18.100C. Problem Set 3

Due date: March 9 (Thursday) in lecture or in my office before noon on due date (except for the writing assignment). Late homeworks will be accepted only with a medical note or for some other MIT approved reason. You may work with others, but the final write-up should be entirely your own and based on your own understanding.

Each problem is worth 10 points.

Problem 1: Rudin: ex. 20 page 44

Problem 2: Let ℓ^2 be the set of sequences of real numbers $\underline{a} = \{a_i\}$ for which $\sup\{\sum_{i=1}^{n} a_i^2 : \ n = 1, 2, 3, \dots\} < \infty$.

Define a distance function on ℓ^2 by:

$$d(\underline{a}, \underline{b}) = \left(\sup\{\sum_{i=1}^{n} (a_i - b_i)^2 : \ n = 1, 2, 3, \dots\} \right)^{\frac{1}{2}}.$$

Check that d is well-defined, and that ℓ^2 (with the metric d) is a metric space.

Problem 3: Let ℓ^∞ be the set of bounded sequences of real numbers, i.e., $\underline{a} = \{a_i\}$ such that $\sup\{|a_i| : \ i = 1, 2, 3, \dots\} < \infty$. Define $d(\underline{a}, \underline{b}) = \sup\{|a_i - b_i| : \ i = 1, 2, 3, \dots\}$.

a) Check that ℓ^∞ is a metric space.

b) Show that the unit ball, $\bar{B}(\underline{0}, 1) = \{\underline{a} : \ d(\underline{0}, \underline{a}) \leq 1\}$, is both closed and bounded.

c) Show that the unit ball is not compact. (Therefore, the Heine-Borel theorem is false in ℓ^∞.) *Hint*: Produce an infinite set in $\bar{B}(\underline{0}, 1)$ with no limit point. (For example, find a sequence of points $\underline{x}_n \in \ell^\infty$, $k \geq 1$, such that $d(\underline{x}_n, \underline{x}_m) = 1$, for all $m \neq n$.)

Problem 4: Rudin: ex. 5, page 78.

Problem 5: Rudin: ex. 20, page 82.

Problem 6: Rudin: ex. 23, page 82.

Writing assignment: Due Wednesday, March 15. Exercises 24 and 25, page 82 in Rudin (the completion of a metric space).

The following problems are recommended for additional practice. They should *not* be turned in with the homework and they will not count towards the homework score. Chapter 2: 18, 21; Chapter 3: 2,3,16,17,19.