THE SHIFT OF ENERGY LEVELS DUE TO RADIATIVE COUPLING

by

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I INTRODUCTION

Recent measurements on the fine structure of hydrogen by Lamb and Retherford [1*] have indicated that the energy levels of the hydrogen atom as calculated in the usual way on the basis of the Dirac theory of the electron are not correct.

At the same time, measurements by Kusch and Foley[2] have indicated that the energy of an electron situated in a magnetic field differs from that calculated using the Dirac theory. If we assume that the spin magnetic moment of the electron is not one Bohr magneton but greater than 1 by an amount $\alpha/2\pi$ ($\alpha = e^2/hc \simeq 1/137$) then we reach agreement with these measurements. This assumption would also remove the greater part of the discrepancy between the calculated and measured hyperfine splitting of the ground states of hydrogen and deuterium as reported by Nafe, Nelson and Rabi[3] and by Nagel, Julian and Zacharias.[4]

These results suggest that calculations on the basis of the Dirac theory should be modified but do not necessarily indicate a failure of the theory. For a stationary state in the absence of fields the Dirac equation is

$$\{ c\vec{\alpha}.\vec{p} + \beta mc^2 - E \}\psi = 0$$

When external fields characterized by $\vec{A}, \phi$ are applied we have

* A list of references is given on page 73.

the usual transformations

$$\vec{p} \longrightarrow \vec{p} - e\vec{A}/c \ , \qquad E \longrightarrow E - e\phi$$

The equation then becomes

$$\{c\,\vec{\alpha}.\vec{p} + \beta mc^2 - E - e\,\vec{\alpha}.\vec{A} + e\phi\}\psi = 0$$

The terms $-e\,\vec{\alpha}.\vec{A}$ and $e\phi$ represent the interaction of the electron with the applied field. For the present let us consider the $-e\,\vec{\alpha}.\vec{A}$ term. In calculating the energy levels of a system of electrons in an electrostatic field whose potential is ϕ it has been customary not to include an $\vec{\alpha}.\vec{A}$ term. On the other hand it has long been appreciated that to calculate for example the rate of spontaneous emission from an excited state of such a system we must include this term where $\vec{A}$ in this case is the vector potential representing the radiation field . This term is effective even when no light quanta are present. It will be remembered that the pure radiation field can be represented as a system of (quantized)oscillators. The wave function for the system of particles plus radiation field now becomes a function not only of the coordinates of the particles but also of the coordinates of the radiation field , for example the occupation numbers of the various modes of oscillation of the field. It now happens that the matrix elements of $\vec{\alpha}.\vec{A}$ are not all zero when the occupation numbers are zero and thus $-e\,\vec{\alpha}.\vec{A}$ is an effective perturbation even when no light quanta are present in the field.

Thus we can see that when the term representing the

radiative coupling is properly inserted in the Dirac equation the energy levels of the system should in general be different from those calculated in the usual way without this interaction term. The difficulty however is that the energy values thus calculated are found to involve divergent integrals. Thus each energy is infinite and moreover the difference of energy between two levels will in general be also infinite. It is for this reason of course that the interaction has been omitted in the usual calculation. It is however very significant that the energy levels calculated with omission of the radiative coupling are only very slightly different from those observed experimentally. Thus, though the coupling gives rise to infinite level shifts, yet in a very real sense it behaves as a small perturbation.

It was suggested some time ago by Schwinger and by Weisskopf that a meaning could be ascribed to the infinite energy which arises because of radiative coupling. Their suggestion is that this infinite energy in the case of a free electron can be regarded as representing an electromagnetic mass which is included, along with the "true" mechanical mass, in the measured mass. The fact that this electromagnetic mass appears to be infinite is not unduly disturbing. The infinity always appears as an integral of the type $\int_{K_0}^{\infty} \frac{dK}{K}$ where cK represents the energy of a light quantum, and thus is due to extremely high values of K. We may reasonably suppose however that electromagnetic theory needs to be revised for extremely

high energies and that if this could be done the integrals occurring would not only be finite but would really show the radiative coupling as a small perturbation, its smallness being characterized by the fact that the electromagnetic mass is of order αm.

So far we have spoken only about the interaction with the radiation field. It should be clear however that we should include in the electromagnetic mass the electrostatic self-energy of the electron, that is the energy due to its Coulomb field.

A separate aspect of the problem is due to the fact that the ordinary Dirac equation for one electron (which we shall call the single electron theory) is incapable of explaining pair production and other phenomena associated with positrons and is moreover plagued by the existence of the negative energy solutions. The Dirac hole theory on the other hand disposes of these difficulties and it is on this theory that the present calculations will be made.

Now however our examination of the self-energy problem must be modified. For we no longer have, when we discuss the self-energy of an electron, a simple system of one electron plus radiation field. Instead we have a many electron problem and we must consider ,as the perturbation to be added to the usual Hamiltonian, the radiative interaction and the electrostatic self-energy of each particle and also the interactions between the particles.

It is well known that the introduction of hole theory results in a logarithmic divergence of the self-energy rather than the linear divergence which is characteristic of the one electron theory. As a consequence of this, when we consider the problem of an electron in an external field and carry through the renormalization of mass in accordance with the idea mentioned above (and also the renormalization of the charge), we shall get finite level shifts due to the radiative coupling. This would not be true in the one electron theory.

The other main result of the introduction of hole theory is the phenomenon of vacuum polarization which leads also to logarithmically divergent expressions. It has however been long recognized that the principal effect of the vacuum polarization is a renormalization of the electron charge in a fashion similar to that suggested above for the electron mass. When this is carried through we get once again finite values for the level shift caused by the polarization of the vacuum. For an electron in an electrostatic field this was first calculated by Uehling[5].

In the present work we shall consider an electron in an external time-independent electromagnetic field. We shall evaluate to the first non-vanishing order the perturbation energy which results when we add to the Hamiltonian of the system terms representing the radiative coupling and the electrostatic self-energy for each electron and the interaction

between the electrons. We shall then develop a procedure for separating out of this result that part which may be regarded as energy due to electromagnetic mass and we shall carry out this separation. We shall also separate out the terms which correspond to a renormalization of the charge. The residue which will be finite will be regarded as the true level shift.

We summarize here the results which emerge from the calculation:

1) A reasonable procedure is developed for separating from the total perturbation energy that part which is due to electromagnetic mass. The separation of the charge renormalization terms is done in the usual way.

2) An application to the case of an electron in a low energy state of an external stationary electrostatic field gives a level shift (where K_0 is given by Bethe[6] and $V = e\phi$)

$$W = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 [\nabla^2 V]_{av}\left\{\int_{K_0}^{\mu}\frac{dk}{k} - \ln 2 + \frac{19}{30}\right\} + \frac{\alpha}{\pi}\frac{e}{2m^2c^2}[\vec{E}\cdot\vec{S}\times\vec{p}]_{av}$$

3) For hydrogenic atoms this gives (where n is the principal quantum number and Ry is the symbol for one Rydberg):

For $L = 0$ $\quad \Delta W = \frac{8}{3\pi}\,\alpha^3\frac{Z^4}{n^3}\,\mathrm{Ry}\left\{\int_{K_0}^{\mu}\frac{dk}{k} - .060\right\}$

For $L \neq 0$ $\quad \Delta W = \frac{1}{\pi}\,\alpha^3\frac{Z^4}{n^3}\,\mathrm{Ry}\,\frac{1}{(L+1)(2L+1)}$ $\quad (J = L + 1/2)$

For $L \neq 0$ $\quad \Delta W = -\frac{1}{\pi}\,\alpha^3\frac{Z^4}{n^3}\,\mathrm{Ry}\,\frac{1}{L(2L+1)}$ $\quad (J = L - 1/2)$

For the $2s_{\frac{1}{2}} - 2p_{\frac{1}{2}}$ separation in hydrogen we get as

the corresponding frequency 1045 megacycles/sec. This agrees within experimental error with the measurement of Lamb and Retherford[1] who give the frequency as being about 1000 megacycles/sec.

4) The spin-orbit term in (2) suggests a surplus magnetic moment due to spin, of magnitude $\alpha/2\pi$. This result (which was previously given by Schwinger) agrees with experiment.

5) An application of the subtraction procedures to the case of an electron in a stationary magnetic field gives a verification of the magnitude of the surplus magnetic moment.

II. GENERAL FORMULAE FOR THE PERTURBATION ENERGY

2.1 We consider an electron in a stationary state of an externally applied time-independent electrostatic or magnetic field. The state of the vacuum in this system will be that where all the negative energy states are filled. Thus the physical situation in which we shall be interested will be that where all the negative energy states and one positive energy state are filled.

We shall regard as a perturbation term the interaction of all the electrons with the radiation field and with each other (including the self-interaction which gives the electrostatic self-energy) and we shall evaluate to the first non-vanishing order the energy which is due to this perturbation; the calculation will be done by the standard perturbation method.

As the significant energy we shall then take the difference between the energies of the two systems: Vacuum plus one electron in a positive energy state, Vacuum.

$$W = W_{vac+1} - W_{vac}$$

We shall call this the perturbation energy.

We describe the electromagnetic field (not including the externally applied field) by potentials $\vec{A}, \phi$ which satisfy

$$\nabla . A + 1/c \dot{\phi} = 0$$

The longitudinal part of this field can now be replaced in the

usual way by the non-retarded Coulomb interaction of the electrons plus the electrostatic self-energy. The transverse part of the field can be represented by a new vector potential $\vec{A}$ such that

$$\nabla \cdot A = 0$$

We shall at first consider the two parts of the field separately. The energy due to the first part we shall call the electrostatic energy and that due to the second the electrodynamic energy.

2.2 The Electrodynamic Energy (W^D)

We expand $\vec{A}$ by assuming periodicity on the surface of a cubic volume V. V is then the normalization volume and should appear at various places throughout the calculation. We shall however not write V explicitly.

$$\vec{A} = \sum_{\substack{\vec{k} \\ \lambda=1,2}} B_k \vec{\epsilon}_\lambda \left\{ e^{i\vec{k}\cdot\vec{r}/\hbar} q_\lambda(\vec{k}) + e^{-i\vec{k}\cdot\vec{r}/\hbar} q_\lambda^*(\vec{k}) \right\}$$

In this expansion $\vec{\epsilon}_\lambda$ is a unit polarization vector and we have

$$\epsilon_\lambda^2 = 1 \qquad \vec{\epsilon}_1 \cdot \vec{\epsilon}_2 = \vec{\epsilon}_1 \cdot \vec{k} = \vec{\epsilon}_2 \cdot \vec{k} = 0$$

We have also $B_k = \left[\frac{2\pi\hbar^2 c}{k}\right]^{\frac{1}{2}}$. The sum over $\vec{k}$ will be replaced eventually by an integral and $e^2 \sum_{\vec{k}} B_k^2 F(\vec{k}) = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{k}}{k} F(\vec{k})$

$$d\vec{k} = k^2 dk d\omega_k$$

The wave function for the system is now a function both of the coordinates of the electrons and of the occupation numbers $n_\lambda(\vec{k})$ which give the degree of excitation of the various

radiation oscillators (i.e. the number of quanta present).

$$\Psi = \psi(\text{electrons})\ \phi(\text{photons})$$

The $\sigma_\lambda(\vec{k})$ and $\sigma^*_\lambda(\vec{k})$ are operators which act on ϕ according to

$$\sigma_\lambda(\vec{k})\ \phi(\ n_\lambda(\vec{k})\ ) = \sqrt{n_\lambda(\vec{k})}\ \phi(\ n_\lambda(\vec{k})-1\ )$$

$$\sigma^*_\lambda(\vec{k})\ \phi(\ n_\lambda(\vec{k})\ ) = \sqrt{n_\lambda(\vec{k})+1}\ \phi(\ n_\lambda(\vec{k})+1\ )$$

Thus $\sigma^*_\lambda(\vec{k})\,\sigma_\lambda(\vec{k})\,\phi(\ n_\lambda(\vec{k})\ ) = n_\lambda(\vec{k})\ \phi(\ n_\lambda(\vec{k})\ )$

The Hamiltonian for the system is H = H(electrons) + H(photons) + H(interaction).

$$H(\text{electrons}) = \sum_{\text{elect.}} c\vec{\alpha}.(\vec{p}-e\vec{A}_0/c) + \beta mc^2 + e\phi_0 \quad = \sum H_0 \text{ say.}$$

where $\vec{A}_0$, ϕ_0 correspond to the external field.

$$H(\text{photons}) = c\sum_{\lambda\vec{k}} k\,\sigma^*_\lambda(\vec{k})\,\sigma_\lambda(\vec{k}) = \sum n_\lambda(\vec{k})\,ck$$

$$H(\text{interaction}) = -e\sum_{\text{elect.}} B_k\,\alpha_\lambda\left\{ e^{i\vec{k}.\vec{r}/\hbar}\,\sigma_\lambda(\vec{k}) + e^{-i\vec{k}.\vec{r}/\hbar}\,\sigma^*_\lambda(\vec{k})\right\}$$

where $\alpha_\lambda = \vec{\alpha}.\vec{\epsilon}^{\vec{k},\lambda}_{\lambda}$.

We shall always concern ourselves with physical situations where no photons are present (all $n_\lambda(\vec{k}) = 0$). We regard H(int.) as a perturbation and shall evaluate the energy $W_{\text{vac}+1}$ for the system in the state : all negative energy states filled, one electron in a positive energy state Ψ_0 ,no photons present. From this we subtract the energy W_{vac} for the system in the state: all negative states filled, no photons.

Since the $b_\lambda(\vec{k})$ have no diagonal elements the first order perturbation energy vanishes. The intermediate states for the second order perturbation energy must be states with one photon ($\vec{k}\ \lambda$) present. But we have (using the symbol $\langle\ \rangle$ for matrix elements)

$$\langle \phi^*(\ 0_\lambda(\vec{k})\ ) b_\lambda(\vec{k}) \phi(\ 1_\lambda(\vec{k})\ \rangle = \langle \phi^*(1_\lambda(\vec{k})\ ) b_\lambda^*(\vec{k})\ \phi\ (0_\lambda(\vec{k})\ )\rangle = 1$$

and these are the only matrix elements of b , b^* which will appear. We may thus conveniently ignore the b , b^* operators and also the dependence of Ψ on the radiation field. We then use

$$H(\text{interaction}) = -e \sum_{\text{elect.}} B_k \alpha_\lambda e^{\pm i\vec{k}\cdot\vec{r}/\hbar}$$

for emission leading into the intermediate state (-), or absorption leading from the intermediate state. In the intermediate state the radiation energy is ck so that the total energy is E(electrons) + ck .In the physical state, the radiation energy is of course zero. We shall always use the symbol E to refer to the energy of the electrons alone.

For the energy due to the perturbation we have

$$W = \sum_i \frac{R_{pi}R_{ip}}{E_p - E_i - ck}$$

R = H(interaction).

p = Physical state.

i = Intermediate state.

It is convenient to introduce the second quantization operators for the electrons. Let ψ_s and E_s be the eigenfunctions and eigenvalues of H_o. We write

$$\chi = \sum_s a_s \Psi_s \qquad \chi^* = \sum_s a_s^* \Psi_s^*$$

We write the wave function for the electrons as $C(\, n_1\, n_2 \ldots)$ where n_i ($= 0$ or 1) is the number of electrons in the i^{th} state. Then

$$a_i C(\ldots, n_i, \ldots) = \pm\, n_i C(\ldots, 1-n_i, \ldots)$$

$$a_i^* C(\ldots, n_i, \ldots) = \pm (1-n_i) C(\ldots, 1-n_i, \ldots)$$

$$a_i^* a_i = n_i \cdot \qquad a_i^* a_i + a_i a_i^* = 1 \ .$$

H(interaction) now transforms as follows;

$$H(\text{inter.}) \rightarrow \langle \chi^* R \chi \rangle = -e \sum_{\substack{r,s \\ \vec{\kappa} \lambda}} a_r^* a_s \, R^{\lambda}_{rs}(\vec{\kappa}) B_\kappa$$

where $R^{\lambda}_{rs}(\vec{\kappa}) = \langle \Psi_r^* \alpha_\lambda e^{\pm i \vec{\kappa}\cdot\vec{r}/\hbar} \Psi_s \rangle$

We write C_o and E^o for the physical state, C_n and E^n for the intermediate states. Then

$$W = e^2 \sum_{\substack{\lambda \vec{\kappa} \\ \kappa e r s, n}} B_\kappa^2 \frac{R^{\lambda}_{\kappa e} R^{\lambda}_{rs}}{E^o - E^n - c\kappa} \langle C_o^* q_\kappa^* q_e C_n \rangle \langle C_n^* q_r^* q_s C_o \rangle$$

$$= \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{\kappa}}{\kappa} \sum_{\substack{\lambda \\ \kappa e r s}} \frac{R^{\lambda}_{\kappa e} R^{\lambda}_{rs}}{E_s - E_r - c\kappa} \langle C_o^* q_\kappa^* q_e q_r^* q_s C_o \rangle$$

The diagonal combinations of the a^*, a operators are $N_k N_r$ ($k = \ell$, $r = s$) and $N_s(1-N_r)$ for $k = s$, $\ell = r$.

