A New Technique for Identification and Control of Systems With Unknown Parameters

by

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Abstract

A new technique for identification and control of systems with unknown parameters is developed. The optimum open-loop control input for identification using an augmented Kalman filter is obtained from a set of necessary conditions that result when a cost function consisting of combinations of the control, the state, and the covariance matrix is minimized. A first-order gradient technique is developed to solve the nonlinear necessary conditions and is applied to simple examples of identification. The improved performance in identification and in the estimates of the states themselves leads to consideration of a new technique for closed-loop control of stochastic nonlinear systems. It is assumed that a linear perturbation estimator-controller combination can keep the system near a nominal trajectory. The given cost function is then expanded in a power-series around the nominal and in taking the expected value a deterministic cost results which is then minimized. The nominal open-loop control is determined from a set of necessary conditions that specify the nominal trajectory as a function of the deterministic cost and covariance matrices. A simple example is then given that shows a significant improvement in performance over the quadratic synthesis approach. Then

Chapter 7 gives the design of entry controller for the uncertain Mars atmosphere using this new control technique. Significant improvements in terminal position and velocity uncertainties result.

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Chapter 1

Introduction

1.1 Statement of the Problem

This thesis addresses the problem of optimum identification and control of systems with unknown parameters. Many systems have characteristics that are either unknown or highly variable. The control-system designer must take this into account in order to achieve satisfactory results.

Examples of systems where identification of unknown parameters is of main concern are the estimation of the center of mass in the LM during lunar ascent and estimation of the activation energy in a nuclear reactor (Wells, 1969). In these systems the controller must not only direct the controlled member to some desired state, but also learn the characteristics of the system.

In some situations a closed-loop control law is desirable for a system with unknown parameters. The controller must identify the unknowns and apply an appropriate response based on the present values of the quantities. Since practically every system has some unknown variables, a practical and efficient control-system design method can be a valuable contribution. Of course, not every system need be optimized with the unknown characteristics in mind; generally, it is a matter of engineering judgement and experience in determining just how critical these effects may be. However, a valuable tool would be the optimum control law that considers these parameters in order to provide a base line for performance of any suboptimal controller.

The basic intent of this thesis, then, may be stated as follows: develop a practical method for the determination of optimal identification and control for systems with unknown parameters. The general solution offered will, in fact, be applicable to a wider range of problems, that of open and closed-loop control for stochastic nonlinear systems since the identification and control of systems with unknown parameters will be shown to be a nonlinear problem.

1.2 Investigation Summary and Relationship to Previous Research

In order to determine optimum identification and control programs for systems with unknown parameters, it is appropriate to first consider the technique that will be used to identify the parameters in the face of noisy, incomplete measurements of the system state. Prior to 1963, identification techniques – such as cross-correlation measurements, sinusoidal response measurements, spectral-density measurements, and numerical deconvolution – were used in a basically off-line manner to estimate system parameters. Sage and Cuenod (1968) give a detailed summary of these approaches and the errors in each.

In 1963 Kopp and Orford used the newly developed Kalman filtering approach to give a practical method of doing the identification, both optimally and on-line. In this approach the Kalman filter state is enlarged to include the unknown parameters and this augmented filter approach has been used by Wells (1969 and 1970), as well as others, in a wide variety of practical identification problems. The on-line capability of the filter as well as its linearity will be useful in the development of a closed-loop control scheme for stochastic systems.

The Kalman filter itself was developed primarily by Kalman (1960 and 1961) although Swerling (1963) claims priority for the filter equations. A history of least-squares estimation from Gauss to Kalman is given by Sorenson (1970).

In Chapter 2 of this thesis it is shown how the Kalman filter is applied to stochastic systems with unknown parameters. Although the filter algorithm was originally derived for linear systems, in practice, since systems with unknown parameters are nonlinear, it is applied by linearization around the current estimate or around nominal conditions. When linearization is used, the covariance matrix of estimation errors becomes a function of the nominal conditions so that estimation performance is directly affected by their choice. In Chapter 2 it is shown how an optimization problem for optimum identification of unknown parameters in a dynamic system can be formulated to determine the best values of the nominal conditions.

By best nominal conditions is meant the optimum open-loop control input signal that will drive the state vector to its desired terminal condition while enabling the estimator to learn the characteristics of the system. The optimization comes about by attempting to minimize an artificial cost function that weights not only the energy put into the system but also the estimation uncertainty in the unknown parameters as reflected by some of the diagonal elements of the covariance matrix.

Other researchers have also formulated problems where estimation performance is of prime importance. In the field of navigation, optimum nominal trajectories can be designed to improve navigation information. The problem is nonlinear because of dependence of the system coefficient matrix or measurement matrix on the nominal trajectory. Vander Stoep (1968) and Sutherland (1966) considered trajectory shaping to improve navigation performance.

In the field of radar-signal design, Athans and Schweppe (1967) formulated an optimization problem to determine modulation signals resulting in minimum variance estimation when control is available over the measurement matrix. The cost function to be minimized was a linear function of the elements of the covariance matrix at the terminal time. Meier, Peschon, and Dressler (1969) treat a more general problem of control over a measurement subsystem within a feedback control system.

In the field of optimum identification, Nahi and Wallis (1969) attempted to find an optimal deterministic input for estimation of system parameters by postulating the existence of an efficient estimator. Analytical difficulties connected with their Bayesian approach are overcome by essentially separating the problem into estimation and control and defining a set of sensitivity equations that result in large computational burden in order to obtain a solution.

In this thesis, the form of the estimator is specified as that of Kalman and the matrix Riccati equation is viewed as another differential equation constraint. This approach leads to an easily formulated optimization problem with the restriction of an optimal linear estimator.

The cost function for identification formulated in Chapter 2 is to be minimized subject to the constraining differential equations describing the propagation of the state and the covariance matrix. In Chapter 3 the necessary conditions for optimization of performance indices involving functions of the state, the control, and a covariance matrix are derived. The approach also considers free and fixed terminal time problems and cases with constraints. It extends the work of Athans and Schweppe (1967), Sutherland (1966), and Vander Stoep (1968) who considered various special cases of the general optimization problem. The filter is not specified as Kalman.

The necessary conditions are shown to result in vector and matrix adjoint differential equations. Controllability conditions are derived for problems involving linear terminal constraints. Since the optimization problem is of necessity nonlinear, a numerical solution procedure is required. However, Chapter 3 presents two examples in which it is possible to obtain an analytical solution. The first example treats a simple integrator system driven by noise that depends on the output of the integrator. The optimum solutions that minimize control expenditure and the mean-squared deviation in the output are presented. The second example minimizes a combination of control expenditure and mean-squared state error at the terminal time for an integrator with a constant but unknown gain.

Chapter 4 considers numerical solution procedures for the vector-matrix two-point boundary value problem that results from the necessary conditions. A first-order gradient method is invented and applied to identification problems in Chapter 5. The original work on the gradient or steepest-descent method was done by Kelly (1960) and Bryson and Denham (1962) and applied to vector two-point boundary value problems. The application of the gradient method to problems involving vectors and matrices could be approached by partitioning the matrices into vectors. However, a much more convenient approach is developed which does not require the partitioning and allows easy computer programming. The result is no more complicated than the conventional gradient method.

Chapter 5 uses the optimization formulation developed in Chapter 2, the necessary conditions developed in Chapter 3, and the numerical solution procedure developed in Chapter 4 in two representative identification problems. These represent the first formal optimal identifiers using the techniques developed in the thesis.

In the first case the optimum identification of the inverse-time constant in a first-order system is formulated by minimizing the uncertainty in the time-constant estimate versus the amount of energy used in meeting the terminal constraints. It is shown that, with measurements proportional to system position, larger excursions of the state away from minimum-energy solutions in the direction of increasing values of the state result in improved estimates of the inverse-time constant at the terminal time. Furthermore, the uncertainty in the state estimate is less than that of the minimum-energy solution at the terminal time because of improved knowledge of the unknown parameter.

The second example considers optimum identification in a second-order system. The example verifies the improved estimation of the unknown parameters and of the system states (position and velocity) when the optimum identification procedure is used. By designing the open-loop control to improve estimation of the unknown parameters, the designer also improves the estimation of the system states at the terminal time. Chapter 5 also shows that rapid convergence to a near-optimum solution is achieved by using the numerical procedure developed in the thesis.

In Chapter 6 attention is directed to developing an optimum closed-loop control law for systems with unknown parameters. The examples in

Chapter 5 showed that it is possible to develop open-loop control laws that minimize an artificial cost function composed of, in part, estimation error and, in turn, decrease the estimation uncertainty in the system states. Such an approach needs modification when developing a closed-loop controller. First, because the designer would like to work with a less artificial cost function and, secondly, because the designer is interested in minimization of actual deviations in the states rather than estimation errors in the states. However, the foundation developed in Chapters 2, 3, and 4 is directly applicable to the problem.

Previous researchers have approached the problem in two ways. First, it is possible to study the effect of the unknown parameters on system performance and to try to design a controller so these effects are minimized. This is called the sensitivity approach -- see Kahne (1968). In this method the cost function is artificially augmented with sensitivity terms that relate how the cost function is affected by changes in the unknown parameters. Differential equations must be developed for each unknown parameter.

Three drawbacks to sensitivity theory are evident. First, how does one justify in physical terms an appropriate weighting to attach to the sensitivity measure? Second, this approach neglects statistical effects; in particular, statistics associated with the unknown parameters which may be available. Third, a vector differential equation must be computed for each unknown parameter.

The second approach is the adaptive approach as represented in the books by Sworder (1966) and Aoki (1967). In this case one attempts to make continuous measurements of system behavior to determine the dynamical characteristics and then adjust the controller parameters based on these measurements. For the general case of a stochastic nonlinear system and nonlinear measurements no practical efficient method exists for solving problems that have a realistic complexity.

The solution offered in Chapter 6 lies in-between the sensitivity and adaptive approaches and represents a practical method of control for systems with unknown parameters. The technique can handle a priori statistical information about the unknown parameters and does not require an artificial augmentation of the cost to cause the controller to consider the unknown parameters. The computation burden is large in that matrix equations must be developed in order to solve the problem. The controller is partially adaptive in the sense that the deviations in the unknown quantities are estimated and control action taken. However, the controller gains are determined from nominal values of the parameters and nominal values of their statistics rather than from the present values of the observed quantities. An advantage of the technique is that all feedback control and estimator gains may be precomputed.

The approach is based on using practical engineering assumptions to achieve a solution to the control problem. The system is nonlinear becase of the unknown parameters and is assumed subject to independent white noise. Some nonlinear measurements corrupted by white noise are available and are related to the state of the system. It is desired to minimize the expected value of a cost function that measures the performance of the system. Three assumptions are then made that permit a solution to this general problem.

It is assumed that a perturbation controller can be built that will keep the actual state vector near a pre-planned value so that the expected values of first-order state deviations are zero. Second, the perturbation controller is to be a linear function of estimates of these deviations. Third, the estimates are to be obtained from a linear filter. This implied separation of perturbation estimation and control is valid for the linearized system.

The first assumption allows an expansion of the cost function to be correct to second-order. Then, in taking the expected value, first-order terms are zero and the second-order terms are covariance matrices, Thus, the cost function is actually evaluated in terms of a deterministic part due to the pre-planned trajectory and calculatable covariance matrices due to the statistical effects. The cost function consisting of covariance matrices and deterministic terms is the same form as used in the open-loop controller design except that now a clear physical interpretation of the terms is available.

The cost function is then minimized using the necessary conditions developed in Chapter 3 and a numerical solution procedure based on Chapter 4 is developed.

Three important results appear in Chapter 6. First, the best linear filter turns out to be formally given as the Kalman filter. Second, the optimal perturbation controller is identical in form to that obtained by quadratic synthesis as given by Bryson and Ho (1969). These first two results are a direct consequence of the second and third assumptions above. The third and most important result shows that the necessary conditions defining the pre-planned trajectory specify the trajectory as a function of the covariance matrices as well as the deterministic part of the cost. This result is different from quadratic synthesis which picks the pre-planned trajectory on deterministic criteria alone and then uses perturbation estimation and control to follow it. Defining the procedure used in the thesis as the combined optimization approach, the control engineer has a set of necessary conditions that can be straightforwardly applied to practical design problems.

The most closely related research on this problem was performed by Denham (1964). He considers a slightly more general nonlinear state, where the noise does not enter additively,but with only terminal costs. His results are not applicable to systems linear in the control, since his expansion is in the Hamiltonian rather than the cost. He retains higher-order terms in the state-vector deviation equations which result in a set of extremely complicated necessary conditions that require calculation of the expectation of first-order quantities. Fitzgerald (1964) considered the same case as Denham with a more general noise model. Feldbaum (1965) calls this approach the dual-control problem. The analysis is in all cases extremely involved and has, to the author's knowledge, never been used on a realistic system. The approach used in Chapter 6 – of immediately transforming the cost function to a deterministic quantity and viewing the covariance matrices as additional constraints – leads to a particularly simple solution with a clear interpretation of the results for a wide variety of optimization problems.

The technique is illustrated by two sample problems. The first problem presents the design of a controller for a first-order system with an unknown time constant. For the criteria used, the quadratic synthesis approach would give 24.2% more cost and 97% more mean-squared terminal error over the combined optimization procedure. It is shown that the controller automatically designs the best controller to minimize the effects of the unknown time constant.

In Chapter 7 the second example is presented. The problem is concerned with achieving a desired set of terminal conditions after entry into the atmosphere of Mars. The unknown parameter is considered to be the atmospheric density. The problem dimension is of order five and represents a challenging test of the theory and the computational algorithms. It is shown that the combined optimization approach results in a substantial decrease in the uncertainty in the desired range over a quadratic-synthesis approach based on a constant-lift-to-drag-ratio flight. The latter results in approximately 25% more range error over the combined optimization results.

Finally, Chapter 8 presents what the author feels are the significant contributions of the thesis and recommendations for further research.

Chapter 2

Formulation of the Optimum Identification Problem

2.1 Introduction

The well-known Kalman linear-filter algorithm has proven to be a practical engineering solution to a wide class of estimation problems. The algorithm was originally derived for linear systems, but in practice, since most dynamic and measurement systems are nonlinear, the optimal filter has been applied to nonlinear systems by linearization around nominal conditions or by updating a linearization around the current estimate. When this technique of linearization is used, the covariance matrix of estimation errors becomes a function of the nominal conditions so that estimation performance is directly affected by their choice. This chapter shows how an optimization problem for optimum identification of unknown parameters in a dynamic system can be formulated to determine the best values of the nominal conditions.

Section 2.2 reviews the basic Kalman filter equations for linear systems; Section 2.3 describes the application of linear filtering to nonlinear systems. In Section 2.4 the identification of unknown parameters using Kalman filtering is shown to result in a nonlinear estimation problem which can then be formulated as an optimization problem in Section 2.5. In Section 2.6 reference is made to other technical problems in which optimization of estimation performance is the design criterion.

2.2 The Kalman Filter

The optimum linear filter developed primarily by Kalman (1960 and 1961) has proven to be a practical solution as well as the theoretical optimum filter for estimation in nonstationary linear stochastic systems. The dynamical system obeys

$$\dot{\underline{x}} = F(t)\,\underline{x} + G(t)\,\underline{u} + \underline{n} \qquad (2.2\text{-}1)$$

where

- $\underline{x}$ is the n-dimensional state vector
- $\underline{u}$ is the m-dimensional control input vector
- $\underline{n}$ is an n-dimensional vector of independent zero-mean white-noise processes
- $F(t)$ is the n x n plant coefficient matrix
- $G(t)$ is the n x m control input matrix

The second-order statistics of the driving noise are represented by

$$< \underline{n}(t) \; \underline{n}(t')^T > \; = \; Q(t) \; \delta(t-t') \qquad (2.2\text{-}2)$$

and the multidimensional probability density for the initial state at time zero is assumed Gaussian with a mean $< \underline{x}(0) >$. And a priori minimum-variance estimate of the state at t = 0 is denoted by a hat and

$$\hat{\underline{x}}(0) \; = \; < \underline{x}(0) > \qquad (2.2\text{-}3)$$

The statistics of this estimate are assumed to be Gaussian with

$$< \underline{x}(0) - \hat{\underline{x}}(0) > \; = \; 0 \qquad (2.2\text{-}4)$$

$$< (\underline{x}(0) - \hat{\underline{x}}(0)) \; (\underline{x}(0) - \hat{\underline{x}}(0))^T > \; = \; E(0) \qquad (2.2\text{-}5)$$

where E(0) is the covariance matrix of the error in the estimate at time zero.

With the start of the physical process described by Eq. 2.2-1, noisy continuous measurements of $\underline{x}$ are obtained from

$$\underline{m} = M(t)\ \underline{x} + \underline{v}(t) \tag{2.2-6}$$

where

$\underline{m}(t)$ is the m-dimensional measurement vector

$M(t)$ is an m x n observation matrix

$\underline{v}(t)$ is an m-dimensional independent white-noise vector with

$$< \underline{v}(t) > = 0 \tag{2.2-7}$$

$$< \underline{v}(t)\ \underline{v}(t')^T > = U(t)\ \delta(t-t') \tag{2.2-8}$$

The propagation of the initial estimate of the state given by Eq. 2.2-3 is according to Kalman:

$$\dot{\hat{\underline{x}}} = F\hat{\underline{x}} + G\underline{u} + K(t)\left[\underline{m} - M\hat{\underline{x}}\right] \tag{2.2-9}$$

The optimal filter is thus a model of the system that is linearly corrected with the difference between the observed measurement and the filter's estimate of the measurement. The gain matrix K(t) is obtained from

$$K(t) = E\ M^T\ U^{-1} \tag{2.2-10}$$

where the covariance matrix is integrated from its initial value given by Eq. 2.2-5 according to

$$\dot{E} = F\,E + E\,F^T + Q - E\,M^T\,U^{-1}\,M\,E \tag{2.2-11}$$

A most valuable property of the filter is that this choice of the gain matrix results in the estimate and the error in the estimate being uncorrelated for all time if the initial correlation is zero. However, if errors are made in modeling the physical plant or in the assumptions for the statistics, the filter is no longer optimal and additional equations can be developed that determine the filter's performance. (See Chapter 4 of Leondes, 1970.)

The estimation error as represented by the covariance matrix is completely independent of the estimate of the state, the control input, and the actual state. It is, therefore, not possible to control the quality of the estimation by changing these variables. In fact, the optimal gains depend only on time and may be precomputed before the start of the process. The Kalman filter can be applied to nonlinear systems and in that case it will be shown that it is possible to control the quality of estimation.

2.3 Application of the Kalman Filter to Nonlinear Systems

The classical application of Kalman filtering to nonlinear stochastic systems of the form

$$\dot{\underline{x}}^a = \underline{f}\,(\underline{x}^a , \underline{u}^a , t) + \underline{n}(t) \tag{2.3-1}$$

with nonlinear measurements

$$\underline{m}^a = \underline{h}\,(\underline{x}^a , \underline{u}^a , t) + \underline{v}(t) \tag{2.3-2}$$

was in the field of astronautical guidance (Smith et al., 1962). However, many other important engineering problems are represented by nonlinear processes and the applications of "quasi-linear" optimal filtering have been extensive.

In this technique a reference trajectory and nominal control are assumed and Eq. 2.3-1 and 2.3-2 are linearized to first-order around these conditions:

$$\delta\dot{\underline{x}} = \underline{f}_x\, \delta\underline{x} + \underline{f}_u\, \delta\underline{u} + \underline{n}(t) \tag{2.3-3}$$

and

$$\delta\underline{m} = \underline{h}_x\, \delta\underline{x} + \underline{h}_u\, \delta\underline{u} + \underline{v}(t) \tag{2.3-4}$$

where the partial derivatives are evaluated on the nominal conditions. Strictly speaking, $\underline{n}(t)$ and $\underline{v}(t)$ should be written as $\delta \underline{n}(t)$ and $\delta \underline{v}(t)$, but the statistics of the variations are the same as those of the quantities so no distinction will be made. Their statistics are

$$< \underline{n}(t)\, \underline{n}(t')^T > = Q\, \delta(t-t')\ ,\quad < \underline{v}(t)\, \underline{v}(t')^T > = U\, \delta(t-t') \qquad (2.3\text{-}5)$$

Also define

$$F = \underline{f}_{\underline{x}} \qquad (2.3\text{-}6)$$

$$G = \underline{f}_{\underline{u}} \qquad (2.3\text{-}7)$$

$$M = \underline{h}_{\underline{x}} \qquad (2.3\text{-}8)$$

$$M' = \underline{h}_{\underline{u}} \qquad (2.3\text{-}9)$$

so that Eq. 2.3-3 and 2.3-4 become

$$\delta \dot{\underline{x}} = F\, \delta \underline{x} + G\, \delta \underline{u} + \underline{n} \qquad (2.3\text{-}10)$$

$$\delta \underline{m} = M\, \delta \underline{x} + M'\, \delta \underline{u} + \underline{v} \qquad (2.3\text{-}11)$$

Assuming the perturbations from the reference trajectory are small allows the construction of a Kalman filter since Eq. 2.3-10 and 2.3-11 are linear equations. The filter equations are

$$\delta \dot{\hat{\underline{x}}} = F\, \delta \hat{\underline{x}} + G\, \delta \underline{u} + K\, (\delta \underline{m} - M\, \delta \hat{\underline{x}} - M'\, \delta \underline{u}) \qquad (2.3\text{-}12)$$

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \qquad (2.3\text{-}13)$$

and

$$K = E M^T U^{-1} \tag{2.3-14}$$

Two important results can be observed from Eq. 2.3-9 – 2.3-14. First, calculation of the gain matrix K depends only on the reference conditions. It is therefore possible, as in the linear estimation case, to compute the optimal gains before the start of the process.

The remainder of this thesis rests upon the second observation that the quality of the estimation, as represented by E(t), depends on the choice of nominal conditions used in evaluating the partial derivatives. Thus, by control of the nominal conditions one can, in fact, control the estimation performance. This is not possible for the linear system discussed in Section 2.2.

In a number of applications it is possible that the optimal filter linearized around a reference solution will suffer degraded performance as time involves if the actual trajectory is not "close" to the nominal. If this is determined during the design of the system, a convenient estimation technique is to linearize around the present estimate of the state and the present control. The filter equations have the form

$$\dot{\hat{\underline{x}}} = \underline{f}\,(\hat{\underline{x}}\,,\,\underline{u}\,,\,t) + K\left[\underline{m} - \underline{h}\,(\hat{\underline{x}}\,,\,\underline{u}\,,\,t)\right] \tag{2.3-15}$$

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \tag{2.3-16}$$

and

$$K = E M^T U^{-1} \tag{2.3-17}$$

where the partial derivative matrices are now evaluated using the best estimate of the present state $\underline{x}$ and the value of the present control. Unfortunately, in this formulation it is not possible to precompute the optimal gains, since the coefficients are evaluated on the present estimate of the actual state. It is still possible to control the quality of the estimation by choice of the nominal control.

Some practical nonlinear systems may not lend themselves to accurate linearized descriptions and the linearized Kalman estimator of either type may not be adequate. A number of researchers have proposed alternate formulations for these situations; unfortunately, these solutions tend to be very impractical, since they typically depend on the calculation of higher-order derivatives or tensors and may result in growing memory filters. (See for example; Phaneuf, 1968 and Leondes, 1970.) They all, however, have associated with them a covariance matrix which depends on nominal conditions. The first part of this thesis will use linear filters only; however, the general results of techniques developed to improve estimation performance by control of the nominal conditions will be applicable to any estimation scheme whose performance is measured by the propagation of the covariance matrix.

2.4 Identification Using Augmented Kalman Filters

The techniques for identification of linear-system parameters are numerously described in the literature. These techniques involve, for example, cross-correlation measurements, sinusoidal response measurements, spectral-density measurements, numerical deconvolution, and learning models. (See for example; Sage and Cuenod, 1968.) Most of the recent attention has been centered around the techniques developed by Kopp and Orford (1963) that enlarge the state space of a Kalman filter to include the unknown parameters. In augmented Kalman estimation the computations are done on-line and the filter has been applied to a wide variety of practical identification problems (Wells, 1969 and 1970). In this section, aspects of the identification problem using augmented Kalman filters are investigated.

For simplicity, assume the system under study may be modeled as

$$\dot{\underline{x}}_1^a = F_1^a \underline{x}_1^a + G_1^a \underline{u}_1^a + \underline{n}_1 \qquad (2.4\text{-}1)$$

The word "identification" is taken to mean the estimation of unknown system parameters; for example, the location of a pole in a linear system. Thus, the identification problem deals with the estimation of some of the elements in F_1^a or G_1^a in Eq. 2.4-1. Let $\underline{b}^a$ represent a p-dimensional vector consisting of the unknown parameters. In augmented filtering the equations (for unknown constant parameters)

$$\dot{\underline{b}}^a = 0 \tag{2.4-2}$$

are considered to be part of a new state vector of dimension s (s = n+p):

$$\underline{x}^a = \begin{bmatrix} \underline{x}_1^a \\ \underline{b}^a \end{bmatrix} \tag{2.4-3}$$

Given noisy measurements of $\underline{x}^a$, it may be possible to estimate the unknown parameters (Ho and Lee, 1964). For example, suppose it is desired to estimate the inverse time constant b^a in the first-order system

$$\dot{x}_1^a = -b^a x_1^a + u_1^a + n_1 \tag{2.4-4}$$

using noisy measurements of the form

$$m = h x_1^a + v_1 \tag{2.4-5}$$

An augmented filter approach is to design a Kalman filter such that the state variables are the deviations of x_1^a and b^a away from their nominal values x and b:

$$\delta\hat{\underline{x}} = \begin{bmatrix} \delta\hat{x}_1 \\ \delta\hat{b} \end{bmatrix} \tag{2.4-6}$$

where the original system is now considered to be nonlinear

$$\dot{\underline{x}}^a = \underline{f}(\underline{x}^a, \underline{u}^a) + \underline{n}(t) = \begin{bmatrix} \dot{x}_1^a \\ \dot{b}^a \end{bmatrix} = \begin{bmatrix} -b^a x_1^a \\ 0 \end{bmatrix} + \begin{bmatrix} u_1^a \\ 0 \end{bmatrix} + \begin{bmatrix} n_1 \\ 0 \end{bmatrix} \tag{2.4-7}$$

The nominal value x is found from Eq. 2.4-4 using nominal values of b and u with the noise set to zero.

Assuming a Gaussian distribution for the initial estimation error, the covariance matrix obeys

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \qquad (2.4\text{-}8)$$

where

$$M = \begin{bmatrix} h & 0 \end{bmatrix} \qquad (2.4\text{-}9)$$

and

$$F = \underline{f}_{\underline{x}} = \begin{bmatrix} -b & -x \\ 0 & 0 \end{bmatrix} \qquad (2.4\text{-}10)$$

Alternately, an estimator of the present value of $\underline{x}^a$ can be designed according to Section 2.3. The identification problem is clearly nonlinear. From Eq. 2.4-4 and 2.4-5 it is evident that more information about b^a is obtained with larger values of x_1^a. If x_1^a were zero, no information at all about b^a could be obtained. Thus, the character of identification is that the quality of estimation of unknown parameters is related to the magnitude of the other state variables. This nonlinear nature of the problem allows the formulation of an optimum identification strategy.

2.5 Formulation of the Optimum Identification Problem

The previous section illustrated the property that identification is basically a nonlinear estimation problem. If the quality of the identification process using a Kalman filter is measured by the diagonal elements of the covariance matrix, it is possible to control the quality by proper choice of the nominal state variables and control inputs since the partial-derivative matrices are evaluated on these nominal conditions.

The nominal s-dimensional augmented state vector obeys

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}\,,\,\underline{u}\,,\,t) \qquad (2.5\text{-}1)$$

with an assumed given initial condition $\underline{x}(0)$. As a somewhat arbitrary distinction, two types of identification are considered. In the first type, which might be called equilibrium identification, some components of the nominal state vector are to be returned to their original value after the identification process. An example might be the deflection of airplane elevators during equilibrium flight in order to determine the dynamic parameters of the aircraft. In the second type of problem, which might be called transient identification, some components of the nominal state are to be driven to some desired terminal conditions during the identification process. Both problems are summarized by the terminal condition at time T:

$$x_i(T) \quad \text{specified} \; ; \quad i = 1, \ldots, q \; ; \; q \leq n \tag{2.5-2}$$

The propagation of the s-dimensional covariance matrix governing the quality of identification of the augmented state deviation vector associated with Eq. 2.5-1 is governed by

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \tag{2.5-3}$$

with

$$E(0) \quad \text{given} \tag{2.5-4}$$

and the partial derivative matrices are evaluated on the nominal conditions for the state vector, the control, and the unknown parameters.

Associated with the identification problem is a cost function J to be minimized. For example,

$$J = \text{trace}\left[C\, E(T)\right] + \int_0^T L(\underline{x}, \underline{u}, E, t)\, dt \tag{2.5-5}$$

The matrix C would determine which elements of E(T) would be minimized in relation to the integral part of the performance index. Typically, L might consist of terms involving $\underline{u}\,\underline{u}^T$ – the amount of energy used in the identification process – and could involve weighting elements of $\underline{x}$ and E. For unspecified terminal-time problems, the cost could involve the amount of time used.

The optimization problem is then to minimize Eq. 2.5-5 subject to the vector and matrix differential constraints Eq. 2.5-1 and Eq. 2.5-3 with a specified terminal condition Eq. 2.5-2. The necessary conditions for optimality for problems of this type are presented in Chapter 3. Since the problem is nonlinear, appropriate numerical solution techniques are developed in Chapter 4. Some illustrative identification problems are presented in Chapter 5. Before addressing these problems, we present some problems from other engineering fields which can be formulated as optimization problems involving covariance matrices.

2.6 Other Optimization Problems

One need not look far from the identification problem to discover fields where optimization of estimation performance is of prime importance. In the field of navigation, optimum nominal trajectories can be designed to improve navigation information. The problem is nonlinear because of dependence of F or M on the nominal trajectory. Vander Stoep (1968) and Sutherland (1966) considered trajectory shaping to improve navigation performance.

In the field of radar-signal design, Athans and Schweppe (1967) formulated an optimization problem to determine modulation signals resulting in minimum variance estimation when control is available over the measurement matrix M. The cost function to be minimized was a linear function of the elements of the covariance matrix at the terminal time.

Techniques developed in this thesis for optimum identification are directly applicable to these areas of research. The presentation of the necessary conditions in the next chapter and numerical solution techniques in the following chapter are derived for problems involving optimization of general performance indices with vector and matrix differential equations as constraints without specification of the particular application. In Chapter 5 specific examples from the field of identification are presented.

2.7 Summary

This chapter has reviewed the application of Kalman filter techniques to linear and nonlinear problems with attention directed to identification in systems with unknown parameters. Since that problem is nonlinear, it is possible to formulate an optimization problem involving minimization of estimation error. The next chapter derives the necessary conditions for optimality using the calculus-of-variations approach.

Chapter 3

Derivation of the Necessary Conditions

3.1 Introduction

In the previous chapter it was shown how optimization problems involving vector and matrix differential equations can be formulated in the study of identification of system parameters. It was indicated that other fields of technical interest also involve problems with optimization of systems of matrices and vectors. The necessary conditions for optimization of performance indices involving functions of the state, the control, and the covariance matrix are derived in this chapter. It is assumed that all differentiability conditions necessary for the application of the minimum principle of Pontryagin are satisfied (Athans and Falb, 1966). The approach is general rather than tied to specific problems and extends the work of Athans and Schweppe (1967), Sutherland (1968), and Vander Stoep (1968) who considered various special cases of the problem.

In Section 3. 2 the necessary conditions for optimality in problems with integral cost functions and terminal cost functions on the state and covariance matrix at a fixed terminal time are derived by means of the calculus-of-variations approach. The necessary conditions are shown to result in vector and matrix adjoint differential equations. For problems involving linear terminal constraints, the controllability conditions and adjoint variable relationships are derived in Section 3. 3. Sections 3. 4 and 3. 5 present two examples in which it is possible to obtain an analytical solution to the optimization problem. The first example treats a simple integrator system driven by noise that depends on the output of the integrator. The optimum solutions that minimize control expenditure and the mean-squared deviation in the output are presented. The second example minimizes a combination of control expenditure and mean-squared state error at the terminal time for an integrator with a constant but unknown gain. Section 3. 6 presents the transversality condition for problems involving free terminal time; Section 3. 7 discusses problems with inequality constraints on the controls and the state variables.

3.2 Terminal Cost Functions and Fixed Terminal Time

The cost function is

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T L\,(\underline{x},\,\underline{u},\,E,\,t)\,dt \qquad (3.2\text{-}1)$$

where:

C is a given constant positive symmetric matrix
k is a terminal cost involving some of the elements of $\underline{x}(T)$
L is the scalar penalty function
T is the fixed terminal time

The state vector is constrained to obey the n-dimensional differential equation

$$\dot{\underline{x}} = \underline{f}\,(\underline{x},\,\underline{u},\,t) \qquad (3.2\text{-}2)$$

where u is an m-dimensional control vector and

$$\underline{x}(0) \quad \text{specified} \qquad (3.2\text{-}3)$$

The symmetric s-dimensional covariance matrix obeys a differential equation of the form

$$\dot{E} = V\,(\underline{x},\,\underline{u},\,E,\,t) \qquad (3.2\text{-}4)$$

with

$$E(0) \quad \text{specified} \qquad (3.2\text{-}5)$$

The covariance matrix may result from any linear or nonlinear estimation or control problem.

The minimization of Eq. 3.2-1 subject to the constraints and boundary conditions Eq. 3.2.2 - 3.2-5 may be accomplished by the calculus-of-variations. Adjoin to the cost function J, the system equations by means of arbitrary multipliers

$\underline{p}$ for the n-dimensional vector $\underline{x}$
P for the s-dimensional matrix E.

Then

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T \left[L + \underline{p}^T(\underline{f} - \dot{\underline{x}}) + \sum_{i=1}^{s}\sum_{j=1}^{s} P_{ij}(V_{ij} - \dot{E}_{ij})\right] dt \qquad (3.2\text{-}6)$$

A more convenient notation is to write the double summation as a trace operation

$$\sum_{i=1}^{s}\sum_{j=1}^{s} P_{ij}(V_{ij} - \dot{E}_{ij}) = \mathrm{tr}\left[P(V - \dot{E})^T\right] \qquad (3.2\text{-}7)$$

Since V and $\dot{E}$ are symmetric, Eq. 3.2-6 becomes

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T \left\{L + \underline{p}^T(\underline{f} - \dot{\underline{x}}) + \mathrm{tr}\left[P(V - \dot{E})\right]\right\} dt \qquad (3.2\text{-}8)$$

Using

$$\mathrm{tr}\left[P(V - \dot{E})\right] = \mathrm{tr}(PV) - \mathrm{tr}(P\dot{E}) \qquad (3.2\text{-}9)$$

define the scalar Hamiltonian H as

$$H = L + \underline{p}^T\underline{f} + \mathrm{tr}(PV) \qquad (3.2\text{-}10)$$

then Eq. 3.2-8 becomes

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T \left[H - \underline{p}^T\dot{\underline{x}} - \mathrm{tr}(P\dot{E})\right] dt \qquad (3.2\text{-}11)$$

Using the fact that

$$\frac{d}{dt} \text{tr}(PE) = \text{tr}\left[\frac{d}{dt}(PE)\right] = \text{tr}(P\dot{E}) + \text{tr}(\dot{P}E) \qquad (3.2\text{-}12)$$

Equation 3.2-11 can be integrated by parts

$$\begin{aligned} J = {} & \text{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \underline{p}(0)^T \underline{x}(0) - p(T)^T \underline{x}(T) \\ & - \text{tr}\left[P(T)\,E(T)\right] + \text{tr}\left[P(0)\,E(0)\right] \\ & + \int_0^T \left[H + \dot{\underline{p}}^T \underline{x} + \text{tr}(P\dot{E})\right] dt \end{aligned} \qquad (3.2\text{-}13)$$

A variation in the control $\delta \underline{u}$ causes first-order changes in the cost δJ, the state $\delta \underline{x}$, the terminal state $\delta \underline{x}(T)$, the covariance matrix δE, and $\delta E(T)$. With fixed initial conditions, E(0) and x(0), Eq. 3.2-13 becomes

$$\begin{aligned} \delta J = {} & \text{tr}\left[C\;\delta E(T)\right] + k_{\underline{x}}\;\delta \underline{x}(T) \\ & - \underline{p}^T\,\delta \underline{x}(T) - \text{tr}\left[P(T)\;\delta E(T)\right] \\ & + \int_0^T \left\{(H_{\underline{x}} + \dot{\underline{p}}^T)\,\delta \underline{x} + \text{tr}\left[(H_E + \dot{P})\;\delta E\right] + H_{\underline{u}}\;\delta \underline{u}\right\} dt \end{aligned} \qquad (3.2\text{-}14)$$

For convenience, require the arbitrary adjoint variables to satisfy

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \qquad (3.2\text{-}15)$$

and

$$\dot{P} = -H_E \quad , \quad P(T) = C \tag{3.2-16}$$

Note, since C and H_E are symmetric, then P is also symmetric.

