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Symmetry-Protected Quantum Spin Hall Phases in Two Dimensions

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Symmetry-protected topological (SPT) states are short-range entangled states with symmetry. Nontrivial SPT states have symmetry-protected gapless edge excitations. In 2 dimension (2D), there are an infinite number of nontrivial SPT phases with $SU(2)$ or $SO(3)$ symmetry. These phases can be described by $SU(2)$ or $SO(3)$ nonlinear-sigma models with a *quantized* topological θ term. At an open boundary, the θ term becomes the Wess-Zumino-Witten term and consequently the boundary excitations are decoupled gapless left movers and right movers. Only the left movers (if $\theta > 0$) carry the $SU(2)$ or $SO(3)$ quantum numbers. As a result, the $SU(2)$ SPT phases have a half-integer quantized spin Hall conductance and the $SO(3)$ SPT phases have an even-integer quantized spin Hall conductance. Both the $SU(2)$ and $SO(3)$ SPT phases are symmetric under their $U(1)$ subgroup and can be viewed as $U(1)$ SPT phases with even-integer quantized Hall conductance.

Gapped quantum states may belong to long-range entangled phases or short-range entangled (SRE) phases [1]. Long-range entangled states have intrinsic topological order and cannot be deformed into direct product states through finite steps of local unitary transformations. Examples of intrinsic topologically ordered phases include fractional quantum Hall liquids [2,3], chiral spin liquids [4,5], and Z_2 spin liquid [6–8]. On the other hand, SRE states are equivalent to direct product states under local unitary transformations. If there is no symmetry, there will be only one SRE phase. If the system has a symmetry, the phase diagram will be much richer. Even SRE states which do not break any symmetry can belong to different phases. Those phases are called SPT phases which stands for symmetry-protected topological phases or symmetry-protected trivial phases. The well-known Haldane phase in $S = 1$ spin chain [9,10] is the first example of bosonic SPT phase in 1 dimension (1D), which is protected by either D_2 spin rotation symmetry or time reversal symmetry. Topological insulators [11–15] are 2 dimension (2D) SPT phases in free fermion systems protected by time reversal symmetry T and $U(1)$ charge conservation symmetry.

Some thought that the topological insulators are characterized by quantum spin Hall effect. However, since spin rotation symmetry is broken by spin-orbital coupling, spin angular momentum is not conserved. Therefore, there is no spin Hall effect in usual topological insulators. Quantum spin Hall effect will be present only if the topological insulators also have an extra $U(1)$ spin rotation symmetry [16].

In this Letter, we will introduce another kind of SPT phases—$SU(2)$ or $SO(3)$ SPT phases in 2D, which are classified by $\mathbb{Z}$. In contrast to topological insulators, these phases are interacting bosonic phases. Owning to the $SU(2)$ or $SO(3)$ symmetry, if the system is open, the boundary excitations will be gapless although the bulk remains gapped. Importantly, different SPT phases can be distinguished experimentally through their linear responses. To this end, we couple the model to external probe field, which is an analogue of the electromagnetic field for spins. We show that spin Hall current will be induced on the boundary with a quantized spin Hall conductance. Different $SU(2)$ SPT phases are characterized by their different half-integer quantized spin Hall conductance, while different $SO(3)$ SPT phases by even-integer quantized spin Hall conductance.

$SU(2)$ principal chiral NLSM.—In 2D, $SU(2)$ SPT phases are classified by group cohomology class $\mathcal{H}^3(SU(2), U(1)) = \mathbb{Z}$ [17]. Owning to the correspondence between the group cohomology class and the topological cohomology class [18], each SPT phase can be described by a principal chiral nonlinear sigma model (NLSM) with quantized topological θ term [which is classified by $H^3(SU(2), Z) = \mathbb{Z}$]. The θ term of the NLSM can be written as [19]

$$S_{\text{top}} = -i\frac{\theta}{24\pi^2} \int_M \text{Tr}(g^{-1}dg)^3, \qquad g \in SU(2), \quad (1)$$

where M is the Euclidian space-time manifold, $g \in SU(2)$ is a 2×2-matrix-valued function of space-time $g(x)$, and $\theta = 2\pi K$ with $K \in \mathbb{Z}$ corresponding to the Kth $SU(2)$ SPT phase. When M has no boundary, S_{top} is quantized into integer times of $-2\pi i$.

