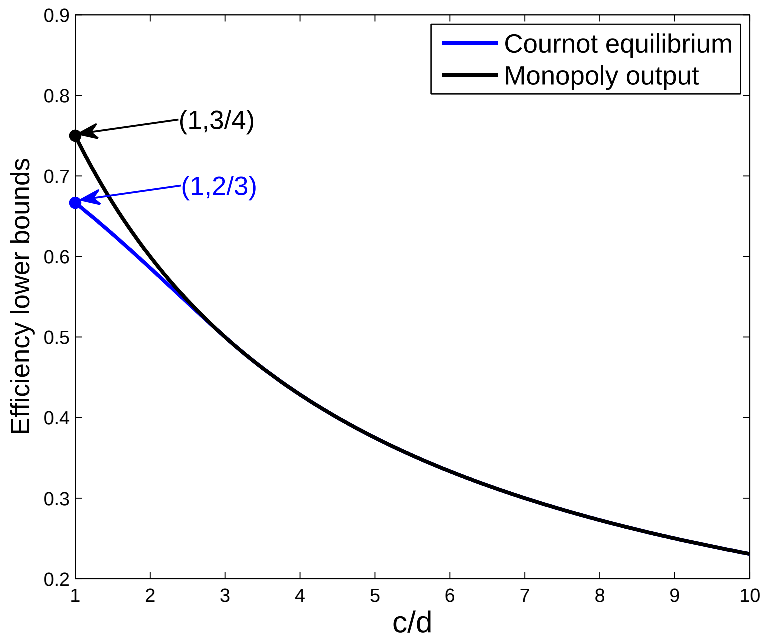


Figure 3-4: Comparison of the lower bounds on the efficiency of Cournot equilibria and monopoly outputs for the case of convex inverse demand functions.

The proof of Theorem 3.3 is given in Appendix B.3. Fig. 3-4 compares the efficiency lower bounds established for Cournot equilibria with that for monopoly outputs. For $\bar{c} = 1$, both efficiency bounds are tight and it is possible for a monopoly output to achieve a higher efficiency than that of a Cournot equilibrium, as shown in the following example.

**Example 3.8.** Consider the model introduced in the proof of part (c) of Theorem 3.1. The inverse demand function is $p(q) = \max\{1 - q, 0\}$. The cost functions are linear:

$$C_1^N(x_1) = 0, \qquad C_n^N(x_n) = \left(1/3 - \frac{1/3}{N-1}\right) x_n, \qquad n = 2, \ldots, N.$$

If all suppliers collude to maximize the total profit, the output will be

$$x_1^P = 1/2, \qquad x_n^P = 0, \quad n = 2, \ldots, N,$$

and $\gamma(\mathbf{x}^P) = 3/4$. On the other hand, it can be verified that the vector

$$x_1 = 1/3, \qquad x_n = \frac{1/3}{N-1}, \quad n = 2, \ldots, N,$$

is a Cournot equilibrium. For any $N \geq 2$, a simple calculation shows that the associated efficiency is $(6N-4)/(9N-9)$. For example, in a model with $N = 10$, the efficiency of the Cournot equilibrium is less than that of the monopoly output, i.e., $\gamma(\mathbf{x}^P) = 3/4 > 56/81 = \gamma(\mathbf{x})$. □

The above example agrees with the observation in earlier works that a monopoly output is not necessarily less efficient than an equilibrium resulting from imperfect competition [21, 24].

We now derive a result similar to Corollary 3.2, and then apply it to a numerical example with the same inverse demand function as in Example 3.5.

**Corollary 3.3.** *Suppose that Assumptions 3.1-3.4 hold, and that $p(\cdot)$ is convex. Let $s$ and $t$ be the real numbers defined in* (3.15)*. If $\partial_- p(s) < 0$, then for any monopoly output $\mathbf{x}^P$, we have*

$$\gamma(\mathbf{x}^P) \geq \frac{\partial_- p(s)}{3\partial_- p(s) + \partial_+ p(t)}.$$

**Proof.** Note that the efficiency of a monopoly output equals the efficiency of a Cournot equilibrium in a modified model with $N = 1$. Therefore, the desired result follows from the proof of Corollary 3.2, except that the general lower bound $f(\cdot)$ is replaced by the tighter one, $3/(3+\bar{c})$, provided by Theorem 3.3. □

**Example 3.9.** Suppose that Assumptions 3.1, 3.3, and 3.4 hold, and that there is a best supplier, whose cost function is linear with a slope $\chi \geq 0$. Consider inverse demand functions of the form in (3.21). Through a simple calculation we have

$$s = \exp\left(\frac{\alpha - \chi}{\beta}\right), \qquad t = \exp\left(\frac{\alpha - \beta - \chi}{\beta}\right),$$

and

$$\frac{p'(t)}{p'(s)} = \frac{\exp\left((\alpha - \chi)/\beta\right)}{\exp\left((\alpha - \beta - \chi)/\beta\right)} = \exp(1).$$

According to Corollary 3.3, for every monopoly output $\mathbf{x}^P$ we have,

$$\gamma(\mathbf{x}^P) \geq 3/(3 + \exp(1)) = 0.525. \tag{3.29}$$

Using the argument in Example 3.5, we conclude that this efficiency bound also applies to the case of nonlinear (convex) cost functions. □

## 3.8 Conclusion

It is well known that Cournot oligopoly can yield arbitrarily high efficiency loss in general; for details, see [44]. For Cournot oligopoly with convex market demand and cost functions, results such as those provided in Theorem 3.2 show that the efficiency loss of a Cournot equilibrium can be bounded away from zero by a function of a scalar parameter that captures quantitative properties of the inverse demand function. With additional information on the cost functions, the efficiency lower bounds can be further refined. Our results apply to various convex inverse demand functions that have been considered in the economics literature.

# Chapter 4

# Profit Loss in Cournot Oligopolies

## 4.1 Introduction

We consider a Cournot oligopoly model where multiple suppliers (oligopolists) compete by choosing quantities. Our objective is to compare the total profit earned at a Cournot equilibrium to the maximum possible total profit, that is, the profit that would be obtained if the suppliers were to collude.

### 4.1.1 Background

It is well known that oligopolists can collude by jointly restricting their output and thereby increase their total profit ([16, 29]). There is a large literature on collusive behavior in oligopolistic markets. For example, the authors of [36] show that in the presence of demand uncertainty, it may be possible for suppliers to form a self-policing cartel to maximize their joint profits. Also, some recent works show that forward trading may raise the prices [62], and may allow suppliers to sustain collusive profits [57].

In this chapter, we focus on the classical static Cournot oligopoly model, and explore the profit loss due to competition. We compare the aggregate profit earned at a Cournot equilibrium to the maximum possible profit, that is, the aggregate profit that would have been achieved if the suppliers were to collude. Oligopolist profit loss due to competition has received some recent attention. The authors of [6] quantify the profit loss in Cournot oligopoly models with concave demand functions. However, most of their results focus on the relation between consumer surplus, producer surplus, and the aggregate social welfare achieved at a Cournot equilibrium, rather than on the relation between the aggregate profit achieved at a Cournot equilibrium and the maximum aggregate profit.The authors of [79] study supply chains with partial positive externalities and show that the profit loss at an equilibrium is at least 25% of the maximum profit.

Other recent works have reported various bounds on the profit loss at an equilibrium for oligopoly models with affine demand functions. For a differentiated oligopoly model, the authors of [28] establish lower and upper bounds on the profit loss at an equilibrium of price (Bertrand) competition. Closer to the results developed in this chapter, Kluberg and Perakis compare the aggregate profit earned by the suppliers under Cournot competition to the corresponding maximum possible [52], for the case where suppliers produce multiple differentiated products and the demand is an affine function of the price. We note however that one of their key assumptions does not hold in the Cournot model studied in this chapter[1].

### 4.1.2 Our contribution

In this chapter, we study the profit loss in a classical Cournot oligopoly model, for a broad class of nonincreasing inverse demand functions that yield concave revenue functions. We establish a lower bound of the form $f^P(c/d, N)$ on the profit ratio of a Cournot equilibrium (the ratio of the aggregate profit earned at the equilibrium to the maximum possible). Here, $f^P$ is a function given in closed form, $c$ is the absolute value of the slope of the line that agrees with the inverse demand function at a profit-maximizing output and at the Cournot equilibrium, $d$ is the absolute value of the slope of the inverse demand function at the Cournot equilibrium, and $N$ is

[1]In our model, the matrix $\mathbf{B}$ in the inverse demand function (a notation used in [52, 53]) is an $N \times N$ matrix ($N$ is the number of suppliers) with all its elements equal to 1 and is therefore not invertible (cf. Chapter 2.2 of [53]).

the number of suppliers. We also derive another form of profit ratio lower bounds, $g^P(c/d, r)$, which does not depend on the number of suppliers, but on the market share of the largest supplier at the equilibrium, $r$.

For Cournot oligopolies with affine inverse demand functions, we have $c/d = 1$, and our lower bounds are tight. More generally, the ratio $c/d$ can be viewed as a measure of nonlinearity of the inverse demand function. As the parameter $c/d$ goes to infinity, the lower bounds converge to zero and arbitrarily high profit losses are possible. Our results allow us to lower bound the profit ratio of Cournot equilibria for a large class of Cournot oligopoly models in terms of qualitative properties of the inverse demand function, without having to restrict to the special case of affine demand functions, and without having to calculate the equilibrium and the profit-maximizing output.

The general methodology used in this chapter is similar in spirit to that used in Chapter 3 to derive lower bounds on the efficiency of Cournot equilibria. Furthermore, the development runs along similar lines. However, the assumptions, the details, and the expressions in the various results are different. For instance, the assumption in Chapter 3 that the inverse demand function is convex is replaced here by an assumption that a monopolist's revenue is a concave function of the price (Assumption 4.1).

### 4.1.3 Outline of the chapter

The rest of the chapter is organized as follows. In the next section, we formulate the model and review available results on the existence of Cournot equilibria. In Section 4.3, we provide some mathematical preliminaries on Cournot equilibria that will be useful later, including the fact that profit ratio lower bounds can be obtained by restricting to linear cost functions. We also show that for the purpose of studying the worst case profit loss, it suffices to restrict to a special class of piecewise linear inverse demand functions. This leads to the main result of this chapter, lower bounds on the profit ratio of Cournot equilibria (Theorems 4.1 and 4.2 in Section 4.4). Based on these theorems, in Section 4.5 we derive a number of corollaries that provide profit ratio lower bounds that can be calculated without detailed information on these equilibria. We apply these results to various commonly encountered inverse demand functions. Finally, in Section 4.6, we make some brief concluding remarks.

## 4.2 Formulation and Background

We consider the same Cournot oligopoly model as the one studied in Chapter 3. We consider a market for a single homogeneous good with inverse demand function $p : [0,\infty) \to [0,\infty)$ and $N$ suppliers. Supplier $n \in \{1, 2, \ldots, N\}$ has a cost function $C_n : [0,\infty) \to [0,\infty)$. Each supplier $n$ chooses a nonnegative real number $x_n$, which is the amount of the good to be supplied by her. The **strategy profile** $\mathbf{x} = (x_1, x_2, \ldots, x_N)$ results in a total supply denoted by $X = \sum_{n=1}^{N} x_n$, and a corresponding market price $p(X)$. Supplier $n$'s payoff is

$$\pi_n(x_n, \mathbf{x}_{-n}) = x_n p(X) - C_n(x_n),$$

where we have used the standard notation $\mathbf{x}_{-n}$ to indicate the vector $\mathbf{x}$ with the component $x_n$ omitted. A strategy profile $(x_1, x_2, \ldots, x_N)$ is a Cournot (or Nash) equilibrium if

$$\pi_n(x_n, \mathbf{x}_{-n}) \geq \pi_n(x, \mathbf{x}_{-n}), \qquad \forall\, x \geq 0, \quad \forall\, n \in \{1, 2, \ldots, N\}.$$

## 4.3 Preliminaries on Cournot Equilibria

In this section, we introduce several main assumptions that we will be working with, some definitions, and several lemmas that will be useful in the proof of our main result. In Section 4.3.1, we present conditions for a nonnegative vector to be a Cournot equilibrium or a monopoly output. Then, in Section 4.3.2, we define the profit ratio of a Cournot equilibrium. In Sections 4.3.3 and 4.3.4, we derive some properties of Cournot equilibria that will be useful later, but which may also be of independent interest. For example, we show that the worst case profit ratio occurs when the cost functions are linear.

In a Cournot oligopoly, the maximum possible profit earned by all suppliers is an optimal solution to the following optimization problem,

$$\begin{aligned} &\text{maximize} \;\; p\left(X\right) \cdot X - \sum_{n=1}^{N} C_n(x_n) \\ &\text{subject to} \;\; x_n \geq 0, \quad n = 1, \ldots, N, \end{aligned} \tag{4.1}$$

where $X = \sum_{n=1}^{N} x_n$. We use $\mathbf{x}^P = (x_1^P, \ldots, x_N^P)$ to denote an optimal solution to (4.1), and let $X^P = \sum\limits_{n=1}^{N} x_n^P$. We will refer to an optimal solution to (4.1) as a **monopoly output**. Under Assumptions 3.1-3.3, there must exist an optimal solution to (4.1). Note however that there may exist multiple optimal solutions to (4.1), which yield different prices. For example, consider a case where $N = 1$ and the cost function of the single supplier is identically zero. The inverse demand function is

$$p(q) = \begin{cases} -q + 1, & \text{if} \;\; 0 \leq q \leq 2/3, \\ \max\{0, -\dfrac{1}{4}(q - 2/3) + 1/3\}, & \text{if} \;\; 2/3 < q. \end{cases}$$

Assumptions 3.1-3.3 are satisfied, and it is not hard to see that $x_1 = 1/2$ and $x_1 = 1$ are two monopoly outputs (optimal solutions to the optimization problem (4.1)), which yield different prices. We define $\mathcal{P}$ as the set of prices resulting from monopoly outputs; that is, a nonnegative real number $v$ belongs to $\mathcal{P}$, if and only if there exists a monopoly output, $\mathbf{x}^P$, with $v = p(X^P)$.

### 4.3.1 Optimality and equilibrium conditions

The following assumption guarantees that the objective function in (4.1) is concave on the interval where it is positive.

**Assumption 4.1.** *On the interval where $p(\cdot)$ is positive, the function $p(q)q$ is concave in $q$.*

Because $p(\cdot)$ is nonincreasing, all concave inverse demand functions satisfy Assumption 4.1. We observe that many convex inverse demand functions also satisfy

Assumption 4.1. For example, inverse demand functions of the form (cf. Eq. (5) in [11])

$$p(q) = \max\{\alpha - \beta q^{\delta}, 0\}, \qquad \alpha,\ \beta,\ \delta > 0, \tag{4.2}$$

and the following class of convex inverse demand functions (cf. Eq. (6) in [11])

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad \alpha,\ \beta > 0, \tag{4.3}$$

satisfy Assumption 4.1. Actually, Assumption 4.1 is weaker than the conditions required on $p(\cdot)$ in the existence result (cf. conditions (b) and (c) in Proposition 3.1).

We observe that under Assumptions 3.1, 3.2 and 4.1, the objective function in (4.1) is concave on the interval where it is positive. Hence, we have the following **necessary and sufficient** conditions for a vector $\mathbf{x}^P$ with $p(X^P) > 0$ to maximize the aggregate profit:

$$\begin{cases} C_n'(x_n^P) \leq p\left(X^P\right) + \partial_- p\left(X^P\right) \cdot X^P, & \text{if } x_n > 0, \\ C_n'(x_n^P) \geq p\left(X^P\right) + \partial_+ p\left(X^P\right) \cdot X^P. & \end{cases} \tag{4.4}$$

In Section 3.3.1, necessary conditions for a nonnegative vector to be a Cournot equilibrium are given in (3.3)-(3.4). Recall that we say that a nonnegative vector $\mathbf{x}$ is a **Cournot candidate**, if it satisfies the necessary conditions (3.3)-(3.4). Note that for a given model, the set of Cournot equilibria is a subset of the set of Cournot candidates. Most of the results obtained in this chapter apply to all Cournot candidates.

**Proposition 4.1.** *Suppose that Assumptions 3.1-3.4 hold. Then, the aggregate profit achieved at a Cournot candidate is nonnegative, and the optimal objective value of the optimization problem (4.1) is positive.*

*Proof.* Proof Let $\mathbf{x}$ be a Cournot candidate. Proposition 3.4 shows that $X > 0$. For every supplier $n$ such that $x_n > 0$, according to the necessary condition (3.3), we have

$C_n'(x_n) \le p(X)$. Hence,

$$\sum_{n=1}^{N} C_n'(x_n)x_n \le p(X) \cdot \sum_{n=1}^{N} x_n.$$

Since each $C_n(\cdot)$ is convex and nondecreasing, we have

$$\sum_{n=1}^{N} C_n(x_n) \le \sum_{n=1}^{N} C_n'(x_n)x_n \le p(X) \cdot X. \tag{4.5}$$

Hence, the aggregate profit achieved at the Cournot candidate, $Xp(X) - \sum_{n=1}^{N} C_n(x_n)$, is nonnegative. We let

$$k \in \arg\min_n \{C_n'(0)\}.$$

Due to Assumption 3.4 and the continuity of the inverse demand and cost functions, there exists some $\varepsilon > 0$ such that

$$\varepsilon p(\varepsilon) - C_k(\varepsilon) > 0,$$

which implies that the optimal objective value in the optimization problem (4.1) is positive. □

### 4.3.2 Profit ratio

Note that if $N = 1$, a Cournot equilibrium must maximize the aggregate profit. We will therefore study the more interesting case where $N \ge 2$. Given a nonnegative vector $\mathbf{x}$, we define its **profit ratio**, $\eta(\mathbf{x})$, by

$$\eta(\mathbf{x}) = \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} C_n(x_n^P)}, \tag{4.6}$$

where $(x_1^P, \dots, x_N^P)$ is an optimal solution to the optimization problem (4.1). Under Assumptions 3.1-3.4, the ratio is well defined, because the denominator is positive. According to Proposition 4.1, a Cournot candidate yields a nonnegative profit, and therefore its profit ratio is nonnegative. For a Cournot candidate $\mathbf{x}$ with $p(X) = 0$, we must have $\eta(\mathbf{x}) = 0$. In Section 4.4, we will establish lower bounds on the profit ratio of Cournot candidates that yield positive prices.

### 4.3.3 Restricting to linear cost functions

In this section, we show that in order to study the worst-case profit ratio of Cournot equilibria, it suffices to consider linear cost functions. This is a counterpart of (but different from) Proposition 3.6 in Chapter 3.

**Proposition 4.2.** *Suppose that Assumptions 3.1-3.4 and 4.1 hold. Let* $\mathbf{x}$ *be a Cournot candidate that is not an optimal solution to (4.1), and let* $\alpha_n = C_n'(x_n)$*. Consider a modified model in which we replace the cost function of each supplier* $n$ *by a new function* $\overline{C}_n(\cdot)$*, defined by*

$$\overline{C}_n(x) = \alpha_n x, \quad \forall x \geq 0.$$

*Then, for the modified model, Assumptions 3.1-3.4, and 4.1 still hold, the vector* $\mathbf{x}$ *is a Cournot candidate and its profit ratio, denoted by* $\overline{\eta}(\mathbf{x})$*, satisfies* $0 \leq \overline{\eta}(\mathbf{x}) \leq \eta(\mathbf{x})$*.*

**Proof.** We first observe that the vector $\mathbf{x}$ satisfies the necessary conditions (3.3)-(3.4) for the modified model. Hence, the vector $\mathbf{x}$ is a Cournot candidate in the modified model. It is not hard to see that Assumptions 3.1, 3.2 and 4.1 hold in the modified model. Finally, since $\alpha_n \geq C_n'(0)$ for every $n$, Assumption 3.3 holds in the modified model.

We now show that Assumption 3.4 holds in the modified model, i.e., that $p(0) > \min_n\{\alpha_n\}$. Since the vector $\mathbf{x}$ is a Cournot candidate in the original model, Proposition 3.4 implies that $X > 0$. Consider a supplier $n$ such that $x_n > 0$. From the necessary conditions (3.3), we have that $\alpha_n \leq p(X)$. If $\alpha_n = p(X) = 0$, then $p(0) > 0 = \min_n\{\alpha_n\}$. If $\alpha_n = p(X) > 0$, then $\partial_- p(X) = 0$. Since $\partial_+ p(X) \leq 0$ and $x_n \leq X$, the necessary conditions (3.3)-(3.4) imply the conditions in (4.4), which are sufficient for $\mathbf{x}$ to maximize the aggregate profit. Since $\mathbf{x}$ is not an optimal solution to the profit maximization problem (4.1), it follows that $\alpha_n < p(X)$, which implies that Assumption 3.4 holds in the modified model.

Let $\mathbf{x}^P$ be an optimal solution to (4.1) in the original model. Proposition 4.1 implies that $p(X^P) > 0$. Since $\mathbf{x}^P$ satisfies the conditions in (4.4) for the modified model, it remains a monopoly output in the modified model. In the modified model, since Assumptions 3.1-3.4 hold, the profit ratio of the vector $\mathbf{x}$ is well defined and given by

$$\overline{\eta}(\mathbf{x}) = \frac{Xp(X) - \sum_{n=1}^{N} \alpha_n x_n}{X^P p(X^P) - \sum_{n=1}^{N} \alpha_n x_n^P}. \tag{4.7}$$

Note that the denominator on the right-hand side of (4.7) is the maximum aggregate profit in the modified model, and the numerator is the aggregate profit achieved at the Cournot candidate $\mathbf{x}$ in the modified model. Proposition 4.1 shows that the denominator is positive, while the numerator is nonnegative.

Since $C_n(\cdot)$ is convex, we have

$$C_n(x_n^P) \geq C_n(x_n) + \alpha_n(x_n^P - x_n), \qquad n = 1, \ldots, N.$$

Adding a nonnegative quantity to the denominator cannot increase the ratio and, therefore,

$$\eta(\mathbf{x}) = \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} C_n(x_n^P)} \geq \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} (\alpha_n(x_n^P - x_n) + C_n(x_n))} \geq 0, \tag{4.8}$$

where the last inequality follows from the fact that $\mathbf{x}$ is a Cournot candidate in the original model and Proposition 4.1. Since $C_n(\cdot)$ is convex and nondecreasing, with $C_n(0) = 0$, we have

$$\sum_{n=1}^{N} C_n(x_n) - \sum_{n=1}^{N} \alpha_n x_n \leq 0. \tag{4.9}$$

Since the right-hand side of (4.8) is in the interval $[0, 1]$, adding a nonpositive quantity to both the numerator and the denominator cannot increase the ratio. Therefore, using (4.9) in the first inequality below we have

$$
\begin{aligned}
\eta(\mathbf{x}) &\geq \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^{N} \left(\alpha_n(x_n^P - x_n) + C_n(x_n)\right)} \\
&\geq \frac{Xp(X) - \sum_{n=1}^{N} C_n(x_n) + \left(\sum_{n=1}^{N} C_n(x_n) - \sum_{n=1}^{N} \alpha_n x_n\right)}{X^P p(X^P) - \sum_{n=1}^{N} \left(\alpha_n(x_n^P - x_n) + C_n(x_n)\right) + \left(\sum_{n=1}^{N} C_n(x_n) - \sum_{n=1}^{N} \alpha_n x_n\right)} \\
&= \frac{Xp(X) - \sum_{n=1}^{N} \alpha_n x_n}{X^P p(X^P) - \sum_{n=1}^{N} \alpha_n x_n^P} \\
&= \overline{\eta}(\mathbf{x}).
\end{aligned} \tag{4.10}
$$

The desired result follows. □

If $\mathbf{x}$ is a Cournot equilibrium, then it satisfies Eqs. (3.3)-(3.4), and therefore is a Cournot candidate. Hence, Proposition 4.2 applies to all Cournot equilibria that do not maximize the aggregate profit. We note that if a Cournot equilibrium $\mathbf{x}$ maximizes the aggregate profit for the original model, then the maximum aggregate profit in the modified model could be zero, in which case $\eta(\mathbf{x}) = 1$, but $\overline{\eta}(\mathbf{x})$ is undefined; see the example that follows.

**Example 4.1.** Consider a model involving a single supplier ($N = 1$). The cost function of supplier 1 is $C_1(x) = x^2$. The inverse demand function is constant, with $p(q) = 1$ for any $q \geq 0$. It is not hard to see that the vector $x_1 = 1/2$ is a Cournot equilibrium, which also maximizes the aggregate profit. In the modified model, we have $\overline{C}_1(x) = x$. The aggregate profit achieved in the modified model is always zero, regardless of the action taken by the supplier. □

Proposition 4.2 shows that a Cournot candidate in the original model remains a Cournot candidate in the modified model. Hence, to lower bound the profit ratio of a Cournot equilibrium in the original model, it suffices to lower bound the profit ratio of a worst Cournot candidate for a modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) restrict to the case of linear cost functions, and study the worst case profit ratio over all Cournot candidates.

### 4.3.4 Other properties of Cournot candidates

In this subsection, we derive several useful and intuitive properties of Cournot candidates. We show that at a Cournot candidate there are two possibilities: either $p(X^P) > p(X)$ and $X^P < X$, or $p(X) \in \mathcal{P}$ (Proposition 4.3); in the latter case, under the additional assumption that $p'(X)$ exists, a Cournot candidate maximizes the aggregate profit (Proposition 4.4). We then show in Proposition 4.5 that to lower bound the worst case profit ratio, it suffices to restrict to a special class of piecewise linear inverse demand functions.

**Proposition 4.3.** *Suppose that Assumptions 3.1-3.4 and 4.1 hold. Let* $\mathbf{x}$ *a Cournot candidate. If* $p(X) \notin \mathcal{P}$*, then for any optimal solution* $\mathbf{x}^P$ *to (4.1), we have* $X > X^P$*.*

**Proof.** Suppose that $p(X) \in \mathcal{P}$. Proposition 4.1 implies that $p(X^P) > 0$ for every monopoly output $\mathbf{x}^P$. If $p(X) = 0$, then we know that $X > X^P$, because $p(\cdot)$ is nonincreasing.

Now consider the case where $p(X) > 0$. Proposition 3.4 shows that $X > 0$. If there is only one supplier that provides a positive quantity at the Cournot candidate, then the necessary conditions (3.3)-(3.4) imply the conditions in (4.4), and we conclude that the Cournot candidate maximizes the aggregate profit, i.e., $p(X) \in \mathcal{P}$, a contradiction. Hence, there are at least two suppliers who produce positive quantities at the Cournot candidate. We therefore have that $X > x_n$, for any $n = 1, \ldots, N$.

Suppose that there exists an optimal solution to (4.1), $\mathbf{x}^P$, such that $0 < X \leq X^P$. Since $p(X) \neq p(X^P)$ and $p(\cdot)$ is nonincreasing, we have $X^P > X$ and $p(X^P) < p(X)$. For every supplier $n$ for which $x_n^P > 0$, we have

$$C_n'(x_n^P) \leq \partial_- p(X^P) X^P + p(X^P) \leq \partial_+ p(X) X + p(X) \leq \partial_+ p(X) x_n + p(X), \quad (4.11)$$

where the first inequality follows from (4.4), the second inequality follows from the fact $X < X^P$ and Assumption 4.1, and the last inequality holds because $\partial_+ p(X) \leq 0$ and $x_n < X$. We now argue that equality cannot hold throughout (4.11). If $\partial_+ p(X) < 0$, then the last inequality is strict; if $\partial_+ p(X) = 0$, the second inequality is strict because

$p(X^P) < p(X)$ and $\partial_- p(X) \leq 0$. Using also the necessary condition (3.4), we have

$$C'_n(x_n^P) < \partial_+ p(X) x_n + p(X) \leq C'_n(x_n).$$

Due to the convexity of the cost functions, it follows that $x_n > x_n^P$ for every $n$ such that $x_n^P > 0$, which contradicts our hypothesis that $X \leq X^P$. □

If the inverse demand function does not satisfy Assumption 4.1, it is possible that the aggregate supply at a Cournot candidate is less than that at a monopoly output, as shown in the following example.

**Example 4.2.** Consider a model involving only one supplier ($N = 1$). The cost function of the supplier is linear with a slope of 2. The inverse demand function is given by

$$p(q) = \begin{cases} -q + 4, & \text{if } 0 \leq q \leq 4/3, \\ \max\{0, -\dfrac{1}{5}(q - 4/3) + 8/3\}, & \text{if } 4/3 < q, \end{cases}$$

which is convex and satisfies Assumption 3.2. It can be verified that the supplier can maximize its profit at $x_1 = 7/3$. It is also easy to check that $x_1 = 1$ is a Cournot candidate. We have $X = 1 < 7/3 = X^P$. □

**Proposition 4.4.** *Suppose that Assumptions 3.1-3.4 and 4.1 hold. Let* $\mathbf{x}$ *be a Cournot candidate. If* $p(X) \in \mathcal{P}$ *and* $p'(X)$ *exists, then* $\eta(\mathbf{x}) = 1$.

**Proof.** From Proposition 4.1 we have that $p(X^P) > 0$ for every monopoly output $\mathbf{x}^P$. Since $p(X) \in \mathcal{P}$, it follows that $p(X) > 0$. Proposition 3.4 implies that $X > 0$. If there is only one supplier $n$ who provides a positive quantity of good at the Cournot candidate, then the necessary conditions (3.3)-(3.4) imply the conditions in (4.4), and we conclude that $\eta(\mathbf{x}) = 1$.

Now consider the case where $\mathbf{x}$ has at least two positive components. Then, $X > x_n$, for any $n = 1, \ldots, N$. Since $p(X) \in \mathcal{P}$, there exists an optimal solution to (4.1), $\mathbf{x}^P$, such that $p(X) = p(X^P)$. We now prove that $p'(X) = 0$. Suppose not, in which case we have $p'(X) < 0$. Since $p(X) = p(X^P)$, we have that $X = X^P$. For every $n$ such that $x_n^P > 0$, from the convexity of $C_n(\cdot)$ and the conditions in (4.4), we have

$$C_n'(0) \le C_n'(x_n^P) = p'\big(X^P\big) \cdot X^P + p(X^P).$$

Since $X = X^P$, $p'(X) < 0$, and $x_n < X$, we have

$$C_n'(0) \le p'\big(X^P\big) \cdot X^P + p(X^P) = p'(X) \cdot X + p(X) < p'(X) \cdot x_n + p(X),$$

which implies that $x_n > 0$, from the necessary condition (3.4). Hence, we have

$$C_n'(x_n^P) = p'\big(X^P\big) \cdot X^P + p(X^P) < p'(X) \cdot x_n + p(X) = C_n'(x_n),$$

which implies that $x_n > x_n^P$, from the convexity of $C_n(\cdot)$. Since $x_n > x_n^P$ for every $n$ such that $x_n^P > 0$, we conclude that $X > X^P$, which contradicts our earlier conclusion that $X = X^P$. The contradiction shows that $p'(X) = 0$.

Since $p'(X) = 0$ and the Cournot candidate $\mathbf{x}$ satisfies the necessary conditions (3.3)-(3.4), it also satisfies the conditions in (4.4). Since $p(X) > 0$, the conditions in (4.4) are sufficient for the Cournot candidate $\mathbf{x}$ to maximize the aggregate profit, i.e., $\eta(\mathbf{x}) = 1$. $\square$

It can be shown that if the inverse demand function is convex, then $p'(X)$ exists for any Cournot candidate $\mathbf{x}$ (cf. Proposition 3 in [91]). On the other hand, for a model satisfying Assumptions 3.1-3.4, if the inverse demand function is not differentiable at $X$, then a Cournot equilibrium $\mathbf{x}$ may yield arbitrarily large profit loss, even if $X = X^P$, and $N$ is held fixed.

**Example 4.3.** Consider a model involving two suppliers ($N = 2$), with $C_1(x) = 0$ and $C_2(x) = x$. The inverse demand function is concave on the interval where it is positive, of the form

$$p(q) = \begin{cases} 1, & \text{if } \ 0 \le q \le 1, \\ \max\{0, -M(q-1)+1\}, & \text{if } \ 1 < q, \end{cases}$$

where $M > 2$. At the vector $(1, 0)$, the maximum aggregate profit, 1, is achieved. The aggregate profit realized at the Cournot equilibrium, $\mathbf{x} = (1/M, 1 - 1/M)$, is

$1/M$. Note that $X = X^P = 1$. However, the profit ratio of $\mathbf{x}$ can be made arbitrarily small, as $M$ grows large.

Based on the preceding propositions, we are now ready to prove the following proposition, which will serve as a basis for the main theorems to be given in the next section.

**Proposition 4.5.** *Let* $\mathbf{x}$ *and* $\mathbf{x}^P$ *be a Cournot candidate and a monopoly output, respectively. Suppose that Assumptions 3.1-3.4 and 4.1 hold,* $p'(X)$ *exists, and that* $p(X) \notin \mathcal{P}$. *Let* $c = |(p(X^P) - p(X))/(X^P - X)|$, $d = |p'(X)|$. *Now consider a modified model in which the inverse demand function is replaced by a piecewise linear function*[2] $p^0(\cdot)$,

$$p^0(q) = \begin{cases} -c(q - X) + p(X), & 0 \le q \le X, \\ \max\{0, -d(q - X) + p(X)\}, & X < q. \end{cases} \tag{4.12}$$

*Let* $\eta^0(\mathbf{x})$ *be the profit ratio of the vector* $\mathbf{x}$ *in the modified model. We have*

$$\eta^0(\mathbf{x}) \le \eta(\mathbf{x}).$$

**Proof.** Since $p^0(X) = p(X)$, the aggregate profit earned at $\mathbf{x}$ is $Xp(X) - \sum_{n=1}^N C_n(x_n)$, in both the original and the modified model. Hence, we have

$$\eta^0(\mathbf{x}) \le \frac{Xp(X) - \sum_{n=1}^N C_n(x_n)}{X^P p^0(X^P) - \sum_{n=1}^N C_n(x_n^P)} = \frac{Xp(X) - \sum_{n=1}^N C_n(x_n)}{X^P p(X^P) - \sum_{n=1}^N C_n(x_n^P)} = \eta(\mathbf{x}),$$

where the inequality holds because the maximum total profit in the modified model is at least $X^P p^0(X^P) - \sum_{n=1}^N C_n(x_n^P)$, and the next equality holds because $p^0(X^P) = p(X^P)$. $\square$

Proposition 4.5 shows that a lower bound on the profit ratio of a Cournot equi-

[2] According to Proposition 4.3, we have $X^P < X$. The first segment of the piecewise linear function $p^0(\cdot)$ agrees with the inverse demand function $p(\cdot)$ at the two points: $(X^P, p(X^P))$ and $(X, p(X))$; the second segment is tangent to the inverse demand curve $p(\cdot)$ at the point $(X, p(X))$.

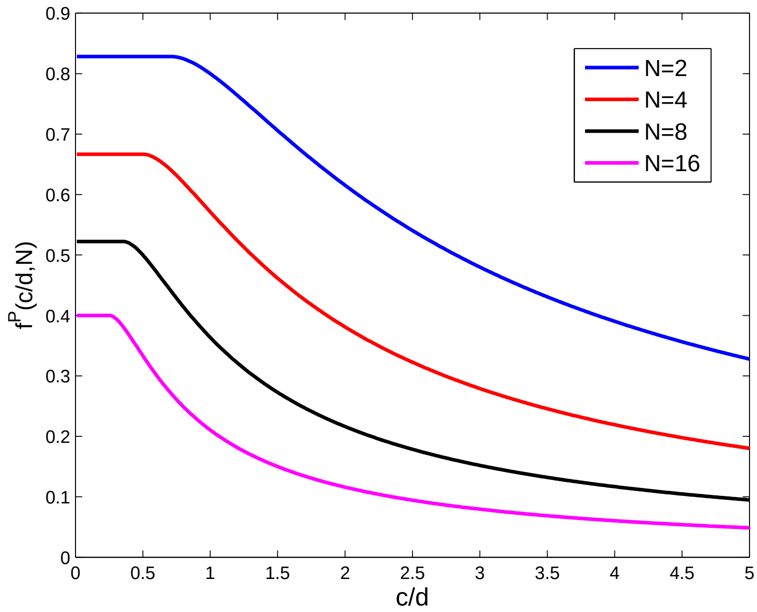


Figure 4-1: A lower bound on the profit ratio of a Cournot equilibrium as a function of the parameter $c/d$, for different values of $N$.

librium can be established by calculating its profit ratio in a modified model with a piecewise linear inverse demand function. This result enables us to derive our main results, given in the next section.

## 4.4 Profit ratio lower bounds

We fist establish a lower bound on the profit ratio of a Cournot candidate as a function of the scalar parameter $c/d$ and the number of suppliers (Theorem 4.1). Through a similar approach, we also provide a lower bound in terms of the scalar parameter $c/d$, and the maximum of the suppliers' market shares at an equilibrium (Theorem 4.2).

**Theorem 4.1.** *Let* $\mathbf{x}$ *and* $\mathbf{x}^P$ *be a Cournot candidate and a monopoly output, respectively. Suppose that Assumptions 3.1-3.4 hold,* $p'(X)$ *exists, and that* $p(X) > 0$*. Let* $c = |(p(X^P) - p(X))/(X^P - X)|$ *and* $d = |p'(X)|$*.*

*(a) If* $p(X) \in \mathcal{P}$*, then* $\eta(\mathbf{x}) = 1$*;*

*(b) If*[3] $p(X) \notin \mathcal{P}$*, then* $p'(X) < 0$*. We have* $\eta(\mathbf{x}) \geq f^P(\overline{c}, N)$*, where* $\overline{c} = c/d$ *and*

$$f^P(\overline{c}, N) = \begin{cases} \dfrac{N - 1 + (\sqrt{N} - 1)^2}{\sqrt{N}(N-1)}, & \text{if} \quad 0 < \overline{c} \leq \sqrt{1/N}, \\ \dfrac{4\overline{c}^3(N-1) + 4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2 N + 2\overline{c} + 1)^2}, & \text{if} \quad \overline{c} > \sqrt{1/N}. \end{cases} \tag{4.13}$$

*(c) If* $\overline{c} = 1$ *(in particular, if* $p(\cdot)$ *is affine), then* $\eta(\mathbf{x}) \geq f^P(1, N) = 4/(N+3)$*. Furthermore, the bound is tight, i.e., for any given* $N \geq 2$*, there exists a model with* $\overline{c} = 1$ *and a Cournot equilibrium whose profit ratio is* $4/(N+3)$*.*

The theorem is proved in Appendix C.1. It can be verified that the function $f^P(\overline{c}, N)$ is strictly decreasing in $N$, as shown in Fig. 4-1. For any given $\overline{c} > 0$, the lower bound, $f^P(\overline{c}, N)$, decreases to zero as the number of suppliers increases to infinity. Also, for any given $N$, the profit ratio lower bound is strictly decreasing in $\overline{c}$, over the interval $[\sqrt{1/N}, \infty)$.

**Theorem 4.2.** *Let* $\mathbf{x}$ *and* $\mathbf{x}^P$ *be a Cournot candidate and a monopoly output, respectively. Suppose that Assumptions 3.1-3.4 and 4.1 hold,* $p'(X)$ *exists, and that* $p(X) > 0$*. Let* $c = |(p(X^P) - p(X))/(X^P - X)|$ *and* $d = |p'(X)|$*. If* $p(X) \notin \mathcal{P}$*, then* $d = |p'(X)| > 0$*, and we have*

*(a)* $\eta(\mathbf{x}) \geq g^P(\overline{c}, r)$*, where* $\overline{c} = c/d$*,* $r$ *is the maximum of the suppliers' market shares*[4]*, i.e.,* $r = \max_n\{x_n/X\}$*, and*

$$g^P(\overline{c}, r) = \begin{cases} r, & \text{if} \quad 0 < \overline{c} \leq r < 1, \\ \dfrac{4\overline{c}r^2}{(\overline{c}+r)^2}, & \text{if} \quad 0 < r < \overline{c}. \end{cases} \tag{4.14}$$

[3]Note that we must have $N \geq 2$; otherwise, a Cournot candidate $\mathbf{x}$ in a model with $N = 1$ satisfies the conditions (3.3)-(3.4), which imply the conditions (4.4). Since $p(X) > 0$, a Cournot candidate for the case $N = 1$ maximizes the aggregate profit and $\mathbf{x} \in \mathcal{P}$.

[4]Proposition 3.4 shows that $X > 0$, and therefore $r$ is well defined. If $r = 1$, then the Cournot candidate satisfies conditions (4.4), and therefore maximizes the aggregate profit. Hence, we have $r \in (0, 1)$.