Equations 3.2-14 now reduces to

$$\delta J = \int_o^T H_{\underline{u}} \, \delta \underline{u} \, dt \tag{3.2-17}$$

and for arbitrary variations in $\underline{u}(t)$, optimality requires

$$H_{\underline{u}} = 0 \tag{3.2-18}$$

If $\underline{u}$ is constrained then, of course, the optimal control is the one that absolutely minimizes H rather than satisfying Eq. 3.2-18.

Thus, the necessary conditions for minimization of

$$J = tr\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_o^T L\,(\underline{x}, \underline{u}, E, t)\,dt \tag{3.2-19}$$

with constraints

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, \underline{u}, t) \quad , \quad \underline{x}(0) \text{ given} \tag{3.2-20}$$

and

$$\dot{E} = V\,(\underline{x}, \underline{u}, E, t) \quad , \quad E(0) \text{ given} \tag{3.2-21}$$

are

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \tag{3.2-22}$$

and

$$\dot{P} = -H_E \quad , \quad P(T) = C \tag{3.2-23}$$

where

$$H = L + \underline{p}^T \underline{f} + tr\,(P\,\dot{E}) \tag{3.2-24}$$

and for optimality

$$H_{\underline{u}} = 0 = L_{\underline{u}} + \underline{p}^T \underline{f}_{\underline{u}} + \left[tr\,(P\,\dot{E})\right]_{\underline{u}} \tag{3.2-25}$$

The optimal control of dimension m is found by solving the m equations represented by Eq. 3.2-25. These values are substituted into Eq. 3.2-20 through 3.2-23 resulting in a two-point boundary value problem (TPBVP) of dimension $2n + 2s^2$. Using the fact that E and P are symmetric, the problem can be reduced to dimension $2n^2 + s^2 + s$. An appropriate numerical procedure to solve the nonlinear TPBVP is presented in the next chapter. Appendix B gives a list for derivatives of traces of matrices.

For illustration, when the covariance matrix represents an optimal estimator

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \tag{3.2-26}$$

and

$$H = L + \underline{p}^T \underline{f} + \operatorname{tr}(P \dot{E}) \tag{3.2-27}$$

the adjoint variables satisfy

$$\dot{\underline{p}} = - H_{\underline{x}}^T = - L_{\underline{x}}^T - \underline{f}_{\underline{x}}^T \underline{p} - \left[\operatorname{tr}(P \dot{E})\right]_{\underline{x}}^T \tag{3.2-28}$$

and

$$\dot{P} = - H_E = - L_E - (F - E M^T U^{-1} M)^T P - P (F - E M^T U^{-1} M) \tag{3.2-29}$$

When the covariance matrix represents a linear filter with a gain K independent of E,

$$\dot{E} = (F - K M) E + E (F - K M)^T + K U K^T + Q \tag{3.2-30}$$

then

$$\dot{\underline{p}} = - L_{\underline{x}}^T - \underline{f}_{\underline{x}}^T \underline{p} - \left[\operatorname{tr}(P \dot{E})\right]_{\underline{x}}^T \tag{3.2-31}$$

and

$$\dot{P} = - L_E - (F - K M)^T P - P (F - K M) \tag{3.2-32}$$

For other filtering processes described by a covariance matrix suitable derivatives can be defined.

In the next section, the necessary conditions for optimization with terminal constraints are considered.

3.3 Linear Terminal Constraints on $\underline{x}(T)$ and Controllability

It is assumed that the first q components of the state vector at the terminal time are specified. Then, the optimization problem is to minimize the performance index J by choosing a control vector $\underline{u}(t)$ that insures the correct terminal state $\underline{x}(T)$. If x_q (the q-th component of the vector $\underline{x}$) is specified at t = T, it follows that the admissable variations must produce $\delta x_q(T) = 0$ in Eq. 3.2-14. Thus, it is not necessary that

$$k_{x_q} - p_q(T) = 0 \qquad (3.3\text{-}1)$$

Essentially, this boundary condition has been traded for another, $x_q(T)$ given, so that the TPBVP still has 2n boundary conditions. The adjoint variables corresponding to the specified values of $x_q(T)$ are unspecified, but are not arbitrary as will be shown.

Besides the performance index J, which is still to be minimized, a q-dimensional vector performance index $\underline{z}$ is defined as

$$\underline{z}(T) = \begin{bmatrix} x_1(T) \\ \cdots \\ x_q(T) \end{bmatrix} = \underline{x}_q(T) \qquad (3.3\text{-}2)$$

As occurred with J, any variation of the control modifies the value of this performance index. Since $\underline{z}(T)$ is specified, it is required that

$$\delta \underline{z}(T) = 0 \quad \text{for matching terminal constraints} \qquad (3.3\text{-}3)$$

and

$$\delta J = 0 \qquad \text{for minimizing } J \qquad (3.3\text{-}4)$$

Because z(T) is specified, $\delta \underline{u}(t)$ is not completely arbitrary; to determine the admissable variations a set of influence functions for $\underline{z}(T)$ are first determined.

Adjoin to Eq. 3.3-2 the system constraints

$$\underline{z}(T) = \underline{x}_q(T) + \int_o^T R^T (\underline{f} - \dot{\underline{x}})\, dt \tag{3.3-5}$$

where R is a matrix of dimension n x q.

Define a q-dimensional vector $\underline{H}$

$$\underline{H} = R^T \underline{f} \tag{3.3-6}$$

and integrate Eq. 3.3-5 by parts

$$\underline{z}(T) = \underline{x}_q(T) - R(T)^T \underline{x}(T) + R(0)^T \underline{x}(0)$$

$$+ \int_o^T (\underline{H} + \dot{R}^T \underline{x})\, dt \tag{3.3-7}$$

A variation in control produces

$$\delta \underline{z}(T) = \delta \underline{x}_q(T) - R(T)\, \delta \underline{x}(T) + R(0)^T \delta \underline{x}(0)$$

$$+ \int_o^T \left[(H_{\underline{x}} + \dot{R}^T)\, \delta \underline{x} + H_{\underline{u}}\, \delta \underline{u} \right] dt \tag{3.3-8}$$

With $\underline{x}(0)$ specified, we choose

$$\dot{R} = - \underline{H}_{\underline{x}}^T \tag{3.3-9}$$

with boundary condition at t = T,

$$R_{ij} = \begin{Bmatrix} 1 \text{ for } i = j \\ \\ 0 \text{ otherwise} \end{Bmatrix} \quad \text{where} \quad \begin{Bmatrix} i = 1, . . . , n \\ \\ j = i, . . . , q \end{Bmatrix} \tag{3.3-10}$$

which results in

$$\delta \underline{z}(T) = \int_o^T \underline{H}_{\underline{u}}\, \delta \underline{u}(t)\, dt \tag{3.3-11}$$

Equation 3.3-11 represents the effects of $\delta \underline{u}$ on the changes in the boundary conditions. Now multiply Eq. 3.3-11 by a row vector $\underline{\ell}^T$ and add the result to Eq. 3.2-17

$$\delta J + \underline{\ell}^T \delta \underline{z}(T) = \int_o^T \left[H_{\underline{u}} + \underline{\ell}^T \underline{H}_{\underline{u}} \right] \delta \underline{u}\, dt \tag{3.3-12}$$

But, by definition

$$\underline{H}_{\underline{u}} = \underline{L}_{\underline{u}} + \underline{p}^T \underline{f}_{\underline{u}} + \left[tr\ (P\ \dot{E}) \right]_{\underline{u}} \tag{3.3-13}$$

and

$$\underline{H}_{\underline{u}} = R^T \underline{f}_{\underline{u}} \tag{3.3-14}$$

Then Eq. 3.3-12 becomes

$$\delta J + \underline{\ell}^T \delta \underline{z}(T) = \int_0^T \left\{ L_{\underline{u}} + (\underline{p} + R\underline{\ell})^T \underline{f}_{\underline{u}} + \left[tr\ (P\dot{E}) \right]_{\underline{u}} \right\} \delta \underline{u}\ dt \tag{3.3-15}$$

Now choose a $\delta \underline{u}$ that decreases J; i.e., produces $\delta J < 0$ and satisfies the q terminal constraints. An appropriate choice for $\delta \underline{u}$ is

$$\delta \underline{u} = - w \left\{ L_{\underline{u}} + (\underline{p} + R\underline{\ell})^T \underline{f}_{\underline{u}} + \left[tr\ (P\ \dot{E}) \right]_{\underline{u}} \right\}^T \tag{3.3-16}$$

where w is a positive scalar constant. Then Eq. 3.3-15 becomes

$$\delta J + \underline{\ell}^T \delta \underline{z}(T) = -w \int_0^T \left\| L_{\underline{u}} + (\underline{p} + R\underline{\ell})^T \underline{f}_{\underline{u}} + \left[tr\ (P\dot{E}) \right]_{\underline{u}} \right\|^2 dt \tag{3.3-17}$$

which is negative unless the integrand vanishes over the whole integration interval. In that case $\delta \underline{u}$ is zero and the optimal control has been found.

Next, determine $\underline{\ell}$ so as to match the terminal constraints by substitution of Eq. 3.3-16 in Eq. 3.3-11.

$$0 = \delta \underline{z}(T) = \int_0^T R^T \underline{f}_{\underline{u}} \left\{ \left[tr\ (P\dot{E}) \right]_{\underline{u}}^T + \underline{f}_{\underline{u}}^T (\underline{p} + R\underline{\ell}) + L_{\underline{u}}^T \right\} dt \tag{3.3-18}$$

Define the q-dimensional vector $\underline{g}$

$$\underline{g} = \int_0^T R^T \underline{f}_{\underline{u}} \left\{ \left[tr\ (P\dot{E}) \right]_{\underline{u}}^T + \underline{f}_{\underline{u}}^T \underline{p} + L_{\underline{u}}^T \right\} dt \tag{3.3-19}$$

and the q x q matrix D

$$D = \int_0^T R^T \underline{f}_{\underline{u}} \underline{f}_{\underline{u}}^T R\ dt \tag{3.3-20}$$

then Eq. 3.3-18 becomes

$$\underline{g} + D\underline{\ell} = 0 \tag{3.3-21}$$

If D^{-1} exists (the controllability condition), it is possible to determine $\underline{\ell}$ such that the terminal constraints can be met.

Thus, a $\delta\underline{u}(t)$ history has been constructed that decreases the performance index and satisfies the terminal constraints. From Eq. 3.3-17 the only time the performance index cannot be decreased is when the integrand is zero. Thus, a stationary solution requires

$$L_{\underline{u}} + (\underline{p}^T + \underline{\ell}^T R^T)\, \underline{f}_{\underline{u}} + \left[\mathrm{tr}\,(P\,\dot{E})\right]_{\underline{u}} = 0 \tag{3.3-22}$$

Since the influence equations are linear, the necessary conditions may be summarized as follows:

For minimization of

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T L\,(\underline{x}, \underline{u}, E, t)\, dt \tag{3.3-23}$$

with

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, \underline{u}, t) \quad , \quad \underline{x}(0) \text{ and } x_1(T) \ldots x_q(T) \text{ specified} \tag{3.3-24}$$

and

$$\dot{E} = V\,(\underline{x}, \underline{u}, E, t) \quad , \quad E(0) \text{ specified} \tag{3.3-25}$$

the necessary conditions are

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad p_i(T) = k_{x_i} + \ell_j \begin{cases} i = 1, \ldots, n \\ j = i \text{ for } i \leq q \\ \ell_j = 0 \text{ for } j > q \end{cases} \tag{3.3-26}$$

$$\dot{P} = -H_E \quad , \quad P(T) = C \tag{3.3-27}$$

where

$$H = L + \underline{p}^T \underline{f} + \mathrm{tr}\,(P\,\dot{E}) \tag{3.3-28}$$

and for optimality

$$H_{\underline{u}} = 0 = L_{\underline{u}} + \underline{p}^T \underline{f}_{\underline{u}} + \left[\mathrm{tr}\,(P\,\dot{E})\right]_{\underline{u}} \tag{3.3-29}$$

Also, ℓ_j can be determined from Eq. 3.3-21, 3.3-20, 3.3-19, 3.3-10 and 3.3-9.

In the next section a simple example will demonstrate the appropriate calculations for a simple problem in which the driving noise is dependent on the nominal state; the following section presents an example of an integrator with unknown gain. Both examples will be solved analytically.

3.4 Illustrative Example 1 - State-Dependent Noise

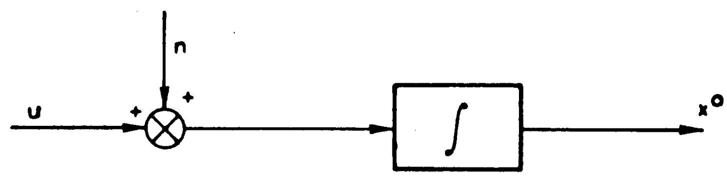

Figure 3-1 Integrator With Driving Noise

The actual system obeys

$$\dot{x}^a = u + n \tag{3.4-1}$$

A nominal system representing Eq. 3.4-1 is

$$\dot{\underline{x}} = u \tag{3.4-2}$$

and the deviation between the two systems is

$$\delta\dot{x} = n \tag{3.4-3}$$

If the white noise has statistics which depend on the nominal solution

$$< n(t)\ n(t') > = Q\ \delta(t-t') = x\ \delta(t-t') \tag{3.4-4}$$

for $x \geq 0$, the covariance matrix of first-order state deviations is given by

$$\dot{E} = x \tag{3.4-5}$$

With assumed nominal conditions $x(0) = 0$, $x(1) = 1$, and $E(0) = 0$, three optimization problems will be formulated to illustrate the effects of including the covariance matrix in the cost. First, the minimum energy solution is presented. Then the integral of the covariance matrix is included in the cost; finally, the covariance matrix at the terminal time is included in the cost.

a) Minimum energy. Minimize

$$J = 0.5 \int_0^1 u^2 \, dt \tag{3.4-6}$$

subject to Eq. 3.4-1 - 3.4-5.

The Hamiltonian is

$$H = 0.5\, u^2 + pu + Px \tag{3.4-7}$$

With

$$H_u = 0 = u + p \longrightarrow u = -p \tag{3.4-8}$$

and

$$\dot{p} = -H_E = 0 \longrightarrow P = \text{constant} \tag{3.4-9}$$

$$P(t) = 0 \longrightarrow P(t) = 0 \tag{3.4-10}$$

$$\dot{p} = -H_x = -P \longrightarrow p = \text{constant} \tag{3.4-11}$$

The optimal control that satisfies the boundary conditions is

$$u = 1 \tag{3.4-12}$$

which results in a nominal state and covariance of

$$x(t) = t \tag{3.4-13}$$

$$E(t) = t^2/2 \tag{3.4-14}$$

b) The cost includes E so that penalty is attached to the integral of the mean-squared deviations during the operating time

$$J = \int_0^1 (0.5\,u^2 + E)\,dt \qquad (3.4\text{-}15)$$

Then

$$H = 0.5\,u^2 + E + pu + Px \qquad (3.4\text{-}16)$$

with

$$H_u = 0 = u + p \longrightarrow u = -p \qquad (3.4\text{-}17)$$

$$\dot{P} = -H_E = -1 \quad , \quad P(1) = 0 \qquad (3.4\text{-}18)$$

$$\dot{p} = -H_x = -P \quad , \quad p(1) \text{ unspecified} \qquad (3.4\text{-}19)$$

Applying the boundary conditions yields for optimal conditions

$$u = -0.5\,t^2 + t + 0.667 \qquad (3.4\text{-}20)$$

$$x = -0.166\,t^3 + 0.5\,t^2 + 0.667\,t \qquad (3.4\text{-}21)$$

$$E = -0.041\,t^4 + 0.166\,t^3 + 0.333\,t^2 \qquad (3.4\text{-}22)$$

$$P = 1 - t \qquad (3.4\text{-}23)$$

$$p = 0.5\,t^2 - t - 0.667 \qquad (3.4\text{-}24)$$

c) A terminal cost. Minimize

$$J = E(1) + \int_0^1 0.5\,u^2\,dt \qquad (3.4\text{-}25)$$

The Hamiltonian is

$$H = 0.5\,u^2 + pu + Px \qquad (3.4\text{-}26)$$

and the necessary conditions are

$$H_u = 0 = p + u \longrightarrow u = -p \qquad (3.4\text{-}27)$$

$$\dot{p} = -H_x = -P \quad , \quad p(1) \text{ unspecified} \qquad (3.4\text{-}28)$$

$$\dot{P} = -H_E = 0 \quad , \quad P(1) = 1 \qquad (3.4\text{-}29)$$

The optimal conditions are

$$u = t + 0.5 \qquad (3.4\text{-}30)$$

$$x = 0.5\, t^2 + 0.5\, t \qquad (3.4\text{-}31)$$

$$E = 0.166\, t^3 + 0.25\, t^2 \qquad (3.4\text{-}32)$$

$$P = 1 \qquad (3.4\text{-}33)$$

$$p = -t - 0.5 \qquad (3.4\text{-}34)$$

Figure 3-2 shows the differences between the three cases. The cases, that involve weighting of the covariance matrix in the cost, attempt to keep x small for as long as possible, since the covariance matrix obeys

$$\dot{E} = x \qquad (3.4\text{-}35)$$

In fact, for

$$J = E(1) + \int_0^1 0.5\, u^2\, dt \qquad (3.4\text{-}36)$$

which could be written as

$$J = \int_0^1 (0.5\, u^2 + x)\, dt \qquad (3.4\text{-}37)$$

it is obvious that the area under x(t) is part of the cost to be minimized.

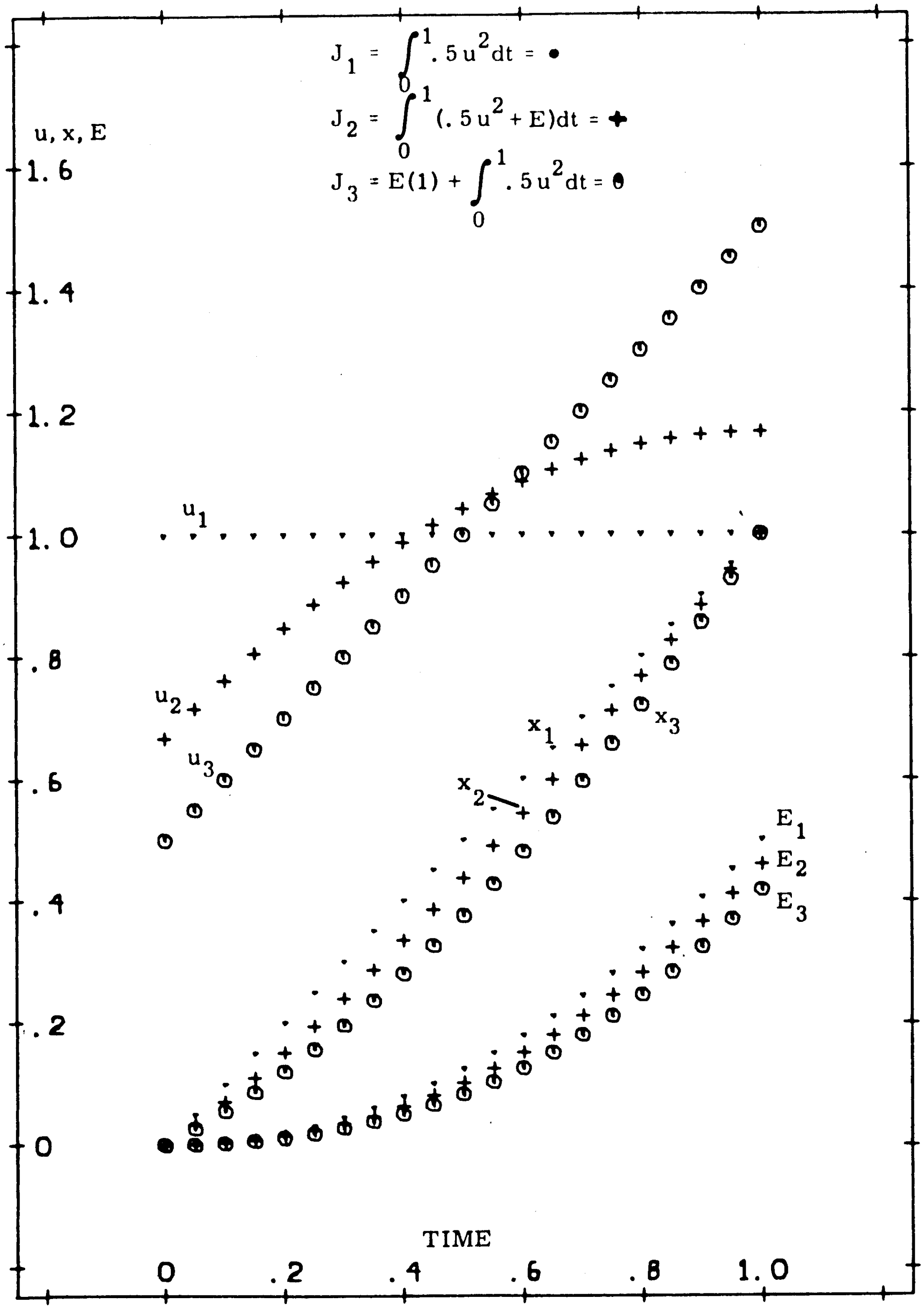

Figure 3-2 **Performance for Different Cost Functions**

3.5 Illustrative Example 2 - Integrator with Unknown Gain

For the simple integrator system with zero initial condition described by the equations

$$\dot{x}^a = b^a u + n \tag{3.5-1}$$

$$< n(t) \; n(t') > = q \; \delta(t - t') \tag{3.5-2}$$

it is desired to generate a nominal open-loop control that takes x from 0 to x(T) and minimizes a combination of energy and the mean-squared miss distance at time T. The actual system cannot meet the terminal condition exactly because of the driving noise and the unknown gain.

The unknown gain is a constant from a Gaussian distribution with a mean of b and variance σ_b^2. If the deviations of the state and of the unknown gain are small, the covariance matrix representing the mean-squared deviations propagates according to

$$\dot{E} = F E + E F^T + Q \tag{3.5-3}$$

where

$$F = \begin{bmatrix} 0 & u \\ 0 & 0 \end{bmatrix} \tag{3.5-4}$$

$$Q = \begin{bmatrix} q & 0 \\ 0 & 0 \end{bmatrix} \tag{3.5-5}$$

and

$$E(0) = \begin{bmatrix} 0 & 0 \\ 0 & \sigma_b^2 \end{bmatrix} \tag{3.5-6}$$

The nominal state obeys

$$\dot{x} = b\,u \quad , \quad x(0) = 0 \,,\; x(T) \text{ specified} \tag{3.5-7}$$

The cost function is

$$J = \mathrm{tr}\left[C\;E(T)\right] + \int_0^T 0.5\,c\,u^2\,dt \tag{3.5-8}$$

where

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \tag{3.5-9}$$

and c weights the amount of energy used for control.

The Hamiltonian becomes

$$H = L + \underline{p}^T \underline{f} + \mathrm{tr}\,(P\,\dot{E}) \tag{3.5-10}$$

$$H = 0.5\,c\,u^2 + p\,b\,u + \mathrm{tr}\,(P\,\dot{E}) \tag{3.5-11}$$

$$H = 0.5\,c\,u^2 + p\,b\,u + 2u\,P_{11}E_{12} + 2u\,P_{12}E_{22} \tag{3.5-12}$$

For optimality

$$H_u = 0 = c\,u + p\,b + 2\,P_{12}\,E_{22} \tag{3.5-13}$$

or

$$u = -\,(p\,b + 2\,P_{11}\,E_{12} + 2\,P_{12}\,E_{22})\,/c \tag{3.5-14}$$

The adjoint variables satisfy

$$\dot{p} = -\,H_x = 0 \tag{3.5-15}$$

$$\dot{P}_{11} = -\,H_{E_{11}} = 0 \quad , \quad P_{11}(T) = C_{11} = 1 \tag{3.5-16}$$

Corrections to:

A New Technique for Identification and Control of Systems with Unknown Parameters

Sc.D. Thesis, Course 16, 1971

George T. Schmidt

Eq. (3.5-25) p. 39

$$u = \frac{\frac{2\sigma_b^2}{c} x(T)}{b\left[e^{2\sigma_b^2 T/c} - 1\right]} e^{2\sigma_b^2 t/c}$$

Eq. (3.5-26) p. 39

$$x(t) = \frac{x(T)}{\left[e^{2\sigma_b^2 T/c} - 1\right]} \left[e^{2\sigma_b^2 t/c} - 1\right]$$

Eq. (3.5-28) p 40

$$u \rightarrow \frac{x(T)}{b} \delta(t')$$

$$\dot{P}_{12} = -2u\,P_{11} \quad , \quad P_{12}(T) = 0 \tag{3.5-17}$$

$$\dot{P}_{22} = -2u\,P_{12} \quad , \quad P_{22}(T) = 0 \tag{3.5-18}$$

Solving Eq. 3.5-15 and 3.5-16 yields

$$p = d \tag{3.5-19}$$

$$P_{11} = 1 \tag{3.5-20}$$

Differentiating Eq. 3.5-14 yields

$$\dot{u} = -(b\,\dot{p} + 2\,P_{11}\dot{E}_{12} + 2\,E_{12}\dot{P}_{11} + 2\,\dot{P}_{12}E_{22} + 2\,P_{12}\dot{E}_{22})/c \tag{3.5-21}$$

and making use of Eq. 3.5-15 through 3.5.17 and

$$\dot{E}_{22} = 0 \longrightarrow E_{22} = \sigma_b^2 \tag{3.5-22}$$

$$\dot{E}_{12} = u\,E_{22} = u\,\sigma_b^2 \tag{3.5-23}$$

gives

$$\dot{u} = \frac{2\,\sigma_b^2}{c}\,u \tag{3.5-24}$$

Solving Eq. 3.5-24 together with Eq. 3.5-7 results in

$$u = \frac{x(T)}{b\,T} - \frac{c}{2\,T\,\sigma_b^2}\left(e^{2\sigma_b^2 T/c} - 1\right) + e^{2\sigma_b^2 t/c} \tag{3.5-25}$$

for the optimal control input and in

$$x(t) = \left[x(T) - \frac{c\,b}{2\,\sigma_b^2}\left(e^{2\sigma_b^2 T/c} - 1\right)\right]\frac{t}{T} + \frac{c\,b}{2\,\sigma_b^2}\left(e^{2\sigma_b^2 T/c} - 1\right) \tag{3.5-26}$$

for the optimal nominal trajectory. As the ratio $c/\sigma_b^2 \to \infty$ (the minimum energy case)

$$u \to \frac{1}{b} \frac{x(T)}{T} \qquad (3.5\text{-}27)$$

and x increases linearly from 0 to x(T). As the ratio $c/\sigma_b^2 \to 0$ (infinite uncertainty in b)

$$u \to \frac{x(T)}{b} \delta(t' - T) \qquad (3.5\text{-}28)$$

where the δ in Eq. 3.5-28 represents the impulse function and x increases instantaneously at $t = t'$ from 0 to x(T). The exact time of the application of the impulse is arbitrary since it does not affect the cost.

3.6 Free Terminal-Time Problems

If the terminal time is not specified, a constraint is now missing and needs to be replaced by another one. Consider the cost function

$$J = \text{tr}\left[C(T)\, E(T)\right] + k\left[\underline{x}(T), t\right] + \int_0^T L(\underline{x}, \underline{u}, E, t)\, dt \qquad (3.6\text{-}1)$$

C(T) and k can be explicit functions of time. For the same class of problems treated in Section 3.3, it is shown in Appendix C that the optimum terminal time is found from the transversality condition

$$k_t + \text{tr}\left[C_t\, E(T)\right] + H(T) = 0 \qquad (3.6\text{-}2)$$

where the subscript t denotes partial differentiation with respect to time evaluated at the terminal time. All other necessary conditions given by Eq. 3.3-24 through 3.3-29 remain the same.

3.7 Problems With Constraints

As was stated in Section 3.1, it has been assumed that all differentiability conditions necessary for the application of Pontryagin's minimum principle are satisfied. Thus, problems involving inequality constraints on the control variables are minimized by finding the absolute minimum of the Hamiltonian with respect to the controls.

Since the Hamiltonian is given by

$$H = L + \underline{p}^T \underline{f} + \text{tr}\,(P\,\dot{E}) \qquad (3.7\text{-}1)$$

and the controls can appear in L, $\underline{f}$ or E, the task of analytically finding an absolute minimum of H is bound to be difficult, especially since the problem is already nonlinear. This problem can be successfully solved by the numerical procedure suggested in the next chapter.

Problems with inequality constraints on the state variables have been extensively treated in the literature. Bryson and Denham (1963) derive the necessary conditions for extremal solutions for a large class of problems involving inequality constraints by means of adjoining the constraints to the performance index by Lagrange multipliers. Further comments on this problem will be given in Chapter 4.

3.8 Summary

This chapter has derived the necessary conditions for optimization of performance indices subject to general vector and matrix differential equations as constraints. By application of the calculus-of-variations it was shown that a set of straightforward equations for optimality result. Two simple analytical examples of the technique were presented; in general, numerical procedures are necessary to solve the nonlinear optimization problem. Appropriate numerical techniques are developed in the next chapter.

Chapter 4

Numerical Solution Techniques

4.1 Introduction

Since only a small number of the type of optimal control problems presented in Chapter 3 may be solved in closed form, it is necessary to consider the technique of solution by computer. There are two general approaches one can take to this task: indirect methods and direct methods.

In the indirect-method approach the TPBVP is converted into a sequence of initial value problems. The unknown initial conditions are guessed and the equations are integrated. In general the resulting final conditions do not match. It is then necessary to change the guesses of the initial conditions in such a way that the final conditions will be met. Unfortunately, this technique results in equations that are always unstable to some degree (Kipiniak, 1961). This instability is associated with the particular difficulty of "getting started" - see Bryson and Ho, 1969 - and this method is usually only practical for finding neighboring extremal solutions after one extremal solution is obtained by some other method. For examples of this approach see Balakrishman and Neustadt (1964).

Various direct methods, which minimize the cost function directly by considering changes in the control, are among the more successful computational approaches to the TPBVP. The best known of these is the Bryson-Kelley-Denham gradient method presented in Kelly (1960) and Bryson and Denham (1962). This technique has been extended in a number of directions, and applied to many problems. One of the first extensions was to the case of bounded controls (Kelly, Kopp, and Mayer, 1961). It has also been modified to include state-variable constraints by various means (Bryson and Denham, 1964). The major difficulty with this method is its slow convergence near the optimum. An effective way of accelerating convergency is through the use of the second variation (Breakwell, Speyer, and Bryson, 1963 and Jacobson and Mayne, 1970) at the expense of an increased computational burden.

If there is a standard method for computing optimal controls, the gradient method is it. It converges slowly but reliably from even extremely poor starting conditions. This technique will be applied to the TPBVP involving vector and matrix differential equations.

The application of the gradient method to problems involving vectors and matrices could be approached by partitioning the matrices into vectors and using the standard approaches. It is more convenient, however, to develop an approach which does not require the partitioning. As will be shown, using first-variation techniques results in a computationally simple extension of the gradient method to the case involving matrices. No simple second-variation approach has been developed that can solve the matrix-vector optimization problem.

In Section 4.2 the gradient method for problems with no terminal constraints and fixed terminal time is presented. Section 4.3 treats a more general problem that includes terminal constraints on some of the state variables; Section 4.4 illustrates the calculations by numerically solving the example from Section 3.4. Section 4.5 deals with problems involving free terminal time; Section 4.6 presents techniques useful for problems with constraints.

4.2 Problems With No Terminal Constraints

The first-order gradient technique for fixed terminal time in problems with no terminal constraints is particularly simple. Suppose it is desired to minimize

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T L(\underline{x}, \underline{u}, E, t)\, dt \qquad (4.2\text{-}1)$$

for the system

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \quad , \quad \underline{x}(0) \text{ given} \qquad (4.2\text{-}2)$$

$$\dot{E} = V(\underline{x}, \underline{u}, E, t) \quad , \quad E(0) \text{ given} \qquad (4.2\text{-}3)$$

With an initial guess for $\underline{u}(t)$ the system, Eq. 4.2-2 and 4.2-3, is integrated forward and the results stored. The adjoint equations are then integrated backward

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \tag{4.2-4}$$

$$\dot{P} = -H_E \quad , \quad P(T) = C \tag{4.2-5}$$

In general,

$$H_{\underline{u}} = 0 \tag{4.2-6}$$

will not be satisfied by the initial guess. If a change in the control ($\delta\underline{u}$) is made, then the incremental first-order change in the cost is

$$\delta J = \int_o^T H_{\underline{u}} \, \delta\underline{u} \; dt \tag{4.2-7}$$

as was demonstrated in Chapter 3. If we wish to make the largest change in δJ, we would calculate the gradient $H_{\underline{u}}$ and then make $\delta\underline{u}$ directed opposite to the gradient

$$\delta\underline{u} = - W(t) \; H_{\underline{u}}^T \tag{4.2-8}$$

If $W(t) > 0$, J becomes smaller with each iteration and the procedure is repeated until either the control or the cost function does not change significantly from iteration to iteration. Convergence is slower as the optimum is approached so that, generally, W must be increased. Sage (1968) suggests W might be picked by using the past value, one-half the past value, twice the past value, and ten times the past value to determine four new values of $\delta\underline{u}$ and J. The value of W which produces the smallest J is then used for the next iteration.

It is also possible to use this technique in problems with linear terminal constraints. For example, if x(T) = 1 is specified, then one might consider augmenting the original cost, Eq. 4.2-1, with a quadratic weighting

$$J_1 = d \left[x(T) - 1 \right]^2 + J \qquad (4.2\text{-}9)$$

where d would be picked such that x(T) approaches 1 to the desired accuracy. However, if d is chosen too large, the algorithm may tend to concentrate more on satisfying the constraint than minimizing the original performance index.

There is a different variation on the first-order gradient method that is no more complicated but has the ability to converge to an exact optimum solution. In this technique, it is assumed that the state and adjoint equations have been integrated using $\underline{u}^*$. Furthermore, it is assumed that $H_{\underline{u}} = 0$ can then be evaluated to yield $\underline{u}_c$, the value of $\underline{u}$ which would cause $\underline{H}_u$ to be 0 on this last iteration. The algorithm is then to choose

$$\delta \underline{u} = - d \, (\underline{u}^* - \underline{u}_c) \qquad (4.2\text{-}10)$$

where d is a decimal fraction between 0 and 1. d = 1 is the best estimate of the $\delta \underline{u}$ that will drive $H_{\underline{u}^*}$ all the way to zero on the next iteration.

In this method $\delta \underline{u}$ is picked to lie along the chord of $H_{\underline{u}^*}$ and 0; in the gradient method $\delta \underline{u}$ lies along the slope of $H_{\underline{u}}$. Of course, if $\underline{H}_{uu}$ is constant, then both methods are identical if $W = \underline{H}_{uu}^{-1}$ and d = 1.

The next section treats the more general optimization problem involving terminal constraints.

