18.100C. Problem Set 7. Solutions

Problem 1: Rudin: Chapter 6, ex. 3.

The functions β_j are defined as follows:

$$\beta_j = \begin{cases} 0, & x < 0 \\ 1, & x > 0 \end{cases},$$

and $\beta_1(0) = 0$, $\beta_2(0) = 1$, $\beta_3(0) = \frac{1}{2}$.

(a) The claim is that f is β_1-integrable if and only if it is continuous from the right at 0, and in that case $\int f \, d\beta_1 = f(0)$.

Let P be the partition of $[-1, 1]$ given by $P = \{x_0 = -1, x_1 = 0, x_2 = x, x_3 = 1\}$ for some x, $0 < x < 1$. Then $U(P, f, \beta_1) = M_2$, and $L(P, f, \beta_1) = m_2$, where M_2, m_2 are the supremum, respectively infimum, of f on the interval $[0, x]$. If f is continuous from the right at 0, one let $x \to 0$, and so $M_2 \to f(0)$ and $m_2 \to f(0)$. (This is similar to the proof of theorem 6.15 in Rudin.)

Conversely, if the integral exists, for every $\epsilon > 0$, there exists a partition P such that $U(P, f, \beta_1) - L(P, f, \beta_1) < \epsilon$. We can assume $0 \in P$ (if not we take a refinement of P). Let x_j be the first point in the partition P to the right of 0. Then $U(P, f, \beta_1) - L(P, f, \beta_1) = M_j - m_j$. Set $\delta = x_j$. For every x, $0 \le x < \delta$, $|f(x_j) - f(0)| < M_j - m_j < \epsilon$. This verifies the definition of continuity (from the right) at 0.

(b) The statement should be: f is β_2-integrable if and only if f is continuous from the left at 0, and in this case $\int f \, d\beta_2 = f(0)$. The proof is analogous to (a).

(c) The proof is analogous to (a) (and (b)).

(d) It follows directly just by applying (a),(b),(c).

Problem 2: Let f be given by

$$f(x) = \begin{cases} 0, & x \text{ irrational} \\ \frac{1}{n}, & x \text{ rational}, \ x = \frac{m}{n} \end{cases}.$$

(For rational numbers $x = \frac{m}{n}$, we assume the representation is in the lowest terms.)

We will show that f is Riemann integrable on the interval $[0, 1]$, and that $\int_a^b f(x)\, dx = 0$. Since $f(x) \ge 0$, and each subinterval in a partition P of $[0, 1]$ contains irrational numbers, the infimum of f on each subinterval is 0, and therefore $L(P, f) = 0$, and $\underline{\int_a^b} f(x)\, dx = 0$. It remains to show that $\overline{\int_a^b} f(x)\, dx = 0$, or equivalently that, for every $\epsilon > 0$, there exists a partition P such that $U(P, f) < \epsilon$.

Only the rational numbers contribute to $U(P, f)$. The idea is that for every natural number $N > 0$, there exist only finitely many (rational) numbers $x = \frac{m}{n}$ in $[0, 1]$, such that $f(x) = \frac{1}{n} > \frac{1}{N}$. (This is because for every such x, $m \le n < N$.) Denote the number of such rationals by $S(N)$. Let N be large, so that $\frac{1}{N} < \frac{\epsilon}{2}$. Let P be a partition of $[0,1]$, so that the distances Δx_i are all smaller than $\epsilon/2S(N)$.

Partition the set of points P into two subsets A and B. In A put all points x_i in P such that the interval $[x_{i-1}, x_i]$ contains a rational point $x = \frac{m}{n}$ with $\frac{1}{n} > \frac{1}{N}$. The subset B is just the complement of A. Note that there are at most $S(N)$ points in A. Then

$$\begin{aligned} U(P,f) &= \sum_{x_i \in A} M_i \Delta x_i + \sum_{x_j \in B} M_j \Delta x_j \\ &\le \sum_{x_i \in A} \Delta x_i + \sum_{x_j \in B} \frac{1}{N} \Delta x_j \\ &\le S(N) \cdot \frac{\epsilon}{2S(N)} + \frac{1}{N} \\ &< \epsilon/2 + \epsilon/2 = \epsilon. \end{aligned}$$

We have seen this kind of proof in Rudin before: the idea is to partition the sum into two sums such that one sum can be made small by using the suprema, and the other sum by using the total length of the subintervals.

Problem 3: Rudin: Chapter 6, ex. 8.

The series $\sum_{n=1}^{\infty} f(n)$ has nonnegative terms, and so it is convergent if and only if it is bounded above. Denote the Nth partial sum by $s_N = \sum_{n=1}^{\infty} f(n)$.