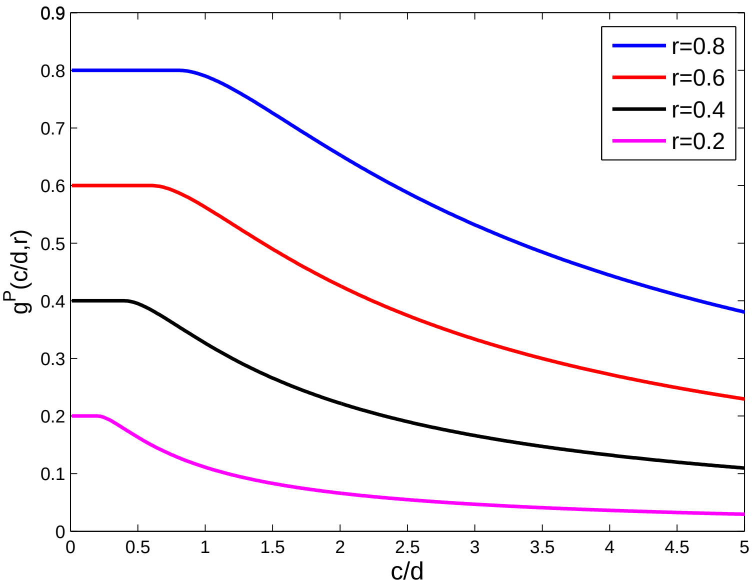


Figure 4-2: A lower bound on the profit ratio of a Cournot equilibrium as a function of the parameter $c/d$, for different values of the largest market share $r$.

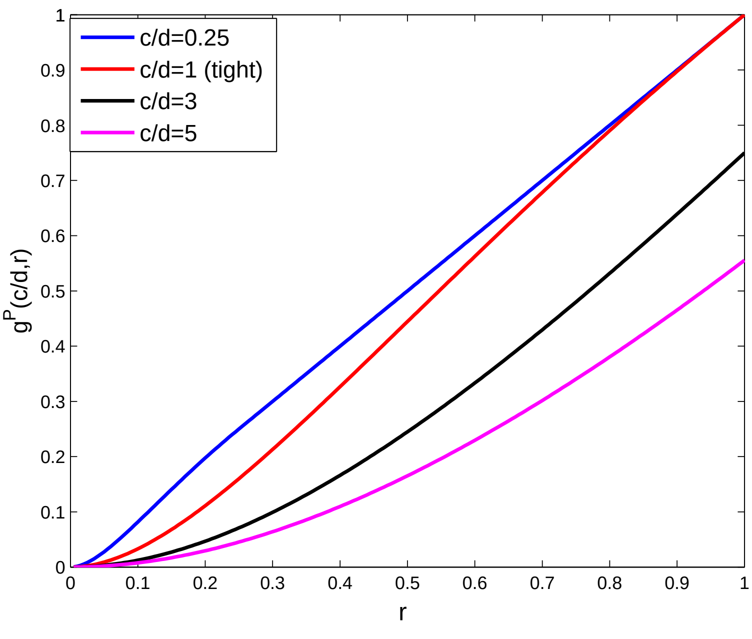


Figure 4-3: A lower bound on the profit ratio of a Cournot equilibrium as a function of the largest market share $r$, for different values of the parameter $c/d$.

*(b) If $\overline{c} = 1$ (in particular, if $p(\cdot)$ is affine), then $\eta(\mathbf{x}) \geq g^P(1, N) = 4r^2/(1+r)^2$. Furthermore, the bound is tight. That is, for every $r \in (0,1)$ and for every $\varepsilon > 0$, there exists a model with $\overline{c} = 1$ and a Cournot equilibrium whose profit ratio is no more than $4r^2/(1+r)^2 + \varepsilon$.*

The theorem is proved in Appendix C.2. The derived lower bounds are illustrated in Fig. 4-2 and Fig. 4-3. For a given $r$, the lower bound is nonincreasing in $\overline{c} = c/d$, and for a given $\overline{c}$, the lower bound increases with $r$. For affine inverse demand functions, we have $\overline{c} = 1$ and the bound is tight (the red curve in Fig. 4-3).

## 4.5 Corollaries and Applications

Given the number of suppliers (or the largest market share at an equilibrium), and the inverse demand function $p(\cdot)$, the lower bounds derived in Theorem 4.1 (Theorem 4.2, respectively) require additional knowledge on the aggregate supply at the Cournot equilibrium and on the monopoly output, i.e., $X$ and $X^P$. For concave inverse demand functions, we first apply Theorems 4.1 and 4.2 to establish a lower bound on the profit ratio of a Cournot equilibrium that depends only on the number of suppliers (or on the largest market share at the equilibrium) in Corollary 4.1. For convex inverse demand functions, in Corollary 4.2 we establish a profit ratio lower bound that depends only on the inverse demand function, and $N$ (or $r$). With a small amount information on the supplier cost functions, we further refine the lower bound in Corollary 4.3. At the end of this section, we apply our results to calculate nontrivial quantitative profit ratio bounds for various inverse demand functions that have been considered in the economics literature.

**Corollary 4.1.** *Suppose that Assumptions 3.1-3.4 and 4.1 hold, and that $p(\cdot)$ is concave and differentiable on the interval where it is positive. For every Cournot candidate $\mathbf{x}$ with $p(X) > 0$, we have $\eta(\mathbf{x}) \geq \max\{f^P(1, N), g^P(1, r)\}$.*

**Proof.** If $p(X) \in \mathcal{P}$, then $\eta(\mathbf{x}) = 1$ and the desired result trivially holds. Otherwise, we have that $X^P < X$ and $\overline{c} \leq 1$. The desired result then follows from Theorems 4.1

and 4.2, and the fact that both $f^P(\cdot, N)$ and $g^P(\cdot, r)$ are nonincreasing. $\square$

**Corollary 4.2.** *Suppose that Assumptions 3.1-3.4 and 4.1 hold, and that the inverse demand function $p(\cdot)$ is convex. If $p(Q) = 0$ for some $Q > 0$, and the ratio, $\mu = \partial_+ p(0)/\partial_- p(Q)$, is finite, then for every Cournot candidate $\mathbf{x}$ with $p(X) > 0$, its profit ratio satisfies $\eta(\mathbf{x}) \geq \max\{f^P(\mu, N), g^P(\mu, r)\}$.*

**Proof.** Since $p(\cdot)$ is nonincreasing and nonnegative, we have $p(q) = 0$ for any $p \geq Q$. Since $p(X) > 0$, we have $X < Q$.

We now argue that $p'(X)$ exists. Proposition 3.4 shows that $X > 0$. The conditions (3.3)-(3.4) applied to some $n$ with $x_n > 0$, imply that

$$p\left(X\right) + x_n \cdot \partial_- p\left(X\right) \geq p\left(X\right) + x_n \cdot \partial_+ p\left(X\right).$$

On the other hand, since $p(\cdot)$ is convex, we have $\partial_- p\left(X\right) \leq \partial_+ p\left(X\right)$. Hence, $\partial_- p\left(X\right) = \partial_+ p\left(X\right)$, as claimed.

If $p(X) \in \mathcal{P}$, then Proposition 4.4 implies that $\eta(\mathbf{x}) = 1 \geq f^P(\mu, N)$. Otherwise, since $p(\cdot)$ is convex and $X < Q$, for any aggregate profit maximizing vector, $\mathbf{x}^P$, we have that $\bar{c} \leq \mu$. The desired result follows from Theorems 4.1 and 4.2, and the fact that both $f^P(\cdot, N)$ and $g^P(\cdot, r)$ are nonincreasing. $\square$

**Corollary 4.3.** *Suppose that Assumptions 3.1-3.4, 4.1 hold, and that $p(\cdot)$ is convex. Let*[5]

$$s = \inf\left\{q \mid p(q) = \min_n C_n'(0)\right\}, \quad t = \inf\left\{q \mid \min_n C_n'(q) \geq p(q) + q\partial_+ p(q)\right\}. \quad (4.15)$$

*If $\partial_- p(s) < 0$, then the profit ratio of a Cournot candidate $\mathbf{x}$ with $p(X) > 0$ is at least*

$$\max\{f^P\left(\partial_+ p(t)/\partial_- p(s), N\right),\ g^P\left(\partial_+ p(t)/\partial_- p(s), r\right)\}.$$

**Proof.** Let $\mathbf{x}$ be a Cournot candidate. For convex inverse demand functions, we have shown in the proof of Corollary 4.2 that $p'(X)$ must exist. If $p(X) \in \mathcal{P}$, Proposition

[5]Under Assumption 3.3, the existence of the real numbers defined in (4.15) is guaranteed, and $t \leq s$.

4.4 shows that $\eta(\mathbf{x}) = 1$, and the desired results trivially hold. Now consider the case $p(X) \notin \mathcal{P}$. Let $\mathbf{x}^P$ maximize the aggregate profit. We first show that the aggregate supply at a Cournot candidate is at most $s$, and then argue that the aggregate supply at a monopoly output is at least $t$. The desired results will follow from the fact that both the functions, $f^P(\cdot, N)$ and $g^P(\cdot, r)$, are nonincreasing.

*Step 1: The aggregate supply at the Cournot candidate, $X$, is no more than $s$.*

We first show that $X \leq s$. Since $p'(X)$ exists, we know that $\mathbf{x}$ satisfies the necessary conditions in (C.1). Proposition 3.4 shows that $X > 0$. For a supplier $n$ with $x_n > 0$, the first equality in (C.1) implies that

$$p(X) \geq C'_n(x_n) \geq C'_n(0) \geq \min_n \{C'_n(0)\},$$

where the first inequality is true because $p(\cdot)$ is nonincreasing and $p'(X) \leq 0$, and the second inequality follows from the convexity of $C_n(\cdot)$.

For any Cournot candidate $\mathbf{x}$ with $p(X) > 0$, we now argue that $p(X) > \min_n\{C'_n(0)\}$. Suppose not. We have $p(X) = \min_n\{C'_n(0)\}$, $p(X) = C'_n(x_n)$, and $p'(X) = 0$. We observe that $\mathbf{x}$ satisfies the conditions in (4.4); since $p(X) > 0$, we know that $\mathbf{x}$ maximizes the aggregate profit. However, since cost functions are convex and $p(X) = \min_n\{C'_n(0)\}$, it is easy to see that the aggregate profit earned at $\mathbf{x}$ cannot be positive, a contradiction with Proposition 4.1. Since $p(X) > \min_n\{C'_n(0)\}$ and $p(\cdot)$ is nonincreasing, we conclude that $X \leq s$.

*Step 2: The aggregate supply at the monopoly output, $X^P$, is at least $t$.*

Proposition 4.1 implies that $X^P > 0$. Applying conditions (4.4) to some $n$ with $x_n^P > 0$, we know that $p'(X^P)$ exists, because $\partial_- p\left(X^P\right) \leq \partial_+ p\left(X^P\right)$. The conditions (4.4) imply that

$$C'_n(x_n^P) \geq p(X^P) + X^P p'(X^P), \qquad \forall n. \tag{4.16}$$

Since $X^P \geq x_n^P$ and the cost functions are convex, we have

$$C'_n(X^P) \geq p(X^P) + X^P p'(X^P), \qquad \forall n,$$

which implies that $X^P \geq t$. Proposition 4.3 shows that $X^P < X$, and therefore we have $t \leq X^P < X \leq s$. Since $\partial_- p(s) < 0$, and $p(\cdot)$ is convex and nonincreasing, we have

$$\bar{c} = c/d \leq \partial_+ p(t)/\partial_- p(s).$$

The desired result follows from Theorems 4.1 and 4.2, as well as the fact that both the functions, $f^P(\cdot, N)$ and $g^P(\cdot, r)$, are nonincreasing. □

If there exists a "best" supplier $n$ such that $C'_n(x) \leq C'_m(x)$, for any other supplier $m$ and any $x > 0$, then the parameters $s$ and $t$ depend only on $p(\cdot)$ and $C'_n(\cdot)$. In the following three examples, we apply Corollary 4.3 to three forms of convex inverse demand functions that appear in the economics literature.

**Example 4.4.** Suppose that Assumptions 3.1, 3.3, and 3.4 hold. Among the $N \geq 2$ suppliers, there is a best supplier that has a linear cost function with a slope $\chi \geq 0$. Consider an inverse demand function of the form in (3.21):

$$p(q) = \max\{0, \alpha - \beta \log q\}, \qquad \alpha,\ \beta > 0.$$

Note that Corollary 4.2 does not apply, because the left derivative of $p(\cdot)$ at 0 is infinite[6]. Since

$$\frac{d^2(qp(q))}{dq^2} = 2p'(q) + qp''(q) = \frac{-2\beta}{q} + \frac{q\beta}{q^2} < 0, \qquad \forall q \in (0, \exp(\alpha/\beta)),$$

we know that Assumption 4.1 holds. Through a simple calculation we have

$$s = \exp\left(\frac{\alpha - \chi}{\beta}\right), \qquad t = \exp\left((\alpha - \beta - \chi)/\beta\right).$$

We have

$$\frac{p'(t)}{p'(s)} = \frac{\exp\left((\alpha - \chi)/\beta\right)}{\exp\left((\alpha - \beta - \chi)/\beta\right)} = \exp\left(1\right),$$

[6]In fact, $p(0)$ is undefined. This turns out to not be an issue: for a small enough $\epsilon > 0$, at a monopoly output and at a Cournot equilibrium, we can guarantee that no supplier chooses a quantity below $\epsilon$. For this reason, the details of the inverse demand function in the vicinity of zero are immaterial as far as the chosen quantities or the resulting aggregate profit are concerned. A similar argument also applies to Example 4.6.

and Corollary 4.3 implies that for every Cournot equilibrium $\mathbf{x}$ with $p(X) > 0$,

$$\eta(\mathbf{x}) \geq \max\{f^P(\exp(1), N),\ g^P(\exp(1), r)\}. \tag{4.17}$$

Now we argue that the lower bound (4.17) holds even without the assumption that there is a best supplier associated with a linear cost function. From Proposition 4.2, the profit ratio of any Cournot equilibrium $\mathbf{x}$ will not increase if the cost function of each supplier $n$ is replaced by

$$\overline{C}_n(x) = C_n'(x_n)x, \qquad \forall x \geq 0.$$

Let $c = \min_n\{C_n'(x_n)\}$. Since the profit ratio lower bound in (4.17) holds for the modified model with linear cost functions, it applies whenever the inverse demand function is of the form (3.21).

**Example 4.5.** Suppose that Assumption 3.1, 3.3, and 3.4 hold. There are $N \geq 2$ suppliers. There exists a best supplier, the cost function of which is linear with a slope $\chi \geq 0$. Consider a group of inverse demand functions in the form in Eq. (3.23):

$$p(q) = \max\{\alpha - \beta q^\delta, 0\}, \qquad \alpha,\ \beta,\ \delta > 0.$$

It is not hard to see that Assumption 3.2 holds. Assumption 3.4 implies that $\alpha > \chi$. Since

$$\frac{d^2(qp(q))}{dq^2} = \frac{d^2(qp(q))}{dq^2} \leq p'(q) + qp''(q) = -\beta\delta q^{\delta-1} - \beta\delta(\delta-1)q^{\delta-1} = -\beta\delta^2 q^{\delta-1} \leq 0,$$

we know that Assumption 4.1 holds. Through a simple calculation we have

$$s = \left(\frac{\alpha-\chi}{\beta}\right)^{1/\delta}, \quad t = \left(\frac{\alpha-\chi}{\beta(\delta+1)}\right)^{1/\delta}.$$

We have

$$\frac{p'(t)}{p'(s)} = \frac{-\beta\delta t^{\delta-1}}{-\beta\delta s^{\delta-1}} = (\delta+1)^{\frac{1-\delta}{\delta}}.$$

From Corollary 4.3 we know that for every Cournot equilibrium $\mathbf{x}$ with $p(X) > 0$,

$$\eta(\mathbf{x}) \geq \max\left\{f^P((\delta+1)^{\frac{1-\delta}{\delta}}, N),\ g^P((\delta+1)^{\frac{1-\delta}{\delta}}, r)\right\}.$$

By the same argument in Example 4.4, we know that the derived profit ratio lower bound holds for general cost functions, as long as the inverse demand function is of the form in Eq. (3.23).

**Example 4.6.** Suppose that Assumptions 3.1, 3.3, and 3.4 hold. Among the $N \geq 2$ suppliers, there is a best supplier that has a linear cost function with a slope $\chi \geq 0$. Consider constant elasticity inverse demand functions, of the form (cf. Eq. (4) in [11])

$$p(q) = \alpha q^{-\beta}, \qquad 0 \leq \alpha, \qquad 0 \leq \beta < 1. \tag{4.18}$$

Assumption 3.4 implies that $\alpha > \chi$. Since

$$2p'(q) + qp''(q) = -2\alpha\beta q^{-\beta-1} + \alpha\beta(\beta+1)q^{-\beta-1} = -\alpha\beta(1-\beta)q^{-\beta-1} \leq 0,$$

we know that Assumption 4.1 holds. Through a simple calculation we have,

$$s = \left(\frac{\chi}{\alpha}\right)^{-1/\beta}, \qquad t = \left(\frac{\chi}{\alpha(1-\beta)}\right)^{-1/\beta}.$$

We have

$$\frac{p'(t)}{p'(s)} = \frac{-\alpha\beta t^{-\beta-1}}{-\alpha\beta s^{-\beta-1}} = (1-\beta)^{\frac{-\beta-1}{\beta}}.$$

From Corollary 4.3 we conclude that a Cournot equilibrium $\mathbf{x}$ with $p(X) > 0$ must satisfy

$$\eta(\mathbf{x}) \geq \max\left\{f^P((1-\beta)^{\frac{-\beta-1}{\beta}}, N),\ g^P((1-\beta)^{\frac{-\beta-1}{\beta}}, r)\right\}.$$

Following the argument in the end of Example 4.4, we conclude that the lower bound on the profit ratio holds for general cost functions, as long as the inverse demand function is of the form in Eq. (4.18).

## 4.6 Conclusions

For Cournot oligopoly models with concave revenue functions, results such as those provided in Theorem 4.2 show that the profit loss of a Cournot equilibrium can be bounded by a function of the largest market share at the equilibrium, and a scalar parameter that captures quantitative properties of the inverse demand function. With a small amount of additional information on the cost functions, the profit ratio lower bounds can be further refined. Our results apply to various inverse demand functions that have been considered in the literature of economics.

# Chapter 5

# Efficiency Loss in a Class of Two-Sided Market Mechanisms

## 5.1 Introduction

The last decade has seen a profusion of results exploring the price of anarchy for various non-cooperative games, e.g., selfish routing games [82, 23], proportional resource allocation games [45, 47], and oligopolist games [52, 91]. However, most existing price of anarchy results are for one-sided markets: it is either that all participants are consumers, or that all participants are suppliers, while in the real world, competition in markets typically occurs on both sides. That is, consumers and suppliers generally compete simultaneously to determine the production, the prices, and the allocation of goods.

In this chapter, we consider a two-sided resource allocation game involving both consumers and suppliers, which is a natural extension of a proportional resource allocation mechanism proposed by Kelly [50]: every consumer (user) submits a payment ("bid") and then a fraction of the resource is allocated to each consumer in proportion to its bid. In the extended two-sided market model, every supplier also submits a bid that reflects its production level, and the market-clearing price is determined by the bids of all participants (both consumers and suppliers).

This resource allocation mechanism for two-sided markets was first studied in [74], where the worst case efficiency loss was shown to occur when utility and cost functions are all linear. Furthermore, the authors of [55] derive a constant bound on the price of anarchy, for the case where every supplier $i$'s marginal cost is nondecreasing and convex with $C_i'(0) = 0$. For a more general model with convex cost functions (every supplier's marginal cost is nondecreasing but is not necessarily convex or zero at 0), we establish a tight lower bound on the efficiency of a Nash equilibrium that depends only a scalar parameter, i.e., the ratio of the maximum marginal cost of suppliers at the equilibrium that provide a positive amount of the good at the equilibrium, to the maximum marginal consumer utility at the equilibrium.

The rest of this chapter is organized as follows. In Section 5.2, we formulate the model, and introduce several main assumptions that we will be working with, and some definitions. In Section 5.3, we prove the main result of this chapter: a tight efficiency lower bound (Theorem 5.1). In Section 5.4, we derive several corollaries based on Theorem 5.1.

## 5.2 Formulation

Consider a market for a single homogeneous good, with $R$ suppliers and $Q$ consumers. Each player $i$, either a consumer or a supplier, submits a nonnegative bid $w_i$. We let $\mathbf{w}$ be the vector consisting of all bids submitted by suppliers and consumers, and $\mathbf{w}_{-i}$ be the vector of all bids submitted by players other than player $i$.

Given a market clearing price $\mu$, the amount of the good received by consumer $j$ is

$$d_j = w_j^C/\mu, \tag{5.1}$$

where the superscript $C$ denotes that the bid is submitted by a consumer. The amount of the good produced by supplier $i$ is

$$s_i = 1 - w_i^S/\mu, \tag{5.2}$$

where the superscript $S$ denotes that the bid is submitted by a consumer. Here, each supplier $i$ actually submits a supply function parameterized by its bid $w_i^S$. Note that it is implicitly assumed in our model that the maximum amount of the good that supplier $i$ could provide is 1.

To clear the market, we have

$$\sum_{j=1}^{Q} d_j = \sum_{i=1}^{R} s_i,$$

and therefore the market clearing price is given by

$$\mu = \frac{\sum_{j=1}^{Q} w_j^C + \sum_{i=1}^{R} w_i^S}{R}. \tag{5.3}$$

Let $U_j(\cdot)$ be the utility function of consumer $j$. Consumer $j$'s payoff is

$$\pi_j^C(\mathbf{w}) = U_j(w_j^C/\mu) - w_j^C. \tag{5.4}$$

Let $C_i(\cdot)$ be the utility function of supplier $i$. The payoff function of supplier $i$ is,

$$\pi_j^S(\mathbf{w}) = \mu(1 - w_i^S/\mu) - C_i(1 - w_i^S/\mu). \tag{5.5}$$

We make the following two assumptions.

**Assumption 5.1.** *For every supplier $i$, its cost function, $C_i : (-\infty, \infty) \to [0, \infty)$, is nondecreasing, continuously differentiable, and convex with $C_i(S) = 0$ for $S \leq 0$.*

**Assumption 5.2.** *For every consumer $j$, its utility function, $U_j : [0, \infty) \to [0, \infty)$, is nondecreasing, continuously differentiable, and concave with $U_j(0) = 0$. We also assume that its right derivative at 0, denoted by $U_j'(0)$ in this chapter, exists.*

**Proposition 5.1.** *Suppose that Assumptions 5.1-5.2 hold. Then, at a Nash equilibrium every supplier produces a nonnegative amount of the good.*

*Proof.* Suppose not, and that there exists a Nash equilibrium $\mathbf{w}$, at which supplier $i$ provides a negative amount of the good, i.e., $s_i < 0$. Since $s_i = 1 - w_i^S/\mu$, we conclude that $w_i^S > \mu > 0$. Since $C_i(s_i) = 0$, from (5.5) we conclude that supplier $i$ receives a negative payoff at the Nash equilibrium.

At the Nash equilibrium, let $W_{-i}$ be the sum of the bids submitted by all players (including both suppliers and consumers) except supplier $i$. With an alternative bid $\widetilde{w}_i = W_{-i}/(R-1)$, supplier $i$ would have produced a zero amount of the good and would have received a zero payoff. Therefore $\mathbf{w}$ cannot be a Nash equilibrium. □

Throughout this chapter we will only consider the nontrivial case where $R \geq 2$. When $R = 1$ (only one supplier exists in the market), there does not exist a nontrivial Nash equilibrium where at least one consumer submits a positive bid. Indeed, if the total bid of the consumers is positive, then the monopolist's payoff is strictly increasing with its own bid, because the monopolist is guaranteed to receive all the payments (bids) submitted by the consumers, and a higher bid results in a higher price, which in turn reduces the amount of the good it needs to supply.

**Proposition 5.2.** *Suppose that Assumptions 5.1 and 5.2 hold, and that $R \geq 2$. Then, there exists a Nash equilibrium.*

Proposition 5.2 is proved in Section 3.4 of [74]. To define the efficiency of a Nash equilibrium $\mathbf{w}$, we first characterize the maximum social welfare that can be achieved.

**Definition 5.1.** *The **optimal social welfare** is the optimal value of the following optimization problem,*

$$\begin{aligned} \text{maximize} \quad & \sum_{j=1}^{Q} U_j(d_j) - \sum_{i=1}^{R} C_i(s_i) \\ \text{subject to} \quad & 0 \leq d_j, \quad j = 1, \ldots, Q \\ & 0 \leq s_i \leq 1, \quad i = 1, \ldots, R, \\ & \sum_{j=1}^{Q} d_j = \sum_{i=1}^{R} s_i. \end{aligned} \tag{5.6}$$

Under Assumptions 5.1 and 5.2, the objective function in (5.6) is concave in the vector $(\{s_i\}_{i=1}^R, \{d_j\}_{j=1}^Q)$. Therefore, the following conditions are necessary and sufficient for a vector $(\{\overline{s}_i\}_{i=1}^R, \{\overline{d}_j\}_{j=1}^Q)$ to maximize the social welfare:

$$\begin{cases} U_j'(\overline{d}_j) = \lambda, & \text{if } \overline{d}_j > 0, \\ U_j'(0) \leq \lambda, & \text{if } \overline{d}_j = 0, \end{cases} \tag{5.7}$$

and

$$\begin{cases} C_i'(\overline{s}_i) = \lambda, & \text{if } \overline{s}_i > 0, \\ C_i'(0) \geq \lambda, & \text{if } \overline{s}_i = 0, \\ C_i'(0) \leq \lambda, & \text{if } \overline{s}_i = 1, \end{cases} \tag{5.8}$$

**Assumption 5.3.** *We assume that* $\min_i\{C_i'(0)\} < \max_j\{U_j'(0)\}$.

Under Assumptions 5.1-5.3, it is not hard to see that the optimal value of the optimization problem (5.6) (the optimal social welfare) is positive. We can therefore define the efficiency of a nonnegative vector $\mathbf{w} = (w_1^S, \ldots, w_R^S, w_1^C, \ldots, w_Q^C)$ as follows.

**Definition 5.2.** *The* ***efficiency*** *of a nonnegative vector* $\mathbf{w} = (w_1^S, \ldots, w_R^S, w_1^C, \ldots, w_Q^C)$ *is defined as*

$$\gamma(\boldsymbol{w}) = \frac{\sum_{j=1}^Q U_j(w_j^C/\mu) - \sum_{i=1}^R C_i(1 - w_i^S/\mu)}{\mathcal{W}}, \tag{5.9}$$

*where* $\mu$ *is the market-clearing price given in* (5.3)*, and* $\mathcal{W}$ *is the optimal value of the optimization problem in (5.6).*

## 5.3 Efficiency Loss of Nash Equilibria

In this section, we first derive some properties of Nash equilibria that will be useful later. We then prove the main result of this chapter: a tight efficiency lower bound.

We will use $W$ to denote the sum of the $R+Q$ components of a given vector $\mathbf{w}$, i.e., $W = \sum_{m=1}^Q w_m^C + \sum_{i=1}^R w_i^S$. The author of [74] has shown that the payoff functions in (5.4) and (5.5) are concave. Hence, the following conditions are necessary and sufficient for a vector **w** (with $\mu$ defined by (5.3)) to be a Nash equilibrium:

$$\begin{cases} U_j'\left(\frac{w_j^C R}{W}\right) \cdot \frac{W - w_j^C}{W} = \mu, & \text{if} \quad w_j^C > 0, \\ U_j'(0) \le \mu, & \text{if} \quad w_j^C = 0, \end{cases} \tag{5.10}$$

and

$$\begin{cases} C_i'\left(1 - \frac{w_i^S R}{W}\right) \cdot \frac{W - w_i^S}{W} = \frac{R-1}{R}\mu, & \text{if} \quad w_i^S < \mu, \\ C_i'(0) \ge \mu, & \text{if} \quad w_i^S = \mu. \end{cases} \tag{5.11}$$

**Proposition 5.3.** *Suppose that Assumptions 5.1-5.3 hold. The marginal cost of a supplier who provides a positive amount of the good at a Nash equilibrium is (strictly) less than the marginal utility of a consumer who obtains a positive amount of the good.*

*Proof.* For a supplier $i$ who provides a positive amount of the good, from (5.2) and (5.3) we have

$$0 \le w_i^S < \mu = \frac{W}{R},$$

which implies that for any consumer $j$ that obtains a positive amount of the good, we have

$$W - w_j^C \le W < \left(W - w_i^S\right) \cdot \frac{R}{R-1}. \tag{5.12}$$

Since $\mu > 0$, according to equilibrium conditions (5.10) and (5.11), we have

$$C_i'(s_i) = \frac{W}{W - w_i^S} \cdot \frac{R-1}{R}\mu < \frac{W}{W - w_j^C} \cdot \mu = U_j'(d_j),$$

which is the desired result. □

Proposition 5.3 leads to the following result.

**Proposition 5.4.** *Suppose that Assumptions 5.1-5.3 hold. The social welfare achieved at a Nash equilibrium is positive.*

*Proof.* Proposition 5.1 shows that at the Nash equilibrium, for every supplier $i$, we have $s_i \geq 0$. We now argue that the total supply at the Nash equilibrium, $\sum_{i=1}^{R} s_i$, is positive. Suppose not. Then, $\sum_{i=1}^{R} s_i = \sum_{j=1}^{Q} d_j = 0$. From the conditions in (5.10)-(5.11), we observe that $U_j'(0) \leq \mu \leq C_i'(0)$ for every pair of supplier $i$ and consumer $j$. This contradicts Assumption 5.3, and therefore we have $\sum_{i=1}^{R} s_i > 0$.

For any pair of supplier $i$ and consumer $j$ such that $s_i d_j > 0$, in Proposition 5.3 we have shown that $C_i'(s_i) < U_j'(d_j)$. Since the cost functions are convex, the utility functions are concave, and $\sum_{i=1}^{R} s_i = \sum_{j=1}^{Q} d_j > 0$, we have

$$\sum_{j=1}^{Q} U_j(d_j) \geq \sum_{j=1}^{Q} U_j'(d_j) d_j > \sum_{i=1}^{R} C_i'(s_i) s_i \geq \sum_{i=1}^{R} C_i(s_i)$$

where the second inequality follows from Proposition 5.3, and the fact that $\sum_{i=1}^{R} s_i > 0$. □

It has been shown in [74] that the worst case efficiency occurs when the utility and cost functions are all linear; we also present a proof for completeness.

**Proposition 5.5.** *Suppose that Assumptions 5.1-5.3 hold, and that $R \geq 2$. Let $\mathbf{w}$ be a Nash equilibrium, and let*[1] $\gamma_i = C_i'(s_i)$, $\beta_j = U_j'(d_j)$. *Consider a modified model where we replace the cost (utility) function of each supplier $i$ (consumer $j$, respectively) by a new function $\overline{C}_i(\cdot)$ ($\overline{U}_j(\cdot)$, respectively), defined by*

$$\overline{C}_i(x) = \max\{\gamma_i x, 0\}, \quad x \in (-\infty, \infty); \qquad \overline{U}_j(x) = \beta_j x, \quad x \in [0, \infty).$$

*Then, for the modified model, Assumptions 5.1-5.3 still hold, the vector $\mathbf{w}$ is a Nash equilibrium, and its efficiency, denoted by $\overline{\gamma}(\mathbf{w})$, satisfies $0 < \overline{\gamma}(\mathbf{w}) \leq \gamma(\mathbf{w})$.*

*Proof.* In the modified model, we observe that Assumptions 5.1 and 5.2 trivially hold. Proposition 5.4 shows that the total supply at the Nash equilibrium is positive, and Proposition 5.3 implies that Assumption 5.3 holds in the modified model (i.e.,

[1]Here, the allocation, $\{s_i\}_{i=1}^{R}$ and $\{d_j\}_{j=1}^{Q}$, is resulting from the vector $\mathbf{w}$.

$\max_j \beta_j > \min_i \gamma_i$). We also note that the vector $\mathbf{w}$ satisfies the equilibrium condition (5.10)-(5.11) for the modified model, and is therefore a Nash equilibrium in the modified model.

Let $(\{\overline{s}_i\}_{i=1}^R, \{\overline{d}_j\}_{j=1}^Q)$ be an optimal solution to the optimization problem (5.6) for the original model. Since the vector, $(\{\overline{s}_i\}_{i=1}^R, \{\overline{d}_j\}_{j=1}^Q)$, satisfies the optimality conditions (5.7)-(5.8) in the modified model, it remains socially optimal. Therefore, the efficiency of $\mathbf{w}$ in the modified model is given by

$$\overline{\gamma}(\mathbf{w}) = \frac{\sum_{j=1}^Q \beta_j d_j - \sum_{i=1}^R \gamma_i s_i}{\sum_{j=1}^Q \beta_j \overline{d}_j - \sum_{i=1}^R \gamma_i \overline{s}_i}. \tag{5.13}$$

Since $\mathbf{w}$ is a Nash equilibrium in the modified model, we have $\overline{\gamma}(\mathbf{w}) > 0$. Since $C_i(\cdot)$ is convex, we have

$$C_i(\overline{s}_i) - C_i(s_i) - \gamma_i(\overline{s}_i - s_i) \geq 0, \quad i = 1, \ldots, R.$$

Similarly, since $U_j(\cdot)$ is concave,

$$U_j(d_j) + \beta_j(\overline{d}_j - d_j) - U_j(\overline{d}_j) \geq 0, \quad j = 1, \ldots, Q.$$

Adding a nonnegative quantity to the denominator cannot increase the ratio and, therefore,

$$\gamma(\mathbf{w}) = \frac{\sum\limits_{j=1}^Q U_j(d_j) - \sum\limits_{i=1}^R C_i(s_i)}{\sum\limits_{j=1}^Q U_j(\overline{d}_j) - \sum\limits_{i=1}^R C_i(\overline{s}_i)} \geq \frac{\sum\limits_{j=1}^Q U_j(d_j) - \sum\limits_{i=1}^R C_i(s_i)}{\sum\limits_{j=1}^Q U_j(d_j) + \beta_j(\overline{d}_j - d_j) - \sum\limits_{i=1}^R (C_i(s_i) + \gamma_i(\overline{s}_i - s_i))}. \tag{5.14}$$

Since $C_i(\cdot)$ is convex, with $C_i(0) = 0$, and $U_j(\cdot)$ is concave, with $U_j(0) = 0$, we have

$$A \triangleq \sum_{i=1}^R C_i(s_i) - \sum_{i=1}^R \gamma_i s_i + \sum_{j=1}^Q \beta_j d_j - \sum_{j=1}^Q U_j(d_j) \leq 0. \tag{5.15}$$

Proposition 5.4 shows that $\gamma(\mathbf{w}) > 0$, and therefore the right-hand side of (5.14) is in the interval $(0, 1]$. Thus, adding the left-hand side of Eq. (5.15) (a nonpositive quantity) to both the numerator and the denominator cannot increase the ratio, as long as the numerator remains nonnegative. The numerator remains nonnegative because it becomes the same as the numerator in the expression (5.13) for $\overline{\gamma}(\mathbf{w})$. We obtain

$$\begin{aligned}\gamma(\mathbf{w}) &\geq \frac{\sum_{j=1}^{Q} U_j(d_j) - \sum_{i=1}^{R} C_i(s_i)}{\sum_{j=1}^{Q} U_j(d_j) + \beta_j(\overline{d}_j - d_j) - \sum_{i=1}^{R}(C_i(s_i) + \gamma_i(\overline{s}_i - s_i))} \\ &\geq \frac{\sum_{j=1}^{Q} U_j(d_j) - \sum_{i=1}^{R} C_i(s_i) + A}{\sum_{j=1}^{Q} U_j(d_j) + \beta_j(\overline{d}_j - d_j) - \sum_{i=1}^{R}(C_i(s_i) + \gamma_i(\overline{s}_i - s_i)) + A} \\ &\geq \frac{\sum_{j=1}^{Q} \beta_j d_j - \sum_{i=1}^{R} \gamma_i s_i}{\sum_{j=1}^{Q} \beta_j \overline{d}_j - \sum_{i=1}^{R} \gamma_i \overline{s}_i} \\ &= \overline{\gamma}(\mathbf{w}) > 0,\end{aligned}$$

where $A$ is defined in (5.15). □

Hence, to lower bound the efficiency of a Nash equilibrium in the original model, it suffices to lower bound the efficiency achieved at a worst Nash equilibrium for a modified model. Accordingly, and for the purpose of deriving lower bounds, we can (and will) consider only linear cost and utility functions to derive a lower bound on the efficiency of Nash equilibria.

For a Nash equilibrium $\mathbf{w}$, let $\alpha$ be the ratio of the maximum marginal cost at the equilibrium of the suppliers who provide a positive amount of the good, to the maximum marginal consumer utility at the equilibrium.[2]

**Theorem 5.1.** *Suppose that Assumptions 5.1-5.3 hold. Then, for a Nash equilibrium* $\mathbf{w}$ *we have*

$$\gamma(\mathbf{w}) \geq \frac{1-\alpha}{1+\alpha}. \tag{5.16}$$

*Furthermore, the bound is tight. That is, for any* $\alpha \in [0, 1)$, *there exists a game with a Nash equilibrium that yields an efficiency of* $(1-\alpha)/(1+\alpha)$.

The proof of Theorem 5.1 is given in Appendix D. The derived lower bound is

[2] According to Proposition 5.4, we know that the total supply at the Nash equilibrium is positive, and Proposition 5.3 implies that $\alpha \in [0, 1)$.

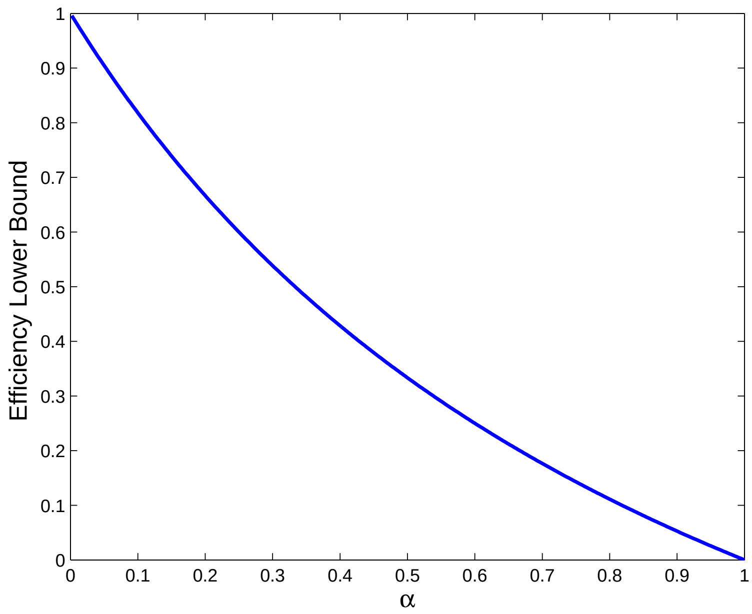


Figure 5-1: A tight lower bound for the efficiency of Nash equilibria of the two-sided market mechanism.

shown in Fig. 5-1, as a function of $\alpha$. Our result is different from that derived in [55], where a constant efficiency lower bound is established for a smaller group of convex cost functions (every supplier $i$'s marginal cost is assumed to be nondecreasing and convex with $C_i'(0) = 0$ in [55]).

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## 5.4 Corollaries and Applications

Given the supplier cost functions and the consumer utility functions, the lower bound of Theorem 5.1 requires additional knowledge on slopes of cost and utility functions at a Nash equilibrium. In this section, we apply Theorem 5.1 to establish several efficiency lower bounds that depend only on the cost and utility functions.

**Corollary 5.1.** *Suppose that Assumptions 5.1-5.3 hold, $R \geq 2$, and that all utility and cost functions are linear, i.e.,*

$$\gamma_i = C_i'(\cdot), \quad \beta_j = U_j'(\cdot).$$

*Let* $\beta = \max_j\{\beta_j\}$*. Then, there exists a Nash equilibrium* $\mathbf{w}$ *such that*

$$\gamma(\mathbf{w}) \geq \frac{1-\overline{\alpha}}{1+\overline{\alpha}},$$

*where*

$$\overline{\alpha} = \max_i\{\gamma_i \mid \gamma_i < \beta\}/\beta.$$

The corollary follows from the fact that $\overline{\alpha} \geq \alpha$, because a supplier $i$ with $\gamma_i \geq \beta$ must produce a zero amount of the good at the Nash equilibrium (Proposition 5.3).

