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ANALYSIS OF TRANSFER LINES CONSISTING OF
TWO UNRELIABLE MACHINES WITH RANDOM PRO-
CESSING TIMES AND FINITE STORAGE BUFFERS

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ABSTRACT

Two Markov process models of transfer lines are presented. In both there are two machines and a single buffer. The machines have exponential failure and repair processes. In one model, the service (manufacturing) process is assumed exponential; in the other, this is generalized to include the Erlang (gamma) distribution. The models are analyzed and a compact solution is obtained for the exponential case. Numerical results are presented for this case which indicate good agreement with intuition. Some theoretical results are obtained for the Erlang case.

ACKNOWLEDGMENTS

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1. INTRODUCTION

An important attribute of the components of systems is their reliability. Components may be unavailable because of routine maintenance or because of repairs for failures. For various reasons (e.g., the difficulty of diagnosing a failure), the length of time that a component is not available to perform its intended task is a random variable.

In many kinds of systems, buffer storages are present. These buffers have the effect of decoupling the system so that changes from normal operating conditions at one part of the system have minimal effect on the operation of other parts of the system. While this is often useful, the precise effect of such storages on system-wide behavior is only partially understood.

This report is concerned with a special class of systems with storage—the two-machine flow shop or transfer line. This class is illustrated in Fig. 1. Workpieces enter the first machine and are processed. They are then stored in the buffer storage and proceed to the second machine, after which they leave the system.
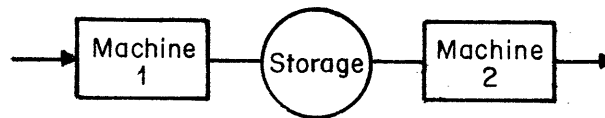
Fig. 1 Two-Machine Transfer Line

Systems of this form, often with more machines and storages, are found in many applications. We use the term "machine" here to describe the site where operations take place. The terms "processor", "stage", or "station" could also be used. The machines can then represent machine tools, chemical reactors, digital computer components, etc. A survey of the literature appears in Schick and Gershwin (1978).

The research reported here is focussed on systems in which the storage is finite, the machines have random failure and repair time distributions, and the processing times of the machines are random. The failure and repair time distributions are exponential. Two kinds of processing time distributions

are considered: exponential and Erlang. The results reported here differ from earlier work in the random nature of the processing times.

Two interpretations are appropriate for the assumption of random processing times. The first is that the system is intended to produce identical parts. The raw pieces have random attributes (such as the amount of metal that must be removed). The processors also have random attributes (such as the quality or amount of wear of tools). The interaction of these effects leads to random processing time.

The second interpretation is that the pieces to be processed are different and require different operations. For example, the pieces may be computer jobs passing through a main processor and an I/O facility.

The second interpretation is particularly significant for flexible manufacturing systems. A flexible transfer line is one in which the machines are capable of a range of operations on different pieces. These operations may take different lengths of time to perform. The randomness arises from the pieces, which arrive at the first machine in random sequence.

In Sections 2 through 7 we analyze systems whose machines have exponentially distributed processing times. Section 2 contains a formal description of the model and its assumptions. Section 3 is a complete statement of a Markov process representation of the system. Performance measures such as efficiency, production rate, and average in-process inventory are described in Section 4. Section 5 contains some theoretical results on this model and Section 6 completely characterizes the steady-state probability vector. Some numerical experiments are described in Section 7.

Systems with Erlang distributed processing times are discussed in Sections 8-11. The formal description of the model is in Section 8; the Markov process is presented in Section 9; and theoretical results appear in Section 10. Here, results are not as well developed as in the expontial case, and a partial characterization of the steady state probability vector appears in Section 11.

Section 12 presents conclusions and outlines needed in future research. Computer programs can be found in the appendices.

2. THE EXPONENTIAL CASE. MODEL DESCRIPTION AND ASSUMPTIONS

The system consists of two machines that are separated by a finite storage buffer. Parts enter machine 1 from the outside. Each part is operated upon in machine 1, then passes to the buffer and then proceeds to machine 2. After being operated on in machine 2, the part leaves the system. It is assumed that a large reservoir of parts is available to machine 1. Figure 1 shows such a system. Each machine can be in two possible states-- operational or under repair. Only when a machine is operational can it perform operations. There are, however, conditions on the storage buffer under which a machine can not operate, even if it is in the operational state. If the storage is full there is no place for parts from machine 1 to go. If the storage is empty there are no pieces available for machine 2 to operate on. A machine can fail only while it operates on a piece.

Service, failure and repair times for machine i are assumed to be exponential random variables with parameters μ_i, p_i, r_i; i=1,2 respectively. The capacity of the storage buffer in N units. We define a binary variable α_i to represent the state of machine i. If $\alpha_i=1$, machine i is operational and if $\alpha_i=0$, machine i is under repair. Let n denote the number of units in the storage plus the number of units in machine 2 (which can be zero or one). Then $0 \leq n \leq N+1$.

In the next section we characterize the steady state balance equations. These steady state probabilities are essential for computing system performance measures such as efficiency, production rate, and average in-process inventory.

3. THE DETAILED BALANCE EQUATIONS

Let the state of the system be represented by

$$s = (n, \alpha_1, \alpha_2)$$

with $n=0,1,\ldots,N$; $\alpha_i=0,1$; $i=1,2$. Whenever $n=0$, machine 2 cannot operate on a piece, and whenever $n=N$, machine 1 cannot operate on a piece.

We distinguish four sets of detailed balance equations, corresponding to the values of α_1 and α_2. For $\alpha_1=\alpha_2=0$ we have

$$p(n,0,0)(r_1+r_2) = p(n,1,0)p_1 + p(n,0,1)p_2, \quad 1 \leq n \leq N-1 \tag{3.1}$$

$$p(0,0,0)(r_1+r_2) = p(0,1,0)p_1 \tag{3.2}$$

$$p(N,0,0)(r_1+r_2) = p(N,0,1)p_2 \tag{3.3}$$

This reflects the fact that the system leaves state $(n,0,0)$ only if repair of one of the two machines takes place. We can reach state $(n,0,0)$ either from state $(n,1,0)$ (unless $n=N$) if machine 1 fails or from state $(n,0,1)$ (unless $n=0$) if machine 2 fails.

The other three sets of equations can be explained in a similar way. $\alpha_1=0$, $\alpha_2=1$:

$$\tag{3.4}$$
$$p(n,0,1)(r_1+\mu_2+p_2) = p(n,0,0)r_2 + p(n,1,1)p_1 + p(n+1,0,1)\mu_2,$$

$$1 \leq n \leq N-1$$

$$p(0,0,1)r_1 = p(0,0,0)r_2 + p(0,1,1)p_1 + p(1,0,1)\mu_2 \tag{3.5}$$

$$p(N,0,1)(r_1+\mu_2+p_2) = p(N,0,0)r_2 \tag{3.6}$$

$\alpha_1=1$, $\alpha_2=0$:

$$p(n,1,0)(p_1+\mu_1+r_2) = p(n-1,1,0)\mu_1 + p(n,0,0)r_1 + p(n,1,1)p_2, \tag{3.7}$$

$$1 \leq n \leq N-1$$

$$p(0,1,0)(p_1+\mu_1+r_2) = p(0,0,0)r_1 \tag{3.8}$$

$$p(N,1,0)r_2 = p(N-1,1,0)\mu_1 + p(N,0,0)r_1 + p(N,1,1)p_2 \tag{3.9}$$

$\alpha_1=1,\ \alpha_2=1:$

$$p(n,1,1)(p_1+p_2+\mu_1+\mu_2) = p(n-1,1,1)\mu_1 + p(n+1,1,1)\mu_2 + \tag{3.10}$$

$$+ p(n,1,0)r_2 + p(n,0,1)r_1, \quad 1\underline{<}n\underline{<}N-1$$

$$p(0,1,1)(p_1+\mu_1) = p(1,1,1)\mu_2 + p(0,1,0)r_2 + p(0,0,1)r_1 \tag{3.11}$$

$$p(N,1,1)(p_2+\mu_2) = p(N-1,1,1)\mu_1 + p(N,1,0)r_2 + p(N,0,1)r_1 \tag{3.12}$$

In Appendix 1 we present a computer program in the APL language for solving the detailed balance equations. The total number of these linear equations is $(2^2)(N+1)$ so that when N is large this becomes costly. In Section 6 we present a more efficient method of finding the steady state probabilities. In the next section we discuss measures of performance, and in Section 5 we derive some theoretical results based on the four sets of detailed balance equations.

4. MEASURES OF PERFORMANCE

There are three measures of performance that are often used as criteria to evaluate the performance of production systems.

The first measure of performance is the <u>efficiency E_i of the i'th</u> <u>machine in the system</u>. Efficiency E_i is defined as the probability that the i'th machine is operating on a piece, or the fraction of time in which the i'th machine produces pieces. We can express the efficiencies in terms of the steady state probabilities as:

$$E_1 = \sum_{n=0}^{N-1} \sum_{\alpha_2=0}^{1} p(n,1,\alpha_2) \qquad (4.1)$$

$$E_2 = \sum_{n=1}^{N} \sum_{\alpha_1=0}^{1} p(n,\alpha_1,1) \qquad (4.2)$$

It is important to distinguish E_i, the efficiency of the i'th machine in the system from e_i (defined in equation (6.53)), the efficiency of machine i if it were operated in isolation. The former is affected by the other machines and the storage while the latter is a characteristic of machine i only.

In Lemma 5 in the next section we show that

$$\mu_1 E_1 = \mu_2 E_2 \quad . \qquad (4.3)$$

The quantity $\mu_i E_i$ can be interpreted as the rate at which pieces emerge from machine i. Equation (4.3) is then a conservation of flow law, and we can define

$$P = \mu_i E_i \quad . \qquad (4.4)$$

This is the production rate of the system.

The efficiency E of the system is defined as

$$E = \frac{\text{actual production rate}}{\text{production rate in the absence of failures}} \qquad (4.5)$$

Since $1/\mu_i$ is the average time a piece spends in machine i when no failure takes place, μ_i is the production rate of machine i in isolation in the absence of failures. The production rate of the system without failures is thus less than $\min(\mu_1,\mu_2)$ and E satisfies

$$E \geq \frac{P}{\min(\mu_1,\mu_2)} \qquad (4.6)$$

Assume $\mu_i = \min(\mu_1,\mu_2)$. Then from (4.4),

$$E \geq E_i \quad .$$

Note also that if $\mu_i < \mu_j$ (i,j = 1 or 2, i≠j) then (4.3) implies that $E_i > E_j$. Therefore, the system's efficiency satisfies

$$E \geq \max(E_1,E_2) \qquad (4.7)$$

Another important measure of system performance is the <u>expected in-process inventory</u>. This can be written

$$\bar{n} = \sum_{n=0}^{N} \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} np(n,\alpha_1,\alpha_2) \quad . \qquad (4.8)$$

5. THEORETICAL RESULTS

In this section we derive some theoretical results. These results are important in providing insight into the model as well as a basis for our discussions in the following sections.

In the first lemma we prove that some of the steady state probabilities are zero.

Lemma 1

$$p(0,0,0) = p(0,1,0) = p(N,0,0) = p(N,0,1) = 0 \qquad (5.1)$$

Proof: Combining equations (3.2) and (3.8) yields:

$$p(0,0,0)r_2 + p(0,1,0)(\mu_1 + r_2) = 0 \qquad (5.2)$$

Since probabilities are non-negative, $p(0,0,0) = p(0,1,0) = 0$.

Combining equations (3.3) and (3.6) yields:

$$p(N,0,0)\ r_1 = p(N,0,1)(r_1 + \mu_2) = 0 \qquad (5.3)$$

Similarly, this implies that $p(N,0,0) = p(N,0,1) = 0$.

Lemma 2 asserts that the rate of transitions from the set of states in which machine 2 is under repair to the set of states in which machine 2 is operational is equal to the rate of transitions in the opposite direction.