Including the dynamic part, the partition function of the NLSM is $Z = \int \mathcal{D}g \, e^{-\int_M d^3x \mathcal{L}}$, where $\mathcal{L}$ is the Lagrangian density,

$$\mathcal{L} = -\frac{1}{4\lambda^2}\,\mathrm{Tr}[(g^{-1}\partial_\mu g)(g^{-1}\partial_\mu g)]$$
$$- i\frac{K}{12\pi}\,\mathrm{Tr}(\varepsilon^{\mu\nu\gamma}g^{-1}\partial_\mu g g^{-1}\partial_\nu g g^{-1}\partial_\gamma g). \quad (2)$$

For large enough λ^2, the renormalization flows to a fixed point where only the topological term remains (λ^2 flows to infinity). The fixed point Lagrangian captures all the physical properties of the SPT phases. So we will focus on the fixed point in the following discussion.

Since the symmetry group is of crucial importance for the physical properties of SPT phases, we stress that the symmetry group of our system is $SU(2)_L$, under which the group element g varies as $g \to \hat{h}g = hg$ for $\hat{h} \in SU(2)_L$. It is easy to check that the Lagrangian equation (2) is invariant under $SU(2)_L$. It can be shown that Eq. (2) has a larger symmetry, it is invariant under the group $SU(2)_L \times SU(2)_R$, where $SU(2)_R$ is the right multiplying group defined as $\hat{\bar{h}}g = gh^{-1}$, $\hat{\bar{h}} \in SU(2)_R$. Furthermore, Eq. (2) also has time reversal symmetry T. Namely, it is invariant under the time reversal transformation, $t \to -t$, $i \to -i$ (consequently $\tau \to \tau$), and $g \to g^{-1}$ [20]. The SPT phases only need the protection of $SU(2)_L$. As will be discussed later, if the extra symmetry $SU(2)_R$ and T is removed by perturbation $\delta\mathcal{L} = \mathrm{Tr}[\partial_\tau g M(x)g^{-1}]$ with $M(x)$ external field, the physical properties of the SPT phases remains unchanged. In the following we will discuss the Lagrangian equation (2) and note $SU(2)_L$ as $SU(2)$ without causing confusion.

If the system has a boundary, the quantized θ term Eq. (1) becomes the Wess-Zumino-Witten term [21,22] in the $1 + 1D$ boundary effective theory. According to Ref. [23], a $1 + 1D$ Wess-Zumino-Witten model with given K may flow to a gapless fixed point $S_{\mathrm{bdr,fix}} = \frac{|K|}{8\pi} \times \int dx^0 dx^1 \mathrm{Tr}[(g^{-1}\partial_\mu g)(g^{-1}\partial_\mu g)] + S_{\mathrm{top}}$, where S_{top} are defined in Eq. (1), $x^0 = \tau$ is the imaginary time, and x^1 is the spacial dimension along the boundary.

If $K > 0$, the boundary excitations at the fixed point are decoupled left mover $J_+ = \frac{K}{2\pi}\partial_+ g g^{-1}$ and right mover $J_- = -\frac{K}{2\pi}g^{-1}\partial_- g$, where $x^\pm = \sqrt{\frac{1}{2}}(x^0 \pm ix^1)$ is the chiral coordinate and $\partial_\pm = \sqrt{\frac{1}{2}}(\partial_0 \mp i\partial_1)$. $J_\pm$ satisfy the equation of motion $\partial_\mp J_\pm = 0$ (which yields gapless dispersion). Importantly, J_+ and J_- behave differently under global $SU(2)$ transformation $g \to hg$. The current J_- is $SU(2)$ invariant $J_- \to J_-$, but J_+ is $SU(2)$ covariant $J_+ \to hJ_+h^{-1}$, so only the left mover J_+ carries $SU(2)$ "charge." This property indicates that the gapless boundary excitations are protected by the $SU(2)_L$ symmetry, because the mass term, such as $\mathcal{L}_{\mathrm{bdr,mass}} \propto (\mathrm{Tr}g)^2$ [24], which gaps out the excitations will mix the left mover and right mover and hence breaks the $SU(2)_L$ symmetry. The bulk perturbation $\delta\mathcal{L} = \mathrm{Tr}[\partial_\tau g M(x)g^{-1}]$, on the other hand, will not cause scattering between the left mover and the right mover since it respects $SU(2)_L$ symmetry; hence, it will leave