**Corollary 5.2.** *Suppose that Assumptions 5.1-5.3 hold, and that* $R > 2$*. There exists a Nash equilibrium* $\mathbf{w}$ *such that*

$$\gamma(\mathbf{w}) \geq \frac{1-\overline{\alpha}}{1+\overline{\alpha}},$$

*where*

$$\overline{\alpha} = \min\left\{1, \frac{\max_i\{C_i'(1)\}}{\mu}\right\}.$$

*Proof.* Proposition 5.4 implies that at the Nash equilibrium, there exists a consumer who obtains a positive amount of the good. The equilibrium conditions in (5.11) imply that for any supplier $i$ with $s_i = 0$ at the Nash equilibrium, its marginal cost at zero must be higher than any other supplier who provides a positive amount of the good. Therefore, we have

$$\alpha \leq \frac{\max_i\{C_i'(s_i)\}}{\max_j\{U_j'(d_j)\}}.$$

From the equilibrium conditions (5.10), we have that $\mu < \max_j\{U_j'(d_j)\}$. Since every cost function is convex, and every supplier can at most provide one unit amount of the good, we further have

$$\alpha \leq \frac{\max_i\{C_i'(s_i)\}}{\max_j\{U_j'(d_j)\}} \leq \frac{\max_i\{C_i'(1)\}}{\mu} = \overline{\alpha}.$$

According to Theorem 5.1, we have

$$\gamma(\mathbf{w}) \geq \frac{1-\alpha}{1+\alpha} \geq \frac{1-\overline{\alpha}}{1+\overline{\alpha}}.$$

□

# Chapter 6

# Pricing of Fluctuations in Electricity Markets

## 6.1 Introduction

This chapter is motivated by the fact that fluctuations in the demand for electricity to be met by conventional thermal generating units typically result in significantly increased, and nontrivial, ancillary costs. Today, such demand fluctuations are mainly due to time-dependent consumer preferences. In addition, in the future, a certain percentage of electricity production is required by law in many states in the U.S. to come from renewable resources [7]. The dramatic volatility of renewable energy resources may aggravate the variability of the demand for conventional thermal generators and result in significant ancillary cost. More concretely, either a demand surge or a decrease in renewable generation may result in (i) higher energy costs due to the deployment of peaking plants with higher ramping rates but higher marginal cost, such as oil/gas combustion turbines, and (ii) the redispatch cost[1] that the system will incur to meet reserve constraints if the increase of demand (decrease of renewable

[1] A certain level of reserve must always be maintained in an electric power system. Local reserve shortages are usually due to the quick increase of system load rather than a capacity deficiency. If the increase of system load makes the system short in reserves, the system will redispatch resources to increase the amount of reserves available. Redispatch generally increases the generation cost and results in higher prices. The redispatch cost can be very high (cf. Section 2.3.2 of [42]).

generation) causes a reserve shortage.

There is general agreement that charging real-time prices (that reflect current operating conditions) to electricity consumers has the potential of reducing supplier ancillary cost, improving system efficiency, and lowering volatility in wholesale prices [93, 86, 17]. Therefore, dynamic pricing, especially real-time marginal cost pricing, is often identified as a priority for the implementation of wholesale electricity markets with responsive demand [41], which in turn raises many new questions. For example, should prices for a given time interval be calculated ex ante or ex post? Does real-time pricing introduce the potential for new types of market instabilities? How is supplier competition affected? In this chapter, we abstract away from almost all of these questions and focus on the specific issue of whether prices should also explicitly encourage consumers to adapt their demands so as to reduce supplier ancillary cost.

To illustrate the issue that we focus on, we note that a basic model of electricity markets assumes that the cost of satisfying a given level $A_t$ of aggregate demand during period $t$ is of the form $C(A_t)$. It then follows that in a well-functioning wholesale market, the observed price should more or less reflect the marginal cost $C'(A_t)$. In particular, prices should be more or less determined by the aggregate demand level. Empirical data do not quite support this view. Fig. 6-1 plots the real-time system load and the hourly prices on February 11, 2011 and on February 16, 2011, as reported by the New England ISO [43]. We observe that prices do not seem to be determined solely by $A_t$ but that the changes in demand, $A_t - A_{t-1}$, also play a major role. In particular, the largest prices seem to occur after a demand surge, and not necessarily at the hour when the load is the highest. We take this as evidence that the total cost over $T + 1$ periods is not of the form

$$\sum_{t=0}^{T} C(A_t),$$

but rather of the form

$$\sum_{t=0}^{T} \big(C(A_t) + H(A_{t-1}, A_t)\big), \tag{6.1}$$

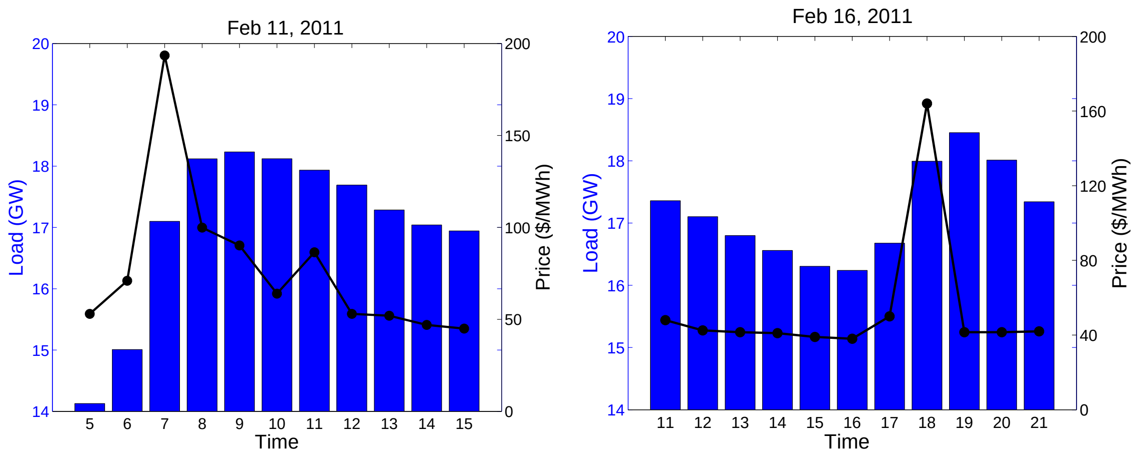


Figure 6-1: Real-time prices and actual system load, ISO New England Inc. Blue bars represent the real-time system loads and the dots connected by a black line represent the hourly prices.

for a suitable function $H$.

We take the form of Eq. (6.1) as our starting point and raise the question of the appropriate prices. A naive view would argue that at time $t$, $A_{t-1}$ has already been realized, and taking its value for granted, a consumer should be charged a unit price equal to

$$C'(A_t) + \frac{\partial}{\partial A_t} H(A_{t-1}, A_t), \tag{6.2}$$

which is the supplier's marginal cost at stage $t$. We refer to this naive approach as "marginal cost pricing" (MCP). However, a simple argument based on standard mathematical programming optimality conditions shows that for system optimality to obtain, the demand $A_{t-1}$ should also incur (after $A_t$ is realized) a cost of

$$\frac{\partial}{\partial A_{t-1}} H(A_{t-1}, A_t). \tag{6.3}$$

In day-ahead markets, suppliers typically carry out an intertemporal optimization, and it is reasonable to expect that the two types of marginal costs, captured by Eqs. (6.2) and (6.3), are both properly accounted for. However, in current real-time balancing markets, once $A_{t-1}$ is realized, a supplier will aim at charging the marginal cost in Eq. (6.2), but will be unable to charge the additional marginal cost in Eq.

(6.3) to the past demand $A_{t-1}$. In contrast, the pricing mechanism that we propose and analyze in this chapter is designed to include the additional marginal cost in Eq. (6.3).[2]

The actual model that we consider will be richer from the one discussed above, in a number of respects. It includes an exogenous source of uncertainty (e.g., representing weather conditions) that has an impact on consumer utility and supplier cost, and therefore the model can incorporate the effects of volatile renewable electricity production. It allows for consumers with internal state variables (e.g., a consumer's demand may be affected by how much electricity she has already used). It also allows for multiple consumer types (i.e., with different utility functions and different internal state dynamics). Consumers are generally modeled as price-takers, as would be the case in a model involving an infinity (a continuum) of consumers. However, we also consider the case of finite consumer populations and explore certain equilibrium concepts that are well-suited to the case of finite but large consumer populations. On the other hand, we ignore most of the distinctions between ex post and ex ante prices. Instead, we assume that at each time step, the electricity market clears. The details of how this could happen are important, but are generic to electricity markets, hence not specific to our models, and somewhat orthogonal to the subject of this chapter. (See however Section 6.7 for some discussion of implementation issues.)

The ancillary cost function $H(A_{t-1}, A_t)$ is of course a central element of our model. How can we be sure that this is the right form? In general, redispatch and reserve dynamics are complicated and one should not expect such a function to capture all of the complexity of the true system costs; perhaps, a more complex functional form such as $H(A_{t-2}, A_{t-1}, A_t)$ would be more appropriate. We believe that the form we have chosen is a good enough approximation, at least under certain conditions. To argue this point, we present in Appendix E.1 an example that involves a more detailed system model (in which the true cost is a complicated function of the entire history

[2] In current two-settlement systems, the real-time prices are charged only on the difference of the actual demand and the estimated demand at the day-ahead market. However, the two-settlement system provides the same real-time incentives to price-taking consumers, as if they were purchasing all of their electricity at the real-time prices (cf. Chapter 3-2 of [87]).

of demands) and show that a function of the form $H(A_{t-1}, A_t)$ can capture most of the cost of ancillary services.

### 6.1.1 Summary of contributions

Before continuing, we provide here a roadmap of the chapter together with a summary of our main contributions.

(a) We provide a stylized (yet quite rich) model of an electricity market, which incorporates the cost of ancillary services (cf. Section 6.2).

(b) We provide some justification of the form of the cost function in our model as a reasonable approximation of more detailed physical models (cf. Appendix E.1).

(c) We propose a pricing mechanism that properly charges for the effects of consumer actions on ancillary services (cf. Section 6.3).

(d) For a continuum model featuring a nonatomic continuum of consumers, we introduce the notion of a Dynamic Oblivious Equilibrium (DOE), in which every consumer maximizes her expected payoff, under the sequence of prices induced by the DOE strategy profile. We show that (under standard convexity assumptions), our mechanism maximizes social welfare in a model involving a continuum of (price-taking) consumers (cf. Theorem 6.2).

(e) We consider the case of a large but finite number of consumers and study it from a game-theoretic perspective. We show that a large population of consumers who act according to a DOE results in asymptotically optimal (as the number of consumers goes to infinity) social welfare (cf. Theorem 6.2), and asymptotically maximizes every consumer's expected payoff (cf. Theorem 6.1).

(f) We illustrate the potential benefits of our mechanism through a simple numerical example. In particular, we show that compared with marginal cost pricing, the proposed mechanism reduces the peak load, and therefore has the potential to reduce the need for long-term investments in peaking plants (cf. Section 6.8).

### 6.1.2 Related literature

There are two streams of literature that are related to our work: the electricity pricing and the game-theoretic literature.

Regarding electricity markets, the impact of supply friction on economic efficiency and price volatility has received some recent attention. Mansur shows that under ramping constraints, the prices faced by consumers may not necessarily equal the true supplier marginal cost [64]. In a continuous-time competitive market model, Cho and Meyn show that the limited capability of generating units to meet real-time demand, due to relatively low ramping rates, does not harm social welfare, but may result in extreme price fluctuations [18]. In a similar spirit, Kizilkale and Mannor construct a dynamic game-theoretic model to study the tradeoff between economic efficiency and price volatility [51]. Closer to the present chapter, Cho and Meyn construct a dynamic newsboy model to study the reserve management problem in electricity markets, where the demand is assumed to be exogenous [19]. The supplier cost in their model depends not only on the overall demand, but also on the generation resources used to satisfy the demand. For example, a quickly increasing demand may require more responsive and more expensive resources (e.g., peaking generation plants).

To study the impact of pricing mechanisms on consumer behavior and load fluctuations, we construct a dynamic game-theoretic model that differs from existing dynamic models for electricity markets, and incorporates both the consumers' responses to real-time price fluctuations and the suppliers' ancillary cost incurred by load swings. Some major differences between our model and existing ones are discussed in the end of Section 6.2.

On the game-theoretic side, the standard solution concept for stochastic dynamic games is **Markov perfect equilibrium** (MPE) [32, 66], where an agent's action depends on the current state of all agents. As the number of agents grows large, the computation of an MPE is often intractable [25]. For this reason, alternative equilibrium concepts for related games featuring a nonatomic continuum of agents (e.g., "oblivious equilibrium" or "stationary equilibrium" for dynamic games without aggregate shocks), have received much recent attention [96, 2].

There is a large literature on a variety of approximation properties of nonatomic equilibrium concepts [65, 3, 4]. Recently, the authors of [2] derive sufficient conditions for a stationary equilibrium strategy to have the **Asymptotic Markov Equilibrium** (AME) property, which requires that a stationary equilibrium strategy asymptotically maximizes every agent's expected payoff, as the number of agents grows large. In their model, the random shocks are assumed to be idiosyncratic across agents. However, in the problem that we are interested in, it is important to incorporate aggregate shocks (such as weather conditions) that have a global impact on all agents in an electricity market. In this spirit, the authors of [97] consider a market model with aggregate profit shocks, and study an equilibrium concept at which every firm's strategy depends on the firm's current state and on the recent history of the aggregate shock. For a general dynamic game model with aggregate shocks, Bodoh-Creed shows that a nonatomic equivalent of an MPE, which we refer to as a **Dynamic Oblivious Equilibrium (DOE)** in this chapter, asymptotically approximates an MPE in the sense that as the number of agents increases to infinity, the actions taken in an MPE can be well approximated by those taken by a DOE strategy of the nonatomic limit game. However, without further restrictive assumptions on the agents' state transition kernel, the approximation property of the actions taken by a DOE strategy does not imply the AME property of the DOE, and we are not aware of any AME results for models that include aggregate shocks. Our work is different in this respect: for a dynamic nonatomic model with aggregate shocks, which is a simplified variation of the general model considered in [12], we prove the AME property of a DOE.

The efficiency of nonatomic equilibria for static games has been addressed in recent research [83, 70, 13]. For a dynamic industry model with a continuum of identical producers and exogenous aggregate shocks, Lucas and Prescott show (under convexity assumptions) that the expected social welfare is maximized at a unique competitive equilibrium [59]. In a similar spirit, in this chapter we show (under convexity assumptions) that the proposed pricing mechanism maximizes the expected social welfare in a model involving a continuum of (possibly heterogeneous) consumers. We also consider the case of a large but finite number of consumers, and show that the expected social welfare can be approximately maximized if all consumers act according to a nonatomic equilibrium (DOE). For large dynamic games, the asymptotic social optimality of nonatomic equilibria (DOEs) established in this chapter seems to be new.

### 6.1.3 Outline of the chapter

The rest of the chapter is organized as follows. In Section 6.2, we introduce the dynamic game model. In Section 6.3, inspired by the observation that the marginal cost pricing may not be socially optimal in a market with friction (see Example 6.1), we propose an alternative dynamic pricing mechanism. In Section 6.4, we give the formal definition of a DOE. In Section 6.5, we show that as the number of consumers increases to infinity, every consumer's expected payoff is approximately maximized by a DOE strategy, if the other consumers follow the DOE strategy. In Section 6.6, we show that under the proposed pricing mechanism, social welfare is maximized at a DOE of a continuum game. Also, for a dynamic game with finitely many consumers, and under a certain convexity assumption, we validate the asymptotic social optimality of the proposed pricing mechanism: as the number of consumers grows large, if all consumers use a DOE strategy, then social welfare is approximately maximized. In Section 6.7, we make some discussions on the implementation of the proposed pricing mechanism. In Section 6.8, we present several numerical examples to compare the proposed pricing mechanism with marginal cost pricing. Finally, in Section 6.9, we make some brief concluding remarks and discuss some directions for future work.

## 6.2 Model

We consider a $(T+1)$-stage dynamic game with the following elements:

1. The game is played in **discrete time**. We index the time periods with $t = 0, 1, \ldots, T$. Each stage may represent a five minute interval in real-time balancing markets where prices and dispatch solutions are typically provided at five

minute intervals.

2. There are $n$ **consumers**, indexed by $1, \ldots, n$.

3. At each stage $t$, let $s_t \in \mathcal{S}$ be an **exogenous state**, which evolves as a Markov chain and whose transitions are independent of the consumer actions. The set $\mathcal{S}$ is assumed to be finite. In electricity markets, the exogenous state may represent time and/or the weather conditions, which impact consumer utility and supplier cost.

4. For notational conciseness, for $t \geq 1$, let $\overline{s}_t = (s_{t-1}, s_t)$, and let $\overline{s}_0 = s_0$. We use $\overline{\mathcal{S}}_t$ to denote the set of all possible $\overline{s}_t$. We refer to $\overline{s}_t$ as the **global state** at stage $t$.

5. Given an initial global state $s_0$, the **initial states (types) of the consumers**, $\{x_{i,0}\}_{i=1}^n$, are independently drawn according to a probability measure $\eta_{s_0}$ over a finite set $\mathcal{X}_0$. We use $X$ to denote the cardinality of $\mathcal{X}_0$.

6. At stage $t$, the **state of consumer** $i$ is denoted by $x_{i,t}$. At $t = 0$, consumer $i$'s initial state, $x_{i,0}$, indicates her type. For $t = 1, \ldots, T$, we have $x_{i,t} = (x_{i,0}, z_{i,t})$, where $z_{i,t} \in \mathcal{Z}$ and $\mathcal{Z} = [0, Z]$ is a compact subset of $\mathbb{R}$. The variables $\{z_{i,t}\}_{i=1}^n$ allow us to model intertemporal substitution effects in consumer $i$'s demand.

7. We use $\mathcal{X}_t$ to denote a **consumer's state space** at stage $t$. In particular, at stage $t \geq 1$, $\mathcal{X}_t = \mathcal{X}_0 \times \mathcal{Z}$.

8. At stage $t$, consumer $i$ takes an **action** $a_{i,t}$, and receives a nonnegative **utility**[3] $U_t(x_{i,t}, s_t, a_{i,t})$.

9. Each consumer's **action space** is $\mathcal{A} = [0, B]$, where $B$ is a positive real number. (In the electric power context, $B$ could reflect a transmission capacity constraint.)

10. We use $A_t = \sum_{i=1}^n a_{i,t}$ to denote the **aggregate demand** at stage $t$.

[3]At $t = 0$, $U_0$ is a mapping from $\mathcal{X}_0 \times \mathcal{S} \times \mathcal{A}$ to $[0, \infty)$, while for $t \geq 1$, $U_t$ is a mapping from $\mathcal{X}_0 \times \mathcal{Z} \times \mathcal{S} \times \mathcal{A}$ to $[0, \infty)$.

11. Given consumer $i$'s current state, $x_{i,t}$, and the next exogenous state $s_{t+1}$, the next state of consumer $i$ is determined by her action taken at stage $t$, i.e., $x_{i,t+1} = (x_{i,0}, z_{i,t+1})$, where $z_{i,t+1} = r(x_{i,t}, a_{i,t}, s_{t+1})$, for a given function $r$.

12. Let $G_t = A_t + R_t$ be the **capacity** available at stage $t$, where $R_t$ is the system **reserve** at stage $t$. For simplicity, we assume that the system reserve at stage $t$ depends only on the current aggregate demand, $A_t$, and the current exogenous state $s_t$. That is, we have $R_t = g(A_t, s_t)$ for a given function of $g$ that reflects the reserve policy of the system operator.

13. At stage $t$, let $\overline{C}(A_t, R_t, s_t)$ denote the sum of the supplier's **generation cost** to meet the aggregate demand $A_t$ through its primary energy resources, e.g., base-load power plants, and the cost to maintain a system reserve $R_t$. Since $R_t$ depends only on $A_t$ and $s_t$, we can write $\overline{C}(A_t, R_t, s_t)$ as a function of $A_t$ and $s_t$, i.e., there exists a **primary cost function** $C : \mathbb{R} \times \mathcal{S} \to [0, \infty)$ such that $C(A_t, s_t) = \overline{C}(A_t, R_t, s_t)$. We assume that for any $s \in \mathcal{S}$, $C(\cdot, s)$ is nondecreasing.

14. At stage $t \geq 1$, let $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$ denote the ancillary cost incurred by load swings[4]. Since $R_t$ depends only on $A_t$ and $s_t$, we can write $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$ as a function of $A_{t-1}$, $A_t$, $s_{t-1}$, and $s_t$, i.e., there exists an **ancillary cost function** $H : \mathbb{R}^2 \times \mathcal{S}^2 \to [0, \infty)$ such that $\overline{H} = H(A_{t-1}, A_t, \overline{s}_t)$. The ancillary cost at stage 0 is assumed to be a function of $s_0$ and $A_0$.

15. At stage 0, the total supplier cost is of the form

$$C(A_0, s_0) + H_0(A_0, s_0), \tag{6.4}$$

and for $t = 1, \ldots, T$, the total supplier cost at stage $t$ is given by

$$C(A_t, s_t) + H(A_{t-1}, A_t, \overline{s}_t). \tag{6.5}$$

[4]In general, the supplier ancillary cost may depend on the entire history of system load and global states. However, a simplified form of ancillary cost functions, $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$, can serve as a good approximation of the supplier ancillary cost (cf. Appendix E.1).

In contrast to existing dynamic models for electricity markets with an exogenous demand process [18, 19], our dynamic game-theoretic model incorporates the consumer reactions to price fluctuations, and allows us to study the impact of pricing mechanisms on consumer behavior and economic efficiency. Through a dynamic game-theoretic formulation, Kizilkale and Mannor study the tradeoff between economic efficiency and price volatility. Our model is different from the one studied in [51] in the following aspects:

1. Our model allows the generation cost to depend on the exogenous state(s), and therefore incorporates supply-side volatility due to uncertainty in renewable electricity generation.

   As an example, consider a case where the exogenous state, $s_t$, represents the electricity generation from renewable resources at stage $t$. Then the demand for conventional generation is $A_t - s_t$. Suppose that the system reserve is proportional to the system load, say, $\delta A_t$ with some $\delta > 0$. The cost function, $\overline{C}(A_t, R_t, s_t)$, then depends only on the output of conventional generating units, $A_t - s_t$, and the system reserve $\delta A_t$. The ancillary cost occurred at stage $t$ depends on the system reserve and the outputs of conventional generating units at stages $t-1$ and $t$, and is therefore a function of $A_{t-1}$, $A_t$, $s_{t-1}$, and $s_t$.

2. More important, instead of penalizing each consumer's attempt to change her action across time, the ancillary cost function in our model penalizes the change in the aggregate demand of all consumers. The change in a single consumer's action may harm or benefit the social welfare, while the volatility of the aggregate demand is usually undesirable.

The main feature of our model is the ancillary cost function $H$, which makes the supplier cost nonseparable over time. In an electric power system, the ancillary cost function models the costs associated with the variability of conventional thermal generator output, such as the energy cost of peaking plants and the redispatch cost. Note that the ancillary cost is not necessarily zero when $A_t \leq A_{t-1}$, because thermal generating units have ramping-down constraints, and because a decrease in renewable electricity production may deploy the system reserve, even if $A_t \leq A_{t-1}$. The presence of the ancillary cost function makes marginal cost pricing inefficient (cf. Example 6.1 in Section 6.3).

To keep the model simple, we do not incorporate any idiosyncratic randomness in consumer state evolution. Thus, besides the randomness of consumer types (initial states), the only source of stochasticity in the model is from the exogenous state $s_t$.

To effectively highlight the impact of pricing mechanisms on consumer behavior, as well as on economic efficiency and demand volatility, we have made the following simplifications and assumptions for the power grid:

(a) As in [18], we assume that the physical production capacity is large enough so that the possible changes of the generation capacity are not constrained.

(b) Transmission capacity is large enough to avoid any congestion. We also assume that the cost of supplying electricity to consumers in different locations is the same. Therefore, a common price for all consumers is appropriate.

(c) We use a simplified form of ancillary cost functions, $\overline{H}(A_{t-1}, A_t, R_{t-1}, R_t, s_t)$, to approximate the supplier ancillary cost. In Appendix E.1, we present a numerical example to justify this approximation.

## 6.3 The Pricing Mechanism

The marginal cost pricing mechanism charges a time-varying unit price on each consumer's demand. As demonstrated in the following example, a time-varying price that equals the supplier's instantaneous marginal cost may not achieve social optimality in a market with friction. For this reason, we propose a new pricing mechanism that takes into account the ancillary cost associated with a consumer's demand at the previous stage.

**Example 6.1.** Consider a two-stage deterministic model with one consumer and one supplier. At stage $t$, the consumer's utility function is $U_t : [0, \infty) \to [0, \infty)$. Let

$a_t$ denote the demand at stage $t$, and let $\mathbf{a} = (a_0, a_1)$. Let $g_t$ denote the actual generation at stage $t$, and let $\mathbf{g} = (g_0, g_1)$. Two unit prices, $p_0$ and $p_1$, are charged on the consumption at stage 0 and 1, respectively. Let $\mathbf{p} = (p_0, p_1)$. The consumer's payoff-maximization problem is

$$\underset{\mathbf{a}}{\text{Maximize}} \quad U_0(a_0) - p_0 a_0 + U_1(a_1) - p_1 a_1. \tag{6.6}$$

Let $H_0$ be identically zero, and let the ancillary cost function at stage 1 depend only on the difference between the supply at the two stages. That is, the ancillary cost at stage 1 is of the form $H(g_1 - g_0)$. The supplier's profit-maximization problem is

$$\underset{\mathbf{g}}{\text{Maximize}} \quad p_0 g_0 + p_1 g_1 - C(g_0) - C(g_1) - H(g_1 - g_0). \tag{6.7}$$

The social planner's problem is

$$\begin{aligned} &\underset{(\mathbf{a},\mathbf{g})}{\text{Maximize}} \quad U_0(a_0) + U_1(a_1) - C(g_0) - C(g_1) - H(g_1 - g_0) \\ &\text{subject to} \quad \mathbf{a} = \mathbf{g}. \end{aligned} \tag{6.8}$$

Now consider a **competitive equilibrium**, $(\mathbf{a}, \mathbf{g}, \mathbf{p})$, at which the vector $\mathbf{a}$ solves the consumer's optimization problem (6.6), the vector $\mathbf{g}$ solves the supplier's optimization problem (6.7), and the market clears, i.e., $\mathbf{a} = \mathbf{g}$. Suppose that the utility functions are concave and continuously differentiable, and that the cost functions $C$ and $H$ are convex and continuously differentiable. We further assume that $H'(0) = 0$, and that for $t = 0, 1$, $U_t'(0) > C'(0)$, $U_t'(B) < C'(B)$. Then, there exists a competitive equilibrium, $(\mathbf{a}, \mathbf{g}, \mathbf{p})$, which satisfies the following conditions:

$$\begin{cases} U_0'(a_0) = p_0, \\ U_1'(a_1) = p_1, \end{cases} \qquad \begin{cases} C'(a_0) - H'(a_1 - a_0) = p_0, \\ C'(a_1) + H'(a_1 - a_0) = p_1. \end{cases} \tag{6.9}$$

We conclude that the competitive equilibrium solves the social welfare maximization problem in (6.8), because it satisfies the following (sufficient) optimality conditions:

$$
\begin{aligned}
&U_0'(a_0) = C'(a_0) - H'(a_1 - a_0), \quad U_1'(a_1) = C'(a_1) + H'(a_1 - a_0), \\
&a_0 = g_0, \quad a_1 = g_1.
\end{aligned} \tag{6.10}
$$

However, we observe that the socially optimal price $p_0$ does not equal the supplier's marginal cost at stage 0, $C'(a_0)$. Hence, by setting the price equal to $C'(a_0)$, we may not achieve social optimality. More generally, in a market with friction, marginal cost pricing need not be socially optimal because it does not take into account the externality conferred by the action $a_0$ on the ancillary cost at stage 1, $H(a_1 - a_0)$. At a socially optimal competitive equilibrium, the consumer should pay

$$
\big(C'(a_0) - H'(a_1 - a_0)\big)a_0 + \big(C'(a_1) + H'(a_1 - a_0)\big)a_1,
$$

where the price on $a_0$ is the sum of the supplier marginal cost at stage 0, $C'(a_0)$, and the marginal ancillary cost associated with $a_0$, $-H'(a_1 - a_0)$. □

Before proposing the pricing mechanism, we introduce a differentiability assumption on the cost functions.

**Assumption 6.1.** *For any $s \in \mathcal{S}$, $C(\cdot, s)$ is continuously differentiable on $[0, \infty)$. For any $(A', \bar{s}) \in \mathcal{A} \times \mathcal{S}^2$, $H(A, A', \bar{s})$ and $H(A', A, \bar{s})$ are continuously differentiable in $A$ on $[0, \infty)$.*[5]

Inspired by Example 6.1, we introduce prices

$$
p_t = C'(A_t, s_t), \quad t = 0, \ldots, T, \tag{6.11}
$$

and

$$
q_t = \frac{\partial H(A_{t-1}, A_t, \bar{s}_t)}{\partial A_{t-1}}, \quad w_t = \frac{\partial H(A_{t-1}, A_t, \bar{s}_t)}{\partial A_t}, \quad t = 1, \ldots, T. \tag{6.12}
$$

At stage 0, we let $q_0 = 0$ and $w_0 = H_0'(A_0, s_0)$. Under the proposed pricing mechanism,

[5] At the boundary of the domain, 0, we require continuity of the right-derivative of $C$, $H_0$, and $H$.

consumer $i$'s payoff at stage $t$ is given by

$$U_t(x_{i,t}, s_t, a_{i,t}) - (p_t + w_t)a_{i,t} - q_t a_{i,t-1}. \tag{6.13}$$

Note that $p_t + w_t$ is the supplier marginal cost at stage $t$ (including the marginal ancillary cost). The proposed pricing mechanism charges consumer $i$ an additional price $q_t$, equal to the marginal ancillary cost with respect to $a_{i,t-1}$, on her previous demand.

To understand the nature of the mechanism, consider a scenario where demand increases quickly. Generally, $H(A_{t-1}, A_t, \overline{s}_t)$ should be nonincreasing in $A_{t-1}$, so that $q_t$ is nonpositive. Similarly, $w_t$ is in general nonnegative because $H(A_{t-1}, A_t, \overline{s}_t)$ is typically nondecreasing in $A_t$. On the other hand, when demand decreases quickly, it is possible to have $q_t$ be nonnegative and $w_t$ be nonpositive. Since $C(\cdot, s)$ is naturally nondecreasing, $p_t$ is always nonnegative.

We now give some notations that will be useful later. For $t = 1, \ldots, T$, let $y_{i,t} = (a_{i,t-1}, x_{i,t})$ be the **augmented state** of consumer $i$ at stage $t$. At $t = 0$, let $y_{i,0} = x_{i,0}$. At stage $t$, let $\mathcal{Y}_t$ be the set of all possible augmented states. In particular, we have $\mathcal{Y}_0 = \mathcal{X}_0$, and $\mathcal{Y}_t = \mathcal{A} \times \mathcal{X}_t$, for $t = 1, \ldots, T$.

For the rest of the chapter, we let $\Delta_n(\cdot)$ denote the set of empirical probability distributions over a given set that can be generated by $n$ samples from the given set. We use $f_t \in \Delta_n(\mathcal{Y}_t)$ to denote the empirical distribution of the augmented state of all consumers at stage $t$, and use $f_{-i,t} \in \Delta_{n-1}(\mathcal{Y}_t)$ to denote the empirical distribution of the augmented state of all consumers (excluding consumer $i$) at stage $t$. Throughout the rest of the chapter, we refer to $f_t$ as the **population state** at stage $t$. Let $u_t \in \Delta_n(\mathcal{A})$ denote the empirical distribution of all consumers' actions at stage $t$, and $u_{-i,t} \in \Delta_{n-1}(\mathcal{A})$ be the empirical distribution of all consumers' (excluding consumer $i$) actions at stage $t$.

For a given $n$, it can be seen from (6.11) and (6.12) that the prices, and thus the stage payoff in (6.13), depend on the current global state, consumer $i$'s current augmented state and current action, as well as the empirical distribution of other consumers' current augmented state and action. Hence, for a certain function $\pi(\cdot)$, we can write the stage payoff in (6.13) as

$$\pi(y_{i,t}, \overline{s}_t, a_{i,t}, f_{-i,t}, u_{-i,t}) = U_t(x_{i,t}, s_t, a_{i,t}) - (p_t + w_t)a_{i,t} - q_t a_{i,t-1}. \tag{6.14}$$

## 6.4 Dynamic Oblivious Equilibrium

To study the aggregate behavior of a large number of consumers in an electricity market, we consider a nonatomic game consisting of a continuum of infinitesimally small consumers indexed by $i \in [0, 1]$. In a nonatomic model, any single consumer's action has no influence on the aggregate demand and the prices. We consider a class of **dynamic oblivious strategies** in which a consumer's action depends only on the history of past exogenous states, $h_t = (s_0, \ldots, s_t)$, and her current state[6], i.e., of the form

$$a_{i,t} = \overline{\nu}_t(x_{i,t}, h_t).$$

Suppose that consumer $i$ uses a dynamic oblivious strategy $\overline{\nu} = (\overline{\nu}_0, \ldots, \overline{\nu}_T)$. Since there is no idiosyncratic randomness, given a history $h_t$, the state $x_{i,t}$ of consumer $i$ at stage $t$ depends only on her initial state $x_{i,0}$. That is, there is a mapping $l_{\overline{\nu},h_t} : \mathcal{X}_0 \to \mathcal{X}_t$, such that $x_{i,t} = l_{\overline{\nu},h_t}(x_{i,0})$. Therefore, we can write the action taken by a dynamic oblivious strategy as

$$a_{i,t} = \nu_t(x_{i,0}, h_t) \triangleq \overline{\nu}_t(l_{\overline{\nu},h_t}(x_{i,0}), h_t). \tag{6.15}$$

Let $\nu = \{\nu_0, \ldots, \nu_T\}$ denote a dynamic oblivious strategy, and $\mathfrak{V}$ be the set of all such strategies.

An alternative form of strategies that depends on consumers' expectations on future prices would lead to a Rational Expectations Equilibrium (REE), an equilibrium

[6]Note that a dynamic oblivious strategy depends only on the consumer's current state, instead of her augmented state. As we will see in Section 6.4.2, in a continuum model, since any single consumer has no influence on the prices, an optimal strategy need not take into account the action taken at the previous stage.

concept based on the rational expectations approach pioneered by [71]. In our continuum model, since the only source of stochasticity is from the exogenous state $s_t$, given a strategy profile, future prices are completely determined by the history $h_t$. Therefore, it is not implausible to expect that the strategy defined in (6.15) will lead to an equilibrium concept (DOE) that is identical in outcomes with a REE (cf. the discussion in Section 6.4.2).

Before formally defining a **Dynamic Oblivious Equilibrium** (DOE), we would like to give some intuition on the idea behind this equilibrium concept. In a continuum model, if all consumers use a common dynamic oblivious strategy $\nu$, the aggregate demand and the prices at stage $t$ depend only on the sequence of exogenous states, $h_t = (s_0, \ldots, s_t)$. A dynamic oblivious strategy $\nu$ is a DOE (cf. the formal definition in Section 6.4.2) if it maximizes every consumer's value function, under the sequence of prices that $\nu$ induces. In Section 6.4.3, we associate a continuum model with a sequence of $n$-consumer models ($n = 1, 2, \ldots$), and describe the relation between the continuum model and the corresponding $n$-consumer model.

### 6.4.1 The sequence of prices induced by a dynamic oblivious strategy

Let $h_t = (s_0, \ldots, s_t)$ denote a history, up to stage $t$, and let $\mathcal{H}_t = \mathcal{S}^{t+1}$ denote the set of all possible histories of length $t + 1$. Note that in a continuum model, given an initial global state $s_0$, the distribution of consumers' initial states is $\eta_{s_0}$. Therefore, under a history $h_t$, if all consumers use the same dynamic oblivious strategy $\nu$, then the average demand is

$$\widetilde{A}_{t|\nu,h_t} = \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) \cdot \nu_t(x, h_t). \tag{6.16}$$

We now introduce the cost functions in a continuum model. Let $\widetilde{C} : \mathbb{R} \times \mathcal{S} \to [0, \infty)$ be a primary cost function in a continuum model. Let $\widetilde{H} : \mathbb{R}^2 \times \mathcal{S}^2 \to [0, \infty)$ be an ancillary cost function at stage $t \geq 1$, and let $\widetilde{H}_0 : \mathbb{R} \times \mathcal{S} \to [0, \infty)$ be an ancillary cost function at the initial stage 0.

Given the cost functions in a continuum model, we define the sequence of prices induced by a dynamic oblivious strategy as follows:

$$\widetilde{p}_{t|\nu,h_t} = \widetilde{C}'(\widetilde{A}_{t|\nu,h_t}, s_t), \quad \widetilde{q}_{0|\nu,h_0} = 0, \quad \widetilde{w}_{0|\nu,h_0} = \widetilde{H}_0'(\widetilde{A}_{0|\nu,h_0}, s_0), \tag{6.17}$$

and for $t \geq 1$,

$$\widetilde{q}_{t|\nu,h_t} = \frac{\partial \widetilde{H}\left(\widetilde{A}_{t-1|\nu,h_{t-1}}, \widetilde{A}_{t|\nu,h_t}, \overline{s}_t\right)}{\partial \widetilde{A}_{t-1|\nu,h_{t-1}}}, \quad \widetilde{w}_{t|\nu,h_t} = \frac{\partial \widetilde{H}\left(\widetilde{A}_{t-1|\nu,h_{t-1}}, \widetilde{A}_{t|\nu,h_t}, \overline{s}_t\right)}{\partial \widetilde{A}_{t|\nu,h_t}}. \tag{6.18}$$

### 6.4.2 Equilibrium strategies

In this subsection we will define the concept of a DOE. Suppose that all consumers other than $i$ use a dynamic oblivious strategy $\nu$. In a continuum model, consumer $i$'s action does not affect the prices. If all consumers except $i$ use a dynamic oblivious strategy $\nu$, consumer $i$'s **oblivious stage value** (stage payoff in a continuum model) under a history $h_t$ and an action $a_{i,t}$, is

$$\widetilde{\pi}_{i,t}(y_{i,t}, h_t, a_{i,t} \mid \nu) = U_t(x_{i,t}, s_t, a_{i,t}) - (\widetilde{p}_{t|\nu,h_t} + \widetilde{w}_{t|\nu,h_t})a_{i,t} - \widetilde{q}_{t|\nu,h_t}a_{i,t-1}, \tag{6.19}$$

where the prices, $\widetilde{p}_{t|\nu,h_t}$, $\widetilde{w}_{t|\nu,h_t}$, and $\widetilde{q}_{t|\nu,h_t}$, are defined in (6.17) and (6.18). Since a single consumer's action cannot influence $\widetilde{q}_t$, the last term in (6.19) is not affected by the action $a_{i,t}$, and the decision at stage $t$ need not take $a_{i,t-1}$ into account, but takes $\widetilde{q}_{t+1}$ into account.

Consumer $i$'s oblivious stage value under a dynamic oblivious strategy $\hat{\nu}$, is[7]

$$\widetilde{\pi}_{i,t}(y_{i,t}, h_t \mid \hat{\nu}, \nu) \triangleq \widetilde{\pi}_{i,t}(y_{i,t}, h_t, \hat{\nu}_t(x_{i,0}, h_t) \mid \nu). \tag{6.20}$$

In particular, we use $\widetilde{\pi}_{i,t}(y_{i,t}, h_t \mid \nu, \nu)$ to denote the oblivious stage value of consumer $i$ at stage $t$, if all consumers use the strategy $\nu$. Given a history $h_t$, and the current augmented state of consumer $i$, $y_{i,t}$, her **oblivious value function** (future expected

[7]Note the initial state (the type) of consumer $i$, $x_{i,0}$, is included in its state $x_{i,t}$, for any $t$.

payoff function in a continuum model) is

$$\widetilde{V}_{i,t}(y_{i,t}, h_t \mid \hat{\nu}, \nu) = \widetilde{\pi}_{i,t}(y_{i,t}, h_t, \mid \hat{\nu}, \nu) + \mathbb{E}\left\{\sum_{\tau=t+1}^{T} \widetilde{\pi}_{i,\tau}(y_{i,\tau}, h_\tau \mid \hat{\nu}, \nu)\right\}, \tag{6.21}$$

where the expectation is over the future global states, $\{s_\tau\}_{\tau=t+1}^{T}$.

**Definition 6.1.** *A strategy $\nu$ is a **Dynamic Oblivious Equilibrium** (DOE) if*

$$\sup_{\hat{\nu}\in\mathfrak{V}} \widetilde{V}_{i,0}(x_{i,0}, s_0 \mid \hat{\nu}, \nu) = \widetilde{V}_{i,0}(x_{i,0}, s_0 \mid \nu, \nu), \quad \forall x_{i,0} \in \mathcal{X}_0, \quad \forall s_0 \in \mathcal{S}.$$

A DOE is guaranteed to exist, under suitable assumptions, and this is known to be the case even for a more general model with idiosyncratic randomness [8]. The DOE defined above, is essentially the same concept as the "dynamic competitive equilibrium" studied in [12], which is defined as the nonatomic equivalent of an MPE in a continuum model. At a DOE, the beliefs of all the consumers on future prices are consistent with the equilibrium outcomes. Therefore, the nonatomic DOE is identical in outcomes with a Rational Expectations Equilibrium (REE). In future electricity markets, consumers may form rational expectations of future prices through an adaptive learning process, or they may receive price estimates from utilities and/or the independent system operator through advanced metering infrastructures. In Section 6.7, we make some discussion on the implementation of the proposed real-time pricing mechanism. We will show in Theorem 6.2 that under the proposed pricing mechanism, and under some certain convexity assumptions, a DOE strategy is socially optimal in a continuum model.

For a dynamic market model with aggregate profit shocks, the authors of [97] introduce a concept of "extended oblivious equilibrium" at which every firm's strategy depends on its current state and on the recent history of the aggregate shock. The extended oblivious equilibrium is computationally tractable; however, an equilibrium strategy may not be an approximate best response for every firm, even if the number of firms is large (cf. the error bounds derived in Section 8.3 of [97]).