Lemma 2

$$r_2 \underbrace{\sum_{n=0}^{N} \sum_{\alpha_1=0}^{1} p(n,\alpha_1,0)} = P_2 \underbrace{\sum_{n=1}^{N} \sum_{\alpha_1=0}^{1} p(n,\alpha_1,1)} \qquad (5.4)$$

probability that machine   probability that machine 2
2 is under repair   can operate

Proof

Add equations (3.1)-(3.3) and (3.7) - (3.9):

$$\sum_{n=0}^{N} p(n,0,0)(r_1+r_2) + \sum_{n=0}^{N-1} p(n,1,0)(p_1+\mu_1+r_2) + r_2 p(N,1,0)$$

$$= p_1 \sum_{n=0}^{N-1} p(n,1,0) + p_2 \sum_{n=1}^{N} p(n,0,1) + \mu_1 \sum_{n=1}^{N} p(n-1,1,0) +$$

$$r_1 \sum_{n=0}^{N} p(n,0,0) + p_2 \sum_{n=1}^{N} p(n,1,1) \tag{5.5}$$

This can be reduced to

$$r_2 \sum_{n=0}^{N} p(n,0,0) + r_2 \sum_{n=0}^{N} p(n,1,0) = p_2 \sum_{n=1}^{N} p(n,0,1) + \tag{5.6}$$

$$p_2 \sum_{n=1}^{N} p(n,1,1)$$

which is equivalent to equation (5.4)

Lemma 3 establishes a corresponding result for machine 1.

Lemma 3

$$r_1 \underbrace{\sum_{n=0}^{N} \sum_{\alpha_2=0}^{1} p(n,0,\alpha_2)}_{\substack{\text{Probability that} \\ \text{Machine 1 is under} \\ \text{repair}}} = p_1 \underbrace{\sum_{n=0}^{N-1} \sum_{\alpha_2=0}^{1} p(n,1,\alpha_2)}_{\substack{\text{Probability that} \\ \text{Machine 1 can} \\ \text{operate}}} \tag{5.7}$$

Proof

Add equations (3.1) - (3.6):

$$\sum_{n=0}^{N} p(n,0,0)(r_1+r_2) + \sum_{n=1}^{N} p(n,0,1)(r_1+\mu_2+p_2) +$$

$$+ p(0,0,1)r_1$$

$$= p_1 \sum_{n=0}^{N-1} p(n,1,0) + p_2 \sum_{n=1}^{N} p(n,0,1)$$

$$+ r_2 \sum_{n=0}^{N} p(n,0,0) + p_1 \sum_{n=0}^{N-1} p(n,1,1) + \mu_2 \sum_{n=0}^{N-1} p(n+1,0,1) \qquad (5.8)$$

This can be reduced to

$$r_1 \sum_{n=0}^{N} p(n,0,0) + r_1 \sum_{n=0}^{N} p(n,0,1) = p_1 \sum_{n=0}^{N-1} p(n,1,0) + p_1 \sum_{n=0}^{N-1} p(n,1,1)$$

$$(5.9)$$

which is equivalent to (5.7).

Lemma 4 shows that the rate of transitions from the set of states with n pieces in storage and machine 1 operational to the set of states with n+1 pieces in storage and machine 2 operational is equal to the rate of transitions in the opposite direction.

Lemma 4

$$\mu_1 \underbrace{\sum_{\alpha_2=0}^{1} p(n,1,\alpha_2)}_{} = \mu_2 \underbrace{\sum_{\alpha_1=0}^{1} p(n+1,\alpha_1,1)}_{}, \qquad 0 \leq n \leq N-1 \qquad (5.10)$$

Probability that   Probability that Machine 2
machine 1 is opera-   is operational with n+1
tional with n pieces   pieces in storage
in storage.

Proof: By induction.

For n=0, add equations (3.2), (3.5), (3.8) and (3.11). Using the results of Lemma 1 we get

$$p(0,0,1)r_1 + p(0,1,1)(p_1+\mu_1) = p(0,1,1)p_1 + p(1,0,1)\mu_2 \qquad (5.11)$$

$$+ p(1,1,1)\mu_2 + p(0,0,1)r_1$$

or

$$p(0,1,1)\mu_1 = p(1,0,1)\mu_2 + p(1,1,1)\mu_2 . \tag{5.12}$$

Since $p(0,1,0) = 0$ this is equivalent to

$$\mu_1 \sum_{\alpha_2=0}^{1} p(0,1,\alpha_2) = \mu_2 \sum_{a_1=0}^{1} p(1,\alpha_1,1) \tag{5.13}$$

which is equation (5.10) with $n=0$.

Let us assume that equation (5.10) holds for $n=m \leq N-2$. We show that this implies (5.10) for $n=m+1$. Add equations (3.1), (3.4), (3.7), and (3.10) with $n = m+1$ ($1 \leq m+1 \leq N-1$). This yields

$$p(m+1,0,0)(r_1+r_2) + p(m+1,1,0)(p_1+\mu_1+r_2) +$$

$$+ p(m+1,0,1)(r_1+\mu_2+p_2) + p(m+1,1,1)(p_1+p_2+\mu_1+\mu_2)$$

$$= p(m+1,1,0)p_1 + p(m+1,0,1)p_2 + p(m,1,0)\mu_1 + p(m+1,0,0)r_1$$

$$+ p(m+1,1,1)p_2 + p(m+1,0,0)r_2 + p(m+1,1,1)p_1 +$$

$$+ p(m+2,0,1)\mu_2 + p(m,1,1)\mu_1 + p(m+2,1,1)\mu_2 + p(m+1,1,0)r_2$$

$$+ p(m+1,0,1)r_1 \tag{5.14}$$

This can be reduced to

$$p(m+1,1,0)\mu_1 + p(m+1,0,1)\mu_2 + p(m+1,1,1)(\mu_1+\mu_2)$$

$$= p(m,1,0)\mu_1 + p(m,1,1)\mu_1 + p(m+2,0,1)\mu_2 + p(m+2,1,1)\mu_2 \tag{5.15}$$

But by induction

$$p(m,1,0)\mu_1 + p(m,1,1)\mu_1 = p(m+1,0,1)\mu_2 + p(m+1,1,1)\mu_2 \tag{5.16}$$

and therefore (5.15) becomes

$$\mu_1 \sum_{\alpha_2=0}^{1} p(m+1,1,\alpha_2) = \mu_2 \sum_{\alpha_1=0}^{1} p(m+2,\alpha_1,1) \tag{5.17}$$

Therefore, equation (5.10) holds for $0 \leq n \leq N-2$. To prove the lemma for

n=N-1 add equations (3.3), (3.6), (3.9), and (3.12) to yield, with Lemma 1,

$$p(N,1,0)r_2 + p(N,1,1)(p_2+\mu_2) = p(N-1,1,0)\mu_1$$

$$+ p(N,1,1)p_2 + p(N-1,1,1)\mu_1 + p(N,1,0)r_2 \tag{5.18}$$

or

$$p(N,1,1)\mu_2 = p(N-1,1,0)\mu_1 + p(N-1,1,1)\mu_1 \tag{5.19}$$

or

$$\mu_1 \sum_{\alpha_2=0}^{1} p(N-1,1,\alpha_2) = \mu_2 \sum_{\alpha_1=0}^{1} p(N,\alpha_1,1) \tag{5.20}$$

since $p(N,0,1) = 0$. This is equation (5.10) for $n = N-1$, so the lemma is

proven.

In the next lemma we prove that the rate of transitions between

the set of states in which machine 1 can produce a piece and the set

of states in which machine 2 can produce a piece are equal.

Lemma 5

$$\mu_1 \underbrace{\sum_{n=0}^{N-1} \sum_{\alpha_2=0}^{1} p(n,1,\alpha_2)}_{\substack{\text{Probability that Machine} \\ \text{1 can produce a piece}}} = \mu_2 \underbrace{\sum_{n=1}^{N} \sum_{\alpha_1=0}^{1} p(n,\alpha_1,1)}_{\substack{\text{Probability that Machine} \\ \text{2 can produce a piece}}} \tag{5.21}$$

Proof: Equation (5.21) is simply equation (5.10) summed for $n = 0,\ldots,N-1$.

This lemma is interpreted in Section 4.

6. CLOSED FORM EXPRESSIONS FOR THE STEADY STATE PROBABILITIES

6.1 Analysis of Internal Equations

We define internal states as states (n, α_1, α_2) where $1 \leq n \leq N-1$.*

Internal equations are the detailed balance equations that do not include any boundary states. The rest are called boundary equations. Following the analysis in Schick and Gershwin (1978) we guess a solution for the internal equations of the form

$$p(n, \alpha_1, \alpha_2) = cX^n Y_1^{\alpha_1} Y_2^{\alpha_2} , \qquad 1 \leq n \leq N-1 \tag{6.1}$$

where c, X, Y_1, Y_2 are parameters to be determined.

By substituting (6.1) into the internal equations we find that those equations are satisifed if X, Y_1, Y_2 satisfy the following three non-linear equations:

$$p_1 Y_1 + p_2 Y_2 - r_1 - r_2 = 0 \tag{6.2}$$

$$\mu_1 (\frac{1}{X} - 1) - p_1 Y_1 + r_1 + \frac{r_1}{Y_1} - p_1 = 0 \tag{6.3}$$

$$\mu_2 (X - 1) - p_2 Y_2 + \frac{r_2}{Y_2} + r_2 - p_2 = 0 \tag{6.4}$$

This is because the internal equations (equations (3.1), (3.4), (3.7), and (3.10)) can be written

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*This is in contrast to the deterministic processing time case (Schick and Gershwin, 1978) in which the internal states are those in which $2 \leq n \leq N-2$.

$$p(n,\alpha_1,\alpha_2) \ (r_1^{1-\alpha_1} p_1^{\alpha_1} + r_2^{1-\alpha_2} p_2^{\alpha_2})$$

$$= (p(n-1,\alpha_1,\alpha_2) - p(n,\alpha_1,\alpha_2))\mu_1\alpha_1$$

$$+ (p(n+1,\alpha_1,\alpha_2) - p(n,\alpha_1,\alpha_2)\mu_2\alpha_2$$

$$+ p(n,1-\alpha_1,\alpha_2)r_1^{\alpha_1} p_1^{1-\alpha_1}$$

$$+ p(n,\alpha_1,1-\alpha_2)r_2^{\alpha_2} p_2^{1-\alpha_2} \tag{6.5}$$

If $p(n,\alpha_1,\alpha_2)$ is given by (6.1), equation (6.5) becomes

$$X^n Y_1^{\alpha_1} Y_2^{\alpha_2} (r_1^{1-\alpha_1} p_1^{\alpha_1} + r_2^{1-\alpha_2} p_2^{\alpha_2})$$

$$= X^n Y_1^{\alpha_1} Y_2^{\alpha_2} \mu_1\alpha_1 (\frac{1}{X} - 1) + X^n Y_1^{\alpha_1} Y_2^{\alpha_2} \mu_2\alpha_2(X-1)$$

$$+ X^n Y_1^{1-\alpha_1} Y_2^{\alpha_2} r_1^{\alpha_1} p_1^{1-\alpha_1} - X^n Y_1^{\alpha_1} Y_2^{1-\alpha_2} r_2^{\alpha_2} p_2^{1-\alpha_2} \tag{6.6}$$

or

$$r_1^{1-\alpha_1} p_1^{\alpha_1} + r_2^{1-\alpha_2} p_2^{\alpha_2}$$

$$= \mu_1\alpha_1 (\frac{1}{X} - 1) + \mu_2\alpha_2 (X-1)$$

$$+ Y_1^{1-2\alpha_1} r_1^{\alpha_1} p_1^{1-\alpha_1} + Y_2^{1-2\alpha_2} r_2^{\alpha_2} p_2^{1-\alpha_2} \tag{6.7}$$

or, finally,

$$0 = \{\mu_1\alpha_1 (\frac{1}{X} - 1) + Y_1^{1-2\alpha_1} r_1^{\alpha_1} p_1^{1-\alpha_1} - r_1^{1-\alpha_1} p_1^{\alpha_1}\}$$

$$+ \{\mu_2\alpha_2 (X-1) + Y_2^{1-2\alpha_2} r_2^{\alpha_2} p_2^{1-\alpha_2} - r_2^{1-\alpha_2} p_2^{\alpha_2}\} . \tag{6.8}$$

Equation (6.2) follows when $\alpha_1 = \alpha_2 = 0$. Equation (6.3) and (6.4) result from $\alpha_1 = 1$, $\alpha_2 = 0$ and $\alpha_1 = 0$, $\alpha_2 = 1$, respectively. When $\alpha_1 = \alpha_2 = 1$, the result is the sum of equation (6.3) and (6.4). This development holds for lines of k machines and k-1 storages, although the boundary behavior has not yet been investigated.

