Bernoulli Numbers and their Applications

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Abstract

The Bernoulli numbers are a set of numbers that were discovered by Jacob Bernoulli (1654-1705). This set of numbers holds a deep relationship with the Riemann zeta function. The Riemann zeta function has been found to have a relationship with prime numbers. The Bernoulli numbers have also been found to be useful for proofs of a restricted version of Fermat's Last theorem. In this paper Bernoulli numbers will be discussed and the properties of this set of numbers as well as different ways to represent the set of Bernoulli numbers. The applications of this set of numbers in number theory will also be discussed such as those in the aforementioned examples.

1 What are Bernoulli Polynomials?

In the 17th century a topic of mathematical interest was finite sums of powers of integers such as the series $1 + 2 + 3 + ... + (n-1)$ or the series $1^2 + 2^2 + 3^2 + ... + (n-1)^2$. The closed form for these finite sums were known, but the sum of the more general series $1^k + 2^k + 3^k + 4^k + ... + (n-1)^k$ was not. It was the mathematician Jacob Bernoulli who would solve this problem with this equality that will be proven later

$$1^k + 2^k + 3^k + 4^k + ... + (n-1)^k = k! \int_0^n \mathbf{B_k(x)}\, dx.$$

In this equality $\mathbf{B_k(x)}$ is the Bernoulli polynomial which we will define next.

Definition 1.1.
A *Bernoulli polynomial* $\mathbf{B_n(x)}$ is a Polynomial that satisfies three properties

(a) $B_0(x) = 1$;

(b) $B_n'(x) = B_{n-1}(x)$;

(c) $\int_0^1 B_n(x)\, dx = 0$ for $n \geq 1$.

One can prove that these Bernoulli polynomials are unique by using the fundamental theorem of calculus. The first three Bernoulli polynomials are $\mathbf{B_0(x)} = 1$, $\mathbf{B_1(x)} = x - \frac{1}{2}$, $\mathbf{B_2(x)} = \frac{1}{2}x^2 - \frac{1}{2}x + \frac{1}{12}$. From this definition we can derive some interesting properties that become useful later in this paper. The first theorem is easily proved from the definition.

Theorem 1.1. *Given a Bernoulli polynomial* $\mathbf{B_n(x)}$ *the following properties are true*

(a) $\mathbf{B_n(1-x)} = (-1)^n\mathbf{B_n(x)}$.

Proof. Given statement is true $n = 1$. Assume statement is true for n case.

$$\mathbf{B_n(1-x)} = (-1)^n\mathbf{B_n(x)} \Rightarrow \int \mathbf{B_n(1-x)}dx = (-1)^n \int \mathbf{B_n(x)}dx$$

$$\Rightarrow -\mathbf{B_{n+1}(1-x)} = (-1)^n\mathbf{B_{n+1}(x)} + C$$

$$\Rightarrow -\int_0^1 \mathbf{B_{n+1}(1-x)}dx = (-1)^n \int_0^1 (\mathbf{B_{n+1}(x)} + C)dx.$$

From definition1.1c $0 = 0 + C \Rightarrow C = 0$.

$$-\mathbf{B_{n+1}(1-x)} = (-1)^n\mathbf{B_{n+1}(x)} \Rightarrow \mathbf{B_{n+1}(1-x)} = (-1)^{n+1}\mathbf{B_{n+1}(x)}.$$

By induction it is true for all n. □

(b) $\mathbf{B_{n+1}(0)} = \mathbf{B_{n+1}(1)}$ *for* $n \geq 1$.

Proof. This proof can be seen in pg.90[2]

$$0 = \int_0^1 \mathbf{B_n(x)}\, dx = \int_0^1 \mathbf{B'_{n+1}(x)}\, dx = \mathbf{B_{n+1}(1)} - \mathbf{B_{n+1}(0)}$$

$$\Rightarrow \mathbf{B_{n+1}(1)} = \mathbf{B_{n+1}(0)}.$$

□

Now we can define Bernoulli numbers from our notion of Bernoulli polynomials.

2 What are Bernoulli Numbers and how are they related to Bernoulli Polynomials?

Bernoulli Numbers are a set of numbers that is created by restricting the Bernoulli polynomials to $x = 0$ and will formally proceed to define.