Then

$$W = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{K}}{K} \sum_{\lambda s,r} \left\{ \frac{R^{\lambda}_{sr} R^{\lambda}_{rs}}{E_s - E_r - cK} N_s(1-N_r) - \frac{R^{\lambda}_{rr} R^{\lambda}_{ss}}{cK} N_s N_r \right\}$$

Applying this result we have $W_{vac+1} - W_{vac}$

$$= W^D = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{K}}{K} \left\{ \sum_{J^+\lambda} \frac{A^{\lambda}_{oJJo}}{E_o - E_J - cK} - \sum_{J^-\lambda} \frac{A^{\lambda}_{oJJo}}{E_J - E_o - cK} \right\} - \frac{\alpha c}{2\pi^2} \int \frac{d\vec{K}}{K^2} \sum_{J^-\lambda} A^{\lambda}_{ooJJ}$$

where we define $A^{\lambda}_{klmn} = \langle \Psi^*_k \alpha_\lambda e^{-i\vec{K}\cdot\vec{r}/\hbar} \Psi_l \rangle \langle \Psi^*_m \alpha_\lambda e^{+i\vec{K}\cdot\vec{r}/\hbar} \Psi_n \rangle$

and $\sum_{J^\pm}$ indicates a sum over the states of $\pm$ energy.

The first two terms have an obvious significance. They indicate that the energy of the vacuum plus one electron is on the one hand increased over that of the vacuum because the added electron can make (virtual) transitions to the positive unoccupied states while on the other hand it is decreased because the presence of the electron in the state Ψ_o eliminates the vacuum transitions to that state.

We shall designate the first two terms together as the electrodynamic exchange energy W^{DX} and the third term as the electrodynamic non-exchange energy W^{DN}. The reason for this separation and nomenclature will become plain later.

We now write W^{DN} in a different form. Expanding the Ψ 's in free electron solutions

$$\Psi_k = \sum a^r_k \phi^r(\vec{\rho}) = \sum a^r_k u^r(\vec{\rho}) e^{i/\hbar \vec{p}\cdot\vec{r}}$$

(where $u^r(\vec{\rho})$ is a normalized Dirac spinor) we have

A^{λ}_{klmn}

$$= \sum q^{*r}_{k}(\vec{p})\, q^{s}_{e}(\vec{p}+\vec{\kappa})\, q^{*t}_{m}(\vec{q})\, q^{u}_{n}(\vec{q}-\vec{\kappa}) \langle u^{*r}(\vec{p})\, \alpha_\lambda\, u^{s}(\vec{p}+\vec{\kappa}) \rangle \langle u^{*t}(\vec{q})\, \alpha_\lambda\, u^{u}(\vec{q}-\vec{\kappa}) \rangle$$

We define $B^{\lambda}_{klmn} = \int \{ \Psi^{*}_{k}(\vec{r})\, \alpha_\lambda\, \Psi_{e}(\vec{r}) \} \{ \Psi^{*}_{m}(\vec{r}')\, \alpha_\lambda\, \Psi_{n}(\vec{r}') \} \frac{d\vec{r}\, d\vec{r}'}{|\vec{r}-\vec{r}'|}$

where $\{\ \}$ denotes the spin scalar product.

Then B^{λ}_{klmn}

$$= \sum q^{*r}_{k}(\vec{p})\, q^{s}_{e}(\vec{q}')\, q^{*t}_{m}(\vec{q})\, q^{u}_{n}(\vec{p}') \langle u^{*r}(\vec{p})\, \alpha_\lambda\, u^{s}(\vec{q}') \rangle \langle u^{*t}(\vec{q})\, \alpha_\lambda\, u^{u}(\vec{p}') \rangle \int e^{i/\hbar(\vec{p}-\vec{q}')\cdot\vec{r} + i/\hbar(\vec{p}'-\vec{q})\cdot\vec{r}'} \frac{d\vec{r}\, d\vec{r}'}{|\vec{r}-\vec{r}'|}$$

The value of the integral is $\frac{4\pi\hbar^2}{\kappa^2} \delta_{\vec{q}',\vec{p}+\vec{\kappa}}$ where $\vec{\kappa} = \vec{q}' - \vec{p}$

Using this we get $B^{\lambda}_{klmn} = \sum_{\vec{\kappa}} \frac{4\pi\hbar^2}{\kappa^2} A^{\lambda}_{klmn} = \frac{1}{2\pi^2\hbar} \int \frac{d\vec{\kappa}}{\kappa^2} A^{\lambda}_{klmn}$

Then $W^{DN} = -e^2 \sum_{J^-,\, \lambda=1,2} B^{\lambda}_{ooJJ}$

To summarize:

$$W^{D} = W^{DX} + W^{DN}$$

$$W^{DX} = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{\kappa}}{\kappa} \sum_{\substack{J \\ \lambda=1,2}} \frac{A^{\lambda}_{oJJo}}{E_o - E_J - c\kappa\delta_J} \qquad \delta_J = \frac{E_J}{|E_J|} = \pm 1$$

$$W^{DN} = -\frac{\alpha c}{2\pi^2} \int \frac{d\vec{\kappa}}{\kappa^2} \sum_{\substack{J^- \\ \lambda=1,2}} A^{\lambda}_{ooJJ} \qquad = -e^2 \sum_{\substack{J^- \\ \lambda=1,2}} B^{\lambda}_{ooJJ}$$

2.3 The Electrostatic Energy (W^S)

When the longitudinal part of the field (derived from ϕ and $\vec{A}$(long.) where $\nabla \times \vec{A}$(long.) = 0) is eliminated the result is to add to the Hamiltonian of the system a term $1/2 \sum \frac{e^2}{r_{ij}}$. For $i \neq j$ the terms of this sum represent simply the non-retarded Coulomb interaction. For $i = j$ the terms are meaningless as they stand but when properly interpreted represent the energy which an electron has because of its Coulomb field. By use of the χ operator introduced previously both parts together can be represented as the average value in the physical state of the operator

$$W_{op} = \frac{1}{2} e^2 \int \{\chi^*(\vec{r}) \chi(\vec{r})\}\{\chi^*(\vec{r}') \chi(\vec{r}')\} \frac{d\vec{r} d\vec{r}'}{|\vec{r}-\vec{r}'|}$$

This is reminiscent of the classical expression $\frac{1}{2}\int \rho(\vec{r})\, \rho(\vec{r}') \frac{d\vec{r} d\vec{r}'}{|\vec{r}-\vec{r}'|}$

But in the present case the χ's are non-commuting quantities and as a result W^S will be found to involve terms of an "exchange"character as well as terms which have a direct classical analogy. As before we shall separate the two types of terms and write

$$W^S = W^{SX} + W^{SN}$$

Introducing the quantity $B^4_{klmn} = \int \{\Phi^*_k(\vec{r})\, \Phi_\ell(\vec{r})\}\{\Phi^*_m(\vec{r}')\, \Phi_n(\vec{r}')\} \frac{d\vec{r} d\vec{r}'}{|\vec{r}-\vec{r}'|}$

in analogy to the B^{λ}_{klmn} introduced previously, we have

$$W_{op} = 1/2\ e^2 \sum a_k^* a_l a_m^* a_n B^4_{klmn}$$

In this case the first order perturbation energy is not zero. We thus need the average value of W_{op} in the physical state.

For any state $C(\ n_1\ n_2\ \ldots\ )$

$$W = 1/2\ e^2 \sum B^4_{klmn} \langle C^*\ a_k^* a_l a_m^* a_n C \rangle$$

Then by the same argument, as before,

$$W = 1/2\ e^2 \sum N_k(1-N_m) B^4_{kmmk} + 1/2\ e^2 \sum N_k N_m B^4_{kkmm}$$

Applying this we have

$$W_{vac+1} - W_{vac} = W^S = 1/2\ e^2 \sum_J B^4_{oJJo} \delta_J + e^2 \sum_{J^-} B^4_{ooJJ}$$

where once again we use the symbol $\delta_J = \pm 1$ according as Ψ_J is a state of $\pm$ energy. As before we label the first term as the electrostatic exchange energy (W^{SX}) and the second as the electrostatic non-exchange energy (W^{SN}).

The relationship derived previously between A_{klmn} and B_{klmn} clearly holds in this case also. Finally we have (defining A^4_{klmn} in the obvious fashion) :

$$W^S = W^{SX} + W^{SN}$$

$$W^{SX} = \frac{\alpha c}{4\pi^2} \int \frac{d\vec{K}}{K} \sum_J A^4_{oJJo} \delta_J = 1/2 \sum_J e^2 B^4_{oJJo} \delta_J$$

$$W^{SN} = \frac{\alpha c}{2\pi^2} \int \frac{d\vec{K}}{K} \sum_{J^-} A^4_{ooJJ} = \sum_{J^-} e^2\ B^4_{ooJJ}$$

2.4 Symetric Form for Electrodynamic and Electrostatic Energy

The expressions for W^D and W^S are in rather different form corresponding to the fact that we have divided the total field into two parts and have treated the two parts in a rather different fashion. This separation is of course not a Lorentz-invariant one. A different procedure is to expand and quantize the total field. The difficulty with this is that the Lorentz condition

$$\nabla . \vec{A} + 1/c \dot{\phi} = 0$$

must be treated as a supplementary condition which connects the two types of longitudinal quanta.

We shall now convert $W^D + W^S$ into the symetrical form which would have emerged from this procedure. This is of little importance in calculating $W^D + W^S$ since the two methods will of course give the same result (the second is however somewhat simpler). But the difference in the two procedures will be found to be of great importance when we later consider the subtraction procedure which will eliminate the energy due to electromagnetic mass.

We consider the sum $1/c\kappa \sum_J A^4_{oJJo} \delta_J$ which occurs in W^S. Let H be the Hamiltonian and E_o, E_J the energies corresponding to Ψ_o, Ψ_J. Then

$$He^{i\vec{K}.\vec{r}/\hbar} - e^{i\vec{K}.\vec{r}/\hbar} H = c\vec{\alpha}.\vec{p} e^{i\vec{K}.\vec{r}/\hbar} - e^{i\vec{K}.\vec{r}/\hbar} c\vec{\alpha}.\vec{p} = c\vec{\alpha}.\vec{K} e^{i\vec{K}.\vec{r}/\hbar}$$

(the other terms of H commute with $e^{i\vec{K}.\vec{r}/\hbar}$.)

Similarly $He^{-i\vec{K}\cdot\vec{r}/\hbar} - e^{-i\vec{K}\cdot\vec{r}/\hbar}H = -c\vec{\alpha}\cdot\vec{K}e^{-i\vec{K}\cdot\vec{r}/\hbar}$

But $\langle \Psi_N^* \left[He^{\pm i\vec{K}\cdot\vec{r}/\hbar} - e^{\pm i\vec{K}\cdot\vec{r}/\hbar}H \right] \Psi_M \rangle = (E_N - E_M)\langle \Psi_N^* e^{\pm i\vec{K}\cdot\vec{r}/\hbar}\Psi_M \rangle$

We have then $(E_N - E_M)\langle \Psi_N^* e^{\pm i\vec{K}\cdot\vec{r}/\hbar}\Psi_M \rangle = \pm cK \langle \Psi_N^* \alpha_K e^{\pm i\vec{K}\cdot\vec{r}/\hbar}\Psi_M \rangle$

where $\alpha_K = 1/K\,\vec{\alpha}\cdot\vec{K}$

We use also the arithmetical identities:

$$\frac{1}{cK} = \left\{ \frac{(E_0 - E_J)^2}{c^2K^2} - 1 \right\} \frac{1}{E_0 - E_J - cK} - \frac{(E_0 - E_J)}{c^2K^2}$$

$$= \left\{ \frac{(E_0 - E_J)^2}{c^2K^2} - 1 \right\} \frac{1}{E_J - E_0 - cK} + \frac{(E_0 - E_J)}{c^2K^2}$$

Applying these results

$$W^{SX} = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{K}}{K} \left[\sum_J \frac{A^4_{0JJ0}}{E_0 - E_J - cK\delta_J} \left\{ \frac{(E_0 - E_J)^2}{c^2K^2} - 1 \right\} - \sum_J A^4_{0JJ0} \frac{(E_0 - E_J)}{c^2K^2} \right]$$

The last term of this gives

$$-\sum_J \frac{\langle \Psi_0^* e^{-i\vec{K}\cdot\vec{r}/\hbar}\Psi_J \rangle \langle \Psi_J^* \vec{\alpha}\cdot\vec{K} e^{i\vec{K}\cdot\vec{r}/\hbar}\Psi_0 \rangle}{cK^2} = \frac{1}{cK^2}\langle \Psi_0^* \vec{\alpha}\cdot\vec{K}\,\Psi_0 \rangle = 0$$

Then

$$W^{SX} = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{K}}{K} \sum_J \frac{1}{(E_0 - E_J - cK\delta_J)} \left\{ A^3_{0JJ0} - A^4_{0JJ0} \right\}$$

where in A^3_{klmn}, α_λ is replaced by $1/K\,\vec{\alpha}\cdot\vec{K} = \alpha_K$

We can now write, using the convention that

$$\sum_{\lambda}{}' F(\lambda) = F(1) + F(2) + F(3) - F(4),$$

$$W^{DX} + W^{SX} = W^{X} = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{k}}{k} \sum_{\lambda J}{}' \frac{A^{\lambda}_{oJJo}}{(E_o - E_J - ck\delta_J)}$$

In evaluating the sum over $\lambda = 1, 2, 3$ we can replace $\vec{\alpha}\cdot\vec{\epsilon}_1$, $\vec{\alpha}\cdot\vec{\epsilon}_2$, $1/k\,\vec{\alpha}\cdot\vec{k}$ by the components of $\vec{\alpha}$ along the axes of any other orthogonal coordinate system. Most simply we can take α_1, α_2, α_3.

To combine W^{DN} and W^{SN} we observe that $A^{3}_{ooJJ} = 0$. For this involves the matrix element

$$\langle \Psi_o^* \alpha_k e^{-i\vec{k}\cdot\vec{r}/\hbar} \Psi_o \rangle = (E_o - E_o)\langle \Psi_o^* e^{-i\vec{k}\cdot\vec{r}/\hbar} \Psi_o \rangle \frac{1}{ck} = 0$$

Then

$$W^{DN} + W^{SN} = W^{N} = -e^2 \sum_{\lambda J}{}' B^{\lambda}_{ooJJ} = \int \left\{ \rho_o(\vec{r}) \rho_{neg}(\vec{r}') - \frac{1}{c^2} \vec{j}_o(\vec{r}) \cdot \vec{j}_{neg}(\vec{r}') \right\} \frac{d\vec{r}\,d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

The last step follows immediately from the definition of B^{λ}_{klmn} and the identification of $c\vec{\alpha}$ as the velocity operator. ρ_o and $\vec{j}_o$ are here the charge and current densities due to the electron in the state Ψ_o; ρ_{neg} and $\vec{j}_{neg}$ are the densities due to the electrons in the negative energy states. As usual the effects of retardation are included in W^X.

Finally we have, as the symetrical form of the perturbation energy:

$$W = W^X + W^N$$

$$W^X = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{K}}{K} \sum_{J\lambda}{}' \frac{A^\lambda_{oJJo}}{(E_o - E_J - cK\delta_J)}$$

$$W^N = -\frac{\alpha c}{2\pi^2} \int \frac{d\vec{K}}{K^2} \sum_{J\lambda}{}' A^\lambda_{ooJJ} = -e^2 \sum_{J\lambda}{}' B^\lambda_{ooJJ} = \int \left\{ \rho_o(\vec{r}) \rho_{neg}(\vec{r}') - \frac{1}{c^2} \vec{j}_o(\vec{r}) \cdot \vec{j}_{neg}(\vec{r}') \right\} \frac{d\vec{r} d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

The definitions of A^λ_{klmn} and B^λ_{klmn} are given below for convenience: (note that we take $\alpha_4 = 1$)

$$A^\lambda_{klmn} = \langle \Psi^*_k \alpha_\lambda e^{-i\vec{K}\cdot\vec{r}/\hbar} \Psi_e \rangle \langle \Psi^*_m \alpha_\lambda e^{+i\vec{K}\cdot\vec{r}/\hbar} \Psi_n \rangle$$

$$B^\lambda_{klmn} = \int \left\{ \Psi^*_k(\vec{r}) \alpha_\lambda \Psi_e(\vec{r}) \right\} \left\{ \Psi^*_m(\vec{r}') \alpha_\lambda \Psi_n(\vec{r}') \right\} \frac{d\vec{r} d\vec{r}'}{|\vec{r} - \vec{r}'|}$$

The terms in W for $\lambda = 1,2$ give the electrodynamic energies; the terms for $\lambda = 3,4$ give the electrostatic energies.

III. EXCHANGE ENERGY (W^X) FOR THE ELECTRON IN AN APPLIED FIELD

3.1 We now take up the problem of calculating the energy W^X, the general expression for which is given in the previous section. We shall leave until later any consideration of the non-exchange energy W^N. We take the case where the unperturbed system includes a stationary external field, which may be electrostatic or magnetic. Thus the unperturbed system will have one electron in a positive state Φ_o ; the vacuum electrons will occupy all the negative states Φ_J- . Φ_o and Φ_J will be eigenfunctions and E_o and E_J the corresponding energy values of the equation

$$\{ \quad c\vec{\alpha}.\vec{p} + \beta mc^2 + U - E \}\Psi = 0$$

Here $U = V = e\phi$, or $U = -e\vec{\alpha}.\vec{A}$ for the electrostatic or magnetic cases respectively where ϕ or $\vec{A}$ describes the applied field. It will not be necessary to consider combined electrostatic and magnetic fields since the results of interest will involve only the first order in the fields.

