Continuous-Time Systems

February 11, 2010
Previously: DT Systems

**Verbal descriptions:** preserve the rationale.

“Next year, your account will contain $p$ times your balance from this year plus the money that you added this year.”

**Difference equations:** mathematically compact.

$$y[n + 1] = x[n] + py[n]$$

**Block diagrams:** illustrate signal flow paths.

![Block diagram](image)

**Operator representations:** analyze systems as polynomials.

$$(1 - pR) Y = RX$$
Analyzing CT Systems

**Verbal descriptions:** preserve the rationale.

“Your account will grow in proportion to the current interest rate plus the rate at which you deposit.”

**Differential equations:** mathematically compact.

\[
\frac{dy(t)}{dt} = x(t) + py(t)
\]

**Block diagrams:** illustrate signal flow paths.

**Operator representations:** analyze systems as polynomials.

\[
(1 - pA)Y = AX
\]
Differential Equations

Differential equations are mathematically precise and compact.

\[
\frac{dr_1(t)}{dt} = \frac{r_0(t) - r_1(t)}{\tau}
\]

Solution methodologies:

- general methods (separation of variables; integrating factors)
- homogeneous and particular solutions
- inspection

Today: new methods based on **block diagrams** and **operators**, which provide new ways to think about systems’ behaviors.
Block Diagrams

Block diagrams illustrate signal flow paths.

**DT**: adders, scalers, and delays – represent systems described by linear difference equations with constant coefficients.

\[ x[n] \overset{+}{\rightarrow} \text{Delay} \overset{\rightarrow}{\rightarrow} y[n] \]

\[ p \]

**CT**: adders, scalers, and integrators – represent systems described by a linear differential equations with constant coefficients.

\[ x(t) \overset{+}{\rightarrow} \int_{-\infty}^{t} (\cdot) \, dt \overset{\rightarrow}{\rightarrow} y(t) \]

\[ p \]
Operator Representation

CT Block diagrams are concisely represented with the $\mathcal{A}$ operator.

Applying $\mathcal{A}$ to a CT signal generates a new signal that is equal to the integral of the first signal at all points in time.

\[ Y = \mathcal{A}X \]

is equivalent to

\[ y(t) = \int_{-\infty}^{t} x(\tau) \, d\tau \]

for all time $t$. 
Evaluating Operator Expressions

As with $\mathcal{R}$, $\mathcal{A}$ expressions can be manipulated as polynomials.

\[ w(t) = x(t) + \int_{-\infty}^{t} x(\tau) d\tau \]

\[ y(t) = w(t) + \int_{-\infty}^{t} w(\tau) d\tau \]

\[ y(t) = x(t) + \int_{-\infty}^{t} x(\tau) d\tau + \int_{-\infty}^{t} x(\tau) d\tau + \int_{-\infty}^{t} \left( \int_{-\infty}^{t_2} x(\tau_1) d\tau_1 \right) d\tau_2 \]

\[ W = (1 + \mathcal{A}) X \]

\[ Y = (1 + \mathcal{A}) W = (1 + \mathcal{A})(1 + \mathcal{A}) X = (1 + 2\mathcal{A} + \mathcal{A}^2) X \]
Evaluating Operator Expressions

Expressions in $\mathcal{A}$ can be manipulated using rules for polynomials.

- **Commutativity:** $A(1 - A)X = (1 - A)AX$

- **Distributivity:** $A(1 - A)X = (A - A^2)X$

- **Associativity:** $\left((1 - A)A\right)(2 - A)X = (1 - A)\left(A(2 - A)\right)X$
Check Yourself

\[ \dot{y}(t) = \dot{x}(t) + p\ddot{y}(t) \]

\[ \dot{y}(t) = x(t) + py(t) \]

\[ \dot{y}(t) = px(t) + py(t) \]

Which best illustrates the left-right correspondences?

1. 2. 3. 4. 5. none
Which best illustrates the left-right correspondences? 4

1. 