4.3 Problems With Linear Terminal Constraints

The optimization problem considered is to minimize during a fixed operating time.

$$J = \mathrm{tr}\left[C\,E(T)\right] + k\left[\underline{x}(T)\right] + \int_0^T L\,(\underline{x}, E, \underline{u}, t)\,dt \qquad (4.3\text{-}1)$$

with constraints

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}, \underline{u}, t) \quad , \quad \begin{cases} \underline{x}(0) \quad \text{specified} \\ \underline{z}(T) = \begin{bmatrix} x_1(T) \\ \cdots \\ x_q(T) \end{bmatrix} \quad \text{specified} \end{cases} \qquad (4.3\text{-}2)$$

$$\dot{E} = V\,(\underline{x}, \underline{u}, E, t) \quad , \quad E(0) \ \text{specified} \qquad (4.3\text{-}3)$$

In the first-order gradient method a nominal control history is guessed. With this guess the system equations 4.3-2 and 4.3-3 are integrated forward. This initial guess will not, in general, satisfy the boundary conditions or result in a minimum cost. Adjoint equations are then determined using the results of the forward integration.

$$h = L + \underline{p}^T \underline{f} + \mathrm{tr}\,(P\,\dot{E}) \qquad (4.3\text{-}4)$$

and

$$\underline{H} = R^T \underline{f} \qquad (4.3\text{-}5)$$

then the adjoint equations are

$$\dot{\underline{p}} = -h_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \qquad (4.3\text{-}6)$$

$$\dot{P} = -h_E \quad , \quad P(T) = C \qquad (4.3\text{-}7)$$

and

$$\dot{R} = -\underline{H}_{\underline{x}}^T \quad , \quad R(T) = \delta_{ij} \quad \begin{cases} i = 1, \ldots, n \\ j = 1, \ldots, q \end{cases} \qquad (4.3\text{-}8)$$

The quantities $\underline{p}(t)$, $P(t)$, and $R(t)$ predict how changes in the control histories, $\delta \underline{u}(t)$, will change the cost and the q terminal conditions $\underline{z}(T)$. From Eq. 3.2-17, the change in cost is

$$\delta J = \int_0^T \left\{ L_{\underline{u}} + \underline{p}^T \underline{f}_{\underline{u}} + \left[tr\,(P\dot{E}) \right]_{\underline{u}} \right\} \delta \underline{u}(t)\; dt \qquad (4.3\text{-}9)$$

and from Eq. 3.3-11, the changes in terminal conditions are

$$\delta \underline{z}(T) = \int_0^T R^T \underline{f}_{\underline{u}}\; \delta \underline{u}(t)\; dt \qquad (4.3\text{-}10)$$

The problem is to pick an appropriate $\delta \underline{u}$ that will satisfy the linearization and which will decrease the cost while constraining the size of $\delta \underline{z}(T)$. Since Eq. 4.3-9 and 4.3-10 are linearized relations, there is no minimum for δJ subject to constraints on the size of $\delta \underline{z}(T)$. A simple way to create a minimum is to add a quadratic integral penalty function in $\delta \underline{u}(t)$ to Eq. 4.3-9:

$$\delta J_1 = \delta J + 0.5 \int_0^T \delta \underline{u}^T\, W\, \delta \underline{u}\; dt \qquad (4.3\text{-}11)$$

where $W(t)$ is an arbitrary m x m positive-definite weighting matrix. The minimization of Eq. 4.3-11, subject to constraints on the change in the terminal conditions Eq. 4.3-10, is solved by adjoining Eq. 4.3-10 to Eq. 4.3-11 with a q-dimensional constant multiplier $\underline{\ell}$:

$$\begin{aligned} \delta J_1 = & \int_0^T \left\{ L_{\underline{u}} + p^T \underline{f}_{\underline{u}} + \left[tr\,(P\dot{E}) \right]_{\underline{u}} \right\} \delta \underline{u}\; dt \\ & + 0.5 \int_0^T \delta \underline{u}^T\, W\, \delta \underline{u}\; dt \\ & + \underline{\ell}^T \left[\int_0^T R^T \underline{f}_{\underline{u}}\; \delta \underline{u}\; dt - \delta \underline{z}(T) \right] \end{aligned} \qquad (4.3\text{-}12)$$

The change in δJ_1 due to a change in $\delta \underline{u}$, neglecting the change in the coefficients, is given by

$$\delta(\delta J_1) = \int_0^T \left\{ \underline{L}_u + \underline{p}^T \underline{f}_u + \left[\text{tr}\,(P\dot{E}) \right]_{\underline{u}} + \underline{\ell}^T R^T \underline{f}_u + \delta \underline{u}^T W \right\} \delta\,(\delta \underline{u})\, dt \qquad (4.3\text{-}13)$$

Clearly, the optimum change occurs when

$$\delta \underline{u} = - W^{-1} \left\{ \underline{L}_u + (\underline{p} + R\,\underline{\ell})^T \underline{f}_u + \left[\text{tr}\,(P\,\dot{E}) \right]_{\underline{u}} \right\}^T \qquad (4.3\text{-}14)$$

or

$$\delta \underline{u} = - W^{-1} \underline{h}_u^T - W^{-1} \underline{H}_u^T \underline{\ell} \qquad (4.3\text{-}15)$$

Substituting Eq. 4.3-15 into Eq. 4.3-10 yields:

$$\delta \underline{z}(T) = - \int_0^T R^T \underline{f}_u W^{-1} \underline{h}_u^T\, dt - \int_0^T R^T \underline{f}_u W^{-1} \underline{H}_u^T \underline{\ell}\, dt \qquad (4.3\text{-}16)$$

With

$$\underline{H}_u = R^T \underline{f}_u \qquad (4.3\text{-}17)$$

define the q-dimensional vector

$$\underline{I}_{kj} = \int_0^T \underline{H}_u W^{-1} \underline{h}_u^T\, dt \qquad (4.3\text{-}18)$$

and the q x q matrix

$$I_{kk} = \int_0^T \underline{H}_u W^{-1} \underline{H}_u^T\, dt \qquad (4.3\text{-}19)$$

Then Eq. 4.3-10 may be written

$$\delta \underline{z}(T) = \underline{I}_{kj} - I_{kk}\, \underline{\ell} \qquad (4.3\text{-}20)$$

If I_{kk}^{-1} exists, Eq. 4.3-20 may be solved for the required $\underline{\ell}$ that will yield the specified $\delta\underline{z}(T)$:

$$\underline{\ell} = - I_{kk}^{-1} \left[\delta\underline{z}(T) + \underline{I}_{kj} \right] \qquad (4.3\text{-}21)$$

Substituting Eq. 4.3-21 and 4.3-15 into Eq. 4.3-9, the predicted change δJ is

$$\delta J = - \int_0^T \underline{h}_u W^{-1} \left\{ \underline{h}_u^T - \underline{H}_u I_{kk}^{-1} \left[\delta\underline{z}(T) + \underline{I}_{kj} \right] \right\} dt \qquad (4.3\text{-}22)$$

Now define a scalar

$$I_{jj} = \int_0^T \underline{h}_u W^{-1} \underline{h}_u^T \, dt \qquad (4.3\text{-}23)$$

and a q-dimensional row vector

$$\underline{I}_{jk} = \int_0^T \underline{h}_u W^{-1} \underline{H}_u^T \, dt = \underline{I}_{kj}^T \qquad (4.3\text{-}24)$$

Then

$$\delta J = - I_{jj} + \underline{I}_{kj}^T I_{kk}^{-1} \left[\delta\underline{z}(T) + \underline{I}_{kj} \right] \qquad (4.3\text{-}25)$$

or

$$\delta J = - (I_{jj} - \underline{I}_{kj}^T I_{kk}^{-1} \underline{I}_{kj}) + \underline{I}_{kj}^T I_{kk}^{-1} \delta\underline{z}(T) \qquad (4.3\text{-}26)$$

As the optimum solution is approached

$$\delta\underline{z}(T) \longrightarrow 0 \qquad (4.3\text{-}27)$$

then, from Eq. 4.3-15,

$$\underline{h}_u + \underline{\ell}^T \underline{H}_u \longrightarrow 0 \qquad (4.3\text{-}28)$$

from Eq. 4.3-26,

$$I_{jj} - \underline{I}_{kj}^T I_{kk}^{-1} \underline{I}_{kj} \longrightarrow 0 \qquad (4.3\text{-}29)$$

and from Eq. 4.3-21

$$\underline{\ell} \longrightarrow - I_{kk}^{-1} \underline{I}_{kj} \qquad (4.3\text{-}30)$$

Note that Eq. 4.3-28 may be interpreted as

$$\underline{H}_u = \underline{h}_u + \underline{\ell}^T \underline{H}_u \qquad (4.3\text{-}31)$$

if

$$H = h + \underline{\ell}^T \underline{H} \qquad (4.3\text{-}32)$$

Now the first-order gradient method for the problem given by Eq. 4.3-1 - 4.3-3 may be summarized as follows:

Step 1.

Estimate a set of control histories $\underline{u}(t)$.

Step 2.

Integrate the system equations forward

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \quad , \quad \underline{x}(0) \text{ given} \qquad (4.3\text{-}33)$$

$$\dot{E} = V(\underline{x}, \underline{u}, E, t) \quad , \quad E(0) \text{ given} \qquad (4.3\text{-}34)$$

and record $\underline{x}(t)$, $E(t)$, $\underline{u}(t)$, and $\underline{x}(T)$.

Step 3.

Determine the influence functions by backward integration using the values obtained in Step 2 with

$$h = L + \underline{p}^T \underline{f} + \mathrm{tr}\,(P\,\dot{E}) \tag{4.3-35}$$

$$\underline{H} = R^T \underline{f} \tag{4.3-36}$$

$$\dot{\underline{p}} = -h_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \tag{4.3-37}$$

$$\dot{R} = -H_{\underline{x}}^T \quad , \quad R(T) = \delta_{ij} \quad \begin{cases} i = 1, \ldots, n \\ j = 1, \ldots, q \end{cases} \tag{4.3-38}$$

$$\dot{P} = -h_E \quad , \quad P(T) = C \tag{4.3-39}$$

Step 4.

Simultaneously, compute using an appropriate W,

$$I_{kk} = \int_o^T \underline{H}_{\underline{u}}\, W^{-1}\, \underline{H}_{\underline{u}}^T\, dt \tag{4.3-40}$$

$$\underline{I}_{kj} = \int_o^T \underline{H}_{\underline{u}}\, W^{-1}\, h_{\underline{u}}^T\, dt \tag{4.3-41}$$

$$I_{jj} = \int_o^T h_{\underline{u}}\, W^{-1}\, h_{\underline{u}}^T\, dt \tag{4.3-42}$$

Step 5.

Choose a value of $\delta \underline{z}(T)$ to cause the nominal solution to be closer to the desired values $\underline{x}_q(T)$.

$$\delta \underline{z}(T) = -d \left[\text{terminal conditions in step 1} - \underline{x}_q(T)\right] \tag{4.3-43}$$

with

$$0 < d < 1 \tag{4.3-44}$$

Then determine

$$\underline{\ell} = - I_{kk}^{-1} \left[\delta \underline{z}(T) + \underline{I}_{kj} \right] \tag{4.3-45}$$

Step 6.

Repeat Steps 2 - 6 using an improved estimate of $\underline{u}(t)$ formed by adding to the previous control the vector

$$\delta \underline{u}(t) = - W^{-1} \left[\underline{h}_u^T + \underline{H}_u^T \; \underline{\ell} \right] \tag{4.3-46}$$

Stop when

$$\delta \underline{z}(T) \longrightarrow 0 \tag{4.3-47}$$

and

$$I_{jj} - \underline{I}_{kj}^T \; I_{kk}^{-1} \; \underline{I}_{kj} \longrightarrow 0 \tag{4.3-48}$$

to the desired degree of accuracy.

The best choices of W and d are not determined a priori. A possible way to choose W is to compare the actual δJ and $\delta \underline{z}(T)$ with the predicted values from Eq. 4.3-25 and 4.3-20. If there is too large a difference, W should be increased; if the difference is small, it is possible to take larger steps and W should be reduced.

4.4 Illustrative Example

Consider the problem of Section 3.4 for which an analytical solution was obtained. The cost function was

$$J = \int_0^T \left(\frac{1}{2} u^2 + E\right) dt \qquad (4.4\text{-}1)$$

with

$$\dot{x} = u \quad , \quad x(0) = 0 \quad , \quad x(1) = 1 \qquad (4.4\text{-}2)$$

and

$$\dot{E} = x \quad , \quad E(0) = 0 \qquad (4.4\text{-}3)$$

the first iteration through the gradient method equations is as follows:

Step 1.

$$\text{Guess } u(t) = 1. \quad \text{Pick } W = 1 \qquad (4.4\text{-}4)$$

Step 2.

Integrating the system equations, 4.4-2 and 4.4-3, forward yields

$$x(t) = t \qquad (4.4\text{-}5)$$

$$E(t) = t^2/2 \qquad (4.4\text{-}6)$$

Step 3.

With

$$h = u^2/2 + E + pu + Px \qquad (4.4\text{-}7)$$

$$\underline{H} = Ru \qquad (4.4\text{-}8)$$

and

$$\dot{p} = -h_x \quad , \quad p(1) = 0 \tag{4.4-9}$$

$$\dot{P} = -h_E \quad , \quad P(1) = 0 \tag{4.4-10}$$

$$\dot{R} = -\underline{H}_x \quad , \quad R(1) = 1 \tag{4.4-11}$$

backward integration results in

$$p = 0.5 - t + t^2/2 \tag{4.4-12}$$

$$P = 1 - t \tag{4.4-13}$$

$$R = 1 \tag{4.4-14}$$

Step 4.

With W = 1

$$I_{kk} = \int_o^1 dt = 1 \tag{4.4-15}$$

$$I_{kj} = \int_o^1 (1 + \frac{1}{2} - t + t^2/2)\, dt = 7/6 \qquad (4.4\text{-}16$$

$$I_{jj} = \int_o^1 (\frac{3}{2} - t + t^2/2)^2\, dt = 83/60 \tag{4.4-17}$$

Step 5.

$\delta z(T)$ is zero since the boundary condition is met and

$$\underline{\ell} = -I_{kk}^{-1}\ (I_{kj}) = -7/6 \tag{4.4-18}$$

Step 6.

Thus, the incremental change in u is

$$\delta u = - W^{-1} \left[h_u + \underline{H_u} \; \underline{\ell} \right] \tag{4.4-19}$$

$$\delta u = - \frac{1}{3} + t - t^2/2 \tag{4.4-20}$$

so

$$u = \frac{2}{3} + t - \frac{t^2}{2} \tag{4.4-21}$$

and

$$I_{jj} - I_{jk} I_{kk}^{-1} I_{kj} = \frac{1}{45} \tag{4.4-22}$$

We have actually found the correct optimal control in only one iteration through Steps 1 - 6. Another pass through Steps 2 - 6 would verify $\delta u = 0$ and Eq. 4.4-22 equals 0.

4.5 Free Terminal Time Problems

For problems with linear terminal constraints and free terminal time, the gradient method is modified to account for the fact that one more free parameter needs to be determined. For the optimization problem

$$J = \mathrm{tr} \left[C(T)\, E(T) \right] + k \left[\underline{x}(T), T \right] + \int_0^T L\,(\underline{x}, \underline{u}, E, t)\, dt \tag{4.5-1}$$

it is shown in Appendix D that the optimum change in the estimate of the best terminal time, at each iteration, is

$$dT = - b^{-1} \left\{ \mathrm{tr} \left[C_t \, E(T) \right] + k_t + H(T) \right\} . \tag{4.5-2}$$

where H(T) is obtained from Eq. 4.3-32. The details of the method are explained in Appendix D; it is important to note that another weighting factor, b, must be chosen a priori.

4.6 Problems With Constraints

The algorithms discussed in this chapter have applied only to problems in which there are no inequality constraints on the control and/or state variables. The simplest approach to such problems is to use integral penalty functions. If the inequality constraint

$$c\,(\underline{x}, \underline{u}, t) \leq 0 \tag{4.6-1}$$

is specified for all time, the performance index may be augmented as follows:

$$J_1 = J + d \int_0^T \left[c\,(\underline{x}, \underline{u}, t)\right]^2 I(c)\,dt \tag{4.6-2}$$

where

$$I(c) = \begin{cases} 0 & c < 0 \\ 1 & c > 0 \end{cases} \tag{4.6-3}$$

If d is picked too large, the gradient algorithm will tend to concentrate more on satisfying the constraint rather than minimizing the cost. As a result, convergence may be slow.

A more effective approach to solving such problems is to join together constrained and unconstrained axes. Unlike the integral penalty approach, this approach is capable of finding the exact solution and uses less computer time. However, one must guess beforehand the sequence of constrained ($c = 0$) and unconstrained ($c < 0$) arcs, and the computer programming is more complicated; the problem is even more difficult for cases in which the inequality constraint is

$$c\,(\underline{x}, t) \leq 0 \tag{4.6-4}$$

because, in general, the adjoint variables are discontinuous at the entry and exit points of any constrained arc. The reader is referred to Bryson and Ho (1969) for further details.

4.7 Summary

This chapter has presented algorithms for the solution of general optimization problems involving vector and matrix differential equations as constraints. These algorithms should provide a convenient and useful technique for solving such general problems as trajectory shaping and open-loop signal design.

In the next chapter, attention is redirected to the specific problem of this thesis: identification and control of systems with unknown parameters.

Chapter 5

Optimum Input Design For Identification

5.1 Introduction

In Chapter 2 the method of using Kalman filtering to identify unknown system parameters was introduced. The technique is amenable to a formal optimization approach because the identification problem is nonlinear. The necessary conditions for optimality and numerical solution techniques were presented in Chapters 3 and 4, respectively. This chapter applies these previous investigations to representative identification problems.

In Section 5.2 the equilibrium identification of the inverse-time constant in a first-order system is investigated; in Section 5.3 the transient identification case is considered. In both cases, performance indices are used that weight uncertainty in the inverse-time constant estimate versus the amount of energy used in meeting the terminal constraints. It is shown that, with measurements proportional to system position, larger excursions of the state away from minimum-energy solutions in the direction of increasing values of the state result in improved estimates of the inverse-time constant at the terminal time. Furthermore, the uncertainty in the state estimate is less than that of the minimum-energy solution at the terminal time in the transient case because of the improved knowledge of the unknown parameter.

In Section 5.4 a more interesting example of identification in a second-order system is presented. The example verifies the improved estimation of the unknown parameters and of the states (position and velocity) when the optimum identification procedure is used. Such an approach is not, however, necessarily the appropriate design for a control system. In that case, the designer is interested in minimization of actual deviations in the states instead of estimation errors in the states. The control problem is the subject of the next chapter.

The second-order system also serves as an ideal test for the usefulness of the numerical procedure. Some particular difficulties, such as violation of the assumed linearization, are discussed and practical solutions presented. It is shown that rapid convergence to a near optimum for the oscillatory system is achieved.

Similar investigations with double integrator plants and plants with two real roots show the same general results when identification is formulated as an optimization problem. Space limitations prohibit presenting these latter results.

5.2 Equilibrium Identification in a First-Order System

In this section the identification of the inverse-time constant in a first-order system is solved as an optimization problem. The system is initially at rest and is to be returned to rest at the end of the identification interval.

The actual system obeys

$$\dot{x}^a = -b^a x^a + u + n \tag{5.2-1}$$

The inverse-time constant b^a is assumed to be a Gaussian distributed random constant that has a mean of b.

An open-loop control input is to be designed for the noise-free system

$$\dot{x} = -bx + u \tag{5.2-2}$$

with specified boundary conditions

$$x(0) = 0 \quad , \quad x(T) = 0 \tag{5.2-3}$$

Thus, the nominal state vector is of dimension 1.

The covariance matrix of estimation errors is associated with the best estimate of the deviations δx and δb away from the nominal values x and b. The matrix differential equation obeys

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \qquad (5.2\text{-}4)$$

where

$$F = \begin{bmatrix} -b & -x \\ 0 & 0 \end{bmatrix} \qquad (5.2\text{-}5)$$

The assumed nominal value of b is 1.

It is assumed that linear measurements of the actual state are available; the matrix M is then

$$M = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (5.2\text{-}6)$$

The assumed values for the measurement and driving-noise matrices are

$$U = 1 \qquad (5.2\text{-}7)$$

and

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (5.2\text{-}8)$$

Finally, the initial covariance matrix of the errors in the estimates of the deviations is

$$E(0) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \tag{5.2-9}$$

The cost function is chosen to trade-off the amount of energy used in moving the nominal system away from 0 versus the uncertainty in the estimate of the deviation in b at the terminal time.

$$J = \operatorname{tr}\left[C\, E(T)\right] + 0.5 \int_0^T u^2 \, dt \tag{5.2-10}$$

where the terminal time equals 10 and

$$C = \begin{bmatrix} 0 & 0 \\ 0 & C_{22} \end{bmatrix} \tag{5.2-11}$$

Defining

$$h = L + \underline{p}^T \underline{f} + \operatorname{tr}(P\dot{E})$$

$$h = 0.5\, u^2 - p b x + \operatorname{tr}(P\dot{E}) + p u \tag{5.2-12}$$

then the adjoint variable p satisfies

$$\dot{p} = -h_x = b p - \left[\operatorname{tr}(P\dot{E})\right]_x$$

$$= b p - \operatorname{tr}\left[P(F_x E + E F_x^T)\right] \tag{5.2-13}$$

where

$$F_x = \begin{bmatrix} 0 & -1 \\ & \\ 0 & 0 \end{bmatrix} \tag{5.2-14}$$

and

$$p(T) = 0 \tag{5.2-15}$$

The adjoint variable P satisfies

$$\dot{P} = -h_E = -(F - EM^T U^{-1} M)^T P - P(F - EM^T U^{-1} M) \tag{5.2-16}$$

and

$$P(T) = \begin{bmatrix} 0 & 0 \\ & \\ 0 & C_{22} \end{bmatrix} \tag{5.2-17}$$

Finally, the influence function R obeys

$$\dot{R} = -\underline{H}_x = bR \tag{5.2-18}$$

and

$$R(T) = 1 \tag{5.2-19}$$

The gradient method for numerical solution proceeds by first guessing an initial u. Then Eq. 5.2-2 and 5.2-4 are integrated forward. Equations 5.2-13, 5.2-16, and 5.2-18 are integrated backward. The remaining steps follow according to Eq. 4.3-36 through 4.3-44.

Figure 5-1 shows the nominal state trajectory for C_{22} = 0.1, 1, and 10. As more weight is attached to the value of the terminal uncertainty, larger excursions of x away from 0 are required to satisfy the optimality conditions. The resulting uncertainty in the estimate is shown in Figure 5-2 and the corresponding control in Figure 5-3.

The performance of the estimator can be better understood by examining the covariance matrix equation (5.2-4) in component form.

$$\dot{E}_{11} = -2E_{11} - E_{11}^2 + 1 - 2x E_{12} \tag{5.2-20}$$

$$\dot{E}_{12} = -E_{12} - x E_{22} - E_{11} E_{12} \tag{5.2-21}$$

$$\dot{E}_{22} = -E_{12}^2 \tag{5.2-22}$$

For x zero, Eq. 5.2-20 shows that E_{11} quickly reaches a steady-state value of 0.414. E_{11} is displayed in Figure 5-4. The measurements quickly reduce the initial uncertainty to the steady-state value; then, as x is driven away from zero, the uncertainty increases only to return to the steady-state value as x returns to 0. In Figure 5-5 the covariance, E_{12}, is displayed. Since it is always negative, the last term in Eq. 5.2-20 shows that the uncertainty in x will always increase due to this term. However, from Eq. 5.2-22, larger values of the covariance result in a decrease in the uncertainty associated with the inverse-time, constant. The final values of E_{22} are 2.7, 0.69, and 0.34. The physical explanation for these results can be found by looking at Eq. 5.2-2 in variational form

$$\delta\dot{x} = -b\,\delta x - x\,\delta b \tag{5.2-23}$$

For x zero, variations in b have no input to the variation in x. However, as x increases, the effect of an unknown time constant becomes a driving term. The variance equations, 5.2-20 – 5.2-22, physically represent this phenomena.

The uncertainty in the state at the terminal time is practically identical for all cases, since as soon as x returns to near zero, the measurements quickly reduce any uncertainty to the steady-state value. As will be shown in the next section, if the terminal condition on x is not 0, but rather 1, substantially different performance is achieved in the estimate of the deviation in x.

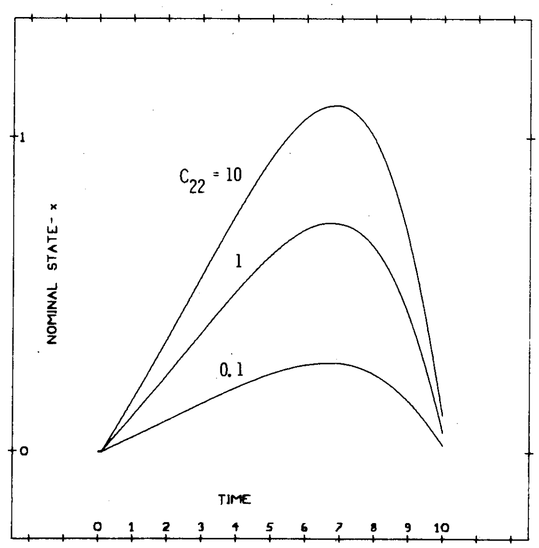

Figure 5-1 State History

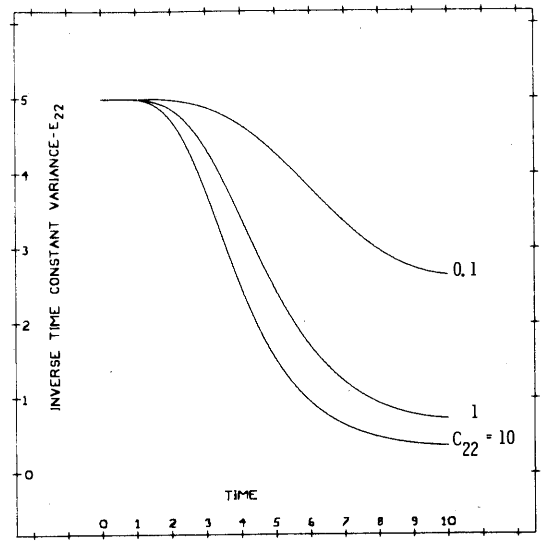

Figure 5-2 Estimation Performance

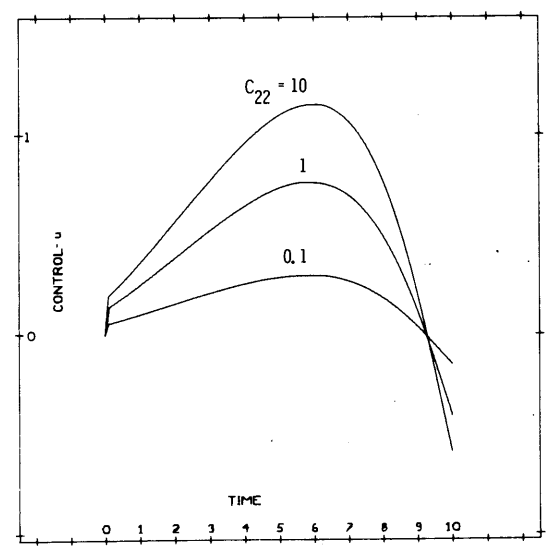

Figure 5-3 Control History

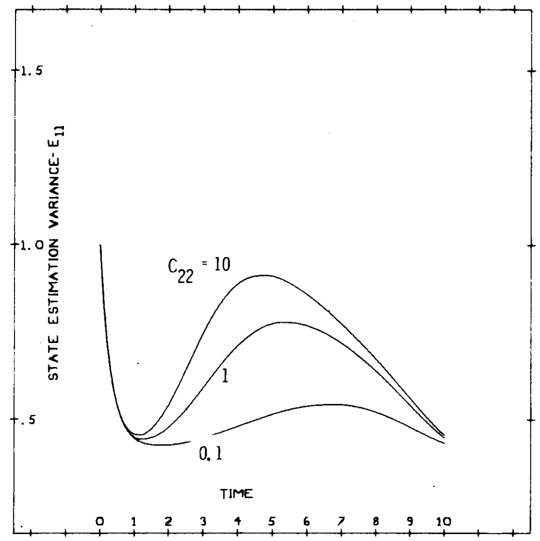

Figure 5-4 State Estimation Performance

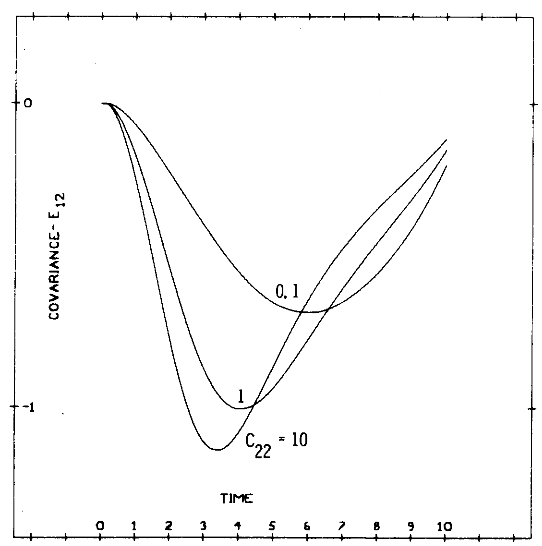

Figure 5-5 Covariance History

5.3 Transient Identification

This section treats the identical problem as in the previous section except that the boundary conditions, Eq. 5.2-3, are

$$x(0) = 0 \quad , \quad x(T) = 1 \tag{5.3-1}$$

The nominal system is to be driven to a particular terminal condition while the estimator is "learning" the unknown parameter.

Figure 5-6 shows the nominal state history for different values of C_{22}. Since,

$$x(T) = -b \int_0^T x \; dt + \int_0^T u \; dt \tag{5.3-2}$$

it is clear that more knowledge about b is gained by increasing the area under the x versus t curve. The corresponding controls are shown in Figure 5-7 with smaller values of C_{22} giving solutions tending towards the minimum energy ($C_{22} = 0$) case.

The variances corresponding to the unknown inverse-time constant and the state are shown in Figures 5-8 and 5-9, respectively. Note the very significant result that, at the terminal time, the uncertainty in the state deviation estimate is less for those cases with increased weighting on the uncertainty in the inverse-time-constant deviation estimate. This is partially due to the decrease in the covariance (Figure 5-10) for those cases. The final values $E_{11}(T)$ are 0.72, 0.85, and 0.91.

Other computer runs indicated obvious trends for different numerical values of the assumptions. For example, if the driving noise is increased, the estimates are all poorer. If the measurement noise is decreased, solutions tend more toward the minimum-energy solution.

In the next section a more complicated identification problem is considered – that of estimating unknown parameters in a second-order system.

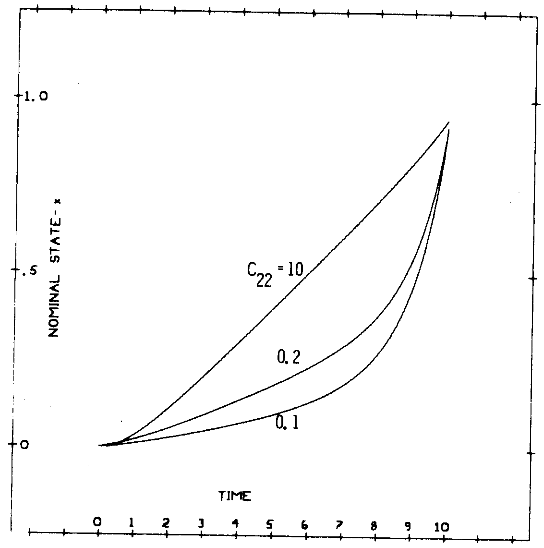

Figure 5-6 State History

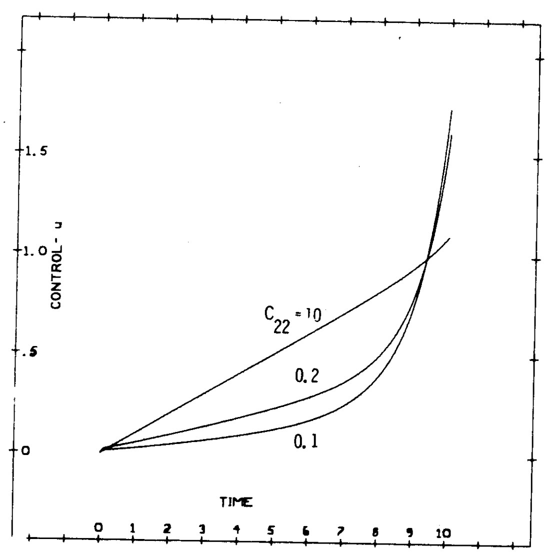

Figure 5-7 Control History

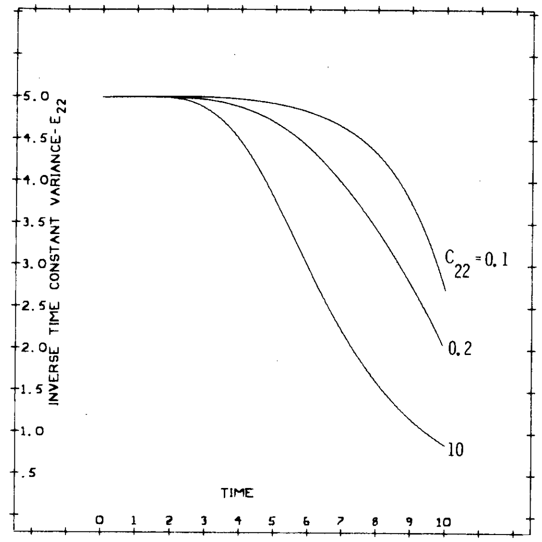

Figure 5-8 Estimation Performance

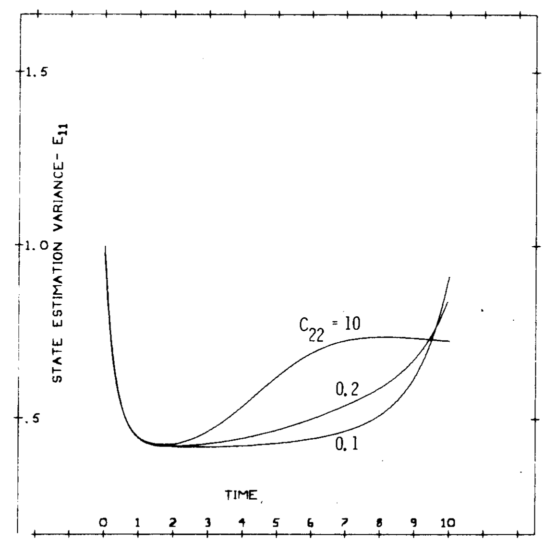

Figure 5-9 State Estimation Performance

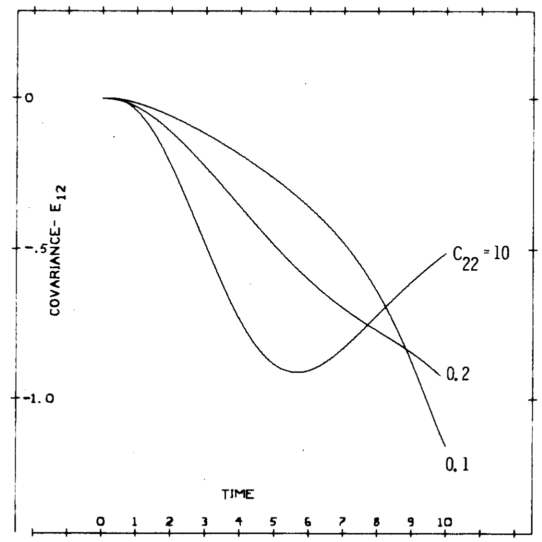

Figure 5-10 Covariance History

5.4 Transient Identification in a Second-Order System

This section considers the optimum transient identification in a second-order system; the following section discusses the performance of the computation algorithm for this example.

The system considered is described by

$$\ddot{y}^a + a^a \dot{y}^a + b^a y^a = u + n \qquad (5.4\text{-}1)$$

The constants a^a and b^a are assumed to be Gaussian distributed with means of a and b, respectively.

An open-loop control is to be designed for the noise-free system that represents Eq. 5.4-1:

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} \qquad (5.4\text{-}2)$$

or

$$\underline{\dot{x}} = \underline{f}\,(\underline{x}, \underline{u}) \qquad (5.4\text{-}3)$$

The boundary conditions are specified

$$x_1(0) = 0 \quad , \quad x_1(T) = 10 \qquad (5.4\text{-}4)$$

$$x_2(0) = 0 \quad , \quad x_2(T) = 0 \qquad (5.4\text{-}5)$$

Thus, the system is to be moved from 0 to 10 and arrive with zero velocity – a very exacting constraint.