Equations (6.2)-(6.4) can be reduced to the following fourth degree polynomial in Y_1:

$$p_1^3 Y_1^4 + (-\mu_2 p_1^2 - 3p_1^2 r_1 - p_1^2 r_2 - p_2 p_1^2 + \mu_1 p_1^2 + p_1^3) Y_1^3$$

$$+ (2\mu_2 p_1 r_1 - r_2 p_1 \mu_1 - 2r_1 p_1 \mu_1 - p_2 p_1 \mu_1 - \mu_2 p_1^2 - 3p_1^2 r_1$$

$$- p_1^2 r_2 - p_2 p_1^2 + \mu_2 r_2 p_1 + 3p_1 r_1^2 + 2p_1 r_1 r_2 + 2p_1 p_2 r_1) Y_1^2$$

$$+ (2p_1 p_2 r_1 + \mu_1 r_1^2 + r_1 r_2 \mu_1 + r_1 \mu_1 p_2 + p_1 r_2 \mu_2 + 2p_1 r_1 \mu_2$$

$$+ 3p_1 r_1^2 + 2r_1 r_2 p_1 - \mu_2 r_1^2 - r_1 r_2 \mu_2 - r_1^2 p_2 - r_1^3$$

$$- r_1^2 r_2) Y_1 + (-r_1 \mu_2 (r_1 + r_2) - r_1^2 r_2 - r_1^3 - r_1^2 p_2) = 0 \qquad (6.9)$$

It is easy to verify that one solution is

$$Y_{11} = \frac{r_1}{p_1} \qquad (6.10)$$

and by substituting (6.10) in (6.2) and (6.3) we obtain:

$$Y_{21} = \frac{r_2}{p_2} \qquad (6.11)$$

and

$$X_1 = 1 \tag{6.12}$$

The other three solutions to (6.9) can be obtained by solving a cubic equation. We can write (6.9) as

$$p_1^2 (p_1 Y_1 - r_1)(Y_1^3 + 3s Y_1^2 + t Y_1 + u) = 0 \tag{6.13}$$

where s, t and u are given by

$$3s = \frac{1}{p_1}(-\mu_2 - 2r_1 - r_2 - p_2 + \mu_1 + p_1) \tag{6.14}$$

$$t = \frac{1}{p_1^2}(\mu_2 r_1 - r_2 \mu_1 - r_1 \mu_1 - p_2 \mu_1 - \mu_2 p_1$$

$$-2p_1 r_1 - p_1 r_2 - p_1 p_2 + \mu_2 r_2 + r_1^2$$

$$+ r_1 r_2 + r_1 p_2) \tag{6.15}$$

$$u = \frac{1}{p_1^2}(\mu_2(r_1 + r_2) + r_1 r_2 + r_1^2 + r_1 p_2) \tag{6.16}$$

The other three values for Y_1 are (Chemical Rubber, 1959)

$$Y_{12} = 2\sqrt{-\frac{a}{3}} \cos\left(\frac{\phi}{3}\right) \tag{6.17}$$

$$Y_{13} = 2\sqrt{-\frac{a}{3}} \cos\left(\frac{\phi}{3} + 2\pi/3\right) \tag{6.18}$$

$$Y_{14} = 2\sqrt{-\frac{a}{3}} \cos\left(\frac{\phi}{3} + 4\pi/3\right) \tag{6.19}$$

where:

$$a = \frac{1}{3}(3t - s^2) \tag{6.20}$$

$$b = \frac{1}{27} (2s^2 - 9st + 27u) \tag{6.21}$$

and

$$\phi = \arcsin \left[\frac{(-\frac{b}{2})}{\sqrt{-\frac{a^3}{27}}} \right] . \tag{6.22}$$

Again Y_{2i}, X_i for $i = 2,3,4$ can be obtained from (6.2) and (6.3).

The conclusion is that for the internal equations:

$$p(n,\alpha_1,\alpha_2) = \sum_{j=1}^{Y} c_j \, x_j^n \, Y_{1j}^{\alpha_1} \, Y_{2j}^{\alpha_2} \tag{6.23}$$

where c_j, $j = 1,2,3,4$, are parameters to be determined.

6.2 Analysis of the Boundary Equations

There are a total of eight boundary states. The probabilities of four of them are specified by Lemma 1. The other four are characterized in the next lemma.

Lemma 6

$$p(0,0,1) = \frac{1}{r_1} \sum_{j=1}^{4} c_j \, (p_1 Y_{1j} Y_{2j} + \mu_2 x_j Y_{2j}) \tag{6.24}$$

$$p(0,1,1) = \sum_{j=1}^{4} c_j Y_{1j} Y_{2j} \tag{6.25}$$

$$p(N,1,0) = \frac{1}{r_2} \sum_{j=1}^{4} c_j x_j^{N-1} (\mu_1 Y_{1j} + p_2 x_j Y_{1j} Y_{2j}) \tag{6.26}$$

$$p(N,1,1) = \sum_{j=1}^{4} c_j X_j^N Y_{1j} Y_{2j} \qquad (6.27)$$

Note that (6.25) and (6.26) are in the internal (6.23) form. Furthermore, the coefficients c_j satisfy

$$c_1 = 0 \qquad (6.28)$$

$$\sum_{j=2}^{4} c_j Y_{1j} = 0 \qquad (6.29)$$

$$\sum_{j=2}^{4} c_j X_j^N Y_{2j} = 0 \qquad (6.30)$$

and the normalization equation,

$$\sum_{n=0}^{N} \sum_{\alpha_1=0}^{1} \sum_{\alpha_2=0}^{1} p(n_1 \alpha_1, \alpha_2) = 1 \qquad (6.31)$$

Proof: The expressions (6.23) - (6.27) and (5.1) satisfy all the detailed balance equations (3.1)-(3.12) identically except for the following.

Equation (3.11) becomes

$$0 = \sum_{j=1}^{4} c_j Y_{2j} (\mu_1 Y_{1j} - \mu_2 X_j Y_{1j} - \mu_2 X_j) \cdot \qquad (6.32)$$

Equation (3.7), for $n = 1$, becomes

$$0 = \sum_{j=1}^{4} c_j X_j \left[(p_1 + \mu_1 + r_2) Y_{1j} - r_1 - p_2 Y_{1j} Y_{2j} \right] \cdot \qquad (6.33)$$

Equation (3.12) becomes

$$0 = \sum_{j=1}^{4} c_j X_j^{N-1} Y_{1j} (\mu_2 X_j Y_{2j} - \mu_1 Y_{2j} - \mu_1) \qquad (6.34)$$

and equation (3.4), for n = N-1, is

$$0 = \sum_{j=1}^{4} c_j X_j^{N-1} [(r_1 + \mu_2 + p_2) Y_{2j} - r_2 - p_1 Y_{1j} Y_{2j}] \qquad (6.35)$$

Equation (6.33) can be transformed by observing that

$$(p_1 + \mu_1 + r_2) Y_{1j} - r_1 - p_2 Y_{1j} Y_{2j}$$

$$= (p_1 Y_{1j} - r_1) + \mu_1 Y_{1j} + Y_{1j} (r_2 - p_2 Y_{2j}) \qquad (6.36)$$

$$= (p_1 Y_{1j} - r_1) + \mu_1 Y_{1j} - Y_{1j} (r_1 - p_1 Y_{1j}) \qquad (6.37)$$

(because of equation (6.2)), and finally,

$$= (p_1 Y_{1j} - r_1)(1 + Y_{1j}) + \mu_1 Y_{1j} \quad . \qquad (6.38)$$

Note that equation (6.3) can be written

$$\mu_1 (\frac{1}{X_j} - 1) = (p_1 Y_{1j} - r_1) \frac{(1 + Y_{1j})}{Y_{1j}} \qquad (6.39)$$

so that expression (6.38) is

$$\mu_1 (\frac{1}{X_j}) - 1) Y_{1j} + \mu_1 Y_{1j} = \mu_1 Y_{1j} / X_j \qquad (6.40)$$

Thus, equation (6.33) is now

$$0 = \sum_{j=1}^{4} c_j Y_{1j} \; . \tag{6.41}$$

The same sequence of steps can be applied to equation (6.35) to yield

$$0 = \sum_{j=1}^{4} c_j X_j^N Y_{2j} \tag{6.42}$$

To analyze (6.32), we first observe that equation (6.39) implies

$$\mu_1 Y_{1j} - \mu_2 X_j (Y_{1j} + 1) =$$

$$\mu_1 Y_{1j} - \frac{\mu_2 X_j \mu_1 \left(\frac{1}{X_j} - 1 \right) Y_{1j}}{p_1 Y_{1j} - r_1} \tag{6.43}$$

if $p_1 Y_{1j} - r_1 \neq 0$. Recall that

$$p_1 Y_{11} - r_1 = 0$$

Equation (6.4) can be written

$$\mu_2 (X_j - 1) = (p_2 Y_{2j} - r_{2j}) \frac{(1 + Y_{2j})}{Y_{2j}} \tag{6.44}$$

so that (6.43) can be transformed to

$$-\mu_1 Y_{1j} / Y_{2j} \; , \tag{6.45}$$

with the use of (6.2), still assuming $p_1 Y_{1j} - r_1 \neq 0$. Equation (6.32) can now be written

$$0 = c_1 Y_{21} (\mu_1 Y_{11} - \mu_2 X_1 Y_{11} - \mu_2 X_1)$$

$$- \mu_1 \sum_{j=2}^{4} c_j Y_{1j} \tag{6.46}$$

or, using (6.10) - (6.12),

$$0 = c_1 \frac{r_2}{p_2} (\mu_1 \frac{r_1}{p_1} - \mu_2 \frac{r_1}{p_1} - \mu_2) - \mu_1 \sum_{j=2}^{4} c_j Y_{1j} \qquad (6.47)$$

Finally, we observe that (6.29) implies that

$$\sum_{j=2}^{4} c_j Y_{1j} = - c_1 \frac{r_1}{p_1} \qquad (6.48)$$

so that (6.47) can be written, after some transformation, as

$$0 = c_1 \left[\frac{\mu_1 r_1}{r_1 + p_1} - \frac{\mu_2 r_2}{r_2 + p_2} \right] \qquad . \qquad (6.49)$$

By the same sort of manipulations, equation (6.34) can also be transformed into equation (6.49).

To complete the lemma, two cases must be considered. If

$$\frac{\mu_1 r_1}{r_1 + p_1} \neq \frac{\mu_2 r_2}{r_2 + p_2} \qquad (6.50)$$

then (6.28) follows from (6.49); and (6.29) and (6.30) follow from (6.28), (6.41) and (6.42).

If

$$\frac{\mu_1 r_1}{r_1 + p_1} = \frac{\mu_2 r_2}{r_2 + p_2} \qquad (6.51)$$

then c_1 is not determined by (6.49). However, in this case $Y_1 = r_1/p_1$ is a __double__ root of (6.9). That is, one of the values of Y_{1j} given by

(6.17) - (6.19) (j = 2,3,4) is r_1/p_1. Then there are only three independent sets of parameters (X_j, Y_{1j}, Y_{2j}) and the coefficients c_j are given by (6.29) - (6.31).

As a final note, the values of c_2, c_3 and c_4 can also be found by solving (5.3) and (5.6) and the normalization equation (6.31).

The quantities

$$p_i = \frac{\mu_i r_i}{r_i + p_i} \tag{6.52}$$

that appear in equation (6.49) have physical significance. We define p_i as the <u>isolated production rate</u> of machine i, the production rate it would have if it were not part of a system with other machines and storages. The ratio

$$e_i = \frac{r_i}{r_i + p_i} \tag{6.53}$$

in the fraction of time it is available (i.e., not under repair) if it were in isolation. This quantity is the <u>isolated efficiency</u>. Since μ_i is the production rate while machine i is operational, $\mu_i e_i$ is the production rate in isolation.

6.3 The Algorithm

Now we can find all the steady state probabilities of the system using the following algorithm:

Step 1

Find Y_{1j}, Y_{2j}, X_j; j = 2,3,4 using (6.17) - (6.19) and (6.2) - (6.3).

Step 2

Solve equations (6.29) - (6.31) to obtain c_j; $j=2,3,4$.

Step 3

Use Lemma 1 and equations (6.23) - (6.27) to evaluate all probabilities. These probabilities can be used to evaluate the measures of performance of Section 4.