2. 

3. 

4. 

5. none
**Elementary Building-Block Signals**

Elementary DT signal: $\delta[n]$.

\[
\delta[n] = \begin{cases} 
1, & \text{if } n = 0; \\
0, & \text{otherwise}
\end{cases}
\]

- shortest possible duration (most “transient”)
- useful for constructing more complex signals

What CT signal serves the same purpose?
Consider the analogous CT signal.

\[ w(t) = \begin{cases} 
0 & t < 0 \\
1 & t = 0 \\
0 & t > 0 
\end{cases} \]

Is this a good choice as a building-block signal?
Consider the analogous CT signal.

\[ w(t) = \begin{cases} 
0 & t < 0 \\
1 & t = 0 \\
0 & t > 0 
\end{cases} \]

Is this a good choice as a building-block signal? No

The integral of \( w(t) \) is zero!
Unit-Impulse Signal

The unit-impulse signal acts as a pulse with unit area but zero width.

\[ \delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t) \]

\[ p_\epsilon(t) \]

\[ \quad \frac{1}{2\epsilon} \quad \text{unit area} \]

\[ p_{1/2}(t) \]

\[ \quad -\frac{1}{2} \quad \frac{1}{2} \quad t \]

\[ p_{1/4}(t) \]

\[ \quad -\frac{1}{4} \quad \frac{1}{4} \quad t \]

\[ p_{1/8}(t) \]

\[ \quad -\frac{1}{8} \quad \frac{1}{8} \quad t \]
Unit-Impulse Signal

The unit-impulse function is represented by an arrow with the number 1, which represents its area or “weight.”

\[ \delta(t) \]

It has two seemingly contradictory properties:

- it is nonzero only at \( t = 0 \), and
- its definite integral \((-\infty, \infty)\) is one!

Both of these properties follow from thinking about \( \delta(t) \) as a limit:

\[
\delta(t) = \lim_{\epsilon \to 0} p_\epsilon(t)
\]
The indefinite integral of the unit-impulse is the unit-step.

\[ u(t) = \int_{-\infty}^{t} \delta(\lambda) \, d\lambda = \begin{cases} 
1; & t \geq 0 \\
0; & \text{otherwise}
\end{cases} \]

Equivalently

\[ \delta(t) \xrightarrow{A} u(t) \]
Impulse Response of Acyclic CT System

If the block diagram of a CT system has no feedback (i.e., no cycles), then the corresponding operator expression is “imperative.”

\[ Y = (1 + A)(1 + A) X = (1 + 2A + A^2) X \]

If \( x(t) = \delta(t) \) then

\[ y(t) = (1 + 2A + A^2) \delta(t) = \delta(t) + 2u(t) + tu(t) \]
Find the impulse response of this CT system with feedback.
Find the impulse response of this CT system with feedback.

\[ x(t) \xrightarrow{+} \mathcal{A} \xrightarrow{\rightarrow} y(t) \]

Method 1: find differential equation and solve it.

\[ \dot{y}(t) = x(t) + py(t) \]

Linear, first-order difference equation with constant coefficients.

Try \( y(t) = Ce^{\alpha t} u(t) \).

Then \( \dot{y}(t) = \alpha Ce^{\alpha t} u(t) + Ce^{\alpha t} \delta(t) = \alpha Ce^{\alpha t} u(t) + C \delta(t) \).

Substituting, we find that \( \alpha Ce^{\alpha t} u(t) + C \delta(t) = \delta(t) + pCe^{\alpha t} u(t) \).

Therefore \( \alpha = p \) and \( C = 1 \) \( \rightarrow \) \( y(t) = e^{pt} u(t) \).
Find the impulse response of this CT system with feedback.