We shall always concern ourselves with states which are "non-relativistic", that is states whose energy, exclusive of the rest mass energy, is small compared with mc^2. Then too, for the average momentum $\bar{p}$ we have

$$\bar{p} \ll \mu \qquad (\mu = mc)$$

We have now

$$W^X = \frac{\alpha c^2}{4\pi^2}\int \frac{d\vec{K}}{K}\sum_{\lambda J}{}' \frac{A^{\lambda}_{oJJ_o}}{(E_o - E_J - cK\delta_J)}$$

$$= \frac{\alpha c^2}{4\pi^2}\int \frac{d\vec{K}}{K}\sum_{\lambda J}{}' \frac{\langle \Psi_o^* \alpha_\lambda e^{-i\vec{K}\cdot\vec{r}/\hbar}\Psi_J\rangle\langle \Psi_J^* \alpha_\lambda e^{i\vec{K}\cdot\vec{r}/\hbar}\Psi_o\rangle}{(E_o - E_J - cK\delta_J)}$$

In evaluating this expression we shall treat Ψ_o exactly. The intermediate states will however be expanded in powers of U, regarding U for these states as a perturbation on a free particle Hamiltonian. Clearly this procedure will not be accurate for low-lying intermediate states but will be satisfactory for higher states in the continuum. To circumvent this difficulty we shall at first restrict ourselves to values of the light quantum momentum K (and we use this term even for the longitudinal field) which are greater than $\delta\mu$. It will be satisfactory to consider $\delta \sim 1/137$. In this way the only effective intermediate states will be those whose average momentum $\bar{p}_i$ is greater than $\delta\mu$.

For $K < \delta\mu$ we shall be able very easily to make the calculation in a manner similar to that of Bethe[6]. The two parts of the result will join correctly so that there will be no doubt concerning the correctness of the procedure.

We expand Ψ_J as

$$\Psi_J = \phi^J(\vec{q}) + \sum \frac{U^{J'J}_{q'q}}{(E^J(q) - E^{J'}(q'))}\phi^{J'}(\vec{q}') + \sum \frac{U^{J''J'}_{q''q'}\, U^{J'J}_{q'q}}{(E^{J''}(q'') - E^J(q))(E^{J'}(q') - E^J(q))}\phi^{J''}(\vec{q}'')$$

where $\phi^J(\vec{q}) = u^J(\vec{q}) e^{i\vec{q}\cdot\vec{r}/\hbar}$ is a Dirac free electron solution.

$$E^J(q) = c\delta_J[\mu^2+q^2]^{1/2} = E(q)\delta_J$$

$$U^{rs}_{pq} = \langle \phi^r(\vec{p})\, U\, \phi^s(\vec{q}) \rangle$$

For brevity we define

$$R^{oJ}_{oq} = \langle \Psi_o^* \alpha_\lambda e^{i\vec{k}\cdot\vec{r}/\hbar} \phi^J(\vec{q}) \rangle$$

$$R^{Jo}_{qo} = \langle \phi^J(q) \alpha_\lambda e^{-i\vec{k}\cdot\vec{r}/\hbar} \Psi_o \rangle$$

$$\delta_J^{\pm} = \tfrac{1}{2}(1 \pm \delta_J)$$

$$B^J(q) = E(q) - E_o\delta_J + ck$$

We insert the intermediate state expansion into W^X and collect together the terms of zero, first and second order in U.

<u>Zero Order</u>

$$W_o^X = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\lambda Jq}{}' R^{oJ}_{oq} R^{Jo}_{qo}\frac{\delta_J}{B^J(q)}$$

<u>First Order</u>

$$W_1^X = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\substack{\lambda \\ J\,q \\ J'\,q'}}{}'\left\{\frac{R^{Jo}_{qo} R^{oJ'}_{oq'} U^{J'J}_{q'q}\delta_J}{(E^J - E^{J'})B^J(q)} + \frac{R^{J'o}_{qo} R^{oJ}_{oq} U^{JJ'}_{qq'}\delta_J}{(E^J - E^{J'})B^J(q)}\right\}$$

In the first term of this we interchange J, J' and q, q'.

We use

$$\frac{1}{E^J(q)+E^{J'}(q')}\left\{\frac{\delta_J}{B^J(q)}-\frac{\delta_{J'}}{B^{J'}(q')}\right\}=\frac{-1}{B^J(q)\,B^{J'}(q')} \qquad J,\ J' \text{ both + or both -}$$

$$=\frac{L}{B^J(q)\,B^{J'}(q')} \qquad J,\ J' \text{ mixed in sign}$$

where $L = 1 + L_0 = 1 + \dfrac{2ck}{E(q)+E(q')}$

Then
$$W_1^X = \frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\substack{\lambda JJ'\\ qq'}}{}' R^{0J}_{0q}\,U^{JJ'}_{qq'}\,R^{J'0}_{q'0}\,\frac{(\delta_J^+\delta_{J'}^+ + \delta_J^-\delta_{J'}^-)}{B^J(q)\,B^{J'}(q')}$$

$$-\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum{}' L\,R^{0J}_{0q}\,U^{JJ'}_{qq'}\,R^{J'0}_{q'0}\,\frac{(\delta_J^+\delta_{J'}^- + \delta_J^-\delta_{J'}^+)}{B^J(q)\,B^{J'}(q')}$$

<u>Second Order</u>

$$W_2^X = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\substack{\lambda\\ JJ'J''\\ qq'q''}}{}'\left\{\frac{U^{JJ'}R^{J'0}R^{0J''}U^{J''J}\delta_J}{(E^J-E^{J'})(E^J-E^{J''})\,B^J(q)}+\frac{R^{J0}R^{0J''}U^{J''J'}U^{J'J}\delta_J}{(E^{J'}-E^J)(E^{J''}-E^J)\,B^J(q)}\right.$$
$$\left.+\frac{U^{J'J''}U^{JJ'}R^{J''0}R^{0J}\delta_J}{(E^{J'}-E^J)(E^{J''}-E^J)\,B^J(q)}\right\}$$

In the first term let $J'', q'' \longrightarrow J, q$; $J, q \to J', q'$; $J', q' \to J'', q''$

In the second term let $J'', q'' \to J, q$; $J, q \longrightarrow J'', q''$

Then
$$W_2^X = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\substack{\lambda\\ JJ'J''\\ qq'q''}}{}' R^{0J}U^{JJ'}U^{J'J''}R^{J''0}\left\{\frac{\delta_{J'}}{(E^{J'}-E^{J''})(E^{J'}-E^J)B^{J'}}\right.$$
$$\left.+\frac{\delta_{J''}}{(E^{J'}-E^{J''})(E^J-E^{J''})B^{J''}}+\frac{\delta_J}{(E^{J'}-E^J)(E^{J''}-E^J)B^J}\right\}$$

We now find that $\left\{\quad\right\} = \dfrac{1}{B^J(q)\,B^{J'}(q')\,B^{J''}(q'')}$ (all J's +)

$= \dfrac{-1}{B^J(q)\,B^{J'}(q')\,B^{J''}(q'')}$ (all J's −)

Then W_2^X

$$= -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{K}}{K}\sum_{\substack{\lambda\\ JJ'J''\\ qq'q''}}{}' R^{oJ}_{oq}\,U^{JJ'}_{qq'}\,U^{J'J''}_{q'q''}\,R^{J''o}_{q''o}\,\frac{\left(\delta_J^+\delta_{J'}^+\delta_{J''}^+ - \delta_J^-\delta_{J'}^-\delta_{J''}^-\right)}{B^J(q)\,B^{J'}(q')\,B^{J''}(q'')}$$

+ terms involving δ_J^+ , δ_J^- mixed.

It will now be convenient to write the expressions for W^X as the expectation values in the state Ψ_o of certain operators.

$$W_o^X = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{K}}{K}\sum_{\lambda}{}'\left\{\sum_{Jq}\frac{\langle\Psi_o^*\alpha_\lambda e^{i\vec{K}\cdot\vec{r}/\hbar}\phi^J(\vec{q})\rangle\langle\phi^{*J}(\vec{q})\alpha_\lambda e^{-i\vec{K}\cdot\vec{r}/\hbar}\Psi_o\rangle\,\delta_J}{E(q) - E_o\delta_J + cK}\right\}$$

We expand Ψ_o as $\Psi_o = \sum a^\rho(\vec{p})\,\phi^\rho(\vec{p})$

The expression in brackets then becomes

$$\sum_{\substack{J\\ \rho i\\ \vec{p}\,\vec{i}}} a^{*\rho}(\vec{p})\,a^i(\vec{p})\,\langle u^{*\rho}(\vec{p})\,\alpha_\lambda\,u^J(\vec{p}-\vec{K})\rangle\langle u^{*J}(\vec{p}-\vec{K})\,\alpha_\lambda\,u^i(\vec{p})\rangle\,\frac{\delta_J}{E(\vec{p}-\vec{K}) - E_o\delta_J + cK}$$

We introduce now the projection operators $G^{\pm}$ which we shall use all throughout the calculation. When operating on free electron states of momentum $\vec{p}-\vec{\kappa}$, these operators select the states of $\pm$ energy.

$$G^{\pm} = \frac{1}{2}\left[1 \pm \frac{c\vec{\alpha}.\vec{p} + \beta mc^2 - c\vec{\alpha}.\vec{\kappa}}{E(\vec{p}-\vec{\kappa})}\right] = \frac{1}{2}\left[1 \pm \frac{H}{E}\right] \quad \text{say.}$$

We write $\left(E(\vec{p}-\vec{\kappa}) \mp E_0 + c\kappa \right) = B^{\pm}$

The expression then becomes $\left\langle \Psi_o^* \alpha_\lambda \left[\frac{G^+}{B^+} - \frac{G^-}{B^-}\right] \alpha_\lambda \Psi_o \right\rangle$

We proceed in the same way for W_1^X, W_2^X. We then get

$$W_o^X = -\frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{\kappa}}{\kappa} \sum_\lambda{}' \left\langle \Psi_o^* \alpha_\lambda \left[\frac{G^+}{B^+} - \frac{G^-}{B^-}\right] \alpha_\lambda \Psi_o \right\rangle$$

$$W_1^X = \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{\kappa}}{\kappa} \sum_\lambda{}' \left\langle \Psi_o^* \alpha_\lambda \left[\frac{\bar{G}U\bar{G}^+}{B^+B^+} + \frac{\bar{G}U\bar{G}^-}{B^-B^-} - \frac{\bar{G}U\bar{G}^-}{B^+B^-} - \frac{\bar{G}U\bar{G}^+}{B^-B^+}\right] \alpha_\lambda \Psi_o \right\rangle$$

$$- \frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{\kappa}}{\kappa} \sum_\lambda{}' \left\langle \Psi_o^* \alpha_\lambda L_o \left[\frac{\bar{G}U\bar{G}^-}{B^+B^-} + \frac{\bar{G}U\bar{G}^+}{B^-B^+}\right] \alpha_\lambda \Psi_o \right\rangle$$

$$W_2^X = -\frac{\alpha c^2}{4\pi^2} \int \frac{d\vec{\kappa}}{\kappa} \sum_\lambda{}' \left\langle \Psi_o^* \alpha_\lambda \left[\frac{\bar{G}U\bar{G}U\bar{G}^+}{B^+B^+B^+} - \frac{\bar{G}U\bar{G}U\bar{G}^-}{B^-B^-B^-}\right] \alpha_\lambda \Psi_o \right\rangle$$

+ Terms involving G^+, G^- mixed.

The second term of W_1^X must be interpreted in a special way. L_o which was originally defined as $\frac{2c\kappa}{E(q)+E(q')}$ now becomes $\frac{2c\kappa}{E(\vec{p}-\vec{\kappa})+E(\vec{p}'-\vec{\kappa})}$. In this the term in p must be understood as standing to the left of U and the term in p' as standing to the right of U. This will however cause no difficulty.

We should note again that the $G^{\pm}$ operators select intermediate free electron states of $\pm$ energy. Thus, for example the term $\alpha_\lambda \overset{+}{G} U \overset{+}{G} \alpha_\lambda$ will correspond to a transition as follows: 1). The electron leaves Φ_o and goes to a positive free state, with emission of a photon. 2). It is transferred to another free state by means of U. 3). It returns to Φ_o with absorption of the photon.

3.2 The Divergent Part of W^X.

We shall not at this stage evaluate explicitly the expressions given above for W_o^X, W_1^X, W_2^X. For the calculation will be considerably easier after we have removed from W^X the part which we will consider as corresponding to the electromagnetic mass.

W^X is of course divergent. It is worth while however to examine the coefficient of this divergence. We shall then find that not only is W^X itself divergent but also the difference between the values of W^X evaluated for two different states is in general divergent. Thus since the divergence depends in general upon the state, a new physical idea will

be needed to give finite values for the difference in energy of two levels.

We shall simply sketch this calculation here. An examination of the general calculation to be given later will make clear the procedure which is being used.

We use the definitions of $B^{\pm}$, and $G^{\pm} = 1/2\left[1 \pm \frac{H}{E}\right]$ where $H = c\vec{\alpha}.\vec{p} - c\vec{\alpha}.\vec{\kappa} + \beta mc^2$, and $E = E(\vec{p}-\vec{\kappa})$. We find also

$$\frac{1}{c}\sum_{\lambda}{}' \alpha_\lambda H \alpha_\lambda = -4\beta\mu + 2\vec{\alpha}.\vec{p} - 2\vec{\alpha}.\vec{\kappa}$$

$$\underline{W_o^X}$$

$$\sum{}' \alpha_\lambda \left[\frac{G^+}{B^+} - \frac{G^-}{B^-}\right]\alpha_\lambda = \frac{2E_o}{B^+B^-} - (2\beta\mu + \vec{\alpha}.\vec{p} - \vec{\alpha}.\vec{\kappa})\frac{c}{E}\left[\frac{1}{B^+} + \frac{1}{B^-}\right]$$

$$\simeq \frac{E_o}{2c^2K^2} - \frac{(2\beta\mu + \vec{\alpha}.\vec{p})}{cK^2} + \frac{c\vec{\alpha}.\vec{\kappa}}{E}\left[\frac{1}{B^+} + \frac{1}{B^-}\right]$$

But expanding B^+, B^-, E, we have

$$c\frac{\vec{\alpha}.\vec{\kappa}}{E}\left[\frac{1}{B^+} + \frac{1}{B^-}\right] \sim \frac{\vec{\alpha}.\vec{\kappa}}{cK^2}\left[1 + \frac{3\vec{\kappa}.\vec{p}}{2K^2}\right]$$

Integrating over $d\omega_\kappa$ gives

$$\int d\omega_\kappa \sum_{\lambda}{}' \alpha_\lambda \left[\frac{G^+}{B^+} - \frac{G^-}{B^-}\right]\alpha_\lambda = \frac{2\pi}{c^2K^2}\left[E_o - 4\beta\mu c - c\vec{\alpha}.\vec{p}\right]$$

The divergent part of W_o^X is then

$$-\frac{\alpha}{2\pi}\left[E_o - 4\beta mc^2 - c\vec{\alpha}.\vec{p}\right]_{av}\int^{\infty}\frac{dK}{K}$$

(There is of course no divergence at $K = 0$.)

$\underline{W_1^X}$. We also have divergent terms in W_1^X.

$$\frac{1}{B^+B^-} \sim \frac{1}{4c^2K^2}\left[1 + \frac{\vec{K}\cdot\vec{p}}{K^2}\right]$$

Thus any terms in $\sum_\lambda{}' \alpha_\lambda GUG\alpha_\lambda$ which behave like 1 or $\vec{\kappa}\cdot\vec{p}$ will diverge.

$$\alpha_\lambda GUG\alpha_\lambda = \frac{1}{4}\alpha_\lambda\left[U \pm \frac{H}{E}U \pm U\frac{H}{E} \pm \frac{H}{E}U\frac{H}{E}\right]\alpha_\lambda$$

The terms in $U\frac{H}{E}$ and $\frac{H}{E}U$ have no divergence, the highest power of κ in these being $1/\kappa\,\vec{\alpha}\cdot\vec{\kappa}$.