7. COMPUTATIONAL EXPERIENCE

In this section, we describe the results of a set of numerical experiments with the model described above.

In Table 1 we vary the parameter μ_1 from .1 to 1000. The other paramters have values: $\mu_2 = 2$, $p_1 = 3$, $p_2 = 4$, $r_1 = 5$, $r_2 = 6$, $N = 4$.

TABLE 1

μ_1	$\bar{n}$	E_1	E_2	PRODUCTION RATE
.1	.0554	.625	.03125	.0625
.5	.358	.622	.1555	.311
1	.919	.5966	.2983	.5966
10	3.73	.1195	.5973	1.1946
100	3.98	.0112	.5999	1.1998
1000	4.	.0012	.6	1.2

We see that as μ_1, the rate of service for machine 1 increases, both E_2 and the production rate increase to a limit of .6 and 1.2 respectively. That is, there is a saturation effect, and no amount of increase in the speed of machine 1 can improve the productivity of the system. Note that as the first machine is speeded up, the amount of material in the storage increases. This is the reason for the increase in production rate.

In Table 2 we vary the parameter μ_2 from .1 to 1000 for the case $\mu_1 = 1$, $p_1 = 3$, $p_2 = 4$, $r_1 = 5$, $r_2 = 6$, $N = 4$.

TABLE 2

μ_2	$\bar{n}$	E_1	E_2	PRODUCTION RATE
.1	3.89	.0599	.59990	.0599
.5	3.19	.2903	.5806	.2903
1	2.08	.4836	.4836	.4836
10	.124	.6247	.06247	.6247
100	.0111	.6249	.006249	.6249
1000	.00109	.625	.000625	.625

In this case we see that although E_2 decreases, the production rate

increases as μ_2 increases. When the second machine is very fast, it

frequently empties the storage and thus spends a lot of time starved

for pieces. Consequently E_2, the fraction of time machine 2 is operating

on a piece, is small. The production rate, in the limit as μ_2 is

large, is simply the isolated production rate of the first machine,

$\mu_1 r_1/(r_1+p_1)$. Here as μ_2 increases, the number of pieces in storage

decreases.

These two tables lead to the following tentative conclusion: If

all other things are equal, it is better to speed up downstream machines

than to speed up upstream machines. Both can increase over all

production rate, but if downstream machines are made faster, the average

in-process inventory is reduced.

In Table 3 we vary the parameter p_1 from .1 to 1000 for the case:

$\mu_1 = 1$, $\mu_2 = 2$, $p_2 = 4$, $r_1 = 5$, $r_2 = 6$, N = 4.

TABLE 3

p_1	$\bar{n}$	E_1	E_2	PRODUCTION RATE
.1	1.61	.846	.423	.846
.5	1.48	.8034	.4017	.8034
1	1.33	.7540	.3770	.7540
10	.396	.3311	.1655	.3311
100	.0418	.0476	.02361	.0476
1000	.00417	.0049	.00249	.0049

As the rate of failure of the first machine increases, the average in-process inventory, production rate, and the efficiencies E_1 and E_2 go to zero together.

In Table 4 we vary the parameter p_2 from .1 to 1000 for the case $\mu_1 = 1$, $\mu_2 = 2$, $p_1 = 3$, $r_1 = 5$, $r_2 = 6$, N = 4.

TABLE 4

p_2	$\bar{n}$	E_1	E_2	PRODUCTION RATE
.1	.464	.6194	.3097	.6194
.5	.507	.6174	.3087	.6174
1	.562	.6158	.3079	.6158
10	1.66	.5322	.2661	.5322
100	3.77	.1131	.0565	.1131
1000	3.98	.0119	.0060	.0119

Again as the rate of failure for the second machine increases, E_1, E_2, and the production rate approach zero, but here $\bar{n}$ approaches N = 4.

In Table 5 we vary the parameter value r_1 from .1 to 1000 for the case $\mu_1 = 1$, $\mu_2 = 2$, $p_1 = 3$, $p_2 = 4$, $r_2 = 6$, N = 4.

TABLE 5

r_1	$\overline{n}$	E_1	E_2	PRODUCTION RATE
.1	.0342	.03225	.01612	.03225
.5	.161	.1426	.07128	.1426
1	.300	.2486	.1243	.2486
10	1.20	.7104	.3552	.7104
100	1.59	.8411	.4206	.8411
1000	1.64	.8554	.4277	.8554

As the rate of repair for the first machine increases, E_1, E_2, $\overline{n}$, and the production rate increase. Note that $\overline{n}$ does not approach N = 4.

In Table 6 we vary the parameter value r_2 from .1 to 1000 for the system $\mu_1 = 1$, $\mu_2 = 2$, $p_1 = 3$, $p_2 = 4$, $r_1 = 5$, N = 4.

TABLE 6

r_2	$\overline{n}$	E_1	E_2	PRODUCTION RATE
.1	3.88	.0486	.02431	.0486
.5	3.36	.2144	.1072	.2144
1	2.74	.3575	.1787	.3575
10	.720	.6088	.3044	.6088
100	.477	.6191	.3095	.6191
1000	.455	.6198	.3099	.6198

Again when the rate of repair for the second machine increases, both the efficiency E_2 and the production rate increase. The in-process inventory decreases.

In Table 7 we vary N from 2 to 100, for the transfer line $\mu_1 = 1$, $\mu_2 = 2$, $p_1 = 3$, $p_2 = 4$, $r_1 = 5$, $r_1 = 5$, $r_2 = 6$.

TABLE 7

N	$\bar{n}$	E_1	E_2	PRODUCTION RATE
2	.599	.5228	.2614	.5228
5	1.01	.6093	.3047	.6093
10	1.16	.6242	.3121	.6242
20	1.18	.625	.3125	.625
50	1.18	.625	.3125	.625
100	1.18	.625	.3125	.625

We see from the table that as N increases, both the efficiencies and the production rate increase up to a limit. This case seems to display the same saturation effect as in Table 1. Furthermore, the average in-process inventory also approaches a limit as the buffer capacity increases.

These examples are not intended to be exhaustive. They are furnished to show the kind of result that is obtainable with this model, and that its behavior agrees with intuition.

8. THE ERLANG CASE. MODEL DESCRIPTION AND ASSUMPTIONS

Until now we have restricted our discussion to a particular service time distribution - the exponential. This assumption however, may become quite troublesome for many systems. In order to have a more realistic model we must allow a more general service time distribution.

In this model we assume that the service time distribution for the two machine is Erlang with $K (K \geq 1)$ phases.* The advantage of this assumption is that a very large class of distributions can be approximated very closely by Erlang distributions (Kleinrock, 1975).

A consequence of the new service time distribution assumption is that we can now find each of the two machines in K+1 states, since in addition to being under repair the machines also can be operational in any one of the K phases of the Erlang distribution.

Let i and j represent the states of each of the two machines, i, j = 0,1,..., K. By i = 0 we mean that machine 1 is under repair and by i = m ($1 \leq m \leq K$) we mean that machine is operational and ready to start the mth Erlangian phase.

Again we assume that machine 1 can operate on a piece only if it is operational and n < N. Machine 2 can operate on a piece only if it is operational and n>0. We also assume that when a machine fails the piece that was being processed when the machine failed must start its service from the beginning, that is from the first phase.

We consider the system in steady state. Due to the Erlang distribution assumption we have a Markovian model.

The quantities r_i, p_i and N have the same meaning as in the exponential case. This implies that the rates of failure and repair are

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*Also known as the gamma distribution, with integer shape parameter.

independent of what phase the system was in when the last failure

occurred. However, μ_i is now the rate that machine i completes each

of its Erlangian phases. Thus, the production rate of machine i,

when it is operational, is μ_i/K.

9. THE DETAILED BALANCE EQUATIONS

Again we denote (n,i,j) to be the state of the system; $n = 0$, $1,...N$; $i, j = 0$, $1,2,...$, K, $K \geq 1$. By our assumptions machine 2 cannot operate on a piece unless $n > 0$. Therefore, the probability of any state with $n = 0$ and $j > 1$ or $n = N$ and $i > 1$ is zero. That is,

$$p(0,i,j) = 0, \qquad j = 2,..., K , \qquad i = 0,1,...,K \qquad (9.1)$$

$$p(N,i,j) = 0, \qquad i = 2,..., K , \qquad j = 0,1,...,K \qquad (9.2)$$

Again we distinguish between four sets of detailed balance equations, to correspond to the values of i and j.

For $i = j = 0$ we have

$$p(n,0,0)(r_1 + r_2) = \sum_{i=1}^{K} p(n,i,0)p_1 + \sum_{j=1}^{K} p(n,0,j)p_2 , \quad 1 \leq n \leq N-1 \qquad (9.3)$$

$$p(0,0,0)(r_1 + r_2) = \sum_{i=1}^{K} p(0,i,0)p_1 \qquad (9.4)$$

$$p(N,0,0)(r_1 + r_2) = \sum_{j=1}^{K} p(N,0,j)p_2 \qquad (9.5)$$

These equations represent the fact that the system enters state $(n,0,0)$ either from state $(n,i,0)$ $(n \neq N, i \neq 0)$ if machine 1 fails or from state $(n,0,j)$ $(n \neq 0, j \neq 0)$ if machine 2 fails.

For $i = 0$, $j \neq 0$,

$$p(n,0,j)(r_1 + \mu_2 + p_2) = p(n,0,j-1)\mu_2 + \sum_{i=1}^{K} p(n,i,j)p_1 , \qquad (9.6)$$

$$2 \leq j \leq K, \; 1 \leq n \leq N-1$$

$$p(n,0,1)(r_1+\mu_2+p_2) = p(n+1,0,K)\mu_2 + \sum_{i=1}^{K} p(n,i,1)p_1 + p(n,0,0)r_2 \quad , \quad (9.7)$$

$$1 \le n \le N-1$$

$$p(0,0,1)r_1 = p(1,0,K)\mu_2 + \sum_{i=1}^{K} p(0,i,1)p_1 + p(0,0,0)r_2 \qquad (9.8)$$

$$p(N,0,j)(r_1+\mu_2+p_2) = p(N,0,j-1)\mu_2, \quad 2 \le j \le k \qquad (9.9)$$

$$p(N,0,1)(r_1+\mu_2+p_2) = p(N,0,0)r_2 \qquad (9.10)$$

For $j = 0$, $\quad i \ne 0$,

$$p(n,i,0)(p_1+\mu_1+r_2) = p(n,i-1,0)\mu_1 + \sum_{j=1}^{K} p(n,i,j)p_2, \qquad (9.11)$$

$$2 \le i \le K, \ 1 \le n \le N-1$$

$$p(n,1,0)(p_1+\mu_1+r_2) = p(n-1,K,0)\mu_1 + \sum_{j=1}^{K} p(n,i,j)p_2 + p(n,0,0)r_1, \qquad (9.12)$$

$$1 \le n \le N-1$$

$$p(0,i,0)(p_1+\mu_1+r_2) = p(0,i-1,0)\mu_1, \ 2 \le i \le K \qquad (9.13)$$

$$p(0,1,0)(p_1+\mu_1+r_2) = p(0,0,0)r_1 \qquad (9.14)$$

$$p(N,1,0)r_2 = p(N-1,K,0)\mu_1 + \sum_{j=1}^{K} p(N,1,j)p_2 + p(N,0,0)r_1 \qquad (9.15)$$

For $i \ne 0$, $j \ne 0$,

$$p(n,i,j)(p_1+p_2+\mu_1+\mu_2) = p(n,i-1,j)\mu_1 + p(n,i,j-1)\mu_2, \qquad (9.16)$$

$$2 \le i \le K; \ 2 \le j \le K, \ 1 \le n \le N-1$$

$$p(n,1,j)(p_1+p_2+\mu_1+\mu_2) = p(n-1,K,j)\mu_1 + p(n,1,j-1)\mu_2 \qquad (9.17)$$

$$+ p(n,0,j)r_1, \ 2 \le j \le K, \ 1 \le n \le N-1$$

$$p(n,i,1)(p_1+p_2+\mu_1+\mu_2) = p(n,i-1,1)\mu_1 + p(n+1,i,K)\mu_2, \qquad (9.18)$$

$$+ p(n,i,0)r_2, \ 2 \le i \le K, \ 1 \le n \le N-1$$

$$p(n,1,1)(p_1+p_2+\mu_1+\mu_2) = p(n-1,K,1)\mu_1 + p(n+1,1,K)\mu_2 \qquad (9.19)$$

$$+ p(n,0,1)r_1 + p(n,1,0)r_2, \ 1 \le n \le N-1$$

$$p(0,i,1)(p_1+\mu_1) = p(0,i-1,1)\mu_1 + p(1,i,K)\mu_2 + p(0,i,0)r_2 \qquad (9.20)$$

$$2 \le i \le K$$

$$p(0,1,1)(p_1+\mu_1) = p(1,1,K)\mu_2 + p(0,0,1)r_1 + p(0,1,0)r_2 \qquad (9.21)$$

$$p(N,1,j)(p_2+\mu_2) = p(N-1,K,j)\mu_1 + p(N,1,j-1)\mu_2 + p(N,0,j)r_1, \qquad (9.22)$$

$$2 \le j \le K$$

$$p(N,1,1)(p_2+\mu_2) = p(N-1,K,1)\mu_1 + p(N,0,1)r_1 + p(N,1,0)r_2 \qquad (9.23)$$

$$\text{for } i = 1; \ j = 1; \ n = N$$

Note that equations (9.14) and (9.20) imply that if the storage is full, the first machine is not allowed to operate on pieces even if it is operational. That is, we do not merely assume that an operation cannot be completed; we assume that an operation cannot be commenced.

