MULTIREGIONAL INPUT-OUTPUT MULTIPLIERS
AND
THE PARTITIONED MATRIX SOLUTION OF THE AUGMENTED MRIO MODEL

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SUBMITTED IN PARTIAL FULFILLMENT
OF THE REQUIREMENTS FOR THE
DEGREE OF

DOCTOR OF PHILOSOPHY

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
(February, 1980)

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Department of Urban Studies and Planning, February 1980

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TO

My Parents,

My Wife

and

My Sons

for their love and patience

ABSTRACT OF THESIS:

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Submitted to the Department of Urban Studies and Planning on October 12, 1979 in partial fulfillment of the requirements for the degree of Doctor of Philosophy

The multiregional input-output (MRIO) model is being used as a working tool for interregional production, income, and employment impact analysis. The model conceptually distinct from the pure interregional input-output model in that it explicitly introduces an interregional trade coefficient matrix.

The objectives of this dissertation are twofold: (a) to systematically explore the multitude of specialized submatrix and scalar output, income, and employment multipliers that can be obtained from, or with the aid of, the open MRIO model multiplier matrix; and (b) to demonstrate the relationship between the augmented and the open MRIO model multipliers, and the relationship between the augmented MRIO model multipliers and the more commonly encountered Keynesian multipliers.

In the course of investigation it was found that the standard method for augmenting the MRIO model is not appropriate for determining the relationship between the augmented model multipliers and both the open model and Keynesian multipliers. For that reason an alternate partitioned matrix solution has been developed for the augmented MRIO model. This solution is generalized, and it proves to be computationally more efficient.

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ACKNOWLEDGEMENTS

This dissertation could not have been realized without the active encouragement and support of many people. To all of them my sincerest thanks even if I have unintentionally omitted mentioning them and the nature of their support.

One person, Professor K. Polenske, was central in the selection and development of the topic. Throughout my years at M.I.T. she was a source of advice and moral support in her capacities as an academic advisor, research supervisor (on the MRIO project) and finally as Thesis supervisor. I have benefitted in many ways from her guidance and assistance.

Many friends and colleagues helped in innumerable ways from general Thesis-related conversations to specific suggestions to the actual reading of portions of the manuscript. These included Dr. R.Bon, E.Bon-Rudin, W.Crown, Professor J. Harris, Professor D.Kresge, Dr.G.Pai, Dr.A.Stadecker, Dr.P.Strigini, and Professor B.Von Rabenau.

Typing this manuscript involved the skills of Ms.M.Ange Le, B.Edwards, N.Gomez, P.Kelly, M.Maginly, P.Tri, and S.Vidanis. I am thankful to all of them and, in particular, to Ms. Betty Bain who was helpful at every stage of the typing.

I am also indebted to the International Bank for Reconstruction and Development for authorizing a nine month Study Leave of Absence and for providing some financial assistance.

Finally, I would like to thank my whole family for assisting me in a multitude of ways and enduring my need for partial withdrawal from normal family life. In particular, I would like to thank my mother, Prita, and my wife, Isa, for their invaluable assistance at all stages of the dissertation's production.

Needless to say, all errors that remain are due to my ignorance or oversight and should not be attributed to any of the persons or institutions mentioned above.

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The multiregional input-output scheme described below is not intended to provide a systematic theoretical description of the many factors and relationships that ultimately determine the pattern of a multiregional economic system; it is designed rather as a rough and ready working tool capable of making effective use of the limited amount of factual information with which, even in the statistically advanced countries, economists have to work.

(W. Leontief and A. Strout, 1963, p. 224)

The intellectual case for I-O analysis rests very largely on the fact that it is a great rarity in economics --- an operational general equilibrium system. It enables us not only to identify but, at the cost of certain rigid assumptions, to <u>measure</u> the interdependence of the economic structure. Since it may be argued that its general equilibrium character is the main virtue of the I-O approach, then this character should not be sacrificed lightly. When we introduce space and distance into the economy, however, as in regional I-O analysis, it is very difficult to retain the general equilibrium features of I-O theory. The most widespread regional I-O model in common use is the single region model. This is a <u>partial</u> model in its preoccupation with economic impacts affecting the study region alone and in its aggregation of the rest of the world into one other region. The interdependence of the local industrial structure is retained, but the model throws no light on the interdependence of economic regions. ... In short, the single region model allows us to take account of local interindustry feedbacks but neglects interregional feedbacks.

(H. W. Richardson, 1972, pp. 53-54)

Introduction

In recent years, it has become increasingly necessary for policy analysts to explicitly outline the ex-post, ex-ante or hypothetical consequences of different public policies and programs. In general, the consequences to be analysed are complex, involving social and political ramifications. However, if the qualitative, or intangible aspects of impacts (and stimuli) are excluded from consideration, analytic economic models can be used to establish stable quantitive relationships between exogenous variables, which are determined outside the models, and endogenous variables, which are determined within the models. These stable relationships can then be used to determine not only the type and direction, but also the magnitude of the changes in the endogenous variables that are consistent with changes in the exogenous variables.

The technical concept used for this purpose, in economic analysis, is the multiplier. The two best known multipliers are the comparative-static Keynesian and Leontief multipliers, both introduced into economic literature in 1936.[1/] They have been used in single economies, at both the national and sub-national levels, to analyze aggregate income formation and industrially disaggregated gross output production respectively. Since their introduction, both types of multipliers have been generalized in two directions. In one direction,

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1/ All formal economic models contain multipliers, including the more complex econometric models which incorporate ad hoc hypotheses to simulate the complexity of actual economies. However, as noted by Hirsch (1977, p. 62) multipliers in dynamic econometric models, which can be calculated through simulation, cannot be derived analytically, since they contain many complex lag structures, as well as non-linearities. Therefore, it is difficult to demonstrate and clarify the properties of the multiplier in these models, despite their greater "realism."

dynamic versions of the less aggregated models have been developed. In the other direction, the level of disaggregation of the comparative-static versions has been increased. This second development is most frequently encountered in operational models that have been developed to analyse interregional industrial impacts.

One well known model of this type is Isard's "pure" Interregional Input-Output model. Because this model bears a tremendous formal analogy to single region Input-Output models (both at the national and sub-national levels), it has been used almost exclusively as the basis for adapting theoretical developments in Input-Output analysis to the interregional context. With the exception of the 1963 Japanese model, however, very few Interregional Input-Output models have been made operational in large and complex economies because of their extensive data demands.

Another type of interregional model which can be used for interregional production, income and employment impact analysis, is the Chenery-Moses Multiregional Input-Output (MRIO) model. This model is conceptually distinct from the "pure" Interregional Input-Output model in that it explicitly introduces an interregional trade coefficient matrix. Many of the theoretical developments in Interregional Input-Output models have not yet been adapted to the special constraints of the MRIO model. An adaptation of some of the technical developments to the MRIO model is more necessary now than in the past because the structure of the model has reduced data requirements by two thirds[2/] and thus, the construction of very large-scale operational Multiregional Input-

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[2/] From (mn x mn) to $(mn^2 + nm^2)$, where m = number of industries and n = number of regions.

Output models has become possible, and the previous need to trade-off regional versus commodity disaggregation has in part been overcome.[3/]

The MRIO model is a linear, comparative-static, demand model, and as such not as theoretically general as non-linear, dynamic and/or supply-constrained models. However, it still provides analytically useful information not available in the more sophisticated single region models. The MRIO model provides a basis for analysing the interrelations between production and trade in various sectors and regions of a large and complex economy. With its aid, it is possible to analyse the differential regional consequences of a national policy or, in the case of each region, to analyse impacts which take into account interregional and inter-industry feedback effects.

In its current form it is already being used as a working tool.[4/] In the past five years numerous studies have been produced[5/] which have analysed multiregional impacts with the aid of the MRIO multiplier matrix. To increase the model's utility an augmented version of the model, incorporating the "household sector", was implemented in 1977.

3/ The present operational U.S. MRIO model is of this type. It has increased regional disaggregation to embrace all 50 states of the Union plus the District of Columbia, while retaining considerable commodity detail (for 79 industries).

4/ Even though the MRIO model does not fully articulate the underlying factors which determine the pattern of trade and the structure of production in a multiregional economic system, it is most likely to form the starting point of other, more ambitious, operational models of interregional interdependence, when the data become available.

5/ A partial list would include Faucett (1975), Kim, Park and Kwak (1975), Rowan (1976), Golladay and Haveman (1977), Rowan (1977), Polenske and Rowan (1977), Holmer (1979),...

Despite the use of the MRIO model for impact analysis, none of the studies have explicitly analysed the structure of the open MRIO multiplier matrix, nor have they systematically explored the multitude of specialized submatrix and scalar output, income and employment multipliers that can be obtained from, or with the aid of, the MRIO multiplier matrix. Even less attention has been paid to the analysis of the augmented MRIO model multipliers. As a consequence, in contrast to single economy Input-Output models, it has not been possible, with the current formulation of the augmented MRIO model, to demonstrate the relationship between the augmented and the open MRIO model multipliers, or the relationship between the augmented MRIO model multipliers and the more commonly encountered Keynesian multipliers.

Therefore, unlike other studies which either estimate the impacts of a specific policy (whether actual or hypothetical), or determine the empirical magnitude or stability of a given set of coefficients or multipliers, this dissertation is motivated by the need to explore methodological issues in MRIO multiplier analysis with the purpose of increasing the operational and theoretical flexibility of the model within the context of the on-going MRIO research project.

In the interest of streamlining the text, while keeping the dissertation reasonably self-contained, a literature review of selected Keynesian and Input-Output comparative-static multipliers, (describing their purpose, capabilities and limitations), has been provided in Annex A.

Chapter 1

MULTIREGIONAL INPUT-OUTPUT ANALYSIS

In 1936, two simplified models of the economy were introduced into economic analysis which over the last four decades have proven to have considerable operational utility. The first was the aggregate Keynesian macroeconomic model which has been used to determine the impact of a change in part of national income on total national income via propensity based multipliers. The second was the disaggregated Leontief interindustry model which made operational the general equilibrium logic of mutually dependent activities. This model has been used to determine the impact of a changed net output on gross output via technical coefficient based multipliers. The introduction of these two types of models led naturally to an investigation of income formation in mutually dependent sub-economies connected by trade. International trade models, whose main purpose was to explain the level of trade between countries, were the first to be used as suitable vehicles for introducing inter-economy multiplier analysis.

One of the earliest theoretical formulations was the two region, single sector, national income model of F. Machlup (1943). This single sector model was later generalized to a multiregional framework by L.A. Metzler in his investigation of "the mechanism by which an expansion or contraction of income in one region or country is transmitted to other regions or countries" (L.A. Metzler, 1950, p.329). In order to include a number of countries these models sacrificed commodity detail, that is, they used national aggregates instead of explicitly introducing the internal structure of production in each country.

Another line of development was introduced by W. Isard (1951 and 1953). This was the 'pure' interregional input-output framework. Unlike the international trade models, this type of model was based on the disaggregation of production and consumption in a single country by both commodity and region. Although formally there was little difficulty in combining the two types of disaggregation, the empirical problems were multiplied because in this model the sources of supply of each commodity input were identified separately for each type of use, for every region. This model has seldom been used since it requires information that is rarely available. The most notable exception is the Japanese interregional model (K. Miyazawa, 1965, and K. Polenske, 1965).

An alternate approach was introduced by W. Leontief (1953). This was the intranational input-output model where commodities were distinguished by whether their supply and demand was balanced at the regional or the national level. Since various commodities balance at different regional levels (for example, sub-regional, regional or supra-regional, but not national), this model could not be used for analysing an individual region's production structure on a basis that was comparable to other similarly defined regions.

The multiregional input-output analysis was introduced by H. Chenery (1953) as an adaptation of the Metzlerian international trade model to a two region national economy (Chenery and Clark, 1959, p. 312). This approach was more data efficient than the pure interregional input-output approach because it did not require identifying the regional source of each input. It also made possible the analysis of the production structures of various regions, something which the intranational input-output model could not do. The multi-regional input-output analysis was subsequently generalized to include more than two regions in a column coefficient trade-flow version by H. Chenery (1956) and L. Moses (1956), and in a gravity coefficient trade-flow version by W. Leontief and A. Strout (1936).

The Theoretical Multiregional Input Output Model[1/]

The pure interregional I-O model is based on a very elaborate technical production matrix in which identical commodities in terms of their physical characteristics are differentiated and treated as different commodities based on their region of origin. Whatever the utility of this approach, it is difficult to implement in very large and complex economies. The multiregional I-O model is based on a more data and cost efficient approach. Despite some loss in information the model is capable of addressing most of the problems that can be addressed by the pure interregional I-O model.

The feature which most clearly distinguishes the MRIO model from the national, regional, and pure interregional I-O models is the interregional trade flow matrix.

This matrix is derived from a trade flow table whose elements refer to gross flows rather than net flows between regions. As a result the MRIO model has a special property. It includes the possibility of the same commodity being cross hauled between any pair of regions. This

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[1/] The reader who is not familiar with the theoretical and empirical foundation of multiregional input-output models in general is advised to refer to Chenery and Clark (1959, pp. 65 - 70, and Chap. 12). More advanced material on multiregional models can be found in Polenske (1970 and 1972a). For a brief review of the conceptual differences between national, regional and interregional models see Richardson (1972, Chap. 4). For a similar brief review of the conceptual differences between interregional, multiregional and intranational accounts, see Polenske (1978a, pp. 8 - 10), and for a slightly more detailed discussion and the policy implications of the same subject, see Round (1974, pp. 43 - 115).

ability to allow a simultaneous flow of the same good in opposite directions is a desirable and useful characteristic, because in actual empirical analysis, the commodities will, as a rule, be defined as aggregates of several similar but not strictly identical items, while regions will often represent more or less extended areas rather than location points. In addition, interregional commodity flows will usually be measured over an interval of time (as long as an entire year) instead of at a point in time. Such aggregation over time will also show cross-hauling where there are shipments in opposite directions in different months. As a result of the three types of aggregation mentioned above, cross-shipments can be expected and are actually observed regularly.

Thus, in constrast to the interregional I-O model "the peculiar theoretical problem of multiregional input-output analysis stems from the simple fact that identical goods can be, and actually are, produced and consumed in different regions" (W. Leontief and A. Strout, 1963, p. 224).

To transform the trade flow data into coefficients which can be used in an analytic model it is necessary to assume that the interregional trade flows are based on certain stable relations. Within the MRIO framework the most general relationship is assumed in the gravity coefficient version. In this version of the model it is assumed that the regional origin of the particular batch of a given kind of good absorbed by its users, is as irrelevant to them as the ultimate regional destination of the output is to the producers. Thus, all interregional movements of a particular commodity or service within a multiregional economy can be treated as shipments from regional supply to regional demand pools of that good.

In the alternate, less general, column-coefficient version of the MRIO model, it is assumed that the sources of supply are fixed for all uses of a given commodity in a given region, rather than depending on the type of use.[2/] Hence, all uses in a region constitute a single market and the supply patterns are determined more by total demand than by the nature of the intended use. By expressing the total production of a given commodity in one region as a function of the total demands in all regions, it is possible to explain the level of exports of each commodity from a region in terms of the demand for imports of that commodity in all regions. With this assumption it is possible, therefore to limit data requirements for each commodity, to total demand rather use-specific interregional flows.

Conceptually there is no problem in increasing the number of regions beyond two, even though the empirical problems increase rapidly with the number of regions involved. The only significant difference between a two region and a multi-region model is that it is no longer possible to assume that an export from one region is an import of the other, because indirect regional interdependence gives rise to what is commonly called triangular or multi-lateral trading patterns. As a result, direct measures of interregional trade are essential.

2/ Whether marginal or average ratios are used in the interregional trade matrix, the sum of the supply coefficients for a given commodity, including intra-regional supply, must equal one (Chenery and Clark, 1959, p. 67).

The Operational MRIO Model

The preceeding developments in multiregional analysis led to the establishment of the multiregional input-output research program by W. Leontief, K. Polenske, et al. (1967), with the objective of developing an operational multiregional input-output (MRIO) model for the U.S. The first operational model was implemented in 1970 (K. Polenske, 1970 and 1972).

The current model was completed in 1973. It is an expanded version of the 1970 model (from 44 regions and 79 industries, to 51 regions and 79 industries). It also incorporates revisions to the 1963 data base. The most disaggregated level of regional data is for individual states (including the District of Columbia), because

> although considerable effort has been made by some analysts to show that an economic spatial unit other than the state would be more appropriate for regional economic analyses, the fact remain that most data are available for states, not for other regional classifications. Also, state data can be easily aggregated into some of the more common regional groupings, such as census regions; or disaggregated; or split into other regional units required for particular regional studies. The desired spatial unit will obviously vary depending upon the economic analysis that is to be made. ... For a sizable number of regional policy decisions, however, the state political body is responsible for implementing the policies; and data compiled by states may be the most appropriate for studies of the economic impact of these policies. (K. Polenske, 1972, p. 4).

The mammoth size of the model, requiring in its initial construction over half a million bits of information, necessitates that careful attention be paid to accounting conventions and data assumptions before the results are actually applied. In this section only some of the conventions will be highlighted. Additional detail on the data estimates, model structure and computational procedures are available in the selected references listed in Annex B.1.

Treatment of External Trade

In input-output accounts imports are often separated into two categories, noncompetitive imports and competitive imports. With slight definitional differences, the corresponding categories in MRIO accounts are directly allocated imports and transferred imports respectively.

An import is classified as directly allocated, or noncompetitive, if a final user purchases the import in a substantially unaltered form, or if there is no domestic production of the good or service or, at least, no close substitute.[3/] These imports are usually incorporated in a row vector of the model, and thus, they are treated as inputs purchased by each industry.

An import is classified as transferred, or competitive, if the product is produced domestically and if the import is not directly consumed by a final user. These imports can be incorporated either as a positive row vector or a negative column vector in the model.

In the operational MRIO model these transferred imports have been recorded as a separate row vector of inputs (Polenske, 1975, p. 12, and Polenske, et al, 1972, p. 19). Thus, the MRIO model is derived from a total supply base transactions table.[4/] As a result the

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3/ Substitutability is generally determined on a judgmental basis "using the following guide: the import should be interchangeable with a domestically produced item without any change in the technology of the consuming industry or the resultant product." (Polenske, 1975, p. 12).

4/ Had the transferred imports been treated as a negative column of final demand, the input coefficients would have been larger, reflecting the smaller denominator of a gross domestic output base table rather than a total supply base table. In addition, element by element combination of the negative column of transferred imports with the positive column of gross exports would result in a column whose elements were either negative or positive depending upon the value of the competitive imports relative to the value of the gross exports for each industry.

elements in the foreign export component of the final demand vector represent gross exports from the United States, with two exceptions: negative entries appear at the intersection of the export column with the two import rows. These elements represent the sums of the two import rows of the national input-output table. They are entered as negative elements, in order to obtain net exports if the entire column of exports is totalled. This net export figure can be negative if total imports exceed total exports in a given year.