But $\frac{H}{E}U\frac{H}{E} \sim \frac{1}{c^2K^2}(H_0 - c\vec{\alpha}\cdot\vec{\kappa})\,U\,(H_0 - c\vec{\alpha}\cdot\vec{\kappa})$

where $H_0 = c\vec{\alpha}\cdot p + \beta mc^2$

The divergent term of this is $\frac{1}{K^2}\vec{\alpha}\cdot\vec{K}\,U\,\vec{\alpha}\cdot\vec{K}$

Let us now, for convenience, restrict ourselves to electrostatic fields, U = V. Then

$$\frac{1}{K^2}\vec{\alpha}\cdot\vec{K}\,U\vec{\alpha}\cdot\vec{K} = V$$

For the divergent parts we have then

$$\sum{}'\alpha_\lambda \bar{G}U\bar{G}^+\alpha_\lambda = \sum{}' 1/2\ V = V = \sum{}'\alpha_\lambda\bar{G}U\bar{G}^-\alpha_\lambda$$

$$\sum{}'\alpha_\lambda \bar{G}U\bar{G}^-\alpha_\lambda = \sum{}'\alpha_\lambda\bar{G}U\bar{G}^+\alpha_\lambda = 0$$

The divergent part of W_1^X is then $\frac{\alpha}{2\pi} [V]_{av} \int \frac{dk}{k}$

It is easy to check that W_2^X and all the terms of higher order are convergent. Using the fact that

$$E_o = [c\vec{\alpha}.\vec{p} + \beta mc^2 + V]_{av}$$

we have the result that, for an electron in the state Ψ_o under the influence of an electrostatic field given by $V = e\phi$, the divergent part of W^X is

$$\frac{3\alpha}{2\pi} [\beta mc^2]_{av} \int \frac{dk}{k}$$

The same result of course holds for the free electron. For this case it was given first by Weisskopf[7]

Since the average value of β depends on the state we see that in general the difference in energy between any two levels will be divergent. There is however a special case of some interest.

For the Dirac equation with electrostatic field $V = e\phi$, we may construct a virial theorem,

$$[\beta mc^2 + V + \vec{r}.\nabla V - E_o]_{av} = 0$$

For the Coulomb field, this reduces to $[\beta mc^2]_{av} = E_o$. (This is however not a gauge-invariant way of stating the result).

We see then that for the special case of the degenerate states of hydrogen (e.g. $ns_{\frac{1}{2}}$, $np_{\frac{1}{2}}$) the interaction with the radiation field does produce a non-infinite splitting of the degeneracy, which in fact will be found to be of the

same order as the Lamb and Retherford splitting. This result is however not very useful. In the light of the subtraction procedure to be described, we may say that it corresponds to the fact that the electromagnetic mass of the electron in the two degenerate states differs by a non-infinite amount.

We should add finally that consideration of the non-exchange energy W^N, which will be made later, will not change the results above.

IV. THE ELECTROMAGNETIC MASS

We must now consider the expression for the perturbation energy and decide what part of it can be regarded as corresponding to an electromagnetic mass to be subtracted away. The essential point of this procedure is easily demonstrated; if we have a system of electrons under the influence of an external field given by $\vec{A}, \phi$, the Hamiltonian, without the radiative coupling, is

$$\sum_{\text{elec.}} c\vec{\alpha}.\vec{p} + e\left[-\vec{\alpha}.\vec{A} + \phi\right] + \beta mc^2$$

Suppose now that when we consider the radiative coupling we find, in the perturbation energy, terms of the form

$$\delta_1 \sum_{\text{elec.}} e\left[-\vec{\alpha}.\vec{A} + \phi\right]_{av} \quad \text{and} \quad \delta_2 \sum_{\text{elec.}} \left[\beta mc^2\right]_{av}$$

where δ_1 and δ_2 are of order $e^2/hc = \alpha$. Insofar as these terms are concerned the effect of the radiative coupling can be described, to a first approximation in α by changing the Hamiltonian to

$$\sum c\vec{\alpha}.\vec{p} + e(1+\delta_1)\left[-\vec{\alpha}.\vec{A} + \phi\right] + \beta m(1+\delta_2)c^2$$

But the effect of the additional terms is unobservable for they indicate that the original charge and mass are effectively increased by the radiative coupling by factors $(1+\delta_1)$ and $(1+\delta_2)$ respectively. The experimentally observed

values would be these increased or "renormalized" values. Thus we should remove and discard the parts of the perturbation energy which correspond to this renormalization.

The restriction that $\delta_1, \delta_2 \sim \alpha$ (or smaller) is very important. For, if this is not so, the renormalization terms cannot be accommodated by the above change in the Hamiltonian. It will turn out in most cases that δ_1 , δ_2 are of the form $\alpha \int \frac{dK}{K}$ and thus divergent. Our fundamental belief however is that the divergence is illusory and that a proper modification of the theory at ultra-relativistic energies would give δ_1, δ_2 as truly of order α. This belief is founded on the facts: 1). That omission of the radiative coupling leads e.g. to energy values in close agreement with experiment.

2). The subtraction procedure based on this belief is successful in giving even closer agreement with experiment.

The subtraction procedure is very simple for the charge renormalization terms and we need give no further discussion of it now. It will be discussed when we encounter such terms in the calculation.

The mass renormalization terms however cause more trouble. We first note that the non-exchange energy W^N contains no mass renormalization energy. For the form of W^N shows that it contains only the energy of interaction between separate electrons and thus cannot be derived from an additional

term in the Hamiltonian involving only one-particle operators (without the radiation field).

As a next step we consider the exchange energy W^X in terms of its expansion in powers of U.

$$W^X = W_o^X + W_1^X + W_2^X + \ldots$$

All the mass renormalization energy is contained in W_o^X . For the higher terms in the expansion correspond to (virtual) transitions in which U has an explicit role. (See page 27 where, as an example, the transition corresponding to the term $\alpha_\lambda \bar{G} U G^+ \alpha_\lambda$ is described).

We now remark that, for a free electron, the perturbation (the self-energy) should manifest itself exclusively as a mass-like term. (This we can see by considering a Lorentz transformation which brings the electron to rest). We have already evaluated the divergent part of the self-energy as

$$\frac{3\alpha}{2\pi} \left[\beta mc^2\right]_{av} \int^{\infty} \frac{dK}{K}$$

This then has the correct form. The finite terms however are found not to have this form. This failure we can regard as a consequence of the divergent nature of our theory. We shall later refer briefly to some recent work of Feynman (as yet unpublished) which defines in a Lorentz-invariant fashion the way in which the integrals over K are to be evaluated. For the moment however we shall ignore this failure and proceed

on the basis that in any case the error should be in the same direction and of about the same magnitude for W_o^X and the quantity we shall subtract from it as the mass energy.

Let us now consider in more detail the physical situation represented by one electron in the (non-relativistic) state Φ_o, all negative energy states filled. In considering intermediate states we shall, as usual, restrict ourselves to light quanta of momentum k such that $k \geqslant \delta\mu$. We can now divide all the states into two groups according as their average momentum is less than $\delta\mu$ (group 1) or greater than $\delta\mu$ (group 2). We may now reasonably neglect the momentum components of Φ_o in region (2). It is then quite easy to prove the reasonable result that we may ignore all states in (1) except Φ_o (whether they are occupied or not). This is because our restriction on k effectively eliminates all transitions between Φ_o and these states. Moreover in the zeroth approximation in U the states in group (2) are free particle states and it is this approximation in U which gives W_o^X which as we have seen contains the electromagnetic mass energy.

Thus W_o^X may be considered as arising from the partial system: one electron in Φ_o, all negative states filled ;both the negative states and the intermediate states are free particle states.

This is completely analogous to the system which gives the self-energy of a free electron, the only difference being that the electron is in a state Φ_o rather than a definite

free electron state. The self-energy for a free electron may be written

$$W_{self} = \frac{-\alpha c^2}{4\pi^2}\int \frac{d\vec{k}}{k}\Big[\sum_{\lambda}{}'\Big\{\alpha_\lambda G^{+}\alpha_\lambda \frac{1}{E(p-k)-H(p)+ck} - \alpha_\lambda G^{-}\alpha_\lambda \frac{1}{E(p-k)+H(p)+ck}\Big\}\Big]_{av}$$

where $H(p) = c\vec{\alpha}.\vec{p} + \beta mc^2$ and this is to be interpreted in the obvious way as $\pm E(p)$.

Following the analogy we therefore define

$$W^{M} = \frac{-\alpha c^2}{4\pi^2}\int \frac{d\vec{k}}{k}\sum_{\lambda}{}'\langle\Psi_o^{*}\Big[\alpha_\lambda G^{+}\alpha_\lambda \frac{1}{E(p-k)-H(p)+ck} - \alpha_\lambda G^{-}\alpha_\lambda \frac{1}{E(p-k)+H(p)+ck}\Big]\Psi_o\rangle$$

and this we shall regard as the electromagnetic mass term to be subtracted from W^{X}, the residue (except for charge renormalization terms) to be regarded as the level shift due to the exchange part of the perturbation energy.

In reaching the expression for W^{M} we have used the symetrical form of the perturbation energy as given in II-4. It was there pointed out that the symetrical (Lorentz gauge) and the non-symetrical ($\nabla.\vec{A} = 0$) formulations give the same results for W^{X} and W^{N}. But this does not hold for W^{M} and consequently we have a certain arbitrariness. The reason for the disagreement is that W^{M} involves not only diagonal matrix elements of the self-energy operator (for which the equivalence does hold) but also non-diagonal elements (for which it does not). These non-diagonal elements correspond to transitions

in which the electron leaves the component $\phi^r(\vec{p})$ of Ψ_o with emission of a quantum but returns on reabsorption to a component $\phi^s(\vec{p})$ where r, s are of opposite energies. (The energy for transitions with r, s of the same energy but opposite spin vanishes.)

The amount of arbitrariness can in fact be evaluated. With some manipulation, we can show that for non-relativistic Ψ_o

$$W^M(\text{Lorentz gauge}) - W^M(\nabla \cdot \vec{A} = 0 \text{ gauge}) = \frac{\alpha}{3\pi m}\langle \Psi_o^* [\beta p^2 - \mu \vec{\sigma} \cdot \vec{p}] \Psi_o \rangle$$

For the electrostatic case U = V this equals (in the notation of V)

$$- \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\frac{1}{4}\nabla^2 V - \frac{1}{2} SV\right]_{av}$$

Then the $\nabla \cdot \vec{A} = 0$ gauge formulation will lead to a surplus magnetic moment $\alpha/6\pi$ evaluated by considering the electrostatic case and to a different value evaluated by considering the magnetic case. The results from this formulation are therefore not Lorentz-invariant.

Thus, as we might expect, the Lorentz gauge formulation, which avoids the non-Lorentz-invariant division of the total field into transverse and longitudinal parts, gives more reasonable results.

Finally we should mention that the work of Feynman referred to above indicates that for a correct result the Lorentz gauge formulation must be used. The essential point of

Feynman's procedure is that the electromagnetic interaction is modified by the introduction of a parameter and the self-energy with the modified interaction is finite. After calcul- of the self-energy the parameter may be allowed to vanish thus returning us to the usual theory but without the diffic- ulty of non-Lorentz-invariance referred to above. It is not certain at present what modification this Lorentz-invariant "cut-off" procedure may make in our present theory.

V. THE CALCULATION OF ΔW^X FOR THE ELECTROSTATIC CASE

5.1 We define as the Level Shift ΔW^X due to the exchange energy

$$\Delta W^X = W^X - W^M = W_0^X + W_1^X + W_2^X .. - W^M$$

We now proceed to calculate ΔW^X for the case $U = V = e\phi$. As emphasized before, we shall concern ourselves with non-relativistic states Ψ_0 whose energy, exclusive of the rest energy, is small compared with mc^2.

In anticipation it may be said that the order of the level shift will be

$$\frac{\alpha}{\mu^2}\left[p^2V\right]_{av} \sim \frac{\alpha}{mc^2}\left[V^2\right]_{av}$$

Thus it will be seen that in the expansion of W^X we should consider only terms up to W_2^X. On the other hand the only terms not smaller than the order of the level shift which can come from W_2^X will be terms in $\left[V^2\right]_{av}$. This is however not a gauge-invariant quantity and thus we should expect these terms to be cancelled exactly. We shall indeed see that this will happen.

It is convenient to combine the four terms, three of which are divergent as they stand, into other terms all of which are convergent.

We combine W_0^X and W^M, and noting that $(E_0 - H(p))\Psi_0 = V\Psi_0$ we get

$$W_0^X - W^M = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{K}}{K}\sum_\lambda{}'\left\langle\Psi_0^*\left[\alpha_\lambda\underline{G}^+\alpha_\lambda\frac{1}{E(\vec{p}-\vec{K})-H(p)+cK} + \alpha_\lambda\underline{G}^-\alpha_\lambda\frac{1}{E(\vec{p}-\vec{K})+H(p)+cK}\right]V\Psi_0\right\rangle$$

To eliminate the operator H(p) in the denominators we introduce the operators $H^{\pm}$ which decompose Ψ_o into its positive and negative energy parts.

$$H^{\pm} = \frac{1}{2}\left[1 \pm \frac{c\vec{\alpha}\cdot\vec{p} + \beta mc^2}{E(p)}\right] = \frac{1}{2}\left[1 \pm \frac{H(p)}{E(p)}\right]$$

Let $D^J = E(\vec{p}-\vec{k}) - E(p)\delta_J + ck$. Then $\frac{1}{D^J} = \frac{1}{B^J} + \frac{(E(p)-E_o)\delta_J}{B^J D^J}$.

Now $W_o^X - W^M =$

$$-\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\lambda}{}' \langle \Psi_o^* \left[\alpha_\lambda \frac{G^+\alpha_\lambda H^+}{B_+^2} + \alpha_\lambda \frac{G^-\alpha_\lambda H^+}{B_-^2} + \frac{H^-}{B^+B^-}\right] V\Psi_o\rangle$$

$$-\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\lambda}{}' \langle \Psi_o^* \left[\alpha_\lambda \frac{G^+\alpha_\lambda H^+}{B_+^2 D^+} - \alpha_\lambda \frac{G^-\alpha_\lambda H^+}{B_-^2 D^-} - \alpha^\lambda \frac{G^+\alpha_\lambda H^-}{B^+B^-D^-} + \alpha_\lambda \frac{G^-\alpha_\lambda H^-}{B^-B^+D^+}\right](E\psi - E_o)V\Psi_o\rangle$$

$= W_1' + W_2'$ say. Since $(E(p) - E_o))V \sim V^2$ we note that W_2' is of the same order as W_2^X . We shall combine W_1' with W_1^X and W_2' with W_2^X .

Since $H^+ = 1 - H^-$,

$$W_1' = \frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\lambda}{}' \langle \Psi_o^* \left[\frac{\alpha_\lambda G^+\alpha_\lambda}{B_+^2} + \frac{\alpha_\lambda G^-\alpha_\lambda}{B_-^2} - \frac{1}{B^+B^-}\right] H^- V\Psi_o\rangle \quad \text{---(1)}$$

$$-\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_{\lambda}{}' \langle \Psi_o^* \left[\alpha_\lambda \frac{G^+\alpha_\lambda}{B_+^2} + \alpha_\lambda \frac{G^-\alpha_\lambda}{B_-^2}\right] V\Psi_o\rangle$$

For the last term we use the simple result

$$\frac{1}{B^J} V = V \frac{1}{B^J} + \frac{1}{B^J} \left[V E(\vec{p}-\vec{k}) - E(\vec{p}-\vec{k}) V \right] \frac{1}{B^J}$$

Then W_1' = (1)

$$+\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_\lambda{}'\langle\Psi_o^*\alpha_\lambda\left[\frac{G^+}{B^+}\left(E(\vec{p}-\vec{k})V-V\,E(\vec{p}-\vec{k})\right)\frac{1}{B^+}+\frac{G^-}{B^-}\left(E(\vec{p}-\vec{k})V-V\,E(\vec{p}-\vec{k})\right)\frac{1}{B^-}\right]\alpha_\lambda\Psi_o\rangle \quad (2)$$

$$-\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_\lambda{}'\langle\Psi_o^*\alpha_\lambda\left[\frac{G^+}{B^+}V\frac{1}{B^+}+\frac{G^-}{B^-}V\frac{1}{B^-}\right]\alpha_\lambda\Psi_o\rangle$$

We finally combine the last term above with W_1^X and we then have

$$\Delta W^X = (1) + (2) + (3) + (4) + W_2^X + W_2'$$

where

$$(3) = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_\lambda{}'\langle\Psi_o^*\alpha_\lambda L_o\left[\frac{G^+}{B^+}V\frac{G^-}{B^-}+\frac{G^-}{B^-}V\frac{G^+}{B^+}\right]\alpha_\lambda\Psi_o\rangle$$

$$(4) = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{k}\sum_\lambda{}'\langle\Psi_o^*\alpha_\lambda\left[\frac{G^+}{B^+}VG^-+\frac{G^-}{B^-}VG^+\right]\left[\frac{1}{B^+}+\frac{1}{B^-}\right]\alpha_\lambda\Psi_o\rangle$$

All these terms are separately convergent. We shall now show that the second order terms give no contribution. After this the other four terms will be calculated separately.

In W_2', since $(E(p)-E_o)V \sim p^2V$, any operators multiplying this which are smaller than ~ 1 (i.e. of order p/μ or p^2/μ^2 may be placed equal to zero. Thus we may neglect

terms arising from the expansions of the denominators. For the same reason we may take $H^{\pm} \simeq G^{\pm}$. Also $\alpha_\lambda G^{\pm} \sim G^{\mp}\alpha_\lambda$ and $G^{+}G^{-} = 0$. Finally $\langle \Psi_o^* G^{-} V^2 \Psi_o \rangle \sim [p^2 V^2]_{av} \sim 0$.