The covariance matrix representing the performance of the optimum estimator is of dimension 4 and corresponds to estimation of the deviations in the augmented state vector with elements x_1, x_2, a and b. Then, the F matrix in the differential equation

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \tag{5.4-6}$$

is

$$F = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -a & -b & -x_1 & -x_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4-7}$$

The nominal values of a and b are chosen to be 2.51 and 3.15 which correspond to a critically damped system and natural period of 3.9 seconds.

It is assumed that linear measurements of the actual position are available; the matrix M is then

$$M = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix} \tag{5.4-8}$$

The assumed values for the measurement and driving-noise matrices are

$$U = 0.05 \tag{5.4-9}$$

and

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0.005 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4-10}$$

The initial covariance matrix of the errors in the estimates of the deviations is chosen as

$$E = \begin{bmatrix} 0.1 & 0 & 0 & 0 \\ 0 & 0.1 & 0 & 0 \\ 0 & 0 & 0.2 & 0 \\ 0 & 0 & 0 & 0.2 \end{bmatrix} \tag{5.4-11}$$

The cost function is again chosen to trade-off the amount of energy put into the system versus uncertainties in the estimates at the terminal times.

$$J = \mathrm{tr}\left[C\,E(T)\right] + 0.5 \int_o^T u^2\,dt \tag{5.4-12}$$

where the terminal time equals 10.

Defining

$$h = L + \underline{p}^T \underline{f} + \mathrm{tr}\left[P\,\dot{E}\right] \tag{5.4-13}$$

or

$$h = 0.5\,u^2 + \underline{p}^T \underline{f} + \mathrm{tr}\left[P\,\dot{E}\right] \tag{5.4-14}$$

where $\underline{f}$ is the right-hand side of Eq. 5.4-2 and $\dot{E}$ is given by Eq. 5.4-6.

The 2-dimensional adjoint variable $\underline{p}$ satisfies

$$\dot{\underline{p}} = -h_{\underline{x}}^T = -F_1^T \underline{p} - \left[\mathrm{tr}\,(P\,\dot{E})\right]_{\underline{x}}^T \tag{5.4-15}$$

or

$$\dot{\underline{p}} = -F_1^T \underline{p} - \begin{bmatrix} \mathrm{tr}\left[P\,(F_{x_1} E + E\,F_{x_1}^T)\right] \\ \\ \mathrm{tr}\left[P\,(F_{x_2} E + E\,F_{x_2}^T)\right] \end{bmatrix} \tag{5.4-16}$$

where

$$F_1 = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} \tag{5.4-17}$$

$$F_{x_1} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4-18}$$

and

$$F_{x_2} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{5.4-19}$$

The 4-dimensional adjoint variable P satisfies

$$\dot{p} = -h_E = -(F - EM^T U^{-1} M)^T P - P(F - EM^T U^{-1} M) \tag{5.4-20}$$

with

$$P(T) = C \tag{5.4-21}$$

Finally, the n x q (2 x 2) influence function R obeys

$$\dot{R} = -\underline{H}_{\underline{x}}^T = -F_1^T R \tag{5.4-22}$$

and

$$R(T) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{5.4-23}$$

The gradient method for numerical solution proceeds by first quessing an initial u. Then Eq. 5.4-2 and 5.4-6 are integrated forward. Then integrate Eq. 5.4-16, 5.4-20, and 5.4-22 backward. The remaining steps follow according to Eq. 4.3-36 through 4.3-44.

Figures 5-11 through 5-17 give the results of the optimization procedure for two cases: the minimum-energy solution C = O and the case corresponding to also weighting the terminal uncertainty in the estimate of the deviation in b where C_{44} is chosen as 1000. Even though the system must meet strict terminal constraints on position and velocity, substantial estimation performance is achieved in all components as opposed to the minimum-energy case. The increase in the energy integral is only from 798 to 830 or approximately 4%. For this small increase in the amount of energy spent, significant improvement is estimation performance is achieved. Table 5-1 compares these results with an additional case involving weighting on the terminal uncertainty in a (C_{33} = 1000).

Table 5-1. Estimation Performance

$$\ddot{y}^a + a^a \dot{y}^a + b^a y^a = u + n$$

	Variance Position Estimate	Variance Velocity Estimate	Variance $\hat{a}$	Variance $\hat{b}$	Energy Integral
Initial Variance T = 0	0.1	0.1	0.2	0.2	
Minimum Energy T = 10 (C = O)	0.126	0.506	0.048	0.123	798
$\hat{a}$ Weighting $C_{33} = 1000$ T = 10	0.115	0.377	0.024	0.075	817
$\hat{b}$ Weighting $C_{44} = 1000$ T = 10	0.093	0.215	0.024	0.049	830

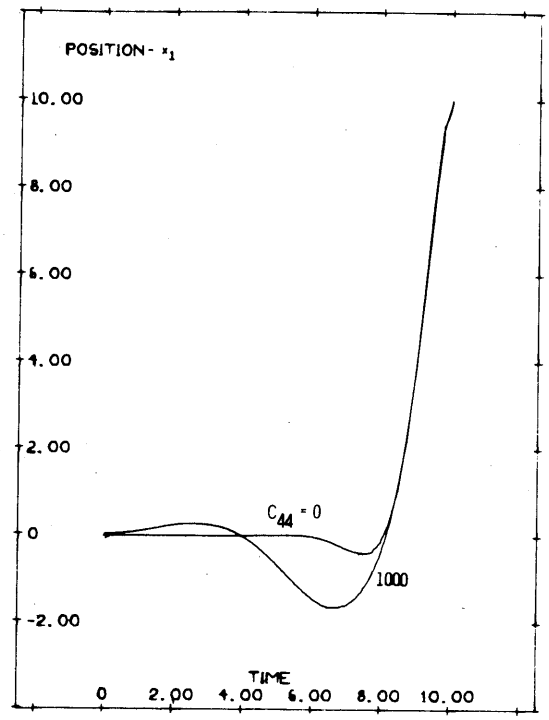

Figure 5-11 Nominal Position Velocity

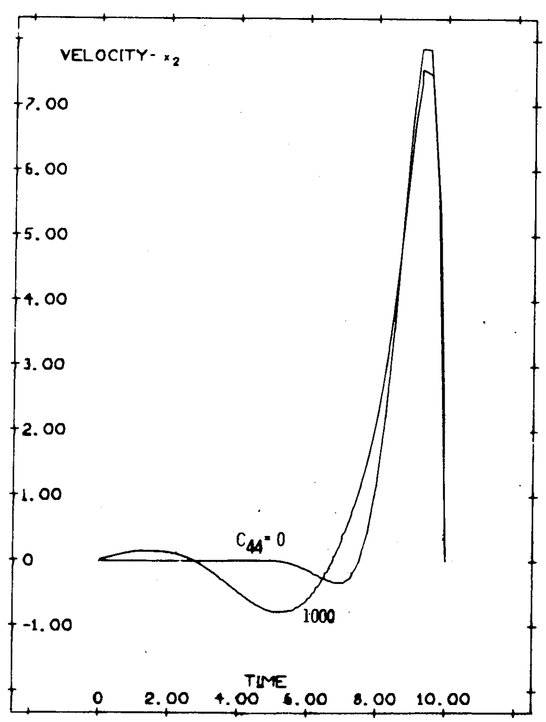

Figure 5-12 Nominal Velocity History

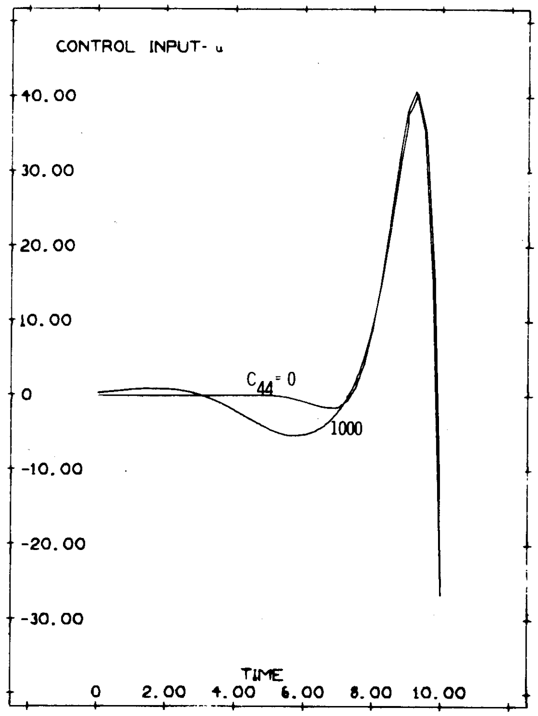

Figure 5-13 Control History

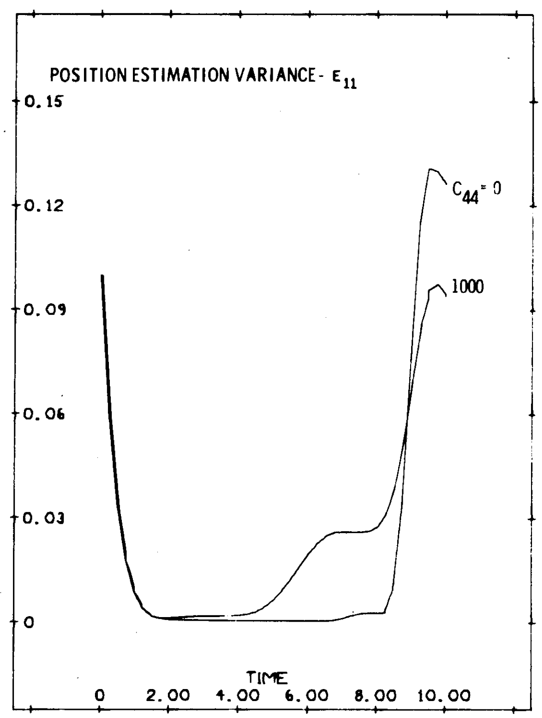

Figure 5-14 Position Estimation Variance

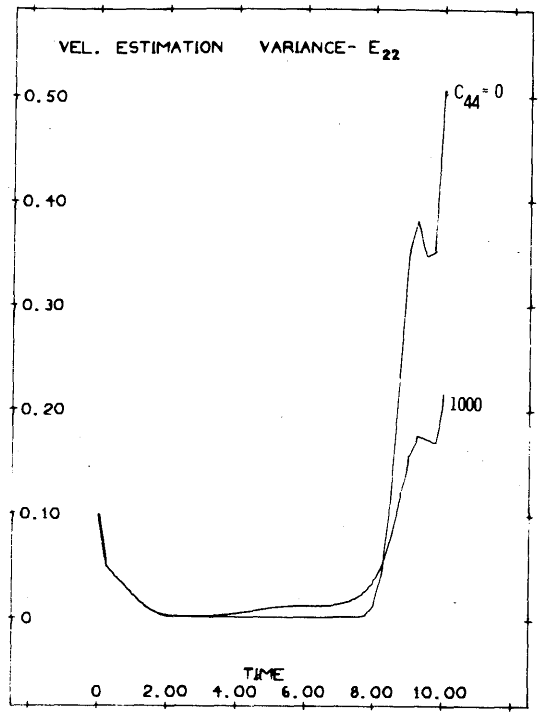

Figure 5-15 Velocity Estimation Variance

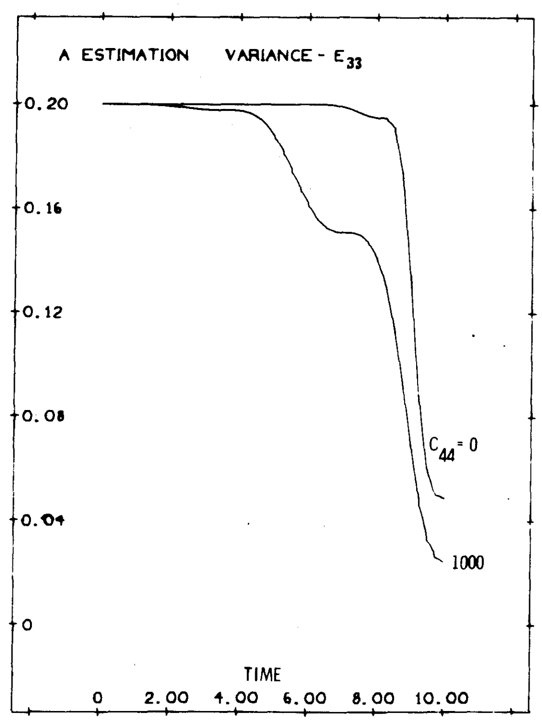

Figure 5-16 a Estimation Variance

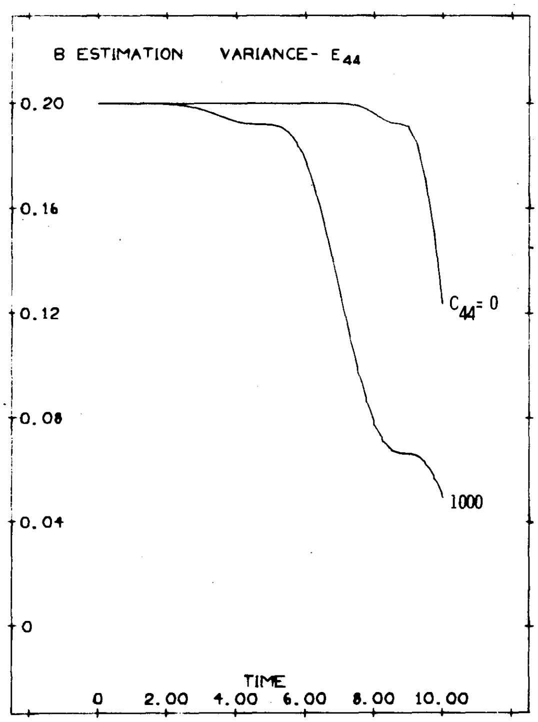

Figure 5-17 b Estimation Variance

5.5 Performance of the Algorithm

The numerical solution to the second-order example involved some typical problems which will now be discussed.

The choice of an initial guess for the control and an associated weighting matrix and constant d must be made with a certain amount of care. If the weighting matrix is chosen too small, large changes in control can occur and the procedure seems to have difficulty in finding an improved control since the linearization inherent in the gradient method is violated. This is not so much a fault of the procedure; the problem itself is very nonlinear. A practical solution was found to be basing the choice for the weighting matrix on its previous value and the value of the change in the control at the terminal time. The previous value of the weighting matrix was scaled so that it would have given a 10% change in the final value of the control. Such a procedure limits the allowable control changes and, while increasing the computer time, helps to guarantee convergence to an optimum.

The choice for an initial u was the solution to the minimum-energy problem as described by Bryson and Ho (1969). The numerical procedure then took seven iterations to reach the solution to the case $C_{33} = 1000$. Using that solution for the next guess, it took only 6 steps to reach the solution for $C_{44} = 1000$. The stopping conditions in each case were the sum of the absolute values of position and velocity errors less than 0.25 and Eq. 4.3-27 less than 0.05.

It was found useful to build into the computer program as many internal checks as possible, while at the same time a most useful test is verifying its capability to find the well-known minimum-energy solution.

It was also convenient to choose as large as possible a time step for integration of the differential equations. Although the second-order system has a natural period of 3.9, the choice of a time step is more involved than simply choosing a smaller number than 3.9. In fact, the governing equation is the covariance matrix equation. With excellent measurements, one must choose a time step such that the term due to the measurements does not make the matrix go negative. For this problem, a time step of 0.25 using a fourth-order Runge-Kutta integration scheme was found to give satisfactory results.

5.6 Summary

This chapter has considered the optimum open-loop identification of unknown parameters. The examples involving first- and second-order systems were efficiently solved using the gradient method developed in Chapter 4. As was previously mentioned, this optimization procedure has been applied to other sample problems with an equal amount of success.

In the next chapter attention is directed to closed-loop control of systems with unknown parameters. In that case the objective is not minimization of estimation errors; but rather minimization of actual state deviations.

Chapter 6

Closed-Loop Control of Stochastic Nonlinear Systems

6.1 Introduction

This chapter considers the closed-loop control of systems with unknown parameters. Since linear systems with unknown parameters may be considered nonlinear systems, the solution offered in this chapter effectively treats a much wider class of problem - control of stochastic nonlinear systems. Problems in this category include optimum guidance and navigation systems for space and terrestrial vehicles and optimum closed-loop process controllers. The examples used to illustrate the control technique in this chapter and the next, however, involve only unknown parameter problems. Many systems have characteristics that are either unknown or highly variable. The control-system designer must take this into account in order to achieve satisfactory results.

There are two ways of approaching the problem which have been found useful. First, it is possible to study the effect of these unknown changes on system performance and to try to design a controller so these effects are tolerable. This is called the sensitivity approach. Second, if it is possible to make continuous measurements of system behavior and determine the dynamical characteristics, the controller parameters can then be adjusted based on these measurements. This is called the adaptive approach.

The adaptive approach is well documented in books by Sworder (1966) and Aoki (1967). The sensitivity approach is generally less well-known. As a simple example, consider a linear system given by

$$\dot{\underline{x}} = F\underline{x} + G\underline{u} \tag{6.1-1}$$

and a quadratic performance index

$$J = \underline{x}(T)^T S(T)\ \underline{x}(T) + \int_0^T (\underline{x}^T A\underline{x} + \underline{u}^T B\underline{u})\, dt \tag{6.1-2}$$

Then a sensitivity vector can be defined by

$$\underline{s} = \frac{d\underline{x}}{da} \qquad (6.1\text{-}3)$$

where a is a parameter of the system in the F matrix. In the sensitivity approach -- see Kahne (1968) -- differential equations are then developed which describe the propagation of $\underline{s}$ with time. The original cost function is augmented by

$$J = J_1 + \int_0^T (\underline{s}^T D \underline{s}) \, dt \qquad (6.1\text{-}4)$$

and minimized by using optimal control theory.

Three drawbacks to sensitivity theory approach are clearly evident. First, how does one justify in physical terms a choice of D? Second, this approach neglects statistical effects; in particular, statistics associated with the unknown parameters which are generally available. Third, an n-dimensional vector must be defined for each parameter, thereby increasing the computational burden.

The solution offered in this chapter lies somewhere in between these two approaches. The technique developed can handle a priori statistical information about the unknown parameters and does not require an artificial augmentation of the cost to cause the controller to consider the unknown parameters. The dimension is the number of state variables and unknown parameters. The controller is partially adaptive in the sense that the unknown quantities are estimated and control action taken. However, the gains used are determined from nominal values of the parameters and nominal values of their statistics rather than basing the gains on the present-observed quantities. Given an infinitely fast computing machine, this could be done but is impractical at the present time.

The approach in this chapter is based on using practical engineering assumptions to achieve a solution to the control problem. The system is assumed nonlinear and subject to independent white noise. Some nonlinear measurements corrupted by white noise are available and are related to the state of the system. It is desired to minimize the expected value of a cost function that measures the performance of the system. The first practical assumption made in Section 6.2 is that a controller can be built that will keep the actual state vector near a pre-planned value during the operation of the system so that the expected value of the first-order state deviations is zero. Second, the assumption is made that the controller that keeps these perturbations small is a linear function of the best estimate of these deviations. Third, the best estimate is to be obtained from a linear filter. The cost function is then expanded in a power series around the pre-planned trajectory. Because the deviations are held to first-order, the expansion is correct to second-order. Then, in taking the expected value, first-order terms in the expansion are zero and the expected value of second-order terms are covariance matrices. Thus, the cost function is actually evaluated in terms of a deterministic part due to the pre-planned trajectory and calculatable covariance matrices due to the statistical effects. The differential equations that describe the propagation of these covariance matrices are derived in Appendix A.

The cost, once evaluated, is to be minimized, subject to the constraining differential equations. In Section 6.3 the calculus-of-variations approach is used to determine the necessary conditions for optimality. It is first shown that the optimal linear filter is a Kalman filter used to estimate the deviations. Second, the optimal perturbation controller is identical in form to that obtained by quadratic synthesis as given by Bryson and Ho (1969). The third and most important result shows that the necessary conditions defining the pre-planned or deterministic trajectory specify the trajectory as a function of the covariance matrices as well as of the deterministic part of the cost. This latter result is different from the quadratic synthesis approach which picks the pre-planned trajectory on deterministic criteria alone and then uses perturbation estimation and control to follow it. The combined optimization procedure defined in this chapter gives a set of necessary conditions that can be straightforwardly applied in practical problems to design the best trajectory considering the statistical nature of the problem. A numerical technique useful in solving the necessary conditions is given in Appendix E.

Sections 6.4 -- 6.10 give the necessary conditions for a class of special problems and Section 6.11 develops the results for evaluating the effects of incorrect a priori statistics used in the design. Section 6.12 presents the design of a controller for a first-order system with an unknown time constant. For the criteria used, the quadratic synthesis approach would give 24.2% more cost and 97% more mean-squared terminal error over the combined optimization procedure. It is shown that this procedure automatically designs the best controller to minimize the effects of the unknown time constant. In Chapter 7 a higher-dimensional problem involving landing on a planet with an unknown atmosphere is considered.

6.2 Transformation of the Performance Index

Consider a stochastic nonlinear system subject to independent zero-mean white noise $\underline{n}$

$$\dot{\underline{x}}^a = \underline{f}\,(\underline{x}^a, \underline{u}^a, t) + \underline{n}(t) \tag{6.2-1}$$

Continuous measurements are available, subject to independent zero-mean white noise $\underline{v}$

$$\underline{m}^a = \underline{h}\,(\underline{x}^a, \underline{u}^a, t) + \underline{v}(t) \tag{6.2-2}$$

Explicit control over the state and the measurements is allowed through $\underline{u}^a$. It is desired to minimize a cost function of the form

$$J_1 = J\,(\underline{x}^a, \underline{u}^a, t) \tag{6.2-3}$$

Because of the stochastic nature of the problem, it is appropriate to consider minimization of the expected cost

$$< J_1 > = < J\,(\underline{x}^a, \underline{u}^a, t) > \tag{6.2-4}$$

Define a system of identical dynamics to that of Eq. 6.2-1 except for the white noise

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \quad , \quad \underline{x}(0) = \langle \underline{x}^a(0) \rangle \tag{6.2-5}$$

and let

$$\delta\underline{x} = \underline{x}^a - \underline{x} \tag{6.2-6}$$

$$\delta\underline{u} = \underline{u}^a - \underline{u} \tag{6.2-7}$$

Assuming continuous first and second derivatives of J with respect to $\underline{x}$ and $\underline{u}$ exist, the cost Eq. 6.2-3 can be expressed in an infinite series around a cost associated with the noise-free dynamics Eq. 6.2-5

$$\begin{aligned} J_1 &= J(\underline{x}, \underline{u}, t) + J_{\underline{x}}\, \delta\underline{x} + J_{\underline{u}}\, \delta\underline{u} \\ &+ 0.5\, \delta\underline{x}^T J_{\underline{xx}}\, \delta\underline{x} + 0.5\, \delta\underline{u}^T J_{\underline{uu}}\, \delta\underline{u} \\ &+ 0.5\, \delta\underline{u}^T J_{\underline{ux}}\, \delta\underline{x} + 0.5\, \delta\underline{x}^T J_{\underline{xu}}\, \delta\underline{u} \\ &+ \text{more terms of higher-order} \end{aligned} \tag{6.2-8}$$

where the partial derivatives are understood to be taken with respect to $\underline{x}^a$ and $\underline{u}^a$ and are evaluated on the noise-free dynamics. In general, an infinite number of terms must be considered for Eq. 6.2-8 to adequately represent the cost function. It will, therefore, be specified that there exist a suitable control law that makes the system with noise approximate the noise-free dynamics; i. e., a controller that guarantees that a first-order representation of $\delta\underline{x}$ is valid where Eq. 6.2-1 is linearized as in Appendix A to give

$$\delta\dot{\underline{x}} = \underline{f}_{\underline{x}}\, \delta\underline{x} + \underline{f}_{\underline{u}}\, \delta\underline{u} + \underline{n} \tag{6.2-9}$$

or

$$\delta \dot{\underline{x}} = F \; \delta \underline{x} + G \; \delta \underline{u} + \underline{n} \qquad (6.2\text{-}10)$$

Representation of $\delta \underline{x}$ to first-order retains J_1 correct to second-order

$$\begin{aligned} J_1 = & \; J(\underline{x}, \underline{u}, t) + \underline{J_x} \; \delta \underline{x} + \underline{J_u} \; \delta \underline{u} \\ & + 0.5 \; \delta \underline{x}^T \; \underline{\underline{J_{xx}}} \; \delta \underline{x} + 0.5 \; \delta \underline{u}^T \; \underline{\underline{J_{uu}}} \; \delta \underline{u} \\ & + 0.5 \; \delta \underline{u}^T \; \underline{\underline{J_{ux}}} \; \delta \underline{x} + 0.5 \; \delta \underline{x}^T \; \underline{\underline{J_{xu}}} \; \delta \underline{u} \qquad (6.2\text{-}11) \end{aligned}$$

This equation is valid for any control system that has the ability to exert tight control such that the effects of noise can be overcome. In the presence of noise this surely requires feedback. Thus "small" noise is not explicitly assumed, but, rather, the existence of a suitable perturbation controller that exerts "reasonable" values of $\delta \underline{u}$ in keeping $\delta \underline{x}$ small. It should be noted that, for those states which are controllable, their perturbations are controllable through Eq. 6.2-9. For uncontrollable states, their perturbations are also uncontrollable, so that their deviations must remain small for Eq. 6.2-11 to be a valid representation of the cost.

At this point two practical constraints are imposed which then provide an elegant solution to this control problem. They are:

(1) The control perturbation to be applied is a linear function of an estimate of the state perturbation

$$\delta \underline{u} = - C \; \delta \hat{\underline{x}} \qquad (6.2\text{-}12)$$

where the gains C depend on the noise-free system and are to be determined in some optimal way. It will be seen that, when the gains are picked in an optimal manner, they are independent of any uncontrollable states, but the control does depend on those states through the estimates of them. Furthermore, it is assumed that $\delta \underline{u}$ can be applied exactly, although the method of analysis to be used can be easily extended to the case where this is not true.

(2) The estimate $\delta \hat{\underline{x}}$ is to be obtained from an unbiased linear estimator that has the property

$$< \underline{e}(t) > = 0 \tag{6.2-13}$$

where the error in the estimate is defined as

$$\underline{e} = \delta \hat{\underline{x}} - \delta \underline{x} \tag{6.2-14}$$

and the form of the perturbation estimator is specified as

$$\delta \dot{\hat{\underline{x}}} = F \, \delta \hat{\underline{x}} + G \, \delta \underline{u} + K \, (\delta \underline{m} - \underline{h}_{\underline{x}} \, \delta \hat{\underline{x}} - \underline{h}_{\underline{u}} \, \delta \underline{u}) \tag{6.2-15}$$

with K to be determined in an optimal fashion.

With the constraint of Eq. 6.2-15, the initial conditions,

$$< \delta \underline{x}(0) > = 0 \tag{6.2-16}$$

$$< \delta \hat{\underline{x}}(0) > = 0 \tag{6.2-17}$$

the linearized measurements as derived in Appendix A,

$$\delta \underline{m} = \underline{h}_{\underline{x}} \, \delta \underline{x} + \underline{h}_{\underline{u}} \, \delta \underline{u} + v \tag{6.2-18}$$

and the perfect knowledge of $\delta \underline{u}$, Eq. 6.2-15, 6.2-12, and 6.2-10 yield for all time:

$$< \delta \hat{\underline{x}}(t) > = 0 \tag{6.2-19}$$

$$< \delta \underline{x}(t) > = 0 \tag{6.2-20}$$

$$< \delta \underline{u}(t) > = 0 \tag{6.2-21}$$

Using these last two conditions in taking the expected value of the cost function Eq. 6.2-11 results in the elimination of the expected values of $\delta \underline{x}$ and $\delta \underline{u}$:

$$\begin{aligned} < J_1 > = {} & J(\underline{x}, \underline{u}, t) + 0.5 < \delta \underline{x}^T \underline{J_{xx}} \delta \underline{x} > \\ & + 0.5 < \delta \underline{u}^T \underline{J_{uu}} \delta \underline{u} > + 0.5 < \delta \underline{x}^T \underline{J_{xu}} \delta \underline{u} > \\ & + 0.5 < \delta \underline{u}^T \underline{J_{ux}} \delta \underline{x} > \end{aligned} \tag{6.2-22}$$

Equation 6.2-22 can be rewritten, using the general relationship for any $\underline{y}$, $\underline{w}$, and V,

$$\underline{y}^T V \underline{w} = \mathrm{tr}\,(V \underline{w}\, \underline{y}^T) \tag{6.2-23}$$

so that

$$\begin{aligned} < J_1 > = {} & J(\underline{x}, \underline{u}, t) + 0.5\ \mathrm{tr}\,(\underline{J_{xx}} < \delta \underline{x}\, \delta \underline{x}^T >) \\ & + 0.5\ \mathrm{tr}\,(\underline{J_{uu}} < \delta \underline{u}\, \delta \underline{u}^T >) \\ & + 0.5\ \mathrm{tr}\,(\underline{J_{ux}} < \delta \underline{x}\, \delta \underline{u}^T >) \\ & + 0.5\ \mathrm{tr}\,(\underline{J_{xu}} < \delta \underline{u}\, \delta \underline{x}^T >) \end{aligned} \tag{6.2-24}$$

Now, using the control law $\delta \underline{u} = -C \ \delta \hat{\underline{x}}$ and defining

$$E = \langle \underline{e} \ \underline{e}^T \rangle \quad = \text{cov. of the estimation error} \tag{6.2-25}$$

$$\hat{X} = \langle \delta \hat{\underline{x}} \, \delta \hat{\underline{x}}^T \rangle \quad = \text{cov. of the estimate} \tag{6.2-26}$$

$$Z = \langle \underline{e} \ \delta \hat{\underline{x}}^T \rangle \quad = \text{cross-cov. of the error and the estimate} \tag{6.2-27}$$

$$X = \langle \delta \underline{x} \, \delta \underline{x}^T \rangle \quad = \text{cov. of the actual state deviation} \tag{6.2-28}$$

where, from $\underline{e} = \delta \hat{\underline{x}} - \delta \underline{x}$ and Eq. 6.2-25 -- 6.2-28

$$X = E + \hat{X} - Z - Z^T \tag{6.2-29}$$

then Eq. 6.2-24 becomes

$$\begin{aligned} \langle J_1 \rangle = {} & J(\underline{x}, \underline{u}, t) + 0.5 \ \mathrm{tr} \left[J_{\underline{xx}} (E + \hat{X} - Z - Z^T) \right] \\ & + 0.5 \ \mathrm{tr} \left[J_{\underline{uu}} C \hat{X} C^T \right] - 0.5 \ \mathrm{tr} \left[J_{\underline{ux}} (\hat{X} - Z) C^T \right] \\ & - 0.5 \ \mathrm{tr} \left[J_{\underline{xu}} C (\hat{X} - Z^T) \right] \end{aligned} \tag{6.2-30}$$

The original expected value of the cost function has now been evaluated in terms of a deterministic part $J(\underline{x}, \underline{u}, t)$ and second moments. This cost is to be minimized, subject to the differential constraints on $\underline{x}$, and the covariance matrices must also obey differential equations. As derived in Appendix A, they are

$$\dot{E} = (F - KM) E + E (F - KM)^T + KUK^T + Q \tag{6.2-31}$$

$$\dot{\hat{X}} = (F - GC) \hat{X} + \hat{X} (F - GC)^T - KMZ - Z^T M^T K^T + KUK^T \tag{6.2-32}$$

$$\dot{Z} = (F - KM)\,Z + Z\,(F - GC)^T - EM^TK^T + KUK^T \qquad (6.2\text{-}33)$$

with given initial conditions and

$$M = \underline{h}_{\underline{x}} \qquad (6.2\text{-}34)$$

$$Q\,\delta\,(t - t') = \langle \underline{n}(t)\ \underline{n}(t')^T \rangle \qquad (6.2\text{-}35)$$

$$U\,\delta\,(t - t') = \langle \underline{v}(t)\ \underline{v}(t')^T \rangle \qquad (6.2\text{-}36)$$

The optimization problem is to minimize Eq. 6.2-30, subject to Eq. 6.2-31, 6.2-32, 6.2-33, and 6.2-5, by finding the optimal control $\underline{u}$, the optimal linear feedback controller gains C, and the optimal linear filter gains K. The original statistical measure of performance is reflected in the cost by the appearance of covariance matrices.

