Stochastic Congruence Equations for Spacetime Fluctuations

by

Antony John Speranza

Submitted to the Department of Physics
in partial fulfillment of the requirements for the degree of

Bachelor of Science in Physics

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2012

Author ....
Department of Physics
May 11, 2012

Certified by ....
Sean P. Robinson
Lecturer
Thesis Supervisor

Accepted by ....
Professor Nergis Mavalvala
Senior Thesis Coordinator, Department of Physics

Stochastic Congruence Equations for Spacetime Fluctuations

by

Antony John Speranza

Submitted to the Department of Physics
on May 11, 2012, in partial fulfillment of the
requirements for the degree of
Bachelor of Science in Physics

Abstract

This work considers some implications of viewing gravity as an emergent force. In such a viewpoint, general relativity arises as the thermodynamic limit of some microscopic theory. As such, one would expect the macroscopic variables such as the curvature tensors to fluctuate about their mean. This thesis presents a method for analyzing the effects of curvature fluctuations on spacetime thermodynamics. This is done by examining the evolution equations for timelike and null congruences, and recasting them as stochastic differential equations. The purpose of viewing the congruence evolution equations as stochastic is in the spirit of nonequilibrium thermodynamics, and may lead to an application of the fluctuation-dissipation theorem to spacetime. It is expected that this reformulation of the congruence equations will lead to further insights on the effects of fluctuations in general relativity.

Thesis Supervisor: Sean P. Robinson
Title: Lecturer

Acknowledgments

There are many who deserve my appreciation for the support they have offered during my time at MIT. As my research advisors and instructors, I thank Mustafa Amin, Alan Guth, Larry Price, Leo Singer, Alan Weinstein and David Litster. As my dedicated lab partner, I thank Dylan Rankin. For their encouragement and aid in my academic pursuits in physics, I thank Javier Duarte and Ian Rousseau. For the support and mentorship they provided, I thank Matthieu Talpe and Gleb Kuznetsov. I thank the brothers of Phi Sigma Kappa for the friendship they have provided.

I am very grateful to my thesis advisor Sean Robinson for engaging conversations, for his patience with this work, and for the time he put in helping me pursue this research.

Finally, I thank my siblings Rosa, Emily, Dominick, Rebecca, and my parents Dominick and Ellen for their unwavering support over the past four years.

Contents

Chapter 1

An Overview of Spacetime Thermodynamics

During the 1970's the study of spacetime singularities in general relativity led to several intriguing results. It was well know that singularities arise generically in physically realistic solutions to Einstein's equations, and that these singularities are often accompanied by causal horizons that prevent events inside the horizon from affecting events on the other side. In studying the properties of horizons and singularities, Hawking noticed that the area of a black hole event horizon can never decrease, provided that the weak energy condition was satisfied [10]. Bekenstein later noted that the fact that area is an extensive, non-decreasing property of the black hole is reminiscent of entropy in thermodynamical systems, and proposed that the area be identified as the entropy of the black hole [2]. Following this line of reasoning, four laws of black hole mechanics were formulated as strong analogies with the laws of thermodynamics, and identifying the area with entropy and the surface gravity κ with the black hole's temperature [1]. This analogy was solidified by Hawking's discovery that quantum fields in the presence of a black hole horizon radiate thermally with a temperature given by $\hbar\kappa/2\pi$ [11].

Remarkably, it was soon discovered that the presence of thermal radiation in spacetime is not confined to situations where a black hole is present. Unruh found that in flat, Minkowski space, a uniformly accelerated observer would perceive a similar thermal radiation spectrum due to the presence of a causal horizon [22]. The thermal radiation arises in this case not due to gravitational fields, since there are none in Minkowski space, but instead due to the existence of a horizon in the spacetime. This motivates the idea that there may be a fundamentally thermal description spacetime.

Further progress was made in this interpretation by Wald, who noted that black hole entropy is not specific to gravitational theories satisfying the Einstein equations. Instead, an expression for entropy can be derived for any stationary spacetime with a Killing horizon, and it is simply the Noether charge associated with the Killing vector on the horizon [24].

The relationship between entropy and horizons has motivated the development of the "holographic principle," which in rough terms states that the true degrees of freedom in a gravitational system lie on the boundary of the region in question [5]. This allowed for a formulation of a bound on the number of degrees of freedom, and hence the entropy, in a region of spacetime [4]. The entropy bound given by the holographic principle is satisfied in spacetimes that are considered physically interesting, but nevertheless general relativity does not require that this principle hold. It is thought that any proposed quantum theory of gravity must imply the holographic principle as one of its features.

An important step in interpreting black hole entropy and spacetime thermodynamics came from Jacobson's paper on the Einstein equation of state [14]. In this paper, Jacobson derived the Einstein equations by assuming that the entropy of a Rindler horizon is proportional to its area. The result of this paper is encouraging since black hole entropy no longer seems to be merely a happy coincidence of the theory. Rather, entropy arises in spacetime thermodynamics as a result of an underlying thermal structure. This analysis has been generalized to nonequilibrium settings [9, 7], where it was noted that shearing effects on horizons tend to lead to entropy production. As was noted in [7], since shearing effects in general relativity are governed by the Weyl curvature tensor, it is likely that this tensor has an important role in quantifying gravitational entropy. Indeed, the Weyl curvature conjecture, attributed to Penrose [19], postulates that areas where Weyl curvature dominates the

Ricci curvature correspond to regions of high spacetime entropy.

Taking the viewpoint advocated by Jacobson that spacetime is an emergent, rather than fundamental, phenomenon, one would expect it to exhibit additional features known to be generic in thermodynamical systems. In particular, macroscopic variables in thermodynamics tend to fluctuate around their mean values even when at thermal equilibrium. These fluctuations give rise to dissipation of work into the internal degrees of freedom of the system, by virtue of the fluctuation-dissipation theorem [15]. The link provided by this theorem between macroscopic fluctuations and the microstructure of the system makes this a powerful probe of these microscopic interactions.

There has been some work applying fluctuation theorems to various aspects of spacetime thermodynamics. Sciama and Candelas applied the fluctuation-dissipation theorem to shearing black hole horizons, and obtained an explanation for why black hole horizons exhibit shear viscosity [6]. Pavon and Rubi have employed simple thermodynamical models to estimate the equilibrium [17] and nonequilibrium [18] fluctuations of macroscopic black hole variables. More recently, Iso and Okazawa have applied new advances in nonequilibrium thermodynamics to explore the effects of black hole horizon fluctuations [13].

Another line of inquiry has involved employing phenomenological stochastic models to spacetime. By assuming that fluctuations exist in the stress-energy tensor or the spacetime curvature tensors, one can view the equations governing motions of particles as Langevin equations, with a stochastic variable modeling the fluctuations. Moffat applied this reasoning to congruences of timelike and null geodesics to provide a means of avoiding singularities and event horizons in certain spacetimes [16]. In a different application of this idea, Borgman and Ford [3] and Thompson and Ford [21] estimated the effects of curvature fluctuations on the focusing and spectral broadening of light rays.

This thesis seeks to examine spacetime fluctuations in a similar manner. The congruence evolution equations have long been a powerful tool for examining properties of a spacetime manifold; this thesis presents a program for applying these equations to fluctuations. Previous works have focused largely on the Raychaudhuri equation, and have neglected all but the lowest order contributions in the evolution equations. Since Chirco and Liberati showed that shearing terms in these equations result in entropy production, neglecting these terms in the congruence equations may mask some important effects in spacetime thermodynamics. This thesis will propose a method for analyzing the congruence evolution equations in the presence of spacetime fluctuations.

This work is organized as follows. In Chapter 2, we review the essential properties of the Rindler spacetime. Rindler space plays a prominent role in local developments of spacetime thermodynamics, so the overview in chapter 2 will be instructive. From there, we develop the theory of timelike and null congruences in Chapter 3, and derive the evolution equations. Then, in Chapter 4, Jacobson's thermodynamical derivation of the Einstein equation of state is presented, as an application of the previous two chapters and in order to further motivate the thermodynamic interpretation of spacetime. In Chapter 5, we present the main results of this thesis, and discuss the further steps that must be taken to complete the analysis of the congruence equations as stochastic equations. Chapter 6 is left for conclusions and discussion. An appendix is included at the end summarizing the notation and conventions used in this work.

Chapter 2

Rindler Spacetime

The Rindler spacetime will play a crucial part in our analysis of the Einstein equation of state in Chapter 4. Additionally, this spacetime bears many of the interesting causal feature of black holes, and hence provides a useful arena in which to understand several concepts arising in black hole thermodynamics. Because of its importance to the rest of this work, we derive some of its central features in this chapter. Parts of this chapter use material presented in Wald [23].

Rindler spacetime is flat, Minkowski space as viewed by a uniformly accelerated observer. Despite the absence of curvature, a family of Rindler observers will perceive a causal horizon, analogous to the horizons that arise in black hole spacetimes. As will be demonstrated, the Rindler observers in fact follow orbits of a Killing vector, and the horizon is a Killing horizon where the Killing vector becomes null. This essential feature of the spacetime will be invoked in our later discussions.