Using these results we have

$$\alpha_\lambda G^{+}\alpha_\lambda G^{+} \sim \alpha_\lambda G^{+} G^{-}\alpha_\lambda = 0$$
$$\alpha_\lambda G^{+}\alpha_\lambda G^{-} \sim G^{-} G^{-} \sim 0$$
$$\alpha_\lambda G^{-}\alpha_\lambda G^{-} \sim \alpha_\lambda G^{-} G^{+}\alpha_\lambda \sim 0$$
$$\alpha_\lambda G^{-}\alpha_\lambda G^{+} \sim G^{+} G^{+} \sim 1$$

Thus only the second term of W_2' need be considered. In this term we may take $D^{-} \sim B^{-}$. We get then, using the fact that $[(E(p)-E_o)V]_{av} = -[V^2]_{av}$,

$$W_2' \simeq -\frac{\alpha c^2}{2\pi^2}\int \frac{d\vec{k}}{k B^{-3}} [V^2]_{av}$$

Now consider W_2^X. The terms involving G^{+} , G^{-} mixed vanish. For

$\alpha_\lambda G^{J} V G^{J'} V G^{J''}\alpha_\lambda \sim \alpha_\lambda V G^{J} G^{J'} G^{J''} V \alpha_\lambda = 0$ if the J's are mixed.

Also $\alpha_\lambda G^{+} V G^{+} V G^{+}\alpha_\lambda \sim V \alpha_\lambda G^{+}\alpha_\lambda V \sim V G^{-} V \sim p^2 V^2 \sim 0$

$\alpha_\lambda G^{-} V G^{-} V G^{-}\alpha_\lambda \sim V G^{+} V \sim V^2$

Then $$W_2^X \simeq \frac{\alpha c^2}{2\pi^2}\int \frac{d\vec{k}}{k B^{-3}} [V^2]_{av}$$

We see that the two second order terms cancel. Thus

ΔW^X = (1) + (2) + (3) + (4), where these terms are given above.

We now proceed to calculate each of the terms. The procedure will be to expand the quantities B, E in powers of $\vec{k}.\vec{p}$ keeping as many terms as will contribute to the effect. We then integrate over directions of $\vec{k}$ and finally over the magnitude of k. To simplify notation we shall often omit writing V; in this case we identify B, p etc. as B, p etc. if they stand to the left of V and as B', p' etc. if they stand to the right of V.

Since we are concerned only with non-relativistic states Ψ_o it will be satisfactory to use at various places the Pauli approximation to the Dirac equation in order to reduce various operators which will occur to their non-relativistic equivalent. The Pauli approximation states that if we write the four-component Dirac wave function Ψ in terms of two-component functions χ , ϕ (where ϕ includes the small components)

$$\Psi = \begin{pmatrix} \chi \\ \phi \end{pmatrix}$$

then for positive energy states we have to a first approximation

$$\phi = \frac{\vec{\sigma}\cdot\vec{p}}{2\mu}\chi \qquad\qquad \phi^* = \chi^* \frac{\vec{\sigma}\cdot\vec{p}}{2\mu}$$

We shall not need the second approximation. Using this to reduce relativistic operators, we shall find that all the

operators occurring will eventually reduce to a combination of the two operators

$$\nabla^2 V \quad , \qquad S\,V \;=\; 1/\hbar\ \nabla V . \vec{\sigma} \times \vec{p}$$

The following identities, which are readily derived by using the commutation rules for the Dirac operators, will be required:

$$1/c \sum_{\lambda}{}' \alpha_\lambda H \alpha_\lambda \;=\; -4\beta\mu \qquad -2\vec{\alpha}.\vec{p} + 2\vec{\alpha}.\vec{k}$$

$$1/c^2 \sum_{\lambda}{}' \alpha_\lambda H\, H' \alpha_\lambda \;=\; 2Q^2 \;-\; 2p^2 + 4\vec{p}.\vec{p}' \;-2\,\vec{\alpha}.\vec{p}\,\vec{\alpha}.\vec{p}' + 2\,\vec{\alpha}.\vec{p}\,\vec{\alpha}.k$$

$$+\ 2\vec{\alpha}.\vec{k}\,\vec{\alpha}.\vec{p}' \;-4\vec{k}.\vec{p} \;-\; 4\vec{k}.\vec{p}'.$$

Here $Q = [\mu^2 + k^2 + p^2]^{1/2}$ We write $A_{\mp} = (Q \mp \frac{E_0}{c} + k)$. We note that Q, cA are simply E and B without the angular dependence.

We need the following expansions, valid for all k.

$$\frac{c}{E} \;=\; \frac{1}{Q}\left[\,1 + \frac{k.p}{Q^2} \;\cdot\;\cdot\,\right] \qquad \frac{c^2}{EE'} \;=\; \frac{1}{QQ'}\left[\,1 + \frac{k.p}{Q^2} + \frac{k.p'}{Q^2}\cdot\;\cdot\,\right]$$

$$\frac{c}{B} \;=\; \frac{1}{A}\left[\,1 + \frac{k.p}{QA} \;\cdot\;\cdot\,\right] \qquad \frac{c^2}{BB'} \;=\; \frac{1}{AA'}\left[\,1 + \frac{k.p}{QA} + \frac{k.p'}{QA'}\cdot\;\cdot\,\right]$$

$$\frac{E}{c} \;=\; Q\left[\,1 - \frac{k.p}{Q^2} - \frac{(k.p)^2}{2Q^4}\cdot\;\cdot\,\right]$$

$$\frac{EE'}{c^2} = QQ'\left[\,1 - \frac{k.p}{Q^2} - \frac{k.p'}{Q^2} + \frac{k^2p^2}{3Q^4} + \frac{k^2p.p'}{3Q^4}\,\right]$$

$$L_0 \;=\; \frac{2ck}{E+E'} \;=\; \frac{k}{Q}\left[\,1 + \frac{k.p}{2Q^2} + \frac{k.p'}{2Q^2}\,\right] \;+\; \frac{k}{4Q^3}\left[\,p^2 \;-\; p'^2\,\right]$$

Except in terms which are larger than the order of the effect we are computing we may write $Q = Q'$ and we have done this above.

We need the following simple theorems:

$$\left[p^2 V - V p^2\right]_{av} = 0$$

$$\left[\vec{\alpha}.p\, V - V \vec{\alpha}.p\right]_{av} = 0$$

$$\vec{\alpha}.\vec{p}\; V\; \vec{\alpha}.\vec{p} = \vec{p}.\, V\, \vec{p} - \hbar^2 \nabla V \qquad \left(\vec{p}.\, V\, \vec{p} = -\hbar^2 \sum \frac{\partial}{\partial x_i} V \frac{\partial}{\partial x_i}\right.$$

$$\left[\vec{p}.\, V\, \vec{p} - p^2 V\right]_{av} = 1/2\, \hbar^2 \left[\nabla^2 V\right]_{av}$$

Finally we need the following integrals. In each case the range of integration is from $k = \delta\mu$ ($\delta \simeq 1/137$) to $k = \infty$. All the integrals are of the form $\int \frac{K^k dK}{Q^m A_+^r A_-^s}$ which we shall write $(k\ m\ r\ s)$. By $\sum \frac{1}{A^2 A'}$ we shall mean $\frac{1}{A_+^3} + \frac{1}{A_-^3} + \frac{1}{A_+^2 A_-} + \frac{1}{A_-^2 A_+}$. Similarly in other cases.

$(1\ 1\ 0\ 2) - (1\ 1\ 2\ 0) = 1/\mu\left[\ln 2 + \ln\delta\right]$. $\sum (1\ 2\ 1\ 1) = -1/\mu^2 \ln\delta$.

$(2\ 3\ 1\ 1) = 1/\mu^2\left[1/2 - 1/2 \ln 2\right]$. $(3230) - (3203) = 1/\mu\left[1/2 \ln 2 + 1/4\right]$

$(3240) + (3204) = 1/\mu^2\left[-3/2 \ln 2 - \ln\delta - 1/4\right]$.

$(3302) - (3320) = 1/\mu\left[\ln 2 - 1\right]$. $(3340) - (3304) = 1/\mu^3\left[-2\ln 2 - \ln\delta\right]$

$\sum (3321) = 1/\mu^2\left[\ln 2\right]$. $\sum (3411) = 1/\mu^2\left[1/2\right]$

$(3430)-(3403) = 1/\mu^3\left[\ln 2 - 1/4\right]$. $(3412)-(3421) = 1/\mu^3\left[\ln 2 - 3/4\right]$

$(4421)+(4412) = 1/\mu^2\left[1/2 \ln 2 - 1/4\right]$. $(4511) = 1/\mu^2\left[5/12 - 1/2 \ln 2\right]$

$(4512)-(4521) = 1/\mu^3\left[2/3 - \ln 2\right]$.

Calculation of (4)

$$(4) = \frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{K}}{K}\left[\sum_\lambda{}'\alpha_\lambda(HH'-EE')\alpha_\lambda\frac{1}{EE'}\left\{\frac{1}{B_+B'_+}+\frac{1}{B_-B'_-}+\frac{1}{B_+B'_-}+\frac{1}{B_-B'_+}\right\}\right]_{AV}$$

$$+\frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{K}}{K}\left[\sum_\lambda{}'\alpha_\lambda(HE'-EH')\alpha_\lambda\frac{1}{EE'}\left\{\frac{1}{B_-B'_+}+\frac{1}{B_-B'_-}-\frac{1}{B_+B'_+}-\frac{1}{B_+B'_-}\right\}\right]_{AV}$$

$$\frac{1}{c^2}\sum_\lambda{}'\alpha_\lambda(HH'-EE')\alpha_\lambda = 2\left\{Q^2 - QQ' - p^2 + 2p.p' - \alpha.p\,\alpha.p' - k.p\right.$$

$$\left. -k.p' - 1/3\,\frac{k^2}{Q^2}(p.p'-p^2) + \alpha.p\,\alpha.k + \alpha.k\,\alpha.p'\right\}$$

$$= \hbar^2\nabla^2V\left\{1-\frac{k^2}{3Q^2}\right\}+2\hbar^2 SV + 2\,\alpha.p\,\alpha.k + 2\,\alpha.k\,\alpha.p' - 2k.p - 2k.p'$$

Then

$$\frac{c^2}{4\pi\hbar^2}\int d\omega_K\sum_\lambda{}'\alpha_\lambda\left(\frac{HH'-EE'}{EE'BB'}\right)\alpha_\lambda$$

$$= \nabla^2V\left\{\frac{1}{\varphi^2AA'} - \frac{1}{3}\frac{K^2}{\varphi^4AA'}\right\} + 2SV\left\{\frac{1}{\varphi^2AA'} - \frac{2}{3}\frac{K^2}{\varphi^4AA'} - \frac{1}{3}\frac{K^2}{\varphi^3A^2A'} - \frac{1}{3}\frac{K^2}{\varphi^3AA'^2}\right\}$$

The first term of (4) therefore gives

$$\frac{\alpha\hbar^2}{4\pi}\left[\nabla^2V\right]_{AV}\left\{\sum\left(\int\frac{K\,dK}{\varphi^2AA'} - \frac{1}{3}\int\frac{K^3\,dK}{\varphi^4AA'}\right)\right\} + \frac{\alpha\hbar^2}{2\pi}\left\{\sum\int\frac{K\,dK}{\varphi^2AA'}\left\{1-\frac{2}{3}\frac{K^2}{\varphi^2}-\frac{2}{3}\frac{K^2}{\varphi A}\right\}\right\}$$

$$= \frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\ \overline{\nabla^2 V}\left\{-\frac{3}{4}\ln\delta\ -\frac{1}{8}\right\} + \overline{SV}\left\{-\frac{3}{2}\ln\delta\ -\frac{1}{2}\ -\ln 2\right\}\right]$$

For the second term of (4) we have

$$\frac{1}{c^2}\sum_\lambda{}'\alpha_\lambda(HE'-EH')\alpha_\lambda = 2e\left[\frac{2\mu}{q^2}(K\cdot p'-K\cdot p)+\frac{\alpha\cdot pK\cdot p'}{q^2}-\frac{\alpha\cdot p'K\cdot p}{q^2}-\frac{\alpha\cdot KK\cdot p'}{q^2}+\frac{\alpha\cdot KK\cdot p}{q^2}\right]$$

$$= \frac{4\mu}{Q}\left[k.p' - k.p\right]$$

The terms in $\alpha.pk.p'$ etc. vanish on integration over $d\omega_K$ while the terms in $\alpha.kk.p'$ etc. lead to $[\alpha.pV - V\alpha.p]_{av} = 0$: then

$$\frac{c^2}{4\pi\hbar^2}\int d\omega_K\sum_\lambda{}'\alpha_\lambda\left(\frac{HE'-EH'}{EE'BB'}\right)\alpha_\lambda = \frac{4\mu K^2}{3q^3AA'}\left(\frac{1}{qA}-\frac{1}{qA'}\right)\frac{1}{\hbar^2}(p\cdot p'-p^2)$$

The second term of (4) gives then

$$\frac{\alpha\hbar^2}{3\pi}[\nabla^2 V]_{Av}\,\mu\left\{\int\frac{K^3dK}{q^4A_-^2A_+}-\int\frac{K^3dK}{q^4A_-A_+^2}\right\}$$

$$= \frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\ \overline{\nabla^2 V}\left\{\ln 2 - \frac{3}{4}\right\}$$

We thus have

$$(4) = \frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\ \overline{\nabla^2 V}\left\{-\frac{3}{4}\ln\delta + \ln 2\ -\frac{7}{8}\right\} + \overline{SV}\left\{-\frac{3}{2}\ln\delta\ -\frac{1}{2}\ -\ln 2\right\}\right]$$

Calculation of (1)

$$(1) = \frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{K}}{K}\left[\left(\frac{1}{B_+^2}+\frac{1}{B_-^2}-\frac{2}{B_+B_-}\right)H^-V\right]_{Av}$$

$$+\frac{\alpha c^2}{8\pi^2}\int\frac{d\vec{K}}{K}\left[\frac{1}{E}\left(\frac{1}{B_+^2}-\frac{1}{B_-^2}\right)\sum_\lambda{}'\alpha_\lambda H\alpha_\lambda H^-V\right]_{Av}$$

We show now that the first term does not contribute since $H^-V \simeq 0$ in N.R. approximation.

$$2H^- = 1-\frac{c\vec{\alpha}\cdot\vec{p}}{E(p)}-\frac{\beta\mu c}{E(p)} \simeq (1-\beta)+\frac{p^2}{2\mu^2}-\frac{\vec{\alpha}\cdot\vec{p}}{\mu}$$

By Pauli approximation $\left[(1-\beta)V\right]_{av} = \frac{1}{2\mu^2}\left[\vec{p}.V\,\vec{p}-\hbar^2 SV\right]_{av}$

$$\left[-\frac{\vec{\alpha}\cdot\vec{p}}{\mu}V\right]_{av} = \frac{1}{2\mu^2}\left[-p^2V-\vec{p}.V\,\vec{p}+\hbar^2 SV\right]_{av}$$

Adding the terms gives $\left[H^-V\right]_{av} \simeq 0$.

For the second term,

$$\frac{c^2}{4\pi}\int d\omega_K\sum_\lambda{}'\alpha_\lambda\frac{H\alpha_\lambda H^-V}{EB^2} = \frac{1}{\varphi A^2}\left[-4\beta\mu-2\vec{\alpha}\cdot\vec{p}+\frac{4}{3}\frac{K^2}{AQ}\vec{\alpha}\cdot\vec{p}+\frac{2}{3}\frac{K^2}{Q^2}\vec{\alpha}\cdot\vec{p}\right]H^-V$$

$$= \frac{1}{\varphi A^2}\left[2+\frac{4}{3}\frac{K^2}{AQ}+\frac{2}{3}\frac{K^2}{Q^2}\right]\vec{\alpha}\cdot\vec{p}\,H^-V$$

The last step comes from the fact that $-\beta\mu H^-V = \vec{\alpha}\cdot\vec{p}\,H^-V$ (N.R)

For $(\vec{\alpha}\cdot\vec{p}+\beta\mu)H^-V = \mu(2H^+-1)H^-V = -\mu H^-V \simeq 0$.

In Pauli approximation we find that

$$\left[\vec{\alpha}\cdot\vec{p}\,H^-V\right]_{av} = \frac{\hbar^2}{4\mu}\overline{\nabla^2 V}-\frac{\hbar^2}{2\mu}\overline{SV}$$

Then

$$(1) = \frac{\alpha}{3\pi}\frac{\hbar^2}{\mu}\left[\frac{1}{4}\overline{\nabla^2 V} - \frac{1}{2}\overline{SV}\right]\left[\int dK \cdot \frac{K}{Q}\left\{\frac{3}{A_+^2} - \frac{3}{A_-^2} + \frac{2K^2}{QA_+^3} - \frac{2K^2}{QA_-^3} + \frac{K^2}{Q^2A_+^2} - \frac{K^2}{Q^2A_-^2}\right\}\right]$$

$$= \frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\overline{\nabla^2 V}\left\{-\frac{3}{4}\ln\delta - \frac{3}{4}\ln 2 + \frac{3}{8}\right\} + \overline{SV}\left\{\frac{3}{2}\ln\delta + \frac{3}{2}\ln 2 - \frac{3}{4}\right\}\right]$$

Calculation of (3)

$$(3) = \frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{K}}{K}\left[\sum_\lambda{}' \alpha_\lambda (HH' - EE')\alpha_\lambda \frac{L_0}{EE'}\left\{\frac{1}{B_+B'_-} + \frac{1}{B_-B'_+}\right\}\right]_{Av}$$

$$+ \frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{K}}{K}\left[\sum_\lambda{}' \alpha_\lambda (HE' - EH')\alpha_\lambda \frac{L_0}{EE'}\left\{\frac{1}{B_-B'_+} - \frac{1}{B_+B'_-}\right\}\right]_{Av}$$

The L_0 term in $(p^2 - p'^2)$ does not contribute. Then except for the terms which arise from the expansion of L_0 we obtain the value of (3) by multiplying the terms of (4) which have mixed denominators (A_+A_- etc.) by k/Q before integrating over k.