the boundary excitations gapless. Under time reversal T, J_+ and J_- exchange their roles $J_+ \leftrightarrow J_-$. If $K < 0$, then the boundary excitations will be redefined as $J_+ = -\frac{K}{2\pi}g^{-1}\partial_+ g$, $J_- = \frac{K}{2\pi}\partial_- g g^{-1}$. In this case, J_+ is $SU(2)$ neutral and J_- carries $SU(2)$ charge.

Following Ref. [23], the boundary excitations of the $SU(2)$ SPT state labeled by K are described by $SU(2)$ Kac-Moody algebra of level $|K|$. In the following we will study how the system (especially the boundary) responds to an external probe field. Without loss of generality, we assume $K > 0$.

Quantized spin Hall conductance.—Now we introduce an external probe field A, which minimally couples to the topological NLSM by replacing every $g^{-1}\partial_\mu g$ term with $g^{-1}(\partial_\mu + A_\mu)g$. Expanding A by three Pauli matrices, $A = \frac{1}{2}\sum_{\mu,a}A_\mu^a \sigma^a dx^\mu$, then we can define a current density operator $J_\mu^a = \frac{\delta\mathcal{L}}{\delta A_\mu^a}\big|_{A_\mu=0}$ with

$$J_\mu^a = -\frac{1}{2\lambda^2}\,\mathrm{Tr}\Big(\partial_\mu g g^{-1}\frac{\sigma^a}{2}\Big)$$
$$+ i\frac{K}{4\pi}\varepsilon_{\mu\nu\gamma}\partial_\gamma\Big[\mathrm{Tr}\Big(\partial_\nu g g^{-1}\frac{\sigma^a}{2}\Big)\Big].$$

J_μ^a is the conserved spin current corresponding to the global $SU(2)$ invariance of the action. The second term on the right-hand side contributes a boundary current since it is a total differential.

At the fixed point $\lambda^2 \to \infty$, only the topological term remains,

$$-i\frac{K}{12\pi}\,\mathrm{Tr}[g^{-1}(d + A)g]^3 = -i\frac{K}{12\pi}\,\mathrm{Tr}[(g^{-1}dg)^3 + A^3$$
$$+ 3(dgg^{-1} \wedge F)$$
$$+ 3d(dgg^{-1} \wedge A)]. \quad (3)$$

Notice that Eq. (3) is invariant under local $SU(2)$ transformation $g \to hg$, if the field A varies as $A \to hAh^{-1} + hdh^{-1}$. If $F = 0$, then A only couples to the edge current via $\mathrm{Tr}(dgg^{-1} \wedge A)$. Notice that only the right moving component J_+ occurs in dgg^{-1}. This means that A only couples to J_+ and does not couple to J_-. When $F \neq 0$, the bulk term $3\,\mathrm{Tr}(dgg^{-1} \wedge F)$ in Eq. (3) is difficult to treat. In order to obtain an effective field theory of the external field A and F, we need to integrate out the group variables g.