2.1 Uniform acceleration in Minkowski space

We begin our discussion by analyzing the trajectory of a uniformly accelerated observer in Minkowski space. By this, we mean that the observer feels a constant acceleration κ in his frame with respect to his proper time τ. Letting $x^\alpha(\tau)$ denote the coordinates of the observer's trajectory, and u^α the unit tangent vector, we have

$$\kappa = \frac{d^2x^1}{d\tau^2} = \frac{d}{d\tau}u^1 = \frac{d}{dT}(\gamma_v v) \tag{2.1}$$

where T is the time coordinate in the Minkowski lab frame, $v \equiv \frac{dx^1}{dT}$ is the Minkowski frame velocity in the direction of the acceleration, and $\gamma_v \equiv (1 - v^2)^{-(1/2)}$ is the instantaneous boost factor associated with this velocity. Noting that $\frac{d\tau}{dT} = \frac{1}{\gamma_v}$, we find

$$\kappa\gamma_v = \frac{d}{d\tau}\gamma_v v \tag{2.2}$$
$$= \gamma_v\frac{dv}{d\tau} + v^2\gamma_v^3\frac{dv}{d\tau} \tag{2.3}$$

from which it follows that

$$\frac{dv}{d\tau} = \frac{\kappa}{\gamma_v^2}. \tag{2.4}$$

The derivative of equation (2.2) is

$$\frac{d^2}{d\tau^2}(\gamma_v v) = \kappa\gamma_v^3 v\frac{dv}{d\tau}$$
$$= \kappa^2\gamma_v v. \tag{2.5}$$

With the choice of initial conditions that the observer be at rest at $\tau = 0$, the solution to (2.5) is

$$\gamma_v v = \sinh(\kappa\tau). \tag{2.6}$$

Since $\gamma_v = \sqrt{1 + \gamma_v^2 v^2}$, the components of u^α in the lab Minkowski frame are

$$u^\alpha \stackrel{*}{=} (\gamma_v, \gamma_v v, 0, 0) \stackrel{*}{=} (\cosh(\kappa\tau), \sinh(\kappa\tau), 0, 0) \tag{2.7}$$

where the symbol $\stackrel{*}{=}$ denotes equality in a particular coordinate system. Integrating with respect to τ, and specifying the initial condition of $x^\alpha(\tau = 0) \stackrel{*}{=} (0, 1/\kappa, 0, 0)$ gives

$$x^\alpha(\tau) \stackrel{*}{=} (z\sinh(\tau/z), z\cosh(\tau/z), 0, 0) \tag{2.8}$$

where $z \equiv 1/\kappa$ is the initial coordinate distance to the origin in Minkowski space for the observer. In fact, z represents the constant spacetime interval between the origin and the accelerated observer:

$$\Delta s^2 = -(z\sinh(\tau/z))^2 + (z\cosh(\tau/z))^2 = z^2. \tag{2.9}$$

2.2 Rindler Metric

The trajectories of the uniformly accelerated observers can be used to define a new coordinate system for Minkowski space. The original Minkowski coordinates will be labeled (T, X, Y, Z). The coordinates Y and Z will remain the same throughout this analysis, so they will be suppressed for the calculations below.

From the trajectory (2.8), it is apparent that the parameter z labels the trajectories of different distances from the Minkowski origin, or equivalently trajectories of different accelerations. This parameter and the time coordinate $t = \tau/z$, can be taken as the new coordinates defining the Rindler metric. The coordinate transformation is given by (2.8), and the transformation matrix for tensors is

$$\Lambda^{\alpha}{}_{\beta'} \stackrel{*}{=} \frac{\partial(T, X)}{\partial(t, z)} = \begin{pmatrix} z\cosh(t) & \sinh(t) \\ z\sinh(t) & \cosh(t) \end{pmatrix}. \tag{2.10}$$

The Minkowski metric is

$$ds^2 = -dT^2 + dX^2, \tag{2.11}$$

and using the transformation (2.10), the new metric is

$$ds^2 = -z^2 dt^2 + dz^2. \tag{2.12}$$

Since the metric is independent of the coordinate t, we see that $\chi^{\alpha} \stackrel{*}{=} (1, 0, 0, 0)$ defines a Killing vector field for this spacetime. In the original Minkowski coordinates, this vector takes the form

$$\chi^\alpha \stackrel{*}{=} \frac{1}{\kappa}(\cosh(\kappa\tau), \sinh(\kappa\tau), 0, 0) = z u^\alpha. \tag{2.13}$$

Thus, up to an overall change in curve parameter, the uniformly accelerated observers in section 2.1 follow orbits of the boost-generating Killing vector χ^α. Note that as $\kappa \to \infty$, which corresponds to $z \to 0$, the magnitude of χ^α goes to zero. However, in Minkowski coordinates, χ^α is given by

$$\chi^\alpha \stackrel{*}{=} (X, T, 0, 0), \tag{2.14}$$

which is well-behaved everywhere. The limit $z \to 0$ corresponds to the lines $X = \pm T$, which are the null rays originating at the origin of the Minkowski coordinates. These lines define a Killing horizon, which also functions as a past causal horizon of the origin.

Chapter 3

Congruence Evolution Equations

This thesis will make extensive use of congruences in a spacetime manifold, so we devote this chapter to developing the essential equations and notation. Our analysis draws from the treatment in Poisson [20] and Hawking and Ellis [12].

A congruence is a family of non-intersecting curves defined in some subset of our spacetime manifold. An intuitive way to conceptualize a congruence is as a bundle of tightly packed wires. In general we will restrict our attention to curves which are everywhere timelike or everywhere null. For timelike congruences, we will denote the tangent vector to the curves by u^α, and for the null case we will denote it by k^α.

3.1 Timelike Congruences

We begin by discussing timelike congruences. Without loss of generality, we can require that u^α is normalized to -1, so that the curves will be parameterized by proper time t. We select a single curve $\gamma(t)$ in our congruence, and note that at each point along the curve there are three independent spacelike directions orthogonal to u^α.

At the point $\gamma(t_0)$, we define a deviation vector field ξ^α to lie tangent to a curve $\zeta(s)$ that cuts across several other curves in our congruence. This creates a one-parameter family of curves in our congruence, labeled by the coordinates s and t. Since s and t define a coordinate system in this one-parameter family, the tangent vectors to $\gamma(t)$ and $\zeta(s)$ are Lie transported along each other,

$$\pounds_\xi u^\alpha = u^\alpha{}_{;\beta}\xi^\beta - \xi^\alpha{}_{;\beta}u^\beta = 0. \tag{3.1}$$

In particular,

$$\frac{D}{dt}\xi^\alpha = \xi^\alpha{}_{;\beta}u^\beta = u^\alpha{}_{;\beta}\xi^\beta. \tag{3.2}$$

Unfortunately, for non-geodesic motion, if ξ^α was initially orthogonal to u^α, it will not remain so as it is parallel transported along $\gamma(t)$, nor will u^α be a parallel transported tangent vector. We would like to quantify the change in the orthogonal components of ξ^α with respect to the accelerated tangent vector u^α. To do this, we first define the projection tensor

$$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta \tag{3.3}$$

which kills any component of a vector parallel to u^α since $h_{\alpha\beta}u^\beta = 0$. Next, we define a linear operator $\frac{D_F}{dt}$ called the Fermi derivative with the following two properties:

$$\frac{D_F}{dt}A^\alpha = h^\alpha{}_\beta \frac{D}{dt}A^\beta \quad \text{for all } A^\alpha u_\alpha = 0 \tag{3.4}$$

$$\frac{D_F}{dt}u^\alpha = 0 \tag{3.5}$$

To determine an explicit expression for the Fermi derivative, we note that this operator must be linear in its argument. Property (3.4) fixes the action of the Fermi derivative on the space orthogonal to u^α, so the general expression for the Fermi derivative will equal this expression up to the addition of some other operator $W^\alpha{}_\beta$ whose row space is spanned by u_β. The most general expression for such an operator is $W^\alpha{}_\beta = V^\alpha u_\beta$ for some arbitrary V^α. So for A^α orthogonal to u^α, the fact that

$u_\beta A^\beta{}_{;\gamma} = -u_{\beta;\gamma}A^\beta$ yields

$$\frac{D_F}{dt}A^\alpha = \frac{D}{dt}A^\alpha + u^\alpha u_\beta A^\beta{}_{;\gamma}u^\gamma + W^\alpha{}_\beta A^\beta$$
$$= \frac{D}{dt}A^\alpha - A^\beta \dot{u}_\beta u^\alpha + V^\alpha u_\beta A^\beta, \tag{3.6}$$

where we have defined the acceleration vector for our curves $\dot{u}^\alpha \equiv \frac{D}{dt}u^\alpha = u^\alpha{}_{;\beta}u^\beta$. We can fix the value of V^α using 3.5:

$$0 = \frac{D_F}{dt}u^\alpha = \dot{u}^\alpha - u^\beta \dot{u}_\beta u^\alpha + V^\alpha u_\beta u^\beta$$
$$V^\alpha = \dot{u}^\alpha. \tag{3.7}$$

Note that since $u^\beta u_\beta = -1$, we must have that $\dot{u}^\beta$ is orthogonal to u^β. Our final expression for the Fermi derivative for an arbitrary vector v^α is thus,

$$\frac{D_F}{dt}v^\alpha = \frac{D}{dt}v^\alpha + v^\beta(u_\beta \dot{u}^\alpha - \dot{u}_\beta u^\alpha) \tag{3.8}$$