6.3 The Necessary Conditions

The derivation of the necessary conditions for optimality proceeds in the usual calculus-of-variations approach. First, for convenience, assume the original cost function was to be minimized over a fixed time and was of the form

$$\langle J_1 \rangle = \langle k\left[\underline{x}^a(T)\right] + \int_0^T L\,(\underline{x}^a\,,\ \underline{u}^a\,,\ t)\ dt \rangle \qquad (6.3\text{-}1)$$

and define

$$S(T) = k_{\underline{xx}} \qquad (6.3\text{-}2)$$

$$A\,(\underline{x}\,,\ \underline{u}\,,\ t) = L_{\underline{xx}} \qquad (6.3\text{-}3)$$

$$B\,(\underline{x}\,,\ \underline{u}\,,\ t) = L_{\underline{uu}} \qquad (6.3\text{-}4)$$

$$N\,(\underline{x}\,,\ \underline{u}\,,\ t) = L_{\underline{xu}} \qquad (6.3\text{-}5)$$

Then Eq. 6.2-30 becomes

$$
\begin{aligned}
< J_1 > = & \; k \left[\underline{x}(T) \right] + \int_o^T L \, (\underline{x} \, , \, \underline{u} \, , \, t) \, dt \\
& + 0.5 \, tr \left\{ S(T) \left[E(T) + \hat{X}(T) - Z(T) - Z(T)^T \right] \right\} \\
& + 0.5 \, tr \left[\int_o^T (AE + A\hat{X} - AZ - AZ^T + BC\hat{X}C^T \right. \\
& \left. - NC\hat{X} + NCZ^T - N^T\hat{X}C^T + N^T ZC^T) \, dt \right] \qquad (6.3\text{-}6)
\end{aligned}
$$

First, viewing K as a control parameter to be picked, the variation in cost due to a change in K is

$$
\begin{aligned}
\delta < J_1 > = & \; 0.5 \, tr \left\{ S(T) \left[\delta E(T) + \delta \hat{X}(T) - \delta Z(T) - \delta Z(T)^T \right] \right\} \\
& + 0.5 \, tr \left[\int_o^T (A \, \delta E + A \, \delta \hat{X} - A \, \delta Z - A \, \delta Z^T \right. \\
& + BC \, \delta\hat{X} \, C^T - NC \, \delta \hat{X} + NC \, \delta Z^T \\
& \left. - N^T \, \delta \hat{X} \, C^T + N^T \, \delta Z \, C^T) \, dt \right] \qquad (6.3\text{-}7)
\end{aligned}
$$

From Eq. 6.2-31, 6.2-32, and 6.2-33

$$
\begin{aligned}
\delta \dot{E} = & \; (F - KM) \, \delta E + \delta E \, (F - KM)^T \\
& - \delta K \, ME - EM^T \, \delta K^T + \delta KUK^T + KU\delta K^T \qquad (6.3\text{-}8)
\end{aligned}
$$

$$
\begin{aligned}
\delta \dot{\hat{X}} = & \; (F - GC) \, \delta \hat{X} + \delta \hat{X} \, (F - GC)^T \\
& - \delta K \, MZ - KM \, \delta Z - Z^T \, M^T \, \delta K^T \\
& - \delta Z^T \, M^T \, K^T + \delta KUK^T + KU\delta K^T \qquad (6.3\text{-}9)
\end{aligned}
$$

$$\delta \dot{Z} = (F - KM)\,\delta Z + \delta Z\,(F - GC)^T - \delta K\,MZ - \delta E\,M^T K^T - E M^T \delta K^T + \delta K\,U\,K^T + K\,U\,\delta K^T \tag{6.3-10}$$

With $\delta Z(0) = \delta \hat{X}(0) = \delta E(0) = 0$ and an assumption that the initial error and the estimate are uncorrelated ($Z(0) = 0$), by choice of

$$K = E M^T U^{-1} \tag{6.3-11}$$

then

$$Z(t) = 0 = \delta E(t) = \delta \hat{X}(t) = \delta Z(t) \tag{6.3-12}$$

and

$$\delta < J_1 > = 0 \tag{6.3-13}$$

The cost is optimized (stationary) with respect to changes in K. Furthermore, this choice of K for the optimal linear filter results in the estimate and the error in the estimate being orthogonal for all time. This K corresponds to the Kalman filter and the cost function now reduces to

$$< J_1 > = k\left[\underline{x}(T)\right] + \int_0^T L(\underline{x}, \underline{u}, t)\,dt + 0.5\,\mathrm{tr}\left[S(T)\,E(T)\right] + 0.5\,\mathrm{tr}\left[S(T)\,\hat{X}(T)\right] + 0.5\,\mathrm{tr}\left[\int_0^T (AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T)\,dt\right] \tag{6.3-14}$$

subject to

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \tag{6.3-15}$$

$$\dot{E} = FE + EF^T + Q - EM^T U^{-1} ME \tag{6.3-16}$$

$$\dot{\hat{X}} = (F - GC)\hat{X} + \hat{X}(F - GC)^T + EM^T U^{-1} ME \tag{6.3-17}$$

The derivation of the necessary conditions for optimality now proceeds in the usual calculus-of-variations approach. Adjoin to the cost the constraints ($\dot{\underline{x}}$, $\dot{E}$, and $\dot{\hat{X}}$) by means of arbitrary multipliers ($\underline{p}$, 0.5 P, 0.5 S) and define a Hamiltonian

$$\begin{aligned} H = {} & L + \underline{p}^T \underline{f} + 0.5 \operatorname{tr}(P\dot{E}) + 0.5 \operatorname{tr}(S\dot{\hat{X}}) \\ & + 0.5 \operatorname{tr}(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T) \end{aligned} \tag{6.3-18}$$

The adjoint variables must satisfy

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \tag{6.3-19}$$

$$\dot{P} = -2H_E \quad , \quad P(T) = S(T) \tag{6.3-20}$$

$$\dot{S} = -2H_{\hat{X}} \quad , \quad S(T) = k_{\underline{xx}} \tag{6.3-21}$$

The optimal control parameters ($\underline{u}$ and C) are determined from

$$H_{\underline{u}} = 0 \tag{6.3-22}$$

$$H_C = 0 \tag{6.3-23}$$

Using Eq. 6.3-21 first, results in

$$\dot{S} = -(F - GC)^T S^T - S^T (F - GC) + NC + C^T N^T - C^T BC - A$$

(6.3-24)

Since the boundary condition Eq. 6.3-21 is symmetric, S is symmetric for all time, or

$$\dot{S} = -(F - GC)^T S - S(F - GC) + NC + C^T N^T - C^T BC - A \quad (6.3\text{-}25)$$

Similarly application of Eq. 6.3-20 yields

$$\begin{aligned} \dot{P} = & -(F - EM^T U^{-1} M)^T P - P(F - EM^T U^{-1} M) \\ & - M^T U^{-1} MES - SEM^T U^{-1} M - A \end{aligned} \quad (6.3\text{-}26)$$

Application of Eq. 6.3-23 yields for arbitrary $\hat{X}$

$$C = B^{-1} (G^T S + N^T) \quad (6.3\text{-}27)$$

and substituting into Eq. 6.3-25 gives

$$\dot{S} = -F^T S - SF + (G^T S + N^T)^T B^{-1} (G^T S + N^T) - A \quad (6.3\text{-}28)$$

The feedback-controller gains C are identical to those that would be obtained by using quadratic synthesis around a given reference trajectory. However, application of Eq. 6.3-19 and 6.3-22 shows quite clearly that the noise-free system must be chosen to include the effects of the stochastic nature of the problem:

$$\begin{aligned} H_{\underline{u}} = 0 = & L_u + \underline{p}^T G + 0.5 \left[\mathrm{tr}(P\dot{E}) \right]_{\underline{u}} + 0.5 \left[\mathrm{tr}(S\dot{\hat{X}}) \right]_{\underline{u}} \\ & + 0.5 \left[\mathrm{tr}(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T \hat{X} C^T) \right]_{\underline{u}} \end{aligned}$$

(6.3-29)

$$\begin{aligned} \dot{\underline{p}} = -H_{\underline{x}}^T = & -L_{\underline{x}}^T - F^T \underline{p} - 0.5 \left[\mathrm{tr}(P\dot{E}) \right]_{\underline{x}}^T - 0.5 \left[\mathrm{tr}(S\dot{\hat{X}} \right]_{\underline{x}}^T \\ & - 0.5 \left[\mathrm{tr}(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T \hat{X} C^T) \right]_{\underline{x}}^T \end{aligned}$$

(6.3-30)

Only for the case of a linear system with linear measurements, quadratic cost, and noises independent of the state or control are the terms involving the derivatives of traces equal to zero, and in that case the noise-free trajectory may be designed without regard for the statistics. This section has shown that, under practical engineering constraints of linear perturbation estimation and feedback control, the overall optimization procedure results in a set of necessary conditions that can be straightforwardly applied in practical design problems.

Finally, the end result of the optimization program will be an optimal control history $\underline{u}(t)$, an optimal trajectory $\underline{x}(t)$, a set of feedback controller gains $C(t)$, and a set of estimator gains $K(t)$. All of these quantities can be calculated a priori and implemented into the system. In the following sections some special cases will be considered.

6.4 Case 1. Free Terminal Time Problems

The transversality condition for optimization problems involving free terminal time is analogous to the conditions in Chapter 3:

$$k_t + 0.5 \ \mathrm{tr} \left\{ S_t \left[E(T) + \hat{X}(T) \right] \right\} + H(T) = 0 \qquad (6.4\text{-}1)$$

from which the optimal terminal time is obtained. S and k are differentiated if they are explicit functions of t evaluated at the terminal time.

6.5 Case 2. Terminal Constraints

In the case where the first q components of $\underline{x}^a(T)$ are specified, a q-dimensional linear constraint vector is defined

$$\underline{z}(T) = 0 \qquad (6.5\text{-}1)$$

and the first q components of $\underline{x}(T)$ must satisfy Eq. 6.5-1. Then, in general, $k\left[\underline{x}^a(T)\right]$ would contain only those unspecified components of $\underline{x}^a(T)$. Thus, the term k_{xx} would not account for the fact that the terminal conditions can not be met exactly in the presence of noise and would not call for any perturbation control on these states. A simple solution is to augment the cost function such that

$$S(T) = P(T) = \underline{k_{xx}} + \underline{z}_x^T \, Y \, \underline{z}_x \qquad (6.5\text{-}2)$$

where Y is a positive-definite symmetric matrix whose elements are selected (by experimentation) to give acceptable values of the mean-squared deviations in those components that the noise-free solution must satisfy exactly.

6.6 Case 3. Differentiability Problems

Throughout this analysis it has been assumed that all necessary derivatives exist. In a number of practical cases this may not be true. For example, if $B^{-1} = \infty$, then no weight is attached to the amount of perturbation control used. A natural approach would again be to augment the cost function with a value of B chosen to give acceptable mean-squared perturbation control by experimentation.

Other difficult cases may arise because of explicit nonlinearities in the state or cost. For example, if

$$< J_1 > = < \int_o^T \left| u^a \right| \, dt > \qquad (6.6\text{-}1)$$

it might be appropriate to consider minimizing

$$< J_1 > = \int_o^T |u| \, dt + 0.5 \, \mathrm{tr} \left[\int_o^T B C \hat{X} C^T \, dt \right] \qquad (6.2\text{-}2)$$

where again B would be chosen to give acceptable experimental performance. The optimal control u would be found by application of the minimum principle to the Hamiltonian.

6.7 Case 4. Quadratic Performance Index, Linear State and Linear Measurements

For this case the Eq. 6.3-29 and 6.3-30 become

$$0 = L_{\underline{u}} + \underline{p}^T G \tag{6.7-1}$$

and

$$\dot{\underline{p}} = - L_{\underline{x}}^T - F^T \underline{p} \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \tag{6.7-2}$$

which means that the optimal deterministic control is designed without regard to the statistical nature of the problem. The perturbation controller and estimator still obey the previous equations

$$\delta \underline{u} = - C \delta \hat{\underline{x}} = - B^{-1} (G^T S + N^T) \delta \hat{\underline{x}} \tag{6.7-3}$$

where S is the solution to the matrix Riccati equation. It is well known, see Bryson and Ho (1969), that the optimal solution to Eq. 6.7-1 and 6.7-2 can be formulated in a closed-loop fashion as

$$\underline{u} = - C \underline{x} \tag{6.7-4}$$

Thus, the actual control applied is

$$\underline{u}^a = - C (\underline{x} + \delta \hat{\underline{x}}) \tag{6.7-5}$$

But $\delta \hat{\underline{x}} + \underline{x}$ is simply the optimal estimate of $\underline{x}^a$, which is obtained from a optimal linear estimator, so

$$\underline{u}^a = - C \hat{\underline{x}}^a \tag{6.7-6}$$

Thus, the solution presented in this chapter reduces to the quadratic synthesis solution which optimizes the deterministic performance index to give a feedback law and uses an optimal filter to generate an estimate of the state.

6.8 Case 5. Nonlinear Criteria, Linear State, and Linear Measurements

In this case the perturbation estimator-controller exists as previously defined, but the optimal deterministic portion of the control is obtained from

$$0 = L_{\underline{u}} + \underline{p}^T G + 0.5 \left[tr\,(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T) \right]_{\underline{u}} \tag{6.8-1}$$

$$\dot{\underline{p}} = - L_{\underline{x}}^T - F^T \underline{p} - 0.5 \left[tr\,(AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T) \right]_{\underline{x}}^T \tag{6.8-2}$$

where the derivatives are taken of the functions with explicit dependence on $\underline{x}$ or $\underline{u}$; that is, A, B, and N. Necessary conditions for problems in this case, without the practical assumptions made in this chapter, have been shown to be partial integrodifferential equations which are extremely difficult to solve (see Deyst, 1966). The assumptions made in this chapter allow solution by using the numerical procedure to be presented in Appendix E.

6.9 Case 6. Terminal Cost Only, Nonlinear State, and Nonlinear Measurements

In this case the cost must again be augmented by a matrix B for the definition of the perturbation controller. The optimum deterministic control is obtained from

$$0 = \underline{p}^T G + 0.5 \left[tr\,(P\dot{E}) \right]_{\underline{u}} + 0.5 \left[tr\,(S\dot{\hat{X}}) \right]_{\underline{u}} \tag{6.9-1}$$

and

$$\dot{\underline{p}} = - F^T \underline{p} - 0.5 \left[tr\,(P\dot{E}) \right]_{\underline{x}}^T - 0.5 \left[tr\,(S\dot{\hat{X}}) \right]_{\underline{x}}^T \tag{6.9-2}$$

The most closely related research on this problem (and to this chapter) was performed by Denham (1964). He considers a slightly more general nonlinear state but with only terminal costs. His results are not applicable to systems linear in the control, since his expansion is in the Hamiltonian rather than the cost. He retains other terms in the expansion for $\delta \dot{\underline{x}}$, resulting in a set of extremely complicated necessary conditions that involve calculation of terms such as $< \underline{e} >$, $< \delta \underline{x} >$ and $< \delta \underline{m} >$. (See for example, his Section VII.) The approach used in this chapter – immediately transforming the cost function to a deterministic quantity and viewing the covariance matrices as additional constraints – lends itself to a particularly simple solution with a clear interpretation of the results for a wide variety of optimization problems. Fitzgerald (1964) considered the same case as Denham with a more general noise model.

6.10 Case 7. Quadratic Cost, Nonlinear State and Nonlinear Measurements

This is the most common case found in engineering since minimization of mean squared control and/or state has a wide variety of physical interpretations. The same perturbation estimator-controller structure results as previously and the optimal deterministic control is determined from

$$0 = L_{\underline{u}} + \underline{p}^T G + 0.5 \left[\text{tr } (P \dot{E}) \right]_{\underline{u}} + 0.5 \left[\text{tr } (S \dot{\hat{X}}) \right]_{\underline{u}} \qquad (6.10\text{-}1)$$

and

$$\dot{\underline{p}} = - L_{\underline{x}}^T - F^T \underline{p} - 0.5 \left[\text{tr } (P \dot{E}) \right]_{\underline{x}}^T - 0.5 \left[\text{tr } (S \dot{\hat{X}}) \right]_{\underline{x}}^T \qquad (6.10\text{-}2)$$

The noise-free and perturbation systems must clearly be designed simultaneously as indicated by Eq. 6.10-1 and 6.10-2.

The "standard approach" to problems of this form has previously been to determine the open-loop control from

$$0 = L_{\underline{u}} + \underline{p}^T G \qquad (6.10\text{-}3)$$

and

$$\dot{\underline{p}} = - L_{\underline{x}}^T - F^T \underline{p} \qquad (6.10\text{-}4)$$

and then use the perturbation controller-estimator combination presented here. Such an approach is not correct, since the terms involving the trace in Eq. 6.10-1 and 6.10-2 are omitted. In Section 12 of this chapter, it is shown for a sample problem that this standard approach costs 24% more than the optimum defined by Eq. 6.10-1 and 6.10-2.

In previous chapters, optimum open-loop control signals have been determined to minimize cost functions composed of, in part, the covariance matrix. In those cases, part of the necessary conditions had as equations

$$0 = L_{\underline{u}} + \underline{p}^T G + 0.5 \left[\mathrm{tr}\,(P\dot{E}) \right]_{\underline{u}} \qquad (6.10\text{-}5)$$

and

$$\dot{\underline{p}} = - L_{\underline{x}}^T - F^T \underline{p} - 0.5 \left[\mathrm{tr}\,(P\dot{E}) \right]_{\underline{x}}^T \qquad (6.10\text{-}6)$$

It was thought that a possible closed-loop controller could be the optimum open-loop signal as defined by Eq. 6.10-5 and Eq. 6.10-6, together with an optimal estimator controller as defined by quadratic synthesis. Such an approach is not correct, since, in a closed-loop system, the interest is in minimization of mean-squared state errors, not in minimization of estimation error. Although comparing Eq. 6.10-5, 6.10-1, and 6.10-3 could lead to the belief that this approach would be a step in the right direction, such a conclusion is unfounded. In fact, for the sample problem in Section 12, the approach is shown to give poorer performance than even using Eq. 6.10-3 and 6.10-4.

One additional comment should be made. The optimal deterministic portion of the control has not been constrained to be open-loop. That is, it may be possible to formulate the deterministic control as a feedback on the noise-free state. Generally, this will not be possible because of the nonlinear nature of the problem. (Of course, it could be constrained to be so by the same method as the perturbation controller.)

6.11 Effect of Incorrect Statistics

Suppose, for a problem under consideration, the optimum control $\underline{u}$, the optimum trajectory $\underline{x}$, and the optimal gains C and K have been determined. The minimum value of the cost Eq. 6.3-14 has been found. Suppose the controller design was based on incorrect statistics; that is, the true values were $E(0)^o$, U^o, and Q^o. The actual cost can be evaluated, providing the system still represents an effective closed-loop controller, with the added complication that the estimate and the error in the estimate are no longer orthogonal because of the now suboptimal gains. The actual expected value of the cost with the incorrect statistics is found by evaluating Eq. 6.3-6

$$\begin{aligned} \langle J_1 \rangle = & \; k\left[\underline{x}(T)\right] + \int_0^T L(\underline{x}, \underline{u}, t)\, dt \\ & + 0.5 \operatorname{tr} \left\{ S(T) \left[E(T) + \hat{X}(T) - Z(T) - Z(T)^T \right] \right\} \\ & + 0.5 \operatorname{tr} \left[\int_0^T (AE + A\hat{X} - AZ - AZ^T + BC\hat{X}C^T \right. \\ & \left. - NC\hat{X} + NCZ^T - N^T\hat{X}C^T + N^T ZC^T)\, dt \right] \end{aligned} \tag{6.11-1}$$

using

$$\dot{\underline{x}} = \underline{f}(\underline{x}, \underline{u}, t) \tag{6.11-2}$$

to evaluate first two terms and integrating the following equations with the correct initial conditions

$$\dot{E} = (F - KM) E + E (F - KM)^T + K U^o K^T + Q^o \quad (6.11\text{-}3)$$

$$\dot{\hat{X}} = (F - GC) \hat{X} + \hat{X} (F - GC)^T - KMZ - Z^T M^T K^T + K U^o K^T \quad (6.11\text{-}4)$$

$$\dot{Z} = (F - KM) Z + Z (F - GC)^T - E M^T K^T + K U^o K^T \quad (6.11\text{-}5)$$

to evaluate the remaining terms, where all matrices S(T), M, Q, A, etc., are evaluated on the model of the system.

6.12 Example. Closed-Loop Control of a First-Order System With Unknown Time Constant

As an illustration of the new control technique, a closed-loop controller will be designed for the stochastic first-order system

$$\dot{y}^a = - b^a y^a + u^a + n \quad (6.12.1)$$

The inverse-time constant b^a is assumed to be an unknown constant picked from a Gaussian distribution with mean b. Thus, the state variable differential equation is of dimension 2.

$$\dot{\underline{x}}^a = \begin{bmatrix} \dot{y}^a \\ \dot{b}^a \end{bmatrix} = \begin{bmatrix} \dot{x}_1^a \\ \dot{x}_2^a \end{bmatrix} = \begin{bmatrix} - x_2^a x_1^a \\ 0 \end{bmatrix} + \begin{bmatrix} u^a \\ 0 \end{bmatrix} + \begin{bmatrix} n \\ 0 \end{bmatrix} \quad (6.12\text{-}2)$$

The noise-free system obeys

$$\underline{\dot{x}} = \begin{bmatrix} \dot{x}_1 \\ \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -x_2 x_1 \\ \\ 0 \end{bmatrix} + \begin{bmatrix} u \\ \\ 0 \end{bmatrix} \tag{6.12-3}$$

with the assumed initial conditions

$$x_2(0) = < b^a > = b = 1 \tag{6.12-4}$$

$$x_1(0) = < x_1^a(0) > = 0 \tag{6.12-5}$$

Furthermore, it is assumed that the expected value of y^a at the terminal time is specified as

$$x_1(T) = < y^a(T) > = 1 \tag{6.12-6}$$

The matrices F and G are

$$F = \begin{bmatrix} -x_2 & -x_1 \\ \\ 0 & 0 \end{bmatrix} , \quad G = \begin{bmatrix} 1 \\ \\ 0 \end{bmatrix} \tag{6.12-7}$$

Linear measurements of y^a corrupted by white by white noise are available to the controller

$$m^a = y^a + v \tag{6.12-8}$$

then

$$M = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad (6.12\text{-}9)$$

The stochastic cost function to be minimized is

$$< J > = 0.5 < \int_o^T (u^a)^2 \, dt > \qquad (6.12\text{-}10)$$

Taking the expected value with the assumption of perturbation estimation and control results in

$$< J > = 0.5 \int_o^T u^2 \, dt + 0.5 \, \mathrm{tr} \left[\int_o^T B C \hat{X} C^T \, dt \right] \qquad (6.12\text{-}11)$$

where

$$B = L_{uu} = 1 \qquad (6.12\text{-}12)$$

From Eq. 6.12-11 it is clear that no penalty would be attached to deviations in $x_1^a(T)$ away from specified nominal $x_1(T)$. Thus, the cost is augmented to weight terminal mean-squared deviations in the perturbation controller

$$< J_1 > = 0.5 \int_o^T u^2 \, dt + 0.5 \, \mathrm{tr} \left[\int_o^T B C \hat{X} C^T \, dt \right] + 0.5 \, \mathrm{tr} \left\{ S(T) \left[E(T) + \hat{X}(T) \right] \right\} \qquad (6.12\text{-}13)$$

This problem may be catagorized as quadratic cost with nonlinear state (Case 7). The necessary conditions are

$$\dot{\underline{x}} = f(\underline{x}, u) \qquad (6.12\text{-}14)$$

$$\underline{x}(0)^T = \begin{bmatrix} 0 & b \end{bmatrix} , \quad \underline{x}(T)^T = \begin{bmatrix} 1 & b \end{bmatrix} \qquad (6.12\text{-}15)$$

$$\dot{\underline{p}} = -F^T \underline{p} - .5 \left[tr\,(P\dot{E})\right]^T_{\underline{x}} - .5 \left[tr\,(S\dot{\hat{X}})\right]^T_{\underline{x}} \tag{6.12-16}$$

$$0 = H_u = u + \underline{p}^T G \tag{6.12-17}$$

$$\dot{E} = FE + EF^T + Q - EM^T U^{-1} ME \tag{6.12-18}$$

$$E(0) \text{ given} \tag{6.12-19}$$

$$\dot{\hat{X}} = (F - GB^{-1} G^T S)\hat{X} + \hat{X}(F - GB^{-1} G^T S)^T + EM^T U^{-1} ME \tag{6.12-20}$$

$$\hat{X}(0) = 0 \tag{6.12-21}$$

$$\dot{P} = -(F - EM^T U^{-1} M)^T P - P(F - EM^T U^{-1} M) - M^T U^{-1} MES - SEM^T U^{-1} M \tag{6.12-22}$$

$$P(T) = S(T) \tag{6.12-23}$$

$$\dot{S} = -F^T S - SF^T + SGB^{-1} G^T S \tag{6.12-24}$$

$$S(T) = S(T) \tag{6.12-25}$$

The numerical values used in the solution to this problem were $T = 10$,

$$E(0) = \begin{bmatrix} 1 & 0 \\ 0 & 5 \end{bmatrix} \tag{6.12-26}$$

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad (6.12\text{-}27)$$

$$U = 1 \qquad (6.12\text{-}28)$$

and

$$S(T) = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad (6.12\text{-}29)$$

The necessary conditions were solved numerically, using the gradient method presented in Appendix E. The results of this combined optimization approach will be compared with the quadratic-synthesis approach. In this latter case the nominal trajectory is determined from the same necessary conditions Eq. 6.12-14 -- 6.12-25 with the exception that the nominal trajectory is picked without regard to the statistics, so that Eq. 6.2-16 becomes

$$\dot{\underline{p}} = - F^T \underline{p} \qquad (6.12\text{-}30)$$

as a result of minimizing

$$0.5 \int_o^T u^2 \, dt \qquad (6.12\text{-}31)$$

without the covariance terms. The time constant is being identified.

The optimal deterministic control signals are shown in Fig. 6-1. The quadratic-synthesis approach results in a control $u = 2 \exp (t-10)$, minimizing the energy integral Eq. 6.12-31 with a value of 1.00. The combined optimization approach yields a value of 1.31 for the energy integral. However, the quadratic-synthesis approach yields a value of 1.51 for the remaining matrix terms in the cost Eq. 6.12-13 as opposed to 0.71 for the combined optimization. The total average cost is thus 2.51 versus 2.02; the quadratic synthesis approach actually costs 24.2% more. Such a substantial improvement in performance in a more practical problem would be significant.

The difference in cost between the two approaches is due primarily to the performance in minimizing the mean-squared deviation in the state at the terminal time. Figures 6-2 and 6-3 show the difference between the two cases in this respect, 1.36 versus 0.69. Figure 6-4 gives the covariances for the inverse-time constant. Note that the estimation of the inverse time constant is poorer in the combined optimization case. (4.42 versus 3.37) This is because the control system tends to minimize the sensitivity to the unknown parameter.

This last statement can be better understood from Fig. 6-5. The final value of x_1 can be written as

$$x_1(T) = -b \int_0^T x_1 \, dt + \int_0^T u \, dt \qquad (6.12\text{-}32)$$

Clearly, variations in $x_1(T)$ with respect to changes in b are minimized, if the area under the x_1 versus t curve is minimized. The combined optimization procedure attempts to do just that, as is shown in Fig. 6-5, completely automatically as opposed to the senstivity-theory design approach to problems of this type.

Furthermore, in Chapter 5 it was shown that the best optimum open-loop control input for identification of the inverse-time constant resulted in a trajectory lying above the minimum-energy solution. Such an input signal maximized the effect of the unknown inverse-time constant; this is not desired in the closed-loop controller and in fact results in performance much worse than simply designing a minimum energy controller. The optimum open-loop control input is not even a local minimum for this problem.

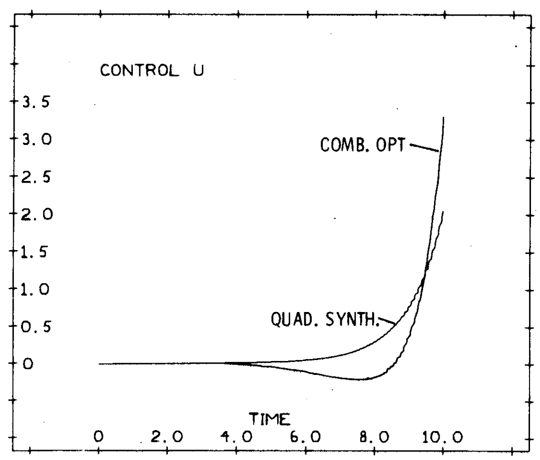

Figure 6-1 Optimal Control Input

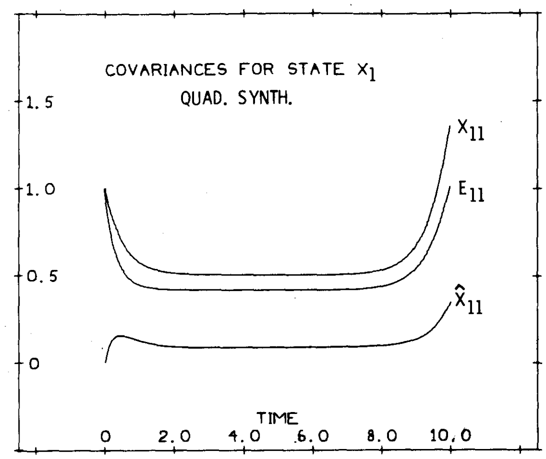

Figure 6-2 Covariances for x_1 - Quadratic Synthesis

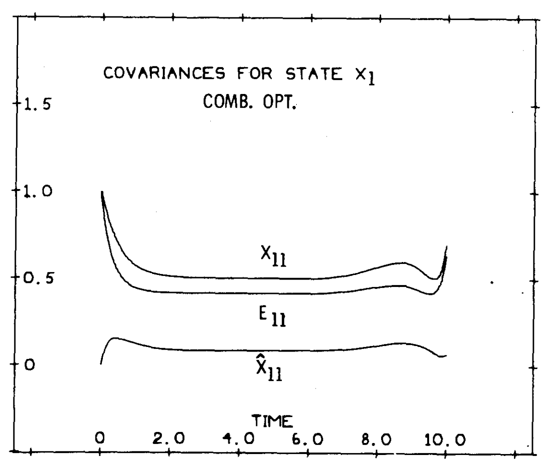

Figure 6-3 Covariances for x_1 - Combined Optimization

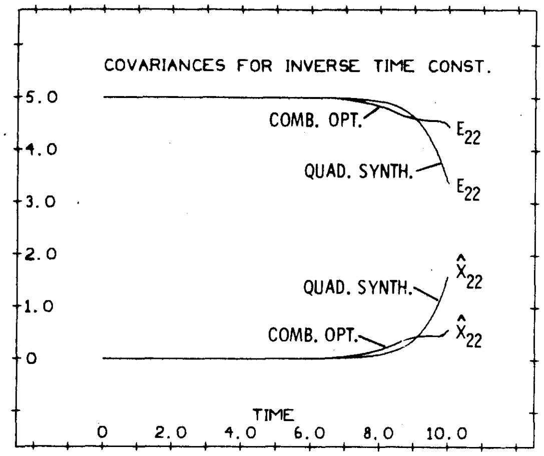

Figure 6-4 Covariances for Inverse Time Constant

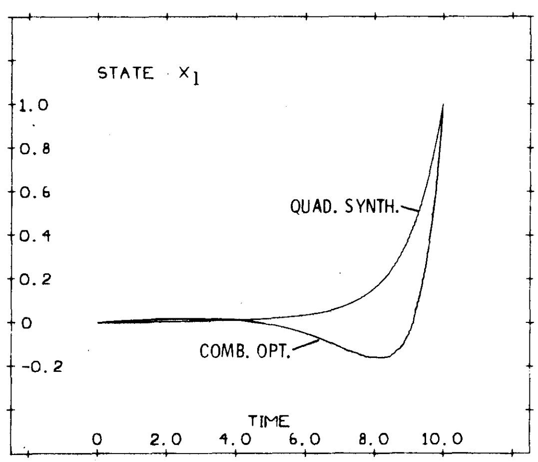

Figure 6-5 Optimum Trajectory for x_1

6.13 Summary

This chapter has presented a new technique for the control of stochastic nonlinear systems. For the sample problem considered, the procedure was seen to offer substantial improvements in system performance as compared to the quadratic synthesis approach. Certainly the main disadvantage of the procedure lies in the fact that it is only appropriate in situations where the reference-trajectory concept is valid. One situation where this is true is in atmospheric-entry problems where the reference-trajectory concept is well-established. The next chapter treats a timely problem - entry into the atmosphere of the planet Mars.

Chapter 7

Optimum Entry Control With An Unknown Atmosphere

7.1 Introduction

The application of the stochastic control theory presented in Chapter 6 depends on being able to use the reference trajectory concept. In entry problems this concept is well-established, as in the Apollo entry control system, and this chapter presents the design for an entry controller when the atmosphere is subject to density uncertainties. The example is based on a Mars entry but the parameters can be changed for another planet.

The Mars entry problem is interesting in that trajectories obtained by applying aerodynamic forces are particularly sensitive to deviations in the parameters of the Mars atmosphere (Shen and Cefola, 1968). Also, the present estimates of the atmospheric parameters cover a wide range of values; for example, scale heights from 3 to 14 miles (Evans, et. al. ,1968) Such a wide variation in parameters would be a problem in designing an entry guidance scheme where the control effectiveness, as well as the trajectory, is subject to large perturbations. Thus an effective guidance, navigation, and control system will be necessary to compensate for atmospheric parameter deviations if the entry vehicle is to operate for all atmospheric possibilities.

The objective of the proposed guidance scheme will be to minimize the errors in a set of pre-specified terminal conditions whatever the atmosphere encountered on Mars. More specifically, the problem of minimizing the errors in range and altitude at a specified range and attitude above the planet's surface will be considered.

The entry vehicle is assumed to be capable of a maximum lift-to-drag ratio of 1.0. This ratio may be achieved either by varying the angle of attack of a winged vehicle or, as in Apollo, flying at constant angle of attack but rolling back and forth around the velocity vector to create less than maximum lift-to-drag ratio. The vehicle is also assumed to have a terminal slow-down capability after the specified final altitude is achieved. The method of aoproach to this problem is summarized in the next paragraph.

A nominal model of the Martian atmosphere and the nominal vehicle parameters are used to generate a trajectory that meets the specified terminal conditions. A convenience choice is a constant lift-to-drag ratio flight of 0.5 which results in a range of 1257 miles. This trajectory gives satisfactory heating and g loading characteristics. Quadratic synthesis, including identification of the density, would then result in a set of rms position errors for the given cost function. The combined optimization approach is then applied to the same cost function; i.e., the nominal trajectory is determined by minimizing the rms deviations. The resultant trajectory is found to give essentially the same heating and g loading as the constant lift-to-drag ratio flight, but the latter case gives approximately 25% more range error than the combined optimization approach.

In Section 7.2 the two-dimensional planar equations of motion are presented for the lifting entry. In Section 7.3 models for the atmosphere and variations in density are derived. Section 7.4 describes the onboard measurements and Section 7.5 gives the constraining differential equations for the nominal state, the covariance matrix of the estimate, and the covariance matrix of the error in the estimate. Section 7.6 presents the cost function to be minimized using the numerical values of the parameters given in Appendix F and Section 7.7 presents the computational results. Additional comments on the performance of the computational algorithm are given in Section 7.8

7.2 Vehicle Dynamics

Assuming the motion of the vehicle may be adequately described as that of a point mass about a spherical non-rotating Mars, the planar equations of motion are

$$\dot{h} = v \sin \gamma \tag{7.2-1}$$

$$\dot{\theta} = \frac{v \cos \gamma}{r + h} \tag{7.2-2}$$

$$\dot{v} = f_v - g \sin \gamma \tag{7.2-3}$$

$$\dot{\gamma} = \frac{f_\gamma}{v} - \frac{g \cos \gamma}{v} + \frac{v \cos \gamma}{r + h} \tag{7.2-4}$$

where

h = altitude above the surface

γ = flight path angle measured positive above the horizon

v = velocity relative to Mars

r = radius of Mars

g = gravity at altitude h calculated from

$$g = g_0 \frac{r^2}{(r+h)^2} \tag{7.2-5}$$

where g_o is the surface gravity

and where f_v and f_γ are the aerodynamic specific forces acting on the vehicle. These forces may be written as

$$f_v = - \frac{\rho a v^2}{2m} c_d \tag{7.2-6}$$

$$f_\gamma = \frac{\rho a v^2}{2 m} c_1 \tag{7.2-7}$$

where

ρ = atmospheric density

a = cross-sectional area of the vehicle

m = mass of the vehicle

c_d = drag coefficient

c_1 = lift coefficient

Generally, lift and drag coefficients are nonlinear functions of angle of attack, sideslip angle, Mach number, Reynolds number, and the angular rates of the vehicle. At the high velocities to be encountered during entry, the coefficients may be assumed dependent only on the angle of attack and sideslip angle. Since this is a planar analysis and range control of the vehicle is likely to be obtained by rolling the vehicle around the velocity vector to achieve less than maximum range, the specific forces are more conveniently written as

$$f_v = - \frac{\rho \, a \, v^2}{2 \, m} c_d \tag{7.2-8}$$

$$f_\gamma = \frac{\rho \, a \, v^2}{2 \, m} c_d \, u \tag{7.2-9}$$

where u is the controlled lift-to-drag ratio.

Defining $c_d \, a/m$ as the parameter b, there results for state equations

$$\dot{h} = v \sin \gamma \tag{7.2-10}$$

$$\dot{\theta} = \frac{v \cos \gamma}{r + h} \tag{7.2-11}$$

$$\dot{v} = - \frac{g_0 \sin\gamma \, r^2}{(r + h)^2} - 0.5 \, b \, \rho \, v^2 \tag{7.2-12}$$

$$\dot{\gamma} = - \frac{g_0 \cos \gamma \, r^2}{v \, (r + h)^2} + \frac{v \cos \gamma}{r + h} + 0.5 \, b \, \rho \, v \, u \tag{7.2-13}$$

It is specifically assumed that the effects of random winds, unsteady motion of the vehicle about the aerodynamic trim, and disturbances in the aerodynamic forces due to unsteady flow around the vehicle are negligible. Effects due to variations in the atmospheric density are discussed in the next section.

7.3 Atmospheric Density Model

A recent model for the atmosphere of Mars given by Evans, Pitts, and Kraus (1967) has found common usage in a variety of studies using aerodynamic braking to achieve a set of desired terminal conditions. Examples of these studies are Garland (1969a, 1969b, 1968) and Repic and Eyman (1969). The density characteristics of this atmosphere are given in Table 7-1. Typically, these studies have designed a guidance scheme based on the mean density model and then the design is tested against both the upper and lower density models. The intent of this section is to develop an appropriate model for the density variations that can be used in the entry controller design.

As can be seen in Table 7-1, density varies considerably among the three models which are assumed to be representative of the whole atmosphere of Mars. That is, variations that depend on season, latitude, and temperature fluctuations are neglected. As a minimum effort, a model for the atmosphere should take into account the spread between the three models of Table 7-1.