To avoid this difficulty, we take the advantage of the local "gauge invariance" of the Lagrangian in Eq. (3). Here the local "gauge transformation" is defined as $g \to h(x)g$ and $A \to hAh^{-1} + hdh^{-1}$. When integrating out the group variables, the effective action of A should also be "gauge" invariant. So we expect the result is the Chern-Simons action (we will see later that this effective action is self-consistent),

$$S_{\text{eff}}(A) = i\frac{K}{4\pi}\int_M \text{Tr}\Big(A\wedge F - \frac{1}{3}A^3\Big),$$

$$= i\frac{K}{8\pi}\int_M d^3x\,\varepsilon^{\mu\nu\lambda}\Big(A_\mu^a\partial_\nu A_\lambda^a + \varepsilon_{abc}\frac{i}{3}A_\mu^a A_\nu^b A_\lambda^c\Big),$$

where $A = \sum_a A_\mu^a dx^\mu \frac{\sigma_a}{2}$. Notice that the trace $\text{Tr}(\frac{\sigma_a}{2}\frac{\sigma_b}{2}) = \frac{1}{2}\delta_{ab}$ contributes an extra coefficient $\frac{1}{2}$. If $F = 0$, $S_{\text{eff}}(A) = -i\frac{K}{12\pi}\int \text{Tr}A^3$, which is consistent with Eq. (3). From the above effective action, we obtain the response current density,

$$\mathcal{J}_\mu^a = \frac{\delta S_{\text{eff}}}{\delta A_\mu^a} = i\frac{K}{4\pi}\varepsilon_{\mu\nu\lambda}\Big(\partial_\nu A_\lambda^a + \frac{i}{2}\varepsilon_{abc}A_\nu^b A_\lambda^c\Big). \quad (4)$$

It will be easier to see the response of the system if the probe field A only contains the spin-z component, $A = \sum_\mu A_\mu^z \frac{\sigma_z}{2}dx^\mu$, which can be viewed as the spin-electromagnetic field that couples to S^z as its charge. Then the responding spin density is proportional to the "magnetic field,"

$$\mathcal{J}_0^z = i\frac{K}{4\pi}(\partial_1 A_2^z - \partial_2 A_1^z) = i\frac{K}{4\pi}b^z.$$

Here we use 0, 1, 2 to label the space-time index and x, y, z to label the spin direction. The spin current is proportional to the "electric field,"

$$\mathcal{J}_1^z = i\frac{K}{4\pi}(\partial_2 A_0^z - \partial_0 A_2^z) = \frac{K}{4\pi}e_2^z,$$

$$\mathcal{J}_2^z = i\frac{K}{4\pi}(\partial_0 A_1^z - \partial_1 A_0^z) = -\frac{K}{4\pi}e_1^z.$$

The direction of the motion of the spin current is orthogonal to the direction of the electric field. This is nothing but a spin Hall effect. Furthermore, the spin Hall conductance is quantized as $\frac{K}{4\pi}$, which is half of the electric integer charge Hall conductance. From this information, we conclude that the $SU(2)$ symmetric topological NLSM model describes a bosonic spin quantum Hall system.

The $SU(2)$ SPT phases can also be viewed as $U(1)$ SPT phases, where $U(1)$ is the S^z spin rotation. The above result implies that the $U(1)$ SPT phases are characterized by a quantized Hall conductance. To understand the value of quantization, let us introduce $A_\mu^c = \frac{1}{2}A_\mu^z$. The charge that A_μ^c couples to is $2S^z$ which is quantized as integers. The effective action for A_μ^c is given by $S_{\text{eff}}(A^c) = i\frac{K}{2\pi}\times\int_M d^3x\,\varepsilon^{\mu\nu\lambda}A_\mu^c\partial_\nu A_\lambda^c$. We see that the charge Hall conductance is $\frac{2K}{2\pi}$. In other words, the Hall conductance for the $U(1)$ SPT phases is quantized as *even* integers $2K$ (in unit of $\frac{1}{2\pi}$), which agrees with a calculation by $U(1) \times U(1)$ Chern-Simons theory [25,26].

In the electric integer quantum Hall system, the boundary excitations are chiral currents. In contrast, the boundary of model (1) contains both left-moving and right-moving gapless excitations. However, only the left mover carries $SU(2)$ charge and couples to the probe field A. In other words, the A field will induce left-moving spin current.

