THE GAMMA FUNCTION

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The Gamma function was discovered during the search for a factorial analog defined on real numbers. This paper will explore the properties of the factorial function and use them to introduce the Gamma function.

1. The Factorial

The factorial function, $n!$, was initially defined over the positive integers as,

$$n! = (n)(n-1)(n-2)\cdots(3)(2)(1).$$

This expresses the number of ways to permute an n-element set. Since a set with 1 element can be permuted only one way, $1! = 1$. With this base case, the factorial may be defined recursively as:

$$n! = n(n-1)!.$$

An interesting property of the factorial function is its rate of growth. The factorial grows in a weak log convex fashion. But to understand this behaviour, we must first understand what it means for a function to be convex, log convex, and weakly convex. These concepts will be explored in the following section.

1.1. **Weak Log Convexity.**

1.1.1. *Convexity.* Consider a real-valued function $f : (a,b) \to \mathbb{R}$. For $x, y \in (a,b)$, define the symmetric function ϕ:

$$\phi(x,y) = \phi(y,x) = \frac{f(x) - f(y)}{x - y}.$$

Then f is convex if $\phi(x,y)$ is monotonically increasing with respect to each variable. This means $\phi(x, y_1) \leq \phi(x, y_2)$ when $y_1 \leq y_2$ and symmetrically, $\phi(x_1, y) \leq \phi(x_2, y)$ when $x_1 \leq x_2$. If in addition, f is differentiable, then f is convex when $f'(x) \leq f'(y)$ for all $a < x < y < b$.

The convexity of f may also be analyzed by considering the function Φ:

$$\Phi(x,y,z) = \frac{\phi(x,z)-\phi(y,z)}{x-y}$$
$$= \frac{f(x)(y-z)+f(y)(z-x)+f(z)(x-y)}{(x-z)(y-z)(x-y)}$$

for $x, y, z \in (a,b)$. Then f is convex if Φ is nonnegative. If f is also twice-differentiable, then f is convex when $f''(x) \geq 0$ for all $a < x < b$.

1.1.2. *Log Convexity.* Log convexity is similarly defined. A function g is log convex if $\log \circ g$ is convex. Suppose g is also twice-differentiable. Then, $(\log \circ g)'' \geq 0$. Differentiating $\log \circ g$ gives us,

$$(\log \circ g)' = \frac{g'}{g}$$
$$(\log \circ g)'' = \frac{g''g - g'^2}{g^2} \geq 0.$$

It follows that $g''g \geq g'^2$. Note that log convex functions are also convex. This is because log is defined for nonnegative reals, so $g \geq 0$. Therefore $g'' \geq g'^2/g \geq 0$ when $g \neq 0$.

1.1.3. *Weak Convexity.* However, convex functions are continuous in $\mathbb{R}$ and the factorial is not. So the factorial is not a convex function, though it behaves like one. Let us define a weaker form of convexity which encompasses noncontinuous functions as well.

There exists certain functions defined on $(a,b) \subseteq \mathbb{R}$ for whom $\Phi(x,y,z)$ may not be nonnegative for all $x,y,z \in (a,b)$, but for whom $\Phi(x,(x+z)/2,z)$ is nonnegative for all $x,y,z \in (a,b)$. Such functions are called weakly convex.

Suppose $f : (a,b) \to \mathbb{R}$ is a weakly convex functions. Consider $a < x < y < z < b$ such that $y = (x+z)/2$ and let,

$$m = (z-y) = (y-x) = \frac{z-x}{2}.$$

This implies,

$$\Phi(x,y,z) = \frac{f(x)(y-z)+f(y)(z-x)+f(z)(x-y)}{(x-z)(y-z)(x-y)}$$
$$= \frac{[-f(x)+2f(y)-f(z)]m}{(x-z)(y-z)(x-y)}.$$

Because $(x-z)(y-z)(x-y)<0$ and $m>0$, it follows that $\Phi(x,y,z)\geq 0$ when $-f(x)+2f(y)-f(z)\leq 0$. Therefore, f is weakly convex if and only if,

$$f\left(\frac{x+z}{2}\right)\leq \frac{1}{2}[f(x)+f(z)].$$

In general, given a weak convex function f defined on (a,b) and $x_1,\ldots,x_n\in (a,b)$,

$$f\left(\frac{x_1+\cdots+x_n}{n}\right)\leq \frac{1}{n}[f(x_1)+\cdots+f(x_n)].$$

We know this statement holds for $n=2$.

