18.100C. Final. Solutions. Spring 2006.

Problem 1.(50 pts): (10; 15; 10; 15)
Let f be the function

$$f(x) = \begin{cases} x^2 \sin(\frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

a) Show that f is continuous for all x.

b) Show that f is differentiable for all x, and find the derivative $f'(x)$.

c) Is $f'(x)$ bounded on the interval $(0, 1)$? Prove your answer carefully.

d) Let g be a differentiable function on $(0, 1)$ such that its derivative is bounded on $(0, 1)$. Prove that $g(x)$ is uniformly continuous on $(0, 1)$.

Solution:

a) For $x \neq 0$, $f(x)$ is a product and composition of elementary continuous functions, therefore it is continuous. We check the limit as x approches 0. Since $|x^2 \sin(\frac{1}{x^2})| \leq |x^2|$, as $x \to 0$, $\lim_{x\to 0} f(x) = 0 = f(0)$. So f is continuous at 0 as well.

b) We'll show that

$$f'(x) = \begin{cases} 2(x \sin \frac{1}{x^2} - \frac{1}{x} \cos \frac{1}{x^2}), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

The formula for $x \neq 0$ is clear by applying the rules of differentiation. We check the derivative at 0:

$$f'(0) = \lim_{x\to 0} \frac{x^2 \sin \frac{1}{x^2} - 0}{x - 0} = \lim_{x\to 0} x \sin \frac{1}{x^2} = 0.$$

The last equality follows as in a) from the fact that $|x \sin \frac{1}{x^2}| \leq |x|$.

c) The derivative f' is unbounded on the interval $(0, 1)$. To see this, consider the sequence $\{x_k\}$ given by $x_k = \sqrt{\frac{1}{2k\pi}}, k \geq 0$. This sequence is clearly in the interval $(0, 1)$. Then $f'(x_k) = \sqrt{2k\pi}$, and $\lim_{k\to\infty} f'(x_k) = \infty$.

d) Let g be a differentiable function on $(0, 1)$, and let $M > 0$ be such that $|g'(x)| \leq M$, for all $x \in (0, 1)$. For every $x, y \in (0, 1)$, by the mean value theorem, there exists c between x and y such that $g(x) - g(y) = g'(c)(x - y)$. But this implies that $|g(x) - g(y)| \leq M|x - y|$, so g is a Lipschitz function, therefore uniformly continuous. (Let $\epsilon > 0$ be given, and set $\delta = \frac{\epsilon}{M}$. Then for all $x, y \in (0, 1)$, such that $|x - y| < \delta$, we have $|g(x) - g(y)| < \epsilon$.)

Problem 2. (60 pts): (10; 10; 15; 10; 15)

a) If $n \geq 1$, find an antiderivative for $e^{-nx}\cos(nx)$. (Hint: use integration by parts.) Check your answer by differentiation.

b) Find $\int_1^{\infty} e^{-nx}\cos(nx)dx$.

c) Consider the series

$$\sum_{n\geq 1}^{\infty} e^{-nx}\cos(nx).$$

Prove that the series converges uniformly on every interval $[a, \infty)$ where $a > 0$.

d) If $f(x)$ denotes the sum of the series in c), show that $f(x)$ is continuous on $(0, \infty)$.

e) Prove that $\left|\int_1^{\infty} f(x)dx\right| \leq 2$, where $f(x)$ is as defined in parts c) and d).

Solution:

a) An antiderivative is $\frac{1}{2n}e^{-nx}(\sin(nx) - \cos(nx))$.

b) Note that, since $|e^{-nx}(\sin(nx) - \cos(nx))| \leq 2e^{-nx}$, and $\lim_{x\to\infty} e^{-nx} = 0$, we have $\lim_{x\to\infty} e^{-nx}(\sin(nx) - \cos(nx)) = 0$. Then, using a), we find that

$$\int_1^{\infty} e^{-nx}\cos(nx)dx = \frac{1}{2ne^n}(\cos n - \sin n).$$

c) Since $|e^{-nx}\cos(nx)| \leq e^{-nx} \leq e^{-na}$, for all $x \in [a, \infty)$, and the series $\sum_{n\geq 1} e^{-na}$ converges (being a geometric series with ratio $0 < e^{-a} < 1$), by the Weierstrass M-test, it follows that the series $\sum_{n\geq 1} e^{-nx}\cos(nx)$ converges uniformly on the interval $[a, \infty)$. (Recall that $a > 0$.)