The coupling of the left-moving current to the A field is consistent with the Chern-Simons action. Remembering that the topological term (3) is local gauge invariant. If space-time is closed, the effective action (4) is gauge invariant as expected. However, if space-time has a boundary, Eq. (4) is no longer gauge invariant. Under local gauge transformation $A \to A' = hAh^{-1} + hdh^{-1}$, the variance of the Chern-Simons term is

$$S_{\text{eff}}(A') - S_{\text{eff}}(A) = i\frac{K}{4\pi}\Big[\int_{\partial M}\text{Tr}(h^{-1}dh\wedge A)$$
$$+ \int_M \frac{1}{3}\text{Tr}(h^{-1}dh)^3\Big]. \quad (5)$$

The first term on the right-hand side depends on the values of A on the boundary, and the second term is independent on A.

Since the gauge anomaly in Eq. (5) is purely a boundary term, it can be canceled by a matter field on the boundary described by $SU(2)$ level-$|K|$ Kac-Moody algebra. To see the cancelation of the anomaly, we may embed the $SU(2)$ level-$|K|$ Kac-Moody algebra into $|K|$ spin-1/2 complex fermions $\psi_I, (I = 1, 2, \ldots, K)$, which leads to the following effective edge theory:

$$S_{\text{bdr}}(\psi, A) = \int dx^0 dx^1 \sum_{I=1}^{K}[\psi_{I-}^\dagger(\partial_0 - i\partial_1)\psi_{I-}$$
$$+ \psi_{I+}^\dagger[(\partial_0 + A_0) + i(\partial_1 + A_1)]\psi_{I+}].$$

Under gauge transformation $\psi'_+ = h\psi_+$, $A' = hAh^{-1} + hdh^{-1}$, the above action has an anomaly [27,28] (for details, see the Supplemental Material [29]) $S_{\text{bdreff}}(A') - S_{\text{bdreff}}(A) = -i\frac{K}{4\pi}\int_{\partial M}\text{Tr}(h^{-1}dh\wedge A)$, which exactly cancels the anomaly of the Chern-Simons action in Eq. (5). This means that the total action of bulk Chern-Simons term and the boundary fermion term is gauge invariant (up to a term which is independent on A).

Since we have K flavors of fermion fields, they also form a representation of $U(K)$ Kac-Moody algebra, which gives rise to extra gapless edge modes. However, only the representation of $SU(2)_K$-Kac Moody algebra are physical degrees of freedom in our model. The extra gapless modes can be gapped out by mass terms which do not break the $SU(2)$ symmetry, or can be removed by performing a projection onto the $U(k)$ singlet at each site [30].

Supposing $\bar{A}$ is the time reversal partner of A, then under T transformation, $\partial_\tau \to \partial_\tau$, $i \to -i$, $g \to g^{-1}$, $A_\mu \to \bar{A}_\mu$, the Lagrangian (3) becomes

$$i\frac{K}{12\pi}\text{Tr}[g(d + \bar{A})g^{-1}]^3 = -i\frac{K}{12\pi}\text{Tr}[(g^{-1}dg)^3 - \bar{A}^3$$
$$+ 3(g^{-1}dg \wedge \bar{F})$$
$$+ 3d(g^{-1}dg \wedge \bar{A})],$$

where $\bar{F} = d\bar{A} + \bar{A}\wedge\bar{A}$. From the above equation, we can see that $\bar{A}$ only couples to J_-, which carries $SU(2)_R$ charge

and is $SU(2)_L$ neutral. Thus the time reversal operation T transforms the $SU(2)_L$ quantities A and J_+ to the $SU(2)_R$ quantities $\bar{A}$ and J_-. This is very different from the model with $-K$, where the right mover J_- carries $SU(2)_L$ charge and is coupling to A.