In Appendix 2 we present a computer program in the APL computer language for solving all the detailed balance equations. The total

number of these equations is $N(K+1)^2 - 2K^2 + 3$. When either N or
K are large the computational effort becomes very great. In Section 11
we present some preliminary work aimed at devising an efficient algorithm
for obtaining the steady state probabilities, similar to that described
above for the exponential case. In the next section we derive some
theoretical results based on the detailed balance equations.

10. THEORETICAL RESULTS

In this section we derive some theoretical results based on the detailed balance equations. These results help us to gain more understanding of the system.

In the following lemma we prove that some of the steady state probabilities are zero.

Lemma 7

$$p(0,i,0) = p(N,0,j) = 0 \quad \text{for all } i \text{ and } j \tag{10.1}$$

Proof: Equation (9.12) and (9.13) imply

$$p(0,i,0) = \left(\frac{\mu_1}{p_1+\mu_1+r_2}\right)^{i-1} \frac{r_1}{p_1+\mu_1+r_2} \, p(0,0,0) \tag{10.2}$$

Equation (9.4) can then be written

$$p(0,0,0)\,(r_1+r_2) = p(0,0,0) \frac{p_1 r_1}{p_1+\mu_1+r_2} \sum_{i=1}^{K} \left(\frac{\mu_1}{p_1+\mu_1+r_2}\right)^{i-1} \tag{10.3}$$

or

$$\frac{p(0,0,0)}{p_1+r_2} \left(p_1 r_2 + r_1 r_2 + r_2^2 + p_1 r_1 \left(\frac{\mu_1}{p_1+\mu_1+r_2}\right)^{K}\right) = 0 \;. \tag{10.4}$$

This implies that $p(0,0,0) = 0$ and (9.22) implies that $p(0,i,0) = 0$.

Similarly,

$$p(N,0,j) = \left(\frac{\mu_2}{r_1+\mu_2+p_2}\right)^{j-1} \frac{r_2}{r_1+\mu_2+p_2} \; p(N,0,0), \quad j = 1,\ldots, K \qquad (10.5)$$

from (9.9) and (9.10). Equation (9.5) can be written

$$p(N,0,0)(r_1+r_2) = p(N,0,0) \frac{r_2 p_2}{r_1+\mu_2+p_2} \sum_{j=1}^{K} \left(\frac{\mu_2}{r_1+\mu_2+p_2}\right)^{j-1} \qquad (10.6)$$

or

$$\frac{p(N,0,0)}{r_1+p_2} \left(r_1^2+r_1 p_2+r_1 r_2+r_2 p_2 \left(\frac{\mu_2}{r_1+\mu_2+p_2}\right)^{K}\right) = 0 \qquad (10.7)$$

and (10.1) follows as before.

Lemmas 8 and 9 establish results which are analogous to Lemmas 2 and 3 above.

Lemma 8

$$r_2 \underbrace{\sum_{n=0}^{N} \sum_{i=0}^{K} p(n,i,0)}_{\substack{\text{probability that} \\ \text{machine 2 is under} \\ \text{repair}}} = p_2 \underbrace{\sum_{n=1}^{N} \sum_{i=0}^{K} \sum_{j=1}^{K} p(n,i,j)}_{\substack{\text{probability that machine} \\ \text{2 can operate on a piece}}} \qquad (10.8)$$

Proof

Let us add equations (9.3) - (9.5) and (9.11) - (9.15).

$$\sum_{n=0}^{N} p(n,0,0)(r_1+r_2) + \sum_{n=0}^{N-1} \sum_{i=1}^{K} p(n,i,0)(p_1+\mu_1+r_2) + p(N,1,0)r_2 \qquad (10.9)$$

$$= \sum_{n=0}^{N-1} \sum_{i=1}^{K} p(n,i,0)p_1 + \sum_{n=1}^{N} \sum_{j=1}^{K} p(n,0,j)p_2$$

$$+ \sum_{n=0}^{N-2} \sum_{i=1}^{K} p(n,i,0)\mu_1 + \sum_{i=1}^{K-1} p(N-1,i,0)\mu_1 + \sum_{n=1}^{N-1} \sum_{i=1}^{K} \sum_{j=1}^{K} p(n,i,j)p_2$$

$$+ \sum_{n=0}^{N-1} p(n,0,0)r_1 + p(N-1,K,0)\mu_1 + \sum_{j=1}^{K} p(N,1,j)p_2$$

$$+ p(N,0,0)r_1$$

This can be reduced to

$$\sum_{n=0}^{N-1} \sum_{i=1}^{K} p(n,i,0)r_2 + p(N,1,0)r_2 + \sum_{n=0}^{N} p(n,0,0)r_2 \qquad (10.10)$$

$$= \sum_{n=1}^{N} \sum_{j=1}^{K} p(n,0,j)p_2 + \sum_{n=1}^{N-1} \sum_{i=1}^{K} \sum_{j=1}^{K} p(n,i,j)p_2 + \sum_{j=1}^{K} p(N,1,j)p_2$$

or

$$r_2 \sum_{n=0}^{N} \sum_{i=0}^{K} p(n,i,0) = p_2 \sum_{n=1}^{N} \sum_{i=0}^{K} \sum_{j=1}^{K} p(n,i,j)$$

since $p(N,i,j) = 0$ for $i > 1$.

Lemma 9

$$r_1 \underbrace{\sum_{n=0}^{N} \sum_{j=0}^{K} p(n,0,j)}_{\substack{\text{Probability that} \\ \text{Machine 1 is under} \\ \text{repair}}} = p_1 \underbrace{\sum_{n=0}^{N-1} \sum_{i=1}^{K} \sum_{j=0}^{K} p(n,i,j)}_{\substack{\text{Probability that Machine} \\ \text{1 can operate on a piece}}} \qquad (10.11)$$

Proof: Let us add equations (9.3) - (9.5) and (9.6) - (9.10):

$$\sum_{n=0}^{N} p(n,0,0)(r_1+r_2) + \sum_{n=1}^{N} \sum_{j=1}^{K} p(n,0,j)(r_1+\mu_2+p_2) + p(0,0,1)r_1$$

$$= \sum_{n=0}^{N-1} \sum_{i=1}^{K} p(n,i,0)p_1 + \sum_{n=1}^{N} \sum_{j=1}^{K} p(n,0,j)p_2 + \sum_{n=2}^{N} \sum_{j=1}^{K} p(n,0,j)\mu_2$$

$$+ \sum_{j=1}^{K-1} p(1,0,j)\mu_2 + \sum_{n=1}^{N-1} \sum_{i=1}^{K} \sum_{j=1}^{K} p(n,i,j)p_1 + \sum_{n=1}^{N} p(n,0,0)r_2$$

$$+ p(1,0,K)\mu_2 + \sum_{i=1}^{K} p(0,i,1)p_1 + p(0,0,0)r_2 \qquad (10.12)$$

This can be reduced to

$$\sum_{n=0}^{N} p(n,0,0)r_1 + \sum_{n=1}^{N} \sum_{j=1}^{K} p(n,0,j)r_1 + p(0,0,1)r_2$$

$$\qquad (10.13)$$

$$= \sum_{n=0}^{N-1} \sum_{i=1}^{K} p(n,i,0)p_1 + \sum_{n=1}^{N-1} \sum_{i=1}^{K} \sum_{j=1}^{K} p(n,i,j)p_1 + \sum_{i=1}^{K} p(0,i,1)p_1$$

or

$$\sum_{n=0}^{N} \sum_{j=1}^{K} p(n,0,j)r_1 = \sum_{n=0}^{N-1} \sum_{i=1}^{K} \sum_{j=0}^{K} p(n,i,j)p_1$$

since $p(0,i,j) = 0$ for $j > 1$.

Lemma 10 is analogous to Lemma 4. Here, however, we must keep track of the phases of the machines. We prove that the rate of transitions between the set of states with machine 1 in the K'th phase and n pieces in storage and the set of states with machine 2 in the K'th phase and n+1 pieces in storage are equal for $0 \leq n \leq N-1$.

Lemma 10

$$\mu_1 \sum_{j=0}^{K} p(n,K,j) = \mu_2 \sum_{i=0}^{L} p(n+1,i,K), \quad 0 \leq n \leq N-1 \qquad (10.14)$$

Proof: First for n=0 let us add all the detailed balance equations with n=0. Using the results of Lemma 7 we get:

$$p(0,0,0)r_1 + \sum_{i=1}^{K} p(0,i,1)(p_1+\mu_1) \qquad (10.15)$$

$$= \sum_{i=1}^{K} p(0,i,1)p_1 + \sum_{i=2}^{K} p(0,i-1,1)\mu_1 + \sum_{i=0}^{K} p(1,0,K)\mu_2 + p(0,0,1)r_1$$

or

$$p(0,K,1)\mu_1 = \sum_{i=0}^{K} p(1,i,K)\mu_2 \qquad (10.16)$$

or

$$\sum_{j=0}^{K} p(0,K,j) = \sum_{i=0}^{K} p(1,i,K)\mu_2 \qquad (10.17)$$

since $p(0,K,0) = p(0,K,j) = 0$ for $j > 1$.

Let us assume now that (10.14) holds for $n = m$, $0 \le m < N-2$. We now prove (10.14) for $n = m+1$. Let us add all the equations with $n = m+1$; $0 \le n \le N-2$.

$$\sum_{j=1}^{K} p(m+1,0,j)(r_1+\mu_2+p_2) + \sum_{i=1}^{K} \sum_{j=1}^{K} p(m+1,i,j)(p_1+p_2+\mu_1+\mu_2)$$

$$= \sum_{i=1}^{K} p(m+1,i,0)p_1 + \sum_{j=1}^{K} p(m+1,0,j)p_2 + \sum_{i=1}^{K-1} p(m+1,i,0)\mu_1$$

$$+ p(m,k,0)\mu_1 + \sum_{i=1}^{K} \sum_{j=1}^{K} p(m+1,i,j)p_2 + p(m+1,0,0)r_1$$

$$+ \sum_{j=1}^{K-1} p(m+1,0,j)\mu_2 + p(m+2,0,K)\mu_2 + \sum_{i=1}^{K} \sum_{j=1}^{K} p(m+1,i,j)p_1$$

$$+ p(m+1,0,0)r_2 + \sum_{i=1}^{K-1} \sum_{j=1}^{K} p(m+1,i,j)\mu_1 + \sum_{j=1}^{K} p(m,k,j)\mu_1$$

$$+ \sum_{i=1}^{K} \sum_{j=1}^{K-1} p(m+1,i,j)\mu_2 + \sum_{i=1}^{K} p(m+2,i,K)\mu_2 + \sum_{j=1}^{K} p(m+1,0,j)r_1$$

$$+ \sum_{i=1}^{K} p(m+1,i,0)r_2 \qquad (10.18)$$

This can be reduced to

$$p(m+1,K,0)\mu_1 + \sum_{i=0}^{K} p(m+1,i,K)\mu_2 + \sum_{j=1}^{K} p(m+1,K,j)\mu_1$$

$$= \sum_{j=0}^{K} p(m,K,j)\mu_1 + \sum_{i=0}^{K} p(m+2,i,K)\mu_2 \qquad (10.19)$$

