18.100C. Quiz 1. Spring 2006.

Name: ____________________ March 14, 2006

Problem 1: _____ /40

Problem 2: _____ /20

Problem 3: _____ /30

Problem 4: _____ /30

Problem 5: _____ /30

Total: _____ /150

Instructions: The exam is closed book, closed notes and calculators are not allowed. You will have 80 minutes for this exam. The point value of each problem is written next to the problem - use your time wisely. Please show all work (and give proofs), unless instructed otherwise. Partial credit will be given only for work shown.

You may use either pencil or ink. Good luck!

Problem 1.(40 points) Decide if the following assertions are true or false. Prove your answer (or give a counterexample).

a) Every collection of disjoint intervals in $\mathbb{R}$ is countable.

b) $\mathbb{R}^2$ contains a countable dense subset.

c) The set of all sequences whose elements are 0 or 1 is countable.

d) Every infinite compact subset of $\mathbb{R}$ must be uncountable.

Problem 2.(20 points) If E is a subset of a metric space X, recall the following notation:

$$E^\circ = \text{ interior of } E, \quad E' = \text{ set of limit points of } E.$$

Suppose that E and F are subsets of X. For each of the following assertions, provide a proof if it is true, or a counterexample if it is false.

a) $(E \cup F)^\circ = E^\circ \cup F^\circ$.

b) $(E \cap F)^\circ = E^\circ \cap F^\circ$.

c) $(E \cap F)' = E' \cap F'$.

d) $(E \cup F)' = E' \cup F'$.

Problem 3.(30 points) A metric space X is said to be *totally bounded* if for every $r > 0$, there exist finitely many points $x_1, \ldots, x_n$ of X so that

$$X = N_r(x_1) \cup \cdots \cup N_r(x_n).$$

(Recall that $N_r(x) = \{y \in X : \ d(y, x) < r\}$.)

a) Prove that every compact metric space is totally bounded.

b) Let X be an infinte set. Define the metric function $d(x, y) = \begin{cases} 1, & \text{if } x \neq y \\ 0, & \text{if } x = y \end{cases}$. (You may assume without proof that d is a metric.) Show that X is bounded, but not totally bounded.

c) For the space X in b), which subsets of X are compact?

Problem 4.(30 points)

a)(20 points) Let $\{x_n\}$ be a Cauchy sequence in a metric space X. Assume that some subsequence $\{x_{n_k}\}$ converges to a point $x \in X$. Prove that the entire sequence $\{x_n\}$ converges to x.

b)(10 points) Let $\{x_n\}$ be a sequence in $\mathbb{R}$ such that

$$|x_{n+1} - x_n| < \frac{1}{2^n}, \text{ for all } n \geq 1.$$

Prove that $\{x_n\}$ converges.

Problem 5.(30 points) Decide if the following sets are open, closed, compact. (The metric functions are the Euclidean ones.) Justify your answers.

a) $E_1 = \{(x, y) \in \mathbb{R}^2 : \; y - 2x \leq 1\}$ in $\mathbb{R}^2$.

b) $E_2 = \{(x, y) \in \mathbb{R}^2 : \; 1 < x^2 + y^2 < 9\}$ in $\mathbb{R}^2$.

c) $E_3 = \{(x, y) \in \mathbb{R}^2 : \; |x| + |y| = 2\}$ in $\mathbb{R}^2$.

d) $E_4 = [0, 1] \cap \mathbb{Q}$ in $\mathbb{R}$.