Suppose the statement is true for an arbitrarily given $n\in\mathbb{N}$. Then it is true for $2n$.

$$\begin{aligned}f\left(\frac{x_1+\cdots+x_{2n}}{2n}\right)&\leq \frac{1}{2}\left[f\left(\frac{x_1+\cdots+x_n}{n}\right)+f\left(\frac{x_{n+1}+\cdots+x_{2n}}{n}\right)\right]\\&\leq \frac{1}{2n}[f(x_1)+\cdots+f(x_{2n})].\end{aligned}$$

And it is true for $(n-1)$. Let $x_1,\ldots,x_{n-1}\in(a,b)$ and define

$$x_n=\frac{x_1+\cdots+x_{n-1}}{n-1}.$$

which is also in (a,b). This gives us,

$$\begin{aligned}f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)=f(x_n)&=f\left(\frac{(n-1)x_n+x_n}{n}\right)\\f(x_n)&\leq\frac{1}{n}[f(x_1)+\cdots+f(x_{n-1})+f(x_n)]\\f\left(\frac{x_1+\cdots+x_{n-1}}{n-1}\right)&\leq\frac{1}{n-1}[f(x_1)+\cdots+f(x_n)].\end{aligned}$$

Since the relation holds for $(n-1)$ when it holds for n, and since it holds for arbitrarily large n, the relation holds for all $n\in\mathbb{N}$ by induction.

1.1.4. *Weak Log Convexity.* Weak log convexity is similarly defined. A function g is weakly log convex if $\log\circ g$ is weakly convex. Suppose g is a weakly log convex function on (a,b). Then for $x,y,z\in(a,b)$,

$$\begin{aligned}\log\circ g\left(\frac{x+z}{2}\right)&\leq\frac{1}{2}[\log\circ g(x)+\log\circ g(z)]\\\log\circ\left[g\left(\frac{x+z}{2}\right)\right]^2&\leq\log\circ g(x)g(z)\\\left[g\left(\frac{x+z}{2}\right)\right]^2&\leq g(x)g(z).\end{aligned}$$

1.2. **The Factorial is Weakly Log Convex.** Having introduced the concept of weak log convexity, we will now show that the factorial, $f(n) = n!$ for $n \in \mathbb{N}$, is a weakly log convex function. This means

$$\left[f\left(\frac{m+n}{2}\right)\right]^2 \leq f(m)f(n), \text{ for all } m, n \in \mathbb{N}.$$

Proof. Let $m = 2u$ and $n = 2v$. Then the above statement becomes,

$$(u+v)!^2 \leq (2u)!(2v)!.$$

Without loss of generality, assume $u \leq v$.

$$\begin{aligned}(u+v)!^2 &= (u+v)!(u+v)\cdots(u+v-(v-u-1))(2u)! \\ &\leq (2v)\cdots(2v-(v-u-1))(u+v)!(2u)! \\ &= (2v)!(2u)!.\end{aligned}$$ □

Then the question is, does a more general version of the factorial function exist? And if it exists, what is it? Such a function f would possess the following properties,

1. $f(1) = 1$,
2. $f(x+1) = (x+1)f(x)$,
3. $f(x)$ is weakly log convex for $x \in \mathbb{N}$,

and would be defined for all nonnegative real numbers.

2. The Existence of the Gamma Function

In the early 18th century, the prolific mathematician Leonhard Euler discovered a function defined for $(0, \infty)$ which mimicked the factorial on the positive integers. It possesses the following properties, which are similar to those of the factorial:

1. $f(1) = 1$,
2. $f(x+1) = xf(x)$,
3. $f(x)$ is log convex for $x \in (0, \infty)$.

This function is known as the gamma function and is defined as,

$$\Gamma(x) = \int_0^\infty e^{-t}t^{x-1}dt.$$

Figure 1. is a graph of the gamma function over both positive and negative values.