The state external export and import estimates are assembled using two separate methodologies (explained in detail in Polenske, 1974, Chapter 4). In the first, exports and imports are allocated to the port of exit and the port of entry respectively. In the second, the exports and imports are allocated to the region of production or the region of consumption respectively. For most analytic purposes the latter approach is preferred and has been incorporated, therefore, into the interregional trade estimates.

Treatment of Interregional Trade

In the MRIO model, the interregional trade flows are meant to account for the complete distribution of each commodity among regions, i.e. they are to account for intra-regional as well as, interregional shipments of each commodity whether supplied from or to domestic sources or foreign sources. Before the interregional trade estimates can be incorporated into the MRIO framework, the estimates have to be reconciled with independent estimates of regional production and regional consumption, as given in the regional input-output matrices. Three factors contribute to the need for consistency adjustments (J. M. Rodgers, 1973, p. 45).

First, all of the MRIO data sets directly related to trade --- regional production, consumption, and interregional commodity shipments --- are estimated from different sources and are therefore subject to estimation errors.

Second, the MRIO model is based on commodity data. Thus, it is designed to describe and operate with interregional flows of specific commodities rather than the interregional movement of goods produced by specific establishments. This requires that the products of an industry that are produced as secondary products by other industries be transferred into the total value of industry shipments and included in the trade flows to the various destinations. Secondary products produced by each industry, however, are not distributed to specific regions of destination, but instead are treated as a shipment to an artificial destination termed "regional transfers-out" (RTRO).[5/] As a result of these conventions, secondary products are double-counted. To highlight the double-counting of the secondary products and to isolate them from the actual commodity flows of the goods (even though they are incorporated in the interregional shipments figures used in the calculation of the trade coefficients), the regional transfers-out (RTRO) are identified separately in a row and column of each interregional trade matrix.

5/ The RTRO accounts also include receipts from activities not directly related to the production and sale of commodities, such as contract research, rents and royalties.

Third, the values of some intra-regional shipments tend to be underestimated because the trade flow estimates, particularly the interregional flows, are based on data for production that is transported. Therefore, they do not include changes in the values of manufacturers' inventories or those intraplant shipments that are considered to be a part of output in the input-output accounts. Production excluded from the intra-regional shipments are particularly important in agriculture, where many commodities are consumed directly on the farms, and in those manufacturing activities where production has a long gestation period resulting in large work-in-process inventories.

The interregional trade and regional production and consumption estimates are reconciled on an iterative basis in light of accounting identities and structural provisions to be discussed in the next section. The final result is a regional trade matrix for each commodity. Each row of the matrix will specify the shipments of the commodity from one producing area (origin) to all consuming areas (destinations). Conversely, each column of the matrix will show the amounts of a commodity received from all producing areas by each consuming area. To distinguish interregional trade within a national economy from external trade with other national economies, the convention has been adopted to refer to the more inclusive imports and exports at the regional level as inflows and outflows respectively.

Treatment of Regional Input-Output Transactions

The MRIO model, as already noted, is commodity based. This requires that establishment data be transformed into commodity data by estimating secondary products separately. In establishment data, the purchases of each industry include the inputs required to produce its primary products, plus the inputs required to produce its other products, normally referred to as secondary products. To obtain an accounting balance between total inputs and total outputs of an industry on a commodity basis, the secondary products are transferred to the industries in which they are produced as primary products. This results in double-counting in both the national control input-output table and in the multitude of regional input-output tables that are used as components of the MRIO model. However, in the MRIO framework, unlike the national framework, the secondary products are isolated in a single exogenous column of transfers-out and a single exogenous row of transfers-in for each regional input-output table[6/].

In the absence of the required survey-based data for all regions on a comparable basis, it was necessary to use non-survey techniques to distribute the transactions of the 453-order national input-output table to all regions. These were then aggregated to 79-order regional input-output tables. Thus, the regional input-output coefficients derived from these transactions tables reflect regional differences in product-mix rather than variations in the technology required to produce the "same" aggregate commodity. In any case, variations in product-mix, rather than in technology, are likely to be the more important sources of differences in regional I-O

[6/] The advantages and alternatives to this approach are noted in Fig. B.2.1, footnote 2.

coefficients at the level of industrial aggregation used in the MRIO model.

A common feature of input-output tables, at both the national and regional levels, is that the sums of corresponding rows and columns of the tables are equal, since by definition total consumption must equal total production for each commodity. This accounting identity is still retained in the complete MRIO table at the national level. However, in the MRIO framework the sums of corresponding rows and columns of the individual regional tables are not necessarily equal for most industries, with the differences being attributable to interregional trade. Only if a commodity produced in a region is not traded, is it possible for the corresponding row and column sums of the commodity to be identical, as occurs, for example, in many of the service industries. In other words, interregionally traded goods are balanced at the national level and non-traded goods at the regional level.

The reconciliation of regional differences in industrial production and consumption, within the national framework, is made possible by a pair of accounting identities and a pair of complementary structural provisions. The accounting identities are:

(1) The nation-wide production (consumption) of each industry is the mathematical sum of the production (consumption) of that industry in all regions;

and (2) The nation-wide production of each industry is equal to the nation-wide consumption of that industry.

The complementary structural provisions are:

(3) The output of an industry produced in a region is not necessarily equal to the output of that industry consumed in the same region,

and (4) Total interregional trade originating in producing sectors is equal to total interregional trade delivered to consuming sectors for each industry.

Provision (4) is necessary for (3) to be consistent with (1) and (2), that is, the difference between the regional production and the regional consumption of the products of an industry in each region has to be made up by interregional trade flows into or out of that region.

Thus, with the MRIO model it is possible to describe and analyse the shipments to and from all regions, as well as the sales and purchases of all industries in the economy, within a unified framework. It is also possible, therefore, for impacts at the regional level to vary with both the degree of regional self-sufficiency, and the degree of internal integration of each region's production structure, and yet remain consistent with the overall impact at the national level.

The formal MRIO model

Mathematically, the standard single economy input-output relationship can be expressed as a set of linear equations:

$$x_i = \sum_{j=1}^{m} a_{ij} x_j + y_i, \qquad \text{for all } i \qquad (1.1)$$

where x_i = total supply of commodity i (or the row sum x_{io});

$a_{ij} = \frac{x_{ij}}{x_j}$ = a technical coefficient representing the amount of input of commodity i required by industry j to produce one unit of output of commodity j;

x_i = total amount of commodity i required by industry j;

x_j = total production of commodity j (or the column sum x_{oj});

y_i = final demand of commodity i;

i,j = 1, ... ,m.

In this model $x_{io} = x_{oj}$, and $\sum_{i=1}^{m} a_{ij} < 1$.

If an economy is assumed to consist of a set of n autarkic regional subeconomies, then assuming no trade between the regions, the input-output model for the m industries and n regions can be represented by an analogous set of linear equations:

$$x_i^g = \sum_{j=1}^{m} a^g_{ij} x_j^g + y_j^g, \qquad \text{for all } i \tag{1.2}$$

where

$a^g_{ij} = \frac{x^g_{ij}}{x^g_j}$ = a technical coefficient representing the amount of input of commodity i required to produce one unit of output of commodity j in region g; and, $\sum_{i=1}^{m} a^g_{ij} < 1$

x_i^g = total supply of commodity i in region g;

x_j^g = total production of commodity j in region g;

y_i^g = final demand of commodity i in region g;

i,j = 1, ..., m; and g = 1, ..., n.

In this formulation the industrial supplies x^g_{jo}, and demand x^g_{oj}, in each region g are still equal since there are no transactions between any of the regions.

If equation (1.2) is to be used to describe a multiregional model, however, it must be modified to account for the commodities traded between the regions. We have already noted at the beginning of this chapter that there are different approaches to incorporating interregional trade, each utilizing a different accounting scheme for trade. Within the multiregional framework three different approaches to estimating trade coefficients have been proposed: the column coefficient, the row coefficient, and the gravity coefficient versions. For a detailed description of their structure and accounting frameworks, see Polenske (1972c, pp. 67 - 76).

The operational MRIO model has been implemented with all three versions of the trade coefficients. Of these versions the Chenery-Moses fixed supply coefficient (i.e., column-coefficient) version of the MRIO model has proved to be conceptually sounder than the present formulation of the row-coefficient version (Bon, 1975), and operationally more convenient than the gravity-coefficient version (Fencl, 1973, and, Fencl and Ng, 1974). Hence, the column-coefficient version of the MRIO model will be used for the subsequent analysis in this dissertation.

Column Coefficient MRIO Model

The column coefficient version of the model has been described in detail by Chenery (1953), Moses (1955), and Polenske (1972a). Hence, only the basic set of equations and the notations to be used throughout the remainder of this chapter, are presented here for n regions and m industries.

Interregional trade is described in the column coefficient model by means of the following relationship:

$$x_i^{gh} = c_i^{gh} x_i^{oh}, \qquad \text{for all } i \qquad (1.3)$$

where

x_i^{gh} = the amount of commodity i produced in region g that is shipped to region h;

x_i^{oh} = the total amount of commodity i consumed in region h;

$c_i^{gh} = \dfrac{x_i^{gh}}{x_i^{oh}}$ = a trade parameter, indicating the fraction of total consumption of commodity i in region h that is produced in and shipped from region g;

and $\sum_{g=1}^{n} c_i^{gh} = 1$

where $i = 1, \ldots, m$; and $g,h = 1, \ldots, n$.

The balancing equations of supply and demand for the open MRIO model can now be stated by combining (1.2) and (1.3) as a set of linear equations:

$$x_i^g = \sum_{h=1}^{n} c_i^{gh} \left(\sum_{j=1}^{m} a_{ij}^h \; x_j^h + y_i^h \right) \qquad (1.4)$$

or in matrix notation as:

$$X = C(\hat{A}X + Y) \qquad (1.5)$$

where

$X = \begin{bmatrix} x_i^g \end{bmatrix}$ = is an (nm × 1) column vector of the total gross outputs of the m-producing industries i, in each of the n-shipping regions g.

$Y = \begin{bmatrix} y_j^h \end{bmatrix}$ [7] = is an (nm × 1) column vector of the final demands of the m-purchasing industries j located in each of the receiving regions h.

---

[7] In equation (1.4) the term y_i^h is used instead of y_j^h. The reason for this will be explained in the next chapter.

$$\hat{A} = \begin{cases} \left[A^{gg}_{jj}\right]^{8/} = \left[A^{g}\right] \\ \left[A^{gh}_{ij}\right] = \left[0\right] \end{cases}$$

is the (mn × mn) expanded block diagonal matrix of regional production coefficients (see Figure B.2.3). There are n sub-matrices A^g, each of dimension (m × m) containing the technical coefficients for the m-industries in a specific region g (see Figure B.2.2). Each of these sub-matrices is located, as a block, along the main diagonal of the matrix $\hat{A}$. Elements of all the (n^2-n) off-diagonal blocks are zero.

$$C = \begin{cases} \left[C^{gh}_{ii}\right] = \left[\hat{C}^{gh}\right] \\ \left[C^{gh}_{ij}\right] = \left[0\right] \end{cases}$$

is the (mn × mn) expanded interregional trade coefficient matrix (see Figure B.2.7) composed of n^2 blocks of (m × m) diagonal submatrices $\hat{C}^{gh}$ with the trade coefficients for all commodities, between each pair of regions, along the principal diagonal of each submatrix and all off-diagonal elements equal to zero. (see Figure B.2.6)

This set of equations can be expanded into the form

$$X = C\hat{A}X + CY \qquad (1.6)$$

or

$$X = \theta X + CY \qquad (1.7)$$

where,

$\theta^{9/} = C\hat{A} = (\theta^{gh}_{ij})$ is the (mn×mn) 'full' matrix[10/] of interregional input coefficients which show the amount of commodity i produced in and shipped from region g for use in the production of one unit of commodity j in region h.

Thus, $X - \theta X = CY$, and the solution of the multiregional input-output set of equations can be written as:

$$X = (I - \theta)^{-1}\, CY \qquad (1.8)$$

8/The symbols (ii) and (ij) in the case of the submatrices of $\hat{A}$ and C refer to the position of the submatrices and not to the coordinates of the technical and trade coefficients.

9/This MRIO trade adjusted production matrix θ must not be confused with the generalized MRIO trade matrix θ used by Bon (1975).

10/The term 'full' is used to distinguish the coefficient density of this matrix from its component matrices, i.e. the sparser block-diagonal matrix $\hat{A}$ and the matrix C with diagonal submatrices.

The matrix $B = (I-\Theta)^{-1}$ can also be approximated by an iterative solution of the form

$$B = I + \Theta + \Theta^2 + \Theta^3 + \ldots., \text{ if } |\Theta|<1 \quad (1.9)$$

where $\sum_{\alpha=0}^{\infty} \Theta^{\alpha}$ can be replaced by $\sum_{\alpha=0}^{n} \Theta^{\alpha}$ with Θ^n reflecting the desired degree of accuracy.

Chapter 2

OPEN MRIO MODEL MULTIPLIERS

In this chapter one of the two formulations of the solution of the open column coefficient MRIO model will be shown to be the more general formulation, which is also more useful for purposes of interpreting the multipliers of the unaugmented and augmented MRIO models. In addition, various types of submatrix and scalar multipliers will be described, including those that can be obtained from the MRIO model which cannot be obtained from the single economy I-O models.

The MRIO Multiplier Matrix D

The solution for the open column coefficient MRIO model can be specified in two forms:

as in equation (1.8), i.e.

$$X = (I - C\hat{A})^{-1}CY$$

or as

$$X = (C^{-1} - \hat{A})^{-1}Y \qquad (2.1)$$

The latter formulation is the one most commonly used in the MRIO research project. However, for purposes of interpreting the MRIO multipliers it is not the preferred formulation. There are two reasons for this.

First, the formulation in (2.1) implies that $|C| \neq 0$, since (1.8) and (2.1) are equivalent only under this condition. This means that, among other things, the expanded trade matrix C cannot have zero columns or zero rows. Bon (1975, p. 10) suggests that these criteria can be ensured if at a minimum all goods are both produced and consumed in every region. This requirement, however, is incomplete, as can be illustrated in a two-commodity, two-region trade matrix. The criterion, as stated, holds for a case such as:

$$\begin{array}{cc} & \begin{array}{cc} \quad R1 \quad & \quad R2 \quad \end{array} \\ \begin{array}{c} R1 \\ \\ R2 \end{array} & \left[\begin{array}{cc|cc} c_1^{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & c_2^{12} \\ \hline 0 & 0 & c_1^{22} & 0 \\ 0 & c_2^{21} & 0 & 0 \end{array}\right] \end{array}$$

Here the first commodity is produced for internal consumption only in both regions, whereas the second commodity is produced in each region for consumption in the other region, an extreme (and irrational) case of cross-hauling. The net result, however, is that both commodities are produced and consumed in both regions.

The criterion does not, on the other hand, rule out the following case

$$\begin{array}{cc} & \begin{array}{cc} \quad R1 \quad & \quad R2 \quad \end{array} \\ \begin{array}{c} R1 \\ \\ R2 \end{array} & \left[\begin{array}{cc|cc} c_1^{11} & 0 & 0 & 0 \\ 0 & c_2^{11} & 0 & 0 \\ \hline 0 & 0 & c_1^{22} & 0 \\ 0 & 0 & 0 & c_2^{22} \end{array}\right] \end{array}$$

Here the C matrix is reduced to the identity matrix I, (with $c_i^{gg}=1$), because by definition <u>all</u> the column sums of C are equal to one, i.e.

$\sum_{g=1}^{n} c_i^{gh} = 1$. Even though both commodities are produced and consumed in both regions, the assumption is inconsistent with the structural requirements of the MRIO model. For regional consumption to differ from regional production, the regions must be interdependent. This requires $c_i^{gg} < 1$ for <u>at least one</u> commodity in each region. Otherwise, there is no rationale for the MRIO model.

Thus, the criterion that every commodity be both produced and consumed in a region has to be qualified with the additional criterion that at least one commodity in every region must be traded.

Despite these minimum criteria, the existence of C^{-1} is at this point based only on empirical observation (similar to the observation noted on p. in Annex A.1 that the aggregate MPC is positive and less than unity) rather than theoretical proof. The repeated use of the formulation in (2.1) has not yet led to any singularity problems in inversion, or non-convergence in iteration. Nonetheless, as Bon noted, "this formulation may be of restricted applicability in regional analysis. More precisely, it is contingent upon the level of aggregation of the data employed" (Bon, ibid.). To date identical results have been obtained from implementing the MRIO model with both formulations, suggesting that both are equivalently general at the existing level of disaggregation of the model. It is conceivable, however, that at a greater level of disaggregation the trade matrix for a specific commodity i (see Fig. B.2.4) will contain (1) a zero row, i.e. that the commodity is consumed but not produced in region h, (2) a zero column, i.e. that the commodity is produced but not consumed in region g, or (3) both a zero row and a zero column for the corresponding region, i.e. that the commodity is neither produced nor consumed in the region. These possibilities could violate the minimum criteria for the existence of C^{-1}.

The first two possibilities can clearly be accommodated in equation (1.8). In addition, the conditions for the existence of (1.8) have been conclusively established theoretically (Bon, loc. cit, p. 25). Therefore, (1.8) is the preferred more general formulation of the open MRIO model solution.

The second reason for preferring equation (1.8) is specific to the focus of this dissertation, which is to analytically demonstrate and interpret the relationship between the multipliers obtained from the augmented and unaugmented versions of the MRIO model.

In annex A.4 it is noted that $B = (I-A)^{-1}$ is the form of the multiplier matrix used for a single economy whether at the national or regional level. For equivalently sectored national and regional I-O models the order of the matrix B will be identical to m, i.e. the number of industries.[1/]

1/ The major difference between the two levels of analysis is likely to be in the magnitude of the coefficients of B. The coefficients of B at the national level will most often be larger than the coefficients of B at the regional level, reflecting the greater complexity and indirectness in the production process at the national level, and the greater amount of leakages out of the more open sub-economy at the regional level. It is highly unlikely that the regional economy will be more complex than the national economy, though it is possible for leakages to be smaller for a remote and isolated regional sub-economy in an otherwise open national economy.

In the case of the 'pure' interregional economy the form of the multiplier matrix B is identical to that of the national and regional I-O models. The order of the matrix, however, will be larger for the same number and type of industries, i.e. nm>m, where m is the number of industries as before and n is the number of regions in the interregional model.

