18.100C. Problem Set 8. Solutions

Problem 1: This problem constructs an example of a continuous function which is nowhere differentiable. For a real number x, let $\{x\}$ denote the distance of x to the nearest integer. Consider the function $f : \mathbb{R} \to \mathbb{R}$, given by the formula

$$f(x) = \sum_{n=0}^{\infty} \frac{\{10^n x\}}{10^n}.$$

(a) Show that the series converges for every $x \in \mathbb{R}$ (and therefore, f is well-defined).

(b) Show that f is continuous at all $x \in \mathbb{R}$.

(c) Prove that for every $x \in \mathbb{R}$, f is not differentiable at x, by showing that the limit

$$\lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

does not exist. (Hint: Consider the decimal expansion of x and take $h_m = \pm 10^{-m}$ depending on the m-th digit after the decimal point in the expansion.)

a) For all $y \in \mathbb{R}$, $0 \le \{y\} < 1$, so $\frac{\{10^n x\}}{10^n} < \frac{1}{10^n}$, and since $\sum_{n=0}^{\infty} \frac{1}{10^n}$ converges, $f(x)$ converges uniformly by Weierstrass' M-test.

b) The function $x \to \{x\}$ is continuous on $\mathbb{R}$ (and periodic with period 1). Then $x \to \{10^n x\}$ is continuous being the composition of two continuous functions. This shows that the series is one of continuous functions, and since it converges uniformly, the sum of the series, $f(x)$, is continuous.

c) Fix $x = a.a_1 a_2 \dots a_m \dots$. Set $h_m = \pm 10^{-m}$ where the choice for + or − will be made later. Then

$$\left| \frac{f(x+h_m) - f(x)}{h_m} \right| = 10^m \left| \sum_{n=0}^{m-1} \frac{\{10^n x + 10^n h_m\} - \{10^n x\}}{10^n} \right|.$$

This is because, if $n \ge m$, $10^n h_m$ is an integer, so $\{10^n (x + h_m)\} = \{10^n x\}$.

For simplicity, denote $\delta_n = \{10^n x + 10^n h_m\} - \{10^n x\}$. For $n < m-1$, $|\delta_n| = |.a_n \dots a_m \dots - .a_n \dots (a_m \pm 1) \dots| \le 10^{-(m-n-1)}$. For $n = m-1$, $|\delta_{m-1}| = |\{.(a_m \pm 1)\dots\} - \{.a_m \dots\}| = 10^{-1}$, if we make the choice of + or − so that both $.(a_m \pm 1)\dots$ and $.a_m \dots$ are on the same side of $\frac{1}{2}$.

In conclusion,

$$\left| \frac{f(x+h_m) - f(x)}{h_m} \right| \ge 10^m (1 - 10^{-1} - 10^{-2} - \dots - 10^{-m+1}) > 10^m (1 - \frac{1}{9}),$$

and this shows that the sequence diverges, and therefore the derivative does not exist.

Problem 2: Rudin: Chapter 6, ex. 13.

(a) This is a straightforward calculation. We just record the result

$$f(x) = \int_x^{x+1} \sin(t^2)dt = \frac{\cos x^2}{2x} - \frac{\cos(x+1)^2}{2(x+1)} - \frac{1}{4}\int_{x^2}^{(x+1)^2} \frac{\cos u}{u^{\frac{3}{2}}}du.$$

Since

$$\left| \int_{x^2}^{(x+1)^2} u^{-\frac{3}{2}}du \right| = \frac{2}{x(x+1)},$$

we conclude that $|f(x)| < \frac{1}{2}(\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x(x+1)}) = \frac{1}{x}$.

(b) Multiply by $2x$ in the previous formula and find that $2xf(x) = \cos(x^2) - \cos(x+1)^2 + r(x)$, where $r(x) = \frac{1}{x}$.