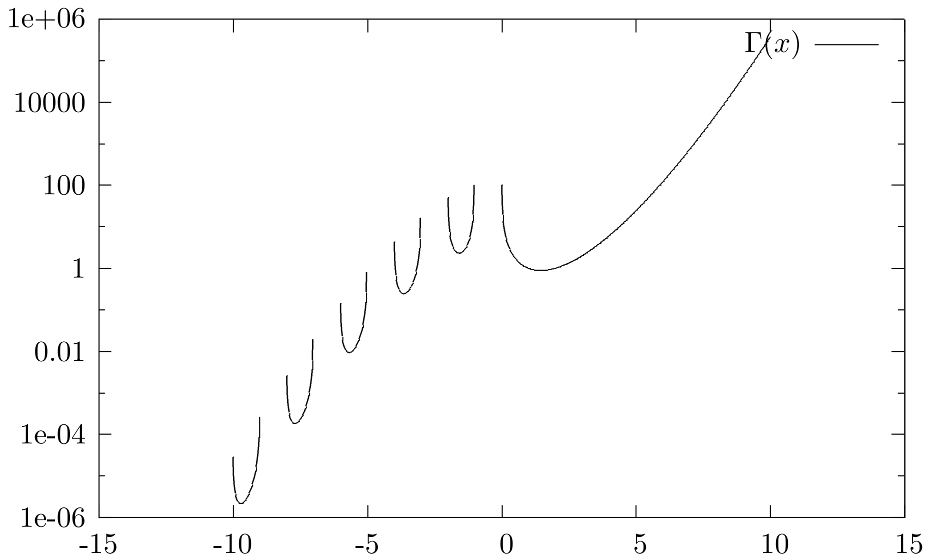

FIGURE 1. The Γ function

It is easy to verify the first two properties of the gamma function.

$$\Gamma(1) = \int_0^\infty e^{-t} dt = | - e^{-t}|_0^\infty = 1.$$

and

$$\begin{aligned}\Gamma(x+1) &= \int_0^\infty e^{-t} t^x dt \\ &= | - t^x e^{-t}|_0^\infty + \int_0^\infty x t^{x-1} e^{-t} dt \\ &= x\Gamma(x).\end{aligned}$$

However, to understand why Γ is log convex, first note that sums of weakly log convex functions are also weakly log convex. Suppose f and g are weakly log convex functions. Then,

$$\left[f\left(\frac{u+v}{2}\right)\right]^2 \le f(u)f(v).$$

$$\left[g\left(\frac{u+v}{2}\right)\right]^2 \le g(u)g(v).$$

E. Artin showed that this implies $\left[f\left(\frac{u+v}{2}\right) + g\left(\frac{u+v}{2}\right)\right]^2 \le [f(u)+g(u)]\,[f(v)+g(v)]$ by considering the polynomial,

$$h(x,y) = a(ax^2 + 2bxy + cy^2) = (ax+by)^2 + (ac-b^2)y^2, \text{ for } a \ge 0.$$

When $b^2 \geq ac$, h is nonnegative.

Let $a_f = f(u)$, $b_f = f\left(\frac{u+v}{2}\right)$, $c_f = f(v)$ and $a_g = g(u)$, $b_g = g\left(\frac{u+v}{2}\right)$, $c_g = g(z)$. Then the convexity of f and g imply $b_f^2 \geq a_f c_f$ and $b_g^2 \geq a_g c_g$, so we have

$$a_f x^2 + 2b_f xy + c_f y^2 \geq 0$$
$$a_g x^2 + 2b_g xy + c_g y^2 \geq 0$$
$$(a_f + a_g)x^2 + 2(b_f + b_g)xy + (c_f + c_g)y^2 \geq 0.$$

Therefore, $(b_f + b_g)^2 \geq (a_f + a_g)(c_f + c_g)$. It follows that $f + g$ is weakly log convex.

Likewise, integrals of weakly log convex functions are weakly log convex. And since log convex functions are also weakly log convex, integrals of log convex functions are log convex.

$f(x) = e^{-t}t^{x-1}$ is a log convex function because it is twice-differentiable and satisfies

$$f''f \geq f'^2$$
$$e^{-2t}t^{2(x-1)}(\ln t)^2 \geq e^{-2t}t^{2(x-1)}(\ln t)^2.$$

Therefore, Γ is a log convex function because it is an integral of a log convex function.

Though the gamma function is not equivalent to the factorial function since $(x+1)! = (x+1)x!$ and $\Gamma(x+1) = x\Gamma(x)$, both functions are similarly defined recursively and both functions grow weak log convexly. On the positive integers, the gamma function may be reduced to

$$\Gamma(x) = (x-1)!.$$

Now that we have found a real analog of $(x-1)!$, we may ask, do other such functions exist?

3. Uniqueness of the Gamma Function

We have shown that there exists a gamma function which behaves like the factorial on the positive integers. We will now show that any function, which is defined for $(0, \infty)$ and behaves like $(x-1)!$ on the positive integers, must be the gamma function.

Let f be such a function. Then,

1. $f(1) = 1$.

2. $f(x+1) = xf(x)$.
3. f is log convex for $x \in (0, \infty)$.