Definition 2.1.
A *Bernoulli number* is a number rational number that satisfies the equality

$$\mathbf{B_n(0)} = \frac{B_n}{n!}.$$

This definition of Bernoulli numbers provides a relationship useful in finding Bernoulli numbers given a Bernoulli polynomial. Given the Bernoulli numbers one would like to be able to build Bernoulli polynomials. Given these relationships one can build the Bernoulli polynomials from Bernoulli Numbers and inversely the Bernoulli numbers from the Polynomials. The following theorem will provide the needed relationship.

Theorem 2.1. *Given the first n Bernoulli numbers the following equality is satisfied:*

$$\mathbf{B_n(x)} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}.$$

Proof. Assume statement is true for n case. From Definition 1.1b.

$$\mathbf{B_n(x)} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k} \Rightarrow \int \mathbf{B_n(x)}dx = \frac{1}{n!}\int\sum_{k=0}^{n}\binom{n}{k}B_k x^{n-k}dx$$

$$\Rightarrow \mathbf{B_{n+1}(x)} = \frac{1}{n!}\sum_{k=0}^{n}\binom{n}{k}\frac{B_k x^{n+1-k}}{n+1-k} + C.$$

From Definition 1.1c we find $C = 0$.

$$\mathbf{B_{n+1}(x)} = \frac{1}{n!}\sum_{k=0}^{n}\frac{n+1}{n+1}\binom{n}{k}\frac{B_k x^{n+1-k}}{n+1-k} = \frac{1}{n!(n+1)}\sum_{k=0}^{n}\frac{n!(n+1)}{k!(n-k)!(n+1-k)}B_k x^{n+1-k}$$

$$\Rightarrow \mathbf{B_{n+1}(x)} = \frac{1}{(n+1)!}\sum_{k=0}^{n}\frac{(n+1!)}{k!(n+1-k)!}B_k x^{n+1-k}.$$

By induction it is true for all n given it is true for $n = 1$.

□

The first 6 Bernoulli numbers are $B_1 = -\frac{1}{2}$, $B_2 = \frac{1}{6}$, $B_4 = -\frac{1}{30}$, $B_3 = B_5 = 0$ and $B_6 = \frac{1}{42}$. From this we observe the beginning of a pattern in the signs of the even Bernoulli numbers and the lack of nonzero Bernoulli numbers after $n = 1$. It is from this suspicion that we shall prove some of the properties of this set of numbers.

3 Properties of Bernoulli Numbers

From the first 6 Bernoulli numbers we began to see a pattern in the Bernoulli numbers in the next theorem we shall try to confirm our suspicions.

Theorem 3.1. *The set of Bernoulli numbers satisfy the following properties*

(i) $B_{2n+1} = 0$ *for* $n \geq 1$.

Proof. From Theorem 1.1b and Definition 2.1

$$\mathbf{B_{2n+1}(1)} = \mathbf{B_{2n+1}(0)} = \frac{B_{2n+1}}{(2n+1)!}.$$

From theorem 1.1a setting $x = 0$

$$\mathbf{B_{2n+2}(1-x)} = (-1)^{2n+2}\mathbf{B_{2n+2}(x)} \Rightarrow \mathbf{B_{2n+2}(1-x)} = \mathbf{B_{2n+2}(x)}.$$

Differentiating and applying Definition1.1b

$$-\mathbf{B_{2n+1}(1-x)} = \mathbf{B_{2n+1}(x)} \rightarrow 0 = \mathbf{B_{2n+1}(1-x)} + \mathbf{B_{2n+1}(x)}.$$

Setting $x = 0$ and applying Theorem1.1b

$$0 = 2(\mathbf{B_{2n+1}(0)}) \Rightarrow \frac{B_{2n+1}}{(2n+1)!} = (\mathbf{B_{2n+1}(0)}) = 0$$

$$\Rightarrow B_{2n+1} = 0.$$

□

(b) B_{2n} and B_{2n+2} have opposite signs for $n \geq 1$.

Proof. From Theorem 1.1a

$$\mathbf{B_{2n+2}(1 - x)} = (-1)^{2n+2}\mathbf{B_{2n+2}(x)} = \mathbf{B_{2n+2}(x)}.$$

Differentiating once using chain rule because $B_{2n+1} = 0$

$$-B_{2n}(1) = (-1)^{2n+2}B_{2n}(0) \Rightarrow B_{2n}(1) = -(-1)^{2n+2}B_{2n}(0) = -B_{2n}(0).$$

Signs of B_{2n} and B_{2n+2} differ by -1. □

Now that we know what a Bernoulli number is, and some of its properties we can begin to discuss the Riemann Zeta Function and find its relationship with the Bernoulli numbers and by doing so perhaps find some more properties of Bernoulli numbers.