Note that the definition of a DOE strategy requires optimality (attaining the supremum in Definition 6.2) only along the equilibrium path [12]. Thus, a DOE is similar in spirit to the "self-confirming equilibria" in [33] and the "subjective equilibria" in [49], in which each agent forms correct beliefs about her opponents only along the equilibrium path.

### 6.4.3 Relation between a continuum model and a corresponding $n$-consumer model

We want the cost functions in a continuum model to approximate the cost functions in an $n$-consumer model. Since the continuum of consumers is described by distributions over $[0, 1]$, the demand given in (6.16) can be regarded as the average demand per consumer. We assume the following relation between the cost functions in a continuum model and their counterparts in the corresponding $n$-consumer model.

**Assumption 6.2.** *For any $n \in \mathbb{N}$, any $s \in \mathcal{S}$, and any $\overline{s}$ in $\mathcal{S}^2$, we have*

$$C^n(A,s) = n\widetilde{C}\left(\frac{A}{n}, s\right), \quad H_0^n(A,s) = n\widetilde{H}_0\left(\frac{A}{n}, s\right), \quad H^n(A, A', \overline{s}) = n\widetilde{H}\left(\frac{A}{n}, \frac{A'}{n}, \overline{s}\right),$$

*where the superscript $n$ is used to indicate that these are the cost functions associated with an $n$-consumer model.*

Assumption 6.2 implies that

$$(C^n)'(A,s) = \widetilde{C}'(A/n, s), \quad (H_0^n)'(A,s) = \widetilde{H}_0'(A/n, s), \quad s \in \mathcal{S},$$

and

$$\frac{\partial H^n(A, A', \overline{s})}{\partial A} = \frac{\partial \widetilde{H}(A/n, A'/n, \overline{s})}{\partial (A/n)}, \quad \frac{\partial H^n(A, A', \overline{s})}{\partial A'} = \frac{\partial \widetilde{H}(A/n, A'/n, \overline{s})}{\partial (A'/n)}, \quad \forall \overline{s} \in \mathcal{S}^2,$$

i.e., the prices of the continuum model at the average demand equal the prices of the corresponding $n$-consumer model.

## 6.5 Approximation in Large Games

In this section, we consider a sequence of dynamic games, and show that as the number of consumers increases to infinity, a DOE strategy for the corresponding continuum game is asymptotically optimal for every consumer (i.e., an approximate best response), if the other consumers follow that same strategy. In the rest of the chapter, we sometimes use a superscript $n$ to indicate quantities associated with an $n$-consumer model.

Suppose that all consumers except $i$ use a dynamic oblivious strategy $\nu$. Given a history $h_t$ and an empirical distribution $f^n_{-i,t}$, we use $v(h_t, f^n_{-i,t}, \nu)$ to denote the empirical distribution of the actions taken by consumers excluding $i$, $u^n_{-i,t}$. In an $n$-consumer model, suppose that consumer $i$ uses a history-dependent strategy $\kappa^n = \{\kappa^n_t\}_{t=0}^T$ of the form

$$a_{i,t} = \kappa^n_t(y_{i,t}, h_t, f^n_{-i,t}), \tag{6.22}$$

while the other consumers use a dynamic oblivious strategy $\nu$. In an $n$-consumer model, let $\mathfrak{K}_n$ denote the set of all possible history-dependent strategies. The stage payoff received by consumer $i$ at time $t$ is

$$\pi^n_{i,t}(y_{i,t}, h_t, f^n_{-i,t} \mid \kappa^n, \nu) = \pi^n \left(y_{i,t}, \overline{s}_t, a_{i,t}, f^n_{-i,t}, v(h_t, f^n_{-i,t}, \nu)\right), \tag{6.23}$$

where $a_{i,t} = \kappa^n_t(y_{i,t}, h_t, f^n_{-i,t})$, and the stage payoff function on the right hand side is given in (6.14). Given an initial global state, $s_0$, and consumer $i$'s initial state, $x_{i,0}$, consumer $i$'s expected payoff under the strategy $\kappa^n$ is

$$V^n_{i,0}\left(x_{i,0}, s_0 \mid \kappa^n, \nu\right) = \mathbb{E}\left\{\sum_{t=0}^{T} \pi^n_{i,t}(y_{i,t}, h_t, f^n_{-i,t} \mid \kappa^n, \nu)\right\}, \tag{6.24}$$

where the expectation is over the initial distribution $f^n_{-i,0}$, and over the future global states, $\{s_t\}_{t=1}^T$. In particular, we use $V^n_{i,0}\left(x_{i,0}, s_0 \mid \nu, \nu\right)$ to denote the expected payoff obtained by consumer $i$ if all consumers use the strategy $\nu$.

**Definition 6.2.** *A dynamic oblivious strategy $\nu$ has the* ***asymptotic Markov equi-***

***librium (AME)*** *property [2], if for any initial global state* $s_0 \in \mathcal{S}$*, any initial state* $x_{i,0} \in \mathcal{X}_0$*, and any sequence of history-dependent strategies* $\{\kappa^n\}$*, we have*

$$\limsup_{n\to\infty} \left(V_{i,0}^n\left(x_{i,0}, s_0 \mid \kappa^n, \nu\right) - V_{i,0}^n\left(x_{i,0}, s_0 \mid \nu, \nu\right)\right) \le 0.$$

We will show that every DOE has the AME property, under the following assumption, which strengthens Assumption 6.1.

**Assumption 6.3.** *We assume that:*

6.3.1. *The four families of functions,* $\{\widetilde{C}'(\cdot, s) : s \in \mathcal{S}\}$, $\{\widetilde{H}_0'(\cdot, s) : s \in \mathcal{S}\}$, $\{\partial\widetilde{H}(A, A', \overline{s})/\partial A : (A', \overline{s}) \in \mathcal{A} \times \mathcal{S}^2\}$*, and* $\{\partial\widetilde{H}(A', A, \overline{s})/\partial A : (A', \overline{s}) \in \mathcal{A} \times \mathcal{S}^2\}$*, are uniformly equicontinuous on* $[0, \infty)$*.*

6.3.2. *The marginal costs are bounded from above, i.e.,*

$$|\widetilde{C}'(A, s)| \le P, \quad |\widetilde{H}_0'(A, s)| \le P, \quad \forall (A, s) \in \mathcal{A} \times \mathcal{S},$$

*and*

$$\left|\frac{\partial\widetilde{H}(A, A', \overline{s})}{\partial A}\right| \le P, \quad \left|\frac{\partial\widetilde{H}(A', A, \overline{s})}{\partial A}\right| \le P, \quad \forall (A', \overline{s}) \in \mathcal{A} \times \mathcal{S}^2,$$

*where* $P$ *is a positive constant.*

6.3.3. *The utility functions,* $\{U_t(x, s, a)\}_{t=0}^T$*, are continuous in* $a$ *and bounded above, i.e.,*

$$U_t(x, s, a) \le Q, \qquad t = 0, \ldots, T, \quad \forall (x, s, a) \in \mathcal{X}_t \times \mathcal{S} \times \mathcal{A},$$

*where* $Q$ *is a positive constant.*

Combining with Assumption 6.2, Assumption 6.3.1 implies that for any $\varepsilon > 0$, there exists a $\delta > 0$ such that for any positive integer $n$, if $|A - \overline{A}| \le n\delta$, then

$$\left|(C^n)'(A, s) - (C^n)'(\overline{A}, s)\right| \le \varepsilon, \quad \left|(H_0^n)'(A, s) - (H_0^n)'(\overline{A}, s)\right| \le \varepsilon, \quad \forall s \in \mathcal{S}, \quad (6.25)$$

and for any $(A', \overline{s}) \in \mathcal{A} \times \mathcal{S}^2$,

$$\left| \frac{\partial H^n(A, A', \overline{s})}{\partial A} - \frac{\partial H^n(\overline{A}, A', \overline{s})}{\partial \overline{A}} \right| \leq \varepsilon, \quad \left| \frac{\partial H^n(A', A, \overline{s})}{\partial A} - \frac{\partial H^n(A', \overline{A}, \overline{s})}{\partial \overline{A}} \right| \leq \varepsilon. \tag{6.26}$$

Note that the boundness of the cost function derivatives implies the Lipschitz continuity of the cost functions. Combining with Assumption 6.2, for any pair of real numbers $(A, \overline{A})$, and any positive integer $n$, we have

$$\left| C^n(A, s) - C^n(\overline{A}, s) \right| \leq P|A - \overline{A}|, \quad \left| H_0^n(A, s) - H_0^n(\overline{A}, s) \right| \leq P|A - \overline{A}|, \quad \forall s \in \mathcal{S}, \tag{6.27}$$

and for any $(A', \overline{s}) \in \mathcal{A} \times \mathcal{S}^2$,

$$\left| H^n(A, A', \overline{s}) - H^n(\overline{A}, A', \overline{s}) \right| \leq P|A - \overline{A}|, \quad \left| H^n(A', A, \overline{s}) - H^n(A', \overline{A}, \overline{s}) \right| \leq P|A - \overline{A}|. \tag{6.28}$$

We argue in the following theorem that a DOE strategy approximately maximizes a consumer's expected payoff in a dynamic game with a large but finite number of consumers, if the other consumers also use that strategy.

**Theorem 6.1.** *Suppose that Assumptions 6.2-6.3 hold. Every DOE has the AME property.*

Theorem 6.1 is proved in Appendix E.2. Various approximation properties of nonatomic equilibrium concepts in a continuum game have been investigated in previous works. Sufficient conditions for a stationary equilibrium (an equilibrium concept for a continuum game without aggregate uncertainty) to have the AME property are derived in [2]. For a continuum game with both idiosyncratic and aggregate uncertainties, [12] shows that as the number agents increases to infinity, the actions taken in an MPE can be well approximated by some DOE strategy of the nonatomic limit game. Note that in an $n$-consumer game, even if all consumers take an action that is close to the action taken by a DOE strategy of the nonatomic limit game, the population states and the prices in the $n$-consumer game can still be very different from their counterparts in the nonatomic limit game. Therefore, without further assumptions on the consumers' state transition kernel (e.g., continuous dependence of consumer states on their previous actions), the approximation property of a DOE on the action space does not necessarily imply the AME property of the DOE.

## 6.6 Asymptotic Social Optimality

In Section 6.6.1, we define the social welfare associated with an $n$-consumer model and with a continuum model. In Section 6.6.2, we show that for a continuum model, the social welfare is maximized at a DOE, and that for a sequence of $n$-consumer models, if all consumers use the DOE strategy of the corresponding continuum model, then the social welfare is asymptotically maximized, as the number of consumers increases to infinity.

### 6.6.1 Social welfare

In an $n$-consumer model, let the vector $\mathbf{a}_t = (a_{1,t}, \ldots, a_{n,t})$ denote the vector of actions taken by all consumers at stage $t$, and let the vector $\mathbf{x}_t = (x_{1,t}, \ldots, x_{n,t})$ denote the vector of consumers' states at stage $t$. Under the current population state and global state, the social welfare realized at stage $t$ is

$$W_t^n(\mathbf{x}_t, \overline{s}_t, \mathbf{a}_t) = -C^n(A_t, s_t) - H^n(A_{t-1}, A_t, \overline{s}_t) + \sum_{i=1}^{n} U_t(x_{i,t}, s_t, a_{i,t}), \quad t = 1, \ldots, T, \tag{6.29}$$

and at stage 0, the social welfare is

$$W_t^n(\mathbf{x}_0, s_0, \mathbf{a}_0) = -C^n(A_0, s_0) - H_0^n(A_0, s_0) + \sum_{i=1}^{n} U_0(x_{i,0}, s_0, a_{i,0}). \tag{6.30}$$

Because of the symmetry of the problem, the social welfare at stage $t$ depends on $\mathbf{x}_t$ and $\mathbf{a}_t$ only through the empirical distribution of state-action pairs. In particular, under a symmetric history-dependent strategy profile $\boldsymbol{\kappa}^n = (\kappa^n, \ldots, \kappa^n)$ (cf. the definition of a history-dependent strategy in Eq. (6.22)), we can write the social welfare at time $t$ (with a slight abuse of notation) as $W_t^n(f_t^n, h_t \mid \boldsymbol{\kappa}^n)$. Given an initial global state $s_0$ and an initial population state $f_0^n$, the expected social welfare achieved under a symmetric history-dependent strategy profile $\boldsymbol{\kappa}^n$ is given by

$$\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n) = W_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n) + \mathbb{E}\left\{\sum_{t=1}^{T} W_t^n(f_t^n, h_t \mid \boldsymbol{\kappa}^n)\right\}, \tag{6.31}$$

where the expectation is over the future global states $\{s_t\}_{t=1}^T$. In particular, we use $\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\nu}^n)$ to denote the expected social welfare achieved by the "symmetric dynamic oblivious strategy profile", $\boldsymbol{\nu}^n = (\nu, \ldots, \nu)$.

In a continuum model, suppose that all consumers use a dynamic oblivious strategy $\nu$. Given an initial global state $s_0$, the expected social welfare is

$$\widetilde{\mathcal{W}}_0(s_0 \mid \nu) = \widetilde{W}_0(s_0 \mid \nu) + \mathbb{E}\left\{\sum_{t=1}^{T} \widetilde{W}_t(h_t \mid \nu)\right\}, \tag{6.32}$$

where the expectation is over the future global states, $\{s_t\}_{t=1}^T$. Here, $\widetilde{W}_t(h_t \mid \nu)$ is the stage social welfare under history $h_t$:

$$\begin{aligned}\widetilde{W}_t(h_t \mid \nu) = -\widetilde{C}(\widetilde{A}_{t|\nu,h_t}, s_t) - \widetilde{H}(\widetilde{A}_{t-1|\nu,h_{t-1}}, \widetilde{A}_{t|\nu,h_t}, \overline{s}_t)& \\ + \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) U_t\left(l_{\nu,h_t}(x), s_t, \nu_t(x, h_t)\right), &\quad t = 1, \ldots, T,\end{aligned} \tag{6.33}$$

where $l_{\nu,h_t}$ maps a consumer's initial state into her state at stage $t$, under the history $h_t$ and the dynamic oblivious strategy $\nu$. The social welfare at stage 0 is given by

$$\widetilde{W}_0(s_0 \mid \nu) = -\widetilde{C}(\widetilde{A}_{0|\nu,h_0}, s_0) - \widetilde{H}_0(\widetilde{A}_{0|\nu,h_0}, s_0) + \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) U_0\left(x, s_0, \nu_0(x, s_0)\right). \tag{6.34}$$

### 6.6.2 Asymptotic social optimality of a DOE

We first define some notation that will be useful in this subsection. Since there is no idiosyncratic randomness, given a history $h_t$, the state of consumer $i$ at stage $t$ depends only on her initial state $x_{i,0}$, and her actions taken at $\tau = 0, \ldots, t-1$. At stage $t \geq 1$, the history $h_t$ and the transition function $z_{i,t+1} = r(x_{i,t}, a_{i,t}, s_{t+1})$ define a mapping $k_{h_t} : \mathcal{X}_0 \times \mathcal{A}^t \to \mathcal{Z}$:

$$z_{i,t} = k_{h_t}(x_{i,0}, a_{i,0}, \ldots, a_{i,t-1}), \quad t = 1, \ldots, T. \tag{6.35}$$

Given an initial state $x_{i,0}$, consumer $i$'s total utility under a history $h_t$ can be written as a function of her actions taken at stages $\tau = 0, \ldots, t$:

$$\overline{U}_{h_t}(x_{i,0}, a_{i,0}, \ldots, a_{i,t}) = U_t(x_{i,0}, s_0, a_{i,0}) + \sum_{\tau=1}^{t} U_t(x_{i,0}, k_{h_\tau}(x_{i,0}, a_{i,0}, \ldots, a_{i,\tau-1}), s_\tau, a_{i,\tau}). \tag{6.36}$$

Before proving the main result of this section, we introduce a series of assumptions on the convexity and differentiability of the cost and the utility functions.

**Assumption 6.4.** *We assume the following.*

6.4.1. *For any $s \in \mathcal{S}$, $\widetilde{C}(\cdot, s)$ is convex; for any $\overline{s} \in \mathcal{S}^2$, $\widetilde{H}(A, A', \overline{s})$ is convex in $(A, A')$.*

6.4.2. *For any $h_T \in \mathcal{H}_T$ and any $x_{i,0} \in \mathcal{X}_0$, the function defined in (6.36) is concave with respect to the vector $(a_{i,0}, \ldots, a_{i,T})$.*

6.4.3. *For any $t \geq 1$, any $h_t \in \mathcal{H}_t$, and any $x_{i,0} \in \mathcal{X}_0$, the function defined in* (6.35) *is monotonic in $a_{i,\tau}$, for $\tau = 0, \ldots, t-1$; further, its left and right derivatives with respect to $a_{i,\tau}$ exist, for $\tau = 0, \ldots, t-1$.*

6.4.4. *For $t \geq 1$, and for any $(x, s, a) \in \mathcal{X}_0 \times \mathcal{S} \times \mathcal{A}$, the left and right derivatives of the utility function $U_t(x, z, s, a)$ in $z$ exist.*

Assumption 6.4.1 is standard. If the utility function is concave in $a$, Assumption 6.4.2 requires that the transition function $k_{h_t}$ preserves concavity (a linear function would be an example). Note that Assumptions 6.4.1 and 6.4.2 guarantee that in both models (a dynamic game with a finite number of consumers, and the corresponding continuum game), the expected social welfare (consumer $i$'s expected payoff) is concave in the actions taken by all consumers (respectively, by consumer $i$). Assumptions 6.4.3 and 6.4.4 ensure the existence of left and right derivatives of the expected social welfare given in (6.32), with respect to the actions taken by consumers. An example where Assumptions 6.4.2-6.4.4 hold is given next.

**Example 6.2.** Consider a category of appliances such as Plug-in Hybrid Electric Vehicles (PHEVs), dish washers, or clothes washers. For such appliances, a customer only cares whether the task is completed before a certain time.

Given an initial state (type) of consumer $i$, $x_{i,0}$, let $D(x_{i,0})$ and $T(x_{i,0})$ indicate her total desired demand and the stage when the task has to be completed, respectively. Under a given history $h_t$, the total utility obtained by consumer $i$ is defined as

$$\overline{U}_{h_t}(x_{i,0}, a_{i,0}, \ldots, a_{i,t}) = Z\left(x_{i,0}, \min\left\{D(x_{i,0}), \sum_{\tau=0}^{\min\{T(x_{i,0}),t\}} a_{i,\tau}\right\}\right).$$

If for any $x_{i,0} \in \mathcal{X}_0$, $Z(x_{i,0}, \cdot)$ is nondecreasing and concave, then Assumption 6.4.2 holds. At stage $t = 0$, we have

$$U_t(x_{i,0}, s_0, a_{i,0}) = Z\left(x_{i,0}, \min\left\{D(x_{i,0}), a_{i,0}\right\}\right).$$

For $t = 1, \ldots, T(x_{i,0})$, we have $z_{i,t} = \sum_{\tau=0}^{t-1} a_{i,\tau}$, and

$$U_t(x_{i,0}, z_{i,t}, s_t, a_{i,t}) = Z\left(x_{i,0}, \min\left\{D(x_{i,0}), a_{i,t} + z_{i,t}\right\}\right) - Z\left(x_{i,0}, \min\left\{D(x_{i,0}), z_{i,t}\right\}\right).$$

For $t \geq T(x_{i,0}) + 1$, we let $z_{i,t} = D(x_{i,0})$, and let $U_t(x_{i,t}, s_t, a_{i,t})$ be identically zero. Suppose that for every $x_{i,0} \in \mathcal{X}_0$, the right and left derivatives of $Z(x_{i,0}, \cdot)$ exist. Then, Assumptions 6.4.3 and 6.4.4 hold. $\square$

**Theorem 6.2.** *Suppose that Assumptions 6.2-6.4 hold. Let $\nu$ be a DOE of the continuum game. Then, the following hold.*

*(a) In the continuum game, the social welfare is maximized at the DOE, i.e.,*[8]

$$\widetilde{\mathcal{W}}_0(s_0 \mid \nu) = \sup_{\vartheta \in \mathfrak{V}} \widetilde{\mathcal{W}}_0(s_0 \mid \vartheta), \quad \forall s_0 \in \mathcal{S},$$

[8]Under Assumption 6.4, the social welfare in a continuum game is concave on consumers' actions taken under different histories. Hence, the optimal social welfare can be achieved by a symmetric dynamic oblivious strategy profile.

*where* $\mathfrak{V}$ *is the set of all dynamic oblivious strategies.*

*(b) For a sequence of n-consumer games, the symmetric DOE strategy profile,* $\boldsymbol{\nu}^n = (\nu, \ldots, \nu)$*, approximately maximizes the expected social welfare, as the number of consumers increases to infinity. That is, for any initial global state* $s_0$*, and any sequence of symmetric history-dependent strategy profiles* $\{\boldsymbol{\kappa}^n\}$*, we have*[9]

$$\limsup_{n\to\infty} \mathbb{E}\left\{ \frac{\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n) - \mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\nu}^n)}{n} \right\} \leq 0,$$

*where the expectation is over the initial population state,* $f_0^n$*.*

The proof of Theorem 6.2 is given in Appendix E.3.

## 6.7 Implementation of the proposed pricing mechanism

To implement a dynamic real-time pricing mechanism, all consumers should be exposed to time-varying prices associated with ex ante estimates of generation costs that reflect system operating conditions (p. 81 of [14]) so that they can adjust their demand in accordance to real-time prices as well as ex ante price estimates. The mechanism proposed in this chapter is not an exception. The ex ante estimates of real-time prices can be developed by evaluating statistical relationships between historical real-time prices and various factors such as load forecast, weather predictions, and expected supply/demand balances [14].

We now provide a brief discussion of the details of a possible implementation of the proposed pricing mechanism:

**Ex ante price estimates.** Suppose that the exogenous state $s_t$ is realized at the beginning of each stage $t$; for every possible realization of the trajectory (scenario)

[9] Under Assumption 6.4, the social welfare in an $n$-consumer game is concave on the actions taken by consumers under different histories. Therefore, $\sup_{\kappa^n \in \mathfrak{K}_n} \mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n)$ is also the maximum social welfare that can be achieved by a (possibly non-symmetric) history-dependent strategy profile.

of future exogenous states $\{s_\tau\}_{\tau=t+1}^{t+T}$, consumers receive corresponding price estimates $\{\hat{p}_\tau\}_{\tau=t}^{t+T}$, $\{\hat{w}_\tau\}_{\tau=t}^{t+T}$, and $\{\hat{q}_\tau\}_{\tau=t}^{t+T}$, from utilities and/or the independent system operator, through advanced metering infrastructures. The consumers also know or receive the probabilities of the different trajectories. With the received price estimates (associated with possible trajectories of future exogenous states) and preset utility functions, each consumer's infrastructure solves a dynamic programming problem to maximize her expected payoff over the horizon from $t$ to $t+T$. (The state at time $\tau$ in this dynamic program is comprised of $y_{i,\tau}$, and the history $(s_t, s_{t+1}, \ldots, s_\tau.)$ The dimension of this state space grows with the time horizon $T$ (because of the exponentially increasing number of histories). Unfortunately, this is unavoidable for models of this type, and might require some further approximations, e.g., in the spirit of [97].

**Ex post prices.** At each stage $t$, after the realization of the system demands $A_{t-1}$ and $A_t$, consumers pay ex post prices $(p_t, w_t, q_t)$ that are determined according to Eqs. (6.11) and (6.12).

**Equilibrium.** In a market with a large number of price-taking consumers, it is possible to make ex ante price estimates (contingent on the realized trajectories) that are close to ex post prices. If every consumer maximizes her own payoff in response to these pretty accurate price estimates, the resulting outcome should be close to that resulting from a Rational Expectations Equilibrium (REE). The results derived in this chapter show that the expected social welfare can be approximately maximized, under the proposed mechanism.

We emphasize here that there remain several challenging implementation issues, e.g., the accuracy of future price estimates and the uncertainty of consumer response to ex ante price estimates. For example, the authors of [81] show that if consumers act myopically to highly inaccurate price estimates, real-time pricing may result in extreme price volatility. However, we note that these challenges are generic to almost all kinds of real-time pricing mechanisms.

## 6.8 Numerical Results

In this section we give a numerical example to compare the proposed pricing mechanism with marginal cost pricing. For the marginal cost pricing mechanism, we first define the DOE for a continuum model in Section 6.8.1. In Section 6.8.2, we consider a two-stage dynamic model in which consumers' marginal utility and demand increase at the second stage. We calculate the equilibria resulting from the two pricing mechanisms, and compare the potential of the two pricing mechanisms to improve social welfare and reduce peak load.

### 6.8.1 Equilibrium under Marginal Cost Pricing

In an $n$-consumer model, at stage $t \geq 1$, the supplier's marginal cost is

$$(C^n)'(A_t^n, s_t) + \frac{\partial H^n(A_{t-1}^n, A_t^n, \overline{s}_t)}{\partial A_t^n} = p_t^n + w_t^n, \quad t = 1, \ldots, T. \tag{6.37}$$

At stage 0, the supplier's marginal cost is

$$(C^n)'(A_0^n, s_0) + (H_0^n)'(A_0^n, s_0) = p_0^n + w_0^n. \tag{6.38}$$

Under marginal cost pricing, each consumer's stage payoff is

$$\pi(y_{i,t}, \overline{s}_t, a_{i,t}, f_{-i,t}^n, u_{-i,t}^n) = U(x_{i,t}, s_t, a_{i,t}) - (p_t^n + w_t^n) \cdot a_{i,t}, \tag{6.39}$$

where the stage marginal cost, $p_t^n + w_t^n$, is given in (6.37) and (6.38), and $y_{i,t} = (x_{i,t}, a_{i,t-1})$.

For marginal cost pricing, we now define the nonatomic equilibrium concept in the corresponding continuum model. Suppose that all consumers other than $i$ use a dynamic oblivious strategy $\nu$. Consumer $i$'s oblivious stage value under marginal cost pricing is given by

$$\widetilde{\pi}_{i,t}(y_{i,t}, \overline{s}_t, f_{t|\nu,h_t}, a_{i,t} \mid \nu) = U_t(x_{i,t}, s_t, a_{i,t}) - (\widetilde{p}_{t|\nu,h_t} + \widetilde{w}_{t|\nu,h_t}) \cdot a_{i,t}, \tag{6.40}$$

where $\widetilde{p}_{t|\nu,h_t}$ and $\widetilde{w}_{t|\nu,h_t}$ are defined in (6.17) and (6.18). Replacing the oblivious stage value function in (6.19) with that given in (6.40), we can define an equilibrium concept for the marginal cost pricing mechanism in a similar way as for the DOE in Section 6.4.

### 6.8.2 Numerical Example

In current wholesale electricity markets, we observe that the highest daily wholesale price usually occurs when the system load increases quickly (cf. Fig. 6-1 in Section 6.1). Inspired by the above observation, we construct a two-stage dynamic model, in which the aggregate demand increases quickly at the second stage, to compare the performance of the proposed mechanism with marginal cost pricing. For simplicity, we assume that there is a continuum of identical consumers indexed by $i \in [0, 1]$. Each consumer would like to consume $1 + x$ and $1.2 - x$ at the two stages, where $x \in [0, E]$. Here, $E \in [0, 0.1]$ (a given constant) is the amount of electricity demand that can be shifted from the second stage to the first stage. The value of $E$ will be called demand substitutability[10].

Formally, consumer $i$'s state at each stage denotes the maximum amount of electricity she could use at the stage[11]. For a given consumer $i$, we have $x_{i,0} = 1 + E$, and her state at stage 1 is determined as follows:

1. if $a_{i,0} \leq 1$, the maximum amount of electricity she could use at the stage 1 is $1.2 - E + E$, i.e., $x_{i,1} = 1.2$;

2. if $1 < a_{i,0} \leq x_{i,0}$, the maximum amount of electricity she could use at the stage 1 is $1.2 - E + (x_{i,0} - a_{i,0}) = 2.2 - a_{i,0} = x_{i,1}$;

3. if $x_{i,0} < a_{i,0}$, the maximum amount of electricity she could use at the stage 1 is

[10] There are two types of elasticity of consumers' demand: (i) consumers may curtail their demand at a high price, and (ii) they may shift their demand to a less expensive time. The first type of demand response is a price elasticity, and the second type is an elasticity of substitution across time. The first type of elasticity is incorporated in our model through the utility functions, and the second type of elasticity is incorporated through $E$.

[11] Since all consumers are in the same type, the consumer state space in this example is a subset of $\mathbb{R}$.

$$1.2 - E = x_{i,1}.$$

To summarize, we have

$$x_{i,1} = 1.2 - E + \max\{0, x_{i,0} - \max\{a_{i,0}, 1\}\}.$$

For each stage $t$, the utility functions are given by

$$U_t(x_{i,t}, s_t, a_{i,t}) = \begin{cases} d_t a_{i,t}, & \text{if} \quad 0 \le a_{i,t} \le x_{i,t}, \\ d_t x_{i,t}, & \text{if} \quad a_{i,t} > x_{i,t}, \end{cases}$$

where the slopes are $d_0 = 10$ and $d_1 = 12$. Here, we assumed that the consumers place a larger value on electricity during peak hours, and that shifting peak load to off-peak hours hurts consumer utility. For example, rescheduling kitchen and laundry activities may cause inconvenience for residential consumers; similarly, industrial consumers may face higher labor cost premiums for off-peak production.

The primary cost function (cf. Section 6.2) is $\widetilde{C}(A, s) = A^2$, for any $s$. We assume that the capacity available at each stage is proportional to the system load, i.e.,

$$G_t = b_t A_t, \quad t = 0, 1,$$

and that the ancillary cost depends only on the difference between the capacity available at two consecutive stages. At the second stage (peak hour), we assume that the system operator maintains a reserve margin of 10%, i.e., $b_1 = 1.1$. We will consider two different system operator policies: (i) the system operator does not forecast the load jump at the second stage, and uses a conservative policy under which $b_0 = 1.12$, and (ii) the system operator predicts the load jump at the second stage, and ramps up the system capacity in advance, by letting $b_0 = 1.2$.

For simplicity, we use a quadratic function to approximate the ancillary cost associated with load fluctuation:

$$\widetilde{H}_0(A_0, s_0) = 10(\max\{b_0 A_0 - 1.12, 0\})^2, \quad \widetilde{H}(A_0, A_1, \overline{s}_1) = 20(\max\{b_1 A_1 - b_0 A_0, 0\})^2,$$

where 1.12 represents the capacity available at the stage before the initial stage[12]. We assumed a higher coefficient, 20, for the ancillary cost at the second stage, due to the increase of the system load.

The preceding two-stage example could also represent a case where renewable electricity generation decreases at the second stage. Consider a scenario where weather conditions at stage 1 result in a decrease $F \in [0, 1]$ in renewable electricity generation. Then, the ancillary cost function at stage 1 is given by

$$\widetilde{H}(A_0, A_1, \overline{s}_1) = 20(\max\{b_1 A_1 - b_0 A_0 + F, 0\})^2.$$

Letting the inelastic demand at stage 1 be $1.2 - E - F$, we obtain an example that is equivalent to the previous two-stage example.

For different levels of demand substitutability $E$, and a variety of system operator policies ($b_0$), we compare the social welfare (in Section 6.8.2) and the peak load (in Section 6.8.2) resulting from the two pricing mechanisms at an equilibrium.

### Social welfare gain

For various levels of demand substitutability ($E \in [0, 0.1]$), and the two different system operator policies, we calculate the equilibria resulting from the two pricing mechanisms. Fig. 6-2 compares the social welfare achieved by the proposed mechanism and the marginal cost pricing mechanism. We observe from Fig. 6-2 the following.

1. **System operator's policy:** When the consumers have a low level of demand substitutability, the policy with $b_0 = 1.2$ achieves a much higher social welfare than the conservative strategy ($b_0 = 1.12$), under both the proposed and the marginal cost pricing mechanisms. For consumers with a high level of demand substitutability, the policy with $b_0 = 1.2$ achieves a slightly smaller social welfare than the conservative strategy ($b_0 = 1.12$), because the policy with $b_0 = 1.2$ results in a lower price at the second stage than the conservative strategy, and

[12]Suppose that the load at stage "$-1$" is 1, and that the capacity available at stage $-1$ is 1.12, under an average reserve margin of 12%.

therefore discourages consumers from shifting their peak load (cf. the discussion in Section 6.8.2).

2. **Social welfare gain at a low level of demand substitutability:** At a low level of demand substitutability, e.g., when $E \leq 0.02$, we observe that the proposed pricing mechanism achieves significantly more social welfare gain (the social welfare achieved by flat rate pricing[13] is used a reference) than marginal cost pricing, under the system operator's conservative strategy ($b_0 = 1.12$); if the system operator ramps up the capacity in advance ($b_0 = 1.2$), both pricing mechanisms achieve approximately the same social welfare as flat rate pricing.

3. **Social welfare gain at a high level of demand substitutability:** If the consumers have a high demand substitutability, e.g., when $E \geq 0.08$, the proposed pricing mechanism achieves approximately 5% more social welfare gain than marginal cost pricing under the system operator's conservative policy ($b_0 = 1.12$); if the system operator ramps up the capacity in advance ($b_0 = 1.2$), the proposed pricing mechanism achieves approximately 50% more social welfare gain than marginal cost pricing.

For a case with zero demand substitutability ($E = 0$) and $b_0 = 1.12$, the one-stage aggregate demand and the social welfare resulting from the three pricing mechanisms are given in Table 6.1. The prices faced by consumers are given in Table 6.2, where the average retail price is the ratio of the total money a consumer pays at an equilibrium to her total demand during the two stages[14]. Note that under the proposed pricing mechanism, a consumer pays

$$(p_0 + w_0 + q_1)a_{i,0} + (p_1 + w_1)a_{i,1},$$

[13] Under flat rate pricing, consumers pay a fixed (time-invariant) retail price for the electricity they consume. Since the average retail price is less than consumers' marginal utility (see Tables 6.2 and 6.4), the payoff-maximizing consumer demand at the two stages is 1 and 1.2, respectively. Since all consumers are identical, the aggregate demand at the two stages is 1 and 1.2.

[14] Note that only consumers under flat rate pricing pay this price. We list the average prices for the two real-time pricing mechanisms to compare the consumers' expense under different pricing mechanisms.

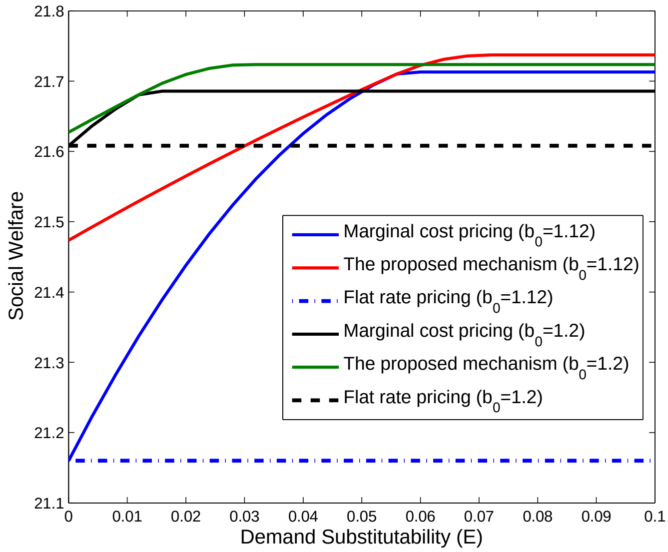


Figure 6-2: The social welfare achieved by the proposed pricing mechanism, the marginal cost pricing mechanism and the flat rate pricing mechanism, as a function of the demand substitutability, $E$.

Table 6.1: Demand and social welfare (per consumer) at $E = 0$ and $b_0 = 1.12$

| | $a_0$ | $a_1$ | Social welfare |
|---|---|---|---|
| Flat rate | 1 | 1.2 | 21.16 |
| Marginal cost | 1 | 1.2 | 21.16 |
| Proposed | 1.0901 | 1.2 | 21.4735 |

while she would pay $(p_0 + w_0)a_{i,0} + (p_1 + w_1)a_{i,1}$ under marginal cost pricing. The negative price $q_1$ is decreasing with the aggregate demand at the second stage. That is, a higher peak load results in a lower price at the first stage, which encourages consumers to increase their demand at the off-peak hour (even if the consumers cannot fully utilize their demand) to avoid the high tariff at the second stage. From Table 6.2 we observe that at the DOE, the proposed pricing mechanism offers each consumer a zero price at the first stage. Because the system operator is committed to a conservative reserve strategy, the proposed pricing mechanism encourages consumers to use more electricity to ramp up the system capacity.

For the case where the system operator ramps up the capacity in advance ($b_0 = 1.2$), and consumers have a high level of demand substitutability ($E = 0.08$), the

Table 6.2: Price fluctuation at $E = 0$ and $b_0 = 1.12$. The price $p_t + w_t$ equals the marginal cost at stage $t$.

| | $p_0 + w_0$ | $p_1 + w_1$ | $q_1$ | Average (retail) price |
|---|---|---|---|---|
| Flat rate | 2 | 11.2 | - | 7.0182 |
| Marginal cost | 2 | 11.2 | - | 7.0182 |
| Proposed | 4.4401 | 6.7609 | -4.4401 | 5.6562 |

Table 6.3: Demand and social welfare (per consumer) at $E = 0.08$ and $b_0 = 1.2$

| | $a_0$ | $a_1$ | Social welfare |
|---|---|---|---|
| Flat rate | 1 | 1.2 | 21.608 |
| Marginal cost | 1.0131 | 1.1869 | 21.6857 |
| Proposed | 1.0308 | 1.1692 | 21.7237 |

one-stage aggregate demand and the social welfare resulting from the three pricing mechanisms are given in Table 6.3. The prices faced by consumers are given in Table 6.4. From Table 6.3 we observe that under the proposed pricing mechanism, consumers would like to shift 0.031 peak load to off-peak hours, while under the marginal cost pricing mechanism, consumers are willing to shift less than 0.014 peak load to off-peak hours. Compared to marginal cost pricing, the more flattened load curve resulting from the proposed pricing mechanism leads to 50% more social welfare gain.

For a given load curve, the proposed pricing mechanism results in a larger price difference between stage 1 and stage 0 than marginal cost pricing, because of the negative price $q_1$. The negative price $q_1$ creates an additional incentive for consumers to shift their load from stage 1 to stage 0. In this way, the proposed pricing mechanism results in a more flattened load curve and a higher social welfare than marginal cost pricing (cf. Table 6.3).

Table 6.4: Price fluctuation at $E = 0.08$ and $b_0 = 1.2$. The price $p_t + w_t$ equals the marginal cost at stage $t$.

| | $p_0 + w_0$ | $p_1 + w_1$ | $q_1$ | Average (retail) price |
|---|---|---|---|---|
| Flat rate | 3.92 | 7.68 | - | 5.971 |
| Marginal cost | 4.324 | 6.324 | - | 5.403 |
| Proposed | 4.868 | 4.505 | -2.363 | 3.568 |

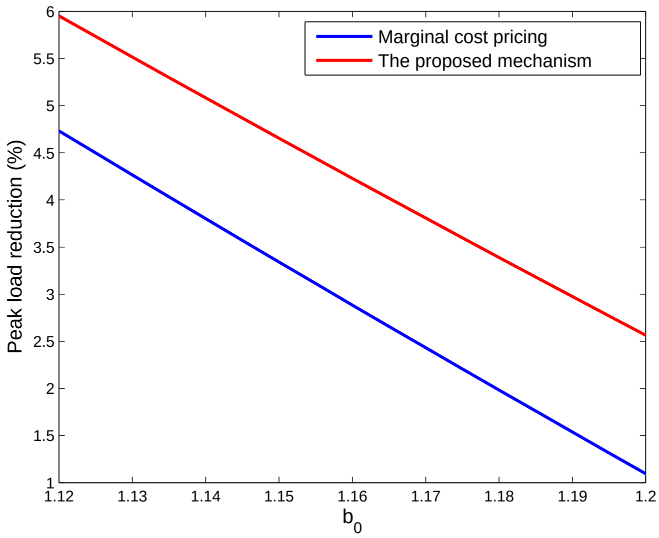


Figure 6-3: Comparison of the percentage of peak load reduction (the peak load under flat rate pricing, 1.2, is used a reference) resulting from the proposed pricing mechanism and the marginal cost pricing mechanism, as a function $b_0$.

### Peak load reduction

Under flat rate pricing, the peak load (the aggregate demand at the second stage) is 1.2, because consumers do not have an incentive to shift their load to off-peak hours. Given a pricing mechanism and a system operator's policy ($b_0$), consumers are willing to substitute across time only up to a certain level. Even with a high level of demand substitutability, consumers prefer not to shift much of their peak load, to avoid the utility loss caused by peak load shifting. For example, with $b_0 = 1.2$ and $E = 0.08$, consumers under marginal cost pricing choose to shift at most 0.013 peak load (cf. Table 6.3). In Fig. 6-3, for different values of $b_0$, we compare the maximum amount of peak load consumers choose to shift under the proposed pricing mechanism and the marginal cost pricing mechanism.