(c) The formula in (b), since $\lim_{x\to\infty} r(x) = 0$, immediately implies that $\limsup_{x\to\infty} xf(x) \leq 1$, and $\liminf_{x\to\infty} xf(x) \geq -1$. In fact equality holds in both these formulas.

(d) The integral converges (see the note about computing the value of this integral). To decide convergence, one can proceed as in (a), but with the integral $\int_x^y \sin(t^2)dt$ and show that $|\int_x^y \sin(t^2)dt| < \frac{1}{x}$.

Problem 3: Rudin: Chapter 7, ex. 4.

(a) Note first that the series is undefined when x is of the form $x = -\frac{1}{m^2}$, for some $m \in \mathbb{Z}$. Also, it is clear that for $x = 0$, the series diverges.

If $x > 0$, $\sum_{n=1}^{\infty} \frac{1}{1+n^2x} = \frac{1}{x}\sum_{n=1}^{\infty} \frac{1}{n^2+\frac{1}{x}}$. Since $\frac{1}{n^2+\frac{1}{x}} < \frac{1}{n^2}$, and $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges, the series $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges (in fact absolutely, as it has positive terms anyway).

If $x < -1$, $\sum_{n=1}^{\infty} \frac{1}{|1+n^2x}| = \frac{1}{|x|}\sum_{n=1}^{\infty} \frac{1}{|n^2+\frac{1}{x}|} = \frac{1}{|x|}\sum_{n=1}^{\infty} \frac{1}{n^2-\frac{1}{|x|}}$. Since $x < -1$, for $n \geq 2$, $\frac{1}{n^2-\frac{1}{|x|}} < \frac{1}{n^2-1}$. The series $\sum_{n=2}^{\infty} 1n^2 - 1$ converges, and so by the comparison test, $\sum_{n=1}^{\infty} \frac{1}{1+n^2x}$ converges absolutely when $x < -1$.

It remains the case $-1 < x < 0$. There exists $m \in \mathbb{Z}$, such that $-\frac{1}{m^2} < x < -\frac{1}{(m+1)^2}$. Then $\sum_{n=m+1}^{\infty} \frac{1}{|1+n^2x|} = \sum_{n=m+1}^{\infty} \frac{1}{n^2|x|-1}$. A similar argument to the other two cases shows that the series converges absolutely in this case as well.

(b) If $x \geq a > 0$, then $\frac{1}{1+n^2x} \leq \frac{1}{1+n^2a}$. Since $\sum \frac{1}{1+n^2a}$ converges, by Weierstrass' M-test, $\sum \frac{1}{1+n^2x}$ converges unformly on every interval $[a,b] \subset (0,\infty)$.

On intervals of the form $(0,b]$ the series does not converge uniformly. Assume by way of contradiction that it does. Then Cauchy's test implies that there exists $N > 0$ such that $\sum_{n=N}^{m} \frac{1}{1+n^2x} < \frac{1}{2}$, for all $x \in (0,b]$. But if we set $x = \frac{1}{N^2}$ we get a contradiction.

If $x \leq b < -1$, $|\frac{1}{1+n^2x}| = \frac{1}{n^2|x|-1} < \frac{1}{n^2|b|-1}$, so similarly the series converges uniformly on all intervals $[a,b] \subset (-\infty,-1)$. But if one considers intervals of the form $[a,-1)$, a similar argument to the case $x > 0$ shows that the series does not converge uniformly.

The case $-1 < x < 0$ is similar. As long as the interval is $[a, b] \subset (-\frac{1}{m^2}, -\frac{1}{(m+1)^2})$, the series converges uniformly, but if one of the endpoints is of the form $-\frac{1}{m^2}$, it does not.

(c) Since all the terms of the series are continuous functions whenever they are defined, $f(x)$ is continuous on all intervals where it converges uniformly. Every point of convergence (see a)) can be put in an interval $[a, b]$ as in b) where the series converges uniformly. Therefore, f is continuous for all values of x for which is converges.

(d) The function f is unbounded though. For example, $\lim_{x \to 0^+} f(x) = \infty$.