At a first glance, it would seem that $(I-\Theta)^{-1}$ in equation (1.8) is the multiplier matrix for the MRIO model and all that remains to complete the formal analogy between the conventional interregional I-O formula and the MRIO formula is to set

$$\overline{Y} = CY \tag{2.2}$$

as the multiplicand, i.e. as an (nm×1) column vector of a particular type of final demand. This cannot be done because when CY is written out in its expanded component form we get:

$$\sum_{h=1}^{n} c_i^{gh} y_i^h = \bar{y}_i^{go} \tag{2.3}$$

or

$$\sum_{h=1}^{n} \left[\frac{x_i^{gh}}{x_i^{oh}}\right] y_i^h = \sum_{h=1}^{n} x_i^{gh} \left[\frac{y_i^h}{x_i^{oh}}\right] \tag{2.4}$$

This element of the potential exogenous vector $\bar{Y}$ does not show the demand for commodity j in the consuming region h which is required by the economic logic of the MRIO demand-model. 2/ **Instead this element shows that portion of the combined final demand of all regions for commodity i that is shipped from a single supplying (producing) region g. This can be illustrated in a two-region, two-commodity context, as:**

$$C = \begin{bmatrix} c_1^{11} & 0 & c_1^{12} & 0 \\ 0 & c_2^{11} & 0 & c_2^{12} \\ c_1^{21} & 0 & c_1^{22} & 0 \\ 0 & c_2^{21} & 0 & c_2^{22} \end{bmatrix} \quad \text{and } Y = \begin{bmatrix} y_1^1 \\ y_2^1 \\ y_1^2 \\ y_2^2 \end{bmatrix}$$

$$\bar{Y} = CY = \begin{bmatrix} c_1^{11}y_1^1 + 0 + c_1^{12}y_1^2 + 0 \\ 0 + c_2^{11}y_2^1 + 0 + c_2^{12}y_2^2 \\ c_1^{21}y_1^1 + 0 + c_1^{22}y_1^2 + 0 \\ 0 + c_2^{21}y_2^1 + 0 + c_2^{22}y_2^2 \end{bmatrix} = \begin{bmatrix} c_1^{11}y_1^1 + c_1^{12}y_1^2 \\ c_2^{11}y_2^1 + c_2^{12}y_2^2 \\ c_1^{21}y_1^2 + c_1^{22}y_1^2 \\ c_2^{21}y_2^1 + c_2^{22}y_2^2 \end{bmatrix} = \begin{bmatrix} c_1^{10}y_1^0 \\ c_2^{10}y_2^0 \\ c_1^{20}y_1^0 \\ c_2^{20}y_2^0 \end{bmatrix}$$

that is, $\bar{y}_i^g = c_i^{go}y_i^o = \sum_{h=1}^{n} c_i^{gh}y_i^h$, **where** $\sum_{h=1}^{n} c_i^{gh} \neq \sum_{g=1}^{n} c_i^{gh} = 1$. **This point is reinforced by the fact that in equation (1.8) the multiplier matrix is** $(C^{-1} - \hat{A})^{-1}$ **and the multiplicand is Y, rather than CY.**

2/ In this model, the elements of the final demand vector are expected to correspond with the elements of the inverse matrix in their coordinate references.

Thus, instead of using the vector $\overline{Y} = CY$ as the exogenous multiplicand and the matrix $B = (I-\Theta)^{-1}$ as the multiplier matrix, it is necessary to incorporate the matrix C as a component of the multiplier matrix $D = (I-\Theta)^{-1}C$ and retain the original vector $Y = [y_j^h]$ as the exogenous multiplicant. The multiplier matrix in the MRIO model requires, therefore, that the matrix $B = (I-\Theta)^{-1}$ be post-multiplied by the matrix C: hence,

instead of $$X = B\overline{Y} \tag{2.5}$$

with $B = (I-\Theta)^{-1}$ and $\overline{Y} = CY$

we have $$X = BCY \tag{2.6}$$

or $$X = DY \tag{2.7}$$

where $D = BC$.

The ability to decompose the multiplier matrix D into the product of two matrices B and C has two advantages over the equivalent formulation of $D = (C^{-1} - \hat{A})^{-1}$.

The first advantage is that it is possible to interpret the two different functions of the interregional trade matrix C in the multiplier matrix D. In its first function C appears in $\theta = C\hat{A}$ as an element of the intermediate use or interindustry production process. Its function is to ensure that the output of an industry in one region becomes a production input in another region. This enables us to interpret the coefficients θ_{ij}^{gh} and b_{ij}^{gh} of the 'full' matrices θ and B, as representing respectively the direct, and the direct plus indirect interindustrial <u>and</u> interregional input requirements per unit of gross output produced by industry j in region h. Thus, θ^{gh}_{ij} and b^{gh}_{ij} reflect production parameters, i.e.

$$\theta = C\hat{A} = \left[c^{gh}_{i} a^{h}_{ij} \right] = \left[\theta^{gh}_{ij} \right] \tag{2.8}$$

and

$$B = (I-\theta)^{-1} = \left[b^{gh}_{ij} \right] \tag{2.9}$$

In its second function, C appears in D = BC as an element of the analytic process in which the output in region g is distributed via all other regions to the final users in region h.

$$D = (I-\theta)^{-1}C = BC = \left[\sum_{k=1}^{n} b^{gk}_{ij} c^{kh}_{j} \right] = \left[d^{gh}_{ij} \right] \tag{2.10}$$

This enables us to interpret the coefficients d^{gh}_{ij} as representing the amount of output industry i in region g has to supply in response to a unit change in the final demand for the output of industry j in region h. It is these two functions of the matrix C which differentiates the interpretation of the interregional I-O model multipliers from the MRIO model multipliers. It is difficult to clearly interpret the alternate formulation of the multiplier matrix D in (2.1) because it is not possible to distinguish between these two functions of the C matrix in the expressions $(C^{-1} - \hat{A})$ and $(C^{-1} - \hat{A})^{-1}$.

The second advantage to using D = BC is that the analogy between the MRIO model's $(I-\theta)^{-1}$ and the pure interregional model's $(I-A)^{-1}$ is necessary to derive the partitioned matrix solution of the augmented model in Annex D and to interpret the matrix Ψ, which is a key element in establishing the analytic relationship between the multipliers of the augmented and unaugmented versions of the MRIO model in Chapter 4.

The MRIO Multiplier Matrix (D) as a Matrix of Comparative-Static Partial Derivatives $\frac{\partial x}{\partial y}$

Before presenting the various types of matrix and scalar multipliers that can be obtained from the multiplier matrix D, it will be useful to show that D is a matrix of comparative-static derivatives. Once this is established it will be possible to show how the various MRIO scalar multipliers can be derived from the submatrix multipliers of D.

Equation (2.6) is the most succinct expression of the solution of the open MRIO model in which X is the vector of endogenous gross outputs, D the multiplier matrix, and Y the multiplicand, or vector of exogenous final demands. This equation can also be written in expanded matrix form as:

$$
\begin{bmatrix}
x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hline \vdots \\ \hline x_1^n \\ \vdots \\ x_{m-1}^n \\ x_m^n
\end{bmatrix}
\quad
\underbrace{
\left[\begin{array}{cccc|c|cccc}
d_{11}^{11} & d_{12}^{11} & \cdots & d_{1m}^{11} & \cdots\cdots & d_{11}^{1n} & \cdots & d_{1,m-1}^{1n} & d_{1m}^{1n} \\
d_{21}^{11} & d_{22}^{11} & \cdots & d_{2m}^{1n} & \cdots\cdots & d_{21}^{1n} & \cdots & {}_{2,m-1}^{1n} & d_{2m}^{1n} \\
\vdots & \vdots & \ddots & & \ddots & & \ddots & \vdots & \vdots \\
d_{m1}^{11} & d_{m2}^{11} & \cdots & d_{mm}^{11} & \cdots\cdots & d_{m1}^{1n} & \cdots & d_{m,m-1}^{1n} & d_{mm}^{1n} \\
\hline
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
\hline
d_{11}^{n1} & d_{12}^{n1} & \cdots & d_{1m}^{n1} & \cdots\cdots & d_{11}^{nn} & \cdots & d_{1,m-1}^{nn} & d_{1m}^{nn} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
d_{m-1,1}^{n1} & d_{m-1,2}^{n1} & \cdots & d_{m-1,m}^{n1} & \cdots\cdots & d_{m-1,1}^{nn} & \cdots & d_{m-1,m-1}^{nn} & d_{m-1,m}^{nn} \\
d_{m1}^{n1} & d_{m2}^{n1} & \cdots & d_{mm}^{n1} & \cdots\cdots & d_{m1}^{nn} & \cdots & d_{m,m-1}^{nn} & d_{mm}^{nn}
\end{array}\right]
}_{n}
\quad
\begin{bmatrix}
y_1^1 \\ y_2^1 \\ \vdots \\ y_m^1 \\ \hline \vdots \\ \hline y_1^n \\ \vdots \\ y_{m-1}^n \\ y_m^n
\end{bmatrix}
$$

Figure 2.1: X = DY in matrix notation

In order to obtain the requisite partial derivatives it is necessary to expand Figure (2.1) further, to represent the following system of equations:

$$
\begin{array}{lcl}
x^1_1 & = & d^{11}_{11}y^1_1 + d^{11}_{12}y^1_2 + \dots + d^{11}_{1m}y^1_m + \dots\dots + d^{1n}_{11}y^n_1 + \dots + d^{1n}_{1,m-1}y^n_{m-1} + d^{1n}_{1m}y^n_m \\
x^1_2 & = & d^{11}_{21}y^1_1 + d^{11}_{22}y^1_2 + \dots + d^{11}_{2m}y^1_m + \dots\dots + d^{1n}_{21}y^n_1 + \dots + d^{1n}_{2,m-1}y^n_{m-1} + d^{1n}_{2m}y^n_m \\
\vdots & & \vdots \\
x^1_m & = & d^{11}_{m1}y^1_1 + d^{11}_{m2}y^1_2 + \dots + d^{11}_{mm}y^1_m + \dots\dots + d^{1n}_{m1}y^n_1 + \dots + d^{1n}_{m,m-1}y^n_{m-1} + d^{1n}_{mm}y^n_m \\
\vdots & & \vdots \\
x^n_1 & = & d^{n1}_{11}y^1_1 + d^{n1}_{12}y^1_2 + \dots + d^{n1}_{1m}y^1_m + \dots\dots + d^{nn}_{11}y^n_1 + \dots + d^{nn}_{1,m-1}y^n_{m-1} + d^{nn}_{1m}y^n_m \\
\vdots & & \vdots \\
x^n_{m-1} & = & d^{n1}_{m-1,1}y^1_1 + d^{n1}_{m-1,2}y^1_2 + \dots + d^{n1}_{m-1,m}y^1_m + \dots\dots + d^{nn}_{m-1,1}y^n_1 + \dots + d^{nn}_{m-1,m-1}y^n_{m-1} + d^{nn}_{m-1,}\ y^n_m \\
x^n_m & = & d^{n1}_{m1}\ y^1_1 + d^{n1}_{m2}\ y^1_2 + \dots + d^{n1}_{mm}\ y^1_m + \dots\dots + d^{nn}_{m1}\ y^n_1 + \dots + d^{nn}_{m,m-1}\ y^n_m + d^{nn}_{mm}\ y^n_m
\end{array}
$$

Figure 2.2: $x^g_i = \sum\limits_{h=1}^{n} (\sum\limits_{j=1}^{m} d^{gh}_{ij}\ y^h_j)$ as a system of equations

From Figure 2.2 it is immediately apparent that the scalar comparative-static partial derivatives of the system of equations are:

$$
\begin{array}{ccccc}
\dfrac{\partial x_1^1}{\partial y_1^1} = d_{11}^{11} & \dfrac{\partial x_1^1}{\partial y_2^1} = d_{12}^{11} & \cdots & \dfrac{\partial x_1^1}{\partial y_{m-1}^n} = d_{1,m-1}^{1n} & \dfrac{\partial x_1^1}{\partial y_m^n} = d_{1m}^{1n} \\
\dfrac{\partial x_2^1}{\partial y_1^1} = d_{21}^{11} & \dfrac{\partial x_2^1}{\partial y_2^1} = d_{22}^{11} & \cdots & \dfrac{\partial x_2^1}{\partial y_{m-1}^n} = d_{2,m-1}^{1n} & \dfrac{\partial x_2^1}{\partial y_m^n} = d_{2m}^{1n} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
\dfrac{\partial x_{m-1}^n}{\partial y_1^1} = d_{m-1,1}^{n1} & \dfrac{\partial x_{m-1}^n}{\partial y_2^1} = d_{m-1,2}^{n1} & \cdots & \dfrac{\partial x_{m-1}^n}{\partial y_{m-1}^n} = d_{m-1,m-1}^{nn} & \dfrac{\partial x_{m-1}^n}{\partial y_m^n} = d_{m-1,m}^{nn} \\
\dfrac{\partial x_m^n}{\partial y_1^1} = d_{m1}^{n1} & \dfrac{\partial x_m^n}{\partial y_2^1} = d_{m2}^{n1} & \cdots & \dfrac{\partial x_m^n}{\partial y_{m-1}^n} = d_{m,m-1}^{nn} & \dfrac{\partial x_m^n}{\partial y_m^n} = d_{mm}^{nn}
\end{array}
$$

Figure 2.3: Scalar partial derivatives of the system of equations X=DY

This set of partial derivatives can be expressed by the general relation

$$\frac{\partial x_i^g}{\partial y_j^h} = d_{ij}^{gh} \qquad \begin{array}{l}(g,h = 1,2,\ldots,n)\\ (i,j = 1,2,\ldots,m)\end{array} \qquad (2.11)$$

which shows most clearly by how much the gross output of industry i located in region g must change in order to be consistent with a unit change in the demand in region h for the output of industry j.

Treating the scalar partial derivatives in each column of the array in Fig. 2.3 as column vectors we get the following vector partial derivatives:

$$\frac{\partial x}{\partial y_1^1} \equiv \frac{\partial}{\partial y_1^1}\begin{bmatrix} x_1^1 \\ x_2^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{m-1}^n \\ x_m^n \end{bmatrix} = \begin{bmatrix} d_{11}^{11} \\ d_{21}^{11} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{m-1,1}^{n1} \\ d_{m1}^{n1} \end{bmatrix}$$

$$\frac{\partial x}{\partial y_2^1} \equiv \frac{\partial}{\partial y_2^1}\begin{bmatrix} x_1^1 \\ x_2^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{m-1}^n \\ x_m^n \end{bmatrix} = \begin{bmatrix} d_{12}^{11} \\ d_{22}^{11} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{m-1,2}^{n1} \\ d_{m2}^{n1} \end{bmatrix}$$

$$\vdots \qquad\qquad \vdots$$

$$\frac{\partial x}{y_{m-1}^n} \equiv \frac{\partial}{\partial y_{m-1}^n}\begin{bmatrix} x_1^1 \\ x_2^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{m-1}^n \\ x_m^n \end{bmatrix} = \begin{bmatrix} d_{1,m-1}^{1n} \\ d_{2,m-1}^{1n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{m-1,m-1}^{nn} \\ d_{m,m-1}^{nn} \end{bmatrix}$$

$$\frac{\partial x}{\partial y_m^n} \equiv \frac{\partial}{\partial y_m^n}\begin{bmatrix} x_1^1 \\ x_2^1 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_{m-1}^n \\ x_m^n \end{bmatrix} = \begin{bmatrix} d_{1m}^{1n} \\ d_{2m}^{1n} \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ d_{m-1,m}^{nn} \\ d_{mm}^{nn} \end{bmatrix}$$

Figure 2.4: Vector derivatives of the system of equations

The vector partial derivatives are simply the columns of the inverse coefficient matrix D. By further consolidation these vectors can be expressed as a single matrix of partial derivatives ($\partial X/\partial Y$):

$$\frac{\partial X}{\partial Y} = \begin{bmatrix} d_{11}^{11} & d_{12}^{11} & \cdots & d_{1,m-1}^{1n} & d_{1m}^{1n} \\ d_{21}^{11} & d_{22}^{11} & \cdots & d_{2,m-1}^{1n} & d_{2m}^{1n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ d_{m-1,1}^{n1} & d_{m-1,2}^{n1} & \cdots & d_{m-1,m-1}^{nn} & d_{m-1,m}^{nn} \\ d_{m1}^{n1} & d_{m2}^{n1} & \cdots & d_{m,m-1}^{nn} & d_{mm}^{nn} \end{bmatrix} = D$$

Fig. 2.5: D as the matrix of partial derivatives $\partial X/\partial Y$.

Thus, the elements of the matrix D show the rate at which the elements of the endogenous vector X must change to be consistent with changes in the exogenous vector Y. Thus, the matrix D is a disaggregated comparative-static multiplier.

As long as the interindustry production structures and the interregional trading patterns can be assumed to remain more or less unchanged, the comparative-static MRIO matrix multiplier can be used to show by how much gross output must change to meet exogenously determined changes in final demand. It cannot be used, however, to determine when the impact will materialize (see Annex A.1).

Output Multipliers in the Open MRIO Model

The capabilities and usefulness of the single economy input-output inverse matrix for impact analysis[3/] is presented in Annex A.3. Yet in the growing literature on I-O multipliers, most authors have tended to neglect the Leontief inverse coefficients in their role as individual multipliers and have concentrated instead on summary measures of them. This is particularly true in regional analysis in the case of output multipliers, where the most common multiplier in use has come to be referred to as the 'column multiplier' or the 'final demand multiplier' (H. Richardson, 1972, p. 39; Jensen, 1976, p. 6; USDA, 1978, p. 29-31). This multiplier is simply the column sum of each column of the Leontief inverse matrix, i.e., b_{oj}. It measures the total output requirements per unit of final demand of a single industry, i.e., y_j. The higher the multiplier the greater the interdependence of that sector with the rest of the economy.

The heavy reliance on these aggregate multipliers, which are of course important in their own right, has tended, nonetheless, to obscure the rich detail of information that is available for specific planning purposes in I-O models.

As already noted, the MRIO model contains even more detailed mutually consistent data than either the national or regional level I-O models. In light of the neglect of Leontief inverse coefficients as individual

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3/ The utility of the direct coefficient and inverse coefficient I-O matrices of course extend well beyond impact analysis to related types of analysis including forecasting, structural analysis, allocation of fixed resources, cost-push price analysis, etc.

multipliers in single economics, it should not be surprising that the same lack of attention has been displayed with regard to the even more disaggregated multipliers available through the MRIO model. Yet these multipliers can be used to determine fairly detailed industrially and regionally differentiated comparative-static impacts which are consistent with proposed or expected changes in national final demand, or the final demand of a single industry or region.

Submatrix Multipliers

To clarify this point, a variety of submatrix multipliers contained in the MRIO multiplier matrix D will be described. Some are unique to the MRIO model and others are related to the national and regional I-O multiplier matrices B^N and B^R. These points will be discussed, and the subvectors of endogenous and exogenous variables that correspond to each of the described submatrix multipliers will be identified. The submatrix multiplier closest to the national I-O inverse coefficients is the (n×n) MRIO interindustry submatrix multiplier D_{ij}.

$$D_{ij} = \begin{bmatrix} d_{ij}^{11} & d_{ij}^{12} & \cdots & d_{ij}^{1n} \\ d_{ij}^{21} & d_{ij}^{22} & \cdots & d_{ij}^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ d_{ij}^{n1} & d_{ij}^{n2} & \cdots & d_{ij}^{nn} \end{bmatrix} \quad i,j = 1,2,\ldots,m$$

Fig. 2.6: The interindustry submatrix multiplier D_{ij}

There are m^2 submatrix multipliers D_{ij} in the MRIO model.