d) Let $f(x)$ denote the sum of the series in c). Then for every $x > 0$, choose a such that $0 < a < x$. The series in c) converges uniformly on $[a, \infty)$, and all the terms of the series are continuous. By a theorem in Rudin, the sum of the series f is continous on $[a, \infty)$, so in particular at x as well.

e) On the interval $[1, \infty)$ the series in c) converges uniformly, and every term in integrable. We can integrate term by term. Moreover, by the triangle inequality:

$$\left|\int_1^{\infty} f(x)dx\right| \leq \sum_{n\geq 1}\left|\int_1^{\infty} e^{-nx}\cos(nx)dx\right| \leq \sum_{n\geq 1}\frac{1}{ne^n}$$
$$\leq \sum_{n\geq 1}\frac{1}{e^n} = \frac{1}{e-1} \leq 1.$$

Problem 3. (35 pts): (15; 20)

a) Define the sequence $\{a_n\}$ by

$$a_{2n} = 2^{2n}, \quad a_{2n+1} = 3^{2n+1}, \quad n \geq 0.$$

Find the radius of convergence of $\sum_{n=0}^{\infty} a_n z^n$.

b) Determine the radius of convergence of $\sum_{n=0}^{\infty} n z^n$, and find a formula for the sum. (Hint: Start with a well-known formula for $\sum_{n=0}^{n} z^n$.) Justify the correctness of your calculations.

Solution:

a) We apply the root test to determine the radius of convergence, and so we need to find $\limsup_{n\to\infty}(a_n)^{1/n}x$. Note that $(a_{2n})^{1/2n} = 2$, and $\lim_{n\to\infty}(a_{2n+1})^{1/(2n+1)} = \lim_{n\to\infty} 3^{\frac{2n}{2n+1}} = 3$. Then $\limsup_{n\to\infty}(a_n)^{1/n} = 3$, and so the radius of convergence is $R = \frac{1}{3}$.

b) Since $\lim_{n\to\infty} n^{\frac{1}{n}} = 1$, the radius of convergence is $R = 1$. So the interval of convergence for $\sum_{n=1}^{\infty} n z^n$ is $(-1, 1)$. Note that $\sum_{n=1}^{\infty} n z^n = z \sum_{n=1}^{\infty} n z^{n-1}$. The series $\sum_{n=0}^{\infty} z^n$ has the same interval of convergence and the sum $f(z) = \frac{1}{1-z}$. The series with the differentiated terms is $\sum_{n=1}^{\infty} n z^{n-1}$, so it converges to $f'(z)$ (by a theorem in Rudin about analytic functions). It follows that

$$\sum_{n=0}^{\infty} n z^n = z f'(z) = \frac{z}{(z-1)^2}, \text{ when } |z| < 1.$$

Problem 4. (50 pts): (15; 15; 15; 5)

Let E be a nonempty closed subset of a metric space X with metric function d. Define the distance from $x \in X$ to E by

$$\rho_E(x) = \inf_{z \in E} d(x, z).$$

a) Prove that $\rho_E(x) = 0$ if and only if $x \in E$.

b) Prove that for all $x \in X, y \in X$,

$$|\rho_E(x) - \rho_E(y)| \leq d(x, y),$$

and therefore $\rho_E : X \to \mathbb{R}$ is uniformly continuous on X.

c) Let K be a compact subset of X, disjoint from E. Prove that there exists $x_0 \in K$ such that $0 < \rho_E(x_0) \leq \rho_E(x)$, for all $x \in K$.

d) If $E \subset \mathbb{R}$ is the Cantor ser, and $x = \frac{5}{6}$, what is the distance $\rho_E(x)$ equal to?