4 Riemman's Zeta Function and using Bernoulli numbers to calculate even values of the Zeta Function

In the 19th century the famous mathematician Bernhard Riemann was attempting to find the values of a function that he called the zeta function, which is defined as follows:

Definition 4.1.
The *Riemman Zeta Function* is a function defined as the following series for the value s.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

This function proves to be useful in approximating the distribution of the prime numbers. It is also found occassionally in physics as well as statistics. In this paper the function when s is an even integer shall be of particular interest because the Bernoulli Numbers prove to be particularly useful when calculating the even values of the zeta function. A discussion on Fourier series is neccessary because it will be useful in proving the theorem that follows the discussion.

4.1 Fourier Series

Joseph Fourier developed a method to approximate a function by using an infinite series of *sine* and *cosine* terms. The following Theorem which will not be proved will describe this method of decomposition of a function.

Theorem 4.1. *Given a function $f(x)$ with period T we will define its* Fourier expansion *to be the series such that*

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi x}{T}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2n\pi x}{T})$$

where

$$a_n = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(x)\cos(\frac{2n\pi x}{T})$$

and

$$b_n = \frac{1}{T}\int_{\frac{-T}{2}}^{\frac{T}{2}} f(x)\sin(\frac{2n\pi x}{T})$$

over the interval $[\frac{-T}{2}, \frac{T}{2}]$.

These Fourier expansions turn out to have various properties that become quite useful.

Theorem 4.2. *Given a Fourier expansion $\mathbf{F(x)} = \sum_{n=1}^{\infty} a_n \cos(\frac{2n\pi x}{T}) + \sum_{n=1}^{\infty} b_n \sin(\frac{2n\pi x}{T})$ for the function $f(x)$ the following properties are satisfied:*

(a) If $f(x)$ is such that $f(x) = -f(-x)$ then $a_n = 0$;

(b) If $f(x)$ is such that $f(x) = f(-x)$ then $b_n = 0$;

(c) $\int_{-\infty}^{\infty} |f(x)|^2 dx = \frac{1}{T}\int_{-\infty}^{\infty} |F(x)|^2 dx$ (Parseval's Theorem).

Given these properties which will not be proved but instead refer the reader to theorem 8.16 [3] or any fourier analysis text, one can proceed to introduce and prove the following theorem which can be used to calculate even values of the zeta function.

Theorem 4.3. *For a given positive integer k the following equality holds*

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{(-1)^{k-1} B_{2k} (2\pi)^{2k}}{2(2k)!}.$$

Proof. Begin by calculating the Fourier expansion for $\mathbf{B_1(x)} = x - \frac{1}{2}$ in interval $[-\frac{1}{2}, \frac{1}{2}]$. For $f(x) = x$ an odd function such that $f(x) = -f(-x)$ the Fourier coefficient is given by $a_0 = \frac{1}{2}$

$$b_n = 4\int_0^{\frac{1}{2}} x \sin(2\pi n x)\, dx = -\frac{(-1)^{n+1}}{n\pi}$$

then for $[-\frac{1}{2}, \frac{1}{2}]$

$$x = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n x) + \frac{1}{2}.$$

Translation from $[-\frac{1}{2}, \frac{1}{2}]$ to $[0, 1]$

$$x = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n(x - \frac{1}{2})) + \frac{1}{2}$$

For $[0,1]$

$$\Rightarrow \mathbf{B_1(x)} = x - \frac{1}{2} = -\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n\pi} \sin(2\pi n(x - \frac{1}{2})).$$

From Definition 1.1b and Fundamental Therorem of Calculus

$$\int_0^x \mathbf{B_1(x)}\,dx = \mathbf{B_2(x)} - \mathbf{B_2(0)} = 2(\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos(2\pi n(x - \frac{1}{2})) - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos(-\pi n))$$

$$\Rightarrow \mathbf{B_2(x)} = 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^2} \cos(2\pi n(x - \frac{1}{2})).$$

Integrating two more times gives

$$\mathbf{B_4(x)} = -2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^4} \cos(2\pi n(x - \frac{1}{2})).$$

This implies the general formula

$$\mathbf{B_{2k}(x)} = 2(-1)^{k+1}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k}} \cos(2\pi n(x - \frac{1}{2}))$$

which, setting $x = 0$, gives

$$\mathbf{B_{2k}(0)} = \frac{B_{2k}}{(2k)!} = 2(-1)^{k+1}\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{(2\pi n)^{2k}} \cos(-n\pi) = 2(-1)^{k+1}\sum_{n=1}^{\infty} \frac{1}{(2\pi n)^{2k}}.$$

$$\Rightarrow (-1)^{k+1}\frac{(2\pi)^{2k}B_{2k}}{2(2k)!} = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \zeta(2k).$$

□

Certain properties for the Bernoulli numbers follow from this Theorem some of which have already have been proved.