SO(3) *SPT phase in* 2 + 1D.—Above we discussed a bosonic spin-1/2 model with quantized spin Hall effect. However, a bosonic particle can never carry spin-1/2. So the $SU(2)$ SPT phases only have theoretical interest. In the following, we will discuss a more realistic bosonic model of integer spins, whose symmetry group is $SO(3)$,

$$S_{\text{top}} = -i\frac{2\pi K}{2 \times 48\pi^2} \int_M \text{Tr}(g^{-1}dg)^3, \qquad g \in SO(3). \quad (6)$$

Here $g \in SO(3)$ is a 3×3 matrix, and $K \in \mathbb{Z}$ is an element of the cohomology $H^3(SO(3), \mathbb{Z}) = \mathbb{Z}$ which is generated by $\frac{1}{48\pi^2}\int_G \text{Tr}(g^{-1}dg)^3$. The factor 2 in the denominator of Eq. (6) is owing to the factor that closed space-time manifold (e.g., $M = S^3$) must cover the group manifold $G = S^3/Z_2$ even times.

Above topological action (6) should be quantized to integer times of $-2\pi i$, even if M is the group manifold itself. To satisfy this condition, K must be an even integer. In other words, only even K belongs to $H^3(SO(3), \mathbb{Z}) = \mathbb{Z}$. Furthermore, only $K = 4r$, $r \in \mathbb{Z}$ give rise to SPT phases. The mathematical reason is that the map from the group cohomology $\mathcal{H}^3(SO(3), U(1))$ to topological cohomology $H^3(SO(3), \mathbb{Z})$ is not onto, only even elements of the latter (namely $K = 4r$) have counterparts of the former [18,31].

The physical reason that K must be $4r$ is the following. We consider space-time with $S^1 \times \Sigma$ topology, but in the limit where the spacial circle S^1 has a very small size. Let us consider the field configuration $g(x^\mu)$ where S^1 maps to the nontrivial element in $\pi_{1[SO(3)]} = \mathbb{Z}_2$: $g(x^\mu) = e^{i\theta\hat{\mathbf{n}}\cdot\mathbf{L}}$ where θ parametrizes the S^1 and L_x, L_y, L_z are the generators of the $SO(3)$ group. In the small S^1 limit, such field configuration is described by the mapping from the space-time Σ to S^2 labeled by the unit vector $\hat{\mathbf{n}}$. Physically, this means that the small S^1 limit, S_{top} can be viewed as the topological θ term in the NLSM of unit vector $\hat{\mathbf{n}}$ with $\theta = 2\pi K$, since if Σ wrap around S^2 once, $g(x^\mu) = e^{i\theta\hat{\mathbf{n}}\cdot\mathbf{L}}$ will wrap around $SO(3)$ twice. In the small S^1 limit the space becomes a thin torus (or a cylinder if it is open) and the system becomes an effective 1D system. We also note that $g(x^\mu) \rightarrow hg(x^\mu)h^{-1}$, $h \in SO(3)$ rotate the unit vector $\hat{\mathbf{n}}$. Such an $SO(3)$ rotation gives rise to an isospin quantum number $\mathbf{S}_{\text{iso}} = \mathbf{S}_L + \mathbf{S}_R$, where $\mathbf{S}_L$ is the spin operator associated with $SO_L(3)$ and $\mathbf{S}_R$ with $SO_R(3)$. The topological θ term with $\theta = 2\pi K$ implies that an open end of the 1D system will carry isospin $\frac{K}{2}$ [32]. This means that a Z_2 vortex (which exists since $\pi_{1[SO(3)]} = \mathbb{Z}_2$) will carry isospin $\frac{K}{2}$. In Ref. [31], it is shown that such a Z_2 vortex (corresponding to the twisted sector in Ref. [31]) carries (S_L, S_R) spins given by $(m + \frac{1}{2}, \frac{K}{2} - m - \frac{1}{2})$, m = integer, if $K = 4r + 2$, and by $(m, \frac{K}{2} - m)$, m = integer, if $K = 4r$.

Thus a Z_2 vortex carries the physical spin (i.e., the S_L spin) given by half integers if $K = 4r + 2$ and by integers if $K = 4r$. Z_2 vortex carrying half-integer spins can happen in the continuous field theory, since the Z_2 vortex is nontrivial in continuous field theory. However, SPT phases are defined on lattice models where space-time are discrete. In this case, the Z_2 vortex can continuously deform into a trivial configuration. Thus the vortex core must be "trivial" and can only carry an integer spin. Consequently, only $K = 4r$ correspond to SPT phases.