To begin with, the nominal atmospheric density model is chosen to be exponential

$$\rho = \rho_0 e^{-\beta h} \tag{7.3-1}$$

where the parameters ρ_0 and β are picked to fit the mean density model. Any density perturbation from the nominal would probably be highly correlated with altitude, yet different values of the random variations could be expected with different altitudes. Since the uncertainty in density would not change rapidly with altitude, it could not be assumed to be white noise, but a shaping filter can be constructed to represent the correlation with altitude as

$$\frac{d}{|dh|}(\delta\rho) = -\frac{1}{h_\rho}\delta\rho + \frac{1}{h_\rho} n_\rho(h) \tag{7.3-2}$$

Table 7-1 Density Versus Altitude

Height (ft)	High Density Model (slug/ft^3)	Mean Model (slug/ft^3)	Low Model (slug/ft^3)
0	2.79×10^{-5}	2.85×10^{-5}	2.40×10^{-5}
16405	2.25×10^{-5}	2.17×10^{-5}	1.66×10^{-5}
32810	1.79×10^{-5}	1.61×10^{-5}	1.09×10^{-5}
65620	1.06×10^{-5}	7.91×10^{-6}	3.29×10^{-6}
98430	5.70×10^{-6}	3.06×10^{-6}	8.07×10^{-7}
131240	2.77×10^{-6}	1.00×10^{-6}	1.86×10^{-7}
164050	1.23×10^{-6}	3.12×10^{-7}	4.01×10^{-8}
246075	1.31×10^{-7}	1.73×10^{-8}	6.03×10^{-10}
328100	1.38×10^{-8}	1.00×10^{-9}	4.73×10^{-12}
492150	1.68×10^{-10}	3.78×10^{-12}	-

Table 7-2 Scale Height Versus Altitude

Height (ft)	High Density Model (ft)	Mean Model (ft)	Low Model (ft)
0	78415	62995	45934
16405	74478	58073	41668
32810	69885	52824	36091
65620	55120	39043	23951
98430	49871	30185	22967
131240	42653	27888	21982
164050	39043	28216	20998
246075	36091	28544	18373
328100	36747	28872	15748
492150	37731	29857	15420

where $\delta\rho$ is the uncertainty in density, h_ρ is the correlation altitude, and n_ρ is a white noise with statistics

$$< n_\rho \, (h) > \; = \; 0 \tag{7.3-3}$$

$$< n_\rho \, (h) \;\; n_\rho \, (h') > \; = \; q_\rho \, (h) \;\; \delta(h-h') \tag{7.3-4}$$

and the absolute value sign is necessary to insure stability of the shaping filter. Steinker (1966) takes this same approach.

Since the model of the vehicle dynamics employs time as the independent variable, Eq. 7.3-2 is transformed to the same independent variable by multiplying by the altitude rate $\dot{h}$ to obtain

$$\delta\dot{\rho} \; = \; |\dot{h}| \; \frac{d}{|dh|} \; \delta\rho \; = \; - \frac{|\dot{h}|}{h_\rho} \; \delta\rho \; + \; \frac{|\dot{h}|}{h_\rho} \; n_\rho \, (h) \tag{7.3-5}$$

With a linearity assumption, the delta function may be transformed to

$$\delta(t-t') \; = \; |\dot{h}| \quad \delta(h-h') \tag{7.3-6}$$

Then, treating the entire forcing function in Eq. 7.3-5 as white noise in time, n(t), we obtain

$$\delta\dot{\rho} \; = \; - \frac{|\dot{h}|}{h_\rho} \; \delta\rho \; + \; n(t) \tag{7.3-7}$$

where

$$< n(t) > \; = \; 0 \tag{7.3-8}$$

$$< n(t) \;\; n(t') > \; = \; q_\rho \, (t) \quad \delta(t-t') \tag{7.3-9}$$

Since we have essentially no knowledge of how density perturbations propagate with altitude, we assume the correlation altitude to be equal to the scale height β^{-1} to obtain

$$\dot{\delta\rho} = - \beta |\dot{h}| \delta\rho + n \qquad (7.3\text{-}10)$$

The forced solution to Eq. 7.3-10 may be written as

$$\delta\rho(t) = \int_{-\infty}^{t} e^{-\beta |\dot{h}| (t-t_1)} n(t_1)\, dt_1 \qquad (7.3\text{-}11)$$

and hence the variance of $\delta\rho$ is

$$\sigma_\rho^2 = \overline{\delta\rho(t)^2} = \int_{-\infty}^{t} dt_1 \int_{-\infty}^{t} dt_2\, e^{-\beta |\dot{h}| (t-t_1)} e^{-\beta |\dot{h}| (t-t_2)} q_\rho(t_1)\, \delta(t_2-t_1)$$

(7.3-12)

On a quasi-stationary basis, we treat $\dot{h}$ and q_ρ as constants to allow integration of this function between 0 and ∞ to obtain

$$\sigma_\rho^2 = \frac{q_\rho}{2 \beta |\dot{h}|} \qquad (7.3\text{-}13)$$

Thus, the amplitude of the covariance of the white-noise needed to produce, on a quasi-stationary basis, a density uncertainty of $\delta\rho^2$ is given by

$$q_\rho = 2 \beta |\dot{h}| \sigma_\rho^2 \qquad (7.3\text{-}14)$$

From Table 7-1 it is clear that the density variance should at least have the characteristic of increasing with decreasing altitude. One way to achieve this is to make the variance proportional to the actual value of the density

$$\sigma_\rho^2 = c_1 \rho \qquad (7.3\text{-}15)$$

Equation (7.3-14) can then be written

$$q_\rho = c\rho \qquad (7.3\text{-}16)$$

where c is to be experimentally chosen to give acceptable mean-squared errors in the nominal density model to correspond to Table 7-1.

7.4 Measurement System

Onboard the vehicle is a measurement system for navigation. For convenience, it is assumed to be an inertial measurement unit. The inertial reference maintained by the gyros is assumed to be aligned at the start of entry with one axis radially directed away from Mars, one axis in the plane of motion perpendicular to the radial direction, and the third axis completing the triad. Accelerometers measure specific force in this inertial frame during the entry. Since planar motion is considered, we only consider the measurements received from the accelerometers in this plane.

The information from the accelerometers, $\underline{m}$, is related to the specific forces f_v and f_γ by

$$\underline{m} = T\ \underline{f}_s + \underline{v} \tag{7.4-1}$$

where

$$T = \begin{bmatrix} \cos(\theta-\gamma) & \sin(\theta-\gamma) \\ -\sin(\theta-\gamma) & \cos(\theta-\gamma) \end{bmatrix} \tag{7.4-2}$$

$$\underline{f}_s = \begin{bmatrix} f_v \\ f_\gamma \end{bmatrix} = \begin{bmatrix} -\dfrac{\rho\, a\, v^2}{2\, m}\, c_d \\ \dfrac{\rho\, a\, v^2}{2\, m}\, c_d\, u \end{bmatrix} \tag{7.3-3}$$

and $\underline{v}$ is an independent white noise vector with statistics

$$< \underline{v}(t) > = 0 \tag{7.4-4}$$

$$< \underline{v}(t)\ \underline{v}(t')^T > = U\ \delta(t-t') \tag{7.4-5}$$

Variations in the measurement vector due to noise and to perturbations in the state vector, which is defined as

$$\underline{x} = \begin{bmatrix} h \\ \theta \\ v \\ \gamma \\ \rho \end{bmatrix} \tag{7.4-6}$$

may be written as

$$\delta \underline{m} = \delta T \; \underline{f}_s + T \; \delta \underline{f}_s + \underline{v} \tag{7.4-7}$$

where

$$\delta T = \begin{bmatrix} -\sin(\theta-\gamma) & \cos(\theta-\gamma) \\ -\cos(\theta-\gamma) & -\sin(\theta-\gamma) \end{bmatrix} (\delta\theta - \delta\gamma) \tag{7.4-8}$$

or

$$\delta T = T \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} (\delta\theta - \delta\gamma) \tag{7.4-9}$$

or

$$\delta T \; \underline{f}_s = (\delta\theta - \delta\gamma) \; T \begin{bmatrix} f_\gamma \\ -f_v \end{bmatrix} \tag{7.4-10}$$

then

$$\delta T \; \underline{f}_s = T \begin{bmatrix} 0 & f_\gamma & 0 & -f_\gamma & 0 \\ 0 & -f_v & 0 & f_v & 0 \end{bmatrix} \delta \underline{x} \tag{7.4-11}$$

The term $\delta \underline{f}_s$ in Eq. 7.4-7 may be written as

$$\underline{\delta f}_s = \begin{bmatrix} 0 & 0 & \frac{2 f_v}{v} & 0 & \frac{f_v}{\rho} \\ 0 & 0 & \frac{2 f_\gamma}{v} & 0 & \frac{f_\gamma}{\rho} \end{bmatrix} \underline{\delta x} \qquad (7.4\text{-}12)$$

Thus, using Eq. 7.4-11 and 7.4-12, Eq. 7.4-7 may be written as

$$\underline{\delta m} = T \; T' \; \underline{\delta x} + \underline{v} \qquad (7.4\text{-}13)$$

or

$$\underline{\delta m} = M \; \underline{\delta x} + \underline{v} \qquad (7.4\text{-}14)$$

where

$$T' = \begin{bmatrix} 0 & f_\gamma & \frac{2 f_v}{v} & -f_\gamma & \frac{f_v}{\rho} \\ 0 & -f_v & \frac{2 f_v}{v} & f_v & \frac{f_\gamma}{\rho} \end{bmatrix} \qquad (7.4\text{-}15)$$

7.5 The Constraining Differential Equations

It is now possible to define the constraining differential equations: the nominal state, the covariance matrix of estimation errors, and the covariance of the estimate.

The nominal state is 5-dimensional:

$$\dot{x}_1 = \dot{h} = v \sin \gamma \qquad (7.5\text{-}1)$$

$$\dot{x}_2 = \dot{\theta} = \frac{v \cos \gamma}{r + h} \qquad (7.5\text{-}2)$$

$$\dot{x}_3 = \dot{v} = - \frac{g_0 \sin \gamma \, r^2}{(r + h)^2} - .5 \; b \; \rho \; v^2 \qquad (7.5\text{-}3)$$

$$\dot{x}_4 = \dot{\gamma} = \frac{- g_0 \cos \gamma \, r^2}{v \, (r + h)^2} + \frac{v \cos \gamma}{r + h} + 0.5 \, b \rho v u \quad (7.5\text{-}4)$$

$$\dot{x}_5 = \dot{\rho} = - \beta \rho \; v \; \sin \gamma \quad (7.5\text{-}5)$$

or

$$\dot{\underline{x}} = \underline{f} \, (\underline{x} \, , \, \underline{u}) \quad (7.5\text{-}6)$$

The covariance matrix of estimation errors propagates according to

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \quad (7.5\text{-}7)$$

where

$$Q = \begin{bmatrix} & & 0 & & \\ & & 0 & & \\ & & 0 & & \\ & & 0 & & \\ 0 & 0 & 0 & 0 & c_\rho \end{bmatrix} \quad (7.5\text{-}8)$$

M is defined in Eq. 7.4-14 and $F = \underline{f}_{\underline{x}}$.

The covariance matrix of the estimate of the deviation in the state propagates according to

$$\dot{\hat{X}} = (F - G C) \hat{X} + \hat{X} (F - G C)^T + E M^T U^{-1} M E \quad (7.5\text{-}9)$$

where C represents the feedback gains for the perturbation controller and

$$G = \underline{f}_{\underline{u}} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ .5 \; b \; \rho \; v \\ 0 \end{bmatrix} . \quad (7.5\text{-}10)$$

The initial conditions for these differential equations, as well as the magnitude of the noises and the values of the constants to be used in this study, are presented in Appendix F.

7.6 Choice of Cost Function

Most of the early work on optimization of trajectories for Earth entry was concerned with cost functions that attempted to minimize the integral of stagnation point heat rate input while constraining the deceleration loading below a tolerable value- see Bryson, et. al., (1962b). For example, one might consider a deterministic cost function

$$J = \int_{o}^{T} \left[c_1 \left(\frac{0.5 \rho v^2 a c_d}{2 m} \right) + c_2 \left(\rho^{1/2} v^3 \right) \right] dt \qquad (7.6\text{-}1)$$

which trades off the deceleration along the flight path versus stagnation point heat rate input (Loh, 1963). By appropriate adjustment of the values of c_1 and c_2 the maximum deceleration can be limited to a tolerable level.

If we were to consider applying the combined optimization technique to Eq. 7.6-1, we would take the expected value of the stochastic cost function identical to Eq. 7.6-1 but with actual values ρ^a and v^a replacing ρ and v to obtain

$$< J > = \int_{o}^{T} \left[c_1 \left(\frac{0.5 \rho v^2 a c_d}{2 m} \right) + c_2 \left(\rho^{1/2} v^3 \right) \right] dt$$

$$+ 0.5 \ \mathrm{tr} \left[\int_{o}^{T} (A E + A \hat{X}) \, dt \right] \qquad (7.6\text{-}2)$$

where $A = L_{xx}$. Since the terminal altitude and range are specified, the cost function Eq. 7.6-2 would have to be augmented to weight deviations in those quantities. Furthermore, additional augmentation of the cost function to weight the amount of perturbation control used would be necessary and thus result in

$$< J > = \int_0^T \left[c_1 \left(\frac{0.5 \rho v^2 a c_d}{2 m} \right) + c_2 (\rho^{1/2} v^3) \right] dt$$

$$+ 0.5 \ tr \left[\int_0^T (A E + A \hat{X} + B C \hat{X} C^T) \right] dt$$

$$+ 0.5 \ tr \left\{ S(T) \left[\hat{X}(T) + E(T) \right] \right\} \qquad (7.6\text{-}3)$$

subject to the constraints Eq. 7.5-6, 7.5-7 and 7.5-9.

While such an artificially constructed cost function could be minimized by the combined optimization procedure, a difficulty arises in attempting to compare that result with the standard quadratic synthesis approach. In the latter case one would minimize the deterministic cost function as given by Eq. 7.6-1 and then design a perturbation controller based on minimizing

$$<\delta^2 J> = 0.5 <\delta \underline{x}(T)^T S(T) \delta \underline{x}(T)> + 0.5 < \int_0^T (\delta \underline{x}^T A \delta \underline{x} + \delta u B \delta u) dt> \qquad (7.6\text{-}4)$$

The first difficulty arises in that there is little justification for using the same values of S(T), A, and B as in Eq. 7.6-3, particularly since A only involves weighting of density and velocity deviations. Another difficulty is in the fact that the two terminal times need not be the same. For some preliminary studies the quadratic synthesis approach actually yielded 500% more range error than the combined optimum approach, yet the comparison is essentially meaningless because the cost functions are different.

The basic problem is that the original cost function Eq. 7.6-1, while perhaps appropriate for Earth entry, is not appropriate for Mars. Because of the thin atmosphere, the assumed vehicle can fly a wide variety of entry trajectories - from ballistic to high lift-to-drag ratios - without occurring decelerations greater than 4 or 5 g's. The heat input load is also significantly less than that of an Earth entry. Therefore, the cost function ought to contain different quantities for a Mars entry. The cost function finally chosen was based on the above considerations and the following reasoning.

Given a desired terminal range and altitude, a deterministic trajectory will be found that meets these constraints. If that trajectory gives acceptable g loading and heat input rate, a quadratic synthesis controller (which includes identification of the density) will be designed that weights terminal miss versus the integral of the perturbation control used, where the weighting will be chosen to keep the sum of the deterministic control and rms perturbation control well below the maximum available u.

Using that same weighting a combined optimization approach will be used to find the best trajectory to minimize

$$\langle J \rangle = 0.5 \ \mathrm{tr} \left\{ S(T) \left[\hat{X}(T) + E(T) \right] \right\} + 0.5 \ \mathrm{tr} \left[\int_0^T B \, C \, \hat{X} \, C^T \, dt \right] \qquad (7.6\text{-}5)$$

which trades off terminal errors versus perturbation control. The result of this optimization will be checked to insure that g loading and heat rate input are acceptable and that no more than maximum u will be called for.

The next section presents numerical results for such a comparison of the two methods.

7.7 Numerical Results

To summarize the problem the combined optimization cost function to be minimized is given as

$$\langle J \rangle = 0.5 \ \mathrm{tr} \left\{ S(T) \left[\hat{X}(T) + E(T) \right] \right\} + 0.5 \ \mathrm{tr} \left[\int_0^T B \, C \, \hat{X} \, C^T \, dt \right] \qquad (7.7\text{-}1)$$

subject to the constraining differential equations

$$\dot{\underline{x}} = \underline{f} \, (\underline{x} \, , \, \underline{u}) \qquad (7.7\text{-}2)$$

$$\dot{E} = F \, E + E \, F^T + Q - E \, M^T \, U^{-1} \, M \, E \qquad (7.7\text{-}3)$$

$$\dot{\hat{X}} = (F - G\,C)\,\hat{X} + \hat{X}\,(F - G\,C)^T + E\,M^T\,U^{-1}\,M\,E \quad (7.7\text{-}4)$$

where $\underline{f}$ is given by Eq. 7.5-1 – 7.5-5, Q by Eq. 7.5-8, G by Eq. 7.5-9, M by Eq. 7.4-14, U by Eq. 7.4-5 and $F = \underline{f}_x$. The actual numerical values are given in Appendix F and the adjoint variables obey Eq. 6.3-19 – 6.3-21.

The solution obtained by the application of the gradient procedure to the problem is compared to quadratic synthesis around a constant lift-to-drag trajectory in Figures 7-1 through 7-3.

The optimum open-loop control shown in Figure 7-1 results in only a slightly different trajectory, as indicated in Figs. 7-2 and 7-3, from a constant lift-to-drag ratio flight, yet the latter trajectory will be shown to give 24.2% more rms range error. During the initial pull-up maneuver a slight decrease in lift-to-drag ratio causes the vehicle to go slightly lower into the dive thus loosing kinetic energy due to drag and not reaching as high an altitude on the up-phase of flight.

The large increase in control near the terminal time causes the vehicle to perform a horizontal maneuver at a higher altitude and then the control rapidly decreases to cause a less steep final descent to the target at a higher velocity as shown in Figures 7-4 and 7-5. As will be shown the higher horizontal maneuver results in a decrease in the rms altitude uncertainty. The change in control is also related to the fact that the terminal constraints must be met at the fixed terminal time of 915 sec. The large terminal velocity (1300 ft/sec) reflects the need to meet these constraints since the horizontal maneuver was performed at a lower velocity. Although the velocity is large, it is in the range of those obtained in a previous study by Garland (1969a).

It was judged that the constant lift-to-drag trajectory gave an acceptable maximum load factor of 3.5 g and acceptable maximum heat input rate where heat rate input = $10^{-8}\ \rho^{1/2}\,v^3$. As shown in Figs. 7-6 and 7-7, the combined optimum approach does not significantly alter these values and is an acceptable alternate trajectory.

Since the cost function for the problem trades-off terminal errors in altitude and range versus the amount of perturbation energy used, the resultant trajectory should physically represent this trade-off except for the final terminal maneuvers in meeting the terminal constraints. A choice of B=0.1 made the rms perturbation control on the order of 10^{-3} which is almost insignificant since it is approximately .2% of the nominal control. Then, the cost is almost entirely made up of rms position errors. As a rule of thumb, one generally picks B to create a 10% rms value for the perturbation control in order to allow better tracking of any perturbations. This approach reflects the iterative nature of optimal designs and could be used in further investigations of this entry problem.

The rms range error shown in Figure 7-8 is always less for the combined optimization approach. The measurements at a lower altitude tend to increase the initial information about range although the more important effect is the changed feedback gains which directly affect the coefficients in the matrix equation for the covariance of the estimate. This later equation, rather than the covariance of the errors equation, shows a large decrease and, thus, when adding diagonal terms to find the rms uncertainty, there results a substantial improvement.(Fig. 7-9)

The rms altitude error shown in Figure 7-10 behaves as expected in that the combined optimization approach gives more error when initially at a lower altitude since the density uncertainty increases and the measurements are not directly sensitive to altitude. The uncertainty later decreases as the combined optimization makes its terminal maneuver at a higher altitude. These effects depend directly on the driving noise and measurement uncertainties and are automatically taken into account by the procedure.

The rms velocity error is shown in Figure 7-11. The combined optimization approach gives a larger uncertainty during all phases of the flight until the final higher speed descent to the target. Although the terminal velocity error is less this is probably not important for the actual design since the terminal landing scheme will of necessity have to compensate for altitude and velocity errors.

The final values of the rms errors are listed in Table 7-3.

Table 7-3 Comparison of Terminal RMS Deviations

	Combined Optimum	Quadratic Synthesis	% Increase
RMS Range	4, 232 ft	5, 942 ft	24. 2%
RMS Altitude	16, 193 ft	16, 937 ft	4. 6%
RMS Velocity	5. 35 ft/sec	28. 7 ft/sec	440%
RMS Flt. Path Ang.	0. 0015 rad	0. 0074 rad	390%

The substantial decrease in range uncertainty is the most important result of this study since it is assumed that the vehicle has a limited lateral movement capability at the terminal time.

The practical implementation of the combined optimization solution should be no more difficult than a quadratic synthesis controller. The feedback gains, two of which are displayed in Figures 7-12 and 7-13, are essentially of the same shape and can probably be approximated by polynomials. The open-loop control could also be fit by a curve during the significantly different portions of the flight but probably a more practical approach would be to fit a constant pulse control to the large change near the end of the flight. The amplitude and width could be adjusted to give acceptable simulation results.

The results of this numerical study are of sufficient interest to justify further investigation of the combined optimization approach. The terminal time can be freed since it was fixed in this problem only to provide a standard basis for comparison. Additionally, one might want to change the weighting on terminal position errors versus perturbation control or perhaps specify some final value of velocity. For any variation of the problem the gradient algorithm should provide a reasonable computational solution. Its performance is discussed in the next section.