Solutions:

a) In one direction, it is clear: if $x \in E$, $\rho_E(x) = 0$. Conversely, assume $0 = \rho_E(x) = \inf_{z \in E} d(x, z)$. This means that there exists a sequence $\{z_k\}$ in E such that $d(x, z_k) < \frac{1}{k}$. This implies that $\lim_{k \to \infty} z_k = x$, and so x is a limit point for E. Since E is closed, $x \in E$.

b) From the definition, it is clear that for every $x \in X$ and $z \in E$, $d(x, z) \geq \rho_E(x)$. Consider the triangle inequality $d(x, y) + d(y, z) \geq d(x, z)$, with $x, y \in X, z \in E$. From the preceeding remark, $d(x, y) + d(y, z) \geq \rho_E(x)$. Then we take the infimum over $z \in E$, and find that $d(x, y) + \rho_E(y) \geq \rho_E(x)$, or equivalently, $\rho_E(x) - \rho_E(y) \leq d(x, y)$. Now we can interchange x and y, and find $\rho_E(x) - \rho_E(y) \geq -d(x, y)$. The claim follows. The uniform continuity follows as in Problem 1 d) (as before, ρ_E is a Lipschitz function).

c)Let K be a compact subset of X, and $K \cap E = \emptyset$. Since the $\rho_E : X \to \mathbb{R}$ is continous, when restricted to K, it is bounded and it attains its minimum (and maximum). Let $x_0 \in K$ be the point where ρ_E attains the minimum on K. Since $x_0 \in K$, necessarily $x_0 \notin E$, so by a), $\rho_E(x_0) > 0$.

d) $\rho_E(x) = \frac{1}{18}$.

Problem 5. (50 pts): (15; 20; 15)

Let $f : [0, 1] \to \mathbb{R}$ be a continuous function.

a) Assume $\int_0^1 f(x)dx = 1$. Show that there exists $c \in (0, 1)$ such that $f(c) = 1$.

b) Now suppose

$$\int_0^1 f(x)x^n dx = \frac{1}{n+1}, \text{ for all } n \geq 0.$$

Prove that $f(x) = 1$ for all $x \in [0, 1]$. (Hint: set $g(x) = f(x) - 1$. You may want to use the Weierstrass theorem.)

c) Prove that if $h : [0, 1] \to \mathbb{R}$ is a continuous nonnegative function and $\int_0^1 h(x)dx = 0$, then $h(x) = 0$, for all $x \in [0, 1]$.

Solutions:

a) This follows immediately by the mean value theorem for integrals (and the fundamental theorem of calculus).

b) Set $g(x) = f(x) - 1$. Then $\int_0^1 g(x)x^n dx = 0$, for all $n \geq 0$. If $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ is any polynomial, this identity immediately implies that $\int_0^1 g(x)P(x) = 0$.

Let $\epsilon > 0$ be given. Since g is continuous on $[0, 1]$, by the Weierstrass theorem, there exists a polynomial $P(x)$ such that $|g(x) - P(x)| \leq \epsilon$. Then

$$\int_0^1 g^2(x)dx = \left| \int_0^1 g(x)(g(x) - P(x))dx \right| \leq \int_0^1 |g(x)||g(x) - P(x)|dx \leq \epsilon \int_0^1 |g(x)|dx.$$

Note that $\int_0^1 |g(x)|dx$ is finite, doesn't depend on ϵ, and since ϵ was arbitrary, necessarily $\int_0^1 g^2(x)dx = 0$.

The function $g^2(x)$ is continous and nonnegative on $[0, 1]$, so the only way the integral can be zero is if the function is zero, which implies $g(x) = 0$, and so $f(x) = 0$.

c) If h is continuous and nonnegative, assume that it is strictly positive at some point x_0. Because of continuity, h must be strictly positive on a subinterval $[a, b]$ containing x_0. Let $m > 0$ denote the minimum of h on $[a, b]$. Then the integral $\int_0^1 h(x)dx \geq (b - a)m > 0$, contradiction!

Problem 6. (55 pts): (25; 15; 15)

Let X be the space of all sequences of real numbers. For any two sequences $\underline{a} = \{a_i\}$ and $\underline{b} = \{b_i\}$, define

$$d(\underline{a}, \underline{b}) = \sum_{i=0}^{\infty} \frac{1}{2^i} \frac{|a_i - b_i|}{1 + |a_i - b_i|}.$$

a) Show that d is well defined, and that it is a metric on X.

b) Prove that, with respect to d, X is bounded, but it is not compact. (Hint: construct a sequence $\{\underline{x}_n\}$ of sequences, such that $d(\underline{x}_n, \underline{x}_m) \geq \frac{1}{2}$ for all n, m.)

c) Prove that the metric space (X, d) is complete.

