# Problem Set 9

Problem 1

We first show that $\{f_n\}$ converges uniformly to the function $f(x) = 0$.
Of course, as the denominator of $f_n$ is strictly positive, the $f_n$ are defined everywhere, and are continuously differentiable. Note that

$$f_n'(x) = \frac{(1+nx^2) - 2nx^2}{(1+nx^2)^2} = \frac{1-nx^2}{(1+nx^2)^2}$$

In particular, $f_n(x)$ has a local maximum at $x = 1/\sqrt{n}$ and a local minimum at $x = -1/\sqrt{n}$, and it's clear that these are actually a global maximum and minimum. So for any $x$,

$$\begin{aligned} |f_n(x)| &\leq \max\{|f_n(1/\sqrt{n})|, |f_n(-1/\sqrt{n})|\} \\ &\leq \frac{1}{2\sqrt{n}} \end{aligned}$$

So for any $\epsilon > 0$, if $N > 1/4\epsilon^2$, we will have $|f_n(x) - f(x)| < \epsilon$ for every $x \in \mathbb{R}$ and $n > N$. So, the $\{f_n\}$ converge uniformly to the function $f(x) = 0$ as claimed.

Then, for any $x \neq 0$,

$$\lim_{n\to\infty} f_n'(x) = 0$$

as the $n^2$ in the denominator will dominate all the other terms. Thus $f'(x) = 0 = \lim_{n\to\infty} f_n'(x)$. However, at 0, we find $f_n'(0) = 1$ for any $n$, so $\lim_{n\to\infty} f_n'(x) = 1$, which is different from $f'(0) = 0$.
In particular, this shows the necessity of the "uniform convergence of $f_n'$" condition in Rudin Theorem 7.17.

Problem 2

Technically, we should check that $f$ is well-defined, but that's clear since for any $x$, $|f(x)| \leq \sum_{n=1}^{\infty} \frac{1}{n^2}$ is finite.

Now, we show that $f$ is discontinuous at every rational point, and continuous at every irrational point.
Suppose that $r \in \mathbb{Q}$, so that $r = a/b$ for some integers $a, b \in \mathbb{Z}$ with $b \neq 0$. We'll show that for any $\delta > 0$ there is some $y$ with $|y - x| < \delta$ and

$$|f(y) - f(r)| \geq \frac{1}{2b^2}$$

In particular, let $y = \max\{r - \delta, r - 1/2b\}$ (note that $y$ is indeed within $\delta$ of $r$). Then,

$$\begin{aligned}
|f(y) - f(r)| &= \sum_{n=1}^{\infty} \frac{|(ny) - (nr)|}{n^2} \\
&\geq \frac{|(by) - (br)|}{b^2} \\
&\geq \frac{(by)}{b^2} \qquad \text{(as } br \text{ is an integer)} \\
&\geq \frac{1}{2b^2} \qquad \text{(as } br > by \geq br - 1/2\text{)}
\end{aligned}$$

So, setting $\epsilon = 1/2b^2$ shows that $f$ is not continuous at $r$.

On the other hand, suppose that $x$ is an irrational number. For any integer $b$, let $\gamma_b$ denote the distance from $x$ to the nearest integer multiple of $1/b$. Note that since $x$ is irrational, each $\gamma_b$ is non-zero. Then, choose $\delta_b$ so that $\delta_b < \min_{i=1}^{b}\{\gamma_i\}$. This is a minimum over a finite set of non-zero numbers, so $\delta_b > 0$.