But by the induction assumption

$$\mu_1 \sum_{j=0}^{K} p(m,K,j) = \mu_2 \sum_{i=0}^{K} p(m+1,i,K) \qquad (10.20)$$

and therefore

$$\mu_1 \sum_{j=0}^{K} p(m+1,K,j) = \mu_2 \sum_{i=j}^{K} p(m+2,i,K) \qquad (10.21)$$

Finally, for n = N-1 add all the detailed balance equations with n=N
(Recall that $p(N,0,0) = p(N,0,j) = 0$, $j \geq 2$.)

$$p(N,1,0)r_2 + \sum_{j=1}^{K} p(N,1,j)(p_2+\mu_2)$$

$$= p(N-1,K,0)\mu_1 + \sum_{j=1}^{K} p(N,1,j)p_2 + \sum_{j=1}^{K} p(N-1,K,j)\mu_1$$

$$+ \sum_{j=2}^{K} p(N,1,j-1)\mu_2 + p(N,1,0)r_2 \qquad (10.22)$$

or

$$p(N,1,K)\mu_2 = \sum_{j=0}^{K} p(N-1,K,j)\mu_1 \qquad (10.23)$$

or

$$\sum_{j=0}^{K} p(N-1,K,j)\mu_1 = \sum_{i=0}^{K} p(N,i,K)\mu_2 \qquad (10.24)$$

since

$$p(N,i,K) = p(N,0,K) \text{ for } i > 1 .$$

Lemma 11, which is analogous to Lemma 5, shows that rate of transitions between the set of states in which machine 1 is in the K'th phase and the storage is not full and the set of states in which machine 2 is in the K'th phase and the storage is nonempty are equal. There is a similar interpretation to that of Lemma 5.

If, as in Section 4, we define E_i to be the fraction of time machine i can produce a piece, then

$$E_1 = \sum_{n=0}^{N-1} \sum_{j=0}^{K} p(n,K,j) \tag{10.25}$$

and

$$E_2 = \sum_{n=1}^{N} \sum_{i=0}^{K} p(n,i,K) \tag{10.26}$$

The rate that parts emerge from machine i is $\mu_i E_i$. Lemma 11 says that these rates are equal so that we can define the system's <u>production rate</u> to be that value. The discussion in Section 4 thus applies to the Erlang service process as well as the exponential.

<u>Lemma 11</u>

$$\mu_1 E_1 = \mu_2 E_2 \tag{10.27}$$

<u>Proof</u>: We proved in Lemma 10 that

$$\mu_1 \sum_{j=0}^{K} p(n-1,K,j) = \mu_2 \sum_{i=0}^{K} p(n,i,K) \quad \text{for } 1 \le n \le N . \tag{10.28}$$

If we sum this equation from n = 1 to n = N we get

$$\mu_1 \sum_{n=1}^{N} \sum_{j=0}^{K} p(n-1,K,j) = \mu_2 \sum_{n=1}^{N} \sum_{i=0}^{K} p(n,i,K) \qquad (10.29)$$

or

$$\mu_1 \sum_{n=0}^{N-1} \sum_{j=0}^{K} p(n,K,j) = \mu_2 \sum_{n=1}^{K} \sum_{i=0}^{K} p(n,i,K) \qquad (10.30)$$

which is (10.27).

11. ANALYSIS OF INTERNAL EQUATIONS

We again define internal equations as all the detailed balance equations that do not include any of the steady state probabilities for $n=0$ or $n=N$. We guess a solution to the steady state probabilities that appear in the internal equations, of the form

$$p(n,\alpha_1,\alpha_2) = c\, X^n\, Y_{11}^{\gamma_1}\, Y_{12}^{\beta_1}\, Y_{21}^{\gamma_2}\, Y_{22}^{\beta_2} \tag{11.1}$$

where for $i = 1,2$,

$$\beta_i = \begin{cases} 0 & \text{if} \quad \alpha_i = 0 \\ 1 & \text{if} \quad \alpha_1 \geq 1 \end{cases} \tag{11.2}$$

$$\gamma_i = \begin{cases} 0 & \text{if} \quad \alpha_i = 0 \\ \alpha_i - 1 & \text{if} \quad \alpha_i \geq 1 \end{cases} \tag{11.3}$$

By substituting (11.1) – (11.3) in the internal equations we get the following five nonlinear equations in the five unknowns X, Y_{11}, Y_{12}, Y_{21}, Y_{22}.

$$Y_{11}Y_{21}(p_1+p_2+\mu_1+\mu_2) = Y_{21}\mu_1+Y_{11}\mu_2 \tag{11.4}$$

$$Y_{11}(p_1+\mu_1+r_2) = \mu_1 + p_2 Y_{22} Y_{11} \frac{1-Y_{21}^K}{1-Y_{21}} \tag{11.5}$$

$$Y_{21}(r_1+\mu_2+p_2) = \mu_2 + p_1 Y_{12} Y_{21} \frac{1-Y_{11}^K}{1-Y_{11}} \tag{11.6}$$

$$XY_{12}Y_{21}(p_1+p_2+\mu_1+\mu_2) = Y_{12}Y_{11}^{K-1}Y_{21}\mu_1 + XY_{12}\mu_2 + XY_{21}r_1 \tag{11.7}$$

$$Y_{11}Y_{22}(p_1+p_2+\mu_1+\mu_2) = Y_{22}\mu_1 + XY_{11}Y_{22}Y_{21}^{K-1}\mu_2 + Y_{11}r_2 \tag{11.8}$$

These five equations in five unknowns can be reduced to a single 2K+2 degree polynomial equation in Y_{11}. A single equation (not in polynomial form) appears in Appendix 3.

This equation has 2K+2 solutions. Thus the internal probabilities are expected to be of the form

$$p(n,\alpha_1,\alpha_2) = \sum_{s=1}^{2K+2} c_s \, X_s^n \, Y_{11s}^{\gamma_1} \, Y_{12s}^{\beta_1} \, Y_{21s}^{\gamma_2} \, Y_{22s}^{\beta_2}$$

where the subscript s refers to solution number, and γ_i and β_i are given by (11.2), (11.3).

This solution is not complete because the boundary probabilities and equations have not been analyzed.

12. CONCLUSIONS AND FUTURE RESEARCH

We have calculated the steady state probabilities for the two-machine transfer line subject to failures and exponentially distributed processing times. These probabilities are used in the calculation of efficiencies, the production rate, and the average in-process inventory. Theoretical and computational results demonstrate that the model behaves in a manner consistent with intuition.

Analysis is somewhat less complete for the transfer line with Erlang distributed processing times. The internal probabilities are well understood, but numerical results cannot be obtained without an understanding of the boundary probabilities, the probabilities of states with storage empty or full. Theoretical results have been obtained that partially characterize the system's behavior.

Future research includes, of course, the completion of the Erlang case. Further numerical experience with these results should be obtained, partially to investigate the differences between the exponential and deterministic processing time systems discussed by Schick and Gershwin (1978). If the differences are small, it may be possible to bypass the Erlang case altogether. Other areas to be investigated include lines and networks of three or more machines.

APPENDIX 1

EXPONENTIAL MODEL COMPUTER PROGRAMS

In this appendix we describe the use of the computer code for the exponential model. The model has been programmed in the APL computer language for use in a time sharing environment. It has been implemented on the MIT IBM 370-168 VM:CMS System.

A. Use of the Computer Code

Step 1:

1.1 Dial 87511

1.2 Type 0 and twice press return

1.3 Type logon gys

1.4 Password:

1.5 Press Return

1.6 Type: apl

Step 2:

Type:)LOAD EXPO

This command means: Load the workspace expo from your private library.

Step 3:

Insert the following inputs:

MEWI ← rate of service for machine 1

MEWI ← rate of service for machine 2

PI ← rate of failure for machine 1

PII ← rate of failure for machine 2

RI ← rate of repair for machine 2

RII   rate of repair for machine 2

M   (storage capacity) <u>plus</u> (one unit)

<u>Step 4</u>:

Type: EFFIC

To compute steady state probabilities by matrix inversions.

Type: COE 1

To compute steady state probabilities by our efficient algorithm. This command displays all steady state probabilities. The command COEO displays only the probabilities $p(0,\alpha_1,\alpha_2)$ and $p(N,\alpha_1,\alpha_2)$ $(\alpha_1,\alpha_2 = 0,1)$.

B. <u>Description of the Computer Code</u>

There are five functions in the workspace oded:

(1)   EFFIC - to compute steady state probabilities by matrix inversion.

(2)   COE - The main function to compute steady state probabilities by our efficient algorithm. In COE we perform the calculations of the four coefficients c_j; j=1,2,3,4 and of the production rate and efficiency.

(3)   ROOT - A function called from COE to compute Y_{1j}, Y_{2j}; j=1,2,3,4.

(4)   ROOT1 - A function called from COE to compute X_j; j = 1,2,3,4.

(5)   SOOK - A function called from COE to generate all the steady state probabilities of the system.

C. <u>Listings of the Computer Code</u>

Figures 2-5 contain the listings of the five functions: EFFIC, COE, ROOT, ROOT1 and SOOK.

```
      ∇EFFIC [□]
      ∇ EFFIC
[1]   A←(M,5)ρ0
[2]   F←((4×M),(4×M))ρ0
[3]   N←0
[4]   SI:N←N+1
[5]   F[N;N+M]←PIx(N<M)
[6]   F[N;N+2×M]←PIIx(N>1)
[7]   F[N;N]←-(PI+PII)
[8]   →(N<M)/SI
[9]   FI:N←N+1
[10]  F[N;N-1]←MEWIx(N>M+1)
[11]  F[N;N+2×M]←PIIx(N>M+1)
[12]  F[N;N-M]←PI
[13]  F[N;N]←(-PII)-(PI+MEWI)x(N<2×M)
[14]  →(N<2×M)/FI
[15]  ZI:N←N+1
[16]  F[N;N-2×M]←PII
[17]  F[N;N+1]←MEWIIx(N<3×M)
[18]  F[N;N+M]←PIx(N<3×M)
[19]  F[N;N]←(-PI)-(MEWII+PII)x(N>1+2×M)
[20]  →(N<3×M)/ZI
[21]  TI:N←N+1
[22]  F[N;N-2×M]←PII
[23]  F[N;N-M]←PI
[24]  F[N;N-1]←MEWIx(N>1+3×M)
[25]  F[N;N]←(-(PI+MEWI))-(PII+MEWII)x(N>1+3×M)
[26]  F[N;N+1]←MEWII
[27]  →(N<(4×M)-1)/TI
[28]  F[4×M;]←(4×M)ρ1
[29]  B←((((4×M)-1)ρ0),1)
[30]  API←B⊟F
[31]  API←APIx(API>1E-6)
[32]  N←0
[33]  LO:N←N+1
[34]  A[N;]←(N-1),API[N,(N+M),(N+2×M),(N+3×M)]
[35]  →(N<M)/LO
[36]  S1←(+/((Mρ0),(Mρ0),(Mρ1),Mρ1)/API)-(+/API[(1+2×M),1+3×M])
[37]  A←A[;1,2,4,3,5]
[38]  A
[39]  S1
      ∇
[40]  ∇
```