We can extend the action of the Fermi derivative to arbitrary tensors by demanding that it commutes with contractions, that its action on tensor products obey the Leibniz rule,

$$\frac{D_F}{dt}(v^\alpha w^\beta) = w^\beta \frac{D_F}{dt}u^\alpha + u^\alpha \frac{D_F}{dt}w^\beta \tag{3.9}$$

and that its action on scalars be the usual derivative,

$$\frac{D_F}{dt}f = \frac{df}{dt}. \tag{3.10}$$

Finally, we note that inner products between Fermi propagated vectors are preserved as we move along the curve. To see this, take X^α and Y^α such that $\frac{D_F}{dt}X^\alpha =$

$\frac{D_F}{dt}Y^\alpha = 0$. Then

$$\begin{aligned}\frac{d}{dt}(X^\alpha Y_\alpha) &= Y_\alpha \frac{D}{dt}X^\alpha + X^\alpha \frac{D}{dt}Y_\alpha \\ &= Y_\alpha\,(\dot{u}_\beta u^\alpha - u_\beta \dot{u}^\alpha)X^\beta + X^\alpha(\dot{u}^\beta u_\alpha - u^\beta \dot{u}_\alpha)Y_\beta \;= 0. \end{aligned} \tag{3.11}$$

Now from the properties of the Fermi derivative and Equation 3.2, we find that

$$\begin{aligned}\frac{D_F}{dt}(h^\alpha{}_\beta \xi^\beta) &= h^\alpha{}_\beta \frac{D}{dt}(h^\beta{}_\gamma \xi^\gamma) \\ &= h^\alpha{}_\beta \left(h^\beta{}_\gamma u^\gamma{}_{;\delta}\xi^\delta + \xi^\gamma(u^\beta{}_{;\delta}u^\delta u_\gamma + u^\beta u_{\gamma;\delta}u^\delta)\right) \\ &= u^\alpha{}_{;\delta}\left(\xi^\delta + (h^\delta{}_\gamma - g^\delta{}_\gamma)\xi^\gamma\right) \\ &= u^\alpha{}_{;\delta}h^\delta{}_\gamma \xi^\gamma \end{aligned} \tag{3.12}$$

From this expression, we may also derive the deviation equation for the second Fermi derivative of ξ^α. It is

$$\begin{aligned}\frac{D_F^2}{dt^2}(h^\alpha{}_\beta \xi^\beta) &= h^\alpha{}_\beta \frac{D}{dt}(u^\beta{}_{;\delta}h^\delta{}_\gamma \xi^\gamma) \\ &= h^\alpha{}_\beta \left[u^\beta{}_{;\delta\mu}h^\delta{}_\gamma \xi^\gamma u^\mu + u^\beta{}_{;\delta}\left(\frac{D_F}{dt}(h^\delta{}_\gamma \xi^\gamma) + h^\mu{}_\gamma \xi^\gamma(\dot{u}_\mu u^\delta - u_\mu \dot{u}^\delta)\right)\right] \\ &= h^\alpha{}_\beta \left[-R^\beta{}_{\nu\delta\mu}u^\nu u^\mu h^\delta{}_\gamma \xi^\gamma + u^\beta{}_{;\mu\delta}h^\delta{}_\gamma \xi^\gamma u^\mu + (u^\beta{}_{;\delta}u^\delta{}_{;\mu} + \dot{u}_\mu \dot{u}^\beta)h^\mu{}_\gamma \xi^\gamma\right] \\ &= \left(-R^\alpha{}_{\nu\mu\delta}u^\nu u^\delta + h^\alpha{}_\beta \dot{u}^\beta{}_{;\mu} + \dot{u}^\alpha \dot{u}_\mu\right)h^\mu{}_\gamma \xi^\gamma. \end{aligned} \tag{3.13}$$

Equation 3.12 and Equation 3.13 take the same form for every deviation vector field along our curve. Thus, we may define a tensor $B^\alpha{}_\beta \equiv h^\alpha{}_\gamma h^\delta{}_\beta u^\gamma{}_{;\delta}$ that describes the instantaneous change in the deviation vector fields along our curve. Then Equation 3.12 becomes

$$\frac{D_F}{dt}(h^\alpha{}_\beta \xi^\beta) = B^\alpha{}_\beta h^\beta{}_\gamma \xi^\gamma \tag{3.14}$$

and we can use Equation 3.13 to find the dynamical equation for $B^\alpha{}_\beta$:

$$\frac{D_F}{dt}(B^\alpha{}_\beta h^\beta{}_\gamma \xi^\gamma) = (-R^\alpha{}_{\delta\beta\mu}u^\delta u^\mu + h^\alpha{}_\delta \dot{u}^\delta{}_{;\beta} + \dot{u}^\alpha \dot{u}_\beta)h^\beta{}_\gamma \xi^\gamma, \tag{3.15}$$

which gives

$$\left[\frac{D_F}{dt}B^\alpha{}_\beta + B^\alpha{}_\delta B^\delta{}_\beta + R^\alpha{}_{\delta\beta\mu}u^\delta u^\mu - h^\alpha{}_\delta \dot{u}^\delta{}_{;\beta} - \dot{u}^\alpha \dot{u}_\beta\right] h^\beta{}_\gamma \xi^\gamma = 0. \tag{3.16}$$

Since this holds for the orthogonal components of all deviation vectors ξ^γ, the orthogonal components of the term in brackets must vanish identically. We then obtain

$$\frac{D_F}{dt}B^\alpha{}_\beta = -B^\alpha{}_\delta B^\delta{}_\beta - R^\alpha{}_{\delta\beta\mu}u^\delta u^\mu + h^\alpha{}_\delta h_\beta{}^\mu \dot{u}^\delta{}_{;\mu} + \dot{u}^\alpha \dot{u}_\beta. \tag{3.17}$$

To extract the geometrical properties of our congruence from this equation, we decompose $B_{\alpha\beta}$ into three parts: the expansion, given by the trace,

$$\theta = h^{\alpha\beta}B_{\alpha\beta}, \tag{3.18}$$

the shear, coming from the symmetric, traceless part,

$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{3}\theta h_{\alpha\beta} \tag{3.19}$$

and the vorticity, from the antisymmetric part,

$$\omega_{\alpha\beta} = B_{[\alpha\beta]}. \tag{3.20}$$

We can reconstruct the original tensor from the expansion, shear and vorticity,

$$B_{\alpha\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{3}\theta h_{\alpha\beta}. \tag{3.21}$$

These three quantities have an intuitive geometrical interpretation. The expansion θ gives the rate that the volume of a unit ball of deviation vectors is increasing as the curve's parameter increases. If δV is the change in the volume of this unit ball, this quantity is related to θ by

$$\theta = \frac{1}{\delta V}\frac{d}{dt}\delta V. \tag{3.22}$$

The shear measures the tendency for the unit ball to distort toward an ellipsoid. This is most readily seen by going to a diagonal basis for $\sigma_{\alpha\beta}$ and noting that since $\sigma_{\alpha\beta}$ has vanishing trace, at least one of its eigenvalues must be negative and at least one must be positive (assuming that $\sigma_{\alpha\beta}$ does not vanish identically. Hence, the eigenvectors of the shear will be elongated in the direction of positive eigenvalues, and shrunk in the direction of negative eigenvalues.

Finally, the vorticity, as an anti-symmetric tensor, is a generator of rotation for the deviation vectors.

The evolution equations for these quantities come directly from Equation 3.17. Taking the trace, we obtain the well-known Raychaudhuri equation for the evolution of the expansion,

$$\frac{d}{dt}\theta = -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - R_{\alpha\beta}u^\alpha u^\beta + \nabla_\alpha \dot{u}^\alpha \tag{3.23}$$

where $\sigma^2 \equiv \sigma_{\alpha\beta}\sigma^{\alpha\beta}$ and $\omega^2 \equiv \omega_{\alpha\beta}\omega^{\alpha\beta}$. To obtain the last term we have used $\nabla_\alpha \dot{u}^\alpha = \dot{u}^\alpha{}_{;\alpha} + (u_\beta \dot{u}^\beta)_{;\alpha}u^\alpha = h^\alpha{}_\beta \dot{u}^\beta{}_{;\alpha} + \dot{u}^2$.

The shear equation involves the Weyl curvature tensor, which in 4 dimensions is given by,

$$C_{\alpha\delta\beta\mu} \equiv R_{\alpha\delta\beta\mu} + g_{\alpha[\mu}R_{\beta]\delta} + g_{\delta[\beta}R_{\mu]\alpha} + \frac{1}{3}Rg_{\alpha[\beta}g_{\mu]\delta} \tag{3.24}$$

The Weyl tensor has the same symmetries as the Riemann tensor, with the additional property that the trace of any pair of its indices vanishes. Thus, the Weyl tensor is associated with the gravitational degrees of freedom that are not fixed by the Ricci tensor. In particular, it describes the effects of long-range gravitational forces due to nonlocal sources, while the Ricci tensor and Einstein's equations give the effects of local matter.