Corollary 4.4. *The even Bernoulli numbers are unbounded or equivalently* $\lim_{k\to\infty} B_{2k} = \infty$.

Proof. The following inequality trivially follows $1 \leq \zeta(2k)$ for positive k because $\zeta(2k)$ begins with 1 in the series followed by positive fractions . Then from Theorem 5.1,

$$1 \leq \frac{(2\pi)^{2k}B_{2k}}{2(2k)!} \Rightarrow \frac{2(2k)!}{(2\pi)^{2k}} \leq B_{2k}$$

since $\frac{2(2k)!}{(2\pi)^{2k}}$ is divergent then B_{2k} must be divergent by comparison. □

Now we begin to use the preceding theorem to calculate the values of the Riemman zeta function for even values.

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2} = (-1)^{1+1}\frac{(2\pi)^2 B_2}{2(2)!} = \pi^2 B_2$$

Recalling $B_2 = \frac{1}{6}$ gives

$$\zeta(2) = \frac{\pi^2}{6}$$

This value of the zeta function was solved for by Euler in the 18th century. The value for $k = 4$ was not known at the time Euler solved for $k = 1$ but the following calculation finds an exact value for $k = 4$.

$$\zeta(8) = \sum_{n=1}^{\infty} \frac{1}{n^8} = (-1)^5 \frac{(2\pi)^8 B_8}{2(8)!} = -\frac{2^7}{40320}\pi^8 B_8 = -\frac{128}{40320}\pi^8 B_8 = -\frac{1}{315}\pi^8 B_8.$$

From $B_8 = -1/30$ gives

$$\zeta(8) = \pi^8 - \frac{1}{315} - \frac{1}{30} = \pi^8 \frac{1}{9450}.$$

From these calculations one sees the power of the previous theorem in calculating values for the zeta function that would require proving individual cases or a similiar theorem.

5 Further reading on Bernoulli numbers and some final notes on Bernoulli numbers

A rigorous introduction to Bernoulli numbers is found in Kenneth Ireland's Number Theory book [1]. In this book various theorems are presented and proved. It also contains an alternate proof to Theorem 5.1 that could be of interest to the reader who does not want to use fourier analysis in proving said theorem. Excursions in Calculus contains the following corollary that Bernoulli sought to answer the question on the sums of powers.

Corollary 5.1. *The sum of the kth powers of the first n-1 integers is given by*

$$1^k + 2^k + 3^k + 4^k + ... + (n-1)^k = k! \int_0^n \mathbf{B_k(x)}\, dx.$$

Proof. Assume statement $\mathbf{B_{n+1}(x+1)} - \mathbf{B_{n+1}(x)} = \frac{x^n}{(n)!}$ is true for n+1;

$$\frac{d}{dx}(\mathbf{B_{n+2}(x+1)} - \mathbf{B_{n+2}(x)}) = \mathbf{B_{n+1}(x+1)} - \mathbf{B_{n+1}(x)} = \frac{d}{dx}(\frac{x^{n+1}}{(n+1)!}) = \frac{x^n}{(n)!}$$

then combining with finite sum

$$1^k + 2^k + 3^k + 4^k + ... + (n-1)^k = k!\sum_{j=0}^{n-1}[\mathbf{B_{n+1}(j+1)} - \mathbf{B_{n+1}(j)}] = k![\mathbf{B_{n+1}(N)} - \mathbf{B_{n+1}(0)}]$$

$$\Rightarrow 1^k + 2^k + 3^k + 4^k + ... + (n-1)^k = k! \int_0^n \mathbf{B_k(x)}\, dx.$$

□

References

[1] Ireland, Kenneth F., A Classical Introduction to Modern Number Theory New York : Springer-Verlag, c1990

[2] Young, Robert M. ,Excursions in calculus : an interplay of the continuous and the discrete Washington, D.C. , Mathematical Association of America, c1992.

[3] Rudin, Walter ,Principles of Mathematical Analysis, McGraw-Hill, 1976.