We observe from Fig. 6-3 that the amount of peak load consumers will shift decreases with $b_0$. This is because a larger reserve at the first stage lowers the price at the second stage, which in turn discourages consumers from shifting their peak load. The proposed pricing mechanism results in a peak load which is approximately 1.5 percent lower than that resulting from marginal cost pricing, regardless of the value

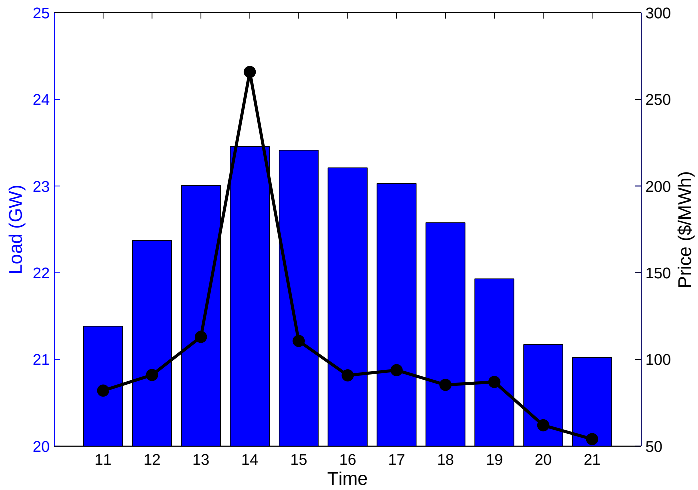


Figure 6-4: Real-time prices and actual system loads on August 01, 2011, ISO New England Inc. Blue bars represent the real-time system loads and the dots connected by a black line represent the hourly prices.

of $b_0$. If the system operator ramps up the system capacity in advance ($b_0 = 1.2$), marginal cost pricing reduces the system peak load resulting from flat rate pricing by approximately one percent. Compared to marginal cost pricing, the negative price $q_1$ in the proposed mechanism encourages consumers to make a larger shift of their peak load (cf. the discussion at the end of Section 6.8.2).

Fig. 6-4 plots the real-time system loads and prices on August 1, 2011, a typical hot summer day in New England[15]. If consumers are able to shift some of their load to the morning (possibly at the expense of losing some utility), the proposed pricing mechanism encourages consumers to shift more of their peak load than marginal cost pricing. Since the highest peak load determines the generation capacity necessary for system reliability, the proposed pricing mechanism has a greater potential to reduce the long-term capacity investment.

[15] `www.ferc.gov/market-oversight/mkt-electric/new-england/2011/08-2011-elec-isone-dly.pdf`

## 6.9 Conclusions and Future Directions

In an electric power system, load swings may result in significant ancillary cost to suppliers. Inspired by the observation that marginal cost pricing may not achieve social optimality in electricity markets, we proposed a new dynamic pricing mechanism that takes into account the externality conferred by a consumer's action on future ancillary cost. Besides proposing a suitable game-theoretic model that incorporates the cost of load fluctuation and a particular pricing mechanism for electricity markets, a main contribution of this chapter was to show that the proposed pricing mechanism achieves social optimality in a dynamic nonatomic game, and approximate social optimality for the case of finitely many consumers, under certain convexity assumptions.

To compare the proposed pricing mechanism with marginal cost pricing, we presented a numerical example in which the demand increases sharply at the last stage. In this example, the proposed pricing mechanism creates a stronger incentive for consumers to shift their peak load than marginal cost pricing, through an additional negative price charged on consumers' demand at off-peak hours. As a result, compared with marginal cost pricing, the proposed pricing mechanism achieves a higher social welfare, and at the same time, reduces the peak load, and therefore has the potential to reduce the need for long-term investments in peaking plants.

We believe that the constructed dynamic game-theoretic model, the proposed pricing mechanism, and more importantly, the insights provided by this work, can be applied to a more general class of markets with friction. As an extension and future work, one can potentially develop and use variations of our framework to a market of a perishable product/service where demand fluctuations incur significant cost to suppliers. Examples of suppliers of such a market with friction include data centers implementing cloud services that suffer from the switching costs to toggle a server into and out of a power-saving mode [56], and large organizations such as hospitals that use on-call staff to meet unexpected demand.

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# Appendix A

# Proof of results in Chapter 2

## A.1 Proof of Theorem 2.5

We first define a bounded variant of the original game in which we restrict the action variables $w_{n,s}$ to lie in the compact set $[0, B]$. We then consider a positive parameter $\varepsilon$ and a perturbed (bounded) game in which the bandwidth allocated to user $n$, when the state vector is $\mathbf{s} = (s_1, \ldots, s_n)$, is

$$b^{\varepsilon}_{n,\mathbf{s}}(\mathbf{w}) = \begin{cases} 0, & \text{if } w_{n,s_n} = 0 \\ \dfrac{w_{n,s_n}}{\varepsilon + \sum_{i=1}^{N} w_{i,s_i}} C, & \text{otherwise.} \end{cases} \tag{A.1}$$

Notice that $b^{\varepsilon}_{n,\mathbf{s}}(\mathbf{w})$ is a continuous function of $(\varepsilon, \mathbf{w})$ at points where either $\varepsilon > 0$ or $\sum_{i=1}^{N} w_{i,s_i} > 0$. The expected payoff of user $n$ at any state $s$ with $p_{n,s} > 0$ is

$$Q^{\varepsilon}_{n,s}(w_{n,s}, \mathbf{w}_{-n}) = -w_{n,s} + \sum_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s\big(b^{\varepsilon}_{n,s}(\mathbf{w})\big)\, \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n} \mid S_n = s). \tag{A.2}$$

The expected payoff of user $n$ is

$$Q^{\varepsilon}_{n}(\mathbf{w}) = \sum_{s \in \mathcal{S}} p_{n,s} Q^{\varepsilon}_{n,s}(w_{n,s}, \mathbf{w}_{-n}). \tag{A.3}$$

Without loss of generality, we assume that $U_s(0) = 0$ for all $s$.

*Step 1: A Nash equilibrium* $\mathbf{w}^\varepsilon$ *exists in the perturbed bounded game.*

From Eqs. (A.1)-(A.3), we see that $Q^\varepsilon_{n,s}$, the payoff of user $n$ at any state $s$, is a concave function of $w_{n,s}$ on $[0, B]$. (This is because we are dealing with the composition of the nondecreasing concave functions $U_s$ and $b^\varepsilon_{n,\mathbf{s}}$.) Furthermore, because $\varepsilon > 0$, the bandwidth allocated to user $n$ (that is, the argument of $U_s$) is a continuous function of $\mathbf{w}$. This, together with the continuity of the $U_s(\cdot)$, implies that $Q^\varepsilon_{n,s}$ is a continuous function of $\mathbf{w}$. Rosen's existence theorem for concave games [80] applies and shows the existence of a BNE, for the perturbed bounded game.

For every $\varepsilon > 0$, we have a BNE $\mathbf{w}^\varepsilon$ of the perturbed bounded game, and these BNEs lie in a compact set. Taking the limit as $\varepsilon \to 0$, using compactness, and by restricting to a convergent subsequence, we conclude that there exists a sequence $\varepsilon_k \to 0$ and a strategy profile vector $\mathbf{w}$ such that $\mathbf{w}^{\varepsilon_k}$ converges to $\mathbf{w}$.

*Step 2. The vector* $\mathbf{w}$ *is a BNE of the unperturbed bounded game.*

We argue by contradiction. Suppose that $\mathbf{w}$ is not a BNE of the bounded game. Then, there exists a user $n$, a state $s_n$ for which $p_{n,s_n} > 0$, and a bid $\overline{w}_{n,s_n} \neq w_{n,s_n}$, such that

$$Q_{n,s_n}(\overline{w}_{n,s_n}, \mathbf{w}_{-n}) > Q_{n,s_n}(w_{n,s_n}, \mathbf{w}_{-n}). \tag{A.4}$$

We will show that for the perturbed bounded game, and sufficiently small $\varepsilon_k > 0$, there exists for user $n$ a profitable deviation from the bid $w^{\varepsilon_k}_{n,s_n}$ at state $s_n$, contradicting the assumption that $\mathbf{w}^{\varepsilon_k}$ is a BNE of the perturbed bounded game.

Using the value of $\overline{w}_{n,s_n}$, we construct a new strategy for user $n$ in the perturbed games, by setting

$$\overline{w}^{\varepsilon_k}_{n,s} = \begin{cases} \overline{w}_{n,s}, & \text{if } \; s = s_n, \\ w^{\varepsilon_k}_{n,s}, & \text{otherwise.} \end{cases} \tag{A.5}$$

First we argue that

$$\lim_{k\to\infty} Q^{\varepsilon_k}_{n,s_n}(\overline{w}^{\varepsilon_k}_{n,s_n}, \mathbf{w}^{\varepsilon_k}_{-n}) = \lim_{k\to\infty} Q^{\varepsilon_k}_{n,s_n}(\overline{w}_{n,s_n}, \mathbf{w}^{\varepsilon_k}_{-n}) = Q_{n,s_n}(\overline{w}_{n,s_n}, \mathbf{w}_{-n}). \tag{A.6}$$

The first equality is immediate from (A.5). To prove the second equality, consider first the case where $\overline{w}_{n,s_n} > 0$. In that case, the bandwidth $C\overline{w}_{n,s_n}/(\varepsilon_k+\overline{w}_{n,s_n}+\sum_{i\neq n} w^{\varepsilon_k}_{i,s_i})$

converges to $C\overline{w}_{n,s_n}/(\overline{w}_{n,s_n} + \sum_{i\neq n} w^{\varepsilon}_{i,s_i})$. Using the continuity of $U_{s_n}$, the second equality follows. For the case where $\overline{w}_{n,s_n} = 0$, all three expressions in (A.6) are equal to zero.

Next we will show that

$$\lim_{k\to\infty} Q^{\varepsilon_k}_{n,s_n}(w^{\varepsilon_k}_{n,s_n}, \mathbf{w}^{\varepsilon_k}_{-n}) = Q_{n,s_n}(w_{n,s_n}, \mathbf{w}_{-n}). \tag{A.7}$$

If $w_{n,s_n} > 0$, Eq. (A.7) follows from the same continuity argument as in the previous paragraph. We therefore assume that $w_{n,s_n} = 0$. Suppose that for every $\mathbf{s}_{-n}$ such that

$$\mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n} \mid S_n = s_n) > 0,$$

we have $\sum_{i\neq n} w_{i,s_i} > 0$. The same continuity argument applies once again to yield Eq. (A.7).

To complete the proof of Eq. (A.7), we are left to consider the case where $w_{n,s_n} = 0$ and there exists some state vector $\bar{\mathbf{s}}_{-n}$ such that $\mathbb{P}(\mathbf{S}_{-n} = \bar{\mathbf{s}}_{-n} \mid S_n = s_n) > 0$ and

$$\lim_{k\to\infty} w^{\varepsilon_k}_{i,\bar{s}_i} = w_{i,\bar{s}_i} = 0, \quad \forall\, i \neq n. \tag{A.8}$$

We will use the BNE property of $\mathbf{w}^{\varepsilon_k}$ to show that this case is impossible. Given the positive number $\mathbb{P}(\mathbf{S} = \bar{\mathbf{s}}) = \mathbb{P}(\mathbf{S}_{-n} = \bar{\mathbf{s}}_{-n} \mid S_n = s_n)p_{n,s_n}$, we define a constant $d$ by

$$d = \frac{1}{2}\min_{s\in\mathcal{S}} \left\{U_s(2C/3) - U_s(C/2)\right\} \cdot \mathbb{P}(\mathbf{S} = \bar{\mathbf{s}}). \tag{A.9}$$

Since the utility functions are strictly increasing, we have $d > 0$. From Eq. (A.8), and for sufficiently large $k$, we have

$$\varepsilon_k \leq \frac{d}{2N}, \qquad w^{\varepsilon_k}_{i,\bar{s}_i} \leq \frac{d}{2N}, \;\; \forall\, i \in \mathcal{N}. \tag{A.10}$$

Since $N \geq 2$, at state $\bar{\mathbf{s}}$, there must exist a user $j$, who obtains a bandwidth no more than $C/2$. We now prove that to bid $d$ is a profitable deviation for user $j$ in the perturbed game with $\varepsilon_k$, when $k$ is sufficiently large. For sufficiently large $k$, (A.10) yields

$$U_{\overline{s}_j}\left(\frac{Cd}{d+\sum_{i\neq j} w^{\varepsilon_k}_{i,\overline{s}_i}}\right) \geq U_{\overline{s}_j}\left(\frac{Cd}{d+Nd/2N}\right) \geq U_{\overline{s}_j}\left(\frac{2C}{3}\right). \tag{A.11}$$

When, user $j$ is at state $\overline{s}_j$, and for any state vector other than $\overline{\mathbf{s}}$, the utility achieved by user $j$ with the bid $d$ cannot be less than that obtained with the original bid $w^{\varepsilon_k}_{j,\overline{s}_j}$. Hence we have

$$Q^{\varepsilon_k}_{j,\overline{s}_j}(d,\mathbf{w}^{\varepsilon_k}_{-j})+d-\left(Q^{\varepsilon_k}_{j,\overline{s}_j}(w^{\varepsilon_k}_{j,\overline{s}_j},\mathbf{w}^{\varepsilon_k}_{-j})+w^{\varepsilon_k}_{j,\overline{s}_j}\right) \geq \left(U_{\overline{s}_j}(\frac{2C}{3})-U_{\overline{s}_j}(\frac{C}{2})\right)\mathbb{P}(\mathbf{S}=\overline{\mathbf{s}}) \geq 2d, \tag{A.12}$$

where the first inequality follows from (A.11), and the second one follows from (A.9). Inequality (A.12) implies that

$$Q^{\varepsilon_k}_{j,\overline{s}_j}(d,\mathbf{w}^{\varepsilon_k}_{-j}) - Q^{\varepsilon_k}_{j,\overline{s}_j}(w^{\varepsilon_k}_{j,\overline{s}_j},\mathbf{w}^{\varepsilon_k}_{-j}) \geq d.$$

This contradicts the assumption that $\mathbf{w}^{\varepsilon_k}$ is a BNE of the perturbed game with $\varepsilon_k$. The validity of the claim (A.7) follows.

From Eqs. (A.6), (A.4), and (A.7), we conclude that

$$\lim_{k\to\infty} Q^{\varepsilon_k}_{n,s_n}(\overline{w}^{\varepsilon_k}_{n,s_n},\mathbf{w}^{\varepsilon_k}_{-n}) = Q^N_{n,s_n}(\overline{w}_{n,s_n},\mathbf{w}_{-n}) > Q^N_{n,s_n}(w_{n,s_n},\mathbf{w}_{-n}) = \lim_{k\to\infty} Q^{\varepsilon_k}_{n,s_n}(w^{\varepsilon_k}_{n,s_n},\mathbf{w}^{\varepsilon_k}_{-n}). \tag{A.13}$$

Thus, for sufficiently large $k$,

$$Q^{\varepsilon_k}_{n,s_n}(\overline{w}^{\varepsilon_k}_{n,s_n},\mathbf{w}^{\varepsilon_k}_{-n}) > Q^{\varepsilon_k}_{n,s_n}(w^{\varepsilon_k}_{n,s_n},\mathbf{w}^{\varepsilon_k}_{-n}),$$

which contradicts the fact that $\mathbf{w}^{\varepsilon_k}$ is a BNE of the perturbed game. Thus, for any user $n$ and any state $s_n$, there cannot exist a profitable deviation $\overline{w}_{n,s_n}$ from $\mathbf{w}$. Therefore, the strategy vector $\mathbf{w}$ must be a BNE of the unperturbed (bounded) Bayesian game, with compact action space $[0,B]$.

*Step 4: Bayesian game with action space* $[0,\infty)$.

To finish the proof for the theorem, we will show that a BNE of the original Bayesian game, with actions restricted to $[0,B]$, which has been proved to exist, must be a BNE of the original Bayesian game with action space $[0,\infty)$. A BNE of the Bayesian game with actions restricted to $[0,B]$ is a strategy profile $\mathbf{w}$ such that,

$$p_{n,s}Q_{n,s}(w_{n,s},\mathbf{w}_{-n}) \geq p_{n,s}Q_{n,s}(\overline{w}_{n,s},\mathbf{w}_{-n}), \quad \forall\, \overline{w}_{n,s}\in[0,B], \quad \forall\, n\,\in\mathcal{N}, \quad \forall\, s\in\mathcal{S}. \tag{A.14}$$

We define $B>0$ by $B=\max_{s\in\mathcal{S}}\{U_s(C)\}$. We will use (A.14) to show that the strategy profile $\mathbf{w}$ is a BNE of the original Bayesian game. Since a user could obtain a zero payoff by bidding zero, and since $\mathbf{w}$ is a BNE of the Bayesian game with action space $[0,B]$, we must have

$$p_{n,s}Q_{n,s}(w_{n,s},\mathbf{w}_{-n}) \geq 0, \quad \forall\, n\in\mathcal{N}, \quad \forall\, s\in\mathcal{S}. \tag{A.15}$$

Furthermore, from the definition of $B$, we have

$$Q_{n,s}(\overline{w}_{n,s},\mathbf{w}_{-n}) \leq U_s(C)-\overline{w}_{n,s}\leq 0, \quad \forall\, \overline{w}_{n,s}\in(B,\infty). \tag{A.16}$$

Combining (A.15) and (A.16), we have

$$p_{n,s}Q_{n,s}(w_{n,s},\mathbf{w}_{-n}) \geq p_{n,s}Q_{n,s}(\overline{w}_{n,s},\mathbf{w}_{-n}), \quad \forall\, \overline{w}_{n,s}\in(B,\infty), \quad \forall\, n\in\mathcal{N}, \quad \forall\, s\in\mathcal{S}.$$

This, together with (A.14) implies that the strategy profile $\mathbf{w}$ is a BNE.

*Step 5: The case of symmetric games.* For the case of symmetric games, one combines the argument given by Nash for symmetric matrix games [73], with Rosen's argument for concave games [80]. The idea is to apply Kakutani's fixed point theorem to the best response map, restricted to the set of symmetric strategy profiles. This argument shows that there exists a symmetric BNE $\mathbf{w}^{\varepsilon}$ for the perturbed bounded game. The limit point $\mathbf{w}$ considered in Step 2 (which was later proved to be a BNE of the original game) is automatically symmetric.

## A.2 Proof of Lemma 2.3

First we provide an expression for the positive constant $\overline{a}$. Since $U_s$ is a concave function with $U_s'(0) = \infty$, it is not hard to show that there exists some $t_0 \leq 1$ such that

$$U_s\left(\frac{7}{6}t\right) - U_s(t) \geq 4Dt, \qquad \forall\, t \in (0, t_0], \quad \forall\, s \in \mathcal{S}, \tag{A.17}$$

where $D$ is the positive constant in Assumption 2.1. We then define

$$\overline{a} = \frac{1}{4} \min_{s\in\mathcal{S}} \min_{t\in[t_0,1]} \left\{ U_s\left(\frac{7}{6}t\right) - U_s(t) \right\}. \tag{A.18}$$

Since $U_s$ is continuous and strictly increasing for every $s$, the constant $\overline{a}$ is well defined and positive.

We now claim that

$$\min_{s\in\mathcal{S}} \min_{t\in[\overline{a}/D,1]} \left\{ U_s\left(\frac{7}{6}t\right) - U_s(t) \right\} \geq 4\overline{a}. \tag{A.19}$$

If $\overline{a}/D \geq t_0$, then (A.19) follows the definition (A.18). On the other hand, if $\overline{a}/D < t_0$, we only need to consider $t$ such that $\overline{a}/D \leq t \leq t_0$. In that case, Eq. (A.17) yields

$$\min_{s\in\mathcal{S}} \left\{ U_s\left(\frac{7}{6}t\right) - U_s(t) \right\} \geq 4Dt \geq 4\overline{a}, \qquad \forall\, t \in [\overline{a}/D, t_0],$$

and the validity of our claim follows.

Suppose now, in order to derive a contradiction, that there exists a BNE $\mathbf{w}$ and some $n$, $s$ such that $p_{n,s} > 0$ and $w_{n,s} < \overline{a}$. Let $(\overline{n}, \overline{s})$ be such that

$$w_{\overline{n},\overline{s}} = \min \left\{ w_{n,s} \mid p_{n,s} > 0 \right\}. \tag{A.20}$$

Due to our hypothesis, we must have $w_{\overline{n},\overline{s}} < \overline{a}$. We will argue that a bid of $3\overline{a}$ is a profitable deviation for user $\overline{n}$ at state $\overline{s}$. Let $W$ be the sum of the bids of all users (a random variable). We consider two separate cases.

(a) Suppose that $W < \overline{a}N$. Then, the utility obtained by user $\overline{n}$ with the new bid of $3\overline{a}$ is at least than $U_{\overline{s}}\left(\frac{3\overline{a}\cdot N}{\overline{a}N+3\overline{a}}\right)$, which, for $N \geq 2$, is strictly larger than $U_{\overline{s}}(7/6)$. Note also that since $w_{\overline{n},\overline{s}}$ is the smallest bid, we have $w_{\overline{n},\overline{s}}N \leq W$ and $U_{\overline{s}}(w_{\overline{n},\overline{s}}N/W) \leq U_{\overline{s}}(1)$. Hence, the difference between the utility achieved by the new bid $3\overline{a}$ and the original bid $w_{\overline{n},\overline{s}}$ satisfies

$$U_{\overline{s}}\left(\frac{3\overline{a}N}{3\overline{a}+W-w_{\overline{n},\overline{s}}}\right) - U_{\overline{s}}\left(\frac{w_{\overline{n},\overline{s}}N}{W}\right) > U_{\overline{s}}\left(\frac{7}{6}\right) - U_{\overline{s}}(1) \geq 4\overline{a},$$

where the last inequality follows from (A.19).

(b) Suppose now that $DN \geq W \geq \overline{a}N$. Then, the difference between the utility achieved by the new bid $3\overline{a}$ and the original bid $w_{\overline{n},\overline{s}}$ satisfies

$$\begin{aligned} U_{\overline{s}}\left(\frac{3\overline{a}N}{3\overline{a}+W-w_{\overline{n},\overline{s}}}\right) - U_{\overline{s}}\left(\frac{w_{\overline{n},\overline{s}}N}{W}\right) &> \min_{w\in[\overline{a},D]}\left\{U_{\overline{s}}\left(\frac{7\overline{a}}{6w}\right) - U_{\overline{s}}\left(\frac{\overline{a}}{w}\right)\right\} \\ &= \min_{t\in[\overline{a}/D,1]}\left\{U_s\left(\frac{7}{6}t\right) - U_s(t)\right\} \geq 4\overline{a}, \end{aligned}$$

where the last inequality follows from (A.19).

We have proved that in either case, the difference between the utility achieved by the new bid $3\overline{a}$ and the original bid $w_{\overline{n},\overline{s}}$ is at least $4\overline{a}$. On the other hand, when giving the new bid $3\overline{a}$, the bid of user $\overline{n}$ has increased by at most $3\overline{a}$ compared to the original bid $w_{\overline{n},\overline{s}}$. Hence, the new bid of $3\overline{a}$ is a profitable deviation for user $\overline{n}$ at state $\overline{s}$. This is a contradiction with the assumption that the original bid $w_{\overline{n},\overline{s}}$ is part of a BNE, and establishes the desired result.

## A.3 Proof of Theorem 2.7

Consider a state $s \in \mathcal{S}$, and two users $m$ and $n$, which will remain fixed throughout this proof. Using the strict concavity of the utility functions, the expected payoff function $Q_{n,s}(w_{n,s}, \mathbf{w}_{-n})$ is a strictly concave function of $w_{n,s}$.

Let $\mathbf{S}_{-nm}$ be a $N-2$ dimensional random vector that represents the states of all users except user $n$ and $m$. For any state vector $\mathbf{s}_{-nm} \in \mathcal{S}^{N-2}$, we use $W(\mathbf{s}_{-nm})$ to denote the total bid given by users other than $n$ and $m$, i.e.,

$$W(\mathbf{s}_{-nm}) = \sum_{i \neq n,m} w_{i,s_i},$$

and also define $f(\mathbf{s}_{-nm}) = W(\mathbf{s}_{-nm})$. Note that if $\mathbf{w}$ is a BNE, then Lemmas 2.2 and 2.3 imply that

$$\bar{a} \leq f(\mathbf{s}_{-nm}) \leq D. \tag{A.21}$$

Since the the states of all users are independent, we can write $Q_{n,s}(w_{s,n}, \mathbf{w}_{-n})$ in the form

$$\begin{aligned} &Q_{n,s}(w_{n,s}^N, \mathbf{w}_{-n}) \\ &= -w_{n,s} + \sum_{s_m \in \mathcal{S}} p_{m,s_m} \sum_{\mathbf{s}_{-nm} \in \mathcal{S}^{N-2}} U_s\left(\frac{w_{n,s}}{\frac{w_{n,s}}{N} + \frac{w_{m,s_m}}{N} + f(\mathbf{s}_{-nm})}\right) \mathbb{P}(\mathbf{S}_{-nm} = \mathbf{s}_{-nm}). \end{aligned} \tag{A.22}$$

From now on, let us assume that $w_{s,n}$ and $w_{s,m}$ are components of a certain BNE $\mathbf{w}$. Since the utility function $U_s$ is strictly concave with $U_s'(0) = \infty$, we have $w_{n,s} > 0$. Since $w_{n,s}$ maximizes the expression in Eq. (A.22), the corresponding derivative must be zero, i.e.,

$$\begin{aligned} \sum_{s_m \in \mathcal{S}} p_{m,s_m} \sum_{\mathbf{s}_{-nm} \in \mathcal{S}^{N-2}} U_s'\left(\frac{w_{n,s}}{\frac{w_{n,s}}{N} + \frac{w_{m,s_m}}{N} + f(\mathbf{s}_{-nm})}\right) \frac{\frac{w_{m,s_m}}{N} + f(\mathbf{s}_{-nm})}{\left(\frac{w_{n,s}}{N} + \frac{w_{m,s_m}}{N} + f(\mathbf{s}_{-nm})\right)^2} \\ \cdot \mathbb{P}(\mathbf{S}_{-nm} = \mathbf{s}_{-nm}) = 1. \end{aligned} \tag{A.23}$$

We define a function $g : [\bar{a}, D] \to \Re$ by

$$g(w) = \sum_{\mathbf{s}_{-nm} \in \mathcal{S}^{N-2}} U_s'\left(\frac{w}{f(\mathbf{s}_{-nm})}\right) \cdot \frac{1}{f(\mathbf{s}_{-nm})} \cdot \mathbb{P}(\mathbf{S}_{-nm} = \mathbf{s}_{-nm}). \tag{A.24}$$

Since $U_s$ is continuously differentiable and strictly concave, it follows that $U_s'$ is continuous and strictly decreasing. It follows that $g$ is also continuous and strictly decreasing. Using also Lemma 2.1, it can be seen that there exists a unique point $\overline{w}$ for which $g(\overline{w}) = 1$.

Having fixed the utility functions $U_s$ (and the corresponding constants $\overline{a}$ and $D$), the fact that the bids are bounded above by $D$, the inequalities (A.21), and the assumption that the $U_s$ are continuously differentiable, we see that Eq. (A.23) implies that

$$\left| \sum_{s_m \in \mathcal{S}} p_{m,s_m} g(w_{n,s}) - 1 \right| \leq \delta(N),$$

where $\delta(N)$ is a constant that depends only on $N$, and which tends to zero as $N$ increases. Since $\sum_{s_m \in \mathcal{S}} p_{m,s_m} = 1$, we obtain

$$|g(w_{n,s}) - g(\overline{w})| = |g(w_{n,s}) - 1| \leq \delta(N).$$

We will now argue that this implies that $w_{n,s}$ is "close" to $\overline{w}$.

We argue by contradiction. Fix some $\varepsilon > 0$. Suppose that

$$\left| w_{n,s}^N - \overline{w} \right| \geq \varepsilon/2.$$

From (A.21), we obtain

$$\left| \frac{w_{n,s}}{f(\mathbf{s}_{-nm})} - \frac{\overline{w}}{f(\mathbf{s}_{-nm})} \right| \geq \frac{\varepsilon}{2D}.$$

Let

$$\gamma = \min_{x \in [\frac{\overline{a}}{D}, \frac{D}{\overline{a}} - \frac{\varepsilon}{2D}]} \left| U_s' \left( x + \frac{\varepsilon}{2D} \right) - U_s'(x) \right|.$$

Since $U_s'$ is strictly decreasing, it follows that $\gamma > 0$. Then, the definition (A.24) of $g$ implies that

$$|g(w_{n,s}) - g(\overline{w})| \geq \frac{\gamma}{2D}.$$

If however, $N$ is large enough so that $\delta(N) < \gamma/(2D)$, we obtain a contradiction. Thus, if $N$ is large enough, we must have $|w_{n,s} - \overline{w}| < \varepsilon/2$. A symmetrical argument yields $|w_{m,s} - \overline{w}| < \varepsilon/2$, and $|w_{n,s} - w_{n,s}| < \varepsilon$, when $N$ is large enough. $\square$

## A.4 Proof of Theorem 2.8

We begin the proof by comparing the Bayesian game with the certainty equivalent optimization and showing that the difference between $W_{BG}(\mathcal{M}^N)/N$ and $W_{CE}(\mathcal{M}^N)/N$ goes to zero. The outline of this part of the proof is as follows. We start with a BNE $\mathbf{w}^N$. Based on Theorem 2.7, as the number of users increases to infinity, the bids given by different users at the same state are almost the same. We can therefore focus on a representative user $n(s)$ for each state $s$. We will first prove that the payoff function of a user changes little if we modify the bids of all other users to coincide with the bids of the representative users. This leads to a symmetric strategy profile $\widetilde{\mathbf{w}}^N$ whose welfare is about the same as that of $\mathbf{w}^N$. We then show that the social welfare associated with $\widetilde{\mathbf{w}}^N$ converges to the social welfare associated with the certainty equivalent formulation.

In the second part of the proof, we argue that the number of users in the ex post optimal formulation is approximately the same as the number of users assumed in the certainty equivalent formulation, to argue that the resulting social welfare is the approximately the same for the two formulations.

*Step 1: The payoff obtained by any user at each state remains almost the same, if the other users bid as the standard user.*

For each state $s \in \mathcal{S}$, we find a user $n(s)$ that $p_{n(s),s} > 0$. Let $\mathbf{w}^N$ be a Nash equilibrium for an $N$-user Bayesian game. We will show that

$$\lim_{N\to\infty} \left| Q_{n,s}^N(w, \mathbf{w}_{-n}^N) - Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N) \right| = 0, \quad \forall w \in [0, D], \tag{A.25}$$

where $\widetilde{\mathbf{w}}_{-n}^N = (\widetilde{\mathbf{w}}_1^N, \ldots, \widetilde{\mathbf{w}}_{n-1}^N, \widetilde{\mathbf{w}}_{n+1}^N, \cdots, \widetilde{\mathbf{w}}_N^N)$ and for each $i \neq n$, $\widetilde{\mathbf{w}}_i^N = (\widetilde{w}_{i,1}^N, \cdots, \widetilde{w}_{i,S}^N)$ with

$$\widetilde{w}_{i,s}^N = \begin{cases} w_{n(s),s}^N, & \text{if } p_{i,s} > 0, \\ 0, & \text{if } p_{i,s} = 0. \end{cases} \tag{A.26}$$

First we write the expected payoff function $Q_{n,s}^N(w, \mathbf{w}_{-n}^N)$ as

$$Q_{n,s}^N(w, \mathbf{w}_{-n}) = -w + \sum\nolimits_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s \left( \frac{w}{\frac{w}{N} + f^N(\mathbf{s}_{-n})} \right) \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n}),$$

where $f^N(\mathbf{s}_{-n})$ is given by

$$f^N(\mathbf{s}_{-n}) = \frac{\sum_{i \neq n} w_{i,s_i}^N}{N}.$$

Given the set of $S$ users, $n(1), \cdots, n(S)$, and a state vector $\mathbf{s}_{-n}$, we define $\widetilde{f}^N(\mathbf{s}_{-n})$ by

$$\widetilde{f}^N(\mathbf{s}_{-n}) = \frac{\sum_{i \neq n} \widetilde{w}_{i,s_i}^N}{N},$$

and write the expected payoff $Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N)$ as

$$Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N) = -w + \sum\nolimits_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s \left( \frac{w}{\frac{w}{N} + \widetilde{f}^N(\mathbf{s}_{-n})} \right) \mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n}).$$

According to Theorem 2.7 we have

$$\lim_{N \to \infty} \left| \widetilde{f}^N(\mathbf{s}_{-n}) - f^N(\mathbf{s}_{-n}) \right| = 0, \quad \forall \mathbf{s}_{-n} \in \mathcal{S}^{N-1}. \tag{A.27}$$

From Lemmas 2.2 and 2.3 we have

$$0 < \bar{a} \leq \widetilde{f}^N(\mathbf{s}_{-n}), f^N(\mathbf{s}_{-n}) \leq D, \quad \forall \mathbf{s}_{-n} \in \mathcal{S}^{N-1},$$

and thus the claim in (A.25) follows (A.27) from the continuity of utility functions.

*Step 2: The limit of* $Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N)$.

We will now show that

$$\lim_{N \to \infty} \left[ Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N) - \left( U_s \left( \frac{w}{\sum_{s \in \mathcal{S}} p_s^N w_{n(s),s}^N} \right) - w \right) \right] = 0, \quad \forall w \in [0, D], \tag{A.28}$$

where

$$p_s^N = \frac{\sum_{n=1}^N p_{n,s}}{N}.$$

In an $N$-user Bayesian game, fixing a user $n$, among the rest $N-1$ users, we let $\overline{M}_s^N$ be the random number of users at state $s$. Given a state vector $\mathbf{s}_{-n}$, we use $\overline{M}_s^N(\mathbf{s}_{-n})$ to denote the number of users at state $s$. Note that $\overline{M}_s^N$ is a random variable while $\overline{M}_s^N(\mathbf{s}_{-n})$ is not. Since the states of different users are independent, Hoeffding's inequality yields,

$$\mathbb{P}\left(\left|\frac{\overline{M}_s^N}{N-1} - p_s^N\right| \geq \varepsilon\right) \leq 2e^{-2\varepsilon^2(N-1)}, \ \forall s \in \mathcal{S}, \quad \forall \varepsilon > 0, \quad \forall N \geq 2. \tag{A.29}$$

We rewrite $Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N)$ as,

$$Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N) = -w + \sum\nolimits_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s(\frac{wN}{w + \sum_{s' \in \mathcal{S}} \overline{M}_{s'}^N(\mathbf{s}_{-n}) w_{n(s'),s'}^N})\mathbb{P}(\mathbf{S}_{-n} = \mathbf{s}_{-n}). \tag{A.30}$$

Since

$$\frac{w}{N} + \frac{\overline{M}_s^N(\mathbf{s}_{-n})}{N} \leq \frac{D}{N} + \frac{\overline{M}_s^N(\mathbf{s}_{-n})}{N-1},$$

for any $w \in [\overline{a}, D]$ and any $\mathbf{s}_{-n} \in \mathcal{S}^{N-1}$, we have

$$Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N) \geq -w + \sum\nolimits_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s\left(\frac{w}{\frac{D}{N} + \frac{\sum_{s' \in \mathcal{S}} \overline{M}_{s'}^N(\mathbf{s}_{-n}) w_{n(s'),s'}^N}{N-1}}\right). \tag{A.31}$$

According to (A.28) and (A.31), for any $\varepsilon > 0$, integer $N \geq 2$, $s \in \mathcal{S}$, $n \in \mathcal{N}$ and $w \in [0, D]$, we have

$$Q_{n,s}^N(w, \widetilde{\mathbf{w}}_{-n}^N) + w \geq U_s\left(\frac{w}{\frac{D}{N} + \sum_{s' \in \mathcal{S}} (p_{s'}^N + \varepsilon) w_{n(s'),s'}^N}\right)(1 - 2e^{-2\varepsilon^2(N-1)}). \tag{A.32}$$

For each $N$, let $\varepsilon_N = (N-1)^{-1/3}$. From (A.32) we have

$$Q^N_{n,s}(w, \widetilde{\mathbf{w}}^N_{-n}) + w \geq U_s \left( \frac{w}{\dfrac{D}{N} + \sum_{s' \in \mathcal{S}} (p^N_{s'} + N^{-1/3}) w^N_{n(s'),s'}} \right) (1 - 2e^{-2(N-1)^{1/3}}). \tag{A.33}$$

The inequality in (A.33) serves as a lower bound for $Q^N_{n,s}(w, \widetilde{\mathbf{w}}^N_{-n}) + w$. To prove (A.28), we also need an upper bound. Since $\overline{M}^N_s(\mathbf{s}_{-n}) \leq N-1$, we have

$$\frac{w}{N} + \frac{\overline{M}^N_s(\mathbf{s}_{-n})}{N} \geq \frac{\overline{M}^N_s(\mathbf{s}_{-n})}{N} \geq \frac{\overline{M}^N_s(\mathbf{s}_{-n})}{N-1},$$

and therefore for any $w \in [\overline{a}, D]$ and any $\mathbf{s}_{-n} \in \mathcal{S}^{N-1}$,

$$Q^N_{n,s}(w, \widetilde{\mathbf{w}}^N_{-n}) \leq -w + \sum\nolimits_{\mathbf{s}_{-n} \in \mathcal{S}^{N-1}} U_s \left( \frac{w}{\sum_{s' \in \mathcal{S}} \dfrac{\overline{M}^N_{s'}(\mathbf{s}_{-n})}{N-1} w^N_{n(s'),s'}} \right). \tag{A.34}$$

Based on (A.29) and (A.34), for any $\varepsilon > 0$, integer $N \geq 2$, $s \in \mathcal{S}$, $n \in \mathcal{N}$ and $w \in [0, D]$, we have

$$Q^N_{n,s}(w, \widetilde{\mathbf{w}}^N_{-n}) + w \leq U_s \left( \frac{w}{\sum_{s' \in \mathcal{S}} (p^N_{s'} - \varepsilon) w^N_{n(s'),s'}} \right) + 2e^{-2\varepsilon^2 (N-1)} U_s(N). \tag{A.35}$$

In particular, for each integer $N$, Eq. (A.35) holds for the defined $\varepsilon_N$, i.e,

$$Q^N_{n,s}(w, \widetilde{\mathbf{w}}^N_{-n}) + w \leq U_s \left( \frac{w}{\sum_{s' \in \mathcal{S}} (p^N_{s'} - N^{-1/3}) w^N_{n(s'),s'}} \right) + 2e^{-2(N-1)^{1/3}} U_s(N).$$

Based on Assumption 2.1, we further have

$$Q^N_{n,s}(w, \widetilde{\mathbf{w}}^N_{-n}) + w \leq U_s \left( \frac{w}{\sum_{s' \in \mathcal{S}} (p^N_{s'} - N^{-1/3}) w^N_{n(s'),s'}} \right) + e^{-2N^{1/3}} D. \tag{A.36}$$

Letting $N \to \infty$ on the right hand side of (A.33) and (A.36), we obtain the desired result in (A.28).