We first compute the term in Equation 3.17 proportional to the Riemann tensor. It will simplify our analysis to note that each term in the equation is orthogonal to u^α, and hence we may multiply by the projection tensor $h_\alpha{}^\gamma h_\beta{}^\rho$ without changing the expression. This gives

$$
\begin{aligned}
-R_{\alpha\delta\beta\mu}u^\delta u^\mu &= -C_{\alpha\delta\beta\mu}u^\delta u^\mu + h_\alpha{}^\gamma h_\beta{}^\rho \left(g_{\gamma[\mu}R_{\rho]\delta} + g_{\delta[\rho}R_{\mu]\gamma} + \frac{1}{3}R g_{\gamma[\rho}g_{\mu]\delta} \right) u^\delta u^\mu \\
&= -E_{\alpha\beta} + \frac{1}{2}h_\alpha{}^\gamma h_\beta{}^\rho \left(-g_{\gamma\rho}R_{\mu\delta} - g_{\delta\mu}R_{\rho\gamma} + \frac{1}{3}R g_{\gamma\rho}g_{\mu\delta} \right) u^\delta u^\mu \\
&= -E_{\alpha\beta} + \frac{1}{2}\left(-h_{\alpha\beta}R_{\mu\delta}u^\mu u^\delta + h_\alpha{}^\gamma h_\beta{}^\rho R_{\gamma\rho} - \frac{1}{3}R h_{\alpha\beta} \right) \\
&= -E_{\alpha\beta} + \frac{1}{2}\left(-\frac{1}{3}h_{\alpha\beta}R_{\mu\delta}h^{\mu\delta} + h_\alpha{}^\gamma h_\beta{}^\rho R_{\gamma\rho} \right) - \frac{1}{3}h_{\alpha\beta}R_{\mu\delta}u^\mu u^\delta \qquad (3.25)
\end{aligned}
$$

where $E_{\alpha\beta} \equiv C_{\alpha\delta\beta\mu}u^\delta u^\mu$ is the electric part of the Weyl tensor.

Now we plug the definition of σ (3.19) into Equation 3.17. Noting that the last term in (3.25) will cancel with the corresponding term coming from $\frac{d}{dt}\theta$, we have

$$
\begin{aligned}
\frac{D_F}{dt}\sigma_{\alpha\beta} =& \frac{D_F}{dt}B_{(\alpha\beta)} - \frac{1}{3}h_{\alpha\beta}\frac{d}{dt}\theta \\
=& -\sigma_{\alpha\delta}\sigma^\delta_\beta - \omega_{\alpha\delta}\omega^\delta{}_\beta - \frac{2}{3}\theta\sigma_{\alpha\beta} - E_{\alpha\beta} + \frac{1}{2}\left(h_\alpha{}^\gamma h_\beta{}^\delta R_{\gamma\delta} - \frac{1}{3}h_{\alpha\beta}R_{\gamma\delta}h^{\gamma\delta} \right) \\
& - \frac{1}{3}h_{\alpha\beta}\left(\omega^2 - \sigma^2 + \nabla_\gamma \dot{u}^\gamma\right) + h_\alpha{}^\delta h_\beta{}^\gamma \dot{u}_{(\delta;\gamma)} + \dot{u}_\alpha \dot{u}_\beta. \qquad (3.26)
\end{aligned}
$$

Finally, the equation for the vorticity also arises from Equation 3.17,

$$
\frac{D_F}{dt}\omega_{\alpha\beta} = -2\sigma_\alpha{}^\delta\omega_{\delta\beta} - \frac{2}{3}\theta\omega_{\alpha\delta}h^\delta{}_\beta + h_\alpha{}^\delta h_\beta{}^\gamma \dot{u}_{[\delta;\gamma]}. \qquad (3.27)
$$

Since $B_{\alpha\beta}$ is transverse to u^α, it is effectively a spacelike 3-tensor. Thus, equations (3.23), (3.26), and (3.27) can be formulated in terms of the components of these tensors in the space orthogonal to u^α. We define a Fermi propagated orthonormal basis e^α_a, $a = 1, 2, 3$, for this subspace. Since inner products of Fermi propagated vectors remain constant, this basis will remain orthonormal and orthogonal to u^α at all points along our curve. Then for an arbitrary transverse tensor $M_{\alpha\beta}$, its projected 3-tensor is defined by

$$
M_{ab} \equiv M_{\alpha\beta}e^\alpha_a e^\beta_b. \qquad (3.28)
$$

In the projected basis, h_{ab} is used to raise and lower Latin indices, and Fermi derivatives of a 4-tensor become ordinary derivatives with respect to t. Since Raychaudhuri's equation only involved scalar quantities, it remains the same in terms of projected tensors. The shear and vorticity evolution equations become

$$\frac{d}{dt}\sigma_{ab} = -E_{ab} + \frac{1}{2}\left(R_{ab} - \frac{1}{3}h_{ab}R_{cd}h^{cd}\right) - \sigma_{ac}\sigma^{c}{}_{b} - \omega_{ac}\omega^{c}{}_{b} - \frac{2}{3}\theta\sigma_{ab}$$
$$-\frac{1}{3}h_{ab}(\omega^2 - \sigma^2 + \nabla_{\gamma}\dot{u}^{\gamma}) + \dot{u}_{(a;b)} + \dot{u}_a\dot{u}_b, \quad (3.29)$$

$$\frac{d}{dt}\omega_{ab} = -2\sigma_{a}{}^{c}\omega_{cb} - \frac{2}{3}\theta\omega_{ab} + \dot{u}_{[a;b]}. \quad (3.30)$$

3.2 Null Congruences

Having derived the evolution equations for timelike congruences, our next goal will be to repeat the analysis for null congruences. Although the end result will be qualitatively similar to the equations derived in the timelike case, there are two subtleties to address in the null case. The first of these deals with the normalization of the tangent vector k^α. Whereas in the timelike case there was a natural choice to parameterize the curves by proper time so that $u^\alpha u_\alpha = -1$, the null case is different because the length of the tangent vector does not depend on the curve's parameter;

$$k^\alpha k_\alpha = 0 \quad (3.31)$$

even after reparameterizing the curve. The reason this is a problem is that one could rescale the parameter of neighboring curves arbitrarily, making it difficult to talk about the change in the separation of nearby curves as a function of the curve's parameter.

The second subtlety deals with the transverse space to our curves. In the timelike case, this was simply the subspace of spacelike vectors orthogonal to the curve's tangent vector. The null case is different because by virtue of (3.31), k^α is orthogonal to itself. This means that the transverse space of deviation vectors of interest is not simply the space of vectors orthogonal to our curve, since moving along a vector parallel to k^α cannot be considered a deviation. The interesting part of the deviation comes from the part that is not expressible in terms of a translation along k^α.

Abstractly, these independent deviations are described as a quotient space. If H_p represents the orthogonal subspace at a point p along the curve, then $S_p = H_p/\{k^\alpha\}$ is the quotient vector space obtained by forming equivalence classes of vectors in H_p that differ by some multiple of k^α.

In practice, we can represent the space S_p by choosing a representative vector in H_p from each equivalence class. This can be done systematically by introducing a particular basis for the tangent space. One basis vector will be k^α. A second null vector N^α, called the auxiliary null vector, is chosen as the second basis element normalized such that $k^\alpha N_\alpha = -1$. Note that N^α is not uniquely determined by k^α; however, our analysis will show that this ambiguity does not effect the evolution of the quantities of interest. The last two spacelike basis vectors e^α_A, $A = 2, 3$, are chosen to be orthonormal and orthogonal to k^α and N^α.

In this basis, $\{N^\alpha, e^\alpha_2, e^\alpha_3\}$ span the subspace of vectors labeling purely transverse deviations, while $\{k^\alpha, e^\alpha_2, e^\alpha_3\}$ span the the orthogonal subspace H_p. Thus, the space of vectors representing the transverse, orthogonal space S_p is spanned by $\{e^\alpha_2, e^\alpha_3\}$. The conclusion is that S_p is 2-dimensional; our deviation vectors ξ^α for a null congruence will lie in a 2-dimensional vector space, as opposed to the 3-dimensional space for timelike congruences. From the point onward, S_p will refer to both the orthogonal quotient space as well as the space spanned by $\{e^\alpha_A\}$.

Having identified the crucial difference for null congruences, we proceed as before to consider the evolution of a transverse deviation vector ξ^α. At this point we restrict our attention to congruences of null geodesics. This restriction does not detract much from our discussion; in the timelike case an accelerated congruence could be interpreted as the worldlines of observers subjected to some non-gravitaional force, while in the null case the congruence cannot represent a family of observers. Furthermore, we will show that when considering hypersurface orthogonal congruences, which are used in the remainder of this thesis, the null congruence is necessarily geodesic.