Besides these we need the terms which arise from $\frac{k}{2Q^3}(\vec{k}\cdot\vec{p} + \vec{k}\cdot\vec{p}')$ in L_0. The second part of (3) has no such terms.
For these terms in the first part of (3),

$$\frac{1}{c^2}\sum_\lambda{}' \alpha_\lambda (HH' - EE')\alpha_\lambda L_0 \equiv \frac{k}{Q^3}(\vec{k}\cdot\vec{p} + \vec{k}\cdot\vec{p}')(\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{k} + \vec{\alpha}\cdot\vec{k}\,\vec{\alpha}\cdot\vec{p}' - \vec{k}\cdot\vec{p} - \vec{k}\cdot\vec{p}')$$

$$\equiv \frac{2}{3}\frac{k^3}{Q^3}(\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{p}' - \vec{p}\cdot\vec{p}') = -\frac{2}{3}\frac{k^3}{Q^3}\hbar^2\,SV$$

Thus the extra terms which arise from the angular dependence of L_0 are

$$-\frac{\alpha}{3\pi}\hbar^2\overline{SV}\int\frac{K^4dK}{Q^5A_+A_-}$$

Then the first part of (3) gives

$$\frac{\alpha\hbar^2}{4\pi}\overline{\nabla^2V}\left\{\int\frac{dK}{A_+A_-Q^3}\frac{K^2}{}\left(2-\frac{2}{3}\frac{K^2}{Q^2}\right)\right\}+\frac{\alpha\hbar^2}{2\pi}\overline{SV}\left\{\int\frac{dK\cdot K^2}{Q^3A_+A_-}\left(2-\frac{2K^2}{Q^2}-\frac{2K^2}{3QA_-}-\frac{2}{3}\frac{K^2}{QA_+}\right)\right\}$$

$$=\frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\quad\overline{\nabla^2V}\left\{\frac{13}{24}-\frac{1}{2}\ln 2\right\}+\overline{SV}\left\{\frac{1}{2}-\frac{1}{2}\ln 2\right\}\right]$$

The second part of (3) gives

$$\frac{\alpha\hbar^2}{3\pi}[\nabla^2V]_{Av}\mu\left\{\int\frac{K^4dK}{Q^5A_+A_-}\left(\frac{1}{A_-}-\frac{1}{A_+}\right)\right\}$$

$$=\frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\overline{\nabla^2V}\left\{\frac{2}{3}-\ln 2\right\}\right]$$

So (3) $=\frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\overline{\nabla^2V}\left\{\frac{29}{24}-\frac{3}{2}\ln 2\right\}+\overline{SV}\left\{\frac{1}{2}-\frac{1}{2}\ln 2\right\}\right]$

Calculation of (2)

$$(2)=\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{K}}{K}\left[(E-E')\left\{\frac{1}{B_+^2B_+'}+\frac{1}{B_-^2B_-'}\right\}\right]_{Av}$$

$$+\frac{\alpha c^2}{8\pi^2}\int\frac{d\vec{K}}{K}\left[\sum_\lambda{}'\alpha_\lambda H\alpha_\lambda(E-E')\frac{1}{E}\left\{\frac{1}{B_+^2B_+'}-\frac{1}{B_-^2B_-'}\right\}\right]_{Av}$$

For the first part $1/c(E-E')=(Q-Q')+\frac{(k.p'-k.p)}{Q}+\frac{(k.p')^2}{2Q^3}-\frac{(k.p)^2}{2Q^3}$

$= 1/Q\ (\ k.p'-k.p\ )$ since the other terms lead to $[Vp^2 - p^2V]_{av}=0$.

Then $\frac{c^2}{4\pi}\int d\omega_K (E-E')\frac{1}{B^2B'} = \frac{1}{3}\frac{K^2}{Q^2A^2A'}\left\{\frac{2p\cdot p'}{A} + \frac{p'^2}{A'} - \frac{2p^2}{A} - \frac{p\cdot p'}{A'}\right\}$

$= \frac{1}{6}\frac{K^2\hbar^2}{Q^2A^4}\nabla^2V$ since we need only terms with $A = A'$.

The first part of (2) gives then

$$\frac{\alpha}{6\pi}\hbar^2[\nabla^2V]_{Av}\left\{\int\frac{K^3dK}{Q^2}\left(\frac{1}{A_+^4}-\frac{1}{A_-^4}\right)\right\}$$

$$= \frac{\alpha}{3\pi}\left(\frac{\hbar}{mc}\right)^2\left[\overline{\nabla^2V}\left\{-\frac{1}{2}\ln\delta - \frac{3}{4}\ln 2 - \frac{1}{8}\right\}\right]$$

For the second part of (2),

$$1/c\sum_\lambda{}'\alpha_\lambda H\alpha_\lambda = -4\beta\mu - 2\vec{\alpha}\cdot\vec{p} + 2\vec{\alpha}\cdot\vec{k} \equiv -4\mu$$

This step follows because $(E-E')\vec{\alpha}\cdot\vec{p} \simeq 0$, $\vec{\alpha}\cdot\vec{k}(E-E')$ will lead to $[\vec{\alpha}\cdot\vec{p}V - V\vec{\alpha}\cdot p]_{av} = 0$, and $\beta\mu = \mu$ in N.R. approximation. Then, just as above,

$$\frac{c^2}{4\pi}\int d\omega_K\sum_\lambda{}'\alpha_\lambda H\alpha_\lambda\frac{(E-E')}{EB^2B'} = -\frac{2}{3}\hbar^2[\nabla^2V]\mu\left\{\frac{K^2}{Q^3A^4}+\frac{K^2}{Q^4A^3}\right\}$$

The second part of (2) gives then

$$-\frac{\alpha}{3\pi}\hbar^2\mu[\nabla^2V]_{Av}\left\{\int dK\frac{K^3}{Q^3}\left(\frac{1}{A_+^4}-\frac{1}{A_-^4}+\frac{1}{QA_+^3}-\frac{1}{QA_-^3}\right)\right\}$$

$$= \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\overline{\nabla^2 V} \left\{ \ln\delta + \ln 2 + \frac{1}{4} \right\} \right]$$

$$\text{So (2)} = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\overline{\nabla^2 V} \left\{ \frac{1}{2} \ln\delta + \frac{1}{4} \ln 2 + \frac{1}{8} \right\} \right]$$

Finally, adding the four terms, we get for the level shift ΔW^X, due to the exchange terms in the perturbation energy and considering only light quanta of energy greater than $\delta\, mc^2$ ($\delta \simeq 1/137$)

$$\Delta W^X = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\overline{\nabla^2 V} \left\{ \int_{\delta_\mu}^{\mu} \frac{dk}{k} - \ln 2 + \frac{5}{6} \right\} + \overline{SV} \left\{ -\frac{3}{4} \right\} \right]$$

This value for the level shift includes of course the shift due to the electrodynamic exchange terms ($\Delta W^{DX} \equiv \lambda = 1,2$) and the shift due to the electrostatic exchange terms ($\Delta W^{SX} \equiv \lambda = 3,4$). We record below the results of the separate calculation of ΔW^{DX} and ΔW^{SX} for $k \geqslant \delta_\mu$.

$$\Delta W^{DX} = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\overline{\nabla^2 V} \left\{ \int_{\delta_\mu}^{\mu} \frac{dk}{k} - \ln 2 + \frac{1}{2} \right\} + \overline{SV} \left\{ \frac{1}{4} \right\} \right]$$

$$\Delta W^{SX} = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\nabla^2 V \left\{ -\frac{1}{6} \right\} + \overline{SV} \left\{ -1 \right\} \right]$$

Before discussing the results we shall first evaluate the contribution to the level shift from light quanta of energy less than $\delta\, mc^2$ and thus remove the restriction on the value of k.

This part of the calculation has already been done by Bethe[6].

As suggested by Bethe, we can ignore retardation for these frequencies. We can also omit, as would be natural in a non-relativistic theory, any consideration of negative energy states. In light of the calculation above this can be immediately verified. For the only terms which can contribute to the lower limit are those with denominators of the type B^+ which behave like k for small k. These terms occur only for positive energy states. Moreover the results above show that only the electrodynamic part of the perturbation energy will contribute for small k. Thus we take only $\lambda = 1,2$. Since $c\vec{\alpha} = \vec{v}$ we make the replacement $\alpha_\lambda = \frac{p_\lambda}{mc}$.

Then from page (20)

$$W^{DX} \simeq \frac{\alpha}{4\pi^2 m^2} \int \frac{d\vec{K}}{K} \sum_{\substack{J \\ \lambda=1,2}} \frac{\langle \Psi_o^* p_\lambda \Psi_J \rangle \langle \Psi_J^* p_\lambda \Psi_o \rangle}{(E_o - E_J - cK)}$$

$$= \frac{2\alpha}{3\pi m^2} \int K dK \sum_J \frac{\langle \Psi_o^* \vec{p} \Psi_J \rangle \langle \Psi_J^* \vec{p} \Psi_o \rangle}{(E_o - E_J - cK)}$$ where this is to be understood as a scalar product.

$$= -\frac{2\alpha}{3\pi m^2 c} \int dK \langle \Psi_o^* p^2 \Psi_o \rangle + \frac{2\alpha}{3\pi m^2 c} \int dK \frac{\langle \Psi_o^* \vec{p} \Psi_J \rangle \langle \Psi_J^* \vec{p} \Psi_o \rangle (E_o - E_J)}{(E_o - E_J - cK)}$$

The first term is easily seen equal to W^M in this approximation. Thus when we subtract W^M this term is cancelled. The second

term gives, exactly as in Bethe's paper

$$\frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \quad \overline{\nabla^2 V} \int_{K_o}^{K_1} \frac{dk}{k}$$

where k_1 is the arbitrary upper limit of integration and k_o, the lower limit, is established as very nearly $[E_J - E_o]_{av}$ for the system. We note that if we take $k_1 = \delta\mu$ then this result joins on correctly to ours as given above.

We shall call ck_o the Bethe lower limit. Its value for the 2s state of hydrogen has been given by Bethe as 17.8 Rydbergs (1 Ry = 1/2 $\alpha^2 mc^2$). It seems probable that a more accurate calculation may change this value slightly. With this value for ck_o we have $\int_{K_o}^{\mu} \frac{dk}{k} = 7.63$ and we shall use this in giving numerical results for the case of hydrogen.

5.2 Discussion of Results.

Before examining the results we shall add to ΔW^X the value for the level shift ΔW^N due to the non-exchange part of the perturbation energy. The calculation of ΔW^N is given in VII.

$$\Delta W = \Delta W^X + \Delta W^N$$

$$W^N = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\overline{\nabla^2 V} \left\{-\frac{1}{5}\right\}\right]$$

Then

$$\Delta W = \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \left[\overline{\nabla^2 V}\left\{\int_{K_o}^{\mu}\frac{dk}{k} - \ln 2 + \frac{5}{6}\ \frac{-1}{5}\right\} + \frac{1}{\hbar}\left[\nabla V . \vec{\sigma} \times \vec{p}\right]_{av}\left\{-\frac{3}{4}\right\}\right]$$

$$= \frac{\alpha}{3\pi}\left[\frac{\hbar}{mc}\right]^2 \quad \left[\nabla^2 V\right]_{av}\left\{\int_{K_o}^{\mu}\frac{dk}{k} - .060\right\} + \frac{\alpha}{\pi}\frac{e}{2m^2c^2}\left[\vec{E}.\vec{S}\times\vec{p}\right]_{av}$$

We may note that $\left\{\int_{K_o}^{\mu}\frac{dk}{k} - .060\right\}$ may be written $\int_{K_o}^{.94\mu}\frac{dk}{k}$

The first term, which we may call the scalar term is similar to the result of Bethe[6] who used a completely non-relativistic theory but arbitrarily excluded light quanta of energy greater than mc^2 in order to obtain a finite result. The equivalent upper limit, according to our theory, is $.94\ mc^2$.

The second term has the characteristic form of a spin-orbit coupling energy and suggests that the electron may be regarded as having a surplus spin magnetic moment due to radiative coupling.

If we regard the electron as possessing a surplus spin magnetic moment $\delta\frac{e\hbar}{2mc}$ (where e is as usual the algebraic charge) we must add to the non-relativistic Hamiltonian a term

$$\delta\frac{e\hbar}{2mc}\vec{\sigma}.\left\{\vec{H} + \frac{1}{c}\vec{E}\times\vec{v}\right\} = \delta\left\{\frac{e}{mc}\vec{S}.\vec{H} + \frac{e}{m^2c^2}\vec{S}.\vec{E}\times\vec{p}\right\}$$

In the present case $\vec{H} = 0$. From the spin-orbit term we identify δ as $\frac{\alpha}{2\pi}$. Thus one effect of the radiative coupling is to increase the effective spin magnetic moment of the electron to

$$\left\{1 + \frac{\alpha}{2\pi}\right\} \quad \text{Bohr Magnetons}$$

This corresponds to a new g factor: $g = 2\left\{1 + \frac{\alpha}{2\pi}\right\}$

In section VI weshall calculate the energy of an electron in a magnetic field and thus obtain an independent identification of δ . The two methods give the same result indicating that in this respect our theory is well behaved. Finally it should be said that the value of the additional magnetic moment was first derived, in a different fashion, by Schwinger.[8]

5.3 Application to Hydrogenic Atoms.

For $L = 0$ (s states) (n = principal quantum no.)

$$\left[\frac{\hbar}{mc}\right]^2 [\nabla^2 V]_{av} = 4\pi\left[\frac{\hbar}{mc}\right]^2 Ze^2 |\Psi_o(0)|^2 = 8\alpha^2 \frac{Z^4}{n^3} Ry$$

$$\frac{e}{2m^2c^2}\left[\vec{E}\cdot\vec{S}\times\vec{p}\right]_{av} = 0$$

For $L \neq 0$

$$\left[\frac{\hbar}{mc}\right]^2 [\nabla^2 V]_{av} = 0$$

$$\frac{e}{2m^2c^2}[\vec{E}\cdot\vec{S}\times\vec{p}]_{av} = \alpha^2 \frac{Z^4}{n^3} Ry \times \begin{cases} \dfrac{1}{(L+1)(2L+1)} & J = L+1/2 \\ \dfrac{-1}{L(2L+1)} & J = L-1/2 \end{cases}$$

We have then

$$\Delta W\ (L=0) = \frac{8}{3\pi}\alpha^3 \frac{Z^4}{n^3} Ry \left\{ \int_{K_o}^{\mu} \frac{dk}{k} - .060 \right\}$$

$$W\ (L\neq 0) = \frac{1}{\pi}\alpha^3 \frac{Z^4}{n^3} Ry \times \begin{cases} \dfrac{1}{(L+1)(2L+1)} & J = L+1/2 \\ \dfrac{-1}{L(2L+1)} & J = L-1/2 \end{cases}$$

We may write these results in terms of the corresponding frequencies ν , where $h\nu = \Delta W$. For $\Delta W = 1/\pi \alpha^3$ Ry we have $\nu = 407.1$ megacycles/sec.

Then for hydrogen, $\nu(2s_{1/2}) = 1030$ mgcs/sec.

$\nu(2p_{1/2}) = -17$ mgcs/sec.

$\nu(2p_{3/2}) = 8$ mgcs/sec.

Thus the $2s_{1/2}$ state, which by the usual Dirac theory is exactly degenerate with the $2p_{1/2}$ state, will on the present theory be higher by an energy corresponding to about 1045 megacycles/sec. This is in agreement with the measurement of Lamb and Retherford[1] who report a value of about 1000 mgcs/sec. subject however to a possible error of perhaps 100 mgcs/sec. Future experimental work will probably provide results which will make possible a closer check of the theory.

VI THE CALCULATION OF ΔW^X FOR THE MAGNETIC CASE

We repeat the calculation for the case of an electron in a stationary magnetic field. Our principal concern will be to verify the magnetic moment result derived from consideration of the electrostatic case and thus check the consistency of our theory.

We have for the applied field $U = \vec{\alpha}.\vec{A}$ where, for convenience, we have taken $\vec{A}$ = -e × Vector potential. The order of magnitude of the additional energy due to radiative coupling will be

$$\Delta W^X \simeq \alpha [\vec{\mu}.\vec{H}]_{av} \simeq e/\hbar c [\vec{\alpha}.\vec{A}]_{av}$$

where $\vec{\mu}$ is the magnetic moment operator. (It is trusted that no confusion will arise between $\alpha = e^2/\hbar c$ and $\vec{\alpha}$ the Dirac velocity operator). We may thus neglect terms in A^2 and p^2A.