*Step 3: Convergence to a solution of the optimization problem in (2.9).*

In this step we aim to show that

$$\lim_{N\to\infty} \frac{w^N_{n(s),s}}{\sum_{s'\in\mathcal{S}} p^N_{s'} w^N_{n(s'),s'}} - \overline{b}^N_s = 0, \quad s = 1,\dots,S, \tag{A.37}$$

where $(\overline{b}^N_1 \cdots, \overline{b}^N_S)$ is the solution to the optimization problem in (2.9). Since $\{w^N_{n(s),s}\}^S_{s=1}$ are the bids given at a Nash equilibrium, we have

$$Q^N_{n(s),s}(w^N_{n(s),s}, \mathbf{w}^N_{-n(s)}) \geq Q^N_{n,s}(w, \mathbf{w}^N_{-n(s)}), \quad \forall w \geq 0, \quad s = 1,\dots,S. \tag{A.38}$$

From Eqs. (A.25), (A.28), and (A.38), we conclude that for any number $\varepsilon > 0$, there exists an $N_1$ such that for any $N \geq N_1$,

$$U_s\left(\frac{w^N_{n(s),s}}{\sum_{s'\in\mathcal{S}} p^N_{s'} w^N_{n(s'),s'}}\right) - w^N_{n(s),s} + \varepsilon \geq U_s\left(\frac{w}{\sum_{s'\in\mathcal{S}} p^N_{s'} w^N_{n(s'),s'}}\right) - w, \quad \forall w \in [0, D]. \tag{A.39}$$

For each $N$, since $U_s(\cdot)$ is strictly concave and continuous, the right hand side of (A.39), as a function of $w$, must have a unique maximum point, $\overline{w}^N_s$, over the compact set $w \in [0, D]$. Now we argue by contradiction that

$$\lim_{N\to\infty} \left|\overline{w}^N_s - w^N_{n(s),s}\right| = 0, \quad s = 1,\dots,S. \tag{A.40}$$

Suppose not. There exists some positive constant $g$ that $\left|\overline{w}^N_s - w^N_{n(s),s}\right| \geq g$ along an increasing subsequence of $N$-user games. We define an interval

$$X(\overline{w}^N_s) = \left\{w \in [0, D] : \left|w - \overline{w}^N_s\right| \geq g\right\},$$

and let

$$\varepsilon^N_s = \min_{w\in X(\overline{w}^N_s)} \left| U_s\left(\frac{w}{\sum_{s'\in\mathcal{S}} p^N_{s'} w^N_{n(s'),s'}}\right) - w - \left(U_s\left(\frac{\overline{w}^N_s}{\sum_{s'\in\mathcal{S}} p^N_{s'} w^N_{n(s'),s'}}\right) - \overline{w}^N_s\right)\right|.$$

Since the function,

$$U_s\left(\frac{w}{\sum_{s'\in\mathcal{S}} p_{s'}^N w_{n(s'),s'}^N}\right) - w,$$

is strictly concave and $\overline{w}_s^N$ is the unique maximum point, we have $\varepsilon_s^N > 0$, for any $N$. Now we find a lower bound for all $\varepsilon_s^N$. We define a group of functions,

$$f_{s,W}(w) \triangleq U_s\left(\frac{w}{W}\right) - w, \quad W \in [\overline{a}, D].$$

Given an $s \in \mathcal{S}$ and some $W \in [\overline{a}, D]$, we let $\overline{w}_{s,W}$ to denote the unique maximum point of the function $f_{s,W}(w)$, over the compact set $[0, D]$. For each function $f_{s,W}(w)$, we define an interval $Y(\overline{w}_{s,W}) = \{w \in [0, D] : |w - \overline{w}_{s,W}| \geq g\}$, and let

$$\delta_{s,W} = \min_{w\in Y(\overline{w}_{s,W})} \left|f_{s,W}(w) - f_{s,W}(\overline{w}_{s,W})\right|.$$

Since $f_{s,W}$ is strictly concave and $\overline{w}_{s,W}$ is its unique maximum point, for every $s \in \mathcal{S}$ and $W \in [\overline{a}, D]$ we have $\delta_{s,W} > 0$. For any $N$, we have

$$\varepsilon_s^N \geq \min_{W\in[\overline{a},D]} \{\delta_{s,W}\} \triangleq \varepsilon_s.$$

Since the $\delta_{s,W} > 0$ and $[\overline{a}, D]$ is compact, we conclude that $\varepsilon_s$ is a positive constant. For the positive constant $\varepsilon_s/2$, there must exist an $N_2$ that for any $N \geq N_2$, (A.39) is valid. However, if (A.40) does not hold, we can find an integer $N \geq N_2$ such that $\left|\overline{w}_s^N - w_{n(s),s}^N\right| \geq g$, and

$$U_s\Big(\frac{\overline{w}_s^N}{\sum_{s'\in\mathcal{S}} p_{s'}^N w_{n(s'),s'}^N}\Big) - \overline{w}_s^N - \left(U_s\Big(\frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}} p_{s'}^N w_{n(s'),s'}^N}\Big) - w_{n(s),s}^N\right) \geq \varepsilon_s,$$

which contradicts the fact expressed in (A.39). It follows that (A.40) is true. Since $\overline{w}_s^N$ maximizes the concave function in the right-hand side of (A.39), we have

$$U_s'\Big(\frac{\overline{w}_s^N}{\sum_{s'\in\mathcal{S}} p_{s'}^N w_{n(s'),s'}^N}\Big) = \sum\nolimits_{s'\in\mathcal{S}} p_{s'}^N w_{n(s'),s'}^N, \quad \forall s \in \mathcal{S}, \quad \forall N. \tag{A.41}$$

Since all utility functions are continuously differentiable, combing (A.40) and (A.41) we have,

$$\lim_{N\to\infty}\left|U_s'\Big(\frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}}p_{s'}^N w_{n(s'),s'}^N}\Big)-\sum\nolimits_{s'\in\mathcal{S}}p_{s'}^N w_{n(s'),s'}^N\right|=0,\quad \forall s\in\mathcal{S}. \tag{A.42}$$

We define

$$\widehat{b}_s^N=\frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}}p_{s'}^N w_{n(s'),s'}^N},\quad \widehat{\lambda}^N=\sum\nolimits_{s'\in\mathcal{S}}p_{s'}^N w_{n(s'),s'}^N, \tag{A.43}$$

and argue that

$$\lim_{N\to\infty}\left|\widehat{b}_s^N-\overline{b}_s^N\right|=0,\ s=1,2,\cdots,S, \tag{A.44}$$

where $(\overline{b}_1^N,\cdots,\overline{b}_S^N)$ is an optimal solution to the optimization problem in (2.9). To prove (A.44), first we argue that

$$\lim_{N\to\infty}\left|\widehat{\lambda}^N-\overline{\lambda}^N\right|=0, \tag{A.45}$$

where $\overline{\lambda}^N=U'(\overline{b}_s^N)$. Suppose not. There must exist some positive number $\delta$ such that for any positive integer $N_3$, we can find some $N\geq N_3$ for which

$$\left|\widehat{\lambda}^N-\overline{\lambda}^N\right|\geq\delta. \tag{A.46}$$

From (A.42), we can find some $N_3$ such that,

$$\left|U_s'(\widehat{b}_s^N)-\widehat{\lambda}^N\right|\leq\frac{\delta}{2},\quad \forall N\geq N_3,\quad \forall s\in\mathcal{S}. \tag{A.47}$$

According to (A.46), we can find a $N\geq N_3$ that $\widehat{\lambda}^N-\overline{\lambda}^N\geq\delta$, or $\overline{\lambda}^N-\widehat{\lambda}^N\geq\delta$. If $\widehat{\lambda}^N-\overline{\lambda}^N\geq\delta$, from we have

$$U_s'(\widehat{b}_s^N)-\overline{\lambda}^N\geq\frac{\delta}{2},\quad s=1,\ldots,S.$$

Since $U_s$ is strictly concave, we have

$$\widehat{b}_s^N<\overline{b}_s^N,\quad s=1,\ldots,S.$$

which contradicts the fact the both $(\widehat{b}_1^N, \ldots, \widehat{b}_S^N)$ and $(\overline{b}_1^N, \cdots, \overline{b}_S^N)$ satisfy the following constraint,

$$\sum_{s=1}^{S} p_s^N \widehat{b}_s^N = \sum_{s=1}^{S} p_s^N \overline{b}_s^N = 1,$$

where the second equality is true because $(\overline{b}_1^N, \ldots, \overline{b}_S^N)$ is an optimal solution to the problem in (2.9). If $\overline{\lambda}^N - \widehat{\lambda}^N \geq \delta$, through a similar way we can prove that

$$\widehat{b}_s^N > \overline{b}_s^N, \ s = 1, \ldots, S,$$

and again, the above second constraint is violated. Hence, we have verified (A.45). Combing (A.42) and (A.45) we have

$$\lim_{N\to\infty} \left| U_s'(\widehat{b}_s^N) - U_s'(\overline{b}_s^N) \right| = 0, \ s = 1, \ldots, S,$$

and the desired result in (A.44) follows from the fact that $U_s'(\cdot)$ is strictly decreasing. From our definition of $\widehat{b}_s^N$, it can be seen that (A.44) is equivalent to the claimed result in (A.37).

*Step 4: Social welfare achieved at a Bayesian Nash equilibrium.*

In this step we will show that

$$\lim_{N\to\infty} \left| \frac{W_{BG}(\mathcal{M}^N)}{N} - \sum_{s\in\mathcal{S}} p_s^N U_s(\widehat{b}_s^N) \right| = 0. \qquad \text{(A.48)}$$

Given a state vector $\mathbf{s}$, we define a subset of the user set $\mathcal{N}^N$,

$$\mathcal{N}_s^N(\mathbf{s}) \triangleq \left\{ n \in \mathcal{N}^N \ \middle| s_n = s \right\}.$$

As a subset of $\mathcal{N}^N$, the set $\mathcal{N}_s^N(\mathbf{s})$ consists of users at state $s$. We write the social welfare achieved at a Nash equilibrium $\mathbf{w}$ as,

$$\frac{W(\mathbf{w}, \mathcal{M}^N)}{N} = \frac{1}{N} \sum_{\mathbf{s}\in\mathcal{S}^N} \sum_{s\in\mathcal{S}} \sum_{n\in\mathcal{N}_s^N(\mathbf{s})} U_s\left(b_n^N(\mathbf{w}, \mathbf{s})\right) \mathbb{P}(\mathbf{S} = \mathbf{s}), \qquad \text{(A.49)}$$

where $W_{BG}(\mathbf{w}, \mathcal{M}^N)$ is the expected social welfare achieved at the Nash equilibrium

$\mathbf{w}$, and $b_n(\mathbf{w}, \mathbf{s})$ is the bandwidth allocated to user $n$, i.e.,

$$b_n^N(\mathbf{w}, \mathbf{s}) = \frac{N w_{n,s}^N}{w_{n,s}^N + \sum_{i \neq n} w_{i,s_i}^N}.$$

According to Theorem 2.7, for any $\varepsilon > 0$, there exists an integer $N_4$ such that

$$\left| b_n^N(\mathbf{w}, \mathbf{s}) - \frac{N w_{n(s),s}^N}{w_{n(s),s}^N + \sum_{i \neq n} w_{n(s_i),s_i}^N} \right| \leq \varepsilon, \quad \forall N \geq N_4, \quad \forall n \in \mathcal{N}^N, \quad \forall \mathbf{s} \in \mathcal{S}^N. \tag{A.50}$$

We use a random variable $M_s^N$ to denote the number of users at state $s$ in an $N$-user Bayesian game. For each $N$-dimensional state vector $\mathbf{s} \in \mathcal{S}$, we use $M_s^N(\mathbf{s})$ to denote the number of users at state $s$. From Eqs. (A.49) and (A.50), for any Nash equilibrium $\mathbf{w}$, we have

$$\lim_{N \to \infty} \left| \frac{W_{BG}(\mathbf{w}, \mathcal{M}^N)}{N} - \frac{1}{N} \sum_{\mathbf{s} \in \mathcal{S}^N} \sum_{s \in \mathcal{S}} M_s^N(\mathbf{s}) U_s \left( \frac{N w_{n(s),s}^N}{w_{n(s),s}^N + \sum_{i \neq n} w_{n(s_i),s_i}^N} \right) \mathbb{P}(\mathbf{S} = \mathbf{s}) \right| = 0. \tag{A.51}$$

Since

$$W_{BG}(\mathcal{M}^N) = \inf_{\mathbf{w} \in BNE} \left\{ W_{BG}(\mathbf{w}, \mathcal{M}^N) \right\},$$

to prove the desired result in (A.48), we only need to show that

$$\lim_{N \to \infty} \left| \frac{1}{N} \sum_{\mathbf{s} \in \mathcal{S}^N} \sum_{s \in \mathcal{S}} M_s^N(\mathbf{s}) U_s \left( \frac{N w_{n(s),s}^N}{w_{n(s),s}^N + \sum\limits_{i \neq n} w_{n(s_i),s_i}^N} \right) \mathbb{P}(\mathbf{S} = \mathbf{s}) - \sum_{s \in \mathcal{S}} p_s^N U_s(\widehat{b}_s^N) \right| = 0. \tag{A.52}$$

Given $N$ and a positive number $\varepsilon$, we define a subset of $\mathcal{S}^N$ as follows,

$$\mathcal{S}^N(\varepsilon) = \left\{ \mathbf{s} \in \mathcal{S}^N : \left| \frac{M_s^N(\mathbf{s})}{N} - p_s^N \right| \leq \varepsilon, \quad s = 1, \ldots, S. \right\}. \tag{A.53}$$

Thus, state vectors in the set $\mathcal{S}^N(\varepsilon)$ are such that each individual state is present with a frequency close to its expected frequency. Since the states of different users are independent, Hoeffding's inequality yields

$$\mathbb{P}(\mathbf{s} \in \mathcal{S}^N(\varepsilon)) \geq (1 - 2e^{-2\varepsilon^2 N})^S, \quad \forall N, \quad \forall \varepsilon > 0. \tag{A.54}$$

Hence, for any $\varepsilon > 0$ and any integer $N$, we have

$$\begin{aligned}
&\sum\nolimits_{\mathbf{s}\in\mathcal{S}^N} \sum\nolimits_{s\in\mathcal{S}} \frac{M_s^N(\mathbf{s})}{N} U_s\left(\frac{N w_{n(s),s}^N}{w_{n(s),s}^N + \sum_{i\neq n} w_{n(s_i),s_i}^N}\right)\mathbb{P}(\mathbf{S} = \mathbf{s})\\
&\geq \sum\nolimits_{\mathbf{s}\in\mathcal{S}^N(\varepsilon)} \sum\nolimits_{s\in\mathcal{S}} \frac{M_s^N(\mathbf{s})}{N} U_s\left(\frac{N w_{n(s),s}^N}{w_{n(s),s}^N + \sum_{i\neq n} w_{n(s_i),s_i}^N}\right)\mathbb{P}(\mathbf{S} = \mathbf{s})\\
&\geq \sum\nolimits_{s\in\mathcal{S}} (p_s^N - \varepsilon) U_s\left(\frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}} (p_{s'}^N + \varepsilon) w_{n(s'),s'}}\right)(1 - e^{-2\varepsilon^2 N})^S\\
&\geq (1 - e^{-2\varepsilon^2 N})^S \sum\nolimits_{s\in\mathcal{S}} p_s^N U_s\left(\frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}} (p_{s'}^N + \varepsilon) w_{n(s'),s'}}\right) - \varepsilon D,
\end{aligned} \tag{A.55}$$

where the second inequality follows (A.54) and the last inequality is valid because each utility function is bounded from above. On the other hand, for every small positive number $\varepsilon$ and every $\mathbf{s} \in \mathcal{S}^N(\varepsilon)$ and every $\forall s \in \mathcal{S}$, we have

$$\widehat{b}_s^N = \frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}} p_{s'}^N w_{n(s'),s'}} \geq \frac{h-\varepsilon}{h} \frac{w_{n(s),s}^N}{\sum_{s'\in\mathcal{S}} (p_{s'}^N - \varepsilon) w_{n(s'),s'}} \geq \frac{h-\varepsilon}{h} \cdot \frac{w_{n(s),s}^N N}{w_{n(s),s}^N + \sum_{i\neq n} w_{n(s_i),s_i}}, \tag{A.56}$$

where $h$ is the positive constant given in Assumption 2.3. Here, the first inequality is true because

$$\frac{p_s^N}{p_s^N - \varepsilon} \leq \frac{h}{h-\varepsilon}, \quad \forall s \in \mathcal{S},$$

and the second inequality follows the definition of $\mathcal{S}^N(\varepsilon)$ in (A.53). For any small positive number $\varepsilon$ and any integer $N$ we have,

$$\begin{aligned}
\sum_{s\in\mathcal{S}} p_s^N U_s(\widehat{b}_s^N) &\geq \sum_{\mathbf{s}\in\mathcal{S}^N(\varepsilon)} \sum_{s\in\mathcal{S}} \left(\frac{M_s^N(\mathbf{s})}{N} - \varepsilon\right) U_s(\widehat{b}_s^N)\mathbb{P}(\mathbf{S} = \mathbf{s})\\
&\geq \sum_{\mathbf{s}\in\mathcal{S}^N(\varepsilon)} \sum_{s\in\mathcal{S}} \left(\frac{M_s^N(\mathbf{s})}{N} - \varepsilon\right) U_s\left(\frac{(h-\varepsilon) w_{n(s),s}^N N}{h(w_{n(s),s}^N + \sum_{i\neq n} w_{n(s_i),s_i})}\right)\mathbb{P}(\mathbf{S} = \mathbf{s})\\
&\geq \sum_{\mathbf{s}\in\mathcal{S}^N(\varepsilon)} \sum_{s\in\mathcal{S}} \frac{M_s^N(\mathbf{s})}{N} U_s\left(\frac{(h-\varepsilon) w_{n(s),s}^N N}{h(w_{n(s),s}^N + \sum_{i\neq n} w_{n(s_i),s_i})}\right)\mathbb{P}(\mathbf{S} = \mathbf{s}) - \varepsilon D,
\end{aligned} \tag{A.57}$$

where the first inequality is true because of the definition of $\mathcal{S}^N(\varepsilon)$ in (A.53), and the second inequality follows from (A.56) and the last one is true due to Assumption 2.1. By letting $\varepsilon_N = N^{-1/3}$ and taking $N$ to infinity on the right hand side of the inequalities in (A.55) and (A.57), we obtain the result in Eq. (A.52), which implies the desired result in (A.48).

*Step 5: Optimality of Nash equilibria in the Bayesian game.*

We can now complete the proof for the first part of the Theorem. We have

$$\begin{aligned} &\lim_{N\to\infty}\left|\frac{W_{BG}(\mathcal{M}^N)}{N} - \frac{W_{CE}(\mathcal{M}^N)}{N}\right| \\ &= \lim_{N\to\infty}\left|\frac{W_{BG}(\mathcal{M}^N)}{N} - \sum_{s\in\mathcal{S}} p_s^N U_s(\bar{b}_s^N)\right| \\ &= \lim_{N\to\infty}\left|\sum_{s\in\mathcal{S}} p_s^N U_s(\widehat{b}_s^N) - \sum_{s\in\mathcal{S}} p_s^N U_s(\bar{b}_s^N)\right| \\ &= 0, \end{aligned}$$

where the second equality is due to Eq. (A.48), and the last one follows from the continuity of utility functions and the relation in (A.44).

*Step 6: Ex post optimal allocation.*

In this step we will show that

$$\lim_{N\to\infty}\left|\frac{W_{PO}(\mathcal{M}^N)}{N} - \frac{W_{CE}(\mathcal{M}^N)}{N}\right| = 0. \tag{A.58}$$

In Lemma 2.1, we have shown that there always exists an ex post optimal allocation such that all users on the same state obtain the same amount of network resource. Given a state vector $\mathbf{s}$, this ex post optimal allocation solves the following optimization problem,

$$\begin{aligned} &\underset{b_1^N,\dots b_S^N}{\text{Maximize}} \ \sum_{s=1}^S M_s^N(\mathbf{s}) U_s(b_s^N) \\ &\text{Subject to} \sum_{s=1}^S M_s^N(\mathbf{s}) b_s^N \le N. \end{aligned} \tag{A.59}$$

We use $W_{PO}(\mathbf{s}, \mathcal{M}^N)$ to denote the optimal value of the above optimization problem, which is the optimal social welfare that can be achieved at state vector $\mathbf{s}$. It is not hard to see that

$$W_{PO}(\mathcal{M}^N) = \sum\nolimits_{\mathbf{s}\in\mathcal{S}} W_{PO}(\mathbf{s}, \mathcal{M}^N) P(\mathbf{S} = \mathbf{s}).$$

For any $\mathbf{s} \in \mathcal{S}^N(\varepsilon)$, we will find an upper bound on the difference between the optimal values of the above optimization problem and the one in (2.9). Both two convex optimization problems must have at least one optimal solution, because their feasible sets are compact. For some $\mathbf{s} \in \mathcal{S}^N(\varepsilon)$, let $(\widetilde{b}_1^N, \ldots, \widetilde{b}_S^N)$ be an optimal solution to the optimization problem in (A.59) and $(\overline{b}_1^N, \ldots, \overline{b}_S^N)$ be an optimal solution to the optimization problem in (2.9). For every $\mathbf{s} \in \mathcal{S}^N(\varepsilon)$, we have

$$\begin{aligned}\sum_{s=1}^{S} \frac{M_s^N(\mathbf{s})}{N} U_s(\widetilde{b}_s^N) \geq \sum_{s=1}^{S} \frac{M_s^N(\mathbf{s})}{N} U_s(\frac{\overline{b}_s^N}{1+\varepsilon/p_s^N}) \geq \sum_{s=1}^{S} (p_s^N - \varepsilon) U_s(\frac{\overline{b}_s^N}{1+\varepsilon/p_s^N}) \\ \geq \sum_{s=1}^{S} p_s^N U_s(\frac{\overline{b}_s^N}{1+\varepsilon/p_s^N}) - \varepsilon SD \geq \sum\nolimits_{s=1}^{S} p_s^N U_s(\frac{\overline{b}_s^N}{1+\varepsilon/h}) - \varepsilon SD.\end{aligned} \tag{A.60}$$

Here, the first inequality is valid because

$$\sum\nolimits_{s=1}^{S} \frac{M_s^N(\mathbf{s})}{N} \frac{\overline{b}_s^N}{1+\varepsilon/p_s^N} \leq \sum\nolimits_{s=1}^{S} (p_s^N + \varepsilon) \frac{\overline{b}_s^N}{1+\varepsilon/p_s^N} = 1.$$

Therefore, $\{\overline{b}_s^N/(1+\varepsilon/p_s^N)\}_{s=1}^S$ is feasible in the optimization problem (A.59). The first inequality in (A.60) then follows the optimality of $(\widetilde{b}_1^N, \cdots, \widetilde{b}_S^N)$. The second inequality is due to the definition of the set $\mathcal{S}^N(\varepsilon)$ in (A.53). The third inequality in (A.60) holds because of Assumption 2.1; the last one follows from Assumption 2.3. Similarly, we have

$$\sum\nolimits_{s=1}^{S} p_s^N U_s(\overline{b}_s^N) \geq \sum\nolimits_{s=1}^{S} p_s^N U_s(\frac{\widetilde{b}_s^N}{1+\varepsilon/p_s^N}) \geq \sum\nolimits_{s=1}^{S} \frac{M_s^N(\mathbf{s})}{N} U_s(\frac{\widetilde{b}_s^N}{1+\varepsilon/h}) - \varepsilon SD. \tag{A.61}$$

For each $N$, let $\varepsilon_N = N^{-1/3}$. According to (A.60) and (A.61), for any positive number

$\delta$, we can find an integer $N_6$ such that

$$\left|\sum_{s=1}^{S} \frac{M_s^N(\mathbf{s})}{N} U_s(\widetilde{b}_s^N) - \sum_{s=1}^{S} p_s^N U_s(\overline{b}_s^N)\right| \le \delta, \quad \forall N \ge N_6, \quad \forall \mathbf{s} \in \mathcal{S}^N(N^{-1/3}). \tag{A.62}$$

Finally we have,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{W_{PO}(\mathcal{M}^N)}{N} - \sum_{s=1}^{S} p_s^N U_s(\overline{b}_s^N) \\
&\le \lim_{N\to\infty} \left| \sum_{\mathbf{s}\in\mathcal{S}^N(N^{-1/3})} \frac{W_{PO}(\mathbf{s},\mathcal{M}^N)}{N} \mathbb{P}(\mathbf{S}=\mathbf{s}) - \sum_{s=1}^{S} p_s^N U_s(\overline{b}_s^N) \right| \\
&\quad + \lim_{N\to\infty} \sum_{\mathbf{s}\notin\mathcal{S}^N(N^{-1/3})} \frac{W_{PO}(\mathbf{s},\mathcal{M}^N)}{N} \mathbb{P}(\mathbf{S}=\mathbf{s}) \\
&\le \lim_{N\to\infty} \left| \sum_{\mathbf{s}\in\mathcal{S}^N(N^{-1/3})} \frac{W_{PO}(\mathbf{s},\mathcal{M}^N)}{N} \mathbb{P}(\mathbf{S}=\mathbf{s}) - \sum_{s=1}^{S} p_s^N U_s(\overline{b}_s^N) \right| + (1 - 2e^{-2N^{-1/3}N})^S D \\
&= 0,
\end{aligned} \tag{A.63}$$

where $W_{PO}(\mathbf{s},\mathcal{M}^N)$ is the optimal social welfare that can be achieved at the state vector $\mathbf{s}$. The first inequality in (A.63) is trivial. The second inequality holds because of Assumption 2.1 and the inequality in (A.54). The last equality follows from (A.62) and the fact that

$$\frac{W_{PO}(\mathbf{s},\mathcal{M}^N)}{N} = \sum_{s=1}^{S} \frac{M_s^N(\mathbf{s})}{N} U_s(\widetilde{b}_s^N).$$

The desired result in (A.58) follows from Eq. (A.63), and the following fact

$$W_{PO}(\mathcal{M}^N) \ge W_{CE}(\mathcal{M}^N) = \sum_{s=1}^{S} p_s^N U_s(\overline{b}_s^N).$$

## A.5 Proof of Proposition 2.2

Let $\alpha_1$ and $\alpha_2$ be the slope of the utility functions at state 1 and 2, respectively. Without loss of generality, we assume that $\alpha_2 > \alpha_1 > 0$. At a symmetric Nash equilibrium, we let $l$ and $h$ be the bid given by users at state 1 and state 2, respectively. Due to Theorem 2.6, we have $h \geq l$. We will prove the desired result by discussing two cases: $l = 0$ and $h \geq l > 0$.

*Step 1: The case $l = 0$.*

Under Assumption 2.2, it is not hard to see that $l = h = 0$ cannot be a Nash equilibrium. We only need to consider the case that $l = 0$, $h > 0$. We argue that a vector $(w_1 = 0,\ w_2 = h > 0)$ cannot be a Nash equilibrium, if there exists a user $n$ such that

$$\mathbb{P}\left(\mathbf{s}_{-n} = (1, 1, \ldots, 1) \mid S_n = 1\right) > 0, \tag{A.64}$$

i.e., given that user $n$ is at the "bad" state, it is possible that all the other users are at the bad state. If (A.64) holds, then player 1 can benefit from biding a small positive number at state 1, e.g.,

$$w_{n,1} = \frac{\mathbb{P}\left(\mathbf{s}_{-n} = (1, 1, \ldots, 1) \mid S_n = 1\right) \cdot \alpha_1 C}{2},$$

instead of bidding 0. Therefore, if a vector $(w_1 = 0,\ w_2 = h > 0)$ is a Nash equilibrium, the we must have

$$\mathbb{P}\left(\mathbf{s}_{-n} = (1, 1, \ldots, 1) \mid S_n = 1\right) = 0, \quad \forall n \in \mathcal{N}, \tag{A.65}$$

Since users' states are independent, Eq. (A.65) implies that there exists a user $m$ such that $p_{m,1} = 0$, i.e., user $m$ is always at state 2. Then, the symmetric Nash equilibrium, $(w_1 = 0, w_2 = h > 0)$, achieves the same social welfare achieved as that achieved by best centralized allocation.

*Step 2: A lower bound for $h$, when $h \geq l > 0$.*

Now we consider the case that $h \geq l > 0$. Since $(w_1 = l, w_2 = h)$ is a symmetric

Nash equilibrium, from the first-order equilibrium conditions we have

$$\begin{aligned}
&\alpha_1 \textstyle\sum_{v=0}^{N-1} \mathbb{P}(M_1 = v+1 \mid S_n = 1) \dfrac{vl + (N-1-v)h}{(l + vl + (N-1-v)h)^2} \\
&= \alpha_2 \textstyle\sum_{v=0}^{N-1} \mathbb{P}(M_1 = v \mid S_n = 2) \dfrac{vl + (N-1-v)h}{(vl + (N-v)h)^2} \\
&= 1,
\end{aligned} \tag{A.66}$$

where $M_1$ is a random variable denoting the number of users at state 1. We now argue that

$$\frac{\alpha_1}{(Nl)^2} \geq \frac{\alpha_2}{((N-1)l + h)^2}. \tag{A.67}$$

Suppose not, and we have

$$\frac{\alpha_1}{(l + vl + (N-1-v)h)^2} < \frac{\alpha_2}{(vl + (N-v)h)^2}, \quad v = 0, 1, \ldots, N-1,$$

which implies that

$$\frac{\alpha_1 \left(vl + (N-1-v)h\right)}{(l + vl + (N-1-v)h)^2} < \frac{\alpha_2 \left(vl + (N-1-v)h\right)}{(vl + (N-v)h)^2}, \quad v = 0, 1, \ldots, N-1.$$

Since the game is symmetric, we have

$$\mathbb{P}(M_1 = v+1 \mid S_n = 1) = \mathbb{P}(M_1 = v \mid S_n = 2), \quad v = 0, 1, \ldots, N-1,$$

and therefore

$$\begin{aligned}
&\alpha_1 \sum_{v=0}^{N-1} \mathbb{P}(M_1 = v+1 \big| S_n = 1) \frac{vl + (N-1-v)h}{(l + vl + (N-1-v)h)^2} \\
&< \alpha_2 \sum_{v=0}^{N-1} \mathbb{P}(M_1 = v \big| S_m = 2) \frac{vl + (N-1-v)h}{(vl + (N-v)h)^2},
\end{aligned}$$

which contradicts (A.66). From (A.67), we have

$$(N-1)l + h \geq \sqrt{\frac{\alpha_2}{\alpha_1}} Nl \Rightarrow Nl + h \geq \sqrt{\frac{\alpha_2}{\alpha_1}} Nl,$$

which implies that

$$h \geq N(\sqrt{\frac{\alpha_2}{\alpha_1}} - 1)l. \tag{A.68}$$

*Step 3: A lower bound for the efficiency of symmetric Nash equilibria, when $h \geq l > 0$.*

The social welfare achieved at any symmetric Nash equilibrium, $w_1 = l, w_2 = h$, $\widetilde{W}_{BG}(\mathcal{M})$, satisfies

$$\begin{aligned}
\frac{\widetilde{W}_{BG}(\mathcal{M})}{W_{PO}(\mathcal{M})} &= \frac{\alpha_1 \mathbb{P}(M_1 = N) + \sum_{v=0}^{N-1} \mathbb{P}(M_1 = v) \dfrac{\alpha_1 vl + \alpha_2 (N-v)h}{vl + (N-v)h}}{\alpha_1 \mathbb{P}(M_1 = N) + \alpha_2 \left(1 - \mathbb{P}(M_1 = N)\right)} \\
&\geq \frac{\sum_{v=0}^{N-1} \mathbb{P}(M_1 = v) \dfrac{\alpha_1 vl + \alpha_2 (N-v)h}{vl + (N-v)h}}{\alpha_2 \left(1 - \mathbb{P}(M_1 = N)\right)} \\
&\geq \sum_{v=0}^{N-1} \mathbb{P}(M_1 = v) \frac{\alpha_1 vl + \alpha_2 (N-v)h}{\alpha_2 (vl + (N-v)h)} \\
&\geq \frac{\alpha_1 (N-1)l + \alpha_2 h}{\alpha_2 \left((N-1)l + h\right)}.
\end{aligned} \tag{A.69}$$

Since $\alpha_2 > \alpha_1 > 0$ and $h \geq l$, it is not hard to see that,

$$\frac{\alpha_1 (N-1)l + \alpha_2 h}{\alpha_2 \left((N-1)l + h\right)},$$

is nonincreasing in $N$ and nondecreasing in $h$. Let $B = \alpha_2/\alpha_1$, and we have

$$\begin{aligned}
&\frac{\alpha_1 (N-1)l + \alpha_2 h}{\alpha_2 \left((N-1)l + h\right)} \geq \frac{\alpha_1 (N-1)l + \alpha_2 (\sqrt{\frac{\alpha_2}{\alpha_1}} - 1)Nl}{\alpha_2 \left((N-1)l + (\sqrt{\frac{\alpha_2}{\alpha_1}} - 1)Nl\right)} \\
&\geq \lim_{N \to \infty} \frac{\alpha_1 (N-1)l + \alpha_2 (\sqrt{\frac{\alpha_2}{\alpha_1}} - 1)Nl}{\alpha_2 \left((N-1)l + (\sqrt{\frac{\alpha_2}{\alpha_1}} - 1)Nl\right)} = \frac{1 + \frac{\alpha_2}{\alpha_1}\sqrt{\frac{\alpha_2}{\alpha_1}} - \frac{\alpha_2}{\alpha_1}}{\frac{\alpha_2}{\alpha_1}\sqrt{\frac{\alpha_2}{\alpha_1}}} = 1 - \frac{1}{\sqrt{B}} + \frac{1}{B\sqrt{B}},
\end{aligned} \tag{A.70}$$

where the first inequality follows from (A.68). Through a simple calculation, we know that the right hand side in (A.70), as a function of $B$, attains a minimum at $B = 3$ over the interval $(1, \infty)$. The desired result follows from (A.69) and (A.70).

# Appendix B

# Proof of results in Chapter 3

## B.1 Proof of Theorem 3.1

We note that part (d) is an immediate consequence of the expression for $g(\beta)$, and we concentrate on the remaining parts. Since the inverse demand function is convex, Proposition 3.3 shows that any Cournot equilibrium satisfies the necessary conditions (C.1). If $X > b/a$, then $p(X) = p'(X) = 0$. In that case, the necessary conditions (C.1) imply the optimality conditions (3.2). We conclude that $\mathbf{x}$ is socially optimal.

We now assume that $X \leq b/a$. Proposition 3.3 shows that $p'(X)$ exists, and thus $X < b/a$. Since $p'(X) = -a < 0$, Proposition 3.8 implies that $p(X) \neq p(X^S)$, for any social optimum $\mathbf{x}^S$. Hence, $\mathbf{x}$ is not socially optimal.

As discussed in Section 3.3.3, to derive a lower bound, it suffices to consider the case of linear cost functions, and obtain a lower bound on the worst case efficiency of Cournot candidates, that is, vectors that satisfy (3.3)-(3.4). We will therefore assume that $C_n(x_n) = \alpha_n x_n$ for every $n$. Without loss of generality, we also assume that $\alpha_1 = \min_n\{\alpha_n\}$. We consider separately the two cases where $\alpha_1 = 0$ or $\alpha_1 > 0$, respectively.

**The case where** $\alpha_1 = 0$

In this case, the socially optimal supply is $X^S = b/a$ and the optimal social welfare is

$$\int_0^{b/a} p(q)\,dq - 0 = \int_0^{b/a} (-ax+b)\,dx = \frac{b^2}{2a}.$$

Note also that $\beta = aX/b$.

Let $\mathbf{x}$ be a Cournot candidate. Suppose first that $x_1 = 0$. In that case, the necessary conditions $0 = \alpha_1 \geq p(X)$ imply that $p(X) = 0$. For $n \neq 1$, if $x_n > 0$, the necessary conditions yield $0 \leq \alpha_n = p(X) - x_n a = -x_n a$, which implies that $x_n = 0$ for all $n$. But then, $X = 0$, which contradicts the fact $p(X) = 0$. We conclude that $x_1 > 0$.

Since $x_1 > 0$, the necessary conditions (C.1) yield $0 = \alpha_1 = b - aX - ax_1$, so that

$$x_1 = -X + \frac{b}{a}. \tag{B.1}$$

In particular, $X < b/a = X^S$, and $\beta < 1$. Furthermore,

$$0 \leq \sum_{n=2}^{N} x_n = X - x_1 = 2X - \frac{b}{a},$$

from which we conclude that $\beta = aX/b \geq 1/2$.

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \neq 1$, whenever $x_n > 0$, we have $\alpha_n = p(X) - ax_n$, so that $\alpha_n x_n = (p(X) - ax_n)x_n$. The social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}
\int_0^X p(q)\,dq - \sum_{n=1}^{N} \alpha_n x_n &= bX - \frac{1}{2}aX^2 - \sum_{n=2}^{N}(p(X) - ax_n)x_n \\
&\geq bX - \frac{1}{2}aX^2 - p(X)\sum_{n=2}^{N} x_n \qquad \text{(B.2)} \\
&= bX - \frac{1}{2}aX^2 - (b - aX)(X - x_1) \\
&= bX - \frac{1}{2}aX^2 - (b - aX)\Big(2X - \frac{b}{a}\Big) \\
&= \frac{3}{2}aX^2 + \frac{b^2}{a} - 2bX.
\end{aligned}$$

We divide by $b^2/2a$ (the optimal social welfare) and obtain

$$\gamma(X) \geq \frac{2a}{b^2}\left(\frac{3}{2}aX^2 + \frac{b^2}{a} - 2bX\right) = 3\beta^2 - 4\beta + 2.$$

This proves the claim in part (b) of the theorem.

**Tightness**

We observe that the lower bound on the social welfare associated with $\mathbf{x}$ made use, in Eq. (B.2), of the inequality $\sum_{n=2}^{N} x_n^2 \geq 0$. This inequality becomes an equality, asymptotically, if we let $N \to \infty$ and $x_n = O(1/N)$ for $n \neq 1$. This motivates the proof of tightness (part (c) of the theorem) given below.

We are given some $\beta \in [1/2, 1)$ and construct an $N$-supplier model ($N \geq 2$) with $a = b = 1$, and the following linear cost functions:

$$C_1^N(x_1) = 0, \qquad C_n^N(x_n) = \left(p(X) - \frac{2X-1}{N-1}\right)x_n, \qquad n = 2, \ldots, N.$$

It can be verified that the variables

$$x_1 = -X + b/a, \qquad x_n = \frac{2X - b/a}{N-1}, \quad n = 2, \ldots, N,$$

form a Cournot equilibrium. A simple calculation (consistent with the intuition given earlier) shows that as $N$ increases to infinity, the sum $\sum_{n=2}^{N} x_n^2$ goes to zero and the associated efficiency converges to $g(\beta)$.

**The case where $\alpha_1 > 0$**

We now consider the case where $\alpha_n > 0$ for every $n$. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$. By Assumption 3.4, we have $b > 1$.

At the social optimum, we must have $p(X^S) = \alpha_1 = 1$ and thus $X^S = (b-1)/a$.

The optimal social welfare is

$$\frac{(b-1)(b+1)}{2a} - \frac{b-1}{a} = \frac{(b-1)^2}{2a}.$$

Note also that $\beta = aX/(b-1)$.

Similar to the proof for the case where $\alpha_1 = 0$, we can show that $x_1 > 0$ and therefore $1 = p(X) - ax_1 = b - aX - ax_1$, so that

$$x_1 = -X + \frac{b-1}{a} > 0,$$

which implies that $\beta < 1$. In particular,

$$X < \frac{b-1}{a} = X^S.$$

Furthermore,

$$0 \leq \sum_{n=2}^{N} x_n = X - x_1 = 2X - \frac{b-1}{a},$$

from which we conclude that $\beta = aX/(b-1) \geq 1/2$.

A calculation similar to the one for the case where $\alpha_1 = 0$ yields

$$\begin{aligned}
\int_0^X p(q)\,dq - \sum_{n=1}^{N} \alpha_n x_n &= bX - \frac{1}{2}aX^2 - x_1 - \sum_{n=2}^{N}(p(X) - ax_n)x_n \\
&\geq bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - p(X)\sum_{n=2}^{N} x_n \\
&= bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - (b - aX)(X - x_1) \\
&= bX - \frac{1}{2}aX^2 + X - \frac{b-1}{a} - (b - aX)\Big(2X - \frac{b-1}{a}\Big) \\
&= \frac{3}{2}aX^2 + \frac{(b-1)^2}{a} - 2(b-1)X.
\end{aligned}$$

After dividing with the value of the social welfare, we obtain $g(\beta)$, as desired.

## B.2 Proof of Theorem 3.2

Let $\mathbf{x}$ be a Cournot candidate. According to Proposition 3.8, if $p(X) = p(X^S)$, then the efficiency of the Cournot candidate must equal one, which proves part (a). To prove part (b), we assume that $p(X) \neq p(X^S)$. By Proposition 3.2, the Cournot candidate $\mathbf{x}$ cannot be socially optimal, and, therefore, $\gamma(\mathbf{x}) < 1$.

We have shown in Proposition 3.6 that if all cost functions are replaced by linear ones, the vector $\mathbf{x}$ remains a Cournot candidate, and Assumptions 3.1-3.4 still hold. Further, the efficiency of $\mathbf{x}$ cannot increase after all cost functions are replaced by linear ones. Thus, to lower bound the worst case efficiency loss, it suffices to derive a lower bound for the efficiency of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each $n$. Without loss of generality, we further assume that $\alpha_1 = \min_n\{\alpha_n\}$. Note that, by Assumption 3.4, we have $p(0) > \alpha_1$. We will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$.

We will rely on Proposition 3.9, according to which the efficiency of a Cournot candidate $\mathbf{x}$ is lower bounded by the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in a model involving the piecewise linear and convex inverse demand function function of the form in the definition of $p^0(\cdot)$. Note that since $p(X) \neq p(X^S)$, we have that $d > 0$. For conciseness, we let $y = p(X)$ throughout the proof.

**The case $\alpha_1 = 0$**

Let $\mathbf{x}$ be a Cournot candidate in the original model with linear cost functions and the inverse demand function $p(\cdot)$. By Proposition 3.3, $\mathbf{x}$ satisfies the necessary conditions (C.1), with respect to the original inverse demand function $p(\cdot)$. Suppose first that $x_1 = 0$. The second inequality in (C.1) implies that $p(X) = 0$. On the other hand, Assumption 3.4 and Proposition 3.4 imply that $X > 0$. Thus, there exists some $n$ such that $x_n > 0$. The first equality in (C.1) yields,

$$0 \leq \alpha_n = p(X) + x_n p'(X) = x_n p'(X) \leq 0,$$

which implies that $p'(X) = 0$. Then, the vector $\mathbf{x}$ satisfies the optimality conditions in (3.2), and is thus socially optimal in the original model. This contradicts the fact that $p(X) \neq p(X^S)$ and shows that we must have $x_1 > 0$.