The derivation from the point onwards mirrors the procedure for the timelike congruence. Once again, ξ^α will be Lie transported along the geodesic, so that

$$\frac{D}{d\lambda}\xi^\alpha = k^\alpha{}_{;\beta}\xi^\beta, \tag{3.32}$$

where λ is an affine parameter along the geodesic. Next we define a projection tensor for the transverse orthogonal subspace:

$$h_{\alpha\beta} = g_{\alpha\beta} + N_\alpha k_\beta + k_\alpha N_\beta. \tag{3.33}$$

In the following calculations, it will be useful to keep in mind several identities: $N_{\beta;\gamma}N^\beta = k_{\beta;\gamma}k^\beta = k_{\beta;\gamma}k^\gamma = 0$ and $N^\beta{}_{;\gamma}k_\beta = -N^\beta k_{\beta;\gamma}$ due to the normalization of N^α and k^α and the geodesic equation. Also, we set $\xi^\gamma k_\gamma = 0$, which remains constant since k^α is geodesic. The deviation of nearby geodesics can be quantified with a transverse derivative $\frac{D_T}{d\lambda}$ defined in analogy with the Fermi derivative:

$$\frac{D_T}{d\lambda}A^\alpha = h^\alpha{}_\beta\frac{D}{d\lambda}A^\beta \quad \text{for all } A^\alpha \in S_p, \tag{3.34}$$

$$\frac{D_T}{d\lambda}k^\alpha = \frac{D_T}{d\lambda}N^\alpha = 0. \tag{3.35}$$

Property (3.34) fixes the action of $\frac{D_T}{d\lambda}$ up to the addition of a linear operator $W^\alpha{}_\beta$ whose null space is S_p. This operator can be expressed generally as $W^\alpha{}_\beta = V^\alpha k_\beta + U^\alpha N_\beta$. Then for $A^\alpha \in S_p$,

$$\begin{aligned}\frac{D_T}{d\lambda}A^\alpha &= \frac{D}{d\lambda}A^\alpha + (N^\alpha k_\beta + N_\beta k^\alpha)A^\beta{}_{;\gamma}k^\gamma + W^\alpha{}_\beta A^\beta \\ &= \frac{D}{d\lambda}A^\alpha - N_{\beta;\gamma}k^\gamma k^\alpha A^\beta + (V^\alpha k_\beta + U^\alpha N_\beta)A^\beta.\end{aligned} \tag{3.36}$$

Property (3.35) determines the values of V^α and U^α:

$$0 = \frac{D_T}{d\lambda}k^\alpha = -N_{\beta;\gamma}k^\gamma k^\alpha k^\beta - U^\alpha \quad \Rightarrow \quad U^\alpha = 0, \tag{3.37}$$

$$0 = \frac{D_T}{d\lambda}N^\alpha = N^\alpha{}_{;\gamma}k^\gamma - N_{\beta;\gamma}K^\gamma k^\alpha N^\beta - V^\alpha \quad \Rightarrow \quad V^\alpha = \dot{N}^\alpha. \tag{3.38}$$

Hence, the transverse derivative operator is given by

$$\frac{D_T}{d\lambda}v^\alpha = \frac{D}{d\lambda}v^\alpha + v^\beta(\dot{N}^\alpha k_\beta - k^\alpha \dot{N}_\beta) \tag{3.39}$$

The action of the transverse derivative extends to tensors by requiring that it commute with contractions and obey the Leibniz rule, and to scalars by requiring that it reduce to the ordinary derivative. It is also apparent from the similarity of this expression to the Fermi derivative (3.8) that transverse transport of vectors preserves inner products.

Noting the expression for the derivative of the projection tensor,

$$\frac{D}{d\lambda}h^\beta{}_\gamma = k_\gamma \dot{N}^\beta + \dot{N}_\gamma k^\beta, \tag{3.40}$$

we compute the transverse derivative of the projected deviation vector,

$$\begin{aligned}\frac{D_T}{d\lambda}(h^\alpha{}_\beta\xi^\beta) &= h^\alpha{}_\beta\frac{D}{d\lambda}(h^\beta{}_\gamma\xi^\gamma)\\ &= h^\alpha{}_\beta\left(h^\beta{}_\gamma k^\gamma{}_{;\delta}\xi^\delta + \xi^\gamma(\dot{N}^\beta k_\gamma + \dot{N}_\gamma k^\beta)\right)\\ &= h^\alpha{}_\gamma k^\gamma{}_{;\delta}h^\delta{}_\beta\xi^\beta.\end{aligned} \tag{3.41}$$

The second transverse derivative is

$$\begin{aligned}\frac{D_T^2}{d\lambda^2}(h^\alpha{}_\beta\xi^\beta) &= h^\alpha{}_\beta\frac{D}{d\lambda}(h^\beta{}_\gamma k^\gamma{}_{;\delta}h^\delta{}_\mu\xi^\mu)\\ &= h^\alpha{}_\gamma\left[k^\gamma{}_{;\delta\nu}k^\nu h^\delta{}_\mu\xi^\mu + k^\gamma{}_{;\delta}\left(\frac{D_T}{d\lambda}(h^\delta{}_\mu\xi^\mu) + h^\nu{}_\mu\xi^\mu(k^\delta\dot{N}_\nu - \dot{N}^\delta k_\nu\right)\right]\\ &= h^\alpha{}_\gamma\left[-R^\gamma{}_{\rho\nu\delta}k^\rho k^\delta + k^\gamma{}_{;\delta\nu}k^\delta + k^\gamma{}_{;\delta}k^\delta{}_{;\nu}\right]h^\nu{}_\mu\xi^\mu\\ &= -h^\alpha{}_\gamma R^\gamma{}_{\rho\nu\delta}h^\nu{}_\mu\xi^\mu.\end{aligned} \tag{3.42}$$

The transverse tensor $B^\alpha{}_\beta \equiv h^\alpha{}_\gamma h^\delta{}_\beta k^\gamma{}_{;\delta}$ once again gives the instantaneous change in the transverse deviation vector,

$$\frac{D_T}{d\lambda}(h^\alpha{}_\beta\xi^\beta) = B^\alpha{}_\beta h^\beta{}_\gamma\xi^\gamma, \tag{3.43}$$

and its evolution equation is found using Equation 3.42:

$$\frac{D_T}{d\lambda}(B^\alpha{}_\beta h^\beta{}_\gamma \xi^\gamma) = -h^\alpha{}_\mu R^\mu{}_{\rho\beta\delta} k^\rho k^\delta h^\beta{}_\gamma \xi^\gamma, \tag{3.44}$$

implying

$$\left[\frac{D_T}{d\lambda}B^\alpha{}_\beta + B^\alpha{}_\gamma B^\gamma{}_\beta + h^\alpha{}_\mu R^\mu{}_{\rho\beta\delta} k^\rho k^\delta\right] h^\beta{}_\gamma \xi^\gamma = 0 \tag{3.45}$$

which holds for all transverse deviation vectors. Thus, the transverse components for the term in brackets vanish identically, giving

$$\frac{D_T}{d\lambda}B^\alpha{}_\beta = -B^\alpha{}_\gamma B^\gamma{}_\beta - h^\alpha{}_\mu h^\gamma{}_\beta R^\mu{}_{\rho\gamma\delta} k^\rho k^\delta \tag{3.46}$$

As before, $B_{\alpha\beta}$ can be separated into irreducible parts. The definitions for θ and $\omega_{\alpha\beta}$ are the same as in (3.18) and (3.20), and the definition of $\sigma_{\alpha\beta}$ requires only a slight modification,

$$\sigma_{\alpha\beta} = B_{(\alpha\beta)} - \frac{1}{2}\theta h_{\alpha\beta} \tag{3.47}$$

since $h^\alpha{}_\alpha = 2$ in the null case, while $h^\alpha{}_\alpha = 3$ in the timelike case. This is a manifestation of the fact that the transverse space S_p is two dimensional. $B_{\alpha\beta}$ is reconstructed from these quantities via the relation

$$B_{\alpha\beta} = \sigma_{\alpha\beta} + \omega_{\alpha\beta} + \frac{1}{2}\theta h_{\alpha\beta}. \tag{3.48}$$

The interpretation of θ is slightly different in the null case. Since the space of transverse deviation vectors is 2-dimensional, θ measure the fractional rate of change in the unit circle of deviation vectors as the curve's affine parameter increases,

$$\theta = \frac{1}{\delta A}\frac{d}{d\lambda}\delta A. \tag{3.49}$$

The action of $\sigma_{\alpha\beta}$ and $\omega_{\alpha\beta}$ are the same as before, just operating on the 2 dimensional space of deviation vectors.

To compute the evolution equation for θ, first note that $h^\alpha{}_\mu h^\gamma{}_\alpha R^\mu{}_{\rho\gamma\delta} k^\rho k^\delta =$

$h^\gamma{}_\mu R^\mu{}_{\rho\gamma\delta} k^\rho k^\delta = R^\mu{}_{\rho\mu\delta} k^\rho k^\delta$. Taking the trace of Equation 3.47 yields Raychaudhuri's equation for null geodesics,

$$\frac{d\theta}{d\lambda} = -\sigma^2 + \omega^2 - \frac{1}{2}\theta^2 - R_{\rho\delta} k^\rho k^\delta. \tag{3.50}$$

For the shear equation, we again substitute the Weyl tensor (3.24) for the curvature term, yielding

$$\begin{aligned}\frac{D_T}{d\lambda}\sigma_{\alpha\beta} = &- \sigma_{\alpha\gamma}\sigma^\gamma{}_\beta - \omega_{\alpha\gamma}\omega^\gamma{}_\beta - \theta\sigma_{\alpha\beta} - h_\alpha{}^\mu h_\beta{}^\gamma \left(C_{\mu\rho\gamma\delta} + \frac{1}{2} g_{\mu\gamma} R_{\delta\rho} \right) k^\rho k^\delta \\ &- \frac{1}{2} h_{\alpha\beta}(\omega^2 - \sigma^2 - R_{\delta\rho} k^\delta k^\rho) \\ = &- \sigma_{\alpha\gamma}\sigma^\gamma{}_\beta - \omega_{\alpha\gamma}\omega^\gamma{}_\beta - \theta\sigma_{\alpha\beta} - h_\alpha{}^\mu h_\beta{}^\gamma C_{\mu\rho\gamma\delta} k^\rho k^\delta - \frac{1}{2} h_{\alpha\beta}(\omega^2 - \sigma^2). \end{aligned} \tag{3.51}$$

Finally, the vorticity equation is

$$\frac{D_T}{d\lambda}\omega_{\alpha\beta} = -2\sigma_\alpha{}^\delta \omega_{\delta\beta} - \theta\omega_{\alpha\delta} h^\delta{}_\beta. \tag{3.52}$$

Equations (3.51) and (3.52) can be re-written using their components in terms of the basis vectors $\{e^\alpha_A\}$, using h_{AB} to raise and lower indices, and noting $e^\alpha_0 = k^\alpha$,

$$\frac{d}{d\lambda}\sigma_{AB} = -\sigma_{AC}\sigma^C{}_B - \omega_{AC}\omega^C{}_B - \theta\sigma_{AB} - C_{A0B0} - \frac{1}{2} h_{AB}(\omega^2 - \sigma^2) \tag{3.53}$$

$$\frac{d}{d\lambda}\omega_{AB} = -2\sigma_A{}^C \omega_{CB} - \theta\omega_{AC} h^C{}_B. \tag{3.54}$$

Chapter 4

The Einstein Equation of State

The mathematical framework developed in previous sections finds many applications in gravitational physics. This chapter explores a particular application which is foundational to the concept of spacetime thermodynamics. This application is the derivation of Einstein's equations as a thermodynamical equation of state.