Except for the constrains of the level $K = 4r$, the remaining discussion is very similar to that of the $SU(2)$ model. We couple the $SO(3)$ NLSM with an external probe field A, $g \rightarrow hg$, $A \rightarrow hAh^{-1} + hdh^{-1}$. Owning to this local gauge invariance, we expect that the effective action for A is a Chern-Simons term (plus a boundary action), $S_{\text{eff}}(A) = i\frac{K}{16\pi}\int_M \text{Tr}(A \wedge F - \frac{1}{3}A^3)$. We can expand $A = \sum_a A^a L_a$, $a = x, y, z$, where L_x, L_y, L_z satisfy $[L_a, L_b] = i\varepsilon_{abc}L_c$ and $\text{Tr}(L_a L_b) = 2\delta_{ab}$. Suppose A is collinear and only contains the z components in spin space, then we obtain the response spin current density, $\mathcal{J}_\mu^z = \frac{\delta S}{\delta A_\mu^z} = i\frac{K}{4\pi}\epsilon^{\mu\nu\lambda}\partial_\nu A_\lambda^z$. The spin Hall conductance is quantized as $\frac{K}{4\pi}$ [the same as the $SU(2)$ case].

We may embed the edge effective theory into $K/2$ flavor free Majorana fermion model,

$$S_{\text{bdr}}(\psi, A) = \int_{\partial M} dx^0 dx^1 \sum_{I=1}^k [\tilde{\psi}_{I-}(\partial_\tau - i\partial_\sigma)\tilde{\psi}_{I-}$$
$$+ \tilde{\psi}_{I+}[(\partial_\tau + A_\tau) + i(\partial_\sigma + A_\sigma)]\tilde{\psi}_{I+}],$$

where $\tilde{\psi}_I$ is a $SO(3)$ triplet Majorana fermion field and $k = K/2$ is the level of $SO(3)$ Kac-Moody algebra. The anomaly of the boundary action cancels the anomaly of the bulk Chern-Simons term. The field A induces a left moving spin current on the edge. Again, the extra $O(k)$ gapless modes can be gapped out by a mass term which does not break the $SO(3)$ symmetry, or can be removed by a projection onto a $O(k)$ singlet per site.

We may also view the $SO(3)$ SPT phases as $U(1)$ SPT phases. From the spin Hall conductance $\frac{K}{4\pi}$ of the $SO(3)$ SPT phases and the fact that $K = 4r$, we see that the $U(1)$ SPT phases have an even-integer quantized Hall conductance (in units of $\frac{1}{2\pi}$).

Conclusion and discussion.—In summary, we study $SU(2)$ and $SO(3)$ symmetry protected topological phases via topological NLSM. These phases have spin quantum Hall effect when they are coupled to external probe fields. The gapless boundary excitations are decoupled left movers and right movers, which are protected by symmetry. When $K > 0$, only the left moving current carries symmetry charge, and can be detected by the probe field. The spin Hall conductance quanta of $SO(3)$ models is 4 times as large as that of the $SU(2)$ models. We also find that the $U(1)$ SPT phases are characterized by an even-integer quantized Hall conductance.

It has been shown that different 2D SPT states with symmetry G are described by Borel group cohomology $\mathcal{H}^3[G, U(1)]$ [17]. In this Letter we show that [for $G = SU(2), SO(3)$] if we gauge the symmetry group, the resulting theory is a Chern-Simons theory with gauge group G which is also classified by $\mathcal{H}^3[G, U(1)]$ [18]. This suggests a very interesting one-to-one duality relation between 2D SPT phases with symmetry G and 2D Chern-Simons theory with gauge group G, for both continuous and discrete groups G [33]. This also suggests that, when we probe the SPT states by "gauging" the symmetry, we can distinguish all the SPT states.

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