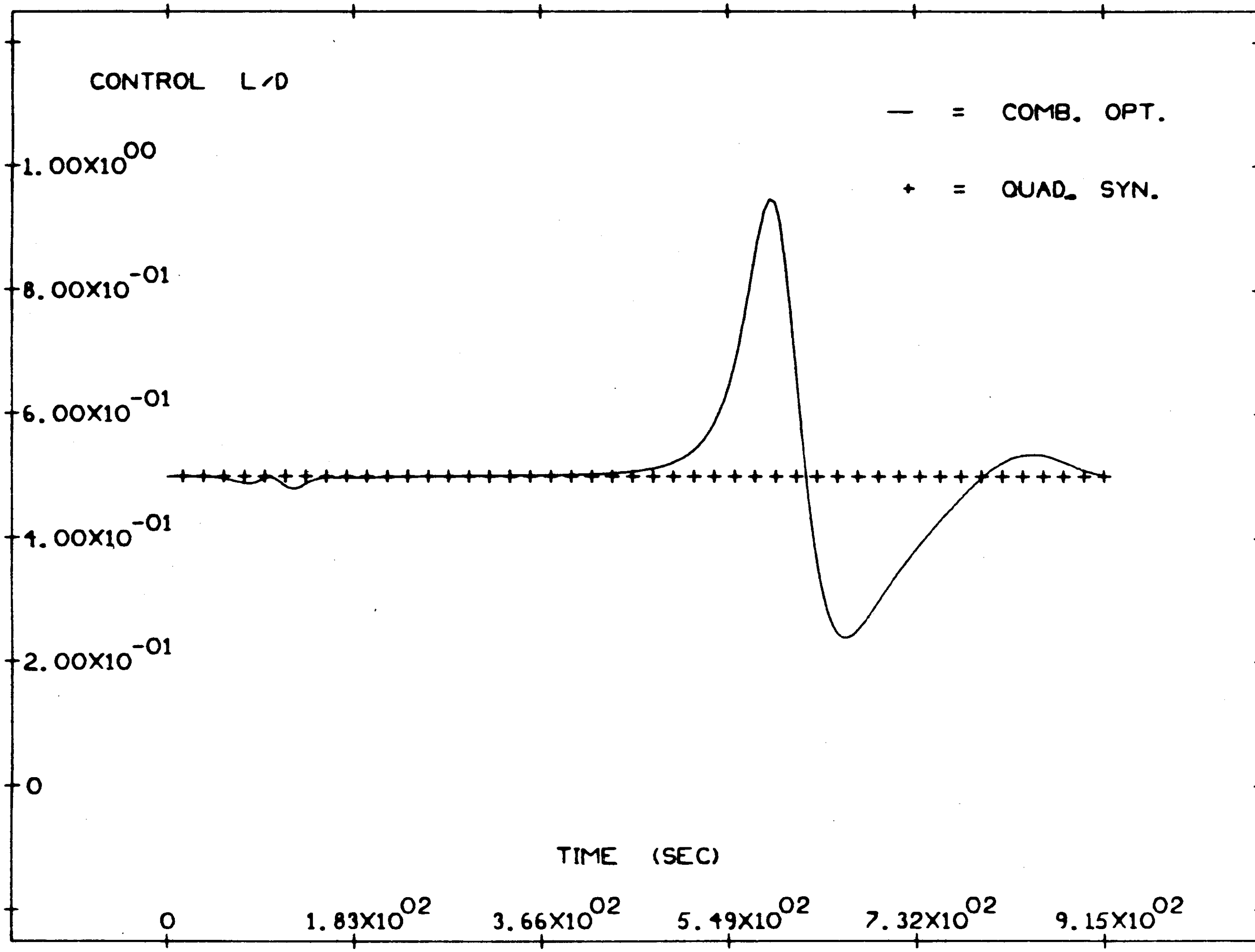

Figure 7-1 Control Input

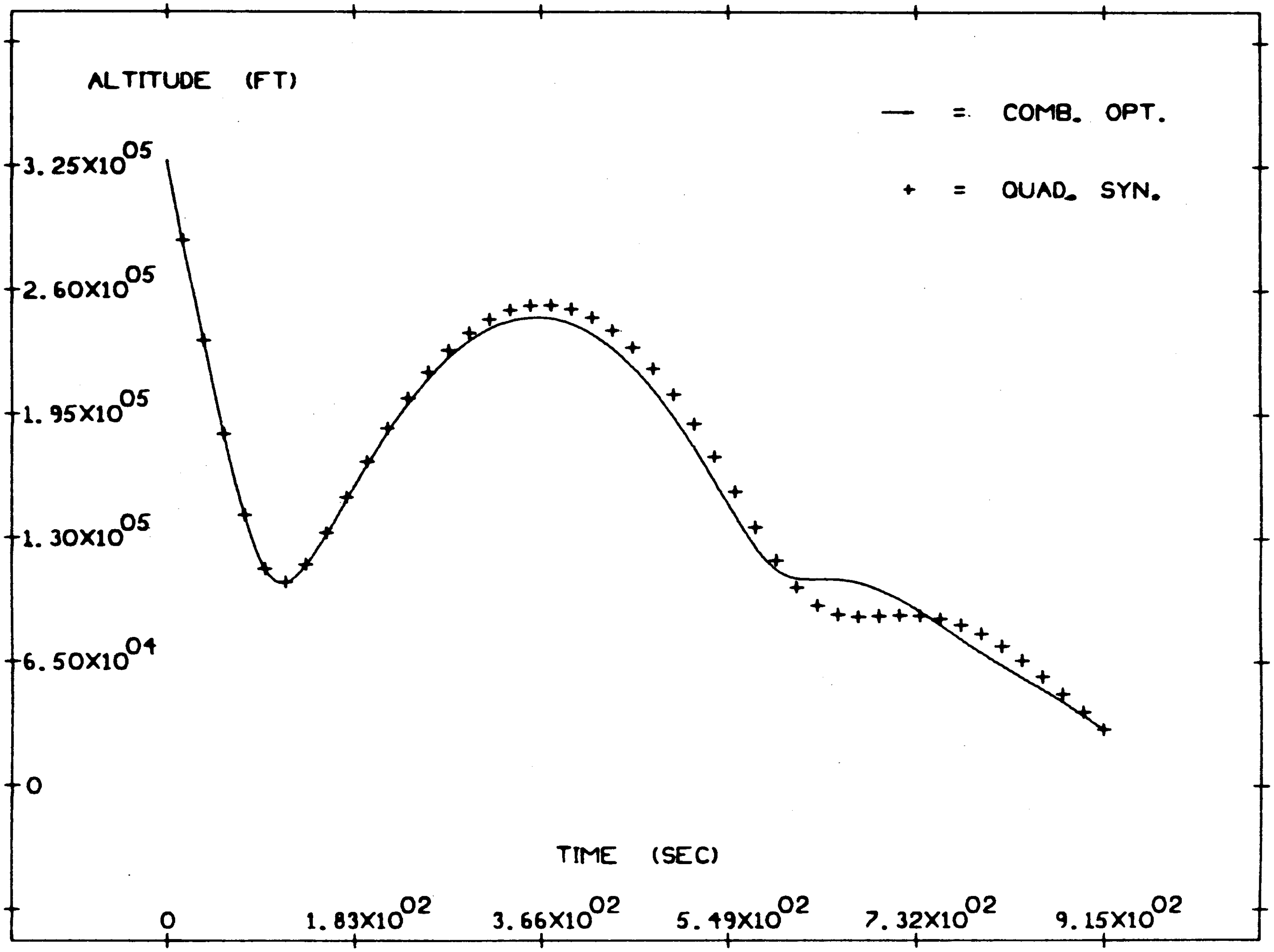

Figure 7-2 Altitude versus Time

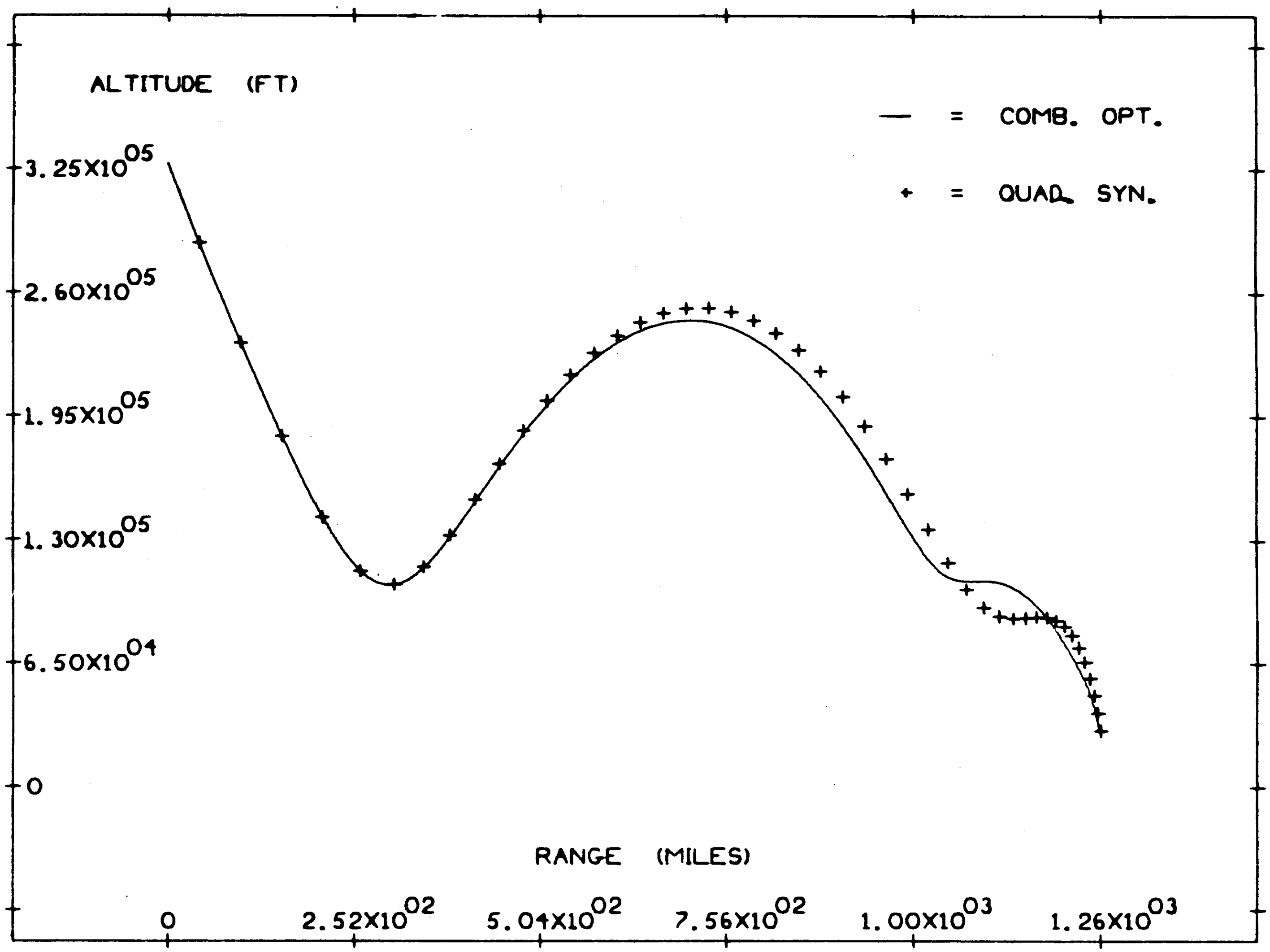

Figure 7-3 Altitude versus Range

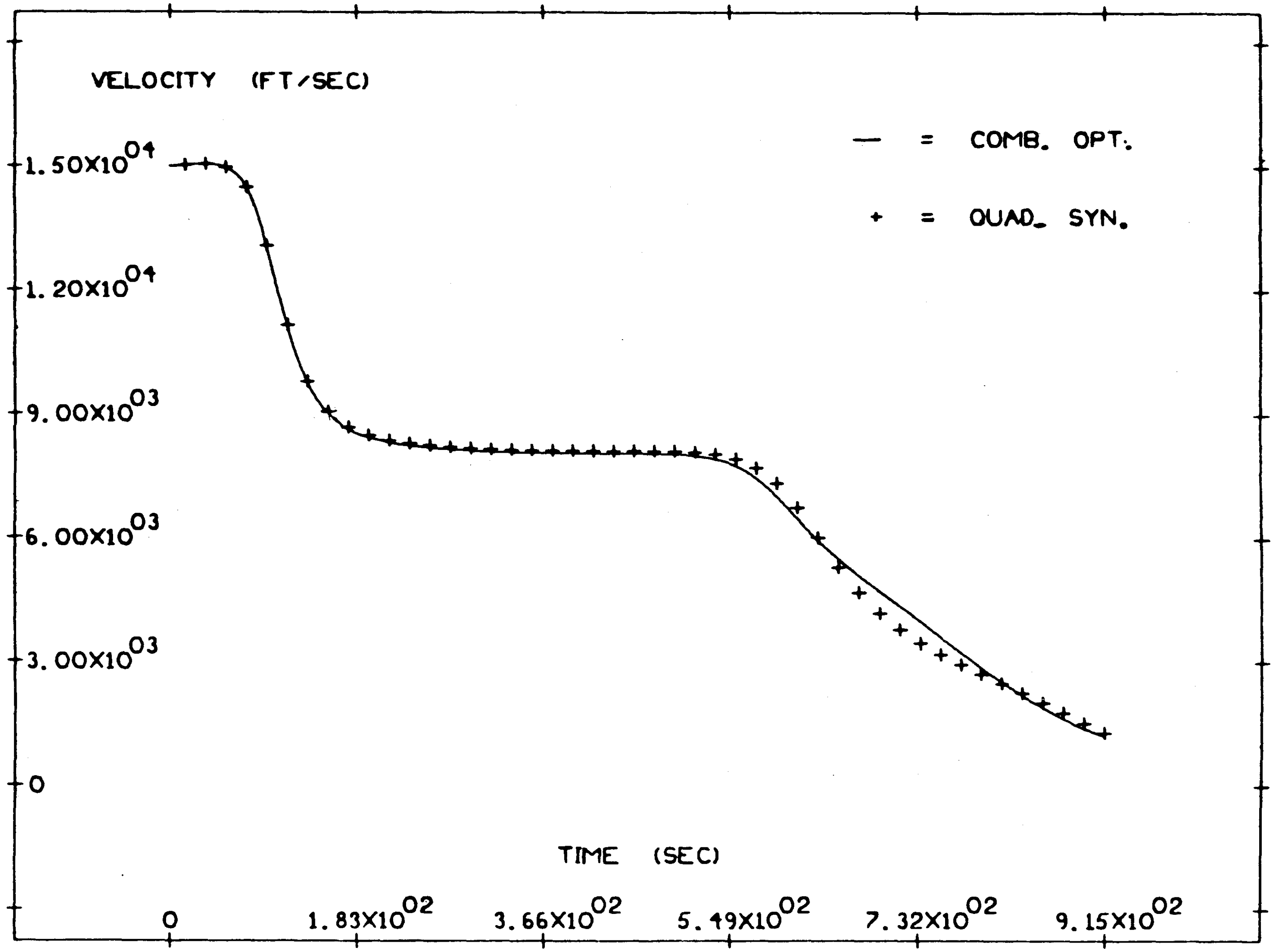

Figure 7-4 Velocity

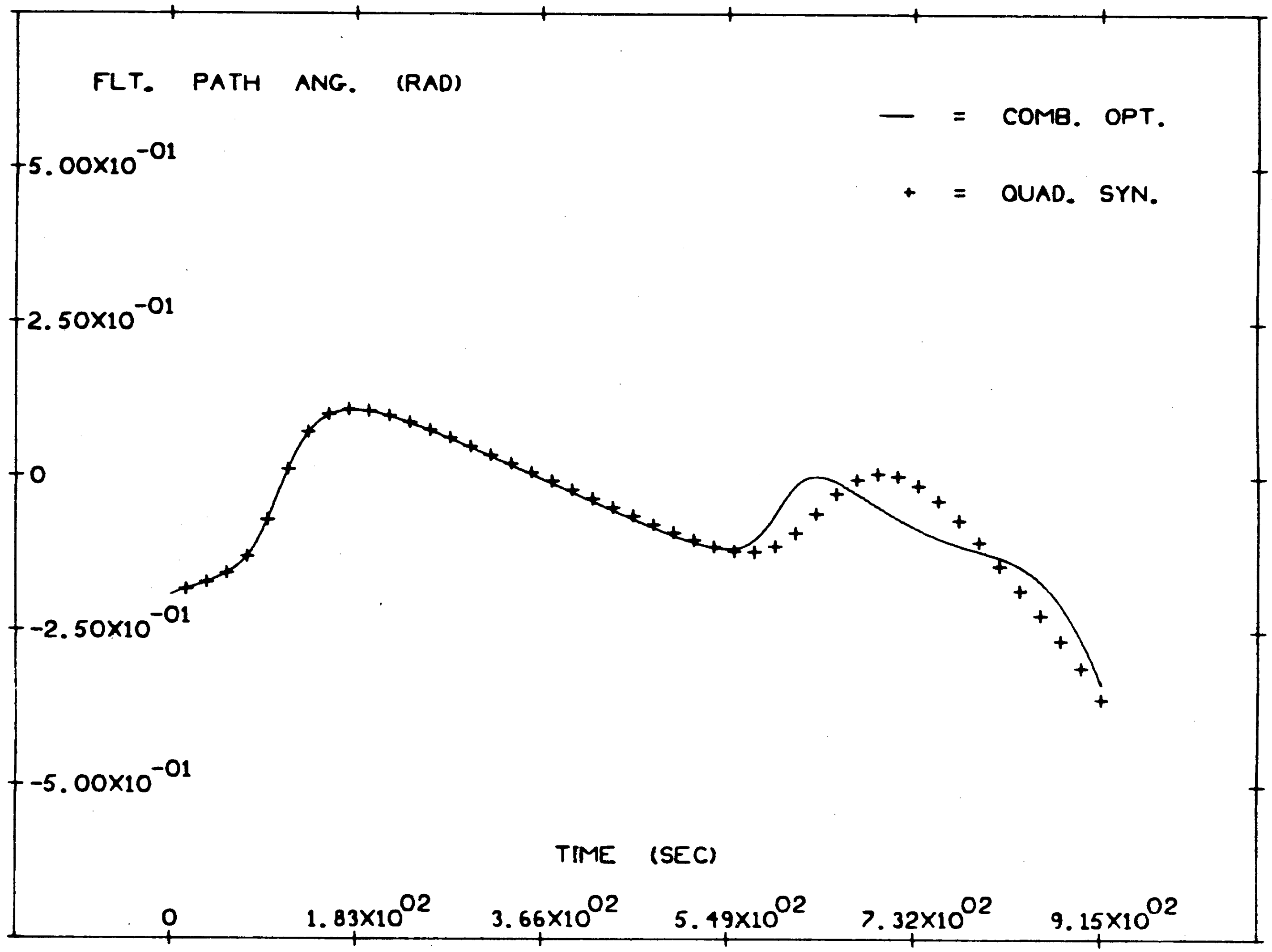

Figure 7-5 Flight Path Angle

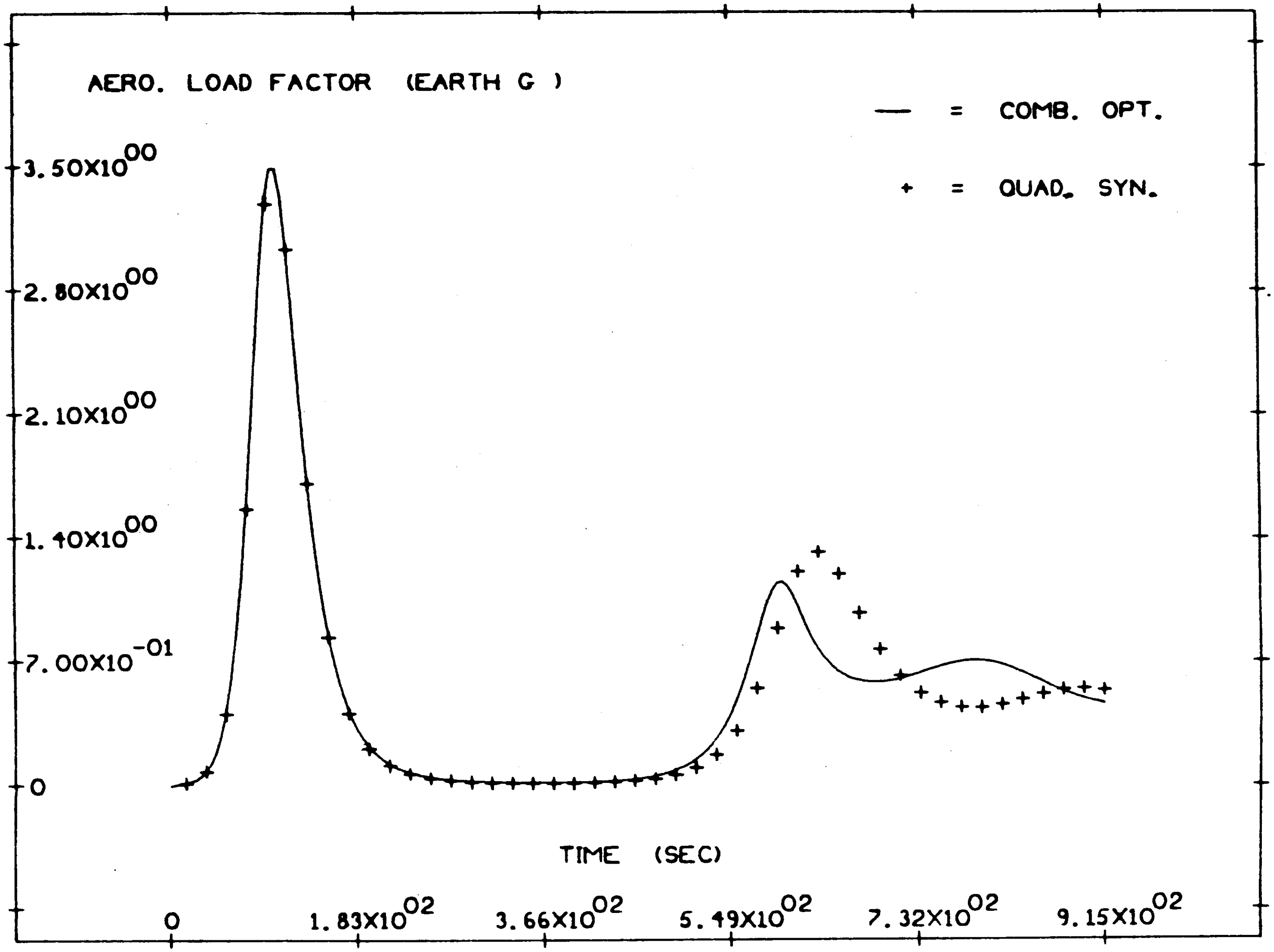

Figure 7-6 Aerodynamic Load Factor

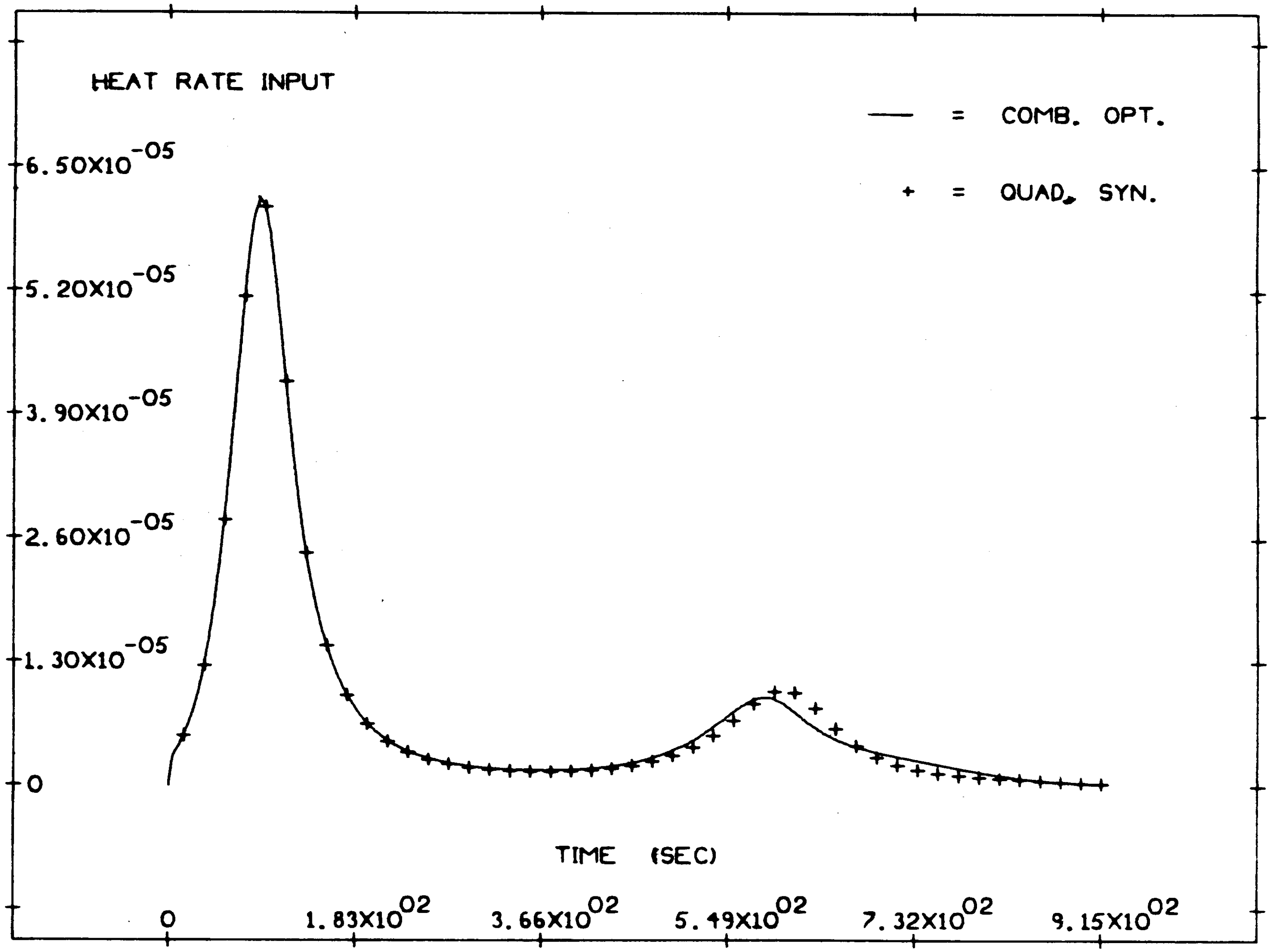

Figure 7-7 Heat Rate Input

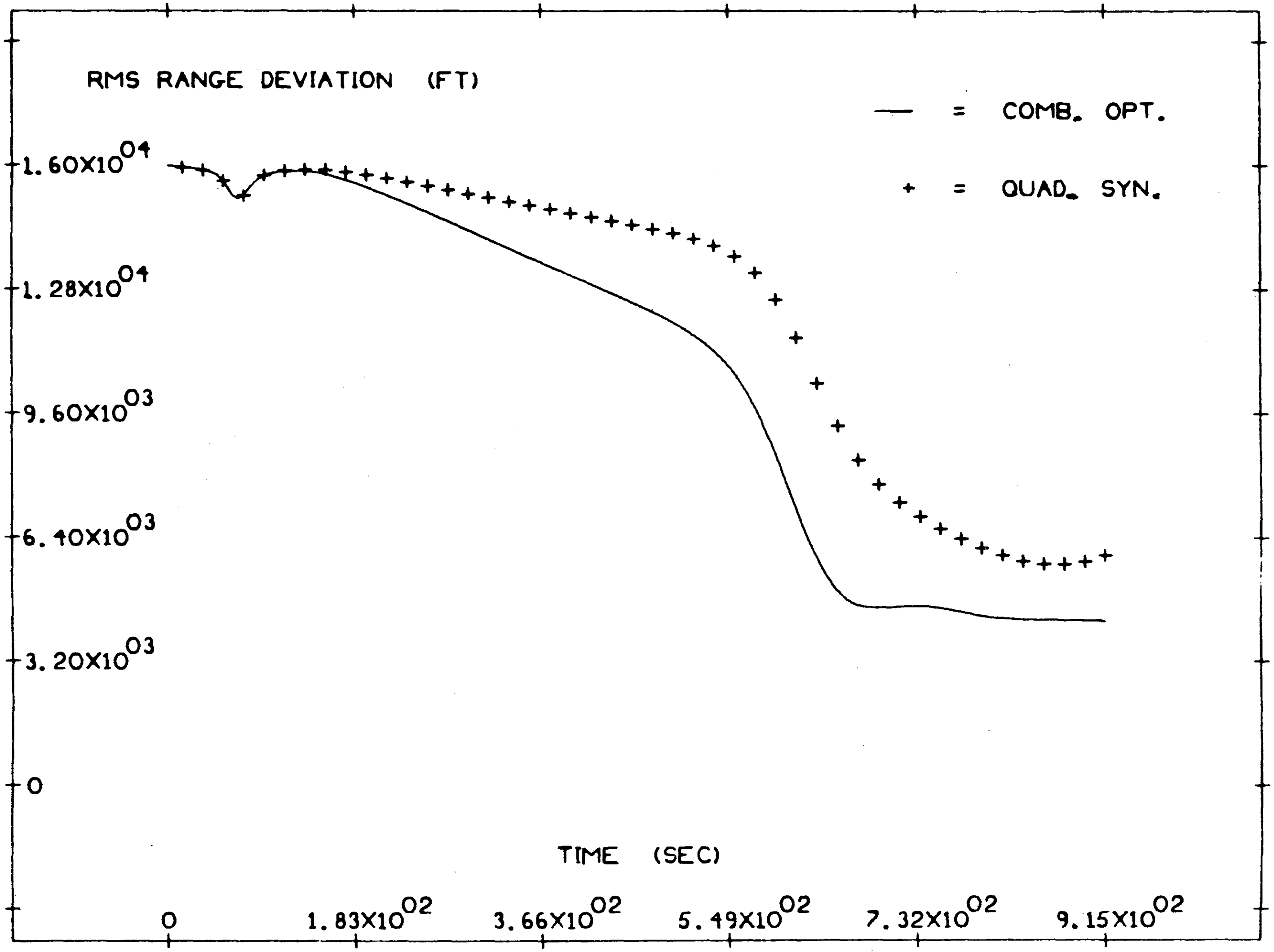

Figure 7-8 RMS Range Deviation

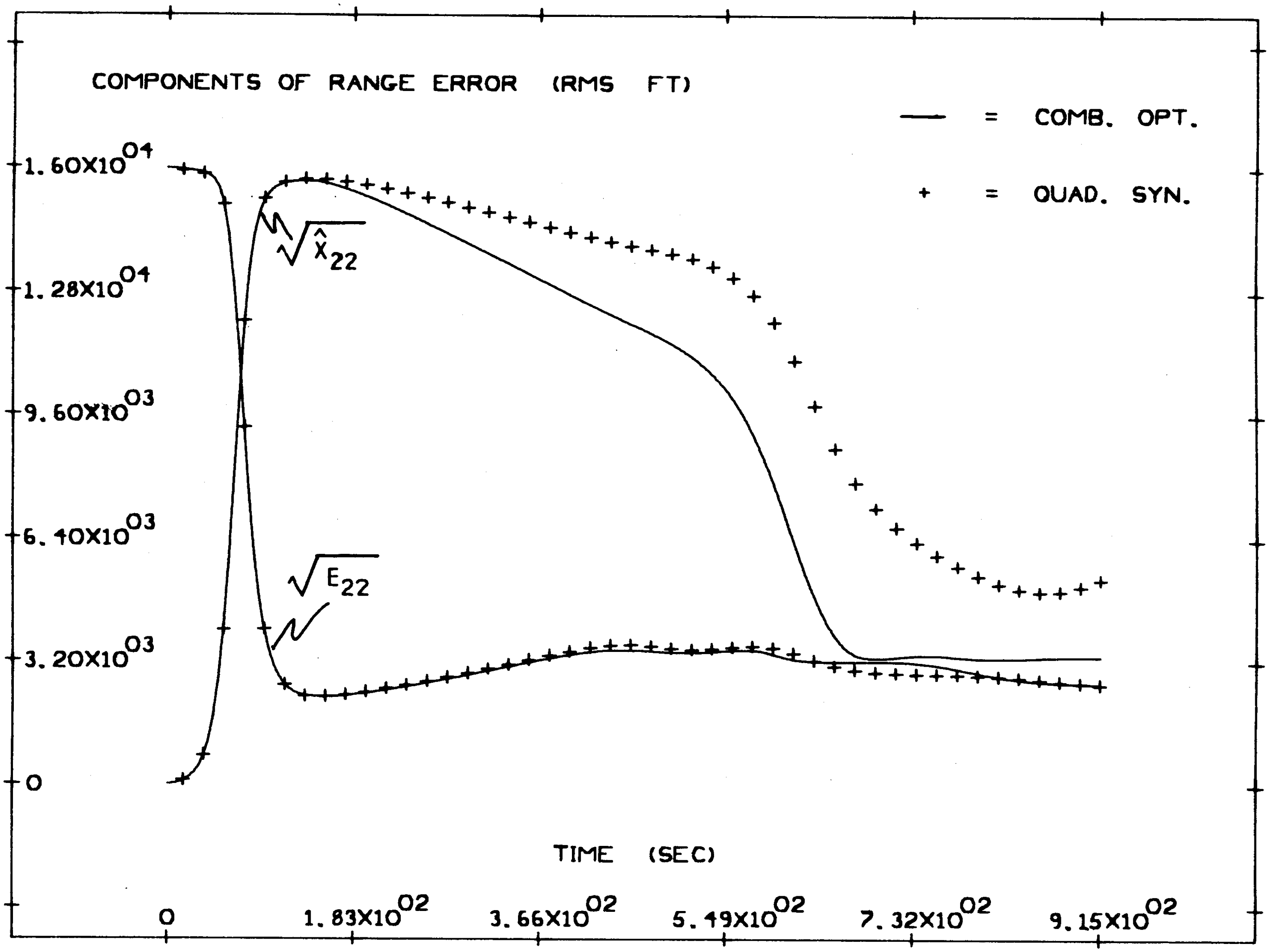

Figure 7-9 Components of Range Error

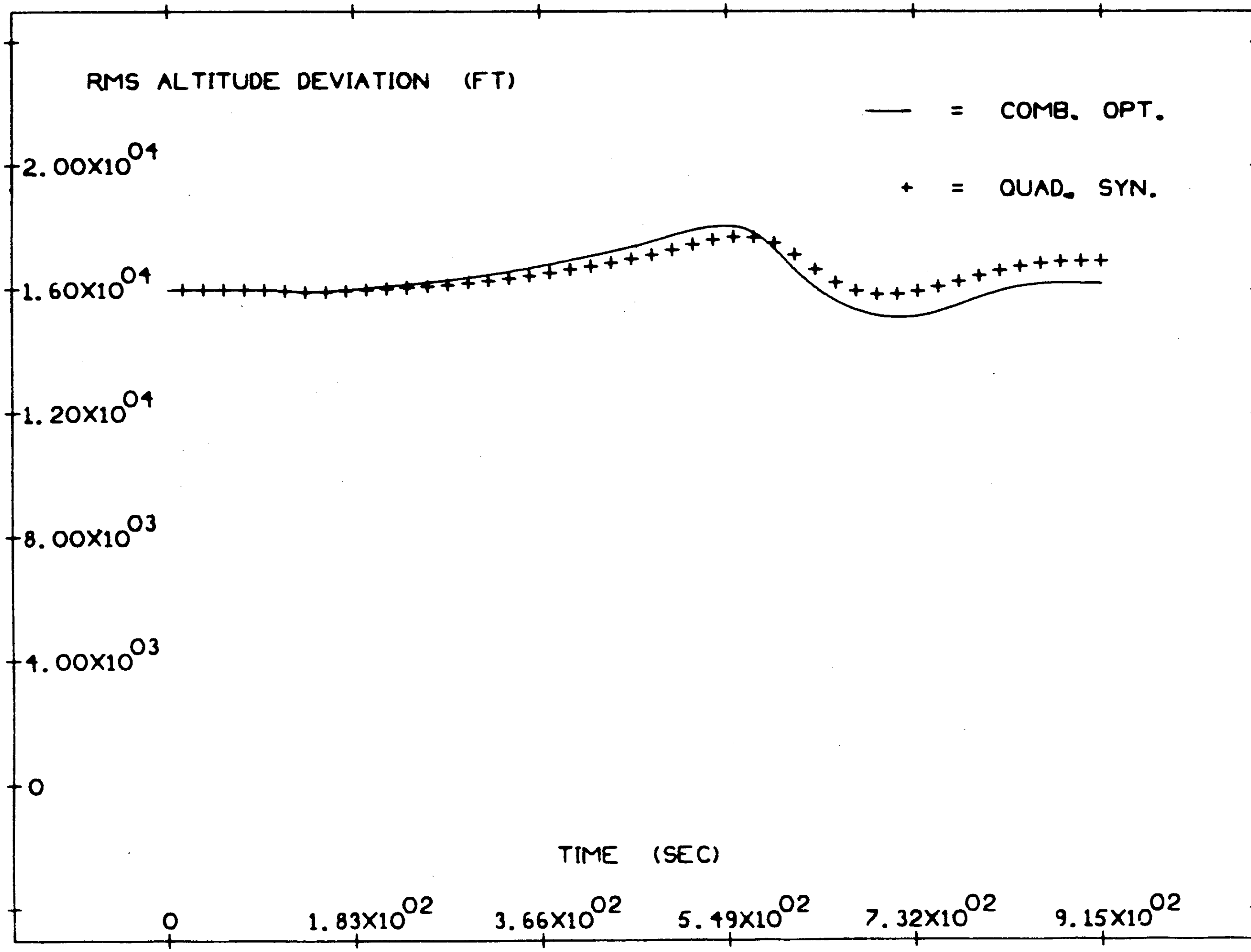

Figure 7-10 RMS Altitude Deviation

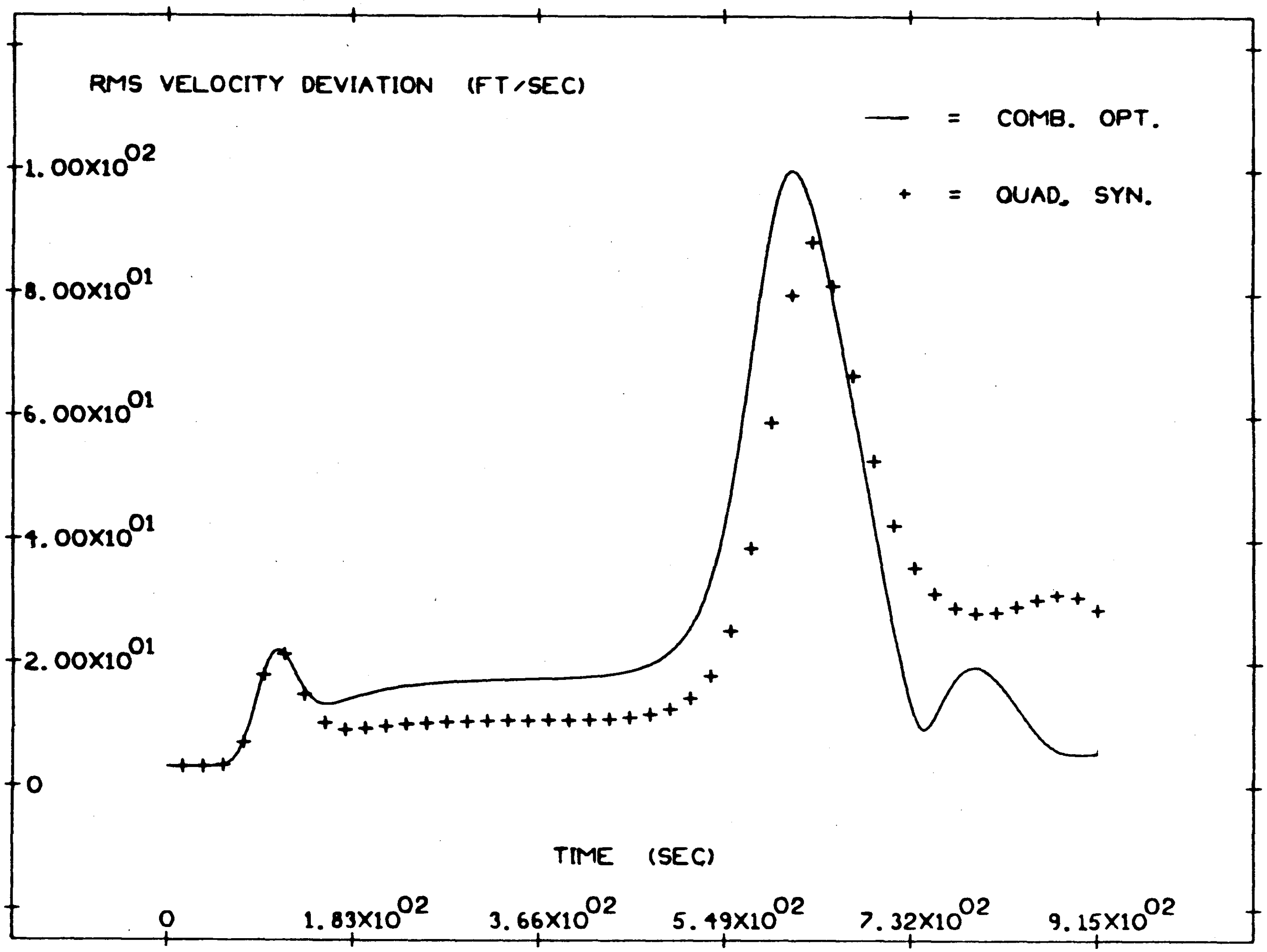

Figure 7-**11** RMS Velocity Deviation

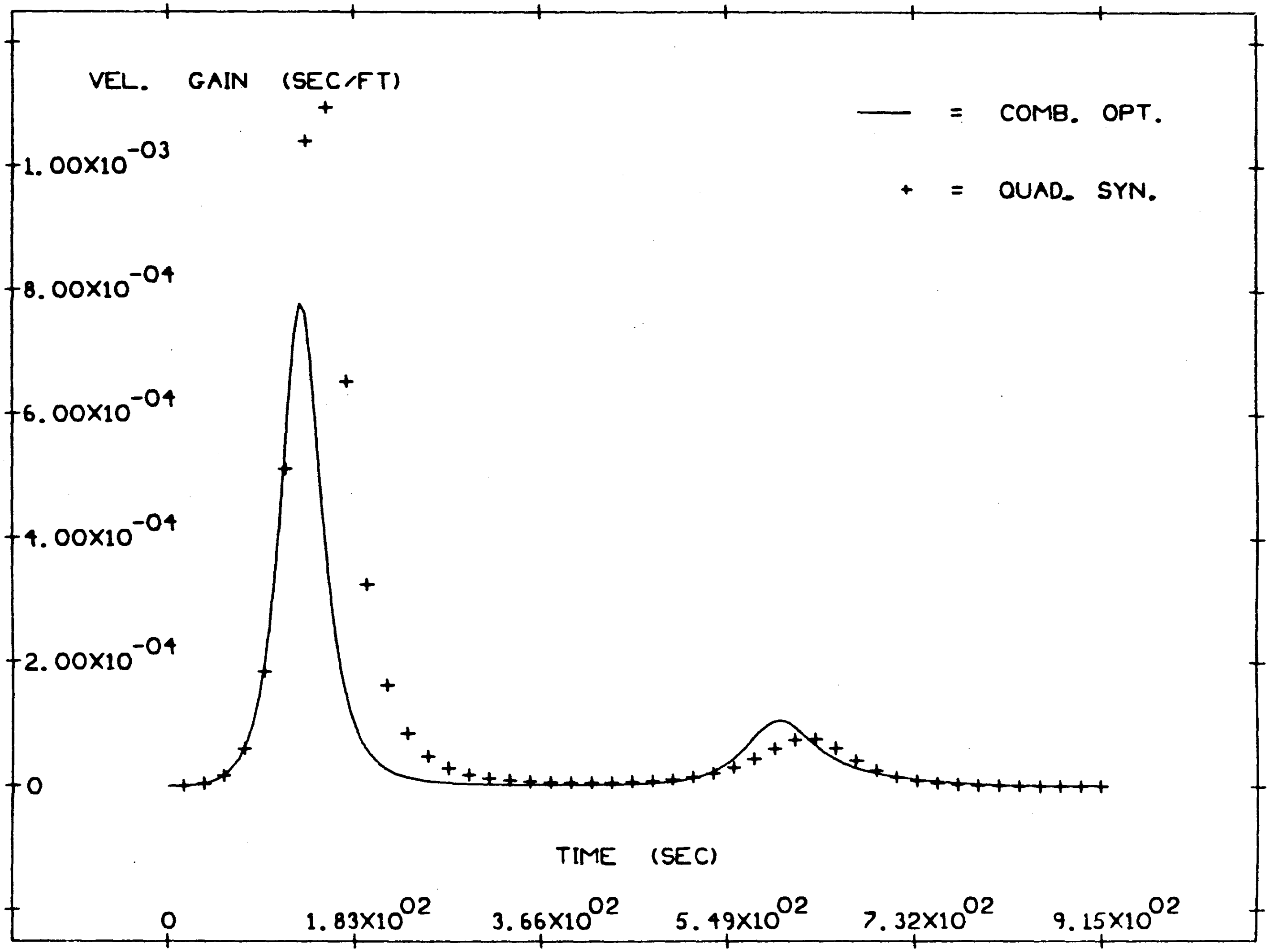

Figure 7-12 Velocity Gain

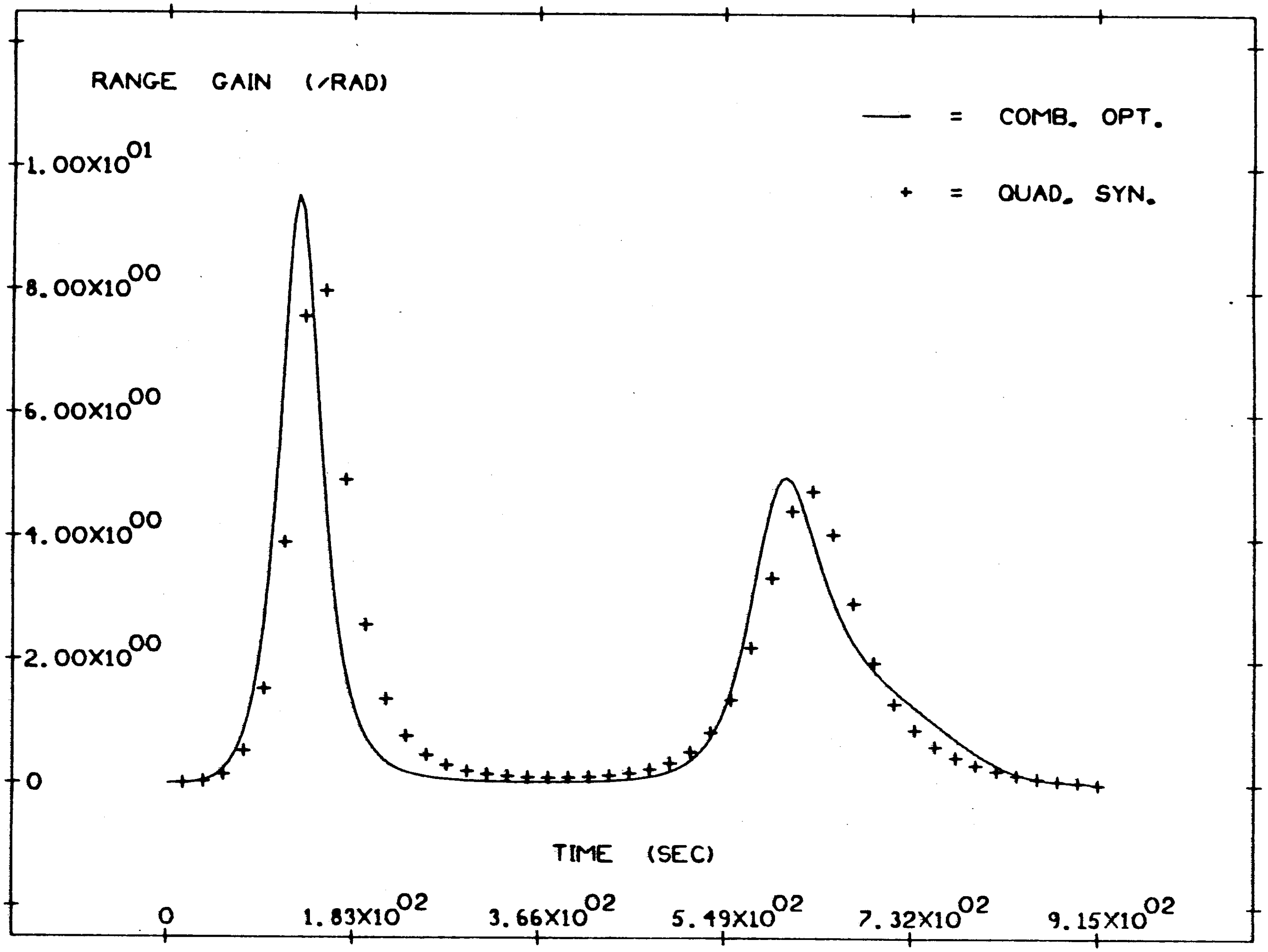

Figure 7-13 Range Gain

7.8 Comments on the Algorithm

The first-order algorithm exhibited the same consistent ability to achieve successive decreases in the cost at each iteration as was evident in the previous examples presented.

The fundamental problem encountered was in integration of the matrix differential equations. Both the forward and backward integrations tend to exhibit instability for the first few time steps; that is, errors due to simply taking too large a first step appear. The choice of an appropriate time step is especially difficult. The entry problem was 915 sec long and the author arbitrarily decided to use an integration step of 5 sec. Unfortunately, in order to prevent instability, a smaller time step of 1 sec was used for the first few steps in each direction as well as during the high g portion of the flight. This latter result came about after many frustrating cases of achieving a few successful iterations and then having a matrix, such as the covariance matrix, become other than positive semi-definite.

As an aid to diagnosing difficulties with the program, a check was made on each matrix diagonal term that should be positive at every time step. If a negative value appeared, the program immediately was made to print out most of the variables involved as well as storing the control history used for the previous successful iteration. In this way the data were saved and allowed the programmer to pick up the iterations at a point close to where the difficulty occurred.

Additional capability was initially built into the program to verify correct reading and writing of data into the data file for the first few and final steps of any differential equation integration. Previous experience indicated an almost inevitable error would occur in this part of the programming, particularly since, in an effort to save computation time, all matrices that must be computed and used repeatedly are calculated with the forward integration of the state and stored in the data file.

Finally, the program required approximately one minute per interation using an IBM 360-75 for the computations.