The idea of the Einstein equation of state is due to Jacobson [14]. In this work, Jacobson considers local causal horizons associated with an arbitrary point p in the spacetime manifold. The equivalence principle allows one to approximate a local neighborhood of p as a Rindler spacetime, with a causal horizon generated null geodesics passing through p and extending to the past of p. This horizon has an associated spacelike cross section, which, following the example of black hole thermodynamics, is associated with an entropy proportional to its area. Raychaudhuri's equation is used to relate the change in horizon area, and hence entropy, to a heat flux across the horizon coming from the stress-energy tensor. Einstein's equation thus arises as an equation of state between the Ricci curvature $R_{\alpha\beta}$ and the stress-energy tensor $T_{\alpha\beta}$.

The formal derivation proceeds as follows. At any point p in spacetime, the equivalence principle allows one to approximate the spacetime near p as being locally Minkowski. In this local approximation, there exist causal horizons extending in the direction of any null vector k^α at p. These causal horizons simply correspond to the past light cone of the point. The causal horizon $\mathcal{H}$ under consideration should actually be thought of as a tube-like null hypersurface generated by k^α orthogonal to a small 2-dimensional spacelike patch $\mathcal{B}$ that contains p. Since the entropy is to be identified with an area element of this surface, we must impose a local equilibrium condition on the generators k^α of the horizon. Since the expansion θ gives the change in area of the spacelike cross sections along the null congruence generating $\mathcal{H}$, the equilibrium condition is that $\theta = 0$ on $\mathcal{B}$.

This space has a boost generating Killing vector χ^α which is null on $\mathcal{H}$ and vanishes at p. According to the Unruh effect [22], uniformly accelerated observers who perceive $\mathcal{H}$ as a causal horizon will assign the horizon a temperature $T = \hbar\kappa/2\pi$, where κ is the magnitude of the observer's acceleration. Motivated by the examples from black hole thermodynamics, we also postulate that the entropy S of the horizon should be propotional to its cross-sectional area A:

$$S = \eta A. \tag{4.1}$$

We saw before that this area is related to the expansion θ of the generating congruence, we have

$$\theta = \frac{1}{\delta A}\frac{d}{d\lambda}dA, \tag{4.2}$$

where λ is an affine parameter along the curve. The Killing vector χ^α does not affinely parameterize the generators of the horizon, but is related to the affinely parameterized vector k^α by

$$\chi^\alpha = -\kappa\lambda k^\alpha. \tag{4.3}$$

Now we have identified a temperature and an entropy, so the Clausius relation states that a change in entropy δS can be related to a heat flux δQ as follows,

$$\delta Q = T\delta S = \frac{\hbar\kappa}{2\pi}\eta\delta A. \tag{4.4}$$

This heat flux can be taken to be the boost energy flowing across the horizon. Thus, it is given by

$$\delta Q = \int_{\mathcal{H}} T_{\alpha\beta} \chi^{\alpha} d\Sigma^{\beta}, \tag{4.5}$$

where $d\Sigma^{\beta}$ is the directed area element of the horizon cross section. The integral is performed over the horizon for some affine parameter distance λ. This equation can be written in terms of affine quantities by

$$\delta Q = -\kappa \int_{\mathcal{H}} T_{\alpha\beta} k^{\alpha} d\lambda k^{\beta} dA. \tag{4.6}$$

Using Equation (3.49), the right hand side of Equation (4.4) can be expressed using equation (4.2) as

$$T\delta S = \frac{\hbar\kappa}{2\pi}\eta \int_{\mathcal{H}} \theta dA. \tag{4.7}$$

Since we demanded $\theta(\lambda = 0) = 0$, the value of θ can be approximated by its Taylor expansion,

$$\theta \approx \theta(\lambda = 0) + \lambda \frac{d\theta}{d\lambda} \tag{4.8}$$

Then employing Raychaudhuri's equation, and neglecting θ^2 and σ^2 terms, we have

$$\theta = -\lambda R_{\alpha\beta} k^{\alpha} k^{\beta}, \tag{4.9}$$

so that equation (4.4) now reads

$$\int_{\mathcal{H}} T_{\alpha\beta} k^{\alpha} k^{\beta} \lambda d\lambda dA = \frac{\hbar\eta}{2\pi} \int_{\mathcal{H}} R_{\alpha\beta} k^{\alpha} k^{\beta} \lambda d\lambda dA. \tag{4.10}$$

Since we can form local Rindler horizons in all null direction around a point p, this leads to the equality

$$\frac{2\pi}{\hbar\eta} T_{\alpha\beta} = R_{\alpha\beta} + C g_{\alpha\beta} \tag{4.11}$$

for some scalar C. We can fix C by demanding local conservation of energy, $T^{\alpha\beta}{}_{;\beta} = 0$, which by the contracted Bianchi identities gives

$$C = -\frac{1}{2} R + \Lambda \tag{4.12}$$

for some constant Λ. The equation of state we arrive at is

$$R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} + \Lambda g_{\alpha\beta} = \frac{2\pi}{\hbar\eta}T_{\alpha\beta} \tag{4.13}$$

which is Einstein's equation with $G = \dfrac{1}{4\hbar\eta}$ and a cosmological constant Λ.

Thus, given the basic thermodynamic assumptions arising from black hole thermodynamics and the Unruh effect, the Einstein equations arise as an equation of state relating curvature to matter.

Eling and Jacobson [9] and Chirco and Liberati [7] have extended this analysis to no longer assume that the σ^2 term in the Raychaudhuri equation is negligible. When that assumption is made, the shear terms from the Raychaudhuri equation give rise to entropy production, as would be expected in a nonequilibrium setting. This is highly suggestive of a relationship between gravitational entropy and shearing in spacetime curvature.

Chapter 5

Stochastic Congruence Equations

Jacobson's derivation of the Einstein equations as thermodynamic equations of state motivates the viewpoint that gravity may be an emergent phenomenon [14]. Although we lack a detailed description of spacetime microstructure, we can still ask questions about its macroscopic properties taken as a thermodynamic system. One property common to all thermodynamic systems is that the values of macroscopic quantities fluctuate around their mean. These fluctuations arise due to essentially random changes in the microscopic degrees of freedom of the system, and as a result the statistical properties of fluctuations are closely related to the system's microstructure.

A classic example of determining microscopic properties from macroscopic fluctuations comes from Einstein's analysis of Brownian motion [8]. In this analysis, the diffusion constant for particles in a gas was related to the linear response of particle velocity to applied force. The relationship is drawn by considering the effects of fluctuations due to stochastic forces applied to the particle, and is a well-known example of the more general fluctuation-dissipation theorem [15].

Fluctuation theory allows a thermodynamic description of a system in a nonequilibrium setting. In such a setting, one expects that the entropy of the system would increase and energy would dissipate as the system approaches equilibrium. The precise description of exactly how much entropy increases in nonequilibrium processes is in general difficult to formulate; however, a number of techniques exist for analyzing near equilibrium processes and linear responses of the system.

If we consider gravity as an emergent, thermodynamical effect, we would expect to observe fluctuations in quantities associated with the gravitational field. The mean values of the metric and curvature tensors would still be governed by Einstein's equations, but we would expect to observe nonzero correlations of the higher moments of these quantities. In this chapter, we present a program for analyzing these fluctuations in terms of congruences of curves in the spacetime. Starting with the congruence evolution equations, which typically are viewed as geometrical identities, we consider a different approach where these are dynamical equations subjected to fluctuations in the spacetime curvature.

Below we write the linear response of the congruence equations to fluctuating terms. From there, one could proceed by promoting the curvature perturbation terms to stochastic variables, and analyze the equation as a Langevin equation. This would hopefully lead to an application of the fluctuation-dissipation theorem to the system. In addition, one would like to formulate the Onsager's relations for the variables, and analyze the linear response of the system in terms of transport coefficients. For this, it will be necessary to identify the appropriate macroscopic variables and associated fluxes.

Chirco and Liberati [7] noted that shear terms in the congruence equations lead to dissipative effects in the derivation of the Einstein equation of state. Below, we keep this result in mind, and consider not only the Raychaudhuri equation, but also the shear equation for the congruence under consideration. We expect that the shear variables of the system would relate to nonequilibrium effects and entropy production. The Weyl tensor plays a prominent role in these equations, and further analysis of these stochastic equations may shed light on the Weyl curvature conjecture [19].