Because $f(x+1) = xf(x)$, it is sufficient to determine the behaviour of f on the interval $(0, 1]$. So consider $0 < x \leq 1$ and let $n \in \mathbb{N}$ such that $n \geq 2$. From the log convexity of f, we have the following,

$$\frac{\log f(-1+n) - \log f(n)}{(-1+n)-n} \leq \frac{\log f(x+n) - \log f(n)}{(x+n)-n} \leq \frac{\log f(1+n) - \log f(n)}{(1+n)-n}.$$

Since f behaves like the factorial on the positive integers and $f(1) = 1$, we know $f(n) = (n-1)!$. This implies that,

$$\begin{array}{rcl} \frac{\log(n-2)! - \log(n-1)!}{-1} \leq & \frac{\log f(x+n) - \log(n-1)!}{x} & \leq \frac{\log n! - \log(n-1)!}{1} \\ \log(n-1) \leq & \frac{\log f(x+n) - \log(n-1)!}{x} & \leq \log n \\ (n-1)^x (n-1)! \leq & f(x+n) & \leq n^x (n-1)!. \end{array}$$

Because $f(x+1) = xf(x)$, it follows that $f(x+n) = (x+n-1)\cdots(x)f(x)$.

$$\begin{array}{rcl} (n-1)^x(n-1)! \leq & (x+n-1)\cdots(x)f(x) & \leq n^x(n-1)! \\ \frac{(n-1)^x(n-1)!}{(x+n-1)\cdots(x)} \leq & f(x) & \leq \frac{n^x n!}{(x+n)(x+n-1)\cdots(x)} \frac{x+n}{n}. \end{array}$$

$(n-1)$ may be replaced with n on the left hand side of the above inequality since the relation holds for all integers $n \geq 2$.

$$\begin{array}{rcl} \frac{n^x n!}{(x+n)\cdots(x)} \leq & f(x) & \leq \frac{n^x n!}{(x+n)\cdots(x)} \frac{x+n}{n} \\ \frac{nf(x)}{x+n} \leq & \frac{n^x n!}{(x+n)\cdots(x)} & \leq f(x). \end{array}$$

Taking the limit of both sides of the inequality as n tends to infinity gives,

$$f(x) \leq \lim_{n\to\infty} \frac{n^x n!}{(x+n)\cdots(x)} \leq f(x).$$

Therefore,

$$f(x) = \lim_{n\to\infty} \frac{n^x n!}{(x+n)\cdots(x)}.$$

Because this limit is unique, f is unique. Thus, there exists only one function which possesses the factorial-like properties discussed. Since the gamma function possesses these properties, the gamma function must be the only such generalization of the factorial to real numbers. Figure 2 illustrates how well Γ approximates factorial.

4. Acknowledgements

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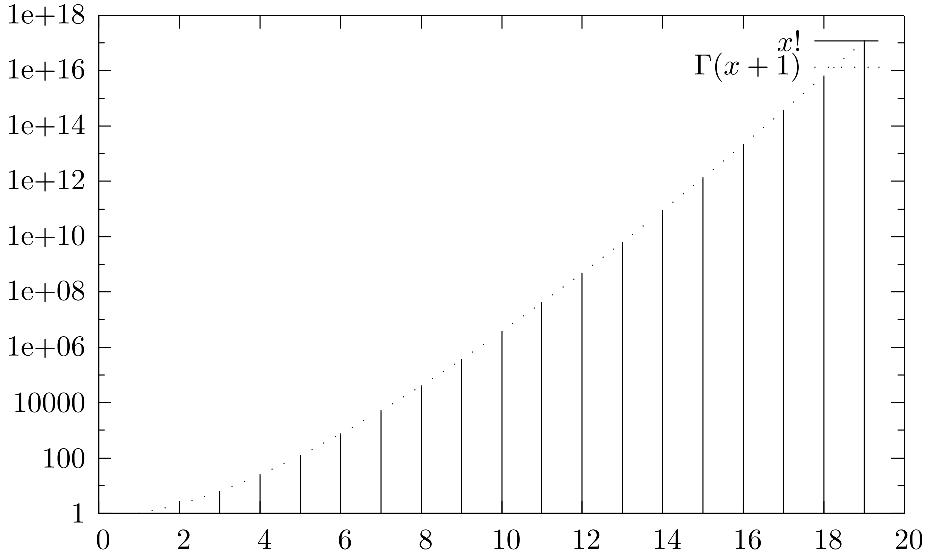

FIGURE 2. A comparison of the factorial and gamma functions

REFERENCES

[1] Artin, Emil. (1964). *The Gamma Function*. New York: Holt, Rinehart, and Winston. Translated by: Michael Butler.