If $p'(X)$ were equal to zero, then the necessary conditions (C.1) would imply the optimality conditions (3.2), and $\mathbf{x}$ would be socially optimal in the original model. Hence, we must have $p'(X) < 0$ and $c > 0$. The first equality in (C.1) yields $y > 0$, $x_1 = y/c$, and $X \geq y/c$. We also have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y}{c}. \tag{B.3}$$

From Proposition 3.9, the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in the modified model cannot be more than its efficiency $\gamma(\mathbf{x})$ in the original model. Hence, to prove the second part of the theorem, it suffices to show that $\gamma^0(\mathbf{x}) \geq f(\overline{c})$, for any Cournot candidate with $c/d = \overline{c}$.

The optimal social welfare in the modified model is

$$\int_0^{\infty} p^0(q)dq - 0 = \int_0^{X+y/d} p^0(q)dq - 0 = \frac{y^2}{2d} + \frac{(2y + cX)X}{2}. \tag{B.4}$$

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \geq 2$, whenever $x_n > 0$, from the first equality in (C.1) we have $\alpha_n = y - x_n c$ and $\alpha_n x_n = (y - x_n c)x_n$. Hence, in the modified model, the social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}
\int_0^{X} p^0(q)dq - \sum\nolimits_{n=1}^{N} \alpha_n x_n &= \frac{(2y + cX)X}{2} - \sum\nolimits_{n=2}^{N} (y - x_n c)x_n \\
&\geq \frac{(2y + cX)X}{2} - y \sum\nolimits_{n=2}^{N} x_n \\
&= \frac{(2y + cX)X}{2} - y(X - y/c) \\
&= cX^2/2 + y^2/c.
\end{aligned}$$

Therefore,

$$\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + y^2/c}{y^2/(2d) + (2y + cX)X/2}. \tag{B.5}$$

Note that $c$, $d$, $X$, and $y$ are all positive. Substituting $\overline{c} = c/d$ and $\overline{y} = cX/y$ in (B.5), we obtain

$$\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + y^2/c}{y^2/(2d) + (2y + cX)X/2} = \frac{c^2X^2/2 + y^2}{y^2c/(2d) + cXy + c^2X^2/2} = \frac{\overline{y}^2 + 2}{\overline{c} + 2\overline{y} + \overline{y}^2}. \tag{B.6}$$

We have shown earlier that $X \geq y/c$, so that $\overline{y} \geq 1$. On the interval $\overline{y} \in [1, \infty)$, the minimum value of the right hand side of (B.6) is attained at

$$\overline{y} = \max\left\{\frac{2 - \overline{c} + \sqrt{\overline{c}^2 - 4\overline{c} + 12}}{2}, 1\right\} \triangleq \phi,$$

and thus,

$$\gamma^0(\mathbf{x}) \geq \frac{\phi^2 + 2}{\phi^2 + 2\phi + \overline{c}} = f(\overline{c}).$$

**The case $\alpha_1 > 0$**

We now consider the case where $\alpha_n > 0$ for every $n$. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$. Suppose first that $x_1 = 0$. The second inequality in (C.1) implies that $p(X) \leq 1$. Proposition 3.4 also implies that $X > 0$ so that there exists some $n$ for which $x_n > 0$. The first equality in (C.1) yields,

$$\alpha_n = p(X) + x_n p'(X) \leq p(X) \leq 1.$$

Since $\alpha_n \geq 1$, we obtain $p(X) = 1$ and $p'(X) = 0$. Then, the vector $\mathbf{x}$ satisfies the optimality conditions in (3.2), and thus is socially optimal in the original model. But this would contradict the fact that $p(X) \neq p(X^S)$. We conclude that $x_1 > 0$.

If $p'(X)$ were equal to zero, then the necessary conditions (C.1) would imply the optimality conditions (3.2), and $\mathbf{x}$ would be socially optimal in the modified game. Therefore, we must have $p'(X) < 0$ and $c > 0$. The first equality in (C.1) yields

$y > 1$, $x_1 = (y-1)/c$, and $X \geq (y-1)/c$. We also have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y-1}{c}, \tag{B.7}$$

from which we conclude that $X \geq (y-1)/c$.

From Proposition 3.9, the efficiency $\gamma^0(\mathbf{x})$ of the vector $\mathbf{x}$ in the modified model cannot be more than its efficiency $\gamma(\mathbf{x})$ in the original model. So, it suffices to consider the efficiency of $\mathbf{x}$ in the modified model. From the optimality conditions (3.2), we have that $p^0(X^S) = 1$, and thus, using the definition of $d$,

$$X^S = X + \frac{y-1}{d}.$$

The optimal social welfare in the modified model is

$$\int_0^{X^S} p^0(q)\, dq - X^S = \frac{y^2-1}{2d} + \frac{(2y+cX)X}{2} - X - \frac{y-1}{d} = \frac{(y-1)^2}{2d} + X(y-1) + \frac{cX^2}{2}.$$

Note that for $n = 1$ we have $\alpha_n x_n = x_1$. For $n \geq 2$ and whenever $x_n > 0$, from the first equality in (C.1) we have $\alpha_n = y - x_n c$ and $\alpha_n x_n = (y - x_n c)x_n$. Hence, in the modified model, the social welfare associated with $\mathbf{x}$ is

$$\begin{aligned}
\int_0^{X} p^0(q)\, dq - \sum\nolimits_{n=1}^{N} \alpha_n x_n &= Xy + cX^2/2 - x_1 - \sum\nolimits_{n=2}^{N} (y - x_n c)x_n \\
&\geq Xy + cX^2/2 - x_1 - y\sum\nolimits_{n=2}^{N} x_n \\
&= X(y-1) + cX^2/2 - (y-1)\sum\nolimits_{n=2}^{N} x_n \\
&= X(y-1) + cX^2/2 - (y-1)(X - (y-1)/c) \\
&= cX^2/2 + (y-1)^2/c.
\end{aligned}$$

Therefore,

$$\gamma^0(\mathbf{x}) \geq \frac{cX^2/2 + (y-1)^2/c}{(y-1)^2/(2d) + X(y-1) + cX^2/2}. \tag{B.8}$$

Note that $c$, $d$, $X$, and $y-1$ are all positive. Substituting $\overline{c} = c/d$ and $\overline{y} =$

$(cX)/(y-1)$ in (B.8), we obtain

$$\gamma^0(\mathbf{x}) \geq \frac{2\overline{y}^2+1}{\overline{c}\overline{y}^2+2\overline{y}+1}. \tag{B.9}$$

From (B.7) we have that $\overline{y} \geq 1$. On the interval $\overline{y} \in [1,\infty)$, the minimum value of the right hand side of (B.9) is attained at

$$\overline{y} = \min\left\{\frac{2-\overline{c}+\sqrt{\overline{c}^2-4\overline{c}+12}}{2}, 1\right\} \triangleq \phi,$$

and thus,

$$\gamma^0(\mathbf{x}) \geq \frac{\phi^2+2}{\phi^2+2\phi+\overline{c}} = f(\overline{c}).$$

## B.3 Proof of Theorem 3.3

According to the discussion in Section 3.7, we only need to lower bound the efficiency of a Cournot equilibrium $\mathbf{x}$ in a model with $N=1$. Since $N=1$, we can identify the vectors $\mathbf{x}$ and $\mathbf{x}^S$ with the scalars $X$ and $X^S$. If $p(X)=p(X^S)$, then according to Proposition 3.8, the efficiency of the Cournot equilibrium, $X$, must equal one, which establishes part (a).

We now turn to the proof of part (b), and we assume that $p(X) \neq p(X^S)$. According to Proposition 3.2, we know that $\mathbf{x}$ cannot be socially optimal. We will consider separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$.

We will again rely on Proposition 3.9, according to which the efficiency of a Cournot candidate $\mathbf{x}$ is lower bounded by the efficiency $\gamma^0(\mathbf{x})$ of $\mathbf{x}$ in a model involving the piecewise linear and convex inverse demand function $p^0(\cdot)$. Note that since $p(X) \neq p(X^S)$, we have that $d > 0$. As shown in the proof of Theorem 3.2, we have $p'(X) < 0$, i.e., $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof.

**The case $\alpha_1 = 0$**

Applying conditions (C.1) to the supplier we have $X = y/c$. From Proposition 3.9, it suffices to show that $\gamma^0(\mathbf{x}) \geq 3/(3+\overline{c})$. The optimal social welfare in the modified model is

$$\int_0^{\infty} p^0(q)\, dq - 0 = \int_0^{X+y/d} p^0(q)\, dq - 0 = \frac{y^2}{2d} + \frac{(2y + cX)X}{2}. \tag{B.10}$$

In the modified model, the social welfare associated with $\mathbf{x}$ is

$$\int_0^{X^P} p^0(q)\, dq - 0 = \frac{(2y + cX)X}{2}.$$

Therefore,

$$\gamma^0(\mathbf{x}) = \frac{(2y + cX)X/2}{y^2/(2d) + (2y + cX)X/2} = \frac{3}{3 + \overline{c}},$$

where the last equality is true because $xc = y$.

### Tightness

Consider the model introduced in the proof of part (c) of Theorem 3.1. The inverse demand function is $p(q) = \max\{1 - q, 0\}$. The supplier's cost function is identically zero, i.e., $C_1(x_1) = 0$. The profit maximizing output is $x_1 = 1/2$. We observe that $\gamma(\mathbf{x}) = 3/4$.

### The case $\alpha_1 > 0$

We now consider the case where $\alpha_1 > 0$. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\alpha_1 = 1$. Applying conditions (C.1) to the supplier, we obtain $X = (y - 1)/c$.

According to Proposition 3.9, it suffices to show that the efficiency of $\mathbf{x}$ in the modified model, $\gamma^0(\mathbf{x})$, is at least $3/(3 + \overline{c})$. From the optimality conditions (3.2) we have that $p^0(X^S) = 1$, and therefore,

$$X^S = X + (y - 1)/d.$$

The optimal social welfare achieved in the modified model is

$$\int_0^{X^S} p^0(q)\,dq - X^S = \frac{y^2-1}{2d} + \frac{(2y+cX)X}{2} - X - \frac{y-1}{d} = \frac{(y-1)^2}{2d} + X^P(y-1) + \frac{c(X)^2}{2}.$$

In the modified model, the social welfare associated with $\mathbf{x}$ is

$$\int_0^{X} p^0(q)\,dq - X = X(y-1) + \frac{cX^2}{2}.$$

Since $cX = y - 1$, we have

$$\gamma^0(\mathbf{x}) = \frac{3}{3+\bar{c}}.$$

# Appendix C

# Proof of results in Chapter 4

## C.1 Proof of Theorem 4.1

According to Proposition 4.4, if $p(X) \in \mathcal{P}$, then the Cournot candidate's profit ratio must equal one.

To prove part (b), we assume that $p(X) \notin \mathcal{P}$. If $p'(X) = 0$, the necessary conditions (3.3)-(3.4) imply the conditions in (4.4). Since $p(X) > 0$, the conditions in (4.4) are sufficient for $\mathbf{x}$ to maximize the aggregate profit. But since $p(X) \notin \mathcal{P}$, this cannot be the case and we must have $p'(X) < 0$ and $d > 0$.

We have shown in Proposition 4.2 that the vector $\mathbf{x}$ remains a Cournot candidate in the modified model with linear cost functions, and Assumptions 3.1-3.4 still hold. Further, to lower bound the worst case profit ratio for Cournot candidates, we only need to derive a lower bound for the profit ratio of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each $n$. Without loss of generality, we further assume that $\alpha_1 = \min_n\{\alpha_n\}$.

Since $p'(X)$ exists, we have the following necessary (and, by definition, sufficient) conditions for a nonzero vector $\mathbf{x}$ to be a Cournot candidate:

$$\begin{cases} C_n'(x_n) = p\,(X) + x_n p'(X), & \text{if } x_n > 0, \\ C_n'(0) \geq p\,(X) + x_n p'(X), & \text{if } x_n = 0. \end{cases} \tag{C.1}$$

Since $p(X) \neq p(X^P)$, we have that $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof. We will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$. According to Proposition 4.5, the profit ratio of the Cournot candidate $\mathbf{x}$ is lower bounded by the profit ratio $\eta^0(\mathbf{x})$ for the case of a piecewise linear function of the form in (4.12).

**The case $\alpha_1 = 0$**

Let $\mathbf{x}$ be a Cournot candidate in the original model, with linear cost functions and the inverse demand function $p(\cdot)$. Suppose first that $x_1 = 0$. The second inequality in (C.1), with $n = 1$, and $C_n'(0) = 0$, implies that $p(X) = 0$. Since $p(X) > 0$, we must have $x_1 > 0$. The first equality in (C.1) yields $y > 0$ and $x_1 = y/d$. We have

$$0 \leq \sum_{n=2}^{N} x_n = X - \frac{y}{d}, \tag{C.2}$$

from which we conclude that $X \geq y/d$.

From Proposition 4.5, the profit ratio of the vector $\mathbf{x}$ in the modified model, $\eta^0(\mathbf{x})$, cannot be more than its profit ratio in the original model, $\eta(\mathbf{x})$. Hence, to prove part (b), it suffices to show that $\eta^0(\mathbf{x}) \geq f^P(\bar{c}, N)$. For the modified model, the maximum aggregate profit is the optimal value of the following optimization problem,

$$\begin{aligned} &\text{maximize} \;\; qp^0(q) \\ &\text{subject to} \;\; q \geq 0. \end{aligned}$$

Since $dX \geq y$, the derivative of the aggregate profit is nonpositive at $q = X$, and so that the aggregate profit is nonincreasing with $q$ on the interval $[X, \infty)$. Hence, in the modified model, the aggregate profit is maximized in the interval $[0, X]$. Through a simple calculation we have:

**(i)** If $cX \geq y$, then the maximum aggregate profit is $(cX + y)^2/(4c)$, achieved at $q = (cX + y)/(2c)$.

**(ii)** If $cX \leq y$, then the maximum aggregate profit is $Xy$, achieved at $q = X$.

Note that for $n = 1$ we have $\alpha_n x_n = 0$. For $n \geq 2$, whenever $x_n > 0$, from the first equality in (C.1) we have $\alpha_n = y - x_n d$ and $\alpha_n x_n = (y - x_n d)x_n$. Since $\alpha_n \geq 0$, we have $y \geq x_n d$, for $n = 2, \ldots, N$. Therefore,

$$(N-1)y \geq d \sum_{n=2}^{N} x_n = dX - y,$$

i.e.,

$$Ny \geq dX. \tag{C.3}$$

In the modified model, the aggregate profit achieved at $\mathbf{x}$ is

$$\begin{aligned} Xp(X) - \sum_{n=1}^{N} \alpha_n x_n &= Xy - \sum_{n=2}^{N} (y - x_n d)x_n \\ &\geq Xy - y\sum_{n=2}^{N} x_n + \frac{(Xd-y)^2}{(N-1)d} \\ &= Xy - y(X - y/d) + \frac{(Xd-y)^2}{(N-1)d} \\ &= \frac{y^2}{d} + \frac{(Xd-y)^2}{(N-1)d}, \end{aligned} \tag{C.4}$$

where the inequality is true because $\sum_{n=2}^{N} x_n^2$ is minimized when $x_2 = x_3 = \cdots = x_N$, subject to the constraint in (C.2). For the case $cX \geq y$, since the maximum aggregate profit is $(cX + y)^2/4c$, we have

$$\eta^0(\mathbf{x}) \geq \frac{y^2/d + (Xd-y)^2/((N-1)d)}{(cX+y)^2/4c}. \tag{C.5}$$

Note that $c$, $d$, and $y$ are positive. Substituting $\overline{y} = cX/y$ and $\overline{c} = c/d$ to (C.5), we have

$$\eta^0(\mathbf{x}) \geq \frac{4\overline{c}^2 + 4(\overline{y} - \overline{c})^2/(N-1)}{\overline{c}(\overline{y}+1)^2}, \quad 1 \leq \overline{y} \leq N\overline{c}, \quad 0 < \overline{c} \leq \overline{y}, \tag{C.6}$$

where the constraints $\overline{y} \geq \overline{c}$ and $\overline{y} \leq N\overline{c}$ follow from (C.2) and (C.3), respectively. For any given $\overline{c} \geq 1/N$, through a simple calculation we obtain that the right hand side of the first inequality in (C.6) is minimized at

$$\overline{y} = \max\left\{\frac{\overline{c}^2 N + \overline{c}}{\overline{c}+1}, 1\right\}.$$

We conclude that if $cX \geq y$, then

$$\eta^0(\mathbf{x}) \geq \begin{cases} \dfrac{\overline{c}^2 + (1-\overline{c})^2/(N-1)}{\overline{c}}, & \text{if} \quad 1/N \leq \overline{c} \leq \sqrt{1/N}, \\ \dfrac{4\overline{c}^3(N-1) + 4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2 N + 2\overline{c} + 1)^2}, & \text{if} \quad \overline{c} > \sqrt{1/N}. \end{cases} \tag{C.7}$$

Note that the case $\overline{c} < 1/N$ cannot happen, by (C.6).

For the case $cX \leq y$, the maximum aggregate profit is $Xy$. From (C.4), we have

$$\eta^0(\mathbf{x}) \geq \frac{y^2/d + (Xd-y)^2/((N-1)d)}{Xy} = \frac{\overline{c}^2(N-1) + (\overline{y}-\overline{c})^2}{\overline{c}\cdot\overline{y}(N-1)}, \quad 0 < \overline{c} \leq \overline{y} \leq 1, \quad \overline{y} \leq N\overline{c}, \tag{C.8}$$

where the constraints $\overline{y} \geq \overline{c}$ and $\overline{y} \leq N\overline{c}$ follow from (C.2) and (C.3), respectively. Through a simple calculation, for any given $\overline{c} \in (0,1]$, we find that the right hand side of (C.8) is minimized at

$$\overline{y} = \min\{\overline{c}\sqrt{N}, 1\}.$$

We conclude that if $cX \leq y$, then

$$\eta^0(\mathbf{x}) \geq \begin{cases} \dfrac{N-1+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, & \text{if} \quad 0 < \overline{c} \leq \sqrt{1/N}, \\ \dfrac{\overline{c}^2 + (1-\overline{c})^2/(N-1)}{\overline{c}}, & \text{if} \quad \sqrt{1/N} \leq \overline{c} \leq 1. \end{cases} \tag{C.9}$$

Through another simple calculation, we conclude that

$$\frac{\overline{c}^2 + (1-\overline{c})^2/(N-1)}{\overline{c}} \geq \frac{(N-1)+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, \qquad \text{if } 0 < \overline{c} \leq \sqrt{1/N}, \tag{C.10}$$

because the left-hand side is nonincreasing in $\overline{c}$, and the left-hand side equals the right-hand side if $\overline{c} = \sqrt{1/N}$. Similarly,

$$\frac{\overline{c}^2 + (1-\overline{c})^2/(N-1)}{\overline{c}} \geq \frac{4\overline{c}^3(N-1) + 4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2 N + 2\overline{c} + 1)^2}, \qquad \text{if } \overline{c} \geq \sqrt{1/N}, \tag{C.11}$$

because the left-hand side equals the right-hand side at $\overline{c} = \sqrt{1/N}$, and the derivative of the left-hand side, with respect to $\overline{c}$, is more than that of the right-hand side, for every $\overline{c} \geq \sqrt{1/N}$. Combining the results in (C.7) and (C.9)-(C.11), we have

$$\eta^0(\mathbf{x}) \geq \begin{cases} \dfrac{N-1+(\sqrt{N}-1)^2}{\sqrt{N}(N-1)}, & \text{if } \; 0 < \overline{c} \leq \sqrt{1/N}, \\ \dfrac{4\overline{c}^3(N-1) + 4\overline{c}(\overline{c}+1)^2}{(\overline{c}^2 N + 2\overline{c} + 1)^2}, & \text{if } \; \overline{c} > \sqrt{1/N}. \end{cases} \tag{C.12}$$

**Tightness**

Given an integer $N \geq 2$, consider a model with an affine inverse demand function $p^0(\cdot)$ of the form (4.12), with $c/d = 1$ and $cX \geq y > 0$. Let the cost of supplier 1 be identically zero and let

$$C_n(x) = \left(y - \frac{d}{N-1}\left(X - \frac{y}{d}\right)\right) x, \qquad n = 2, \ldots, N. \tag{C.13}$$

It is not hard to see that the vector with components

$$x_1 = \frac{y}{d}, \quad x_n = \frac{1}{N-1}\left(X - \frac{y}{d}\right), \qquad n = 2, \ldots, N, \tag{C.14}$$

satisfies the conditions (3.3)-(3.4). It can be verified that $\mathbf{x}$ is a Cournot equilibrium.

For the case where $cX \geq y$ and $\min_n\{C_n'(\cdot)\} = 0$, we have shown that the maximum total profit is $(cX+y)^2/4c$, and the aggregate profit achieved at $\mathbf{x}$ is given by the right-hand side of (C.4). When $\overline{y} = cX/y = (N+1)/2$, the profit ratio of the Cournot equilibrium $\mathbf{x}$ is given by

$$\eta(\mathbf{x}) = \frac{y^2/d + (Xd-y)^2/(N-1)d}{(cX+y)^2/4c} = \frac{4 + 4(\overline{y}-1)^2/(N-1)}{(\overline{y}+1)^2} = \frac{4}{N+3},$$

which is the profit ratio lower bound in (C.12), for the case $\overline{c} = 1$.

**The case $\alpha_1 > 0$**

We now consider the case where $\alpha_n > 0$ for every $n$. By rescaling the cost coefficients and permuting the supplier indices, we can assume that $\min_n\{\alpha_n\} = \alpha_1 = 1$.

Let $\mathbf{x}$ be a Cournot candidate in the original model with linear cost functions and an inverse demand function $p(\cdot)$. Suppose first that $x_1 = 0$. The second inequality in (C.1) implies that $p(X) \leq 1$. Proposition 3.4 implies that $X > 0$, so that there exists some $n$ such that $x_n > 0$. The first equality in (C.1) yields,

$$\alpha_n = p(X) + x_n p'(X) \leq 1.$$

Since $\alpha_n \geq 1$, we have that $p(X) = 1$ and $p'(X) = 0$. We observe that $\mathbf{x}$ satisfies the conditions in (4.4), and since $p(X) > 0$, we know that $\mathbf{x}$ maximizes the aggregate profit. However, since the cost functions are convex and $p(X) = \min_n\{\alpha_n\} = 1$, it is easy to see that the aggregate profit earned at $\mathbf{x}$ cannot be positive, a contradiction with Proposition 4.1. We therefore have $p(X) > 1$ and $x_1 > 0$.

As argued earlier, since $p(X) \notin \mathcal{P}$, we have $p'(X) < 0$. The first equality in (C.1), for $n = 1$, yields $y > 0$ and $x_1 = (y-1)/d$. We have

$$0 \leq \sum_{n=2}^{N} x_n = X - (y-1)/d, \tag{C.15}$$

from which we conclude that $X \geq (y-1)/d$.

From Proposition 4.5, the profit ratio of the vector $\mathbf{x}$ in the modified model, $\eta^0(\mathbf{x})$, cannot be more than its profit ratio in the original model, $\eta(\mathbf{x})$. Hence, to prove part (b), it suffices to show that $\eta^0(\mathbf{x}) \geq f^P(\overline{c}, N)$. For the modified model, the maximum aggregate profit is the optimal value of the following optimization problem,

$$\begin{aligned} &\text{maximize} \;\; qp^0(q) - q \\ &\text{subject to} \;\; q \geq 0. \end{aligned}$$

Since $dX \geq y - 1$, the derivative of the aggregate profit at $q = X$ is nonpositive, and so the aggregate profit is nonincreasing with $q$ on the interval $[X, \infty)$. Again, the aggregate profit is maximized in the interval $[0, X]$. Through a simple calculation we have:

**(i)** If $cX \geq y-1$, then the maximum aggregate profit is $(cX+y-1)^2/(4c)$, achieved at $Q_1 = (cX + y - 1)/(2c)$.

**(ii)** If $cX \leq y - 1$, then the maximum aggregate profit is $X(y - 1)$, achieved at $Q_2 = X$.

Note that for $n = 1$ we have $\alpha_n x_n = x_n$. For $n \geq 2$, whenever $x_n > 0$, from the first equality in (C.1) we have $\alpha_n = y - x_n d$ and $\alpha_n x_n = (y - x_n d)x_n$. Since $\alpha_n \geq 1$, we have $y - 1 \geq x_n d$, for $n = 2, \ldots, N$. Therefore

$$(N-1)(y-1) \geq d \sum_{n=2}^{N} x_n = dX - (y-1),$$

which implies that

$$N(y-1) \geq dX. \tag{C.16}$$

Hence, in the modified model, the aggregate profit achieved at $\mathbf{x}$ is

$$\begin{aligned} X(y-1) - \sum\nolimits_{n=1}^{N} \alpha_n x_n &= X(y-1) - (y-1)/d - \sum\nolimits_{n=2}^{N} (y - x_n d)x_n \\ &\geq X(y-1) - y\sum\nolimits_{n=2}^{N} x_n + \frac{(Xd-y-1)^2}{(N-1)d} \\ &= X(y-1) - (y-1)(X - (y-1)/d) + \frac{(Xd-y-1)^2}{(N-1)d} \\ &= \frac{(y-1)^2}{d} + \frac{(Xd-y-1)^2}{(N-1)d}, \end{aligned} \tag{C.17}$$

where the inequality is true because $\sum_{n=2}^{N} x_n^2$ is minimized when $x_2 = x_3 = \ldots = x_N$, subject to the constraint in (C.15). Note that $c$, $d$ and $y-1$ are positive. If $y-1$ is replaced by $y$, then

1. The aggregate profit achieved at $\mathbf{x}$, which is given in (C.17), is the same as that in (C.4).
2. The maximum aggregate profit is the same as that in the case where $\alpha_1 = 0$.
3. The constraints in (C.15) and (C.16) are equivalent to those in (C.2) and (C.3), respectively.

Let $\overline{y} = cX/(y-1)$ and $\overline{c} = c/d$. The desired results in (C.12) can be obtained by repeating the proof for the case $\alpha_1 = 0$.

## C.2 Proof of Theorem 4.2

Without loss of generality, let supplier 1 have the largest market share, i.e., $x_1 = \max_n\{x_n\}$ and $r = x_1/X$. According to Proposition 4.2, the vector $\mathbf{x}$ remains a Cournot candidate in the modified model with linear cost functions, and Assumptions 3.1-3.4 still hold. Therefore, to lower bound the worst case profit ratio for Cournot candidates, we only need to derive a lower bound for the profit ratio of Cournot candidates for the case of linear cost functions. We therefore assume that $C_n(x_n) = \alpha_n x_n$ for each $n$. From the conditions (C.1), it is not hard to see that $\alpha_1 = \min_n\{\alpha_n\}$.

Since $p(X) \neq p(X^P)$, we have that $c > 0$. For conciseness, we let $y = p(X)$ throughout the proof. Through an approach similar to that used in the proof of Theorem 4.1, we will prove the theorem by considering separately the cases where $\alpha_1 = 0$ and $\alpha_1 > 0$. According to Proposition 4.5, the profit ratio of the Cournot candidate $\mathbf{x}$ is lower bounded by the profit ratio $\eta^0(\mathbf{x})$ for the case of a piecewise linear inverse demand function of the form in (4.12).

**The case $\alpha_1 = 0$**

In part (b) of Theorem 4.1 we have shown that $p'(X) < 0$. The first equality in (C.1) yields $y > 0$, $x_1 = y/d$ and $r = y/dX$. For the case $cX \geq y$, in the proof of

Theorem 4.1 (cf. Eq. (C.5)) we have shown that

$$\eta^0(\mathbf{x}) \geq \frac{y^2/d + (Xd - y)^2/((N-1)d)}{(cX+y)^2/4c}. \tag{C.18}$$

Note that $c$, $d$ and $y$ are positive. Substituting $r = y/dX$ and $\bar{c} = c/d$ to (C.18), we have

$$\eta^0(\mathbf{x}) \geq \frac{4\bar{c}^2 r^2 + 4(\bar{c} - \bar{c}r)^2/(N-1)}{\bar{c}(\bar{c}+r)^2} \geq \frac{4\bar{c}^2 r^2}{\bar{c}(\bar{c}+r)^2} = \frac{4\bar{c}r^2}{(\bar{c}+r)^2}, \qquad 0 \leq r \leq \min\{\bar{c}, 1\}, \tag{C.19}$$

where the constraint $r \leq \bar{c}$ follows from $cX \geq y$.

For the case $cX \leq y$, we have $r \geq \bar{c}$. In the proof for Theorem 4.1 (cf. Eq. (C.8)) we have shown that

$$\eta^0(\mathbf{x}) \geq \frac{y^2/d + (Xd-y)^2/((N-1)d)}{Xy} = \frac{\bar{c}^2 r^2 (N-1) + (\bar{c} - \bar{c}r)^2}{\bar{c}^2 r(N-1)} \geq r, \quad 0 < \bar{c} \leq r < 1. \tag{C.20}$$

**Tightness**

Given some $r \in (0,1)$, consider a model with $N \geq \lceil 1/r \rceil + 1$, and an affine inverse demand function $p^0(\cdot)$ of the form in (4.12), where $c/d = 1$ and $rdX = y$. The cost of supplier 1 is identically zero and[1]

$$C_n(x) = \left( y - \frac{d}{N-1}(X - rX) \right) x, \qquad n = 2, \dots, N.$$

It is not hard to see that the vector with components

$$x_1 = rX, \qquad x_n = \frac{1}{N-1}(X - rX), \qquad n = 2, \dots, N,$$

satisfies the conditions (3.3)-(3.4). It can be verified that $\mathbf{x}$ is a Cournot equilibrium. The maximum total profit is $(cX+y)^2/4c$, and is achieved at the monopoly output $\mathbf{x}^P = ((cX+y)/2c, 0, \dots, 0)$. On the other hand, the aggregate profit achieved at $\mathbf{x}$

[1] Since $N > 1/r$ and $rdX = y$, we have that $C'_n(\cdot) > 0$.

is given in (C.4). We have

$$\eta^0(\mathbf{x}) = \frac{y^2/d + (Xd - y)^2/((N-1)d)}{(cX+y)^2/4c} = \frac{4r^2 + 4(1-r)^2/(N-1)}{(1+r)^2},$$

and as the number of suppliers increases to infinity, the profit ratio of the Cournot equilibrium converges to $4r^2/(1+r)^2$.

**The case $\alpha_1 > 0$**

The proof is similar to the case that $\alpha_1 = 0$ and is omitted.

# Appendix D

# Proof of Theorem 5.1

We have shown in Proposition 5.5 that if all cost and utility functions are replaced by linear ones, the vector $\mathbf{w}$ remains a Nash equilibrium, and Assumptions 5.1-5.3 still hold. Further, the efficiency of $\mathbf{w}$ cannot increase after all cost and utility functions are replaced by linear ones. Thus, to lower bound the worst case efficiency loss, it suffices to derive a lower bound for the efficiency of Nash equilibria for the case of linear cost and utility functions. We therefore assume that $\gamma_i = C_i'(\cdot)$ and $\beta_j = U_j'(\cdot)$, for each supplier $i$ and consumer $j$.

Without loss of generality, we further assume that $\beta_1 = \max_j\{\beta_j\} = 1$, and that every supplier provides a positive amount of good at the Nash equilibrium[1]. Proposition 5.3 shows that $\gamma_i \in [0, 1)$, for every $i$. For a given $\alpha \in [0, 1)$, without loss of generality, we assume that $\alpha_1 = \max_i\{\gamma_i\} = \alpha$. Fixing $Q$ and $R$, the optimal social welfare is achieved when every supplier $i$ provides 1 unit, the maximum amount of good it could produce, to the consumer with the highest marginal utility. Therefore, the optimal social welfare is

$$\mathcal{W} = R - \sum_{i=1}^{R} \gamma_i.$$

Applying the equilibrium condition (5.10) to consumer 1, with $U_1(x) = x$, we obtain $\mu = 1 - d_1/R$. For a given $\alpha \in [0, 1)$, the worst case efficiency loss is bounded below

[1]A supplier that produces zero amount of good bids $\mu$ at a Nash equilibrium; the price and the social welfare realized at the Nash equilibrium will not change, if the supplier is "removed" from the model.

by the optimal value of the following optimization problem,

$$
\begin{aligned}
\text{minimize} \quad & \frac{d_1 + \sum_{j=2}^{Q} \beta_j d_j - \sum_{i=1}^{R} \alpha_i s_i}{R - \sum_{i=1}^{R} \alpha_i}, \\
\text{subject to} \quad & \beta_j (1 - d_j/R) = 1 - d_1/R, \qquad \text{if } d_j > 0, \\
& \beta_j \leq 1 - d_1/R, \qquad \text{if } d_j = 0, \\
& \gamma_i (s_i + R - 1) = (R-1)(1 - d_1/R), \\
& 0 \leq \beta_j \leq 1, \quad j = 2, \dots, Q; \qquad 0 \leq \gamma_i \leq \alpha_1 = \alpha, \quad i = 2, \dots, R, \\
& \sum_{j=1}^{Q} d_j = \sum_{i=1}^{R} s_i, \\
& 0 \leq d_j \leq R, \quad j = 1, \dots, Q; \qquad 0 \leq s_i \leq 1, \quad i = 1, \dots, R.
\end{aligned} \tag{D.1}
$$

Here, the first two constraints follow from the conditions in (5.10) and the next constraint follow from the conditions in (5.11). The fourth constraint follows from our assumption that the maximum marginal utility is 1, and that all suppliers have a marginal cost no more than $\alpha$. The fifth constraint comes from the mechanism itself, i.e., the total supply equals with the total demand in the market. The last constraint is also due to the mechanism, i.e., each supplier can at most produce one unit amount of good.

Note that the optimal value of the above optimization problem will not change if we let $\beta_j(1 - d_j/R) = 1 - d_1/R$ for every consumer $j$. With the following relations,

$$
\beta_j = \frac{1 - d_1/R}{1 - d_j/R}, \quad \gamma_i = \frac{(R-1)(1 - d_1/R)}{s_i + R - 1}, \tag{D.2}
$$

the optimization problem in (D.1) is equivalent to the following optimization problem over $\{d_j\}_{j=1}^{Q}$ and $\{s_i\}_{i=1}^{R}$,

$$\begin{aligned} \text{minimize} \quad & \frac{d_1 + \sum_{j=2}^{Q} \frac{1-d_1/R}{1-d_j/R} d_j - \sum_{i=1}^{R} \frac{(1-d_1/R)(R-1)}{s_i+R-1} s_i}{R - \sum_{i=1}^{R} \frac{(1-d_1/R)(R-1)}{s_i+R-1}}, \\ \text{subject to} \quad & 0 \leq d_j \leq d_1 \leq R, \ \ j=2,\ldots,Q, \quad 0 \leq s_i \leq 1, \ \ i=1,\ldots,R, \\ & \sum_{j=1}^{Q} d_j = \sum_{i=1}^{R} s_i, \\ & \frac{(R-1)(1-d_1/R)}{s_i+R-1} \leq \alpha. \end{aligned} \tag{D.3}$$

For any given $d_1$ and $R$, the optimization problem in (D.3) is convex in $(\{s_i\}_{i=1}^{R}, \{d_j\}_{j=2}^{Q})$, and therefore there exists a symmetric optimal solution,

$$s_i = s, \quad i=1,\ldots,R; \qquad d_j = d, \quad j=2,\ldots,Q. \tag{D.4}$$

From (D.2) and (D.4), it is not hard to see that at the symmetric optimal solution, all suppliers have the same cost function, i.e., $\gamma_i = \alpha$ for every $i$. Substituting the relations in (D.4) to the second constraint of the the optimization problem in (D.3), we have

$$d_1 + (Q-1)d = Rs.$$

We will discuss two possible cases, i.e., $Q = 1$ and $Q \geq 2$.

**The case $Q = 1$**

In this case we have

$$d_1 = Rs.$$

With the relation $d_1 = Rs$, the objective function of the optimization problem in (D.3) can be simplified as

$$\frac{sR - \frac{(1-s)(R-1)}{s+R-1} sR}{R - \frac{(1-s)(R-1)}{s+R-1} R} = s.$$

According to (D.2), we have

$$s = \frac{R - 1 + \alpha - \alpha R}{R - 1 + \alpha},$$

which is strictly increasing in $R$. When $R = 2$, the optimal (minimum) value of the optimization problem in (D.1) is

$$\frac{1 - \alpha}{1 + \alpha}.$$

**The case $Q \geq 2$**

In this case we have

$$d = \frac{Rs - d_1}{Q - 1}.$$

Since $\beta_1 = \max_j\{\beta_j\}$, the equilibrium conditions (5.10) imply that $d \leq d_1$. The optimization problem in (D.3) can be simplified as an optimization problem over $d_1$ and $s$,

$$\begin{aligned} \text{minimize} \quad & \frac{d_1 + \dfrac{1 - d_1/R}{1 - (Rs - d_1)/\left(R(Q-1)\right)} \cdot (Rs - d_1) - \dfrac{(1 - d_1/R)(R-1)}{s + R - 1} sR}{R - \dfrac{(1 - d_1/R)(R-1)}{s + R - 1} R}, \\ \text{subject to} \quad & 0 \leq d_1 \leq R, \\ & d_1/R \leq s \leq \min\{Qd_1/R, 1\}, \end{aligned} \tag{D.5}$$

where the constraint $s \leq Qd_1/R$ follows from the fact that $d \leq d_1$. The value of the objective function in (D.5) is decreasing in $Q$, and a larger value of $Q$ can only result in a larger feasible set of the variable $s$. Therefore, the worst case efficiency occurs when the number of consumers increases to infinity. For any $Q$, the optimal (minimum) value of the optimization problem in (D.5) is no less than the optimal (minimum) value of the following optimization problem,

$$\begin{aligned}
\text{minimize} \quad & \frac{d_1 + (1 - d_1/R) \cdot (Rs - d_1) - \dfrac{(1 - d_1/R)(R - 1)}{s + R - 1} sR}{R - \dfrac{(1 - d_1/R)(R - 1)}{s + R - 1} R}, \\
\text{subject} \quad & \text{to } 0 \le d_1 \le R, \\
& d_1/R \le s \le 1.
\end{aligned} \tag{D.6}$$

Let $y = d_1/R$. From (D.2) we have

$$s = \frac{(R - 1)(1 - y - \alpha)}{\alpha}. \tag{D.7}$$

Substituting the above relation to the objective function in (D.6), we obtain an optimization problem over $y$ and $\alpha$,

$$\begin{aligned}
&\text{Minimize } \frac{(y + \alpha - 1)^2 (R - 1) + y^2 \alpha}{\alpha - \alpha^2}, \\
&\text{Subject to } (1 - \frac{R\alpha}{R - 1})^+ \le y \le \frac{(R - 1)(1 - \alpha)}{R - 1 + \alpha},
\end{aligned} \tag{D.8}$$

where $(\cdot)^+ = \max\{0, \cdot\}$. It can be verified that the objective function in (D.8) is nonincreasing in $y$, and therefore, the minimum is attained at

$$y = \frac{(R - 1)(1 - \alpha)}{R - 1 + \alpha}.$$

For a given $\alpha \in [0, 1)$, and an $R \in \{2, 3, \ldots\}$, the optimal (minimum) value of the optimization problem (D.8) is

$$\frac{\dfrac{b(1 - \alpha)^2 \alpha^2}{R - 1 + \alpha} + b^2 (1 - \alpha)^2 \alpha}{\alpha - \alpha^2} = \frac{b(1 - \alpha)^2 \alpha}{\alpha - \alpha^2} = b(1 - \alpha) = y,$$

where $b = (R - 1)/(R - 1 + \alpha)$. The minimum value is increasing in $b$, and $b$ is increasing in $R$. Therefore, the worst case efficiency occurs when $R = 2$, and for a given $\alpha \in [0, 1)$, $(1 - \alpha)/(1 + \alpha)$ is the minimum value of the optimization problem (D.8), which also serves as a lower bound for the optimal value of the optimization problem in (D.5).

**Tightness**

For a given $\alpha \in [0, 1)$, consider a model with one consumer ($Q = 1$) and two identical suppliers ($R = 2$). Every supplier has a linear cost function $C(s) = \alpha s$, and the consumer has a linear utility function of slope 1. The optimal social welfare, $2(1-\alpha)$, is achieved when both suppliers provide one unit of the good. On the other hand, it can be verified that

$$w_1^S = w_2^S = \frac{\alpha}{1+\alpha}, \qquad w_1^C = \frac{1-\alpha}{1+\alpha},$$

satisfies the equilibrium conditions in (5.10) and (5.11), and is therefore a Nash equilibrium. It yields an output of

$$s_1 = s_2 = \frac{1-\alpha}{1+\alpha}, \qquad d_1 = \frac{2(1-\alpha)}{1+\alpha},$$

and $2(1-\alpha)^2/(1+\alpha)$ social welfare. Its efficiency is $(1-\alpha)/(1+\alpha)$.

