18.100C. Problem Set 8

Due date: May 4 (Thursday) in recitation or in my office before 11 on due date. Late homeworks will be accepted only with a medical note or for some other MIT approved reason. You may work with others, but the final write-up should be entirely your own and based on your own understanding.

Each problem is worth 15 points.

Problem 1: This problem constructs an example of a continuous function which is nowhere differentiable. For a real number x, let $\{x\}$ denote the distance of x to the nearest integer. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{\{10^n x\}}{10^n}.$$

(a) Show that the series converges for every $x \in \mathbb{R}$ (and therefore, f is well-defined).

(b) Show that f is continuous at all $x \in \mathbb{R}$.

(c) Prove that for every $x \in \mathbb{R}$, f is not differentiable at x, by showing that the limit

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

does not exist. (Hint: Consider the decimal expansion of x and take $h_m = \pm 10^{-m}$ depending on the m-th digit after the decimal point in the expansion.)

Problem 2: Rudin: Chapter 6, ex. 13.

Problem 3: Rudin: Chapter 7, ex. 4.

The following problems are recommended for additional practice. They should *not* be turned in with the homework and they will not count towards the homework score. Chapter 7: 3, 5, 6, 7.