Method 2: use operators.

\[ Y = A(X + pY) \]
\[ \frac{Y}{X} = \frac{A}{1 - pA} \]

Now expand in ascending series in \( A \):

\[ \frac{Y}{X} = A(1 + pA + p^2A^2 + p^3A^3 + \cdots) \]

If \( x(t) = \delta(t) \) then

\[ y(t) = A(1 + pA + p^2A^2 + p^3A^3 + \cdots) \delta(t) \]
\[ = (1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots) u(t) = e^{pt}u(t) . \]
We can visualize the feedback by tracing each cycle through the cyclic signal path.

\[ y(t) = (\mathcal{A} + p\mathcal{A}^2 + p^2\mathcal{A}^3 + p^3\mathcal{A}^4 + \cdots) \delta(t) \]
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Making $p$ negative makes the output converge (instead of diverge).

\[
y(t) = (A - pA^2 + p^2 A^3 - p^3 A^4 + \cdots) \delta(t) \\
= (1 - pt + \frac{1}{2}p^2 t^2 - \frac{1}{6}p^3 t^3 + \cdots) u(t)
\]
Making $p$ negative makes the output converge.

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Convergent and Divergent Poles

The fundamental mode associated with $p$ diverges if $p > 0$ and converges if $p < 0$.

![Diagram of system with X, A, and Y nodes, and feedback loop with p.](image)

$p = 1$

$p = -1$
Convergent and Divergent Poles

The fundamental mode associated with $p$ diverges if $p > 0$ and converges if $p < 0$. 

![Diagram showing convergent and divergent poles]

Re $p$  \[\begin{array}{c}
\text{Convergent} \\
\text{Divergent}
\end{array}\]

Im $p$
In CT, each cycle adds a new integration.

\[ y(t) = (A + pA^2 + p^2A^3 + p^3A^4 + \cdots) \delta(t) \]
\[ = (1 + pt + \frac{1}{2}p^2t^2 + \frac{1}{6}p^3t^3 + \cdots) u(t) = e^{pt} u(t) \]
In DT, each cycle creates another sample in the output.

\[ y[n] = \left(1 + p\mathcal{R} + p^2\mathcal{R}^2 + p^3\mathcal{R}^3 + p^4\mathcal{R}^4 + \cdots \right) \delta[n] \]

\[ = \delta[n] + p\delta[n - 1] + p^2\delta[n - 2] + p^3\delta[n - 3] + p^4\delta[n - 4] + \cdots \]
Comparison of CT and DT representations

Locations of convergent poles differ for CT and DT systems.

\[
\begin{align*}
X & \rightarrow + \rightarrow A \rightarrow Y \\
\frac{A}{1 - pA} & = e^{pt}u(t) \\
X & \rightarrow + \rightarrow p \rightarrow Y \\
\frac{1}{1 - p\mathcal{R}} & = p^n u[n]
\end{align*}
\]
Mass and Spring System

Use the $A$ operator to solve the mass and spring system.

\[ F = K(x(t) - y(t)) = M\ddot{y}(t) \]

\[ \frac{Y}{X} = \frac{K A^2}{1 + \frac{K}{M} A^2} \]
Mass and Spring System

Factor system functional to find the poles.

\[ \frac{Y}{X} = \frac{\frac{K}{M}A^2}{1 + \frac{K}{M}A^2} = \frac{\frac{K}{M}A^2}{(1 - p_0 A)(1 - p_1 A)} \]

\[ 1 + \frac{K}{M}A^2 = 1 - (p_0 + p_1)A + p_0 p_1 A^2 \]

The sum of the poles must be zero.
The product of the poles must be \( K/M \).

\[ p_0 = j \sqrt{\frac{K}{M}} \quad p_1 = -j \sqrt{\frac{K}{M}} \]
Alternatively, find the poles by substituting $A \rightarrow \frac{1}{s}$. The poles are then the roots of the denominator.