Expanding the intermediate states in powers of U it will be sufficient to consider the terms of order zero and one. Since α_λ does not now commute with U, the manipulations used for the electrostatic case in order to write ΔW^X as the sum of individually convergent terms will not be convenient. This however will cause no difficulty.

In the same notation as before we have $W_1' = W_0^X - W^M$.

$$W_1' = -\frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{K}}{K}\sum_\lambda{}' \left\langle \Psi_0^* \left[\alpha_\lambda \frac{G^+\alpha_\lambda}{B_+^2} H^+ + \alpha_\lambda \frac{G^-\alpha_\lambda}{B_-^2} H^+ + \frac{H^-}{B_+B_-} \right] \vec{\alpha}.\vec{A}\, \Psi_0 \right\rangle$$

$$W_1^X = \frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{\kappa}\sum_\lambda{}' \langle \Psi_0^* \alpha_\lambda \left[\frac{G^+}{B^+}\alpha\cdot A\frac{G^+}{B^+} + \frac{G^-}{B^-}\alpha\cdot A\frac{G^-}{B^-} - \frac{G^+}{B^+}\alpha\cdot A\frac{G^-}{B^-} - \frac{G^-}{B^-}\alpha\cdot A\frac{G^+}{B^+}\right]\alpha_\lambda \Psi_0\rangle$$

$$- \frac{\alpha c^2}{4\pi^2}\int\frac{d\vec{k}}{\kappa}\sum_\lambda{}' \langle \Psi_0^* \alpha_\lambda L_0 \left[\frac{G^+}{B^+}\alpha\cdot A\frac{G^-}{B^-} + \frac{G^-}{B^-}\alpha\cdot A\frac{G^+}{B^+}\right]\alpha_\lambda \Psi_0\rangle$$

$$\Delta W^X = W_1' + W_1^X$$

We now calculate each of these terms. We shall not use the Pauli approximation in the course of the calculation. We shall often use the identity

$$\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{A} + \vec{\alpha}\cdot\vec{A}\,\vec{\alpha}\cdot\vec{p} = \vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A} + \hbar\vec{\sigma}\cdot\vec{H}$$

(where $\vec{H} = \nabla\times\vec{A}$). Besides the integrals given previously, we shall need the following (the notation is as before):

$$(1020)+(1002) = \left[1/2\int\frac{dk}{Q} - \ln 2 - \ln\delta - 1/4\right]$$

$$(1020)+(1002)-2(1011) = \left[-\ln 2 - \ln\delta - 1/2\right]$$

$$(2111) = \left[1/4\int\frac{dk}{Q} - 1/8\right], \quad (3221)-(3212) = 1/\mu\left[1/2\ln 2 - 1/4\right]$$

$$(4311) = \left[1/4\int\frac{dk}{Q} + 1/2\ln 2 - 5/8\right], \quad (4321)-(4312) = 1/\mu\left[-\ln 2 + 3/4\right]$$

$$\sum(3211) = \left[\int\frac{dk}{Q} - \ln 2\right].$$

The integral $\int\frac{dk}{Q}$ which appears above is divergent but will eventually cancel out.

Calculation of W_1'.

Using the definitions of $G^\pm$, $H^\pm$ we write

$$W_1' = \frac{\alpha c^2}{4\pi^2}\int \frac{d\vec{k}}{k}\left[\left\{\frac{1}{B_+^2}+\frac{1}{B_-^2}-\frac{2}{B_+B_-}\right\}H^-\vec{\alpha}\cdot\vec{A}\right]_{Av}$$

$$-\frac{\alpha c^2}{4\pi^2}\int \frac{d\vec{k}}{k}\left[\left\{\frac{1}{B_+^2}+\frac{1}{B_-^2}\right\}\vec{\alpha}\cdot\vec{A}\right]_{Av}$$

$$-\frac{\alpha c^2}{8\pi^2}\int \frac{d\vec{k}}{k}\left[\frac{1}{E}\left\{\frac{1}{B_+^2}-\frac{1}{B_-^2}\right\}\sum_\lambda{}' \alpha_\lambda H \alpha_\lambda H^+\vec{\alpha}\cdot\vec{A}\right]_{Av}$$

The first term gives then

$$\frac{\alpha}{\pi}\left[H^-\vec{\alpha}\cdot\vec{A}\right]_{av}\left\{\int k\,dk\left[\frac{1}{A_+^2}+\frac{1}{A_-^2}-\frac{2}{A_+A_-}\right]\right\} = \frac{\alpha}{\pi}\left[H^-\vec{\alpha}\cdot\vec{A}\right]_{av}\left\{-\ln 2-\ln\delta-\frac{1}{2}\right\}$$

The second term gives

$$-\frac{\alpha}{\pi}\left[\vec{\alpha}\cdot\vec{A}\right]_{av}\left\{\int k\,dk\left[\frac{1}{A_+^2}+\frac{1}{A_-^2}\right]\right\} = \frac{\alpha}{\pi}\left[\vec{\alpha}\cdot\vec{A}\right]_{av}\left\{-\frac{1}{2}\int^{\infty}\frac{dk}{Q}+\ln 2+\ln\delta+\frac{1}{4}\right\}$$

For the third term we have $1/c\sum_\lambda{}' \alpha_\lambda H\alpha_\lambda = -4\beta\mu - 2\alpha\cdot p + 2\alpha\cdot k$

$$H^+ \simeq 1/2\left[1+\beta+\frac{\alpha\cdot p}{\mu}\right]$$

Then

$$\frac{1}{c}\sum_\lambda{}' \alpha_\lambda H\alpha_\lambda H^+\vec{\alpha}\cdot\vec{A} \equiv -(1+\beta)(2\mu\alpha\cdot A+\alpha\cdot p\,\alpha\cdot A)+(1-\beta)\alpha\cdot k\,\alpha\cdot A$$

$$= -\left\{4\mu\alpha\cdot A+(\beta-1)\alpha\cdot p\,\alpha\cdot A-4H^-\mu\alpha\cdot A\right\}+(1-\beta)\alpha\cdot k\,\alpha\cdot A$$

The first part of this gives

$$\frac{\alpha}{2\pi}\left[4\alpha\cdot A+(\beta-1)\frac{\alpha\cdot p\,\alpha\cdot A}{\mu}-4H^-\alpha\cdot A\right]_{av}\left\{-\ln 2-\ln\delta\right\}$$

For the second part

$$\frac{c^3}{4\pi}\int d\omega_K\frac{\alpha\cdot K\alpha\cdot A}{EB^2} = \frac{\alpha\cdot p\,\alpha\cdot A}{QA^2}\cdot\frac{1}{3}\cdot K^2\left(\frac{1}{Q^2}+\frac{2}{QA}\right)$$

This then gives

$$\frac{\alpha}{6\pi}\left[(\beta-1)\,\alpha\cdot\frac{p}{\mu}\alpha\cdot A\right]_{Av}\mu\left\{\int\frac{K^3dK}{\varphi^2}\left[\frac{1}{\varphi A_+^2}-\frac{1}{\varphi A_-^2}+\frac{2}{A_+^3}-\frac{2}{A_-^3}\right]\right\}$$

$$=\frac{\alpha}{4\pi}\left[(\beta-1)\frac{\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{A}}{\mu}\right]_{av}$$

Collecting the terms we have

$$W_1'=\frac{\alpha}{2\pi}\left[\vec{\alpha}\cdot\vec{A}\right]_{av}\left\{-\int_0^\infty\frac{dk}{Q}\;-2\ln 2\;-2\ln\delta+\frac{1}{2}\right\}$$

$$+\frac{\alpha}{2\pi}\left[H^-\vec{\alpha}\cdot\vec{A}\right]_{av}\left\{2\ln 2+2\ln\delta\;-1\right\}$$

$$+\frac{\alpha}{2\pi}\left[(\beta-1)\frac{\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{A}}{\mu}\right]_{av}\left\{-\ln 2\;-\ln\delta+\frac{1}{2}\right\}$$

Calculation of W_1^X

We write the first term of W_1^X in the form

$$\frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda R_1\alpha_\lambda\left\{\frac{1}{B^+B'^+}+\frac{1}{B^-B'^-}-\frac{1}{B^+B'^-}-\frac{1}{B^-B'^+}\right\}\right]_{Av}$$

$$+\frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda(R_2+R_3)\alpha_\lambda\left\{\frac{1}{B_+B'_+}-\frac{1}{B_-B'_-}\right\}\right]_{Av}$$

$$+\frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda(R_2-R_3)\alpha_\lambda\left\{\frac{1}{B_-B'_+}-\frac{1}{B_+B'_-}\right\}\right]_{Av}$$

$$+\frac{\alpha c^2}{16\pi^2}\int\frac{d\vec{k}}{k}\left[\sum_\lambda{}'\alpha_\lambda R_4\alpha_\lambda\left\{\frac{1}{B_+B'_+}+\frac{1}{B_-B'_-}+\frac{1}{B^+B'^-}+\frac{1}{B^-B'^+}\right\}\right]_{Av}$$

In this expression,

$$R_1 = \vec{\alpha}\cdot\vec{A}$$

$$R_2 = \frac{H}{E}\vec{\alpha}\cdot\vec{A} \equiv \frac{H_0\vec{\alpha}\cdot\vec{A}}{Q} - \frac{\vec{\alpha}\cdot\vec{k}\,\vec{\alpha}\cdot\vec{A}}{Q} - \frac{k^2}{3Q^3}\vec{\alpha}\cdot p\,\vec{\alpha}\cdot A$$

$$R_3 = \vec{\alpha}\cdot\vec{A}\frac{H}{E} \equiv \frac{\vec{\alpha}\cdot\vec{A}H_0}{Q} - \frac{\vec{\alpha}\cdot\vec{A}\,\vec{\alpha}\cdot\vec{k}}{Q} - \frac{k^2}{3Q^3}\vec{\alpha}\cdot\vec{A}\,\vec{\alpha}\cdot\vec{p}$$

$$R_4 = \frac{H}{E}\vec{\alpha}\cdot\vec{A}\frac{H}{E} \equiv \frac{1}{Q^2}\left\{ H_0\vec{\alpha}\cdot\vec{A}H_0 - \frac{1}{3}k^2\vec{\alpha}\cdot\vec{A} - \frac{2k^2}{3Q^2}\beta\mu[\vec{A}\cdot\vec{p}+\vec{p}\cdot\vec{A}] - 2\beta\mu\vec{A}\cdot\vec{k}\right\}$$

where $H_0 = \vec{\alpha}\cdot\vec{p} + \beta\mu$. The equivalent forms for the R's are found by expanding E and keeping terms which will contribute to ΔW. Then

$$R_2+R_3 \equiv \frac{1}{Q}\left\{ \left(1 - \frac{k^2}{3Q^2}\right)\left(A\cdot p + p\cdot A + \hbar\,\sigma\cdot H\right)\right\} - \frac{2A\cdot k}{Q}$$

$$R_2-R_3 \equiv \frac{1}{Q}\left\{ \alpha\cdot A\,\alpha\cdot k - \alpha\cdot k\,\alpha\cdot A\right\}$$

We perform the integrations over $\vec{k}$ and write each term as

$$\frac{\alpha}{4\pi}\left[\sum_\lambda{}' \alpha_\lambda T \alpha_\lambda\right]_{av}$$

<u>R_1 terms</u>

$$T_1 = \alpha\cdot A\int k\,dk\left[\frac{1}{A_+^2} + \frac{1}{A_-^2} - \frac{2}{A_+A_+}\right] = \alpha\cdot A\left\{ -\ln 2 - \ln\delta - \frac{1}{2}\right\}$$

<u>R_2+R_3 terms</u>

$$\frac{e^2}{4\pi}\int -\frac{2A\cdot K}{QBB'}\,d\omega_K = -\frac{2}{3}\frac{K^2}{Q^2}\left\{\frac{p\cdot A}{A^2A'} + \frac{A\cdot p}{AA'^2}\right\}$$

Thus the terms in A.k give

$$T_2 = -\frac{2}{3}\left[A\cdot p + p\cdot A\right]\int d\kappa \frac{\kappa^3}{Q^2}\left\{\frac{1}{A_+^3} - \frac{1}{A_-^3}\right\}$$

$$= -\frac{2}{3\mu}\left[A\cdot p + p\cdot A\right]\left\{\frac{1}{2}\ln 2 + \frac{1}{4}\right\}$$

The other $(R_2 + R_3)$ terms give

$$T_3 = \left[A\cdot p + p\cdot A + \hbar\sigma\cdot H\right]\int \frac{\kappa}{Q}\left(1 - \frac{1}{3}\frac{\kappa^2}{Q^2}\right)\left(\frac{1}{A_+^2} - \frac{1}{A_-^2}\right)d\kappa$$

$$= \left[A\cdot p + p\cdot A + \hbar\,\sigma\cdot H\right]\frac{1}{\mu}\left\{-\frac{2}{3}\ln 2 \quad -\ln\delta \quad -\frac{1}{3}\right\}$$

<u>$R_2 - R_3$ terms</u>

$$\frac{c^2}{4\pi}\int \frac{d\omega_\kappa}{QBB'}\left(\alpha\cdot A\,\alpha\cdot\kappa - \alpha\cdot\kappa\,\alpha\cdot A\right) = \frac{2\kappa^2}{3Q^2AA'}\left\{-\alpha\cdot\underset{A}{p}\,\alpha\cdot A + \alpha\cdot\underset{A'}{A}\,\alpha\cdot\underset{A}{p} + \underset{A}{p}\cdot\underset{A'}{A} - \underset{A'}{A}\cdot\underset{A}{p}\right\}$$

These terms give $T_4 = \frac{2}{3}\hbar\sigma\cdot H\int \frac{d\kappa\cdot\kappa^3}{Q^2A_+A_-}\left\{\frac{1}{A_+} - \frac{1}{A_-}\right\}$

$$= \frac{2}{3}\hbar\,\frac{\sigma\cdot H}{\mu}\left\{\frac{1}{2}\ln 2 \quad -\frac{1}{4}\right\}$$

<u>R_4 terms</u>

$$\frac{c^2}{4\pi}\int \frac{-2\beta A\cdot\kappa\, d\omega_\kappa}{Q^2BB'} = -\frac{2}{3}\frac{\kappa^2}{Q^3}\left\{\beta\underset{A^2A'}{p\cdot A} + \beta\underset{AA'^2}{A\cdot p}\right\}$$

Then $T_5 = -\frac{2}{3}\left[\beta A\cdot p + \beta p\cdot A\right]\mu\int d\kappa\sum\frac{\kappa^3}{Q^3A^2A'}$

$$= \frac{-2}{3\mu}\left[\beta A\cdot p + \beta\, p\cdot A\right]\left\{\ln 2\right\}$$

For the other R_4 terms we use

$$H_0\,\alpha\cdot A\,H_0 = \beta\mu A\cdot p + \beta\mu p\cdot A + \mu\hbar\beta\sigma\cdot H - \mu^2\alpha\cdot A$$

$$\text{So } T_6=\int\left\{\mu\hbar\beta\sigma\cdot H+\left(\beta\mu A\cdot p+\beta\mu p\cdot A\right)\left(1-\tfrac{2}{3}\tfrac{K^2}{\varphi^2}\right)-\alpha\cdot A\left(\mu^2+\tfrac{1}{3}K^2\right)\right\}\sum\frac{K\,dK}{\varphi^2 A A'}$$

$$=\frac{\hbar}{\mu}\beta\sigma\cdot H\left\{-\ln\delta\right\}+\left[\beta A\cdot p+\beta\, p\cdot A\right]\frac{1}{\mu}\left\{-\ln\delta\ -\tfrac{1}{3}\right\}$$

$$+\alpha\cdot A\left\{\ \ln\delta\ -\tfrac{1}{3}\int^{\infty}\frac{dk}{q}+\frac{1}{3}\ln 2\right\}$$

To sum over λ we use the following easily derived identities:

$$\sum_\lambda{}'\alpha_\lambda\alpha\cdot A\,\alpha_\lambda = -2\,\alpha\cdot A \qquad \sum_\lambda{}'\alpha_\lambda A\cdot p\,\alpha_\lambda = 2A\cdot p$$

$$\sum_\lambda{}'\alpha_\lambda\sigma\cdot H\,\alpha_\lambda = -2\,\sigma\cdot H \qquad \sum_\lambda{}'\alpha_\lambda\beta\, A\cdot p = -4\beta\, A\cdot p$$

$$\sum_\lambda{}'\alpha_\lambda\beta\sigma\cdot H = 0$$

Then, collecting terms, we have for the first part of W_1^X.

$$\frac{\alpha}{2\pi}\left[\alpha\cdot A\right]_{av}\left\{\frac{2}{3}\ln 2+\frac{1}{2}+\frac{1}{3}\int^{\infty}\frac{dk}{Q}\right\}+\frac{\alpha}{2\pi}\left[\frac{\hbar\,\sigma\cdot H}{\mu}\right]_{av}\left\{\frac{1}{3}\ln 2+\ln\delta+\frac{1}{2}\right\}$$

$$+\frac{\alpha}{2\pi}\left[\frac{A\cdot p+p\cdot A}{\mu}\right]_{av}\left\{-\ln 2-\ln\delta\ -\frac{1}{2}\right\}+\frac{\alpha}{2\pi}\left[\frac{\beta A\cdot p+\beta\, p\cdot A}{\mu}\right]_{av}\left\{\frac{4}{3}\ln 2+2\ln\delta+\frac{2}{3}\right\}$$