Fig. 2. EFFIC Program for Exponential Systems

```apl
      ∇COE[□]∇
      ∇ COE FLAG
[1]   COEF←4ρ0
[2]   M←M-1
[3]   'MEW1 IS: ',▼MEWI
[4]   'MEW2 IS: ',▼MEWII
[5]   'P1 IS: ',▼PI
[6]   'P2 IS: ',▼PII
[7]   'R1 IS: ',▼RI
[8]   'R2 IS: ',▼RII
[9]   'STORAGE CAPACITY IS: ',▼M
[10]  ROOT
[11]  'Y1 IS: ',▼YY1
[12]  'Y2 IS: ',▼YY2
[13]  'X IS: ',▼XX
[14]  LJ←ρ0
[15]  MJ←ρ0
[16]  AJ←ρ0
[17]  →((M×⍟XX[3])>65)/SOD
[18]  A←((XX[4]*M)×YY2[4]×YY1[2])-((XX[2]*M)×YY2[2]×YY1[4])
[19]  A←-(A÷((XX[4]*M)×YY2[4]×YY1[3])-((XX[3]*M)×YY2[3]×YY1[4]))
[20]  →SOR
[21]  SOD:A←0
[22]  SOR:→((M×⍟XX[4])>65)/SOP
[23]  →(A≠0)/SOU
[24]  B←-(YY2[3]×YY1[2])÷(YY2[3]×YY1[4])
[25]  →SOPP
[26]  SOU:B←((XX[3]*M)×YY2[3]×YY1[2])-((XX[2]*M)×YY2[2]×YY1[3])
[27]  B←-(B÷((XX[3]*M)×YY2[3]×YY1[4])-((XX[4]*M)×YY2[4]×YY1[3]))
[28]  →SOPP
[29]  SOP:B←0
[30]  SOPP:J←1
[31]  KI:J←J+1
[32]  AJJ←0
[33]  BJJ←0
[34]  →((M×⍟XX[J])>65)/FLA
[35]  →(XX[J]=1)/SAL
[36]  AJJ←(1+YY1[J])×(1+YY2[J])×XX[J]÷(1-XX[J])
[37]  SAL:AJJ←AJJ+YY2[J]×((PI×YY1[J])+(MEWII×XX[J]))÷RI
[38]  AJJ←AJJ+YY1[J]×YY2[J]
[39]  →(XX[J]=1)/TAL
[40]  BJJ←-(1+YY1[J])×(1+YY2[J])×XX[J]÷(1-XX[J])
[41]  TAL:BJJ←BJJ+XX[J]×YY1[J]×YY2[J]
[42]  BJJ←BJJ+YY1[J]×(MEWI+PII×XX[J]×YY2[J])÷RII
[43]  →(XX[J]=1)/FAL
[44]  AJJ←AJJ+(XX[J]*M-1)×BJJ
[45]  →FLA
[46]  FAL:AJJ←AJJ+((XX[J]*M-1)×BJJ)+(M-1)×(1+YY1[J])×(1+YY2[J])
[47]  FLA:OJJ←((PI÷RI)×YY1[J]×YY2[J])+((MEWII÷RI)×XX[J]×YY2[J])
[48]  LJ←LJ,OJJ
[49]  →((M×⍟XX[J])>65)/SLA
[50]  GK:OOJJ←((MEWI÷RII)×YY1[J]×(XX[J]*M-1))+((PII÷RII)×YY1[J]×YY2[J]×(XX[J]*M))
[51]  →BLA
[52]  SLA:AJJ←OOJJ←0
[53]  BLA:MJ←MJ,OOJJ
[54]  AJ←AJ,AJJ
[55]  →(J<4)/KI
[56]  COEF[2]←1÷((AJ[1]×1)+(AJ[2]×A)+(AJ[3]×B))
[57]  COEF[3]←COEF[2]×A
[58]  COEF[4]←COEF[2]×B
[59]  'COEFFICIENTS ARE: ',▼COEF
[60]  M←M+1
[61]  COEF←COEF[2 3 4]
[62]  SOOK
[63]  'E1 IS:',▼ZON
[64]  'E2 IS:',▼(ZON×MEWI÷MEWII)
[65]  'PRODUCTION RATE IS:',▼MEWII×ZON
      ∇
[66]
```

Fig. 3. COE Program for Exponential Systems

```
        ∇ROOT[□]
        ∇ ROOT
[1]     A1←(RI÷PI)+((MEWII×PI*2)+(PI*3)÷((MEWII×PI*2)+(3×PI×PI*2)+(PI*3)÷((FIIXPI*2)+(FIIXPI*2))÷(PI*3)
[2]     A2←(2×MEWII×PI×RI)+(MEWII×PI×RII)+(3×PI×PI×RII)+(2×PI×PI×RI)
[3]     A2←A2-(PI×PI×MEWI)+(2×PI×PI×FI)+(FII×PI×MEWI)+(MEWII×PI*2)+(3×PI×PI×MEWII)+(FIIXPI*2)+(3×FII×PI×MEWII)+(3×PI×PI×MEWII)+(2×PI×PI×MEWII)+(2×PI×PI×PI))
[4]     A2←(A2-(PI*3))÷(A1×PI÷PI)
[5]     A3←(2×PI×PI×FII)+(MEWII×PI*2)+(PI×RII×RI*2)+(PI×MEWII×FII)+(PI×RII×MEWII)+(2×PI×RII×MEWII)+(3×PI×PI×MEWII)+(2×PI×PI×MEWII))
[6]     A3←A3-(MEWII×PI×FI*2)+(PI×RII×MEWII)+(PI×RI*2)+(PI×3)+(FII×PI×RI*2)
[7]     A3←(A3÷(PI*3))÷(A2×PI÷PI)
[8]     B1←(1÷3)×(3×A2)-(A1*2))
[9]     B2←(1÷27)×(2×A1*3)-(9×A1×A2)-27×A3)
[10]    +(((((B2*2)÷4)+((B1*3)÷27)÷(0)/ROOT
[11]    C1←((-B2×0.5))×(1÷3)
[12]    C2←C1
[13]    X1←C1+C2
[14]    X2←(((C1-C2)÷2)×(-3×0.5))-((C1+C2)÷2)
[15]    X3←(-((C1-C2)÷2)×(-3×0.5))-((C1+C2)÷2)
[16]    BOOM:YY1←((RII÷PI))÷(X1-(A1÷3))÷(X2-(A1÷3))×(X3-(A1÷3)))
[17]    YY2←((RII÷PI)),((RI÷RII-PI×YY1[2])÷PII)
[18]    YY2←YY2,((RI÷RII-PI×YY1[3])÷PII))
[19]    YY2←YY2,((RI÷RII-PI×YY1[4])÷PII))
[20]    YI←YY1[2]
[21]    XX←1,X1
[22]    YI←YY1[3]
[23]    ROOT1
[24]    XX←XX,X1
[25]    YI←YY1[4]
[26]    ROOT1
[27]    XX←XX,X1
[28]    →0
[29]    BOOM:TETA←¯2○(((-B2÷2)÷(((-B1*3)÷27)×0.5))
[30]    X1←2×((-B1÷3)*0.5)×(2○(TETA÷3))
[31]    X2←2×((-B1÷3)*0.5)×(2○((TETA÷3)+((○1)×(2÷3))))
[32]    X3←2×((-B1÷3)*0.5)×(2○((TETA÷3)+((○1)×(4÷3))))
[33]    →BOOM
[34]
[35]    ∇

        ∇ROOT1[□]
        ∇ ROOT1
[1]     X1←1÷1+(PI×YI÷MEWI)-((RI÷MEWI)+(FI÷(MEWI×YI))-(PI÷MEWI))
        ∇
        ∇
```

Fig. 4. ROOT and ROOT1 Programs for Exponential Systems

```
        ∇SOOK[□]
        ∇ SOOK
[1]     Q←(M,6)ρ0
[2]     ZN←1
[3]     AL1←0
[4]     AL2←0
[5]     G←Mρ0
[6]   LOOS:N←0
[7]     ZN←ZN+1
[8]   FON:N←N+1
[9]     AJ←ιO
[10]    J←1
[11]  FOM:J←J+1
[12]    →((|COEF[J-1])>0)/SAN
[13]    AJ←AJ,0
[14]    →SAND
[15]  SAN:AJ←AJ,(YY1[J]*AL1)×(YY2[J]*AL2)×(XX[J]*N)
[16]  SAND:→(J<4)/FOM
[17]    G[N+1]←+/COEF×AJ
[18]    →(N<(M-2))/FON
[19]    Q[;ZN]←G
[20]    →(ZN=2)/ZOOR
[21]    →(ZN=3)/FOOR
[22]    →(ZN=4)/KOOR
[23]    →COOK
[24]  FOOR:AL1←1
[25]    AL2←0
[26]    G[1]←0
[27]    G[M]←+/COEF×MJ
[28]    →LOOS
[29]  ZOOR:AL1←0
[30]    AL2←1
[31]    G[1]←+/COEF×LJ
[32]    G[M]←0
[33]    →LOOS
[34]  KOOR:AL1←1
[35]    AL2←1
[36]    G[1]←+/COEF×((YY1×YY2)[2 3 4])
[37]    J←1
[38]  SL:J←J+1
[39]    →((M×⍟XX[J])<65)/SLD
[40]    XX[J]←0
[41]  SLD:→(J<4)/SL
[42]    G[M]←+/COEF×((YY1×YY2×(XX*M-1))[2 3 4])
[43]    →LOOS
[44]  COOK:Q[;1]←((ιM)-(Mρ1))
[45]    Q[;6]←+/(0 1 1 1 1 0 /Q)
[46]    'STEADY STATE PROBABILITIES ARE:
[47]    '   NO        -00          -01          -10          -11'
[48]    3 0 16 ¯6 16 ¯6 16 ¯6 16 ¯6 16 ¯6 ▼Q[(1,(NρFLAG),1)/ιN+2;]
[49]    ' TOTAL ', 12 ¯6 16 ¯6 16 ¯6 16 ¯6 ▼+/≠Q[; 2 3 4 5]
[50]    'NBAR IS: ',▼NBAR←+/Q[;6]×Q[;1]
[51]    'ON THE AVERAGE, THE STORAGE IS ',(▼100×NBAR÷(M-1)),' PERCENT FULL'
[52]    ZOR←(RII÷PII)×+/Q[;2]+Q[;4]
        ∇
[53]  ∇
```

Fig. 5. SOOK Program for Exponential Systems

D. <u>Computer Outputs</u>

Outputs for the following two cases are shown in Figs. 6 and 7.

(1) $\mu_1 = 1$, $\mu_2 = 2$, $p_1 = 3$, $p_2 = 4$, $r_1 = 5$, $r_2 = 6$, $N = 6$

(2) $\mu_1 = 5$, $\mu_2 = 2$, $p_1 = 4$, $p_2 = 2$, $r_1 = 4$, $r_2 = 7$, $N = 6$

COE 1
MEW1 IS: 1
MEW2 IS: 2
P1 IS: 3
P2 IS: 4
R1 IS: 5
R2 IS: 6
STORAGE CAPACITY IS: 6
Y1 IS: 1.66667 5.27582 -1.11235 1.83653
Y2 IS: -1.5 -1.20687 3.58426 -1.3726
X IS: 1 0.0720474 6.33153 0.559578
COEFFICIENTS ARE: 0 0.0214335 -1.12368E-8 0.0615724
STEADY STATE PROBABILITIES ARE:

NO	-00	-01	-10	-11	TOTAL
0	0.00000E00	1.94673E-01	0.00000E00	2.91685E-01	4.86358E-01
1	3.29102E-02	4.91558E-02	5.51298E-02	9.66866E-02	2.33883E-01
2	1.91683E-02	2.65965E-02	3.48218E-02	4.93117E-02	1.29898E-01
3	1.07778E-02	1.48080E-02	1.97746E-02	2.72588E-02	7.26192E-02
4	6.01847E-03	8.22252E-03	1.11044E-02	1.52942E-02	4.06395E-02
5	3.26385E-03	4.22721E-03	6.33118E-03	8.97205E-03	2.27943E-02
6	0.00000E00	0.00000E00	6.15628E-03	7.65162E-03	1.38079E-02
TOTAL	7.21387E-02	2.97683E-01	1.33318E-01	4.96860E-01	

MEAN IS: 1.07091
ON THE AVERAGE, THE STORAGE IS 17.8486 PERCENT FULL
E1 IS: 0.308185
E2 IS: 0.154093
PRODUCTION RATE IS: 0.61637

Fig. 6. Output of COE 1 for Exponential Systems

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```
COE 0
MEW1 IS: 5
MEW2 IS: 2
P1 IS: 4
P2 IS: 2
R1 IS: 4
R2 IS: 7
STORAGE CAPACITY IS: 6
Y1 IS: 1  3.33925  -1.66907  0.829826
Y2 IS: 3.5  -1.1785  8.83815  3.84035
X IS: 1  0.29139  6.94214  1.42897
COEFFICIENTS ARE: 0  -0.000917738  -1.22052E-7  0.00369276
STEADY STATE PROBABILITIES ARE:
```

NO	-00	-01	-10	-11	TOTAL
0	0.00000E00	2.56678E-02	0.00000E00	1.53815E-02	4.10494E-02
6	0.00000E00	0.00000E00	1.01591E-01	3.01728E-01	4.03320E-01
TOTAL	5.83165E-02	2.40024E-01	1.54784E-01	5.46876E-01	

```
MEAN IS: 4.47993
ON THE AVERAGE, THE STORAGE IS 74.6655 PERCENT FULL
E1 IS: 0.74585
E2 IS: 1.86463
PRODUCTION RATE IS: 1.4917
```

Fig. 7. Output of COE 0 for Exponential Systems

APPENDIX 2

ERLANG MODEL COMPUTER PROGRAM

In this appendix we describe the use of a program for obtaining the steady state probabilities for the Erlangian model by matrix inversion. The model has been programmed in the APL computer language. We discuss how to use the program.

A. Use of the Program

Step 1

Get into the CMS computer system.

Step 2

Type:) LOAD ABA

This command means: load the workspace ABA from your private library.

Step 3

Insert the following inputs:

MEWI ← rate of service for machine 1

MEWII ← rate of service for machine 2

PI ← rate of failure for machine 1

PII ← rate of failure for machine 2

RI ← rate of repair for machine 1

RII ← rate of repair for machine 2

M ← (storage capacity) plus (one unit)

K ← number of service phases

Step 4

Type: EFFIC

To compute steady state probabilities by matrix inversion.

B. Listing of the Program

The listing of the computer program EFFIC is shown in Fig. 8.

C. Computer Output

A computer output for the case:

$$\mu_1 = 2, \ \mu_2 = 2, \ p_1 = 9, \ p_2 = 7, \ r_1 = 3, \ r_2 = 6, \ N = 6, \ K = 2$$

is shown in Fig. 9. It should be noted that the efficiency printed out by this program is E_2.