$$\frac{Y}{X} = \frac{\frac{K}{M}A^2}{1 + \frac{K}{M}A^2}$$

Substitute $A \rightarrow \frac{1}{s}$:

$$\frac{Y}{X} = \frac{\frac{K}{M}}{s^2 + \frac{K}{M}}$$

$$s = \pm j\sqrt{\frac{K}{M}}$$
Mass and Spring System

The poles are complex conjugates.

\[ s = \pm \sqrt{\frac{K}{M}} \equiv \omega_0 \]

The corresponding fundamental modes have complex values.

fundamental mode 1: \( e^{j\omega_0 t} = \cos \omega_0 t + j \sin \omega_0 t \)

fundamental mode 2: \( e^{-j\omega_0 t} = \cos \omega_0 t - j \sin \omega_0 t \)
Mass and Spring System

Real-valued inputs always excite combinations of these modes so that the imaginary parts cancel.

Example: find the impulse response.

\[
\frac{Y}{X} = \frac{\frac{K}{M} A^2}{1 + \frac{K}{M} A^2} = \frac{K}{M} \left( \frac{A}{1 - p_0 A} - \frac{A}{1 - p_1 A} \right)
\]

\[
= \frac{\omega_0^2}{2j \omega_0} \left( \frac{A}{1 - j \omega_0 A} \right) - \frac{\omega_0}{2j} \left( \frac{A}{1 + j \omega_0 A} \right)
\]

makes mode 1

makes mode 2

The modes themselves are complex conjugates, and their coefficients are also complex conjugates. So the sum is a sum of something and its complex conjugate, which is real.
Mass and Spring System

The impulse response is therefore real.

\[
\frac{Y}{X} = \frac{\omega_0}{2j} \left( \frac{A}{1 - j\omega_0 A} \right) - \frac{\omega_0}{2j} \left( \frac{A}{1 + j\omega_0 A} \right)
\]

The impulse response is

\[
h(t) = \frac{\omega_0}{2j} e^{j\omega_0 t} - \frac{\omega_0}{2j} e^{-j\omega_0 t} = \omega_0 \sin \omega_0 t; \quad t > 0
\]
Alternatively, find impulse response by expanding system functional.

\[
\frac{Y}{X} = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} = \omega_0^2 A^2 - \omega_0^4 A^4 + \omega_0^6 A^6 - + \cdots
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - + \cdots, \quad t \geq 0
\]
Look at successive approximations to this infinite series.

\[
\frac{Y}{X} = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} = \omega_0^2 A^2 \sum_{l=0}^{\infty} \left(-\omega_0^2 A^2\right)^l
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \sum_{l=0}^{\infty} \omega_0^2 \left(-\omega_0^2\right)^l A^{2l+2} \delta(t) \\
= \omega_0^2 t
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\]

\[
y(t)
\]

\[
\begin{array}{c}
0 \\
\hline
\end{array}
\]

\[
t
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Mass and Spring System

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y(t)

\[0\quad t\]
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\[
0 \quad t
\]
Mass and Spring System

Look at successive approximations to this infinite series.

\[
\frac{Y}{X} = \frac{\omega_0^2 A^2}{1 + \omega_0^2 A^2} = \omega_0^2 A^2 \sum_{l=0}^{\infty} (-\omega_0^2 A^2)^l
\]

If \( x(t) = \delta(t) \) then

\[
y(t) = \sum_{l=0}^{\infty} \omega_0^2 (-\omega_0^2)^l A^{2l+2} \delta(t)
\]

\[
= \omega_0^2 t - \omega_0^4 \frac{t^3}{3!} + \omega_0^6 \frac{t^5}{5!} - \omega_0^8 \frac{t^7}{7!} + \omega_0^{10} \frac{t^9}{9!} - + \cdots = \omega_0 \sin \omega_0 t
\]
Comparison of CT and DT representations

Important similarities and important differences.

\[ \frac{A}{1 - pA} \]
\[ e^{pt}u(t) \]

\[ \frac{1}{1 - pR} \]
\[ p^n u[n] \]