Solution:

a) Since $\frac{|a_i - b_i|}{1+|a_i - b_i|} < 1$, and the series $\sum_{i=0}^{\infty} 2^{-i}$ is a convergent (geometric) series, by the comparison test, the series used to define $d(\underline{a}, \underline{b})$ is convergent. Therefore $d(\underline{a}, \underline{b}) < \infty$ is well-defined.

We need to check the axioms of the metric. The only one which is not obvious is the triangle inequality. This follows from the inequality

$$\frac{|x - z|}{1 + |x - z|} \leq \frac{|x - y|}{1 + |x - y|} + \frac{|y - z|}{1 + |y - z|}, \quad x, y, z \in \mathbb{R},$$

which, in turn, can be proved by a direct calculation.

b) Consider, for example, the sequence $\underline{x}_n$ defined by $\underline{x}_n = (n, 0, \dots, 0, \dots)$. For two such sequences, $\underline{x}_n$ and $\underline{x}_m$, $n \neq m$, we have

$$d(\underline{x}_n, \underline{x}_m) = \frac{|n - m|}{1 + |n - m|} \geq \frac{1}{2}.$$

(Another good example to consider would be $\underline{x}_n = (n, n, \dots, n, \dots)$.) But this implies that the sequence $\{\underline{x}_n\}$ in X does not have any convergent subsequences. Therefore X cannot be compact.

c) Let $\{\underline{x}_n\}$ be a Cauchy sequence in (X, d). Each $\underline{x}_n$ is a sequence of real numbers, let us denote it by $\underline{x}_n = \{x_{n,i}\}$.

Fix $j \geq 0$. The first claim is that the sequence $\{x_{n,j}\}_n$ is Cauchy in $\mathbb{R}$. Let $\epsilon > 0$ be given. Since $\{\underline{x}_n\}$ is Cauchy in X, we can choose $N > 0$ such that $d(\underline{x}_n, \underline{x}_m) < 2^{-j} \frac{\epsilon}{1+\epsilon}$, for all $n, m > N$. Clearly $\frac{|x_{n,j} - x_{m,j}|}{1+|x_{n,j} - x_{m,j}|} \leq 2^j d(\underline{x}_n, \underline{x}_m) < \frac{\epsilon}{1+\epsilon}$, for all $n, m > N$, which implies that $|x_{n,j} - x_{m,j}| < \epsilon$, for all $n, m > N$. This proves the claim that $\{x_{n,j}\}_n$ is Cauchy in $\mathbb{R}$. Since $\mathbb{R}$ is complete (with the Euclidean metric), $\{x_{n,j}\}_n$ is convergent, and denote its limit by y_j.

To summarize, for each $j \geq 1$, $x_{n,j} \to y_j$, as $n \to \infty$. Let us prove that $\{\underline{x}_n\}$ converges to $\underline{y}$ in (X, d), where $\underline{y} = \{y_j\}$. Let $\epsilon > 0$ be given. There exists $M > 0$, such that

(1)
$$\sum_{i \geq M} \frac{1}{2^i} \frac{|x_{n,i} - y_i|}{1 + |x_{n,i} - y_i|} \leq \sum_{i \geq M} 2^{-i} < \frac{\epsilon}{2}.$$

(This is because the geometric series $\sum_{i \geq 0} 2^{-i}$ is convergent.)

For every $j \in \{0, \ldots, M-1\}$, $\{x_{n,j}\}_n$ converges to y_j, so we can find N_j such that $\frac{|x_{n,j}-y_j|}{1+|x_{n,j}-y_j|} < 2^j \frac{\epsilon}{2M}$, for all $n \geq N_j$.

Let N be the maximum of all N_j, $j = 0, \ldots, M-1$. Then for all $n \geq N$,

$$\sum_{i=0}^{M-1} \frac{1}{2^i} \frac{|x_{n,i}-y_i|}{1+|x_{n,i}-y_i|} < M \cdot \frac{\epsilon}{2M} = \frac{\epsilon}{2}. \tag{2}$$

Now combining equations (1) and (2), we find that $d(\underline{x}_n, \underline{y}) < \epsilon$, for all $n \geq N$. This proves that $\{\underline{x}_n\}$ converges to $\underline{y}$.