If $(i \neq j)$, the submatrix multiplier D_{ij} can be used to determine the impact on industry i's output in all regions, i.e. on the (1×n) subvector (X_i):

$$(X_i) = \begin{bmatrix} x_i^1 \\ x_i^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_i^n \end{bmatrix} \qquad i = 1,2,\ldots,m$$

consequent upon a changed final demand for the output of industry j in all regions, i.e. the multiplicand subvector (Y_j):

$$(Y_j) = \begin{bmatrix} y_j^1 \\ y_j^2 \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_j^n \end{bmatrix} \qquad j = 1,2,\ldots,m$$

If $(i = j)$ the submatrix multiplier D_{ij} can be used to determine the impact of a changed final demand in all regions for a specific commodity, upon its own industry's gross output in all regions.

To obtain the impact on the output of industry i in a specific region g, i.e. the g^{th} scalar element $(X_i)^g$ of the industrial gross output subvector (X_i), it is necessary only to post-multiply the g^{th} row vector of the submatrix multiplier D_{ij} by the column subvector of final demand (Y_j).

The submatrix multiplier D_{ij} can be interpreted as the (n×n) regional expansion of <u>each</u> element of the national I-O inverse matrix. When $(i = j)$, D_{ij} represents an expansion of the i^{th} diagonal element of the matrix B^N, and when $(i \neq j)$, D_{ij} represents an expansion of the $(i,j)^{th}$ non-diagonal element of B^N. Thus, this submatrix multiplier can be used with the j^{th} element of the national final demand vector Y^N, i.e. y_j^N, instead of the MRIO subvector (Y_j) to obtain the regionalized impact on the gross output of industry i. Therefore, instead of the MRIO equation

$$X_i = D_{ij}Y_j \tag{2.12}$$

the equation

$$X_i = D_{ij}(e^T)y_j^N \tag{2.13}$$

can be substituted, where e^T is the transpose of a summing row vector e, all of whose elements are equal to unity. Of course in using the summing vector e^T and the national final demand scalar y_j^N it is implicitly assumed that the regional composition of the final demand for industry j's output is either unavailable or unimportant, otherwise it is preferable to use the MRIO subvector (Y_j).

By interpreting the submatrix multiplier D_{ij} as the regional expansion of each element of the national multiplier matrix B^N, it is possible to demonstrate a key difference between the inverse coefficients of the MRIO model and those of the national I-O model. This difference has a bearing on policy analysis.

In the national I-O matrix B the diagonal elements must be greater than or equal to one, i.e. $b_{ii}>1$, because a $1 increase in the final demand for commodity i will always generate at least $1 of gross output by the same industry, otherwise gross output would be insufficient to meet the demand of final users. However, as noted by DiPasquale and Polenske (1977, p.), and as is apparent in the tables of Annex E, the diagonal elements of the MRIO multiplier matrix can be less than one. This is a direct consequence of regionally expanding each diagonal element of B^N into a submatrix multiplier D_{ij}, with i=j. In fact, the MRIO multiplier matrix D will have very few coefficients greater than one, when the model is implemented at its full level of disaggregation. Thus, whereas the matrix D clearly multiplies the effect of the vector Y, the coefficients of the multiplier matrix D are more in the nature of dividers than multipliers. This special characteristic of the coefficients of multiplier matrices has been noted by R.M. Goodwin (1949, p. 534) in his discussion of disaggregated Keynesian multipliers.

The submatrix multiplier closest to a regional multiplier matrix with comparable industrial sectors, is a special case of the (m×m) MRIO interregional submatrix multiplier D^{gh}.

$$D^{gh} = \begin{bmatrix} d_{11}^{gh} & d_{12}^{gh} & \cdots\cdots & d_{1m}^{gh} \\ d_{21}^{gh} & d_{22}^{gh} & \cdots\cdots & d_{2m}^{gh} \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1}^{gh} & d_{m2}^{gh} & \cdots\cdots & d_{mm}^{gh} \end{bmatrix}$$

Fig. 2.7. The interregional submatrix multiplier D^{gh}.

If $(g \neq h)$, the submatrix multiplier D^{gh} can be used to determine the impact on region g's output of all industries, i.e. the $(1 \times m)$ subvector (X^g):

$$(X^g) = \begin{bmatrix} x_1^g \\ x_2^g \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ x_m^g \end{bmatrix} \qquad g = 1,2,\ldots,n$$

consequent upon a changed final demand for the output of all industries in region h, i.e. the multiplicand subvector (Y^h) :

$$(Y^h) = \begin{bmatrix} y_1^h \\ y_2^h \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ \cdot \\ y_m^h \end{bmatrix} \qquad h = 1,2,\ldots,n$$

There are n^2 submatrix multipliers D^{gh} corresponding to the n^2 contiguous blocks of dimension $(m \times m)$ in the MRIO multiplier matrix D. The off-diagonal blocks are unique to interregional models. They can be referred to as 'cross regional submatrix multipliers' to highlight the fact that they can be used to determine the impacts in one region consequent upon final demand changes in another region.

To obtain the impact on a specific industry i in region g, i.e. the i^{th} scalar element $(X^g)_i$ of the regional gross output subvector (X^g), it is necessary only to post-multiply the i^{th} row vector of the submatrix multiplier D^{gh} by the column subvector of final demand (Y^h). In analogy to the previous case, this submatrix multiplier can be used with the scalar sum of region h's final demand, i.e. $y_o^{R(h)}$ (suitably adjusted to exclude the region's exports not destined to meet foreign demand requirements), instead of the MRIO subvector (Y^h). Thus, instead of the MRIO equation

$$X^g = D^{gh}Y^h \tag{2.14}$$

the equation

$$X^g = D^{gh}(e^T)y_o^{R(h)} \tag{2.15}$$

can be substituted. In using equation (2.15) there is of course the implicit assumption that the industrial composition of the final demand in region h is either unavailable or unimportant, otherwise it is preferable to use the MRIO subvector (Y^h).

In the special case where (g=h), i.e. the (m×m) block submatrices along the principal diagonal of matrix D, the submatrix multiplier D^{gh} can be used to determine the impact on a region g's gross output consequent upon a change in the region's own final demand. In this case the submatrix multiplier is akin to the multiplier matrix B^R for region g. For purposes of regional analysis, however, the coefficients of this submatrix multiplier are likely to be more accurate than the inverse coefficients of comparably sectored and estimated national and regional

I-O matrices. This is because national coefficients that are scaled down for a region, do not reflect the region's specific production structure. In particular they will tend to overestimate impacts by not taking account of larger imports (from other regions) in the more open regional sub-economics. The regional coefficients, on the other hand, are derived from models which do not show the region's interconnections with other regions in the nation even though they reflect the region's specific production structure. Hence, the inverse coefficients will tend to underestimate impacts because they do not take account of interregional feedbacks. The MRIO inverse coefficients take account of both problems, and their estimates of impacts are somewhere between the other two estimates and therefore likely to be more accurate.

Whether or not the MRIO inverse coefficients, or the regional, or scaled down national coefficients should be used in a specific regional context will depend on the degree of that region's self-sufficiency and the amount by which that region's production structure differs from the national production structure. As already noted, however, there is one very important difference between using a regional I-O model and the regional component of the MRIO model for the same region. The final demand vector of the MRIO model is constructed along the lines of the national model. Hence, the gross export component of regional final demand must exclude exports not eventually destined for foreign demand, because intranational exports are already incorporated in interregional trade either in the form of net regional exports, i.e., as the difference between regional production and consumption, or as gross regional exports in the rows of the interregional trade flow tables.

There are two additional types of submatrix multipliers that can be obtained from the MRIO multiplier matrix D which have no counterpart in non-regionally differentiated single economy models. These are the submatrix multipliers D_i^h and D_j^g.

In order to determine the effect of a changed regional final demand subvector for region (h), i.e. (Y^h), on the industrial gross output subvector for a specific industry (i), i.e. (X_i), the appropriate submatrix multiplier to use would be the (n×m) MRIO region-to-industry submatrix multiplier (D_i^h):

$$D_i^h = \begin{bmatrix} d_{i1}^{1h} & d_{i2}^{1h} & \cdots\cdots & d_{im}^{1h} \\ d_{i1}^{2h} & d_{i2}^{2h} & \cdots\cdots & d_{im}^{2h} \\ \vdots & \vdots & \ddots & \vdots \\ d_{i1}^{nh} & d_{i2}^{nh} & \cdots\cdots & d_{im}^{nh} \end{bmatrix} \qquad \begin{array}{l} h = 1,2,\ldots,n \\ i = 1,2,\ldots,m \end{array}$$

Fig. 2.8: The region-to-industry submatrix multiplier D_i^h

To obtain the multiplier impact on a specific scalar element $(X_i)^g$ of the industrial gross output subvector (X_i), it is necessary only to post-multiply the g^{th} row-vector of the multiplier submatrix (D_i^h) by the column sub-vector of regional final demand (Y^h).

Similarly, it is possible to determine the impact on the subvector of regional gross output for a specific region (g), i.e. (X^g), consequent upon a changed final demand subvector for a specific industry j, i.e. (Y_j), by using the (mxn) MRIO industry to region submatrix multiplier D_j^g:

$$D_j^g = \begin{bmatrix} d_{1j}^{g1} & d_{1j}^{g2} & \cdots\cdots & d_{1j}^{gn} \\ d_{2j}^{g1} & d_{2j}^{g2} & \cdots\cdots & d_{2j}^{gn} \\ \vdots & \vdots & \ddots & \vdots \\ d_{mj}^{g1} & d_{mj}^{g2} & \cdots\cdots & d_{mj}^{gn} \end{bmatrix} \qquad \begin{aligned} g &= 1,2,\ldots,n \\ j &= 1,2,\ldots,m \end{aligned}$$

Fig. 2.9: The industry-to-region submatrix multiplier D_j^g

In analogy to the previous cases it is possible to determine the multiplier effect on a specific industry or scalar element $(X^g)_i$ of the regional gross output subvector (X^g), by post-multiplying the i^{th} row-vectors of the multiplier submatrix (D_j^g) by the column sub-vector of industrial final demand (Y_j).

The four types of sub-matrix multipliers just described are useful when the exogenously determined multiplicands are in the form of subvectors, that is, when exogenous changes are disaggregated by industries or regions. In this case it is possible to determine the effect of changes in the industrial or regional distribution of a given level of final demand expenditures, as well as the effect of changes in the net level of total final demand expenditures.

Scalar Multipliers

Often, however, the various multiplicands whose impact a policy analyst is interested in are scalar sums of the final demand for an industry or region, or they are scalar components of industrial or regional final demand. The analyst may, nonetheless, be interested in a more specific (or less specific) impact than that corresponding to the level of specificity of the multiplicand. In these cases it is often more convenient to use scalar rather than submatrix multipliers.

For these different analytic purposes it is possible to derive scalar multipliers from the MRIO model. With the exception of the individual inverse coefficients d_{ij}^{gh}, all the other MRIO scalar multipliers involve summing the most detailed partial effect ($\partial X_i^g / \partial Y_j^h = d_{ij}^{gh}$) along one, two, three, or all four of its coordinate dimensions. These scalar multipliers must only be used with appropriate corresponding scalar multiplicands. Thus, if the aggregation of d_{ij}^{gh} involves summing along the dimension of the purchasing industry j, or the region h in which the demand is located, or both, then the corresponding scalar multiplicand must be respectively, a scalar sum y_o^h, or y_j^o, or y_o^o. The use of these multiplicands presume that the effect of changes in the industrial or regional distribution of exogenous expenditures can be ignored. This of course is a common enough assumption encountered in aggregate macro-economic analysis. However, in the context of MRIO impact analysis, this assumption must be used judiciously, as it undermines one of the principal advantages of the MRIO model, i.e. its ability to determine the industrial and regional impact that is consistent with a change in the composition of final demand expenditures, even if there is no net change in total final demand expenditures. However, there are occasions when it might be necessary or desirable to ignore composition effects and use aggregate scalar multipliers. Here we will demonstrate the procedure for obtaining scalar multipliers from the MRIO model.

Based on the expanded equation form of the MRIO solution in Fig.(2.2), each element x_i^g of the gross output vector X can be written in the form of the inner product of the elements of each row-vector of the inverse matrix D and the final demand column vector Y;

$$x_i^g = \sum_{h=1}^{n} \left(\sum_{j=1}^{m} d_{ij}^{gh} y_j^h \right) \tag{2.16}$$

Equation (2.16) can also be written as:

$$x_i^g = \sum_{j=1}^{m} d_{ij}^{g1} y_j^1 + \sum_{j=1}^{m} d_{ij}^{g2} y_j^2 + \ldots + \sum_{j=1}^{m} d_{ij}^{gn} y_j^n \tag{2.17}$$

The inner product terms $\left(\sum_{j=1}^{m} d_{ij}^{gh} y_j^h \right)$, for each h, are inseparable. In this form the weighting of each inverse coefficient d_{ij}^{gh} by an appropriate y_j^h ensures that the composition effect is an integral part of the final impact. Hence, for example, we cannot obtain a scalar partial derivative for the sub-vector (Y^h) unless the sub-vector is reduced to a scalar sum. Thus, the inner product terms $\left(\sum_{j=1}^{m} d_{ij}^{gh} j_j^h \right)$ can also be written in the form:

$$\left(\sum_{j=1}^{m} d_{ij}^{gh} \right) \left(\sum_{j=1}^{m} y_j^h \right) = \left(\sum_{j=1}^{m} d_{ij}^{gh} \right) y_o^h \tag{2.18}$$

In this form $(\sum_{j=1}^{m} y_j^h)$ is an independent scalar element and not a part of an inseparable inner product term. With this reformulation of the multiplicand it is possible to specify a scalar multiplier for the i^{th} element of the subvector (X^g) as the partial derivative of the scalar x_i^g, with respect to the multiplicand y_o^h.

$$\frac{\partial x_i^g}{\partial y_o^h} = \sum_{j=1}^{m} d_{ij}^{gh} \qquad h = 1,2,\ldots,n \tag{2.19}$$

Similarly, to derive the scalar multiplier for the multiplican y_j^o it is necrssary to separate out the first component of each term in the inner product $(\sum_{j=1}^{m} d_{ij}^{gh} y_j^h)$ to get:

$$x_i^g = (d_{i1}^{g1} y_1^1 + \sum_{j=2}^{m} d_{ij}^{g1} y_j^1) + (d_{i1}^{g2} y_1^2 + \sum_{j=2}^{m} d_{ij}^{g2} y_j^2) + \ldots + (d_{i1}^{gn} y_1^n + \sum_{j=2}^{m} d_{ij}^{gn} y_j^n) \tag{2.20a}$$

Collecting the $(d_{i1}^{gh} y_1^h)$ terms, we get:

$$x_i^g = (d_{i1}^{g1} y_1^1 + d_{i1}^{g2} y_1^2 + \ldots + d_{i1}^{gn} y_1^n) + (\sum_{j=2}^{m} d_{ij}^{g1} y_j^1 + \sum_{j=2}^{m} d_{ij}^{g2} y_j^2 + \ldots + \sum_{j=2}^{m} d_{ij}^{gn} y_j^n) \tag{2.20b}$$

which can be rewritten as:

$$x_i^g = (\sum_{h=1}^{n} d_{i1}^{gh}) (\sum_{h=1}^{n} y_1^h) + \sum_{h=1}^{n} (\sum_{j=2}^{n} d_{ij}^{gh} y_j^h) \tag{2.20c}$$

The first part of the expression on the right hand side of equation (2.20c) is:

$$(\sum_{h=1}^{n} d_{ij}^{gh}) (\sum_{h=1} y_1^h) = (\sum_{h=1} d_{ij}^{gh}) y_1^o \tag{2.21}$$

A similar procedure can be used to break down the second part of the expression on the right-hand side of equation (2.20c) to obtain products containing the scalar sums y_2^o, y_3^o, ..., y_m^o. Thus, the partial derivative of each component of the gross output vector X, i.e. x_i^g, as:[4/]

$$\frac{\partial x_i^g}{\partial y_j^o} = \sum_{h=1}^{n} d_{ij}^{gh} \qquad j = 1,2,\ldots,m \tag{2.22}$$

It is also possible, using an analogous line of reasoning, to sum the inverse coefficients along the endogenous industries or regions, i.e. those that must respond to the changes in the scalar multiplicands.

There are, therefore, four basic types of scalar multiplicands, and four basic types of scalar components that are affected. In both cases the types can be industry and region-specific (or detailed), industry-specific, region-specific, and non-specific (or total). Together they give rise to sixteen basic MRIO scalar output-multipliers. These are presented in tabular form in Table A with the initiating scalar stimuli, that is, cases I - IV, as column headings and the types of scalar components affected, that is sub-cases a - d, as row headings. The numbers of

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[4/] This same result can also be obtained by rewriting equation (2.16) as:

$$x_i^g = \sum_{j=1}^{m} \left(\sum_{h=1}^{n} d_{ij}^{gh} y_j^h \right) \tag{2.16'}$$

and following the same reasoning that led to equation (2.19).

each type of scalar multiplier that can be obtained from an m-industry and n-region MRIO model are provided in parenthesis below each multiplier.

In the interest of continuity, the definitions and interpretations of the sixteen basic scalar multipliers have been relegated to Annex B.3. Graphic illustrations are also provided in the annex to clarify the relationship of the various scalar multipliers to the submatrix or subvector multipliers from which they are derived.[5/]

In Table A all the scalar multipliers IIIc can be arranged in the form of a submatrix multiplier analogous to the submatrix multiplier D^{gh}. Except in this case the multiplicand and the affected variable will be vectors with no industrial dimension. This would be a pure 'interregional submatrix multiplier'. Similarly, all the scalar multipliers in IIb, IIc and IIb can be arranged into submatrix multipliers analogous to D_{ij}, D_j^g and D_i^h respectively.

It is important to note that the magnitude of all the MRIO submatrix and scalar output multipliers remain unchanged as long as the MRIO inverse matrix D is assumed to be stable, i.e., as long as the fixed coefficient technology matrix A and the fixed coefficient trade matrix C, from which the D matrix is derived, can be assumed to be adequate for purposes of generating a first-order approximation solution.

5/ The basic MRIO submatrix multipliers can be used even when some components of the multiplicand subvector do not change. This is a moot point in the case of the scalar multipliers since the scalar multiplicands do not have any components by definition.