We have chosen $\delta_b$ so that, for any integer $i$ with $1 \leq i \leq b$, both $iy$ and $ix$ lie between the same consecutive pair of integers. That is, for any $y$ with $|y - x| < \delta_b$ and for $1 \leq i \leq b$ we have

$$|(iy) - (ix)| = i|y - x|$$

Now, fix an $\epsilon > 0$. Choose $b \in \mathbb{Z}$ such that

$$\sum_{n=b+1}^{\infty} \frac{1}{n^2} < \frac{\epsilon}{4}$$

(this is always possible as the sum of inverse squares converges) and choose $\delta$ such that

$$\delta < \min\left\{ \delta_b \ , \ \frac{\epsilon}{2}\left(\sum_{n=1}^{b} \frac{1}{n}\right)^{-1} \right\}$$

Then, for $y$ with $|y - x| < \delta$, we have

$$\begin{aligned}
|f(y) - f(x)| &= \sum_{n=1}^{\infty} \frac{|(ny) - (nx)|}{n^2} \\
&= \sum_{n=1}^{b} \frac{n|y - x|}{n^2} + \sum_{n=b+1}^{\infty} \frac{|(ny) - (nx)|}{n^2} \\
&\leq \delta \left( \sum_{n=1}^{b} \frac{1}{n} \right) + \sum_{n=b+1}^{\infty} \frac{2}{n^2} \\
&\leq \epsilon/2 + \epsilon/2
\end{aligned}$$

After this painful check, we find that the only discontinuities of $f$ are at rational numbers. This is a countable dense set. In fact, $f$ is Riemann-integrable on every bounded interval: on such intervals, the set of rational numbers has measure 0 and $f$ is bounded, so we can apply Rudin Theorem 11.33.

Problem 3

We need to show that $\{f_n\}$ converges uniformly on $K$. We'll use the Cauchy criterion for uniform convergence (Rudin Theorem 7.8), since it's a bit more convenient. The strategy will be to use the triangle inequality to express the quantity we are interested in, $|f_m(x) - f_n(x)|$, as a sum of other absolute values, each of which can be made arbitrarily small by other considerations.

One good way to use the fact that a space is compact is to use the fact that any open cover has a finite subcover, and that's what we'll do here. Of course, for any $\delta > 0$ and any $x \in K$, the open ball $B(x, \delta)$ contains $x$. So, for any $\delta$, the set of open neighborhoods

$$S_\delta = \{B(x, \delta) | x \in K\}$$

is an open cover of $K$. Thus, it has a finite subcover

$$T_\delta = \bigcup_{i=1}^{m} B(x_i, \delta)$$

for some set of points $\{x_1, \ldots, x_m\}$ of $K$.

Now we must find some large $N$ such that the $|f_n - f_m|$ are small for $n, m > N$. We'll use equicontinuity to show that we only need to do this for finitely many points, and then use pointwise convergence to do this for each of those points simultaneously.
Fix some $\epsilon > 0$. By assumption the family $\{f_n\}$ is equicontinuous. That is, there is some $\delta$ such that for any $n \in \mathbb{Z}$, and $x, y \in K$ with $|x - y| < \delta$ we have $|f_n(x) - f_n(y)| < \epsilon$. This $\delta$ yields a finite set of points $\{x_1, \ldots, x_m\}$ and a $T_\delta$ as above. By construction each point $x \in K$ is no more than a distance of $\delta$ away from some $x_i$.
Now we use the pointwise convergence of the $f_n$ to bound the convergence at all of these $x_i$ simultaneously. That is, for each of the $x_i$ in our finite set, there is some $N_i$ such that for all $n, m > N_i$, we have $|f_m(x_i) - f_n(x_i)| < \epsilon$. If we set $N$ to be the maximum of these (finitely many) $N_i$, then the $f_n$ are small on all of the $x_i$ simultaneously.

Let $x$ be any point of $K$. Pick one of the points $x_i$ with $|x - x_i| < \delta$. Then, for $n, m > N$, we have

$$\begin{aligned} |f_m(x) - f_n(x)| &\leq |f_m(x) - f_m(x_i)| + |f_m(x_i) - f_n(x_i)| + |f_n(x_i) - f_n(x)| \\ &\leq \epsilon + \epsilon + \epsilon \end{aligned}$$

so the $f$ converge uniformly by the Cauchy criterion for uniform convergence.