# Appendix E

# Proof of results in Chapter 6

## E.1 Approximation of the supplier cost

In this appendix, we show via simulation that at least in some cases, the supplier cost (including the cost of ancillary service) can be captured by a simplified cost function of the form in (6.5). We consider a $(T+1)$-stage dynamic model with two energy resources, a primary energy resource and an ancillary energy resource. It is assumed that the forecast demand is met by the primary energy resource (e.g., coal-fired or nuclear power generators), and that at stage $t = 1, \ldots, T$, the deviations from the forecast demand, $\{w_t\}_{t=1}^{T}$, are independent random variables uniformly distributed on $[-\omega, \omega]$. At the initial stage 0, we assume that the forecast error is zero, i.e., $w_0 = 0$.

At stage $t$, let $b_t$ denote the difference between the actual output of the primary energy resource and the forecast demand, and let $d_t$ denote the output of the ancillary energy resource (e.g., oil/gas combustion turbines). For simplicity, we will assume that the cost of a positive primary energy resource (respectively, ancillary energy resource) is $b_t^2$ (respectively, $10d_t^2$).

Let $r_b$ be the ramping rate of the primary energy resource, and $r_d$ be the ramping rate of the ancillary energy resource. At the initial stage 0, we assume that $b_0 = w_0 = 0$, and $d_0 = 0$. At stage $t \geq 1$, if $w_t < 0$, then $d_t = 0$, and we assume that $b_t = 0$, that is, the system operator maintains a high level of (potential) output in order to be able to deal with a possible unexpected demand surge in the future; if $w_t > 0$, we assume that $b_t = \min\{w_t, b_{t-1} + r_b\}$, where $b_{t-1} + r_b$ is the maximum possible output of the primary energy resource at stage $t$, and that $d_t = \min\{w_t - b_t, d_{t-1} + r_d\}$. The total supplier cost (excluding the cost to meet the forecast demand) is

$$C = \sum_{t=1}^{T} \left(b_t^2 + 10d_t^2\right). \tag{E.1}$$

For notational convenience, we let $(\cdot)^+ = \max\{\cdot, 0\}$. We use the following function to approximate the supplier cost:

$$\widetilde{C} = \sum_{t=1}^{T} \left(\widetilde{b}_t^2 + 10\widetilde{d}_t^2\right), \tag{E.2}$$

where $\widetilde{d}_t = \min\left\{r_d, \left(0, w_t - (w_{t-1})^+ - r_b\right)^+\right\}$, and $\widetilde{b}_t = (w_t - \widetilde{d}_t)^+$.

The function in (E.2) well approximates the supplier cost in (E.1), if for an unexpected demand surge at stage $t$, the system load at the previous stage, $w_{t-1}$, is met by the primary energy resource, and load shedding rarely occurs (so that $(w_t)^+$ typically equals $b_t + d_t$). Note that in (E.2), for each stage $t$, the approximated cost depends only on $w_{t-1}$ and $w_t$. Therefore, the approximated cost in (E.2) can be written as

$$\widetilde{C} = \sum_{t=1}^{T} \left(((w_t)^+)^2 + H(w_{t-1}, w_t)\right), \tag{E.3}$$

where $H(w_{t-1}, w_t) = \left(\widetilde{b}_t^2 + 10\widetilde{d}_t^2 - ((w_t)^+)^2\right)^+$.

For different values of the parameters, $r_b$, $r_d$, and $\omega$, we evaluate the performance of the approximation via simulation. Fig. E-1 depicts some numerical results of a simulation experiment and we can make the following observations:

1. The main source of approximation error is from the following scenario: at stage $t-1$, the deviation in demand $w_{t-1}$ is nonpositive, $w_t > r_b$, and $w_{t+1} > 2r_b$. In this scenario, the output of the primary energy source at stage $t$ is $r_b$, which is less than $w_t$. When $\omega/r_b \leq 2$, this scenario never occurs and we observe from Fig. E-1 that the approximation error is close to zero, regardless of the value of

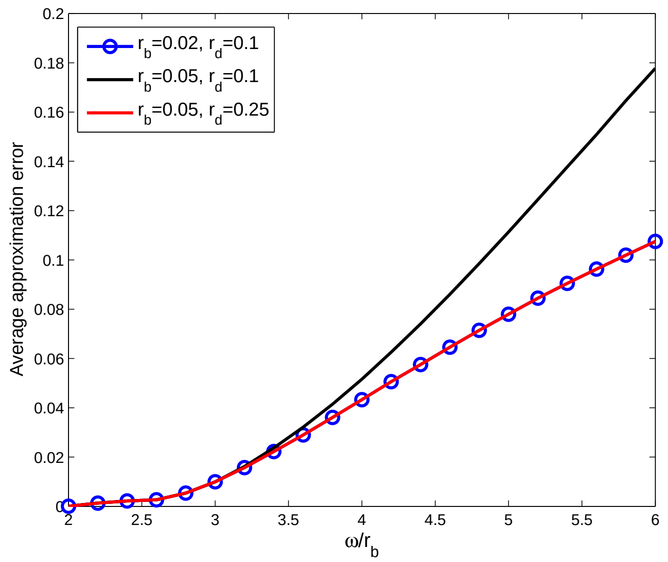


Figure E-1: A simulation experiment with $T = 24$, and $500,000$ trajectories for each $\omega/r_b$ on the horizontal axis. The approximation error is defined by $|C - \widetilde{C}|/C$. The average approximation error (vertical axis) is the mean value of the approximation errors of the $500,000$ trajectories.

$r_d$.

2. Comparing the black curve with the red curve in Fig. E-1, we observe that when $\omega/r_b > 3$ (when $r_b = 0.05$ and $\omega > 0.15$), the approximation error for the case where $r_d = 0.1$ is larger than that for the case where $r_d = 0.25$. This is because for the case with $r_d = 0.1$, as $\omega/r_b$ increases from 3 to 6 (as $\omega$ increases from 0.15 to 0.3), the probability of load shedding increases, which deteriorates the performance of the approximation.

3. Finally, and perhaps most importantly, when the ramping rate of the ancillary energy resource is high enough to prevent any load shedding, the approximation error is an increasing function of the single parameter $\omega/r_b$ (e.g., the blue curve with circle markers for $r_b = 0.02, r_d = 0.1$ and the red curve for $r_b = 0.05, r_d = 0.25$ merge together in Fig. E-1); in this case, we observe from Fig. E-1 that the approximation error is less than 10% for a wide range of parameter values.

## E.2 Proof of Theorem 6.1

We consider a sequence of $n$-consumer models where $n-1$ consumers (all except for consumer $i$) use a DOE strategy $\nu$. As the number of consumers increases to infinity, the randomness of consumer initial states averages out. Thus, in Step 1 we show that the aggregate demand (in an $n$-consumer model) at a history $h_t$ is close to $n\widetilde{A}_{t|\nu,h_t}$ (defined in Eq. (6.16)), with high probability. As a consequence, we show in Step 2 that as $n \to \infty$, consumer $i$'s expected payoff associated with any sequence of actions can be approximated by her oblivious value defined in (6.21). Since the DOE strategy $\nu$ maximizes consumer $i$'s oblivious value among all possible strategies, we argue in Step 3 that as $n \to \infty$, the maximum expected payoff consumer $i$ can obtain is asymptotically no larger than the optimal oblivious value. In Step 4, we show that consumer $i$'s optimal oblivious value can be approximately achieved if she uses the DOE strategy $\nu$. We finally conclude with the AME property of the DOE strategy $\nu$ (the result in Theorem 6.1).

In what follows, we will be using the uniform metric over the set of probability distributions on the finite set $\mathcal{X}_0$. Specifically, if $f$ and $f'$ are two distributions on $\mathcal{X}_0$, we let

$$d(f, f') \triangleq \|f - f'\|_\infty = \max_{x \in \mathcal{X}_0} |f(x) - f'(x)| \,. \tag{E.4}$$

**Step 1:** *With high probability, the aggregate demand under a history $h_t$ is close to $n\widetilde{A}_{t|\nu,h_t}$.*

Given an initial distribution $f^n_{-i,0}$, and if all consumers (excluding $i$) use a dynamic oblivious strategy $\nu$, we write their aggregate demand at a history $h_t$ as

$$A^n_{-i,t} = (n-1) \sum_{x \in \mathcal{X}_0} f^n_{-i,0}(x) \nu_t(x, h_t).$$

Recall that (cf. (6.16))

$$\widetilde{A}_{t|\nu,h_t} = \sum_{x \in \mathcal{X}_0} \eta_{s_0}(x) \cdot \nu_t(x, h_t).$$

We observe that if $d(f^n_{-i,0}, \eta_{s_0}) \le \delta/(XB)$, then at any history $h_t$ we have

$$\left| A^n_{-i,t} - (n-1)\widetilde{A}_{t|\nu,h_t} \right| \le \delta(n-1), \tag{E.5}$$

with probability at least $1-O(e^{-n})$. More precisely, since the consumers' initial states are independently drawn according to $\eta_{s_0}$, Hoeffding's inequality ([40]) yields,

$$\mathbb{P}\left( d(F^{n-1}_{s_0}, \eta_{s_0}) \ge \delta/(XB) \right) \le 2X \exp\left\{ -2(n-1)\delta^2/(X^2B^2) \right\}, \quad \forall s_0 \in \mathcal{S}, \quad \forall \delta > 0, \tag{E.6}$$

where $X$ is the cardinality of the set $\mathcal{X}_0$ and $F^{n-1}_{s_0}$ is an $X$-dimensional random vector denoting the distribution of the initial states of the $n-1$ consumers (excluding $i$).

**Step 2:** *Under a given history $h_T$, consumer $i$'s expected payoff can be approximated by a corresponding oblivious value, defined in* (E.9).

In an $n$-consumer model, suppose that all consumers other than $i$ use a dynamic oblivious strategy $\nu$. Given a complete history $h_T = (s_0, \ldots, s_T)$, and consumer $i$'s initial state $x_{i,0}$, we define her expected payoff under a history-dependent strategy $\kappa^n$ by

$$V^n_{i,0}(x_{i,0}, h_T \mid \kappa^n, \nu) = \mathbb{E}\left\{ V^n_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu) \right\},$$

where the expectation is over the initial distribution, $f^n_{-i,0}$, and $V^n_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu)$ is consumer $i$'s payoff under the given initial distribution $f^n_{-i,0}$,

$$V^n_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu) = \sum_{t=0}^{T} \pi^n_{i,t}(y_{i,t}, h_t, f^n_{-i,t} \mid \kappa^n, \nu), \tag{E.7}$$

and where the stage payoff function, $\pi^n_{i,t}(\cdot)$, has been defined in (6.23). Note that given $f^n_{-i,0}$, and since all consumers other than $i$ use a dynamic oblivious strategy, the distribution of their augmented states, $f^n_{-i,t}$, is completely determined by the history $h_t$. Therefore, given $f^n_{-i,0}$, consumer $i$'s history-dependent strategy $\kappa^n$ is equivalent to a dynamic oblivious strategy: the action it takes at stage $t$ depends only on $x_{i,0}$

and $h_t$. We can therefore define an oblivious strategy $\widetilde{\nu}^n(\kappa^n, f^n_{-i,0})$ such that

$$\widetilde{\nu}_t(\kappa^n, f^n_{-i,0})(x_{i,0}, h_t) = \kappa^n_t(y_{i,t}, h_t, f^n_{-i,t}),$$

where $f^n_{-i,t}$ is the distribution of the $n-1$ consumers' augmented states under the history $h_t$, induced from the initial distribution $f^n_{-i,0}$ by the symmetric oblivious strategy profile $(\nu, \dots, \nu)$, and $y_{i,t}$ is consumer $i$'s augmented state under the history $h_t$, induced from her initial state $x_{i,0}$ by the strategy $\kappa^n$.

In the corresponding continuum model, suppose that all consumers other than $i$ use a dynamic oblivious strategy $\nu$. For a given complete history $h_T$, we define consumer $i$'s oblivious value under an initial distribution $f^n_{-i,0}$, her initial state $x_{i,0}$, and the history-dependent strategy $\kappa^n$:

$$\widetilde{V}_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu) = \sum_{t=0}^{T} \widetilde{\pi}_{i,t}(y_{i,t}, h_t \mid \widetilde{\nu}(\kappa^n, f^n_{-i,0}), \nu), \tag{E.8}$$

where the oblivious stage value function $\widetilde{\pi}_{i,t}(\cdot)$ is given in (6.20). We define the expected oblivious value for consumer $i$ under the history-dependent strategy $\kappa^n$, as[1]

$$\widetilde{V}_{i,0}(x_{i,0}, h_T \mid \kappa^n, \nu) = \mathbb{E}\left\{\widetilde{V}_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu)\right\}, \tag{E.9}$$

where the expectation is over the initial distribution, $f^n_{-i,0}$. For any $\varepsilon > 0$, in this step we aim to show that there exists a positive integer $N$ such that for any sequence of history-dependent strategies $\{\kappa^n\}$,

$$\left|\widetilde{V}_{i,0}(x_{i,0}, h_T \mid \kappa^n, \nu) - V^n_{i,0}(x_{i,0}, h_T \mid \kappa^n, \nu)\right| \le \varepsilon, \quad \forall n \ge N, \quad \forall h_T \in \mathcal{H}_T, \quad \forall x_{i,0} \in \mathcal{X}_0. \tag{E.10}$$

For a given $s_0$, let $\mathfrak{F}^{n-1}_{s_0}(\delta)$ be the set of $f^n_{-i,0}$ such that $d(f^n_{-i,0}, \eta_{s_0}) \le \delta$. To verify (E.10), we first argue that for any $\varepsilon > 0$, there exists an positive integer $N_1$ and some

[1] This is actually the oblivious value achieved by a mixed strategy under the complete history $h_T$. In the continuum model, under a history $h_t$, the mixed strategy takes an action $\widetilde{\nu}_t(\kappa^n, f^n_{-i,0})(x_{i,0}, h_t)$, if the distribution of the $n-1$ consumers' (excluding $i$'s) initial states in the corresponding $n$-consumer model is realized as $f^n_{-i,0}$.

$\delta > 0$ such that for any $f^n_{-i,0} \in \mathfrak{F}^{n-1}_{s_0}(\delta/(XB))$ and any $n \geq N_1$,

$$\left|\widetilde{V}_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu) - V^n_{i,0}(x_{i,0}, h_T, f^n_{-i,0} \mid \kappa^n, \nu)\right| \leq \varepsilon/2, \quad \forall h_T \in \mathcal{H}_T, \quad \forall x_{i,0} \in \mathcal{X}_0. \tag{E.11}$$

Under the uniform equicontinuity assumption for the derivatives of the cost functions (see Eqs. (6.25) and (6.26)), we know that a small deviation of the aggregate demand from $\widetilde{A}_{t|\nu,h_t}$ will result in prices that are only slightly different from the prices in the continuum model. We also note that consumer $i$ cannot take an action larger than $B$, and her payoff is influenced by other consumers only through the prices. For any $\varepsilon > 0$, we can find some $\delta > 0$ and a positive integer $N_1$ such that for any given $(x_{i,0}, h_t)$, if $f^n_{-i,0} \in \mathfrak{F}^{n-1}_{s_0}(\delta/(XB))$, then the inequality in (E.5) holds for any history $h_\tau$, which implies that for any history $h_t$,

$$\left|\widetilde{\pi}_{i,t}(y_{i,t}, h_t, \kappa^n_t(y_{i,t}, h_t, f^n_{-i,t}) \mid \nu) - \pi^n_{i,t}(y_{i,t}, h_t, f^n_{-i,t} \mid \kappa^n, \nu)\right| \leq \varepsilon/(2T+2), \quad \forall n \geq N_1, \tag{E.12}$$

i.e., consumer $i$'s stage payoff (under the action $\kappa^n_t(y_{i,t}, h_t, f^n_{-i,t})$) in the $n$-consumer model is close to her oblivious stage value (under the same action $\kappa^n_t(y_{i,t}, h_t, f^n_{-i,t})$) in the continuum model, if the initial distribution in the $n$-consumer model, $f^n_{-i,0}$, is close to its expectation. The result in (E.11) follows from Eq. (E.12) and the definitions in (E.7) and (E.8). Note that $Q + 2BP$ is an upper bound on the stage payoff that consumer $i$ could obtain, and $-2BP$ is a lower bound on consumer $i$'s stage payoff, under Assumption 6.3. The desired result in (E.10) follows from (E.11), and the fact that the probability that $f^n_{-i,0} \notin \mathfrak{F}^{n-1}_{s_0}(\delta/(XB))$ decays exponentially with $n$ (cf. Eq. (E.6)).

**Step 3:** *The maximum expected payoff consumer $i$ can obtain is asymptotically no larger than the optimal oblivious value.*

In this step, we consider the case where all consumers in an $n$-consumer model except for $i$ use a DOE strategy $\nu$, and argue that for any sequence of history-dependent strategies $\{\kappa^n\}$,

$$\limsup_{n\to\infty}\left(V^n_{i,0}\left(x_{i,0},s_0\mid\kappa^n,\nu\right)-\widetilde{V}_{i,0}(x_{i,0},s_0\mid\nu,\nu)\right)\le 0,\quad\forall s_0\in\mathcal{S},\quad\forall x_{i,0}\in\mathcal{X}_0,\tag{E.13}$$

where consumer $i$'s expected payoff, $V^n_{i,0}\left(x,s\mid\kappa^n,\nu\right)$, is given in (6.24), and $\widetilde{V}_{i,0}(x,s\mid\nu,\nu)$ is the oblivious value function in (6.21). We first observe that

$$V^n_{i,0}\left(x_{i,0},s_0\mid\kappa^n,\nu\right)=\sum_{h_T\in\mathcal{H}_T(s_0)}\mathbb{P}(h_T\mid s_0)\cdot V^n_{i,0}\left(x_{i,0},h_T\mid\kappa^n,\nu\right),\tag{E.14}$$

where $\mathcal{H}_T(s_0)$ is the set of complete histories commencing at state $s_0$, and $\mathbb{P}(h_T\mid s_0)$ is the probability that the history $h_T$ is realized, conditional on the initial global state being $s_0$. We define

$$\widetilde{V}_{i,0}(x_{i,0},s_0\mid\kappa^n,\nu)=\sum_{h_T\in\mathcal{H}_T(s_0)}\mathbb{P}(h_T\mid s_0)\cdot\widetilde{V}_{i,0}(x_{i,0},h_T\mid\kappa^n,\nu).\tag{E.15}$$

Note that if $\kappa^n$ happens to be a dynamic oblivious strategy, this definition is consistent with the definition of oblivious value function in (6.21).

For any $\varepsilon>0$, let $N$ be the integer defined in Eq. (E.10); for any sequence of history-dependent strategies $\{\kappa^n\}$, we argue that

$$\begin{aligned}\widetilde{V}_{i,0}(x_{i,0},s_0\mid\nu,\nu)&\ge\sum\nolimits_{h_T\in\mathcal{H}_T(s_0)}\mathbb{P}(h_T\mid s_0)\cdot\widetilde{V}_{i,0}\left(x_{i,0},h_T\mid\kappa^n,\nu\right)\\&\ge\sum\nolimits_{h_T\in\mathcal{H}_T(s_0)}\mathbb{P}(h_T\mid s_0)\cdot\left(V^n_{i,0}(x_{i,0},h_T\mid\kappa^n,\nu)-\varepsilon\right)\\&=V^n_{i,0}\left(x_{i,0},s_0\mid\kappa^n,\nu\right)-\varepsilon,\qquad\forall n\ge N,\qquad\forall x_{i,0}\in\mathcal{X}_0.\end{aligned}\tag{E.16}$$

The DOE strategy $\nu$, by definition, maximizes consumer $i$'s oblivious value function among all possible dynamic oblivious strategies. The first inequality in (E.16) follows from the fact that $\widetilde{V}_{i,0}(x_{i,0},s_0\mid\kappa^n,\nu)$ is a weighted sum of the oblivious values achieved by a family of dynamic oblivious strategies[2]. The second inequality in (E.16) is due

[2]Note that for a given $f^n_{-i,0}$, the action taken by $\kappa^n$ depends only on $x_{i,0}$ and $h_t$, and that $\widetilde{V}_{i,0}\left(x_{i,0},h_T\mid\kappa^n,\nu\right)$ is the oblivious value achieved by a mixed strategy; cf. the footnote associated with (E.9).

to (E.10), and the last equality in (E.16) follows from (E.14). The desired result, (E.13), follows.

**Step 4:** *Consumer i's optimal oblivious value can be asymptotically achieved at an n-consumer game under a DOE strategy.*

In this step, we consider the case where all consumers in an $n$-consumer model use a DOE strategy $\nu$, and show that

$$\lim_{n\to\infty}\left(\widetilde{V}_{i,0}(x_{i,0},s_0 \mid \nu,\nu) - V_{i,0}^{n}\left(x_{i,0},s_0, \mid \nu,\nu\right)\right) = 0, \quad \forall s_0 \in \mathcal{S}, \quad \forall x_{i,0} \in \mathcal{X}_0. \quad \text{(E.17)}$$

According to (E.10), with $\kappa^n = \nu$, for any $\varepsilon > 0$, we can find some $N$ such that

$$\left|\widetilde{V}_{i,0}(x_{i,0},h_T \mid \nu,\nu) - V_{i,0}^{n}(x_{i,0},h_T \mid \nu,\nu)\right| \le \varepsilon, \quad \forall n \ge N, \quad \forall h_T \in \mathcal{H}_T, \quad \forall x_{i,0} \in \mathcal{X}_0.$$

The desired result in (E.17) follows from (E.14) and (E.15). Theorem 6.1 follows from (E.13) and (E.17).

# E.3 Proof of Theorem 6.2

## E.3.1 Proof of Part (a)

We will show that in a continuum model, a DOE strategy maximizes the expected social welfare among all possible symmetric dynamic oblivious strategy profiles (part (a) of the theorem), i.e., that if $\nu$ is DOE, then

$$\widetilde{\mathcal{W}}_0(s_0 \mid \nu) = \sup_{\vartheta\in\mathfrak{V}} \widetilde{\mathcal{W}}_0(s_0 \mid \vartheta), \qquad \forall s_0 \in \mathcal{S}. \quad \text{(E.18)}$$

Let $S$ and $X$ be the cardinality of $\mathcal{S}$ and $\mathcal{X}_0$, respectively. Given the initial global state $s_0$, the number of possible histories of length $t+1$ is $S^t$. Hence, the number of all possible histories commencing at state $s_0$ is $\sum_{t=0}^{T} S^t$. Given an initial global state $s_0$, the expected social welfare defined in (6.32) is a deterministic function of the following $(X \sum_{t=0}^{T} S^t)$-dimensional action vector:

$$\{\nu_t\,(x, h_t)\}_{x\in\mathcal{X}_0,\ h_t\in\mathcal{H}(s_0)}\,, \tag{E.19}$$

where $\mathcal{H}(s_0)$ is the set of positive probability histories commencing at state $s_0$. Under Assumption 6.4, the expected social welfare defined in (6.32) is a concave function of the vector in (E.19). Therefore, the following conditions are necessary and sufficient for the action vector (in the form of (E.19)) associated with the DOE strategy $\nu$ to maximize the expected social welfare, among all possible dynamic oblivious strategies[3]:

$$\begin{cases} \dfrac{\partial_+ U_t(l_{\nu,h_t}(x_{i,0}), s_t, \nu_t(x_{i,0}, h_t))}{\partial_+ \nu_t(x_{i,0}, h_t)} \leq \widetilde{p}_{t|\nu,h_t} + \widetilde{w}_{t|\nu,h_t} + 1_{\tau<T}\cdot g^{+}_{t|\nu,h_t}(x_{i,0}), \\ \hfill \text{if } \ \nu_t(x_{i,0}, h_t) < B, \\ \dfrac{\partial_- U_t(l_{\nu,h_t}(x_{i,0}), s_t, \nu_t(x_{i,0}, h_t))}{\partial_- \nu_t(x_{i,0}, h_t)} \geq \widetilde{p}_{t|\nu,h_t} + \widetilde{w}_{t|\nu,h_t} + 1_{\tau<T}\cdot g^{-}_{t|\nu,h_t}(x_{i,0}), \\ \hfill \text{if } \ \nu_t(x_{i,0}, h_t) > 0, \end{cases} \tag{E.20}$$

where $l_{\nu,h_t}(x_{i,0})$ is consumer $i$'s state, $x_{i,t}$, under a (positive probability) history $h_t$ and the strategy $\nu$ (cf. p.15), the prices, $\widetilde{p}_{t|\nu,h_t}$ and $\widetilde{w}_{t|\nu,h_t}$ are given in (6.17) and (6.18), and where, if $k_{h_\tau}(\cdot)$ (cf. the definition in (6.35)) is nondecreasing in $a_{i,t}$ for any $t < \tau \leq T$, then $g^{+}_{t|\nu,h_t}(x_{i,0})$ is given by[4]

$$g^{+}_{t|\nu,h_t}(x_{i,0}) = \mathbb{E}\left\{ \widetilde{q}_{t+1|\nu,h_{t+1}} - \sum_{\tau=t+1}^{T} \frac{\partial_+ U_\tau(x_{i,0}, z_{i,\tau}, s_\tau, a_{i,\tau})}{\partial_+ z_{i,\tau}} \cdot \frac{\partial_+ z_{i,\tau}}{\partial_+ a_{i,t}} \right\}, \qquad \forall x_{i,0}\in\mathcal{X}_0, \tag{E.21}$$

where the price, $\widetilde{q}_{t+1|\nu,h_{t+1}}$, is defined in (6.18), the expectation is over the future global states, $\{s_\tau\}_{t+1}^{T}$, $z_{i,\tau} = k_{h_\tau}(x_{i,0}, a_{i,0}, \ldots, a_{i,\tau-1})$ for $\tau > t$, and $a_{i,\tau} = \nu_\tau(x_{i,\tau}, h_\tau)$ for $\tau \geq t$. The expression (E.21) is the part of the right derivative of the expected social welfare (6.32) with respect to the action $a_{i,t}$, which reflects the influence of

[3]We use the notations $\partial_+ f$ and $\partial_- f$ to denote the right and left, respectively, derivatives of a function $f$.

[4]If for some $\tau > t$, $k_{h_\tau}(\cdot)$ is decreasing in $a_{i,t}$, then the right partial derivative of $U_\tau(x_{i,\tau}, s_\tau, a_{i,\tau})$ with respect to $z_{i,\tau}$ in (E.21) should be replaced by its left partial derivative.

consumer $i$'s action at stage $t$ on the ancillary cost $\widetilde{H}(\widetilde{A}_t, \widetilde{A}_{t+1}, \overline{s}_{t+1})$ at the next stage, and on her future utility (due to the influence of the action $a_{i,t}$ on the future state $z_{i,\tau}$, through the functions $k_{h_\tau}(\cdot)$). In (E.20), $g^-_{t|\nu,h_t}(x_{i,0})$ can be defined by replacing the right (left) partial derivatives in (E.21) with left (respectively, right) partial derivatives.

Given an initial global state $s_0$, and the initial state of consumer $i$, $x_{i,0}$, her oblivious value, defined in (6.21), is a deterministic, concave function of the vector

$$\{\nu_t\left(x_{i,0}, h_t\right)\}_{h_t \in \mathcal{H}(s_0)} \tag{E.22}$$

of actions that she would take at any given stage and for any given history. Since the DOE strategy $\nu$ maximizes consumer $i$'s oblivious value, it is easily checked that the vector in (E.22) must satisfy the conditions (E.20). Since this is true for any $x_{i,0} \in \mathcal{X}_0$, we conclude that the action vector (E.19) (which is comprised by putting together the vectors in (E.22), for different types in the set $\mathcal{X}_0$) satisfies the conditions (E.20). Thus, the DOE $\nu$ satisfies the sufficient condition for optimality and the result (E.18) follows.

### E.3.2 Proof of Part (b)

For a given initial global state $s_0$, let us fix some initial distribution $f_0^n$ with $d(f_0^n, \eta_{s_0}) \leq \delta$, where $\delta$ is small. In Step 1, we compare the social welfare achieved by various strategy profiles and show that

$$\frac{1}{n}\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n) \leq \frac{1}{n}\mathcal{W}_0^n(s_0 \mid \boldsymbol{\vartheta}^{n,f_0^n}) \approx \widetilde{\mathcal{W}}_0(s_0 \mid \vartheta^{n,f_0^n}).$$

Here, $\kappa^n$ is a general history-dependent strategy profile for the $n$-consumer model (cf. (6.22)). The symmetric strategy profile $\boldsymbol{\vartheta}^{n,f_0^n} = (\vartheta^{n,f_0^n}, \ldots, \vartheta^{n,f_0^n})$ is one that maximizes expected social welfare given the initial population state $f_0^n$. In Step 1, we will argue that $\vartheta^{n,f_0^n}$ can be identified with a dynamic oblivious strategy. In the approximate equality we are comparing the expected (over future global states, $\{s_t\}_{t=1}^T$)

social welfare under the same oblivious strategy $\vartheta^{n,f_0^n}$ (hence the same sequence of actions for each consumer type $x \in \mathcal{X}_0$) under two different initial population states (initial distributions of consumer types), $f_0^n$ and $\eta_{s_0}$.

Since $\nu$ is a DOE, part (a) of the theorem implies that

$$\widetilde{\mathcal{W}}_0(s_0 \mid \vartheta^{n,f_0^n}) \leq \widetilde{\mathcal{W}}_0(s_0 \mid \nu).$$

Note that as the number of consumer grows large, with high probability the initial population state $f_0^n$ is close to its expectation, $\eta_{s_0}$. In Step 2, we complete the proof of part (b) by showing that

$$\widetilde{\mathcal{W}}_0(s_0 \mid \nu) \approx \frac{1}{n}\mathbb{E}\{\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\nu}^n)\},$$

where the expectation is over the initial population state $f_0^n$.

**Step 1:** *If the initial population state is close to its expectation, the optimal social welfare in an n-consumer model can be approximated by the social welfare achieved by a dynamic oblivious strategy in the corresponding continuum model.*

In this step, we aim to show that in an $n$-consumer model, for any given initial global state $s_0$ and any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any initial distribution $f_0^n$ with $d(f_0^n, \eta_{s_0}) \leq \delta$, we can find a dynamic oblivious strategy $\vartheta^{n,f_0^n}$ that satisfies

$$\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n) \leq n\widetilde{\mathcal{W}}_0(s_0 \mid \vartheta^{n,f_0^n}) + \varepsilon n, \tag{E.23}$$

for all symmetric history-dependent strategy profiles, $\boldsymbol{\kappa}^n = (\kappa^n, \ldots, \kappa^n)$. Given an initial global state $s_0$ and an initial population state $f_0^n$, we observe that the social welfare, $\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n)$, is a deterministic, concave function of the following vector of consumers' actions under different histories,

$$\left\{\kappa_t^n\left(m_{i,\kappa^n,h_t}(x_{i,0}), h_t, f_{-i,t}^n\right)\right\}_{h_t \in \mathcal{H}(s_0),\ x_{i,0} \in \mathcal{X}_0,\ i=1,\ldots,n}, \tag{E.24}$$

where $m_{i,\kappa^n,h_t} : \mathcal{X}_0 \to \mathcal{Y}_t$ maps consumer $i$'s initial state into her augmented state at the history $h_t$, under the strategy profile $\boldsymbol{\kappa}^n$, and $f^n_{-i,t}$ is the distribution of other consumers' augmented states at the history $h_t$, under the strategy profile $\boldsymbol{\kappa}^n$. Note that given the initial population state $f^n_0$, the strategy profile $\boldsymbol{\kappa}^n$, and a history $h_t$, the augmented state of consumer $i$ at stage $t$ depends only on her initial state $x_{i,0}$.

Since the social welfare $\mathcal{W}^n_0(f^n_0, s_0 \mid \boldsymbol{\kappa}^n)$ is concave in the action vector in (E.24), there exists a symmetric solution, $\vartheta^{n,f^n_0} = \{\vartheta^{n,f^n_0}_t\}^T_{t=0}$, such that if at any history $h_t \in \mathcal{H}(s_0)$, all consumers with the same initial state take the same action according to

$$a_{i,t} = \vartheta^{n,f^n_0}_t(x_{i,0}, h_t), \quad i = 1, \ldots, n, \tag{E.25}$$

then the expected social welfare, $\mathcal{W}^n_0(f^n_0, s_0 \mid \boldsymbol{\kappa}^n)$, is maximized among all possible symmetric history-dependent strategy profiles[5]. In (E.25) we have defined a dynamic oblivious strategy $\vartheta^{n,f^n_0}$ that maximizes the expected social welfare in the $n$-consumer model, conditional on the initial global state being $s_0$, and the initial population state being $f^n_0$. That is, in an $n$-consumer model, for any given $s_0$ and $f^n_0$, there exists a dynamic oblivious strategy $\vartheta^{n,f^n_0}$ such that

$$\sup_{\kappa^n} \mathcal{W}^n_0(f^n_0, s_0 \mid \boldsymbol{\kappa}^n) = \mathcal{W}^n_0(f^n_0, s_0 \mid \boldsymbol{\vartheta}^{n,f^n_0}), \tag{E.26}$$

where $\boldsymbol{\vartheta}^{n,f^n_0} = (\vartheta^{n,f^n_0}, \ldots, \vartheta^{n,f^n_0})$ is the corresponding symmetric dynamic oblivious strategy profile. To verify (E.23), it suffices to show that for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $f^n_0$ with $d(f^n_0, \eta_{s_0}) \leq \delta$,

$$\left| \mathcal{W}^n_0(f^n_0, s_0 \mid \boldsymbol{\vartheta}^{n,f^n_0}) - n\widetilde{\mathcal{W}}_0(s_0 \mid \vartheta^{n,f^n_0}) \right| \leq \varepsilon n, \tag{E.27}$$

i.e., if all consumers use the strategy $\vartheta^{n,f^n_0}$, the difference between the optimal social welfare achieved in an $n$-consumer model and the social welfare achieved in the corresponding continuum model can be made arbitrarily small, if the initial population state is close enough to its expectation, $\eta_{s_0}$. We next argue that the result in (E.27) holds for any dynamic oblivious strategy $\vartheta$.

[5]The fact that the supremum is attained is a consequence of our continuity assumption and the fact that the various variables of interest can be restricted to be in a compact set.

To prove (E.27), we first upper bound the difference between the supplier cost in an $n$-consumer model and that in the corresponding continuum model. Since all cost functions are Lipschitz continuous (see Eqs. (6.27) and (6.28)), for any $\varepsilon > 0$, there exists some $\delta_1 > 0$ such that if

$$\left| A_t^n - n\widetilde{A}_{t|\vartheta,h_t} \right| \leq X\delta_1 Bn, \qquad t = 0, \ldots, T, \qquad \forall h_t \in \mathcal{H}(s_0), \tag{E.28}$$

then

$$\left| C^n(A_t^n, s_t) - C^n(n\widetilde{A}_{t|\vartheta,h_t}, s_t) \right| \leq n\varepsilon/(3T+3), \qquad t = 0, \ldots, T, \qquad \forall h_t \in \mathcal{H}_t(s_0), \tag{E.29}$$

$$\left| H_0^n(A_0^n, s_0) - H_0^n(n\widetilde{A}_{0|\vartheta,h_0}, s_0) \right| \leq n\varepsilon/(3T+3), \tag{E.30}$$

and for $t = 1, \ldots, T$,

$$\left| H^n(A_{t-1}^n, A_t^n, \overline{s}_t) - H^n(n\widetilde{A}_{t-1|\vartheta,h_{t-1}}, n\widetilde{A}_{t|\vartheta,h_t}, \overline{s}_t) \right| \leq n\varepsilon/(3T+3), \qquad \forall h_t \in \mathcal{H}_t(s_0), \tag{E.31}$$

where $\mathcal{H}_t(s_0)$ is the set of all histories of length $t+1$ commencing at state $s_0$. Given an initial population state $f_0^n$, if all consumers use the strategy $\vartheta$, the aggregate demand under a history $h_t$ is

$$A_t^n = n \sum_{x \in \mathcal{X}_0} f_0^n(x) \vartheta_t(x, h_t).$$

From (6.16) we observe that if $d(f_0^n, \eta_{s_0}) \leq \delta_1$, the condition in (E.28) holds, and then Eqs. (E.29)-(E.31) are verified.

We now show that if the initial population state is close to its expectation, the total utility obtained by all consumers is close to its counterpart in the corresponding continuum model. Given an initial population state $f_0^n$, we write the total utility obtained by all consumers under a history $h_t$ as

$$\sum_{i=1}^{n} U_t(x_{i,t}, s_t, a_{i,t}) = n \sum_{x \in \mathcal{X}_0} f_0^n(x) U_t\left(l_{\vartheta,h_t}(x), \vartheta_t(x, h_t), s_t\right).$$

On the other hand, the utility achieved in the corresponding continuum model is given by

$$\widetilde{U}_{t|\vartheta,h_t} \triangleq \sum_{x\in\mathcal{X}_0} \eta_{s_0}(x) U_t\left(l_{\vartheta,h_t}(x), \vartheta_t(x,h_t), s_t\right).$$

We have that if $d(f_0^n, \eta_{s_0}) \le \varepsilon/(3XQ(T+1))$, then for any $n \in \mathbb{N}^+$,

$$\left|\sum\nolimits_{i=1}^{n} U_t(x_{i,t}, s_t, a_{i,t}) - n\widetilde{U}_{t|\vartheta,h_t}\right| \le n\varepsilon/(3T+3), \qquad t = 0,\dots,T, \qquad \forall h_t \in \mathcal{H}_t(s_0), \tag{E.32}$$

Let $\delta = \min\{\delta_1, \varepsilon/(3XQ(T+1))\}$. If $d(f_0^n, \eta_{s_0}) \le \delta$, from (E.29)-(E.32) we have

$$\left|\widetilde{W}_t(h_t \mid \vartheta) - W_t^n(f_t^n, h_t \mid \boldsymbol{\vartheta}^n)\right| \le n\varepsilon/(T+1), \quad t = 0,\dots,T, \quad \forall h_t \in \mathcal{H}_t(s_0).$$

Eq. (E.27) follows from the definition of expected social welfare in an $n$-consumer model (6.31), and in a continuum model (6.32). The desired result in (E.23) follows.

**Step 2:** *Asymptotic social optimality of a DOE.*

In this step, we complete the proof of part (b) of the theorem, using the fact that as the number of consumers grows large, with high probability the initial population state is close to its expectation. Note that the action space is $[0, B]$, so that $|A_t| \le nB$. Using Assumption 6.3, $C^n(\cdot)/n$ is therefore bounded. A similar argument holds for $H_0^n(\cdot)/n$ and $H^n(\cdot)/n$. Furthermore, the total utility per consumer is also bounded. Thus, there exists some constant $D$ that upper bounds $|\mathcal{W}_0^n/n|$. We define $\mathfrak{F}_{s_0}^n(\delta)$ as the set of initial population states such that $d(f_0^n, \eta_{s_0}) \le \delta$. By the law of large numbers, for any pair of positive real numbers, $\varepsilon$ and $\delta$, we can find an integer $N$ such that

$$\sum_{f_0^n \notin \mathfrak{F}_{s_0}^n(\delta)} \mathbb{P}\Big(F_{s_0}^n = f_0^n\Big) \cdot \sup_{\kappa^n} |\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n)| \le D\mathbb{P}(d(f_0^n, \eta_{s_0}) > \delta) \le \varepsilon n, \qquad \forall n \ge N. \tag{E.33}$$

For any $\varepsilon > 0$, let $\delta$ be the positive real number defined in (E.23), and let $N$ be the positive integer given in (E.33); for any $n \ge N$ and any symmetric history-dependent strategy profile $\boldsymbol{\kappa}^n$, we have

$$
\begin{aligned}
&\mathbb{E}\left\{\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n)\right\} \\
&\leq \sum\nolimits_{f_0^n \in \mathfrak{F}_{s_0}^n(\delta)} \mathbb{P}\big(F_{s_0}^n = f_0^n\big) \cdot \mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\kappa}^n) + \varepsilon n \\
&\leq \sum\nolimits_{f_0^n \in \mathfrak{F}_{s_0}^n(\delta)} \mathbb{P}\big(F_{s_0}^n = f_0^n\big) \cdot \Big(n\widetilde{\mathcal{W}}_0(s_0 \mid \vartheta^{n,f_0^n}) + \varepsilon n\Big) + \varepsilon n \\
&\leq \sum\nolimits_{f_0^n \in \mathfrak{F}_{s_0}^n(\delta)} \mathbb{P}\big(F_{s_0}^n = f_0^n\big) \cdot \Big(n\widetilde{\mathcal{W}}_0(s_0 \mid \nu) + \varepsilon n\Big) + \varepsilon n \\
&\leq \sum\nolimits_{f_0^n \in \mathfrak{F}_{s_0}^n(\delta)} \mathbb{P}\big(F_{s_0}^n = f_0^n\big) \cdot (\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\nu}^n) + 2\varepsilon n) + \varepsilon n \\
&\leq \mathbb{E}\left\{\mathcal{W}_0^n(f_0^n, s_0 \mid \boldsymbol{\nu}^n)\right\} + 4\varepsilon n,
\end{aligned}
$$

where the first inequality follows from (E.33), the second inequality is due to (E.23), the third inequality follows from the optimality property of the DOE $\nu$ (part (a) of the theorem), the fourth inequality follows similar to (E.27) (the proof of Eq. (E.27) remains valid for any dynamic oblivious strategy), and the last inequality follows from (E.33).