TABLE A: TABULAR PRESENTATION OF SCALAR MULTIPLIERS DERIVED FROM THE MRIO MODEL INVERSE MATRIX $D = (I-C\hat{A})^{-1}C$

(with the number of each type of multiplier in brackets)

Scalar Sum Of Gross Output Vector Components Affected By The Multiplier	Scalar Multiplicand, or Scalar Sum of the Components of the Final Demand Vector			
	I (mn) Final Demand-Components (fdc): for each industry in each region y_j^h	II (m) Industrial Demands (id): for each industry in all regions $y_j^o = \sum_{h=1}^{n} y_j^h$	III (n) Regional Demands (rd): for all industries in each region $y_o^h = \sum_{j=1}^{m} y_j^h$	IV (1) National Demand (nd): for all industries in all regions $y_o^o = \sum_{h=1}^{n} \sum_{j=1}^{m} y_j^h$
a. (mn) 'Detailed': for each industry in each region x_i^g	${}^{x}M_{ij}^{gh} = d_{ij}^{gh}$ [mn x mn]	${}^{x}M_{ij}^{go} = \sum_{h=1}^{n} d_{ij}^{gh}$ [m x mn]	${}^{x}M_{io}^{gh} = \sum_{j=1}^{m} d_{ij}^{gh}$ [n x mn]	${}^{x}M_{io}^{go} = \sum_{h=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh}$ [mn]
b. (m) 'Industry-Specific': for each industry in all regions $x_i^o = \sum_{g=1}^{n} x_i^g$	${}^{x}M_{ij}^{oh} = \sum_{g=1}^{n} d_{ij}^{gh}$ [mn x m]	${}^{x}M_{ij}^{oo} = \sum_{g=1}^{n} \sum_{h=1}^{n} d_{ij}^{gh}$ [m x m]	${}^{x}M_{io}^{oh} = \sum_{g=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh}$ [n x m]	${}^{x}M_{io}^{oo} = \sum_{g=1}^{n} \sum_{h=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh}$ [m]
c. (n) 'Region-Specific': for all industries in each region $x_o^g = \sum_{i=1}^{m} x_i^g$	${}^{x}M_{oj}^{gh} = \sum_{i=1}^{m} d_{ij}^{gh}$ [mn x n]	${}^{x}M_{oj}^{go} = \sum_{i=1}^{m} \sum_{h=1}^{n} d_{ij}^{gh}$ [m x n]	${}^{x}M_{oo}^{gh} = \sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij}^{gh}$ [n x n]	${}^{x}M_{oo}^{go} = \sum_{i=1}^{m} \sum_{h=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh}$ [n]
d. (1) 'Total': for all industries in all regions $x_o^o = \sum_{g=1}^{n} \sum_{i=1}^{m} x_i^g$	${}^{x}M_{oj}^{oh} = \sum_{g=1}^{n} \sum_{i=1}^{m} d_{ij}^{gh}$ [mn]	${}^{x}M_{oj}^{oo} = \sum_{g=1}^{n} \sum_{i=1}^{m} \sum_{h=1}^{n} d_{ij}^{gh}$ [m]	${}^{x}M_{oo}^{oh} = \sum_{g=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij}^{gh}$ [n]	${}^{x}M_{oo}^{oo} = \sum_{g=1}^{n} \sum_{i=1}^{m} \sum_{h=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh}$ [1]

Income and Employment

Multipliers in the Open MRIO Model

In complete analogy with the single economy I-O models (see Annex A.3), the open MRIO model has a set of primary inputs. Thus, the complete model must also satisfy a secondary condition, that is

$$v_o = \sum_{g=1}^{n} \left(\sum_{i=1}^{m} \overset{*}{v}{}_i^g x_i^g \right) + {}^v y \tag{2.23}$$

or

$$v_o = \overset{*}{V}_o X + {}^v y \tag{2.24}$$

where

v_o is the scalar of total primary inputs

$\overset{*}{V}_o$ is the (1xmn) row vector of primary input coefficients: $\overset{*}{v}{}_i^g$

X is the (mnx1) column vector of gross outputs: x_i^g

and ${}^v y$ is the scalar representing the direct interaction between primary supply and final demand.

The scalar ${}^v y$ does not enter into the open model's solution of the endogenous gross output variables. As a result, the interaction between primary supply and final demand is generally treated as a zero element in theoretical models. In operational models, however, it cannot be treated as a zero element because it includes some import items, as well as, direct payments to households and government employees. It can also function as a residual balancing element to absorb the non-distributed errors associated with empirically estimated operational models. In the operational U.S. MRIO model the scalar ${}^v y$ (or the sum of the transactions in the southeast quadrant of the transactions table in Fig. B.2.1), accounts for 6-10% of the total value of final demand, depending on whether both the Service Industry Residual (SIR) and the

State Transfers out (STRO) are included or excluded from total final demand respectively.

It is evident from equation (2.24) that interregional trade affects the magnitude of total primary supply v_o only indirectly via the gross output vector X, and that the magnitude of primary supply reflects only direct plus indirect production requirements.

Substituting the solution for X from equation (2.6) into equation (2.24) v_o can be written as a function of the exogenously determined variables Y and ${}^{v}y$:

$$v_o = \overset{*}{V}_o DY + {}^{v}y \tag{2.25}$$

Substituting the full expression for D from equation (2.10) into equation (2.25) it is possible to write

$$v_o = \overset{*}{V}_o(I - C\hat{A})^{-1}CY + {}^{v}y \tag{2.26}$$

as an alternative form of the solution of total primary inputs.

In this formulation

$${}^{v}D = \overset{*}{V}_o(I - C\hat{A})^{-1}C \tag{2.27}$$

is the multiplier vector of primary inputs.

Instead of determining total primary supply with the aid of equation (2.24) or (2.26), it may be desirable to determine industry and region-specific, industry-specific, or region-specific primary supply requirements. This can be done in one of two ways:

$$v_o = (\overset{*}{v}{}_1^1 \quad \overset{*}{v}{}_2^1 \dots \overset{*}{v}{}_m^1 \mid \overset{*}{v}{}_1^2 \quad \overset{*}{v}{}_2^2 \dots \overset{*}{V}{}_m^2 \mid \dots \mid \overset{*}{v}{}_1^n \quad \overset{*}{v}{}_2^n \dots \overset{*}{v}{}_m^n) \left[\begin{array}{c} x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hline x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \\ \hline \vdots \\ \hline x_1^n \\ x_2^n \\ \vdots \\ x_m^n \end{array}\right]$$

$$v_o = \sum_g \sum_i \overset{*}{v}{}_i^g \, x_i^g$$

or $$v_o = \sum_h \sum_j \overset{*}{\nu}{}_j^h \, y_j^h$$

where $$\overset{*}{\nu}{}_j^h = \sum_g \sum_i \overset{*}{v}{}_i^g \, d_{ij}^{gh}$$

in matrix notation the coefficients $\overset{*}{\nu}{}_j^h$ form a vector ${}^{d}\overset{*}{V}_o$ of dimension (1 x mn).

Figure 2.10a The primary supply multiplier vector ${}^{d}\overset{*}{V}_o = \overset{*}{V}_o D$, with k = 1.

In the case where the analyst is interested only in the impact on a specific component of primary supply, it is necessary only to select the corresponding direct primary supply coefficient (or coefficients) and to post-multiply it by the solution for the corresponding component of X in equation (2.24), or of DY in equation (2.26). Thus

$$v_i^g = \overset{*}{v}{}_i^g x_i^g + {}^v y_i^g \tag{2.28}$$

or

$$v_i^g = v_i^g \left(\sum_{h=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh} \right) + {}^v y_i^g \tag{2.29}$$

These are the industry and region-specific primary supply requirements, where all the variables are scalars. Restricting the presentation to equation (2.24), it is possible to determine

$$v_i = \sum_{g=1} \overset{*}{v}{}_i^g x_i^g + {}^v y_i \tag{2.30}$$

$$= \overset{*}{V}_i X_i + {}^v y_i \tag{2.31}$$

where v_i is the total industry-specific primary supply requirement for industry i, $\overset{*}{V}_i$ a row sub-vector of dimension (lxn), and X_i a column sub-vector of dimension (nxl). Similarly,

$$v^g = \sum_{i=1}^{m} \overset{*}{v}{}_i^g x_i^g + {}^v y^g \tag{2.32}$$

$$= \overset{*}{V}{}^g X^g + {}^v y^g \tag{2.33}$$

where v^g is the total region-specific primary supply requirement for region g, $\overset{*}{V}{}^g$ a row-subvector of dimension (lxm) and X^g a column sub-vector of dimension (mxl).

In the second approach if the analyst is interested in determining the primary supply requirements for each of the m industries in each of the n regions simultaneously, i.e., an (mnxl) column vector V, it is preferable to treat the vector $\overset{*}{V}_o$ of primary supply coefficients as a diagonal matrix $\hat{\overset{*}{V}}$ of dimension (mnxmn) [as in Fig. 2.10b], since

$$
\begin{bmatrix} v_1^1 \\ v_2^1 \\ \vdots \\ v_m^1 \\ \hline v_1^2 \\ v_2^2 \\ \vdots \\ v_m^2 \\ \hline \vdots \\ \hline v_1^n \\ v_2^n \\ \vdots \\ v_m^n \end{bmatrix}
=
\begin{bmatrix}
\overset{*}{v}{}_1^1 & & & & & & & & & & \\
& \overset{*}{v}{}_2^1 & & & & & & & & & \\
& & \ddots & & & & & & & & \\
& & & \overset{*}{v}{}_m^1 & & & & & & & \\
& & & & \overset{*}{v}{}_1^2 & & & & & & \\
& & & & & \overset{*}{v}{}_2^2 & & & & & \\
& & & & & & \ddots & & & & \\
& & & & & & & \overset{*}{v}{}_m^2 & & & \\
& & & & & & & & \ddots & & \\
& & & & & & & & & \overset{*}{v}{}_1^n & & \\
& & & & & & & & & & \overset{*}{v}{}_2^n & \\
& & & & & & & & & & & \ddots \\
& & & & & & & & & & & & \overset{*}{v}{}_m^n
\end{bmatrix}
\begin{bmatrix} x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hline x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \\ \hline \vdots \\ \hline x_1^n \\ x_2^n \\ \vdots \\ x_m^n \end{bmatrix}
$$

$$v_i^g = \overset{*}{v}{}_i^g \; x_i^g$$

or $$v_i^g = \sum_h \sum_j \overset{*}{\bar{\nu}}{}_{ij}^{gh} \; y_j^h$$

where $$\overset{*}{\bar{\nu}}{}_{ij}^{gh} = \overset{*}{v}{}_i^g \; d_{ij}^{gh}$$

in matrix notation the coefficients $\bar{\nu}_{ij}^{gh}$ form a square matrix ${}^{d}\overset{*}{V}$ of dimension (mn x mn)

Figure 2.10b The primary supply multiplier matrix ${}^{d}\overset{*}{V} = \hat{\overset{*}{V}} D$, with k = 1.

$$\overset{*}{V} = e\hat{\overset{*}{V}} \qquad (2.34)$$

or

$$\overset{*}{V}I = \hat{\overset{*}{V}} \qquad (2.35)$$

where e is a summing row vector of dimension (1 mn) and I the identity matrix. Then all primary supply requirements for each region and industry can be expressed as

$$V = \hat{\overset{*}{V}}X + {}^{v}Y \qquad (2.36)$$

where ${}^{v}Y$ is a column vector of dimension (mn × 1).

To determine only the (nx1) column subvector V^g of region-specific primary supply requirements, it is also possible to use a block diagonal matrix $\hat{\overset{*}{V}}{}^g$ of dimension (n×mn) [analogous to the matrix $\hat{\overset{*}{W}}$ in Fig. C.2a.1], in which case

$$V^g = \hat{\overset{*}{V}}{}^g X + {}^{v}Y^g \qquad (2.37)$$

where X is a full column vector and ${}^{v}Y^g$ a column subvector of dimension (n × 1). The coefficient matrix required to determine the (m × 1) subvector V_i of industry-specific primary supply requirements is more complicated to set up. Therefore, it is preferable to obtain this subvector directly from the vector V in (2.36).

Total primary supply v_o can also be decomposed into a vector representing its functional components, i.e. domestic factor income (or value added) and imports, with the former decomposed further into the income for the different factors labor, capital, etc. To represent v_o as a k order column vector V_k of functionally differentiated primary inputs, requires that the vector $\overset{*}{V}_o$ be replaced by a full matrix $\overset{*}{V}_k$ of primary input-specific coefficients, where $\overset{*}{V}_k$ is of dimension (k × mn) [as in Fig. 2.10c]. Then

$$\begin{bmatrix} v_{o(1)} \\ v_{o(2)} \\ \vdots \\ v_{o(k)} \end{bmatrix} = \left[\begin{array}{cccc:c:cccc} \overset{*}{v}{}^{1}_{1(1)} & \overset{*}{v}{}^{1}_{2(1)} & \cdots & \overset{*}{v}{}^{1}_{m(1)} & \cdots & \overset{*}{v}{}^{n}_{1(1)} & \overset{*}{v}{}^{n}_{2(1)} & \cdots & \overset{*}{v}{}^{n}_{m(1)} \\ \overset{*}{v}{}^{1}_{1(2)} & \overset{*}{v}{}^{1}_{2(2)} & \cdots & \overset{*}{v}{}^{1}_{m(2)} & \cdots & \overset{*}{v}{}^{n}_{1(2)} & \overset{*}{v}{}^{n}_{2(2)} & \cdots & \overset{*}{v}{}^{n}_{m(2)} \\ \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \overset{*}{v}{}^{1}_{1(k)} & \overset{*}{v}{}^{1}_{2(k)} & \cdots & \overset{*}{v}{}^{1}_{m(k)} & \cdots & \overset{*}{v}{}^{n}_{1(k)} & \overset{*}{v}{}^{n}_{2(k)} & \cdots & \overset{*}{v}{}^{n}_{m(k)} \end{array}\right] \left[\begin{array}{c} x^1_1 \\ x^1_2 \\ \vdots \\ x^1_m \\ \hdashline \vdots \\ \hdashline x^n_1 \\ x^n_2 \\ \vdots \\ x^n_m \end{array}\right]$$

$$v_{o(k)} = \sum_g \sum_i \overset{*}{v}{}^{g}_{i(k)} x^g_i$$

$$= \sum_h \sum_j \overset{*}{\nu}{}^{h}_{j(k)} y^h_j$$

The elements $\overset{*}{\nu}{}^{h}_{j(k)}$ form a matrix ${}^{d*}V_k$, with k rows and mn column .

Figure 2.10c The primary supply multiplier matrix ${}^{d*}V_k = \overset{*}{V}_k\, D$, with $k > 1$.

$$V_k = \overset{*}{V}_k X + {}^{v}Y_k \qquad (2.38)$$

where ${}^{v}Y_k$ is a column vector of the same dimension as V_k. The primary requirements for the k^{th} input can then be determined as

$$v_k = \sum_{g=1}^{n} \left[\sum_{i=1}^{m} \overset{*}{v}{}^{g}_{(k)i} x^{g}_{i} \right] + {}^{v}y_k \qquad (2.39)$$

or

$$v_k = \overset{*}{V}_k X + {}^{v}y_k \qquad (2.40)$$

where v_k and ${}^{v}y_k$ are scalars, and $\overset{*}{V}_k$ a row-vector of dimension (1xmn).

It is possible to adapt equations (2.28), 2.31), (2.33) and (2.36) to the determination of industry and/or region-specific requirements for the k^{th} input.

In Annexes A.3 and A.4 some of the issues involved in selecting and estimating direct income and employment coefficients for an I-O model are discussed. The same issues are relevant in the case of the MRIO model. An additional complication is introduced in the MRIO context as a result of the regional disaggregation of each industry's income and employment coefficient. Nevertheless, formally equation (2.40) can be used to determine income and employment levels whether the income concept used for v_k and $\overset{*}{V}_k$ refers to wages, wages and salaries, profit, disposable income, etc. in value units, and whether the employment concept used for v_k and $\overset{*}{V}_k$ refers to person years, machine years, etc. in physical units.

In (2.40) v_k is a function of endogenously determined output X and exogenously determined primary supplies ${}^{v}y$, i.e.

$$v_k = f(X, {}^{v}y_k) \qquad (2.41)$$

By substituting the solution for X in equation (2.6) into equation (2.40), v_k can be expressed as a function exclusively of exogenously determined variables Y and ${}^{v}y_k$,

$$v_k = g(Y, {}^{v}y_k) \qquad (2.42)$$

where
$$v_k = \overset{*}{V}_k DY + {}^{v}y_k \qquad (2.43a)$$

or
$$v_k = \overset{*}{V}_k (I - C\hat{A})^{-1} CY + {}^{v}y_k \qquad (2.43b)$$

In the latter formulation

$${}^{v}D_k = \overset{*}{V}_k (I - C\hat{A})^{-1} C \qquad (2.44)$$

can be used as either an income or an employment multiplier for the k^{th} income or employment category. In the case of income multipliers, the affected variable v_k and the multiplicand Y are measured in the same units, because all the coefficients in the multiplier ${}^{v}D_k$ are dimensionless (since both the numerators and denominators of the coefficients in the matrices $\overset{*}{V}_k$, C and $\hat{A}$ are measured in value units). In the case of employment multipliers, v_k and Y are measured in different units reflecting the dual dimension of the employment coefficients $\overset{*}{V}_k$, in which the numerator is measured in physical units of the primary input and the denominator in value units of output sales.

Total income or total employment v_k can also be expressed in the form of an (mnx1) column vector V_k (not to be confused with the column vector of primary supply V, which is also of the same dimension, or the column vector V_k of functionally differentiated primary inputs, which has the dimension $(k \times 1)$. The vector $\overset{*}{V}_k$ would then have to be expressed in the form of a diagonal matrix $\hat{\overset{*}{V}}_k$ of dimension $(mn \times mn)$, then

$$V_k = \hat{\overset{*}{V}}_k DY + {}^{v}Y_k \qquad (2.45)$$

where ${}^{v}Y_k$ is a column vector of the same dimension as V_k. In this formulation it is clear that, with the MRIO model, it is possible to determine 'detailed' industry and region-specific income or employment effects. On the basis of the income or employment multiplier matrix $\hat{\overset{*}{V}}_k D$ a variety of aggregate scalar multipliers can be constructed in analogy to the output multipliers already discussed. For example, substituting the expression for x_i^g from equation (2.16) into equation (2.46) we get:

$$\overset{*}{v}{}^{g}_{(k)i} = \sum_{h=1}^{n} \sum_{j=1}^{m} \overset{*}{v}{}^{g}_{(k)i} d^{gh}_{ij} y^{h}_{j} + {}^{v}y^{g}_{(k)i} \qquad (2.46)$$

where the scalar $(\overset{*}{v}_{(k)i} d^{gh}_{ij})$ is the detailed income or employment multiplier which can be used to determine the income or employment generated in the i^{th} industry in region g, consequent upon a changed final demand for commodity j in region h.

It can be shown on the basis of equation (2.46) and the fact that the detailed income or employment multiplier can be expressed as

$\overset{*g}{v}_{(k)i}$ M_{ij}^{gh} that for each of the scalar output-multiplier ${}^{x}M_{ij}^{gh}$, ${}^{x}M_{io}^{gh}$, ..., ${}^{x}M_{io}^{oo}$, ${}^{x}M_{oo}^{oo}$ in Table A, there is a corresponding scalar income or employment multiplier, ${}^{w}M_{ij}^{gh}$, ${}^{w}M_{io}^{gh}$, ..., ${}^{w}M_{io}^{oo}$, ${}^{w}M_{oo}^{oo}$, and ${}^{e}M_{ij}^{gh}$, ${}^{e}M_{io}^{gh}$, ..., ${}^{e}M_{io}^{oo}$, ${}^{e}M_{oo}^{oo}$ respectively. A selected set of these multipliers are defined in Annex B.4.