We now evaluate the second part of W^X. For the terms which do not arise from the angular dependence of L_0 we multiply the A_+A_- terms above by k/Q before integrating. For the

terms arising from the expansion of L_o we need only consider the $-2\beta\, A.k$ term in R_4. The other terms contribute nothing; but

$$\frac{1}{4\pi}\int \frac{-2\beta A\cdot K}{Q^2 A_+ A_-} L_o\, d\omega_K = -\frac{1}{3}\left[\beta A\cdot p + \beta p\cdot A\right]\frac{K^3}{Q^5 A_+ A_-}$$

These give $$T_7 = -\frac{2}{3}\left[\beta A.p + \beta p.A\right]\mu\int \frac{k^4\, dk}{Q^5 A_+ A_-}$$

For the second part of W_1^X we have then

$$T = \alpha\cdot A\int \frac{dK\cdot K^2}{QA_+A_-}\left\{-2-2\frac{\mu^2}{Q^2}-\frac{2}{3}\frac{K^2}{Q^2}\right\} + \frac{2}{3}\hbar\sigma\cdot H\int \frac{dK\, K^4}{Q^3 A_+ A_-}\left\{\frac{1}{A_+}-\frac{1}{A_-}\right\}$$

$$+\left[\beta\mu A\cdot p + \beta\mu p\cdot A\right]\int \frac{dK\cdot K^2}{Q^3 A_+ A_-}\left\{-\frac{2}{3}\frac{K^2}{QA_+}-\frac{2}{3}\frac{K^2}{QA_-}+2-2\frac{K^2}{Q^2}\right\}$$

Integrating and summing over λ gives

$$\frac{\alpha}{2\pi}\left[\vec{\alpha}.\vec{A}\right]_{av}\left\{\frac{2}{3}\int^{\infty}\frac{dk}{Q}-\frac{2}{3}\ln 2+\frac{1}{3}\right\}+\frac{\alpha}{2\pi}\left[\frac{\hbar\vec{\sigma}.\vec{H}}{\mu}\right]_{av}\left\{\frac{2}{3}\ln 2-\frac{1}{2}\right\}$$

$$+\frac{\alpha}{2\pi}\left[\frac{\beta\vec{A}.\vec{p}+\beta\vec{p}.\vec{A}}{\mu}\right]_{av}\left\{\frac{2}{3}\ln 2-\frac{2}{3}\right\}$$

For the energy shift we thus get

$$\Delta W^X = \frac{\alpha}{2\pi}\left[\vec{\alpha}.\vec{A}\right]_{av}\left\{-2\ln 2-2\ln\delta+\frac{4}{3}\right\}+\frac{\alpha}{2\pi}\left[\frac{\hbar\vec{\sigma}.\vec{H}}{\mu}\right]_{av}\left\{\ln 2+\ln\delta\right\}$$

$$+\frac{\alpha}{2\pi}\left[\frac{\vec{A}.\vec{p}+\vec{p}.\vec{A}}{\mu}\right]_{av}\left\{-\ln 2-\ln\delta-\frac{1}{2}\right\}+\frac{\alpha}{2\pi}\left[\frac{\beta\vec{A}.\vec{p}+\beta\vec{p}.\vec{A}}{\mu}\right]_{av}\left\{2\ln 2+2\ln\delta\right\}$$

$$+\frac{\alpha}{2\pi}\left[H^{-}\vec{\alpha}.\vec{A}\right]_{av}\left\{2\ln\delta+2\ln 2-1\right\}+\frac{\alpha}{2\pi}\left[(\beta-1)\frac{\vec{\alpha}.\vec{p}\,\vec{\alpha}.\vec{A}}{\mu}\right]_{av}\left\{-\ln\delta-\ln 2+\frac{1}{2}\right\}$$

We may now apply the Pauli approximation to reduce this to its non-relativistic equivalent. Then

$$\mu[\vec{\alpha}\cdot\vec{A}]_{av} = 1/2\left[\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A} + \hbar\,\vec{\sigma}\cdot\vec{H}\right]_{av}$$

$$[\beta\vec{A}\cdot\vec{p}]_{av} = [\vec{A}\cdot\vec{p}]_{av} \qquad [\beta\,\vec{p}\cdot\vec{A}]_{av} = [\vec{p}\cdot\vec{A}]_{av}$$

$$[H\vec{\alpha}\cdot\vec{A}]_{av} = 0 \qquad [(\beta-1)\vec{\alpha}\cdot\vec{p}\,\vec{\alpha}\cdot\vec{A}]_{av} = 0$$

and $$\Delta W^{X} = \frac{\alpha}{2\pi}\left[\frac{\hbar\,\vec{\sigma}\cdot\vec{H}}{\mu}\right]_{av}\left\{\frac{2}{3}\right\} + \frac{\alpha}{2\pi}\left[\frac{\vec{A}\cdot\vec{p}+\vec{p}\cdot\vec{A}}{\mu}\right]_{av}\left\{\frac{1}{6}\right\}$$

It will be remembered that we have defined $\vec{A}$ = -e × Vector potential (e= algebraic charge) and $\vec{H} = \nabla\times\vec{A}$. Returning now to the usual notation $\vec{A}$ = Vector potential, $\vec{H}$ = Magnetic field, we have

$$\Delta W^{X} = -\frac{\alpha}{2\pi}\,\frac{e\hbar}{2mc}[\vec{\sigma}\cdot\vec{H}]_{av} - \frac{\alpha}{2\pi}\,\frac{e}{mc}\,\frac{1}{6}\left[\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A} + \hbar\,\vec{\sigma}\cdot\vec{H}\right]_{av}$$

We immediately note that this result is not gauge-invariant and we can recognize by this the fact that it contains an energy corresponding to a charge renormalization. If one effect of introducing the radiative perturbation is to change the effective charge to e+δe the energy of the system increases on this account by an amount $-\delta e\,\overline{\alpha\cdot A}$ (which is of course not gauge-invariant). In terms of the reduced equation the additional energy is

$$-\frac{\delta e}{2mc}\left[\vec{A}\cdot\vec{p} + \vec{p}\cdot\vec{A} + \hbar\,\vec{\sigma}\cdot\vec{H}\right]_{av}$$

In the case above we therefore identify $\delta e = \alpha/6\pi\, e$. Subtracting out the energy corresponding to this in accordance with our interpretation outlined in IV we get for the true energy shift

$$\Delta W = \Delta W^{X} = -\frac{\alpha}{2\pi}\frac{e\hbar}{2mc}\left[\vec{\sigma}.\vec{H}\right]_{a_v}$$

We anticipate here the fact that the non-exchange terms will give no contribution in this order.

The energy can be interpreted as due to an additional magnetic moment, the total spin magnetic moment being now

$$\left\{1 + \frac{\alpha}{2\pi}\right\} \text{ Bohr magnetons.}$$

This agrees with the result of the electrostatic calculation. We may note finally that the orbital magnetic moment is not affected.

VII. THE NON-EXCHANGE ENERGY (W^N)

It will be remembered that we have separated the total perturbation energy, which results from consideration of the radiative coupling, into two parts W^X and W^N corresponding respectively to exchange and non-exchange phenomena. We have considered in V and VI the effects due to W^X and we now consider W^N. We have pointed out that W^N contains no mass renormalization terms. It will be found however to contain (divergent) charge renormalization terms and when these are subtracted out the residue will give a finite level shift.

This phenomenon has been considered by various authors, originally for the electrostatic case by Uehling[5]. In particular, Weisskopf [9] has given a treatment using a procedure closely akin to that we have used above for the non-exchange terms. For the sake of completeness we shall sketch the treatment.

$$W^N = \int \left\{ \rho_0(\vec{r})\,\rho_{neg}(\vec{r}') - \frac{1}{c^2}\vec{j}_0(\vec{r})\cdot\vec{j}_{neg}(\vec{r}') \right\} \frac{d\vec{r}\,d\vec{r}'}{|\vec{r}-\vec{r}'|}$$

where ρ_0, $\vec{j}_0$ are the charge and current densities due to the electron in the state Ψ_0 and ρ_{neg}, $\vec{j}_{neg}$ the densities due to the vacuum electrons.

ρ_{neg} and $\vec{j}_{neg}$ will be expanded in powers of U (U = V = $e\phi$ or = $-e\vec{\alpha}\cdot\vec{A}$ where ϕ, $\vec{A}$ = external field). The terms independent of U will represent the densities of the

vacuum electrons unperturbed by the applied field and, following the usual procedure of hole theory, the energy corresponding to these will be discarded as being without physical interest. We shall omit these terms from the start.

7.2 The electrostatic case, U = V.

Expanding Ψ_J as before, we have

$$\rho_{neg} = e\sum_{J^-}\{\Psi_J^*\,\Psi_J\}$$

$$= e\sum_{\substack{J^- \\ \vec{q}}}\{\phi^{*J}(\vec{q})\,\phi^{J}(\vec{q})\} + e\sum_{\substack{J^-J' \\ \vec{q}\,\vec{q}'}}\frac{1}{E^J(\vec{q}) - E^{J'}(\vec{q}')}\left[\{\phi^{*J}(\vec{q})\,\phi^{J'}(\vec{q}')\}V^{J'J}_{\vec{q}'\vec{q}} + \{\phi^{*J'}(\vec{q}')\,\phi^{J}(\vec{q})\}V^{JJ'}_{\vec{q}\vec{q}'}\right]$$

We omit the first term (independent of U). In the third term we interchange J, J' and q, q'. Then

$$\rho_{neg} = -e\Big(\sum_{\substack{J^-J'_+ \\ \vec{q}\,\vec{q}'}} + \sum_{\substack{J^+J'^- \\ \vec{q}\,\vec{q}'}}\Big)\frac{\{\phi^{*J}(\vec{q})\,\phi^{J}(\vec{q}')\}}{[E(\vec{q}) + E(\vec{q}')]}V^{J'J}_{\vec{q}'\vec{q}}$$

$$= -e\Big(\sum_{\substack{J^-J'_+ \\ \vec{q}\,\vec{g}}} + \sum_{\substack{J^+J'^- \\ \vec{q}\,\vec{g}}}\Big)\frac{e^{-i\vec{g}\cdot\vec{r}/\hbar}}{[E\vec{q} + E(\vec{q}-\vec{g})]}V(\vec{g})\langle u^{*J}(\vec{q})\,u^{J'}(\vec{q}-\vec{g})\rangle\langle u^{*J'}(\vec{q}-\vec{g})\,u^{J}(\vec{q})\rangle$$

where we have written $V = \sum_{\vec{g}} V(\vec{g})\,e^{-i\vec{g}\cdot\vec{r}/\hbar}$

The sum over J, J' is readily done (eg. by the method of spurs) and we get

$$\rho_{neg} = -2e \sum_{\vec{q}\,\vec{g}} \frac{V(\vec{g})\, e^{-i/\hbar\, \vec{g}\cdot\vec{r}}}{E(\vec{q}) + E(\vec{q}-\vec{g})} \left\{ 1 - \frac{c^2(q^2 - \vec{q}\cdot\vec{g} + \mu^2)}{E(q)\, E(\vec{q}-\vec{g})} \right\}$$

For our purpose we shall be concerned only with $g \sim p$. For W^N involves the integrals $\int e^{i/\hbar [(\vec{p}'-\vec{p})\cdot\vec{r} - \vec{g}\cdot\vec{r}']} \frac{d\vec{r}\, d\vec{r}'}{|\vec{r}-\vec{r}'|}$

where $\vec{p}$, $\vec{p}'$ are momenta in the spectrum of Ψ_o. The integrals vanish unless $\vec{g} = \vec{p}' - \vec{p} \sim p$. But in summing over $\vec{q}$ there will be no contribution for $q \ll \mu$. We may therefore take $q \gg g$. As a more convenient notation we write $\vec{q} = \vec{k}$. Then

$$\rho_{neg} = \frac{-e}{4\pi^3\hbar^3} \sum_{\vec{g}} V(\vec{g})\, e^{-i/\hbar\, \vec{g}\cdot\vec{r}} \int \frac{d\vec{k}}{E(k) + E(k-g)} \left\{ 1 - \frac{c^2(k^2 - k\cdot g + \mu^2)}{E(k) E(k-g)} \right\}$$

Expanding the integrand in powers of g and integrating over the directions of $\vec{k}$ gives

$$\frac{2\pi k^2}{E^3(k)} \left\{ c^2 g^2 \left[\frac{1}{2} - \frac{1}{6} \frac{c^2 k^2}{E^2(k)} \right] + c^4 g^4 \left[-\frac{1}{2} \frac{1}{E^2(k)} + \frac{25\, c^2 k^2}{24\, E^4(k)} - \frac{21\, c^4 k^4}{40\, E^6(k)} \right] + O(g^6) \right\}$$

For the g^2 terms (which are divergent) we note that

$$\sum_{\vec{g}} V(\vec{g})\, e^{-i/\hbar\, \vec{g}\cdot\vec{r}}\, g^2 = -\hbar^2 \nabla^2 V = 4\pi\hbar^2 e\, \rho(e)$$

where $\rho(e)$ is the external charge density which produces V, (e.g. the proton charge density in the case of the hydrogen atom). These terms then correspond to the field-independent

part of the vacuum polarization. We may regard them as charge renormalization terms (they lead to an energy term of the type $\langle \Psi_o^* V \Psi_o \rangle$).

For the g^4 terms,

$$\sum_{\vec{g}} V(\vec{g}) e^{-i/\hbar\, \vec{g}.\vec{r}}\, g^4 = \hbar^4 \nabla^2 (\nabla^2 V) = -4\pi\hbar^4 e\, \nabla^2 \rho(e)$$

$$c^4 \int \frac{k^2\, dk}{E^5(k)} \left\{ -\frac{1}{2} + \frac{25}{24}\frac{c^2k^2}{E^2(k)} - \frac{21}{40}\frac{c^4k^4}{E^4(k)} \right\} = \frac{1}{\mu^2 c} \left\{ -\frac{1}{6} + \frac{5}{24} - \frac{3}{40} \right\} = \frac{-1}{30\mu^2 c}$$

Then for the physically interesting induced charge we have

$$\rho_{neg} = \frac{e\hbar}{60\pi^2\mu^2 c} \nabla^2 (\nabla^2 V) = -\frac{\alpha}{15\pi} \left(\frac{\hbar}{mc}\right)^2 \nabla^2 \rho(e)$$

This induced charge causes a finite level shift which we shall call ΔW^N.

$$\Delta W^N = \frac{\alpha}{60\pi^2} \left(\frac{\hbar}{mc}\right)^2 \int \left\{ \Psi_o^*(\vec{r})\, \Psi_o(\vec{r}) \right\} \nabla^2 (\nabla^2 V)_{(\vec{r}')} \frac{d\vec{r}\, d\vec{r}'}{|\vec{r}-\vec{r}'|}$$

$$= -\frac{\alpha}{15\pi} \left(\frac{\hbar}{mc}\right)^2 \overline{\nabla^2 V}$$

in agreement with the result of Uehling[5].

For the electrostatic case the induced current density vanishes as would be expected and thus ΔW^N given above is the total level shift due to the non-exchange terms. This can be verified in many ways ; e.g. if we replace $V^{J'J}_{q'q}$ by $(\vec{\alpha}.\vec{A})^{J'J}_{q'q}$ then the expressions vanish when we sum over J, J' as above.

7.3 The Magnetic Case.

When the external field is a stationary magnetic field the induced charge density vanishes and we need only consider the energy arising from the interaction of the current due to the electron in the state Ψ_o and the physically meaningful part of the induced vacuum current.

We shall not outline the calculation here. It proceeds exactly as in the electrostatic case. After subtracting out the current due to the unperturbed vacuum electrons and that corresponding to the (divergent) field-independent magnetic polarizability of the vacuum, we get as we might expect

$$\frac{1}{c}\vec{j}_{neg} = \frac{\alpha}{60\pi^2}\left(\frac{\hbar}{mc}\right)^2 \nabla^2(\nabla^2\vec{A})$$

On interaction with $\vec{j}_o$ this leads to an energy involving $[\nabla^2 \mathbf{A}]_{av}$ which is smaller in order than the energy corresponding to the surplus magnetic moment in which we are interested. For our purposes we may therefore omit the non-exchange energy in the case of an external stationary magnetic field .

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BIOGRAPHY

The author was born in St. John's, Newfoundland on Nov. 13, 1921. After attending St. Bonaventure's College he entered Memorial University College, St. John's in 1938 and Dalhousie University in 1940. He graduated from Dalhousie in 1942. After a short period with the National Research Council of Canada he entered the Royal Canadian Navy from which he was discharged in October, 1945. Since November, 1945 he has attended M.I.T., first as a full-time student and later as Teaching Fellow and Research Associate. Two years ago he married Helen G. Harquail of New Brunswick, Canada.