```
        ∇EFFIC
[133]   [□]
        ∇ EFFIC
[1]     'MEW1 IS: ',⍕MEWI
[2]     'MEW2 IS: ',⍕MEWII
[3]     'P1 IS: ',⍕PI
[4]     'P2 IS: ',⍕PII
[5]     'R1 IS: ',⍕RI
[6]     'R2 IS: ',⍕RII
[7]     'STORAGE CAPACITY IS: ',⍕M-1
[8]     'K IS: ',⍕K
[9]     P←(((1+(2×K)+K*2)×M),((1+(2×K)+K*2)×M))⍴0
[10]    A←(M,((1+(2×K)+K*2)+1))⍴0
[11]    N←0
[12] SI:N←N+1
[13]    J←0
[14] SJ:J←J+1
[15]    P[N;N+J×M]←PI×(N<M)
[16]    P[N;N+(K+J)×M]←PII×(N>1)
[17]    →(J<K)/SJ
[18]    P[N;N]←-(RI+RII)
[19]    →(N<M)/SI
[20]    NN←N
[21]    N←0
[22]    J←0
[23] TOJ:J←J+1
[24] TI:N←N+1
[25]    NN←NN+1
[26]    →(N>M)/BOR
[27]    P[NN;N]←RIX(N≤M)
[28]    P[NN;(K×M)+N-1]←MEWIX((N>1)∧(N≤M))
[29]    L←0
[30] TJ:L←L+1
[31]    P[NN;N+(L+2K)×M]←PIIX((N>1)∧(N≤M))
[32]    →(L<K)/TJ
[33]    →(N≤M)/POL
[34] BOR:→((N=(J+1)×M)∧(N<K×M))/TP
[35]    →(N=K×M)/FOS
[36]    P[NN;N]←MEWIX((N≥1+J×M)∧(N<(J+1)×M))
[37]    JJ←0
[38] TBJ:JJ←JJ+1
[39]    P[NN;(((((3+(J-1))×K)+JJ-1)×M)-M×(J-1))+N]←PIIX((N≥(J×M)+2)∧(N<(J+1)×M))
[40]    →(JJ<K)/TBJ
[41] POL:P[NN;NN]←(-RII)-(PI+MEWI)×(N≠M)
[42]    →(N≤M)/FOS
[43] TP:→(N<(J+1)×M)/TI
[44]    →(J<(K-1))/TOJ
[45] FOS:→(N<(K×M))/TI
[46]    N←0
[47]    J←0
[48] ZOJ:J←J+1
[49] ZI:N←N+1
[50]    NN←NN+1
[51]    →(N>M)/COR
[52]    P[NN;N]←RIIX(N≤M)
[53]    P[NN;1+N+(2×K)×M]←MEWIIX(N<M)
[54]    →(N=M)/FOK
[55]    L←-1
[56] ZL:L←L+1
[57]    P[NN;((1+(2+L)×K)×M)+N]←PI
[58]    →(L<K-1)/ZL
[59]    →(N≤M)/FOK
[60] COR:→(N=1+J×M)/NOS
[61]    P[NN;(K×M)+N]←MEWIIX((N>(J×M)+1)∧(N≤(J+1)×M))
[62]    JJ←-1
[63] ZBJ:JJ←JJ+1
[64]    P[NN;((1+(JJ+2)×K)×M)+N]←PIX((N>1+J×M)∧(N<(J+1)×M))
[65]    →(JJ<(K-1))/ZBJ
[66] FOK:P[NN;NN]←((-RI)-(PII+MEWII)×(N>1))
```

Fig. 8. EFFIC Program for Erlang Systems (page 1 of 2)

```
[67]    →((N=M)∧(K=1))/NOS
[68]    →(N<((J+1)×M)/ZI
[69]    →(J<(K-1))/ZOJ
[70] NOS:→(N<K×M)/ZI
[71]    I←0
[72] KOI:I←I+1
[73]    J←0
[74] KOJ:J←J+1
[75]    N←0
[76] KON:N←N+1
[77]    NN←NN+1
[78]    →((I>1)∧(N=M))/SOF
[79]    →((J>1)∧(N=1))/SOF
[80]    →((K=1)∧(N=M))/HO
[81]    F[NN;N+M+(I-1)×M]←RII×(N≤M)×(J=1)
[82]    F[NN;((K+J)×M)+N]←RI×(N≤M)×(I=1)
[83]    →(J>1)/SEF
[84]    F[NN;(3×K×M)+((I-1)×K×M)+1+N]←MEWII×(J=1)×(N<M)
[85]    →(I>J)/NEF
[86]    →DEF
[87] SEF:F[NN;(2×K×M)+((J-1)×M)+((I-1)×K×M)+N]←MEWII×(J>1)×(N≤M)
[88]    →(I>1)/NEF
[89] DEF:F[NN;((1+(K+1)×K)×M)+((J-1)×M)+N-1]←MEWI×(I=1)×(N>1)
[90]    →KEF
[91] NEF:F[NN;((1+K)×M)+((I-1)×K×M)+((J-1)×M)+N]←MEWI×(I>1)×(N<M)
[92] KEF:F[NN;NN]←-(MEWI+FI)×(((⌊(N÷M))-(N÷M))≠0)
[93]    F[NN;NN]←-(MEWII+FII)×(((⌊((N-1)÷M))-((N-1)÷M))≠0))+F[NN;NN]
[94]    →SOF
[95] SOF:→(N<M)/KON
[96]    →(J<K)/KOJ
[97]    →(I<K)/KOI
[98]    F[((1+(2×K)+K*2)×M)-1;]←((1+(2×K)+K*2)×M)ρ1
[99]    →HOB
[100] HO:F[NN;]←J
[101] HOB:E←((1+(2×K)+K*2)×M)ρ1
[102]    E←((⌈/F)>0)
[103]    F←E/F
[104]    F←E/⍉F
[105]    B←(+/E)ρ0
[106]    B[ρB]←1
[107]    AFI←B⌹F
[108]    →(K=1)/SOL
[109]    DS←((1+(2×K)+K*2)×M)ρ0
[110]    I←0
[111]    II←1
[112] TOS:I←I+1
[113]    →(E[I]=0)/TOS
[114]    DS[I]←AFI[II]
[115]    II←II+1
[116]    →(I<((1+(2×K)+K*2)×M)-1)/TOS
[117]    AFI←DS
[118] SOL:AFI←AFI×(AFI>1E¯6)
[119]    N←0
[120] LO:N←N+1
[121]    A[N;]←(N-1),AFI[N,N+(ι((2×K)+K*2))×M]
[122]    →(N<M)/LO
[123]    'STEADY STATE PROBABILITIES ARE:'
[124]    '        '
[125]    A
[126]    S1←0
[127]    KK←0
[128] SO:KK←KK+1
[129]    S1←(+/A[(1+ιM-1);2+K+KK×K])+S1
[130]    →((1+K+KK×K)<1+(2×K)+K*2)/SO
[131]    'EFFICIENCY IS: ',⍕S1
[132]    'PRODUCTION RATE IS: ',⍕MEWII×S1
        ∇
[133] ∇
```

Fig. 8 (continued)

```
        EFFIC
MEW1 IS: 2
MEW2 IS: 2
P1 IS: 9
P2 IS: 7
R1 IS: 3
R2 IS: 6
STORAGE CAPACITY IS: 6
K IS: 2
STEADY STATE PROBABILITIES ARE:
```

N	00	10	20	01	02	11	12	21	22
0	0	0	0	0.411	0	0.113	0	0.0207	0
1	0.0988	0.0279	0.00513	0.0691	0.0154	0.021	0.00442	0.00368	0.000809
2	0.0462	0.0134	0.00242	0.0319	0.00716	0.00928	0.00208	0.00167	0.000376
3	0.0215	0.00626	0.00113	0.0148	0.00333	0.00431	0.000968	0.000778	0.000175
4	0.01	0.00291	0.000525	0.0069	0.00155	0.00201	0.000451	0.000362	0.0000813
5	0.00464	0.00137	0.000247	0.00318	0.000718	0.000969	0.000213	0.000171	0.0000384
6	0	0.00297	0	0	0	0.00202	0.000456	0	0

```
EFFICIENCY IS: 0.0383
PRODUCTION RATE IS: 0.0765
```

Fig. 9   Output of EFFIC for an Erlang System.

EQUATION FOR Y_{11}—

Equation (A.1) for Y_{11} was obtained from equations (11.4) - (11.8) by means of the MACSYMA system (MACSYMA, 1977). The following notation applies to this equation.

$$A = Y_{11}$$

$$M1 = \mu_1$$

$$M2 = \mu_2$$

$$P1 = p_1$$

$$P2 = p_2$$

$$R1 = r_1$$

$$R2 = r_2$$

The computer printout of the equation is shown on the page following.

$$
(1 - A)\ A^{K-1}\ M1\ M2\ (A\ (R1 - P1 - M1) + M1)
$$

$$
/((1 - A^K)\ P1\ (A\ (P2 + P1 + M2 + M1) - M1)
$$

$$
(\frac{(1 - A)\ M2\ (P2 + P1 + M2 + M1)\ (A\ (R1 - P1 - M1) + M1)}{(1 - A^K)\ P1\ (A\ (P2 + P1 + M2 + M1) - M1)}
$$

$$
- \frac{(1 - A)\ M2\ (A\ (R1 - P1 - M1) + M1)}{A\ (1 - A^K)\ P1} - \frac{A\ M2\ R1}{A\ (P2 + P1 + M2 + M1) - M1}))
$$

$$
- P2\ (\frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})^{1-K}\ (1 - (\frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})^K)
$$

$$
((P2 + P1 + M2 + M1)\ (A\ (P2 + P1 + M2 + M1) - M1)
$$

$$
(1 - \frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})\ (\frac{A\ M2\ (R2 - P2 - M2)}{A\ (P2 + P1 + M2 + M1) - M1} + M2)
$$

$$
/(M2\ P2\ (1 - (\frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})^K))
$$

$$
- M1\ (A\ (P2 + P1 + M2 + M1) - M1)\ (1 - \frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})
$$

$$
(\frac{A\ M2\ (R2 - P2 - M2)}{A\ (P2 + P1 + M2 + M1) - M1} + M2)/(A\ M2\ P2
$$

$$
(1 - (\frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})^K)) - A\ R2)
$$

$$
/((A\ (P2 + P1 + M2 + M1) - M1)\ (1 - \frac{A\ M2}{A\ (P2 + P1 + M2 + M1) - M1})
$$

$$
(\frac{A\ M2\ (R2 - P2 - M2)}{A\ (P2 + P1 + M2 + M1) - M1} + M2))
$$

$$
= 0 \qquad\qquad (A.1)
$$

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