The setup for a congruence of timelike curves goes as follows. We let u^α denote the tangent vector for our congruence. For this analysis, we will consider our congruence to be irrotational, and hence u^α will be hypersurface orthogonal at all points. Doing so simplifies our analysis by setting $\omega_{\alpha\beta} = 0$. It also allows us to formulate our equations in terms of intrinsic properties of the hypersurfaces.

Recall that for a congruence we are interested in the separation of nearby curves as we advance the curve's parameter. Since our curves are hypersurface orthogonal, we can characterize the congruence in terms of its expansion θ and shear $\sigma_{\alpha\beta}$. The evolution of the expansion as we move along a curve is governed by the Raychaudhuri equation,

$$\frac{d\theta}{dt} = -\frac{1}{3}\theta^2 - \sigma_{\alpha\beta}\sigma^{\alpha\beta} - R_{\mu\nu}u^\mu u^\nu + \nabla_\alpha \dot{u}^\alpha \tag{5.1}$$

where $\dot{u}^\alpha \equiv u^\beta \nabla_\beta u^\alpha$ is the spacelike vector that describes the acceleration of the curves in our congruence. For geodesics, this term vanishes. It will also be convenient to define the scalar $Q = R_{\mu\nu}u^\mu u^\nu$. The equation for the shear is

$$\frac{D_F}{dt}\sigma_{\alpha\beta} = -E_{\alpha\beta} + \frac{1}{2}(h_\alpha{}^\gamma h_\beta{}^\delta R_{\gamma\delta} - \frac{1}{3}h_{\alpha\beta}R_{\gamma\delta}h^{\gamma\delta}) - \sigma_{\alpha\gamma}\sigma^\gamma{}_\beta - \frac{2}{3}\theta\sigma_{\alpha\beta} + \frac{1}{3}h_{\alpha\beta}\sigma^2 + \kappa_{\alpha\beta}. \tag{5.2}$$

Here, $E_{\alpha\beta} = C_{\alpha\gamma\beta\delta}u^\gamma u^\delta$ is the electric part of the Weyl tensor which describes tidal forces due to nonlocal gravitational sources, $h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$ is the projection tensor onto our hypersurfaces, and $\kappa_{\alpha\beta} = h_\alpha{}^\gamma h_\beta{}^\delta \dot{u}_{(\gamma;\delta)} + \dot{u}_\alpha \dot{u}_\beta - \frac{1}{3}h_{\alpha\beta}\nabla_\gamma \dot{u}^\gamma$ describes the contribution of acceleration to the shear.

We note that Equation 5.2 is tensorially valid; however, each term in the equation is purely transverse, i.e. orthogonal to u^α. We may thus write this equation in terms of intrinsic 3-tensors defined on our hypersurfaces orthogonal to the congruence. To do this, we let e^α_a denote three orthogonal basis vectors in our hypersurface. Then we may define our induced metric

$$h_{ab} = g_{\alpha\beta}e^\alpha_a e^\beta_b = h_{\alpha\beta}e^\alpha_a e^\beta_b \tag{5.3}$$

and extrinsic curvature

$$K_{ab} = u_{\alpha;\beta}e^\alpha_a e^\beta_b. \tag{5.4}$$

We also define the intrinsic covariant derivative

$$V_{a|b} = V_{\alpha;\beta}\, e^\alpha_a e^\beta_b \tag{5.5}$$

which corresponds with the covariant derivative with respect to the induced metric

h_{ab}. The intrinsic curvature tensors of the hypersurface can be expressed in terms of the projected Riemann curvature and the extrinsic curvature. We have for the intrinsic Riemann tensor,

$$^3R_{abcd} = R_{\alpha\beta\gamma\delta}e^\alpha_a e^\beta_b e^\gamma_c e^\delta_d + K_{ad}K_{bc} - K_{ac}K_{bd}, \tag{5.6}$$

and taking the trace of this gives the intrinsic Ricci tensor

$$^3R_{ab} = R_{ab} + K_a{}^c K_{cb} - KK_{ab}, \tag{5.7}$$

where $K \equiv K^a{}_a$ and R_{ab} is the projected Ricci 4-tensor. Finally, our expression for intrinsic scalar curvature is

$$^3R = h^{ab}R_{ab} + K_{ab}K^{ab} - K^2. \tag{5.8}$$

Also, since $\sigma_{\alpha\beta}$, $E_{\alpha\beta}$ and $\kappa_{\alpha\beta}$ are transverse tensors, their projections σ_{ab}, E_{ab} and κ_{ab} are well-defined.

From its definition (5.4), we see that the extrinsic curvature K_{ab} is exactly the object that gives the evolution of a deviation vector along a curve in our congruence. Hence, we can relate it to the expansion and shear as follows:

$$K_{ab} = \sigma_{ab} + \frac{1}{3}\theta h_{ab} \tag{5.9}$$

Using these definitions, we are ready to write our shear equation in terms of intrinsic quantities.

$$\begin{aligned}\frac{d}{dt}\sigma_{ab} =& -E_{ab} + \frac{1}{2}\left(R_{ab} - \frac{1}{3}(R^a{}_a + Q)h_{ab}\right) - \sigma_{ac}\sigma^c{}_b - \frac{2}{3}\theta\sigma_{ab} + \frac{1}{3}h_{ab}\sigma^2 + \kappa_{ab} \\ =& -E_{ab} + \frac{1}{2}\left({}^3R_{ab} + KK_{ab} - K_a{}^cK_{cb} - \frac{1}{3}({}^3R + Q + K^2 - K_{cd}K^{cd})h_{ab}\right) - \\ & \sigma_{ac}\sigma^c{}_b - \frac{2}{3}\theta\sigma_{ab} + \frac{1}{3}h_{ab}\sigma^2 + \kappa_{ab}\end{aligned}$$

$$\frac{d}{dt}\sigma_{ab} = -E_{ab} + \frac{1}{2}\left({}^3R_{ab} - \frac{1}{3}({}^3R + Q)h_{ab}\right) - \frac{3}{2}\sigma_{ac}\sigma^c{}_b - \frac{1}{2}\theta\sigma_{ab} + \frac{1}{2}h_{ab}\sigma^2 + \kappa_{ab}. \quad (5.10)$$

This for of the shear equation, along with Equation 5.1, will be the starting point for analyzing the effects of fluctuations on our geometry.

We now consider small fluctuations in our curvature, and write equations for their effect on the expansion and shear. Denoting these curvature fluctuations by δE_{ab}, $\delta^3 R_{ab}$, $\delta^3 R$ and δQ, the corresponding fluctuations $\delta\theta$ and $\delta\sigma_{ab}$ obey the equations,

$$\frac{d}{dt}\delta\theta = -\delta Q - \frac{2}{3}\theta\delta\theta - 2\sigma_{ab}\delta\sigma^{ab} + \dot{u}^\alpha\delta\dot{u}_\alpha + h^{\alpha\beta}\nabla_\beta\delta\dot{u}_\alpha \quad (5.11)$$

$$\begin{aligned}\frac{d}{dt}\delta\sigma_{ab} = -\delta E_{ab} + \frac{1}{2}\left(\delta^3 R_{ab} - \frac{1}{3}h_{ab}(\delta^3 R + \delta Q)\right) + \delta\kappa_{ab} + \\ -3\sigma^c{}_{(a}\delta\sigma_{b)c} - \frac{1}{2}\sigma_{ab}\delta\theta - \frac{1}{2}\theta\delta\sigma_{ab} + h_{ab}\sigma^{cd}\delta\sigma_{cd}.\end{aligned} \quad (5.12)$$

Here, we have assumed that fluctuations in the induced metric h_{ab} are small compared to other fluctuating values. Since the curvature fluctuations come from the Riemann tensor which is composed of second derivatives of the metric, it is consistent to have large curvature fluctuations with comparatively small metric fluctuations. At this point, however, this does represent a loss of generality.

Equations 5.11 and 5.12 compose a set of seven linear equations for the fluctuations of the expansion and shear. Since the shear must be a trace free tensor, only six of these equations will be independent. Thus, we may choose any six of the variables $\{\delta\theta, \delta\sigma_{ab}\}$, $a \le b$, and compose a vector $\boldsymbol{\vartheta}$ that satisfies the differential equation,

$$\frac{d}{dt}\boldsymbol{\vartheta} = L\boldsymbol{\vartheta} + \boldsymbol{\gamma} + \boldsymbol{\alpha}. \quad (5.13)$$

Here, L is a 6×6 matrix dependent on the values of θ, σ_{ab} and h_{ab}. The vector $\boldsymbol{\gamma}$ encapsulates the curvature fluctuations of our system, and depends only on δE_{ab}, $\delta^3 R_{ab}$, $\delta^3 R$, δQ, and h_{ab}. Finally, $\boldsymbol{\alpha}$ gives the contributions of acceleration fluctuations to this equation, and is composed of the quantities $\delta\kappa_{ab}$, $\dot{u}^\alpha$, $\delta\dot{u}^\alpha$ and h_{ab}.

From here, we would like to analyze equation (5.13) and relate the correlation functions of $\boldsymbol{\vartheta}$ to the correlation functions of $\boldsymbol{\gamma}$ and $\boldsymbol{\alpha}$. Then, we would proceed as described at the beginning of this section, and look to derive any interesting relationships among the fluctuating variables.