As was the case with the scalar output-multipliers, the greatest amount of information on employment and income impacts is determined through the use of the 'detailed' multipliers and the least through the use of 'total' multipliers. As before, the use of scalar sum multiplicands with these scalar multipliers is equivalent to assuming that composition effects are insignificant.

For regional analysis the 'detailed' MRIO income and employment multipliers have an advantage over equivalently sectored 'detailed' national I-O income and employment multipliers because leakages resulting from inter-regional trade are explicitly taken into acoount in the former. Hence they are less likely to overestimate impacts. This can be demonstrated as follows.

At the national level the 'detailed' wage and salary-income multiplier can be expressed as:

$$ {}^{w}M_{ij}^{N} = \overset{*N}{v}_{(w)i} \, ({}^{x}M_{ij}^{oo}) \tag{2.47} $$

where $\overset{*N}{v}_{(w)i}$ is the wage and salary earned in the i^{th} industry nation-wide, and ${}^{x}M_{ij}^{oo} = b_{ij}^{N}$, in multiregional notation. On the other hand, the multiregional detailed wage and salary-income multiplier is expressed as:

$$ {}^{w}M_{ij}^{gh} = \overset{*g}{v}_{(w)i} \, ({}^{x}M_{ij}^{gh}) \tag{2.48} $$

Clearly the national detailed interindustry income multiplier must be larger than the detailed MRIO income multiplier. That is,

$$ {}^{w}M^{N}_{ij} > {}^{w}M^{gh}_{ij} \tag{2.49}$$

because, by definition the national wage and salary coefficient is the weighted sum of the regional coefficients for all regions, i.e.

$$ {}^{w}M^{N}_{ij} = \sum_{h=1}^{n} (\sum_{g=1}^{n} \overset{*}{v}{}^{g}_{(w)i} \; {}^{x}M^{gh}_{ij}) \tag{2.50}$$

However, even if instead of $\overset{*}{v}{}^{N}_{(w)i}$ the regional income coefficients $\overset{*}{v}{}^{g}_{(w)i}$ of the MRIO model were used with the scaled down national inverse coefficient matrix for region g the resulting 'detailed' income multiplier would still be larger than the detailed MRIO income multiplier because the scaled down national inverse coefficients, or output multipliers, would not incorporate, leakages to regions h via interregional trade.

$$ {}^{w}M^{g(N)}_{ij} = \overset{*}{v}{}^{g}_{(w)i} \; ({}^{x}M^{g(N)}_{ij}) = \overset{*}{v}{}^{g}_{(w)i} \; (\sum_{h=1}^{n} {}^{x}M^{gh}_{ij}) > \overset{*}{v}{}^{g}_{(w)i} \; ({}^{x}M^{gh}_{ij}) = {}^{w}M^{gh}_{ij} \tag{2.51}$$

or

$$ {}^{w}M^{g(N)}_{ij} > {}^{w}M^{gh}_{ij} \tag{2.52}$$

A similar argument can be made with regard to the advantage of 'detailed' MRIO income and employment multipliers over 'detailed' regional I-O income and employment multipliers since the latter will tend to underestimate impacts by not including interregional feedback in the production of gross outputs. From the preceeding examples it should be clear that output-multipliers form an important component of both income and employment multipliers[6/] and will, in fact, form an important component of other types of MRIO multipliers, including export multipliers, government spending multipliers, etc. Thus, even though it is very important to carefully specify and estimate the direct income or employment coefficients of the diagonal matrix $\hat{\overset{*}{V}}_k$, it is also important that the analyst be aware of the structure and limitations of the output multipliers that are the constituent parts of the income and employment multipliers that are being used. Variations in detailed income and employment multipliers are as much a function of the differences in regional industrial technology, and the national pattern of interregional trade, as they are of direct estimates of industry and region-specific income and employment coefficients. It is unlikely that any improvement in the specification and estimation of the direct income-to-output and employment-to-output coefficients can totally offset weaknesses in the output multiplier component of the income and employment multipliers.

As is pointed out in Annex A it is also necessary that the multiplier be established on a basis that is consistent with the exogenously determined data by which it is to be multiplied. In recent studies by Golladay and Haveman (1977), Rowen (1977a), and D. DiPasquale and K. Polenske (1977), the multiregional output, employment, or income multipliers have been combined with projected changes in final demand to determine the

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6/ For example, both income and employment multipliers reflect <u>direct and indirect</u> production requirements because the gross output multipliers are incorporated in them. However, since the coefficients $\overset{*}{v}^g_{(k)i}$ are by construction less than one, the income and employment multipliers must of necessity always be smaller than the gross-output multipliers for the corresponding industry and region.

corresponding output, employment, or income effects of those expenditures. However, in many other studies, particularly regional studies, Type I multipliers are used (see Annex A.4), in which the income and employment multipliers of an open I-O model are divided by the direct income or employment coefficients in the industry and region in which the demand is located. The resulting multipliers are then combined with projected changes in these direct incomes, or employment to calculate the total impact.

As noted by DiPasquale and Polenske (1977), I-O income and employment multipliers can be adapted to suit the needs of the analyst who encounters problems in obtaining projections for specific variables, whether final demand, income or some other variable. Many regional analysts have used direct income, in part because this measure is more easily projected at the regional level. If only direct income projections are available, instead of final demand projections, it is then necessary that the MRIO income-multipliers be transformed to an equivalent basis.

Chapter 3

CLOSING THE MRIO MODEL WITH RESPECT TO CONSUMPTION

There are both theoretical and operational reasons for wanting to incorporate the personal income formation process into the MRIO model. From a theoretical point of view the omission of the income formation process is often not justified in interindustry analysis (see Annex A.3), particularly in a multiregional context, because the total amount of income generated depends on the distribution of production in the various regions, since the marginal propensity to save is likely to differ between regions (Chenery and Clark, 1959, p. 70), and "the location of production depends on the location of consumption, and the latter cannot be determined separately from the calculation of income generated in each sector and region." (ibid. p. 68)[1/]

Another reason often cited as a justification for closing an interindustry model is in essence an operational reason, though it is not noted as such in the literature. This reason is based on the observation that the open MRIO model income multiplier takes into account only the income generated by the total production requirements of one dollar of final demand. It is argued that the repercussions of the initial change in final demand, does not terminate there, since the additional income generated in the process of producing an extra dollar

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[1/] The last point has also been given by K. Miyazawa (1968, p. 40) as the fundemental reason for closing an interregional model with respect to consumption.

of final demand is bound to induce more consumption, more production and consequently more income. In order to evaluate the overall income effect (direct, plus indirect, plus induced) within an interindustry framework it is then argued that an augmented model should be used. What this argument does not note is that for equivalent multiplicands [see equation (5.10)], the gross output and total incomes will be identical whether an open model, or a model augmented with respect to consumption, is used.[2/] Thus, the larger multipliers incorporating induced effects are a consequence of reducing the base through which injections enter the income flow, i.e. by excluding consumption from the exogenously determined vector of final demand. This reason for closing the model is, therefore, really analogous to that which led to the introduction of the compound multiplier in macroeconomic analysis (see Annex A.2). That is, if, for a variety of reasons, the induced effects of a major component[3/] of the more inclusive

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2/ The induced expenditures are not a new series of injections that could lead to a different overall income effect as is implicit in the argument that is paraphrased above.

3/ At the national level personal consumption expenditures (PCE) have averaged approximately 65 percent of total GNP (K. Polenske, 1972a, p. 9) and more than 75 percent for some sectors (L. Taylor, 1975, p. 48). Even though the share of PCE as a proportion of State final demand varies from State to State it still outweighs all other components of final demand combined.

exogenously determined variable (or multiplicand) cannot be projected independently, then it is desirable to incorporate the induced effects directly into the multiplier.

In Annex A.3 it is pointed out that there are two basic approaches to closing the I-O model: an iterative approach using the open model and a simultaneous approach using the augmented model. It is only in the latter approach that the multiplier matrix itself incorporates the induced income effects, at the cost of restricting consumption demand to a linear function of income (though the functional relationship can be either homogenous or non-homogenous).

Thus, only with the augmented model is it possible to compare analytically the difference between multipliers incorporating induced effects and those multipliers derived from the open model which do not include induced effects. In the next section the standard approach to closing the MRIO model will be reviewed and its problems discussed as a prelude to introducing an alternate approach to closing the model which will make possible the analytic comparison of the augmented model multipliers with the open model multipliers.

3.1: Standard Approach to Augmenting the MRIO Matrix Θ and the Solution of the Augmented Model

The standard procedure for augmenting the multiregional trade-adjusted technical coefficient matrix Θ is related to that used in augmenting a national or regional technical coefficient matrix. It is important to note, however, that unlike the single economy I-O models, the (mn×nm) MRIO matrix Θ cannot be augmented directly by appending to it the (nm×1) column vector of consumption coefficients and the (1×nm) row vector of income coefficients to obtain an $[(nm+1) \times (nm+1)]$ augmented matrix ${}^{a}\Theta$, as might superficially appear to be analogous to augmenting the national or regional I-O models.

Instead in order to obtain the correctly dimensioned augmented matrix ${}^{a}\Theta$ it is necessary to augment each of the component matrices of Θ, that is C and $\hat{A}$ separately. To augment $\hat{A}$, each of the regional technical coefficient matrices A^{g} must be augmented individually by adding an additional row and column of income and consumption coefficients respectively to obtain an $[(m+1) \times (m+1)]$ augmented technical coefficient matrix ${}^{a}A^{g}$. The (n) matrices of this type are then incorporated as blocks along the principal diagonal of the expanded technical coefficient matrix ${}^{a}A$ to obtain a square matrix[4/] with dimensions $[(nm+n) \times (nm+n)]$ (see Figure C.1.1).

The consumption coefficients that are used to augment the model must be derived from the functional relationship between consumption expenditures and that part of income which is assumed to be the source of the expenditures.

4/ In order for the augmented matrix to be invertible (that is, for both right-hand and left-hand inverse matrices to exist and be equal) it is necessary that the matrix be square.

There is no one correct set of income and consumption coefficients. Generally, it is preferable that if 'personal consumption expenditures' are made endogenous that the 'disposable personal income' on which these personal consumption expenditures are based be fully incorporated. It is also possible, if the value-added data cannot be disaggregated in order to obtain disposable personal income, to incorporate the whole value-added row-vector.

Alternately, and this is the procedure commonly used in I-O and MRIO analysis, it is possible to treat the household sector as a 'fictious' industry in which output supplied is equal to output demanded, i.e. in which the row sum and the column sum of a given industry are equal at the national level. This is equivalent to setting consumption expenditures equal to, rather than, less than income received.

Given the data base of the MRIO model the endogenous income variable is 'wage and salary' income. Hence, the income coefficients represent wage and salary income coefficients and are defined as:

$$\overset{*}{w}{}^{g} = \frac{x^{g}_{(w)j}}{x^{g}_{oj}} \qquad (3.1)$$

where $x^{g}_{(w)j}$ is the income generated in industry j, whose output is x^{g}_{oj}. The 'personal consumption expenditure' coefficients take the form

$$\overset{*}{c}{}^{h}_{i} = \frac{{}^{c}y^{h}_{i}}{{}^{c}y^{h}} = \frac{{}^{c}y^{h}_{i}}{w^{h}}, \quad \text{since } {}^{c}y^{h} \equiv w^{h} \qquad (3.2)$$

where ${}^{c}y^{g}_{i}$ represents personal consumption expenditures allocated for the i^{th} commodity in region h, ${}^{c}y^{h}$ total personal consumption expenditures in region h, and w^{h} total wage and salary income in region h. These coefficients will be used throughout the analysis in chapters 3 and 4.

Before the augmented MRIO matrix Θ can be obtained it is necessary to create an expanded C matrix of interregional trade flows which is of the same dimension as the augmented technical coefficient matrix ${}^{a}A$. This requires that, in analogy to the other MRIO industries (see Figure C.1.3), an (n×n) trade coefficient matrix C_k be created for the 'fictitious' industry and its coefficients distributed along the diagonals of each of the n^2 diagonal submatrices ${}^{a}\hat{C}^{gh}$ representing the trade of all commodities between each pair of regions (see Figure C.1.4). These diagonal submatrices, of dimension $[(m+1)\times(m+1)]$ each, can then be arranged as the blocks of the expanded trade flow matrix ${}^{a}C$ to obtain another $[(mn+n) \times (mn+n)]$ matrix (see Figure C.1.5a). In the absence of trade flow data for this 'fictitious' industry, it is possible to treat its output as not traded, in which case the matrix $C_k = I$, the identity matrix. Then only the $(m+1)^{th}$ diagonal elements of the submatrices ${}^{a}\hat{C}^{gg}$ will be expanded with a positive element equal to unity, while the $(m+1)^{th}$ diagonal element of the submatrices ${}^{a}\hat{C}^{gh}$ (with $g \neq h$) will be expanded with zero elements only (see Figure C.1.6).

It is only after the augmented matrices $({}^{a}C)$ and $({}^{a}\hat{A})$ have been created that the augmented matrix ${}^{a}\Theta = {}^{a}C\,{}^{a}\hat{A}$ can be derived with the correct dimensions $[(mn+n)\times(mn+n)]$ (see Figure C.1.7).

The balancing equations and solution to this formulation of the partially closed MRIO model are in form strictly analogous to the balancing equations and solution of the open MRIO model, though the interpretation and magnitude of the elements of the multiplier matrix ${}^{a}D$ and the multiplicand vector ${}^{a}\tilde{Y}$ will be different. The complete augmented MRIO model can be written as:

$$ {}^{a}X = {}^{a}C({}^{a}\hat{A} + {}^{a}\tilde{Y}) \qquad (3.3) $$

$$ \tilde{v}_o = {}^{a}\tilde{V}_o\,{}^{a}X + {}^{v}\tilde{y} \qquad (3.4) $$

where a designates augmented vectors and matrices.

$\sim$ designates vectors and scalars of lesser magnitude than their corresponding vectors and scalars in the open MRIO model.

${}^{a}X$ is a column vector of dimension $[(nm+n)\times 1]$, in which the first (m) elements of each of the subvectors ${}^{a}X^{g}$ of dimension (m+1) are the same as the corresponding elements of the subvectors X^{g} of dimension (m) in the gross output vector, X, and the $(m+1)^{th}$ element of each subvector ${}^{a}X^{g}$ represents total 'wages and salary' income in the g^{th} region.

${}^{a}\tilde{Y}$ is a column vector of dimension $[(nm+n)\times 1]$ representing the final demand vector (excluding personal consumption expenditures), in which the first m elements of each of the subvectors ${}^{a}\tilde{y}^{h}$ of dimension (m+1) correspond to the same elements of the subvectors y^{h} of dimension m in the final demand vector Y though the magnitude of the elements of ${}^{a}\tilde{y}^{h}$ are smaller than their counterparts y^{h} by the amount ${}^{c}y^{h}$, and the $(m+1)^{th}$ element of each subvector ${}^{a}\tilde{y}^{h}$ represents exogeneously determined wage and salary income.

$\tilde{v}$ is the scalar of total primary supply excluding wage and salary income.

${}^{a}\tilde{V}^{*}_{o}$ is the $[1\times(nm+n)]$ vector of primary supply coefficients excluding wage and salary income coefficients.

and $\tilde{v}_{\tilde{y}}$ is the scalar of the direct payments (excluding wage and salary income) to primary suppliers by the non-consumption final demand sector.

The simultaneous solution for gross outputs and 'wage and salary' income in this model is represented by the equations

$${}^{a}X = ({}^{a}I - {}^{a}C\,{}^{a}\hat{A})^{-1}\,{}^{a}C\,{}^{a}\tilde{Y} \qquad (3.5)$$

$$= {}^{a}D\,{}^{a}\tilde{Y} \qquad (3.6)$$

where

$${}^{a}D = {}^{a}B\,{}^{a}C \qquad (3.7)$$

$${}^{a}B = ({}^{a}I - {}^{a}\Theta)^{-1} \qquad (3.8)$$

and

$${}^{a}\Theta = {}^{a}C\,{}^{a}\hat{A} \qquad (3.9)$$

The multiplier matrix ${}^aD$ will consist of n^2 block submatrices of dimension $[(m+1) \times (m+1)]$. The first (m×m) set of coefficients in each block correspond to the same set of coefficients in the blocks of the open model multiplier matrix D, except they now include induced effects.

The same number and types of submatrix and scalar output multipliers can be obtained from the matrix ${}^aD$ as were obtained from the matrix D with suitable adjustments in interpretation and noting that the first m rows of each block contain an $(m+1)^{th}$ element to represent the effect of exogenously determined income on gross outputs.

It is also possible in this case to obtain regional income multipliers directly from the augmented inverse. Empirical results from implementing this model (see, for example, DiPasquale and Polenske, 1977) are consistent with apriori expectations based on the results obtained from closing single economy I-O models. That is, the output and income multipliers obtained with (3.5) are larger than the corresponding output and income multipliers obtained with (2.6) and (2.45) respectively. The difference in the magnitude of the multipliers is attributed to induce effects. That the difference is in fact due to induced effects cannot be demonstrated in this formulation of the multiplier matrix. It is also not possible to show theoretically if the output and income multipliers of the augmented model differ systematically from those of the open model[4/], and if they do, why and by how much. Finally with this formulation of the model it is not possible to demonstrate the theoretical relationship of the

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[4/] As has been done in the case of aggregated income multipliers in single economy I-O models (See Annex A.4).

augmented MRIO model multipliers to the aggregate Keynesian model multipliers.

An alternate approach to augmenting the MRIO matrix Θ is presented in the next section. As a result of this reformulation of the model it is possible with the aid of a partitioned matrix solution to address the above mentioned problems analytically. The partitioned matrix solution also turns out to be computationally more efficient that the standard solution of the augmented MRIO model.

3.2: Proposed Approach to Augmenting the MRIO Matrix and the partitioned Matrix Solution of the Augmented Model

To obtain the augmented MRIO matrix ${}^{a}\Theta$ in the preceeding section, each of the regional submatrices of the open model matrices C and $\hat{A}$ were augmented separately and then entered into the expanded matrices ${}^{a}C$ and ${}^{a}\hat{A}$. An alternate procedure, based on an adaptation of a method introduced by Miyazawa (1968) to augment the 'pure' interregional model[5/] is presented in this section. This procedure also requires that the component matrices of Θ, i.e., C and A, be augmented separately,[6/] but not by directly augmenting each of the regional submatrices ${}^{a}A^{g}$ and ${}^{a}\hat{C}^{gh}$. A detailed step-by-step description of the proposed procedure is provided in Annex C.2. In outline it involves the following adjustments:

To augment the expanded technical coefficient matrix A, it is necessary to use block diagonal matrices, where the blocks refer to region-specific

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5/ His analysis (Miyazawa, 1963 and 1968) is in fact less general than the one developed in this dissertation because it is predicated on the assumption of a closed economy and no direct transactions between the final demand and primary supply sectors of the open or augmented model, i.e. he assumes $\hat{Z}$, $\hat{U}$, ${}^{w}\tilde{Y}$ and ${}^{v}\tilde{y}$ are all equal to zero.