Define $F(b) = \int_1^b f(x)\, dx$. Since $f(x) \ge 0$, F is increasing. Therefore F has a limit as $b \to \infty$ if and only if F is bounded above. From these two remarks, it suffices to prove that F is bounded above if and only if $\{s_N\}$ is bounded above.

Let $N > 0$ be given. If P is the partition $\{1, 2, \ldots, N\}$ of $[1, N]$, $U(P,f) = \sum_{n=1}^{N-1} f(n)$, and $L(P,f) = \sum_{n=1}^{N-1} f(n+1)$. (We used here the fact that $f(x)$ is decreasing.) In terms of s_N,

$$U(P,f) = s_{N-1}, \quad L(P,f) = s_N - f(1).$$

Clearly $L(P,f) \le F(N) \le U(P,f)$, and so

$$s_N - f(1) \le F(N) \le s_{N-1}.$$

But this double inequality implies that F is bounded if and only if s_N is bounded.

Problem 4: Rudin: Chapter 6, ex. 10.

(a) Consider the function $f : \mathbb{R} \to \mathbb{R}$, $f(x) = e^x$. The second derivative of f is positive, so by a previous homewrok exercise, $f(x)$ is convex. Apply the convexity inequality

$$f(\lambda x + (1-\lambda)y) \le \lambda f(x) + (1-\lambda) f(y),$$

to $\lambda = \frac{1}{p}$ (so $1 - \lambda = \frac{1}{q}$), $x = p \ln u$, and $y = q \ln v$. We get

$$uv = e^{(\frac{1}{p} p \ln u + \frac{1}{q} q \ln v)} \le \frac{1}{p} e^{p \ln u} + \frac{1}{q} e^{q \ln v} = \frac{u^p}{p} + \frac{v^q}{q}.$$

(b) For every x, part (a) implies that

$$\frac{f(x)^p}{p} + \frac{g(x)^q}{q} \geq f(x)g(x).$$

Integrate both sides of this inequality, and by property 6.12 (b), we find that

$$\int_a^b fg \, d\alpha \leq \int_a^b \frac{f^p}{p} \, d\alpha + \int_a^b \frac{g^q}{q} \, d\alpha = \frac{1}{p} + \frac{1}{q} = 1.$$

(c) Set $\nu = \left(\int_a^b |f|^p \, d\alpha\right)^{\frac{1}{p}}$ and $\mu = \left(\int_a^b |g|^q \, d\alpha\right)^{\frac{1}{q}}$. Then the hypothesis of part (b) apply to the functions $\frac{|f|}{\nu}$ and $\frac{|g|}{\mu}$. From (b) we get then $\int_a^b \frac{|f|}{\nu} \frac{|g|}{\mu} \, d\alpha \leq 1$, and equivalently $\int_a^b |fg| \, d\alpha \leq \nu\mu$. To obtain Holder's inequality, we only need to remark that by the result in theorem 6.13(b): $|\int_a^b fg \, d\alpha| \leq \int_a^b |fg| \, d\alpha$.

(d) Since the improper integrals exist, one can just take $lim_{b \to \infty}$ in the inequality in (c).

Problem 5: Rudin: Chapter 6, ex. 16.

For $1 < s < \infty$, the Riemann's zeta function is defined as

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

Since $s > 1$, we know from Chapter 3 that the series is convergent, so this definition makes sense.

(a) Consider the integral $\int_1^\infty \frac{[x]}{x^{s+1}} \, ds$. By the integral test, this improper integral is convergent, since the associated series is just $\zeta(s)$. We want to show that this integral is actually equal to $\frac{1}{s}\zeta(s)$. Set $F(N) = s\int_1^N \frac{[x]}{x^{s+1}} \, dx$, and $s_N = \sum_{n=1}^N \frac{1}{n^s}$.

The calculation is as follows:

$$\begin{aligned}
F(N) &= s \sum_{n=1}^{N-1} \int_n^{n+1} \frac{[x]}{x^{s+1}} dx = s \sum_{n=1}^{N-1} n \int_n^{n+1} \frac{d}{x} x^{s+1} \\
&= s \sum_{n=1}^{N-1} [-\frac{1}{s}\frac{1}{x^s}]_n^{n+1} = \sum_{n=1}^{N-1} \left(\frac{1}{n^{s-1}} - \frac{n}{(n+1)^s}\right) \\
&= \sum_{n=1}^{N-1} \left(\frac{1}{n^{s-1}} - \frac{1}{(n+1)^{s-1}} + \frac{1}{(n+1)^s}\right) \\
&= 1 - \frac{1}{N^{s-1}} + s_N - 1 = s_N - \frac{1}{N^{s-1}}.
\end{aligned}$$

Taking the limit $N \to \infty$, we obtain the result.

(b) This follows from (a) by integration by writing $\frac{[x]}{x^{s+1}} = \frac{1}{x^s} - \frac{x-[x]}{x^{s+1}}$.