Since this description involves a timelike congruence, it bears large differences with Jacobson's [14] and Chirco and Liberati's work [7] whose analysis involves null generators of a horizon. Spacetime thermodynamics often require a horizon and null rays, so it would be interesting to repeat this derivation for a congruence of null geodesics. An attractive feature the null congruence equation for shear (3.53) is that the only curvature contribution comes from the Weyl tensor, while the Ricci tensor affects only the Raychaudhuri equation. This may lead to an interpretation of the shear equation where the Weyl tensor leads to nonequilibrium entropy production, and may shed light on the validity of the Weyl curvature conjecture. This analysis is left to future work on this topic.

Chapter 6

Conclusions

The results of black hole and spacetime thermodynamics suggest that the origin of gravity and spacetime could be emergent in nature. If this is the case, the classical variables such as curvature and energy in general relativity would be expected to fluctuate. This work has explored a new way for analyzing thermodynamical fluctuations in spacetime. After rigorously developing the equations for expansion and shear, a paradigm for analyzing this equations stochastically was proposed. In summary, this program will treat the stochastic equations similarly to a Langevin equation, and look to derive several nonequilibrium properties. These include applying the fluctuation-dissipation theorem to the system, formulating Onsager's relations for relevant generalized forces and fluxes, and analyzing the linear response and transport coefficients in this system. A complete analysis of this method has yet to be performed, but it has shown some promising features.

By reinterpreting the geometrical properties of the expansion and shear of a congruence of geodesics, this work represents a new perspective for probing fluctuation phenomena in spacetime. Furthermore, it provides a possible way of examining the separate contributions to the spacetime entropy of the Weyl and Ricci tensors, by separating out the respective evolution equations for the shear and expansion. The form of the stochastic equation (5.13) shows that the linear response to these geometrical variables can be divided neatly into terms depending on the values of σ_{ab} and θ themselves, terms depending only on the curvature tensors, and terms depending only on the acceleration vector of the congruence. This separation is encouraging for analyzing examples of fluctuating spacetimes in the future.

For our analysis, it was taken as an assumption that our congruence is hypersurface orthogonal, and hence the vorticity can be neglected. However, it may make sense to analyze the equation for ω_{ab} as well. Note that the curvature tensors play no part in the evolution of ω_{ab}, except through the indirect influence of σ_{ab} and θ. This may make analyzing the fluctuation properties of the voriticity simpler than the expansion and shear.

As mentioned at the end of Chapter 5, formulating the stochastic equations for a null congruence would also be an interesting line of research. Null congruences are better suited for probing the holographic nature of spacetime, since deviations for null congruences are essentially two dimensional. Since the holographic principle bounds the entropy content of a surface using light sheets [5], which are null congruences, examining the fluctuation properties of these sheets may enhance our understanding of how and why this principle manifests itself in general relativity.

If nothing else, this work provides a new, albeit incomplete, perspective on the link between thermodynamics and the geometry of spacetime. One would hope that such a perspective would help further illuminate the still developing field of spacetime thermodynamics.

Appendix A

Notation and Conventions

Throughout this work, the metric signature employed is $(-,+,+,+)$. Greek indices will take on values $\alpha = 0,1,2,3$, latin indices take on $a = 1,2,3$, and capital latin indices will take on $A = 2,3$. The Riemann tensor is defined such that $-R^{\alpha}{}_{\beta\gamma\delta}V^{\beta} = V^{\alpha}{}_{;\gamma\delta} - V^{\alpha}{}_{;\delta\gamma}$. The Ricci tensor is then $R_{\alpha\beta} = R^{\mu}{}_{\alpha\mu\beta}$, so that Einstein's equations are $R_{\alpha\beta} - \frac{1}{2}Rg_{\alpha\beta} = 8\pi T_{\alpha\beta}$. This work employs units where $c = G = 1$.

$M_{(\alpha\beta)} = \frac{1}{2}(M_{\alpha\beta} + M_{\beta\alpha})$	Symmetrized tensor.
$M_{[\alpha\beta]} = \frac{1}{2}(M_{\alpha\beta} - M_{\beta\alpha})$	Antisymmetrized tensor.
$\overset{*}{=}$	Equality in a particular coordinate system.
$V^2 = V^{\alpha}V_{\alpha}$	Norm squared of a vector.
$M^2 = M_{\alpha\beta}M^{\alpha\beta}$	Tensor scalar.
u^{α}	Timelike unit tangent vector.
k^{α}	Null tangent vector.
$V^{\alpha}{}_{;\beta} = \nabla_{\beta}V^{\alpha}$	Covariant derivative of a vector.
$\dot{u}^{\alpha} = u^{\alpha}{}_{;\beta}u^{\beta}$	Acceleration vector along a curve.
$\frac{D}{dt}V^{\alpha} = V^{\alpha}{}_{;\beta}\,u^{\beta}$	Derivative of V^{α} along a curve.
$\frac{D_F}{dt}V^{\alpha} = \frac{D}{dt}V^{\alpha} + V^{\beta}(u_{\beta}\dot{u}^{\alpha} - \dot{u}_{\beta}u^{\alpha})$	Fermi derivative of V^{α} along a curve.

$\frac{D_T}{dt}V^\alpha = \frac{D}{dt}V^\alpha + V^\beta(\dot{N}^\alpha k_\beta - k^\alpha \dot{N}_\beta)$	Transverse derivative of V^α along a null curve.
$h_{\alpha\beta} = g_{\alpha\beta} + u_\alpha u_\beta$	Projection tensor, timelike curve.
$h_{\alpha\beta} = g_{\alpha\beta} + k_\alpha N_\beta + N_\alpha k_\beta$	Projection tensor, null curve.
$C^{\alpha\delta}{}_{\beta\mu} = R^{\alpha\delta}{}_{\beta\mu} + 2g^{[\alpha}{}_{[\mu}R^{\delta]}{}_{\beta]} + \frac{1}{3}Rg^\alpha{}_{[\beta}g_{\mu]}{}^\delta$	Weyl tensor.
$E_{\alpha\beta} = C_{\alpha\mu\beta\delta}u^\mu u^\delta$	Electric part of the Weyl tensor.

References

[1] J. M. Bardeen, B. Carter, and S. W. Hawking. The four laws of black hole mechanics. *Comm. in Math. Phys.*, 31(2), 1973.

[2] Jacob D. Bekenstein. Black holes and entropy. *Phys. Rev. D*, 7(8), 1973.

[3] J. Borgman and L. H. Ford. Effects of stress tensor fluctuations upon focusing. *Phys. Rev. D*, 70(6), 2004.

[4] Raphael Bousso. A covariant entropy conjecture. *J. High Energy Phys.*, 1999(7), 1999.

[5] Raphael Bousso. The holographic principle. *Rev. Mod. Phys.*, 74:825–874, 2002.

[6] P. Candelas and D. W. Sciama. Irreversible thermodynamics of black holes. *Phys. Rev. Lett.*, 38(23), 1977.

[7] G. Chirco and S. Liberati. Nonequilibrium thermodynamics of spacetime: the role of gravitational dissipation. *Phys. Rev. D*, 81(024016), 2010.

[8] A. Einstein. Investigations on the theory of the brownian movement. *Ann. der Physik*, 17:549–560, 1905.

[9] Christopher Eling and Ted Jacobson. Nonequilibrium thermodynamics of spacetime. *Phys. Rev. Lett.*, 96(121301), 2006.

[10] S. W. Hawking. Gravitational radiation from colliding black holes. *Phys. Rev. Lett.*, 26(21), 1971.

[11] S. W. Hawking. Black hole explosions? *Nature*, 248:30–31, 1974.

[12] S. W. Hawking and G. F. R. Ellis. *The large scale structure of space-time.* Cambridge University Press, Cambridge, 1973.

[13] Satoshi Iso and Susumu Okazawa. Stochastic equations in black hole backgrounds and non-equilibrium fluctuation theorems. *Nucl. Phys. B*, 851:380–419, 2011.

[14] Ted Jacobson. Thermodynamics of spacetime: The Einstein equation of state. *Phys. Rev. Lett.*, 75(7), August 1995.

[15] R. Kubo. The fluctuation-dissipation theorem. *Rep. Prog. Phys.*, 25(255), 1966.

[16] J. W. Moffat. Stochastic gravity. *Phys. Rev. D*, 56(10), 1997.

[17] D. Pavon and J. M. Rubi. Thermodynamical fluctuations of a massive Schwarzschild black hole. *Phys. Lett. A*, 99(5), 1983.

[18] D. Pavon and J. M. Rubi. Nonequilibrium thermodynamic fluctuations of black holes. *Phys. Rev. D*, 37(8), 1988.

[19] R. Penrose. *General Relativity, An Einstein centenary survey*, chapter 12–Singularities and time-asymmetry, pages 581–638. Cambridge University Press, Cambridge, 1979. Edited by S. W. Hawking and W. Israel.

[20] Eric Poisson. *A Relativist's Toolkit: The Mathematics of Black-Hole Mechanics.* Cambridge University Press, Cambridge, 2004.

[21] R. T. Thompson and L. H. Ford. Spectral line broadening and angular blurring due to spacetime geometry fluctuations. *Phys. Rev. D*, 74(2), 2006.

[22] W. G. Unruh. Notes on black-hole evaporation. *Phys. Rev. D*, 14(4), 1976.

[23] Robert M. Wald. *General Relativity.* The University of Chicago Press, Chicago, 1984.

[24] Robert M. Wald. Black hole entropy is the Noether charge. *Phys. Rev. D*, 48(8), 1993.