6/ Of course, once the logic of the procedure has been demonstrated it is possible for Θ to be augmented directly.

subvectors of income and/or consumption coefficients. These block-diagonal matrices cannot be obtained by post-multiplying the row-vector of income coefficients of the open model, or premultiplying the column-vector of consumption coefficients, by the identity matrix I. Instead the new matrices must be constructed in the same way in which the matrix $\hat{A}$ is constructed. In addition, the scalar at the intersection of the two vectors must be disaggregated into the form of a diagonal matrix. These three new matrices can then be used to augment the matrix $\hat{A}$.

The expanded interregional trade coefficient matrix must also be augmented in order to be consistent with the reformulated expanded technical coefficient matrix. However, instead of augmenting each of the diagonal submatrices of C with the elements of the newly incorporated interregional trade coefficient matrix C_k, as in the standard MRIO procedure, it is now necessary that the full matrix C_k be inserted as a second block on the principal diagonal of a new block-diagonal matrix, which has the open model's expanded trade matrix C as the other block. A reformulated 'augmented matrix Θ' is then obtained by postmultiplying the reformulated trade coefficient matrix by the reformulated technical coefficient matrix.

The reformulated augmented MRIO system of equations can now be written in partitioned matrix notation as

$$\left[\begin{array}{c} X \\ \hline W \end{array}\right] = \left[\begin{array}{c:c} C & O \\ \hline O & C_k \end{array}\right] \left[\begin{array}{c:c} A & \hat{\overset{*}{C}} \\ \hline \hat{\overset{*}{W}} & \hat{\overset{*}{Z}} \end{array}\right] \left[\begin{array}{c} X \\ \hline W \end{array}\right] + \left[\begin{array}{c:c} C & O \\ \hline O & C_k \end{array}\right] \left[\begin{array}{c} \tilde{Y} \\ \hline w_{\tilde{Y}} \end{array}\right] \qquad (3.10)$$

The secondary condition which the model must fulfill for the non-incorporated primary inputs (i.e. imports plus other value added components excluding wage and salary income) is

$$\tilde{v}_o = \left[\begin{array}{c:c} \tilde{\overset{*}{V}} & \overset{*}{U} \end{array}\right] \left[\begin{array}{c} X \\ \hline W \end{array}\right] + \tilde{v}_{\tilde{y}} \qquad (3.11)$$

In the above equations, X, C and $\hat{A}$ are the same vector and matrices encountered in the open MRIO model. The vector $\tilde{Y}$ is the same dimension as the vector X and represents the exogenous final demand expenditures excluding personal consumption expenditures (PCE). The elements of the (n×1) vector W represent region-specific total wage and salary incomes. The coefficients of the (n×nm) block-diagonal matrix $\hat{\overset{*}{W}}$ for wage and salary income and the (1×nm) vector $\overset{*}{V}$ for primary inputs (excluding wage and salary income) are linear functions of gross outputs. The region-specific consumption coefficients are linear functions of the region-specific wage and salary incomes. These include the coefficients of the (nm×1) block-diagonal matrix $\hat{\overset{*}{C}}$ for each commodity i, the (n×n) diagonal matrix $\hat{\overset{*}{Z}}$ for wage and salary income, and the (1×n) row vector $\overset{*}{U}$ for 'other primary inputs'. Finally the (n×1) vector ${}^{w}\tilde{Y}$, and the scalar ${}^{v}\tilde{y}$, represent exogenously determined region-specific wage and salary income, and exogenously determined region-specific other primary inputs, respectively. In the operational MRIO model the former represents basically the wage and salary income of government employees and the latter represents basically the imports delivered directly to the non-PCE final demand sector. The scalar ${}^{v}\tilde{y}$, however, also contains a number of other residual categories. Some additional detail on the definition of the coefficients is available in Annex C.2.

Equation (3.11) is a straight forward solution of the scalar $\tilde{v}_o$ The results only differ in interpretation from the primary supply scalar v_o

in equation (2.24), as a result of the exclusion of wage and salary income.[7/] Hence, the subsequent analysis will be limited to solving and interpeting the results of equation (3.10). Before equation (3.10) can be solved it is necessary for convinience to introduce a new set of notation after post-multiplying the 'augmented matrix C' by the 'augmented matrix $\hat{A}$':

$$\left[\begin{array}{c} X \\ \hline W \end{array}\right] = \left[\begin{array}{c|c} CA & C\hat{\overset{*}{C}} \\ \hline C_k\hat{\overset{*}{W}} & C_k\hat{\overset{*}{Z}} \end{array}\right]\left[\begin{array}{c} X \\ \hline W \end{array}\right] + \left[\begin{array}{c|c} C & 0 \\ \hline 0 & C_k \end{array}\right]\left[\begin{array}{c} \tilde{Y} \\ \hline w_{\tilde{Y}} \end{array}\right] \qquad (3.12)$$

The new notation involves setting:

$$\left[\begin{array}{c|c} CA & C\hat{\overset{*}{C}} \\ \hline C_k\hat{\overset{*}{W}} & C_k\hat{\overset{*}{Z}} \end{array}\right] = \left[\begin{array}{c|c} \Theta & \Gamma \\ \hline \Upsilon & \Lambda \end{array}\right] \qquad (3.13)$$

Thus,

(1) $C\hat{A} = \Theta = \left[\Theta^{gh}_{ij}\right]$ represents the (mn×mn) matrix of interregional input coefficients per unit of output as in the open MRIO model, where

$\Theta^{gh}_{ij} = c^{gh}_{i}a^{h}_{ij}$ represents the amount of the i^{th} commodity produced in region g which is supplied to region h for use in the production of one unit of output of the j^{th} industry

(g,h = 1,2,...,n) and (i,j = 1,2,...,m)

7/ The scalar $\tilde{v}$ can also be written in the form of a vector $\tilde{V}$ in strict analogy to the primary supply vector V in equation (2.36).

(2) $\hat{\overset{*}{C}}C = \Gamma = \left[\gamma_i^{gh}\right]$ represents the (mn×n) matrix of input-specific PCE coefficients, where

$\gamma_i^{gh} = c_i^{gh} \overset{*}{c}_i^{h}$ represents the amount of personal consumption expenditures allocated out of each unit of income earned by the 'household' sector in region h for the purchase of the i^{th} commodity produced in region g.

(3) $C_k \hat{\overset{*}{W}} = T = \left[\upsilon_j^{gh}\right]$ represents the (n×mn) matrix of direct 'wage and salary' income-to-output coefficients, where

$\upsilon_j^{gh} = c_{(k)}^{gh} \overset{*}{w}{}_{j(k)}^{h}$ represents the amount of wage and salary income earned in region g from each unit of production of the j^{th} industry in region h.

(4) $C_k \hat{\overset{*}{Z}} = \Lambda = \left[\lambda^{gh}\right]$ represents the (n×n) matrix of inter-regional wage and salary income coefficients per unit of total PCE, where

$\lambda^{gh} = c_{(k)}^{gh} \overset{*}{z}{}_{(k)}^{h}$ represents the wage and salary income earned in region g directly from each unit of total personal consumption expenditures in region h.

The correctness of these interpretations can be confirmed by referring to the augmented matrix Θ in Figure C.2a.8. The subscript (k) in $c_{(k)}^{gh}$, $\overset{*}{w}{}_{j(k)}^{h}$ and $\overset{*}{z}{}_{(k)}^{h}$ has been used at this point only to distinguish the use of the coefficients of the matrix C_k from the coefficients of the matrix C. In the conclusion, however, it will become apparent that (k) can be used to designate the k^{th} income group in each region.

The solution of the system of equations in (3.12) can now be written as:

$$\left[\begin{array}{c} X \\ \hline W \end{array}\right] = \left[\begin{array}{c|c} I-\Theta & -\Gamma \\ \hline -T & I-\Lambda \end{array}\right]^{-1} \left[\begin{array}{c|c} C & O \\ \hline O & C_k \end{array}\right] \left[\begin{array}{c} \tilde{Y} \\ \hline \tilde{w}_{\tilde{Y}} \end{array}\right] \qquad (3.14)$$

The partitioned matrix approach is used in Annex D.2 to solve the inverse matrix in equation (3.14). Incorporating the results obtained there, the solution of the augmented MRIO model can be expressed as

$$\left[\begin{array}{c} X \\ \hline W \end{array}\right] = \left[\begin{array}{c|c} B(I+\Gamma\Psi TB) & B\Gamma\Psi \\ \hline \Psi TB & \Psi \end{array}\right] \left[\begin{array}{c|c} C & 0 \\ \hline 0 & C_k \end{array}\right] \left[\begin{array}{c} \tilde{Y} \\ \hline {}^{w}\tilde{Y} \end{array}\right] \qquad (3.15)$$

where $B = (I-\Theta)^{-1}$ as in the open MRIO model

$$\Psi = \bar{\Psi}(I-\Lambda\bar{\Psi})^{-1} \qquad (3.16)$$

$$\bar{\Psi} = (I-\Phi)^{-1} \qquad (3.17)$$

and $\Phi = TB\Gamma$ 8/ $\qquad$ (3.18)

Using the open MRIO multiplier matrix D = BC, the multiplier matrix of the augmented MRIO model can finally be expressed as

$$\left[\begin{array}{c} X \\ \hline W \end{array}\right] = \left[\begin{array}{c|c} D(I+\hat{\mathring{C}}\tilde{\Psi}WD) & D\hat{\mathring{C}}\tilde{\Psi} \\ \hline \tilde{\Psi}\hat{\mathring{W}}D & \tilde{\Psi} \end{array}\right] \left[\begin{array}{c} Y \\ \hline {}^{w}\tilde{Y} \end{array}\right] \qquad (3.19)$$

where $\tilde{\Psi} = \Psi C_k$

This set of equations can also be written as

$$X = D_{11}\tilde{Y} + D_{12}({}^{w}\tilde{Y}) \qquad (3.20)$$

$$W = D_{21}\tilde{Y} + D_{22}({}^{w}\tilde{Y}) \qquad (3.21)$$

8/ Φ can also be expressed as a function of the open MRIO multiplier matrix D, i.e.,

$$\Phi = C_k\hat{\mathring{W}}D\hat{\mathring{C}} \qquad (3.18)$$

where

$$D_{11} = D(I + \hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D) \text{ 9/} \qquad (3.22)$$

$$D_{12} = D\hat{\overset{*}{C}}\tilde{\Psi} \qquad (3.23)$$

$$D_{21} = \tilde{\Psi}\hat{\overset{*}{W}}D \qquad (3.24)$$

and

$$D_{22} = \tilde{\Psi} \qquad (3.25)$$

In the special case where $C_k = I$, then $\tilde{\Psi}$ reduces to Ψ and the above relations become

$$D_{11} = D(I + \hat{\overset{*}{C}}\Psi\hat{\overset{*}{W}}D) \qquad (3.22)$$

$$D_{12} = D\hat{\overset{*}{C}}\Psi \qquad (3.23)$$

$$D_{21} = \Psi\hat{\overset{*}{W}}D \qquad (3.24)$$

$$D_{22} = \Psi \qquad (3.25)$$

and the Φ component of Ψ reduces to

$$\Phi = \hat{\overset{*}{W}}D\hat{\overset{*}{C}} \qquad (3.26)$$

This is the formulation used in Annex E to show the equivalance of the results obtained by using the partitioned matrix solution with the results obtained, by DiPasquale and Polenske (1977), using the standard MRIO procedure in a three region, four commodity model using actual 1963 data.

The subsequent analysis will be based, however, on the general case where $C_k \neq I$. To highlight the formal differences between the MRIO direct coefficient and multiplier matrices, and Miyazawa's formulation (1968) of the same matrices in the interregional I-O context a comparison is presented in Table B.

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$$D_{11a} = D \qquad (2.73'a)$$

$$D_{11b} = D\overset{*}{C}\tilde{\Psi}\hat{\overset{*}{W}}D \qquad (2.73'b)$$

TABLE B: Comparison of the Forms of the unaugmented and augmented Interregional and Multiregional I-O Model direct coefficient and multiplier matrices

	Interregional I-O Model		Multiregional I-O Model		
	Open	Augmented	Open	Augmented	
				Standard	Alternate
(dimension)	$(nm \times nm)$	$\begin{bmatrix} (nm \times nm) & (nm \times n) \\ (n \times nm) & (n \times n) \end{bmatrix}$	$(nm \times nm)$	$(nm+n) \times (nm+n)$	$\begin{bmatrix} (nm \times nm) & (nm \times n) \\ (n \times nm) & (n \times n) \end{bmatrix}$
Direct Coefficient Matrices	A	${}^{a}A = \begin{bmatrix} A & \overset{*}{C} \\ \overset{*}{V} & 0 \end{bmatrix}$	$\Theta = C\hat{A}$	${}^{a}\Theta = {}^{a}C\,{}^{a}\hat{A}$	${}^{a}\Theta = \begin{bmatrix} C\hat{A} & C\hat{\overset{*}{C}} \\ C_k\hat{\overset{*}{W}} & C_k\hat{\overset{*}{Z}} \end{bmatrix} = \begin{bmatrix} \Theta & \Gamma \\ T & \Lambda \end{bmatrix}$
Direct-and Indirect-Coefficient Matrices	$B = (I-A)^{-1}$		$D = (I-\Theta)^{-1}C$ or $D = (C^{-1}-\hat{A})^{-1}$		
Direct, Indirect and Induced Coefficient Matrices		${}^{a}B = \begin{bmatrix} I-A & -\overset{*}{C} \\ -\overset{*}{V} & I \end{bmatrix}^{-1} = \begin{bmatrix} B(I+CKVB) & BCK \\ KVB & K \end{bmatrix}$ with $K = (I-L)^{-1}$, $L = \overset{*}{V}B\overset{*}{C}$ $(n \times n)$		${}^{a}D = (I-{}^{a}\Theta)^{-1}\,{}^{a}C$ or ${}^{a}D = ({}^{a}C^{-1}-{}^{a}\hat{A})^{-1}$	${}^{a}D = \begin{bmatrix} I-\Theta & -\Gamma \\ -T & \Lambda \end{bmatrix}^{-1} \begin{bmatrix} C & 0 \\ 0 & C_k \end{bmatrix} = \begin{bmatrix} B(I+\Gamma\Psi TB) & B\Gamma\Psi \\ \Psi TB & \Psi \end{bmatrix} \begin{bmatrix} C & 0 \\ 0 & C_k \end{bmatrix} = \begin{bmatrix} D(I+\hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D) & D\hat{\overset{*}{C}}\tilde{\Psi} \\ \tilde{\Psi}\hat{\overset{*}{W}}D & \tilde{\Psi} \end{bmatrix}$ with $\tilde{\Psi} = \Psi C_k$, $\Psi = \bar{\Psi}(I-\Lambda\bar{\Psi})^{-1}$, $\bar{\Psi} = (I-\Phi)^{-1}$, $\Phi = TB\Gamma = C_k\hat{\overset{*}{W}}D\hat{\overset{*}{C}}$ $(n \times n)$

CHAPTER 4

AUGMENTED MRIO MODEL MULTIPLIERS

In Chapter 3 it has been pointed out that the empirically estimated coefficients of the augmented MRIO model inverse are larger than the inverse coefficients of the open MRIO model which is consistent with a priori expectation. However, since the form of the standard solution of the augmented MRIO model is identical to the form of the open model solution, the theoretical relationship between the two sets of multipliers cannot be demonstrated analytically. Instead, the observed differences between the two sets of multipliers have been imputed to induced effects based only on reasoning by analogy with the single economy I-O models. With the aid of the partitioned matrix solution presented in the previous chapter, it is now possible to rigorously demonstrate that the augmented MRIO model multipliers incorporate induced effects and that the relationship between the two sets of multipliers is systematic.

Interpretation of the MRIO matrix $\tilde{\Psi}$ as an interregional multiplier matrix

In order to interpret the partitioned matrix components of the 'augmented multiplier matrix' ${}^{a}D$, it is necessary to first interpret the matrix $\tilde{\Psi}$ which appears in all four quadrants of ${}^{a}D$.

$$
{}^{a}D = \begin{bmatrix} D(I+\hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D) & D\hat{\overset{*}{C}}\tilde{\Psi} \\ \tilde{\Psi}\,\hat{\overset{*}{W}}D & \tilde{\Psi} \end{bmatrix} \tag{4.1}
$$

The most general formulation of $\tilde{\Psi}$ is

$$\tilde{\Psi} = \Psi C_k = [\overline{\Psi}(I-\Lambda\overline{\Psi})^{-1}]C_k \tag{4.2}$$

This (n x n) matrix can be interpreted in stages. First, it is necessary to ignore C_k and focus on interpreting Ψ. The formulation of Ψ provided in equation (3.16) presumes that $\Lambda > 0$.

In general	$\Lambda > 0$	if $\hat{\overset{*}{Z}} > 0$	(4.3)
since	$\Lambda = \hat{C}_k\hat{\overset{*}{Z}}$	if $C_k \neq I$	(4.4a)
and	$\Lambda = \overset{*}{Z}$	if $C_k = I$	(4.4b)
However,	$\Lambda = 0$	if $\hat{\overset{*}{Z}} = 0$	(4.5)

This latter assumption is normally encountered in non-operational models, in which case equation (3.16) reduces to

$$\Psi = \overline{\Psi} \tag{4.6}$$

where $\overline{\Psi} = (I-\phi)^{-1}$. Thus, $\Psi > 0$, even if $\hat{\overset{*}{Z}} = 0$.1/ Therefore, $\overline{\Psi}$, which is unaffected by whether or not Λ is positive, will be interpreted first. Equation (3.17) shows that $\overline{\Psi}$ is a function of ϕ and according to equation (3.18)

$$\Phi = TB\Gamma$$

where $B = (I-\theta)^{-1}$.

The elements of ϕ can be expressed as

$$\phi^{th} = \sum_{r=1}^{n}\sum_{j=1}^{m}\sum_{s=1}^{n}\sum_{i=1}^{m} \nu_j^{gr} b_{ji}^{rs} \gamma_i^{sh} \tag{4.7}$$

where ν_j^{gr} is the wage and salary income earned in region g from each unit of output of the j^{th} industry in region r

1/ However, if $\hat{\overset{*}{Z}}=0$, then the commodity-specific coefficient of $\hat{\overset{*}{C}}$ must be larger than the coefficients of $\overset{*}{C}$ when $\overset{*}{Z} > 0$. This point will be demonstrated in the next chapter (see equation (5.12)).

b_{ji}^{rs} is the amount of output the jth industry in region r must produce per unit of industry i's output in region s

and

γ_{i}^{sh} is the demand for the output of the i^{th} industry in region s per unit of wage and salary income in region h

By summing over all intermediate regions, s and r, and over all intermediate industries, i and j, the coefficients ϕ^{gh} represent the amount of wage and salary income induced in region g (via the interregional production process) as a consequence of commodity-specific consumption expenditures from each unit of income in region h.