7.9 Summary

This chapter has shown the ability of the combined optimization approach to achieve substantial improvement in operating performance over the quadratic synthesis approach for a problem of realistic complexity. The computational algorithm provided a reasonably efficient method of achieving a near-optimal solution. Because the entry problem was of large dimension and extremely nonlinear, this chapter shows the combined optimization concept deserves attention as a possible approach for practical problems in stochastic nonlinear control.

Chapter 8

Contributions, Recommendations and Conclusion

8.1 Contributions of the Thesis

It is with some hesistancy that the author writes this section. What constitutes a contribution is probably better answered in the future rather than at the time of this writing since the thesis is clearly a engineering thesis rather than a theoretical one. As such, it should be judged by the use of its ideas and techniques presented, as well as the further research it inspires. With this thought, the author believes he has made the following contributions in order of presentation in the thesis.

In Chapter 2 the formulation of the identification problem as an optimization problem is new. Since one of the usual proposed applications of optimal control theory is process control, and judging the importance that unknown parameters play by the work of Wells (1969 and 1970), the formulation that allows the identification process to be optimally achieved should be useful to the design engineer.

Recognizing that in other fields minimization of estimation error is also important, the author feels that the set of necessary conditions derived in Chapter 3 as well as the algorithms of Chapter 4 should enable design engineers in other specialties to use the concepts and techniques in solving their problems in a practical manner.

The examples of Chapter 5 contribute to an overall understanding of the physical processes involved when one includes covariance matrices in the cost function.

Chapters 6 and 7 are probably the most important in the thesis in that they give a practical solution and realistic examples in the field of control of stochastic nonlinear systems. The physical connection between the deterministic cost function minimized and the original cost function helps in the understanding of the problem. Furthermore, the ease with which problems can be formulated, and solutions obtained for this very complicated problem, lead the author to believe that this is the most significant result of the thesis.

8.2 Suggestions for Further Research

An essential piece of research that would aid the application of the ideas presented in this thesis would be the development of improved computational algorithms for vector-matrix optimization problems. The first-order method presented here achieves the desired optimum solution at the expense of a substantial number of iterations. Perhaps a second-variation technique as given by Bryson and Ho (1969) or a technique similar to that of Jacobson and Mayne (1970) could be developed to speed convergence. Of course, the technique must be able to work with vectors and matrices without any partitioning.

The extension of both the open-loop and closed-loop design problems to discrete systems could be a valuable piece of future research. This would represent a straightforward, but not necessarily trivial, extension of the ideas presented in this thesis.

The optimization technique as applied to estimation performance can be used with any filtering scheme whose performance is judged by a covariance matrix. Application to various nonlinear filters, as described in Chapter 7 of Leondes (1970), might show an even greater dependence on the nominal conditions. Similarly, there is no theoretical reason why these results can not be extended and applied to any linear or nonlinear smoothing technique as described in Chapters 8 and 9 of Leondes (1970).

Simple extensions of this work to closed-loop control systems that involve correlated noises, cross-correlation between plant and measurement noise, and control-application induced noises are possible and can be derived using the techniques developed in this thesis.

A important contribution could be a method for predicting and correcting the difficulties associated with integrating the matrix Riccati equations as discussed in this thesis.

Finally, one may consider the question: when is optimal control theory useful? As seen in this thesis, any attempt at optimization results in a complicated process that involves computer time, money, and manpower. In many cases the optimal designs may not even be used other than to give a baseline for performance. Is it possible to develop a figure of merit to tell when the combined optimization procedure will be useful?

8.3 Conclusion

This thesis has presented a new formulation for the optimum identification and control of systems with unknown parameters by using a practical design approach based on linear estimation and linear feedback controllers.

In the chapters on optimum identification the mathematical procedures and numerical solution techniques were developed and then shown to be effective in producing optimal control signals for the examples in Chapter 5. Furthermore, these techniques were directly applicable to the closed-loop control problem considered in Chapter 6 where an example demonstrated the significant improvement in performance over the quadratic synthesis approach.

The entry problem of Chapter 7 further demonstrated the applicability of the design technique to practical problems of significant difficulty such that the application of the techniques presented should be considered by the design engineer involved with stochastic nonlinear systems.

Appendix A

Statistical Properties of the Estimator - Controller Combination

This Appendix develops the statistical properties of the perturbation estimator - controller combination discussed in Chapter 6.

The dynamical system obeys

$$\dot{\underline{x}}^a = \underline{f}\,(\underline{x}^a\,,\ \underline{u}^a\,,\ t) + \underline{n}(t) \qquad \text{(A-1)}$$

and, when linearized around a nominal, the perturbation equations are

$$\delta\dot{\underline{x}} = F\ \delta\underline{x} + G\ \delta\underline{u} + \underline{n}\ ,\ < \delta\underline{x}(0) > \ = 0 \qquad \text{(A-2)}$$

$\underline{n}$ is independent zero-mean white noise with

$$< \underline{n}(t)\ \underline{n}(t')^T > \ = Q\ (t-t') \qquad \text{(A-3)}$$

The measurements are given by

$$\underline{m}^a = \underline{h}\,(\underline{x}^a\,,\ \underline{u}^a\,,\ t) + \underline{v}(t) \qquad \text{(A-4)}$$

where $\underline{v}$ is independent zero-mean white noise with statistics

$$< \underline{v}(t) \; \underline{v}(t')^T > \; = \; U \; \delta\,(t-t') \tag{A-5}$$

Linearizing the measurements around a nominal

$$\delta \underline{m} = \underline{\underline{h}}_x \, \delta \underline{x} + \underline{\underline{h}}_u \, \delta \underline{u} + \underline{v} \tag{A-6}$$

Also, let

$$M = \underline{\underline{h}}_x \tag{A-7}$$

The linear estimator is required to have the form

$$\delta \dot{\hat{\underline{x}}} = F \, \delta \hat{\underline{x}} + G \, \delta \underline{u} + K \, (\delta \underline{m} - M \, \delta \hat{\underline{x}} - \underline{\underline{h}}_u \, \delta \underline{u}) \tag{A-8}$$

with

$$< \delta \hat{\underline{x}}(0) > \; = \; 0 \tag{A-9}$$

It is assumed that $\delta \underline{u}$ is known, so that substitution of Eq. A- 6 into A- 8 results in

$$\delta \dot{\hat{\underline{x}}} = F \, \delta \hat{\underline{x}} + G \, \delta \underline{u} + K \left[M \, (\delta \underline{x} - \delta \hat{\underline{x}}) + \underline{v} \right] \qquad \text{(A-10)}$$

The perturbation controller is required to have the form

$$\delta \underline{u} = - \, C \, \delta \hat{\underline{x}} \qquad \text{(A-11)}$$

Substituting this into Eq. A-2 and Eq. A-10, the perturbation state and estimator are coupled

$$\delta \dot{\underline{x}} = F \, \delta \underline{x} - G \, C \, \delta \hat{\underline{x}} + \underline{n} \qquad \text{(A-12)}$$

$$\delta \dot{\hat{\underline{x}}} = F \, \delta \hat{\underline{x}} - G \, C \, \delta \hat{\underline{x}} + K \left[M \, (\delta \underline{x} - \delta \hat{\underline{x}}) + \underline{v} \right] \qquad \text{(A-13)}$$

or, in terms of $\underline{e} = \hat{\delta \underline{x}} - \delta \underline{x}$,

$$\dot{\underline{e}} = (F - K \, M) \, \underline{e} + K \, \underline{v} - \underline{n} \qquad \text{(A-14)}$$

$$\delta \dot{\hat{\underline{x}}} = (F - G \, C) \, \delta \hat{\underline{x}} - K \, M \, \underline{e} + K \, \underline{v} \qquad \text{(A-15)}$$

It is desired to develop differential equations for the covariance of the error in the estimate and the covariance of the estimate, i. e.,

$$\dot{E} = d \, (< \underline{e} \, \underline{e}^T >) \, / \, dt \qquad \text{(A-16)}$$

and

$$\dot{\hat{X}} = d \, (< \delta \hat{\underline{x}} \; \delta \hat{\underline{x}}^T >) \, / \, dt \qquad \text{(A-17)}$$

as well as for the cross-covariance

$$\dot{Z} = d \, (< \underline{e} \; \delta \hat{\underline{x}}^T >) \, / \, dt \qquad \text{(A-18)}$$

With

$$\dot{E} = <\underline{e}\,\dot{\underline{e}}^T> + <\underline{e}\,\dot{\underline{e}}^T> \qquad \text{(A-19)}$$

Substituting Eq. A-14

$$\dot{E} = (F - K M) E + E (F - K M)^T + K <\underline{v}\,\underline{e}^T> + <\underline{e}\,\underline{v}^T> K^T - <\underline{n}\,\underline{e}^T> - <\underline{e}\,\underline{n}^T> \qquad \text{(A-20)}$$

Using $\underline{e} = \delta\hat{\underline{x}} - \delta\underline{x}$, it is well known – see Brock (1965) or Denham (1964)– that the last two terms combine to give **Q**. In an analogous calculation given by Denham and Speyer (1964), the third and fourth terms result in KUK^T, so that

$$\dot{E} = (F - K M) E + E (F - K M)^T + K U K^T + Q \qquad \text{(A-21)}$$

Using a similar calculation for $\dot{\hat{X}}$

$$\dot{\hat{X}} = <\delta\dot{\hat{\underline{x}}}\;\delta\hat{\underline{x}}^T> + <\delta\hat{\underline{x}}\;\delta\dot{\hat{\underline{x}}}^T> \qquad \text{(A-22)}$$

and substituting from Eq. A-15

$$\dot{\hat{X}} = (F - G C)\hat{X} + \hat{X}(F - G C)^T - K M <\underline{e}\;\delta\hat{\underline{x}}^T> - <\delta\hat{\underline{x}}\;\underline{e}^T> M^T K^T + K <\underline{v}\;\delta\hat{\underline{x}}^T> + <\delta\hat{\underline{x}}\;\underline{v}^T> K^T \qquad \text{(A-23)}$$

The last two terms combine to give KUK^T (Denham and Speyer, 1964) and using the definition of Z in Eq. A-18

$$\dot{\hat{X}} = (F - G C)\hat{X} + \hat{X}(F - G C)^T - K M Z - Z^T M^T K^T + K U K^T \qquad \text{(A-24)}$$

The expression for $\dot{Z}$ is similarly evaluated.

$$\dot{Z} = < \dot{\underline{e}} \ \delta\hat{\underline{x}}^T > + < \underline{e} \ \delta\dot{\hat{\underline{x}}}^T > \tag{A-25}$$

Using Eq. A-14 and A-15

$$\begin{aligned} \dot{Z} &= (F - K M) Z + Z (F - G C)^T - E M^T K^T \\ &\quad + K < \underline{v} \ \delta\hat{\underline{x}}^T > - < \underline{n} \ \delta\hat{\underline{x}}^T > + < \underline{e} \ \underline{v}^T > K^T \end{aligned} \tag{A-26}$$

The fourth and sixth terms give KUK^T and the fifth term is zero. Thus,

$$\dot{Z} = (F - K M) Z + Z (F - G C)^T - E M^T K^T + K U K^T \tag{A-27}$$

Finally, an expression for the mean-squared deviation in the state is

$$X = < \delta\underline{x} \ \delta\underline{x}^T > = < (\delta\hat{\underline{x}} - \underline{e}) (\delta\hat{\underline{x}} - \underline{e})^T >$$

$$X = < \delta\hat{\underline{x}} \ \delta\hat{\underline{x}}^T > - < \underline{e} \ \delta\hat{\underline{x}}^T > - < \delta\hat{\underline{x}} \ \underline{e}^T > + < \underline{e} \ \underline{e}^T >$$

$$X = \hat{X} - Z - Z^T + E \tag{A-28}$$

It should be noted that in most cases

$$Z(0) = < \underline{e}(0) \ \delta\hat{\underline{x}}(0)^T > = 0 \tag{A-29}$$

that is, the error in the estimate and the estimate at t = 0 are orthogonal. Then, if the linear filter gains are chosen as

$$K = E M^T U^{-1} \tag{A-30}$$

$Z(t) = 0$ for all time. The error and the estimate are uncorrelated. For this case

$$X = \hat{X} + E \tag{A-31}$$

$$\dot{E} = F E + E F^T + Q - E M^T U^{-1} M E \tag{A-32}$$

and

$$\dot{\hat{X}} = (F - G C) \hat{X} + \hat{X} (F - G C)^T + E M^T U^{-1} M E \tag{A-33}$$

Appendix B

A List of Gradient Matrices

The following operations hold when the elements of X are independent.

	The derivative of	with respect to	is
1.	$tr(X)$	X	I
2.	$tr(AX)$	X	A^T
3.	$tr(AX^T)$	X	A
4.	$tr(AXB)$	X	$A^T B^T$
5.	$tr(AX^T B)$	X	BA
6.	$tr(AX)$	X^T	A
7.	$tr(AX^T)$	X^T	A^T
8.	$tr(AXB)$	X^T	BA
9.	$tr(AX^T B)$	X^T	$A^T B^T$
10.	$tr(AXBX)$	X	$A^T X^T B^T + B^T X^T A^T$
11.	$tr(AXBX^T)$	X	$A^T X \; B^T + AXB$

(This list is from Athans, 1968.)

Appendix C

Necessary Conditions for Free Terminal-Time Problems

This problem is identical in character to the class of problems treated in Chapter 3 except that the terminal time is free. One constraint is now missing and needs to be replaced by another one. For convenience, choose to optimize

$$J = \mathrm{tr}\left[C(T)\ E(T)\right] + k\left[\underline{x}(T),\ T\right] + \int_0^T L(\underline{x}, \underline{u}, E, t)\,dt \tag{C-1}$$

where there are no terminal constraints. C(T) and k can be explicit functions of time. The subscript t with C(T) and k denotes partial differentiation with respect to t evaluated at the terminal time. If C(T) and k are not explicit functions of time, these partial derivatives would be zero.

Adjoin to the cost the system constraints and use the definition of the Hamiltonian to obtain

$$\begin{aligned} J = {} & \mathrm{tr}\left[C(T)\ E(T)\right] + k\left[\underline{x}(T),\ T\right] \\ & + \int_0^T \left[H - \underline{p}^T \mathring{\underline{x}} - \mathrm{tr}\,(\dot{P}\,E)\right] dt \end{aligned} \tag{C-2}$$

The differential of Eq. C-2, taking into account differential changes in the terminal time T, is

$$\begin{aligned} dJ = {} & \mathrm{tr}\left[C\ d\,E(T)\right] + \mathrm{tr}\left[C_t\ E(T)\right] dT \\ & + k_{\underline{x}}\ \underline{dx} + k_t\ dt + L(T)\ dT \\ & + \int_0^T \Big[H_{\underline{x}}\ \delta\underline{x} + H_{\underline{u}}\ \delta\underline{u} + \mathrm{tr}\ (H_E\ \delta E) \\ & \qquad - \underline{p}^T\ \delta\mathring{\underline{x}} - \mathrm{tr}\ (P\ \delta\dot{E})\Big]\ dt \end{aligned} \tag{C-3}$$

Integrating Eq. C-3 by parts, with E(0) and $\underline{x}(0)$ fixed, gives

$$
\begin{aligned}
dJ = & \ \mathrm{tr}\left[C\, d\, E(T)\right] + \mathrm{tr}\left[C_t\, E(T)\right] dT \\
& - \mathrm{tr}\left[P(T)\, \delta E(T)\right] + k_{\underline{x}}\, \underline{dx} - \underline{p}(T)^T\, \delta\underline{x}(T) \\
& + k_t\, dT + L(T)\, dT \\
& + \int_o^T \left\{ (H_{\underline{x}} + \dot{\underline{p}}^T)\, \delta\underline{x} + \mathrm{tr}\left[(H_E + \dot{P})\, \delta E\right] + H_{\underline{u}}\, \delta\underline{u} \right\} dt
\end{aligned}
\qquad \text{(C-4)}
$$

As previously, choose $\underline{p}$ and P such that

$$\dot{\underline{p}} = -H_{\underline{x}}^T \quad , \quad \underline{p}(T) = k_{\underline{x}}^T \qquad \text{(C-5)}$$

$$\dot{P} = -H_E \quad , \quad P(T) = C \qquad \text{(C-6)}$$

and make use of the facts that

$$\underline{dx}(T) = \delta\underline{x}(T) + \dot{\underline{x}}(T)\, dT \qquad \text{(C-7)}$$

and

$$dE(T) = \delta E(T) + \dot{E}(T)\, dT \qquad \text{(C-8)}$$

so that Eq. C-4 becomes

$$
\begin{aligned}
dJ = & \left\{ \mathrm{tr}\left[C_t\, E(T)\right] + k_t + L(T) + \mathrm{tr}\left[C\, \dot{E}(T)\right] \right. \\
& \left. \quad + k_{\underline{x}}\, \dot{\underline{x}}(T) \right\} dT \\
& + \int_o^T H_{\underline{u}}\, \delta\underline{u}\, dt
\end{aligned}
\qquad \text{(C-9)}
$$

For optimality, it is required that

$$H_{\underline{u}} = 0 \qquad \text{(C-10)}$$

and

$$\text{tr}\left[C_t\, E(T)\right] + k_t + L(T) + \text{tr}\left[C\, \dot{E}(T)\right] + k_{\underline{x}}\, \dot{\underline{x}}(T) = 0 \qquad \text{(C-11)}$$

Equation C-11 can also be written as

$$k_t + \text{tr}\left[C_t\, E(T)\right] + H(T) = 0 \qquad \text{(C-12)}$$

which is usually called the transversality condition.

For the case with linear terminal constraints on the first q state variables, Eq. C-12 is still the transversality condition, but the boundary condition on $\underline{p}(T)$ is given by Eq. 3.3-26.

Appendix D

Gradient Method for Free Terminal Time Problems

This algorithm is quite similar to the algorithm derived in Chapter 4 for fixed terminal time. From Eq. C-9, the differential change in cost for changes in the control and in the terminal time is

$$dJ = \left\{ tr \left[C_t \; E(T) \right] + k_t + H(T) \right\} dT + \int_o^T H_{\underline{u}} \; \delta \underline{u} \; dt \qquad (D\text{-}1)$$

Define the q-dimensional constraint vector $\underline{z}(T)$ corresponding to the specified linear terminal constraints and use Eq. 3.3-11 to write the changes in the boundary conditions for changes in the control as

$$\underline{dz} = \int_o^T R^T \underline{f}_{\underline{u}} \; \delta \underline{u} \; dt \qquad (D\text{-}2)$$

As in Chapter 4, quadratic penalty functions in $\delta \underline{u}(t)$ and dT are added to Eq. D-1. Then Eq. D-2 is adjoined to Eq. D-1 with constant multipliers $\underline{\ell}$ to obtain:

$$dJ_1 = dJ + 0.5 \; b \; (dT)^2 + 0.5 \int_o^T \delta \underline{u}^T \; W \; \delta \underline{u} \; dt$$
$$+ \underline{\ell}^T \left[\int_o^T R^T \underline{f}_{\underline{u}} \; \delta \underline{u} \; dt - \underline{dz} \right] \qquad (D\text{-}3)$$

The first variation of Eq. D-3, neglecting the change in coefficients, is

$$d(dJ_1) = \left\{ tr \left[C_t \; E(T) \right] + k_t + H(T) + b \; dT \right\} d(dT)$$
$$+ \int_o^T \left\{ L_{\underline{u}} + \underline{p}^T \underline{f}_{\underline{u}} + \left[tr \; (P \dot{E}) \right]_{\underline{u}} + \underline{\ell}^T R^T \underline{f}_{\underline{u}} + \delta \underline{u}^T \; W \right\} \delta (\delta \underline{u}) \, dt$$
$$(D\text{-}4)$$

from which it is clear that the minimum in dJ_1 occurs if

$$\delta \underline{u} = - W^{-1} \left\{ L_{\underline{u}} + (\underline{p} + R \underline{\ell})^T \underline{f}_{\underline{u}} + \left[tr\ (P \dot{E}) \right]_{\underline{u}} \right\}^T \qquad \text{(D-5)}$$

and

$$dT = - \frac{1}{b} \left\{ tr \left[C_t\ E(T) \right] + k_t + H(T) \right\} \qquad \text{(D-6)}$$

Substitute Eq. D-5 and D-6 into Eq. D-2 and use the definitions of $\underline{I}_{kj}$ and I_{kk} in Chapter 4 to obtain the changes in the terminal conditions:

$$\underline{dz} = - \underline{I}_{kj} - I_{kk} \underline{\ell} \qquad \text{(D-7)}$$

Provided the required inverse exists, $\underline{\ell}$ can now be determined:

$$\underline{\ell} = - I_{kk}^{-1} (\underline{dz} + \underline{I}_{kj}) \qquad \text{(D-8)}$$

The predicted change in dJ can be found by substitution of Eq. D-6 and D-7 into Eq. D-3:

$$dJ = - \frac{1}{b} \left\{ tr \left[C_t\ E(T) \right] + k_t + H(T) \right\}^2 - I_{jj} - \underline{I}_{kj}^T \underline{\ell} \qquad \text{(D-9)}$$

with I_{jj} defined as in Chapter 4 and $\underline{\ell}$ determined by Eq. D-8.

As the optimum is approached, it follows:

from Eq. D-5, that

$$H_u = h_{\underline{u}} + \underline{\ell}^T H_{\underline{u}} \longrightarrow 0 \qquad \text{(D-10)}$$

from Eq. D-6, that

$$tr \left[C_t\ E(T) \right] + H(T) + k_t \longrightarrow 0 \qquad \text{(D-11)}$$

from Eq. D-8, that

$$\underline{\ell} \longrightarrow - \; I_{kk}^{-1} \; \underline{I}_{kj} \tag{D-12}$$

and from Eq. D-9, that

$$I_{jj} - \underline{I}_{kj}^{T} \; I_{kk}^{-1} \; \underline{I}_{kj} \longrightarrow 0 \tag{D-13}$$

Then the gradient method for free terminal time problems is as follows:

Step 1.

Guess a nominal control $\underline{u}(t)$ and a terminal time T.

Step 2.

Identical to Step 2 in fixed T problem. Also record tr $\left[C_t E(T) \right]$, and k_t.

Step 3.

Determine the influence functions as in Step 3 of the fixed T problem.

Step 4.

Identical to Step 4 of the fixed T problem.

Step 5.

Identical to Step 5 of the fixed T problem except $\delta \underline{z}(T)$ is replaced by $\underline{dz}(T)$ and

$$\underline{\ell} = - \; I_{kk}^{-1} \; (\underline{dz} + \underline{I}_{kj}) \tag{D-14}$$

Step 6.

Repeat Steps 2 - 6 using an improved estimate of $\underline{u}(t)$ formed by adding to the previous control the vector

$$- W^{-1} \left\{ L_{\underline{u}} + (\underline{p} + R\underline{\ell})^T \underline{f}_{\underline{u}} + \left[tr\,(P\dot{E}) \right]_{\underline{u}} \right\}^T \qquad \text{(D-15)}$$

Also, improve the estimate of the terminal time with

$$dT = - b^{-1} \left\{ tr \left[C_t\; E(T) \right] + k_t + H(T) \right\} \qquad \text{(D-16)}$$

where

$$H(T) = h(T) + \underline{\ell}^T\; \underline{H}(T) \qquad \text{(D-17)}$$

Stop when

$$\underline{dz} \longrightarrow 0 \qquad \text{(D-18)}$$

$$tr \left[C_t\; E(T) \right] + k_t + H(T) \longrightarrow 0 \qquad \text{(D-19)}$$

and

$$I_{jj} - \underline{I}_{kj}^T\; I_{kk}^{-1}\; \underline{I}_{kj} \longrightarrow 0 \qquad \text{(D-20)}$$

The choice of b and W can be made to limit the size of the first step in the algorithm by comparing the actual $\underline{dz}$ and dJ with the predicted values from Eq. D-7 and D-9. If there is a large discrepancy, b and W should be increased; if there is a small discrepancy, it is possible to take larger steps, and b and W can be reduced.

Appendix E

Gradient Method for Closed-Loop Controller

The first-order gradient method for solution of the optimization problem

$$< J_1 > = k \left[\underline{x}(T)\right] + \int_o^T L\,(\underline{x}\,,\,\underline{u}\,,\,t)\,dt$$

$$+ 0.5 \text{ tr } \left[\int_o^T (A\,E + A\,\hat{X} + B\,C\,\hat{X}\,C^T - N\,C\,\hat{X} - N^T\,\hat{X}\,C^T)\,dt\right]$$

$$+ 0.5 \text{ tr } \left[S(T)\ E(T)\right] + 0.5 \text{ tr } \left[S(T)\ \hat{X}(T)\right] \qquad \text{(E-1)}$$

with the first q components of $\underline{x}(T)$ specified, proceeds directly as in Chapter 4 with the addition of two matrix differential equations.

Step 1.

Guess a control history $\underline{u}(t)$. Pick a weighting matrix W.

Step 2.

Integrate forward

$$\dot{\underline{x}} = \underline{f}\,(\underline{x}\,,\,\underline{u}\,,\,t) \quad , \quad x(0) \text{ given} \qquad \text{(E-2)}$$

$$\dot{E} = F\,E + E\,F^T + Q - E\,M^T\,U^{-1}\,M\,E \quad , \quad E(0) \text{ given} \qquad \text{(E-3)}$$

Step 3.

Integrate backward

$$\dot{S} = -\,S\,F - F^T\,S + (G^T\,S + N^T)^T\,B^{-1}\,(G^T\,S + N^T) - A \qquad \text{(E-4)}$$

$$\dot{P} = -\,(F - E\,M^T\,U^{-1}\,M)^T\,P - P\,(F - E\,M^T\,U^{-1}\,M)$$
$$-\,M^T\,U^{-1}\,M\,E\,S - S\,E\,M^T\,U^{-1}\,M - A \qquad \text{(E-5)}$$

$$S(T) = P(T) = \underline{k_{xx}} + \underline{z}_x^T \, Y \, \underline{z}_x \qquad \text{(E-6)}$$

where $\underline{z}(T) = 0$ is a q vector that represents the linear terminal constraints.

Step 4.

Integrate forward

$$\dot{\hat{X}} = \left[F - GB^{-1}(G^T S + N^T)\right] \hat{X} + \hat{X} \left[F - GB^{-1}(G^T S + N^T)\right]^T + E M^T U^{-1} M E \quad , \quad \hat{X}(0) = 0 \qquad \text{(E-7)}$$

Step 5.

Define

$$h = L + \underline{p}^T \underline{f} + 0.5 \text{ tr } (P \dot{E}) + 0.5 \text{ tr } (S \dot{\hat{X}}) + 0.5 \text{ tr } (AE + A\hat{X} + BC\hat{X}C^T - NC\hat{X} - N^T\hat{X}C^T) \qquad \text{(E-8)}$$

$$\underline{H} = R^T \underline{f} \qquad \text{(E-9)}$$

Integrate backward

$$\dot{\underline{p}} = - \underline{h}_x^T \quad , \quad \underline{p}(T) = \underline{k}_x^T \qquad \text{(E-10)}$$

$$\dot{R} = - \underline{H}_x^T = - F^T R \quad , \quad R_{ij}(T) = \delta_{ij} \begin{cases} i = 1, \ldots, n \\ j = 1, \ldots, q \end{cases} \qquad \text{(E-11)}$$

Step 6.

Compute

$$I_{kk} = \int_0^T \underline{H}_u \, W^{-1} \, \underline{H}_u^T \, dt \quad \text{(q x q matrix)} \qquad \text{(E-12)}$$

$$\underline{I}_{kj} = \int_0^T \underline{H}_u \, W^{-1} \, \underline{h}_u^T \, dt \quad \text{(q row vector)} \qquad \text{(E-13)}$$

$$I_{jj} = \int_0^T \underline{h}_u \, W^{-1} \, \underline{h}_u^T \, dt \quad \text{(scalar)} \qquad \text{(E-14)}$$

Step 7.

Choose a value of $\delta \underline{z}$ to cause the next nominal solution to be closer to the desired values $\underline{z}(T) = 0$. Pick

$$\delta \underline{z} = - d \; \underline{z}(T) \qquad \text{(E-15)}$$

with

$$0 < d < 1 \qquad \text{(E-16)}$$

Step 8.

Then determine the incremental change in $\underline{u}$

$$- W^{-1} \left[\underline{h}_u^T - \underline{H}_u \; I_{kk}^{-1} \; (\delta \underline{z} + \underline{I}_{kj}^T) \right] \qquad \text{(E-17)}$$

Step 9.

Repeat Steps 2 through 8 using an improved estimate of $\underline{u}(t)$ from Step 8 until

$$\delta \underline{z} \longrightarrow 0 \qquad \text{(E-18)}$$

and

$$I_{jj} - \underline{I}_{kj}^T \; I_{kk}^{-1} \; \underline{I}_{kj} \longrightarrow 0 \qquad \text{(E-19)}$$

to the desired degree of accuracy.

The best choices for W and d can only be determined by experimentation. Unfortunately, their choice strongly affects the covergence rate of the algorithm, so that the numerical solution procedure is largely an art.

It has usually been found to be worthwhile to compare the cost from Eq. E-1, once the optimal has been found, with the cost

$$< J_1 > = k \left[\underline{x}(T) \right] + \int_0^T L(\underline{x}, \underline{u}, t)\, dt + 0.5\ \mathrm{tr} \left[S(0)\ E(0) \right] + 0.5\ \mathrm{tr} \left[\int_0^T (SQ + C^T BCE) dt \right] \quad \text{(E-20)}$$

using the same optimal solution. This latter expression should be equal to Eq. E-1. If the two costs do not agree, then a programming problem exists. Equation E-20 is from Bryson & Ho (1969).

Another check on the programming can be made using the fact that the mean-squared deviation in uncontrollable states cannot change unless those states are noise-driven or have dynamics associated with them. Thus, the sum of the corresponding diagonal elements of E and $\hat{X}$ must remain constant.

The extension to variable terminal time cases proceeds as in Appendix D.

Appendix F

Numerical Values for the Mars Entry Problem

The numerical values used in the Mars entry problem of Chapter 7 are presented in this appendix.

Mars surface gravity and radius. (Shen and Cefola, 1968)

$$g_0 = 12.3 \text{ ft/sec}^2 \tag{F-1}$$

$$r = 10.86\ (10)^6 \text{ ft} \tag{F-2}$$

Vehicle Parameter. The vehicle parameter was chosen to be representative of the vehicles studied by Garland (1969a, 1969b, 1968).

$$\frac{c_d\, a}{m} = 1.023 \text{ ft}^2/\text{slug} \tag{F-3}$$

Surface density and inverse scale height. A surface density value of

$$\rho_0 = 4.8 \times 10^{-5} \text{ slug/ft}^3 \tag{F-4}$$

was used in the nominal model. This value gives the same density as the mean model of Table 7-1 at 32,810 ft when

$$\beta^{-1} = 30,000 \text{ ft} \tag{F-5}$$

The latter value is approximate for altitudes greater than 65,000 ft as seen in Table 7-2. These altitudes include all the significant maneuvers of the vehicle.

Nominal Initial Conditions. These are chosen as representative of the entry problems studied by Garland

$$h(0) = 328{,}000 \text{ ft} \tag{F-6}$$

$$\theta(0) = 0 \tag{F-7}$$

$$v(0) = 15{,}000 \text{ ft/sec} \tag{F-8}$$

$$\gamma(0) = -0.19 \text{ rad} \tag{F-9}$$

$$\rho(0) = 8 \times 10^{-10} \text{ slug/ft}^3 \tag{F-10}$$

Initial Covariance Matrix. The position and velocity errors are representative of presently achievable accuracies in navigation.

$$E_{11}(0) = 25 \text{ km}^2 = 2.56 \times 10^8 \text{ ft}^2 \tag{F-11}$$

$$E_{22}(0) = \frac{2.56 \times 10^8}{[r + h(0)]^2} \text{ rad}^2 \tag{F-12}$$

$$E_{33}(0) = 1.0 \text{ (m/sec)}^2 = 10 \text{ (ft/sec)}^2 \tag{F-13}$$

$$E_{44}(0) = 4.44 \times 10^{-8} \text{ rad}^2 \tag{F-14}$$

$$E_{55}(0) = 8.55 \times 10^{-12} \text{ (slug/ft}^3)^2 \tag{F-15}$$

The uncertainty in flight path angle is determined by assuming the velocity errors are isentropic.

Terminal Conditions. For convenience the terminal range was chosen as

$$\theta(T) = 0.5983 \text{ rad (1257.1 miles)} \tag{F-16}$$

since it can be reached by flying at a constant lift-to-drag ratio of 0.5. The terminal altitude was chosen as

$$h(T) = 5.6 \text{ miles} = 29,600 \text{ ft} \quad \text{(F-17)}$$

to give sufficient remaining altitude for the terminal landing scheme.

Density Driving noise. The constant in Eq. 7.3-16

$$q_\rho = c\rho \quad \text{(F-18)}$$

was experimentally determined to be

$$c = 4.5 \times 10^{-6} \frac{\text{slug}}{\text{ft}^3 \text{ sec}} \quad \text{(F-19)}$$

which gives acceptable density errors along the trajectory.

Measurement Noise. The noise in the accelerometers was assumed to generate a noise covariance.

$$U = \begin{bmatrix} 0.02 & 0 \\ 0 & 0.02 \end{bmatrix} \text{ ft}^2/\text{sec}^3 \quad \text{(F-20)}$$

For example, if the accumulated velocity outputs of the accelerometers have a mean-squared uncertainty of 0.1 ft/sec over one second, and the correlation time, τ, associated with this uncertainty is 0.1 sec, then U_{11} could be evaluated as

$$U_{11} = 2\tau\,(0.1) = 0.02 \text{ ft}^2/\text{sec}^3 \quad \text{(F-21)}$$

Cost function weighting. The terminal weighting on altituae and range deviations was identical (note, range is in radians)

$$
S(T) = \begin{bmatrix} 2.78 \times 10^{3} & 0 & 0 & 0 & 0 \\ 0 & 1.39 \times 10^{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \qquad \text{(F-22)}
$$

The weighting on the perturbation control was chosen as

$$
B = 0.1 \qquad \text{(F-23)}
$$

By trading-off B versus S(T) different weighting is attached to terminal error versus mean-squared perturbation control.

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Biography

George Thomas Schmidt was born December 1, 1942 in Jersey City, New Jersey. He attended public schools in Union City, New Jersey and graduated from Union Hill High School in June 1960. He entered MIT and received his Bachelor and Master of Science degrees in the Honors Program of the Department of Aeronautics and Astronautics. As an undergraduate he was an Alfred P. Sloan National Scholar and was elected a member of Tau Beta Pi, Sigma Gamma Tau, and Sigma Xi.

Mr. Schmidt joined the Charles Stark Draper Laboratory in October 1961 and worked part-time during the school years and full-time summers in the Polaris Electronics Laboratory. In June 1964, he received a staff appointment in the Apollo System Test Division and was responsible for maintaining and calibrating various Apollo ground support equipment and for the layout and installation procedures for Apollo Guidance, Navigation and Control System field site laboratories. Beginning in September 1964 he received an appointment as a Research Assistant in the same group and investigated the application of Kalman filtering techniques to the alignment and calibration of inertial navigation systems for his Master's Thesis. Later continuing this work as a Principal Engineer, Mr. Schmidt was responsible for the development of the Apollo prelaunch alignment and calibration procedures as well as the system analysis for various inertial navigation projects. He is the author of several technical papers and reports as well as a contributor to the text "Theory and Applications of Kalman Filtering".

Mr. Schmidt has had extensive teaching experience as an assistant in MIT courses on Atmospheric Entry, Astronautical Guidance, Space Dynamics and Gyroscopic Instruments and Inertial Guidance and also as a part-time faculty member in Systems Engineering at Boston University. He has also consulted for two industrial companies.

Mr. Schmidt entered MIT full-time in September 1969 to perform his doctoral thesis research and since completion of the research he has returned to staff responsibilities at the Draper Laboratory.

Mr. Schmidt is a member of the MIT Club of Boston, The American Society for Engineering Education, the AIAA and the IEEE. He is married to the former Sylvia E. Raymond of Waterville, Maine. Mrs. Schmidt graduated from Boston University and Boston College and is presently a guidance consellor in the Stoneham Public School System. The couple presently reside in Brookline, Mass.