Problem 4

By assumption each function $f(x)x^i$ is Riemann-integrable. Suppose that $p(x) = \sum_{i=0}^{m} a_i x^i$ is a polynomial. Then, since $p(x)$ is a finite sum of terms $x^i$, $f(x)p(x)$ is Riemann-integrable and

$$\int_0^1 f(x)p(x)\ dx = \sum_{i=0}^{m} a_i \int_0^1 f(x)x^i\ dx = 0$$

Now, let $P_n$ be a sequence of polynomials on the interval $[0,1]$ such that the $P_n$ converge uniformly to $f(x)$. Since each $f(x)P_n(x)$ is Riemann-integrable on $[0,1]$ and the convergence is uniform, by Rudin Theorem 7.16 the limit $f(x)^2$ is Riemann-integrable and

$$\int_0^1 f(x)^2\ dx = \lim_{n\to\infty} \int_0^1 f(x)P_n(x)\ dx = 0$$

Note that $f(x)^2 \geq 0$. Since the integral evaluates to 0, we must have that $f(x) = 0$.

(I'm assuming you guys have covered this last result already. If not, the way to prove it is to show the converse. Assume that a Riemann-integrable function $f$ is non-negative everywhere, and is strictly positive at some point. Then, there is an obvious partition with strictly positive lower sum. As the integral is greater than any lower sum, this implies that the integral is greater than 0.)

Problem 5

This problem really belongs in a complex analysis class. Rudin does this all the time *sigh*.
Anyways, the way to think about this is to note that the algebra $\mathscr{A}$ consists of all the polynomial functions on the unit circle. So, this problem is designed to show that you really need self-adjointness of $\mathscr{A}$ to apply the Stone-Weierstrass theorem (since the "conjugate polynomials" are not in $\mathscr{A}$, it need not approximate every continuous function).

It's clear that $\mathscr{A}$ separates points and vanishes nowhere. Note that for any $n \in \mathbb{Z}$, the integral

$$\int_0^{2\pi} e^{ni\theta} e^{i\theta} \, d\theta = \frac{1}{(n+1)i} \left[ e^{(n+1)i\theta} \right]_0^{2\pi} = 0$$

Since $\mathscr{A}$ consists of polynomial functions, the same integral is zero for any $f \in \mathscr{A}$, and hence for any function in the uniform closure of $\mathscr{A}$.

However, there are continuous functions on the unit circle such that the integral

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta \neq 0$$

One example is suggested by the failure of Stone-Weierstrass: we should look at conjugation $f(e^{i\theta}) = e^{-i\theta}$. Then,

$$\int_0^{2\pi} f(e^{i\theta}) e^{i\theta} \, d\theta = \int_0^{2\pi} 1 \, d\theta = 2\pi$$

and so this continuous function is not in the uniform closure of $\mathscr{A}$.

Problem 6

We first must prove that $|f_p(x)| \leq d(a,p)$ for all $x \in X$. This is just the triangle inequality twice:

$$f_p(x) = d(x,p) - d(x,a) \leq d(x,a) + d(a,p) - d(x,a) = d(a,p)$$

$$-f_p(x) = d(x,a) - d(x,p) \leq d(x,p) + d(p,a) - d(x,p) = d(a,p)$$

In particular, this shows that each $f_p$ is bounded on $X$, and so is a member of $\mathscr{C}(X)$.

Now, we must show that $\|f_p - f_q\| = d(p,q)$.

$$f_p(x) - f_q(x) = d(x,p) - d(x,a) - d(x,q) + d(x,a) = d(x,p) - d(x,q)$$

By same reasoning as before, this implies that for any $x \in X$, $|f_p(x) - f_q(x)| \leq d(p,q)$. Then,

$$\|f_p - f_q\| = \sup_{x \in X} |f_p(x) - f_q(x)| \leq d(p,q)$$

Of course, $|f_p(p) - f_q(p)| = d(p,q)$, so we obtain equality. So, if we define the map of vector spaces $\Phi : X \to \mathscr{C}(X)$ taking $p$ to $f_p$, we see that the map $\Phi$ is an isometry.

Finally, we must see that the closure $Y$ of $\Phi(X)$ in $\mathscr{C}(X)$ is complete. However, we know that $\mathscr{C}(X)$ is complete, so $Y$ (being a closed subset of a complete metric space) is complete as well.