The matrix Φ can also be expressed as a function of the open model multiplier matrix D instead of its component matrix B. Then

$$\Phi = \Upsilon D\hat{\overset{*}{C}} \tag{4.8}$$

where

$$\phi^{gh} = \sum_{r=1}^{n} \sum_{j=1}^{m} \sum_{i=1}^{m} \upsilon_{j}^{gr} d_{ji}^{rh} c_{i}^{*h} \tag{4.9}$$

all of whose components have been defined previously.

The matrix Φ is of dimension (n x n). That is, it is of the same order as the number of regions in the model. Its coefficients ϕ^{gh}, therefore show the interdependence between regional incomes. They are, thus, similar to the coefficients of the MRIO matrix θ which shows the interdependence between regional outputs.

The correctness of this interpretation of the matrix ϕ can also be shown more clearly by tracing, step by step, in the manner of Miyazawa (1963, p.94)[2/] the effect on region g's income of the commodity-specific consumption expenditure allocations per unit of region h's income:

Using the multiplier matrix D the analytic propagation process associated with the non-consumption multiplicand $\tilde{Y}$ can be represented as:

(1)	$D\tilde{Y}$	which shows how much the output in each industry must change to be consistent with the exogenously determined changes in the non-consumption components of final demand

This effect, however, is only the initial effect of $\tilde{Y}$ because the output changes will in turn generate changes in income. For region h, induced income changes can be represented as

(2)	$(T^h)D\tilde{Y}$	which shows how much income in the h^{th} region must change to be consistent with the change in output $D\tilde{Y}$

2/ Miyazawa's procedure is presented in the context of a single economy I-O model with different income categories. However, his subsequent use of a similar type of matrix in the context of the pure interregional I-O model (Miyazawa, 1968) suggests that the adaptation of this procedure to the interpretation of the MRIO matrix Ψ is warranted.

Since in the augmented model the income to consumption loop is closed, the changed income in Step 2 requires a change in consumption in region h equivalent to

(3) $(\Gamma^h)(T^h)D\tilde{Y}$ which shows the induced change in the h^{th} region's consumption expenditures due to the changed income in the region

These induced consumption changes in region h will in turn necessitate additional changes in output equal to

(4) $B(\Gamma^h)(T^h)D\tilde{Y} = D\hat{\overset{*}{C}}{}^h T^h D\tilde{Y}$ which shows the changed output in all industries due to the changed consumption expenditures in region h

The output changes induced by region h in Step 4 will in turn generate changes in income in region g equivalent to

(5) $(T^g)D(\hat{\overset{*}{C}}{}^h)(T^h)D\tilde{Y}$ which shows the changed income in the g^{th} region due to the changed output resulting from the changed income in region h

Thus, each element of the matrix Φ, that is, ϕ^{gh}, can be written as

$$\phi^{gh} = (T^g)D(\hat{\overset{*}{C}}{}^h) = \frac{(T^g)D(\hat{\overset{*}{C}}{}^h)(T^h)D\tilde{Y}}{(T^h)D\tilde{Y}} = \frac{\text{step 5}}{\text{step 2}} \qquad (4.10)$$

From which it is possible to interpret the coefficient ϕ^{gh} as the amount of income in the g^{th} region generated by the consumption expenditure from each unit of induced income in the h^{th} region and to interpret Φ as a matrix of 'direct inter-regional income' coefficients.

The matrix Φ represents only the direct effect on income in region g per unit of additional income in region h. The effect of region h's income on region g does not stop here. There will also be indirect effects as the changed income in region h effects the income of other regions which in turn effect the income of region g. The total direct plus indirect effect of income changes in one region on income changes in another region can, therefore, be represented as[3/]

$$\bar{\Psi} = (I-\Phi)^{-1} \tag{4.11}$$

The coefficients $\bar{\Psi}^{gh}$ show how much income is induced directly and indirectly in region g by commodity-specific consumption expenditures per unit of income in region h.

Thus, the matrix $\bar{\Psi}$ is to matrix Φ of direct interregional income coefficients what the matrix B is to matrix Θ of direct interregional production coefficients.

3/ Alternately, $\bar{\Psi}$ can be written as the sum of a converging series:

$$\bar{\Psi} = \sum_{\alpha=0}^{\infty} \Phi^{\alpha} \tag{4.11a}$$

$$= I + \Phi + \Phi^{1} + \Phi^{2} + \ldots \tag{4.11b}$$

This formulation is used in Annex D.3 to translate Miyazawa's subjoined inverse into submatrix D_{11} of the MRIO partitioned matrix solution.

Returning to the general case where $\Lambda > 0$[4/] the matrix $\overline{\Psi}$ is post-multiplied by the multiplier matrix $(I-\Lambda\overline{\Psi})$. Setting

$$\overset{*}{\Lambda} = \Lambda\Psi \tag{4.12}$$

the coefficients of the matrix $\overset{*}{\Lambda}$ can be written as

$$\overset{*}{\lambda}{}^{gh} = \lambda^{gr}\psi^{rh} \tag{4.13}$$

ψ^{rh} has just been defined and λ^{gr} was defined in equation (3.13). Hence $\overset{*}{\lambda}{}^{gh}$ can be interpreted as the amount of income earned in region g out of each unit of induced income in region h as a result of direct 'intra-household transactions'. Then the coefficients of the matrix

$$\overline{\lambda} = (I-\overset{*}{\lambda})^{-1} \tag{4.14}$$

i.e. the coefficients $\overline{\lambda}{}^{gh}$ represent direct plus indirect wage and salary income earned in region g per unit of induced income in region h via direct intra-household transactions.

Finally the coefficients of the matrix

$$\Psi = \overline{\Psi}\overline{\Lambda} \tag{4.15}$$

can be written as

$$\psi^{gh} = \overline{\psi}{}^{gr}\overline{\lambda}{}^{rh} \tag{4.16}$$

to show how much income is induced directly and indirectly in region g by consumption expenditures per unit of income in region h as modified

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4/ That is, when $\hat{\overset{*}{Z}} > 0$. It should be noted at this point that the coefficients of the matrix $\overset{*}{Z}$ are often not estimated as carefully as the other coefficients in the model. The reason for this is that the coefficients of the IVth (or southeast) quadrant of I-O models do not enter into the solution of gross outputs in the open version of the model. Hence, it is possible to treat them as balancing residual elements. However, the potential inclusion of the coefficients $\overset{*}{Z}$ (which are obtained from IV quadrant data) in the solution of gross outputs in the augmented version of the model suggests the need for more careful estimation of the elements of the IVth quadrant than might at first seem warranted by their size. They represent between 2-10% of the column sums of the consumption variable. However, repeated use of ψ in the subcomponents of the partitioned matrix solution suggests that errors will be compounded through multiplication.

to take into account the effect of direct wage and salary payments per unit of total consumption in region h. In this case

$$\Psi > \overline{\Psi}, \text{ because } \Lambda > 1 \tag{4.17}$$

The full MRIO form of the direct plus indirect interregional income matrix is not Ψ but $\tilde{\Psi} = \Psi C_k$, if $C_k > I$. Since C_k is a matrix of interregional flows of factor payments (for example, workers remittances, interest on investments, etc.), its inclusion will not affect the interpretation of $\tilde{\Psi}$. The function of the matrix is to ensure that the regions involved in the interregional income multiplier $\tilde{\Psi}$ are consistent with the regions of the vectors or matrices by which $\tilde{\Psi}$ is pre- or post-multiplied.

<u>Augmented model output multipliers</u>

In equation (3.20) the gross output vector is shown to be a linear function of the two exogenously determined vectors $\tilde{Y}$ and ${}^{w}\tilde{Y}$:

$$X = D_{11}\tilde{Y} + D_{12}({}^{w}\tilde{Y})$$

The component multiplier matrix D_{11} is the sum of two separate propagation processes

$$D_{11} = D + \Xi \tag{4.18}$$

where $$\Xi = D\hat{\overset{*}{C}}\,\tilde{\Psi}\,\hat{\overset{*}{W}}D$$

In this formulation D is the output multiplier resulting from the standard Leontief-type propagation process which excludes feedback from the income formation process, and Ξ the output multiplier resulting from induced consumption consequent upon incorporating the income formation process into the augmented MRIO model. The component $D\hat{\overset{*}{C}}$ of Ξ represents the output induced by commodity-specific consumption expenditures per unit of induced regional income, and the component $\hat{\overset{*}{W}}D$ is the income generated per unit of output. These two components are linked by the direct plus indirect interregional income coefficients of matrix $\tilde{\Psi}$. Thus, Ξ represents the induced output multiplier.

In the open MRIO model the vector of exogenously determined wage and salary income ${}^{w}\tilde{Y}$ does not enter into the solution of gross outputs. However, in the augmented model there are two possibilities; (1) if ${}^{w}\tilde{Y} = 0$, then Ξ represents total induced output; (2) if ${}^{w}\tilde{Y} > 0$, then the exogenously determined income ${}^{w}\tilde{Y}$ will also induce a certain amount of output equivalent to

$$\tilde{\Xi} = D\hat{\overset{*}{C}}\tilde{\Psi} \tag{4.19}$$

Thus, total induced output will be

$$\overline{X} = \Xi\,\tilde{Y} + \tilde{\Xi}\,({}^{w}\tilde{Y}) \tag{4.20}$$

By incorporating the vector of exogenously determined wage and salary income ${}^{w}Y$ into the solution of gross outputs it becomes possible to estimate the production consequences of changes in the level and/or regional composition[5/] of government payments to its employees. This type of analysis cannot be done in the open model.

The matrices D and Ξ are of the same order. Both have the dimensions (mn x mn). Hence, a set of submatrix and scalar multipliers can be obtained from Ξ which will be similar in type and number to that obtained from D in Chapter 2. Therefore, the multiplier matrix D, or its components, can be used to determine the direct plus indirect effects of all or some elements of the multiplicand vector $\tilde{Y}$, while the multiplier matrix Ξ, or its components, can be used to determine the induced

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5/ This could include changes in state and local government payrolls, or the distribution of Federal payrolls (for example, consequent upon policy changes regarding regional centralization or decentralization of Federal agencies).

effects of the same elements of the multiplicand $\tilde{Y}$. The combination of the two results will give the direct plus indirect plus induced impacts consistent with a change in the multiplicand vector $\tilde{Y}$, or a subset of its elements.

The matrix $\tilde{\tilde{\Xi}}$, however, is of dimension (mn x n) because the exogenously determined incomes of the multiplican vector ${}^{w}\tilde{Y}$ are only region-specific with no industry dimension. Hence, submatrix multipliers of the type D^{gh} and D_i^h can be obtained from $\tilde{\tilde{\Xi}}$, but not the submatrix multipliers of the type D_{ij} and D_j^g. The variety of scalar multipliers that can be obtained from $\tilde{\tilde{\Xi}}$ are also reduced by half in that the only types of scalar multiplicands that can be obtained from ${}^{w}Y$ are types III and IV, i.e., those which reflect 'regional demand' and 'total demand' respectively, Types I and II scalar multiplicands, reflecting 'final-demand components' and 'industrial demand' respectively cannot be obtained from the (n x 1) vector ${}^{w}\tilde{Y}$. For types III and IV multiplicands, however, the full complement of types of impacts (cases a-d) can be derived from the multiplier matrix $\tilde{\tilde{\Xi}}$.

If an analyst is interested in estimating the impact on a single element, or a subset of elements, of the gross output vector it is then necessary to use the criteria 'type of impact' to ensure that the results obtained from using components of D, Ξ and $\tilde{\tilde{\Xi}}$ with the appropriate elements of the vectors $\tilde{Y}$ and ${}^{w}\tilde{Y}$ correctly reflect the direct, indirect and induced impacts for the desired components of the output vector X. For example, the 'detailed' impact of the 'final-demand-components' consists of three parts:

(1) $$x_i^g = d_{ij}^{gh}\tilde{y}_j^h \tag{4.21}$$

for the direct and indirect impact of non-consumption 'final-demand-components'

(2) $$x_i^g = \xi_{ij}^{gh} \tilde{y}_j^h \qquad (4.22)$$

for the induced impact of non-consumption final demand components, and

(3) $$x_i^g = \tilde{\xi}_i^{gh} \, {}^w\tilde{y}^h \qquad (4.23)$$

for the induced impact of exogenously determined income components.

Combining all three results will produce a 'detailed' impact including direct, indirect and induced effects, for each element of the exogenously determined vectors $\tilde{Y}$ and ${}^w\tilde{Y}$:

(a) $$x_i^g = d_{ij}^{gh} y_j^h + \xi_{ij}^{gh} y_j^h + \tilde{\xi}_i^{gh} \, {}^w y^h \qquad (4.24)$$

Alternately,

(b) $$x_i^o = \sum_{g=1}^{n} d_{ij}^{gh} y_j^h + \sum_{g=1}^{n} \xi_{ij}^{gh} y_j^h + \sum_{g=1}^{n} \tilde{\xi}_i^{gh} ({}^w y^h) \qquad (4.25)$$

for industry-specific impacts

(c) $$x_o^g = \sum_{i=1}^{m} d_{ij}^{gh} y_j^h + \sum_{i=1}^{m} \xi_{ij}^{gh} y_j^h + \sum_{i=1}^{m} \tilde{\xi}_i^{gh} ({}^w y^h) \qquad (4.26)$$

for region-specific impacts

(d) $$x_o^o = \sum_{g=1}^{n} \sum_{i=1}^{m} d_{ij}^{gh} y_j^h + \sum_{g=1}^{n} \sum_{i=1}^{m} \xi_{ij}^{gh} y^h{}_j + \sum_{g=1}^{n} \sum_{i=1}^{m} \tilde{\xi}_i^{gh} ({}^w y^h)$$

for total impacts. (4.27)

Augmented Model Income and Employment Multipliers

In equation (3.21) the total wage and salary income multiplier is shown to be a linear function of the two exogenously determined vectors Y and ${}^wY$, that is

$$W = D_{21}\tilde{Y} + D_{22}({}^{w}\tilde{Y}) \tag{4.28a}$$

$$= \Omega\tilde{Y} + \tilde{\Psi}({}^{w}\tilde{Y}) \tag{4.28b}$$

where $\Omega = \tilde{\Psi}\overset{*}{W}D$ from (3.24). Thus,

$$W = \tilde{\Psi}\hat{\overset{*}{W}}D\tilde{Y} + \tilde{\Psi}({}^{w}\tilde{Y}) \tag{4.29}$$

The income-multiplier matrix for non-consumption final demand is the (n x nm) matrix Ω. Unlike the open model solution the exogenously determined vector of wage and salary income, ${}^{w}\tilde{Y}$, also has an income-multiplier matrix $\tilde{\Psi}$. Factoring out the common expression equation (4.29) can be rewritten as

$$W = \tilde{\Psi}(\hat{\overset{*}{W}}D\tilde{Y} + {}^{w}\tilde{Y}) \tag{4.30}$$

Whose coefficients can be expressed as

$$w^g = \tilde{\psi}^{gh}\left(\sum_{r=1}^{n}\sum_{i=1}^{m}\sum_{j=1}^{m} \hat{\overset{*}{w}}{}^{h}_{i} d^{hr}_{ij} \tilde{y}^{r}_{j} + {}^{w}\tilde{y}^{h}\right) \tag{4.31}$$

where w^g, ψ^{gh} and all the other elements are scalars. These coefficients can also be written as

$$w^g = \tilde{\psi}^{gh}\bar{w}^h \tag{4.32}$$

where $\bar{w}^h$ is a scalar representing the sum of the exogenously determined income for region h, i.e. ${}^{w}\tilde{y}^{h}$, plus the income induced in region h by the non-consumption final demand expenditures in all regions, or

$\sum_{r=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} \overset{*}{w}{}_i^h d_{ij}^{hr} y_j^r$. The complete set of equations of the type

represented by equation (4.32) can be written in matrix notation as

$$W = \tilde{\Psi} \overline{W} \qquad (4.33)$$

In this formulation, in contrast to that in equation (4.29), the income induced by non-consumption final demand expenditures, $\hat{\overset{*}{W}} D \tilde{Y}$, is a part of the multiplicand rather than the multiplier. Thus, equations (4.33) and (4.29) are analogous to the simple and compound macroeconomic multipliers respectively in Annex A.2. The utility of equation (4.33) is that it demonstrates in another way the correctness of our previous interpretation of the matrix $\tilde{\Psi}$ as an 'interregional income-multiplier matrix'.

From equation (4.29) it is clear that if an analyst is only interested in determining the differential regional income impacts incorporating induced effects (as a consequence of using a smaller base of autonomous injections) it is not necessary to invert the full [(mn + n) x (mn + n)] matrix ${}^{a}B$ of the augmented MRIO model, as it is when using the standard MRIO procedures. Instead, it is sufficient to obtain the matrix by inverting the much smaller (n x n) matrix $\overline{\Psi} = (I - \Phi)^{-1}$. This represents a considerable computational savings over the standard procedure. In the currently operational MRIO model with 51 regions and 79 industries this partitioned matrix solution results in reducing the size of the matrix to be inverted from (4080 x 4080) to (51 x 51) if the model were to be implemented at its full level of disaggregation.

It is important to note that in the standard solution, as well as in the partitioned matrix solution described in Chapter 3, only region-specific income impacts can be determined within the augmented model solution. The reason for this can be more clearly demonstrated in the context of the partitioned matrix solution where the dimension of the (n x 1) vector of wage and salary income is constrained by the dimension of the (n x n) matrix $\tilde{\Psi}$ which is in turn constrained by the dimension of the (n x n) matrix C_k of interregional flows of factor payments. The matrix C_k enters into the solution of $\tilde{\Psi}$ directly, since $\tilde{\Psi} = \Psi C_k$, and indirectly via the matrix $\Phi = C_k\hat{\overset{*}{W}}D\hat{\overset{*}{C}}$. The order of the matrix C_k is determined by the number of regions in the model.

Therefore, unlike the open model solution for wage and salary income in equation (2.46), i.e.

$$W = \hat{\overset{*}{W}}DY + {}^{w}Y \qquad (4.34)$$

where W, Y and ${}^{w}Y$ are column vectors of dimension (mn x 1) and the diagonal matrix $\hat{\overset{*}{W}}$ is of the same dimension (mn x mn) as the matrix D, in the augmented model solution in (4.29)

$$W = \tilde{\Psi}\hat{\overset{*}{W}}DY + \tilde{\Psi}({}^{w}Y)$$

where the (n x mn) block-diagonal matrix $\hat{\overset{*}{W}}$ is not of the same dimension as the matrix D. Hence, in the open model solution the coefficients of the (mn x 1) vector W can be represented as

$$w_i^g = \left(\sum_{h=1}^{n}\sum_{j=1}^{m} \overset{*}{w}{}_i^g d_{ij}^{gh} y_j^h\right) + {}^{w}y_i^g \qquad (4.35)$$

where the income-multiplier coefficient (see Annex B.4) is

$$ {}^{w}M_{ij}^{gh} = \overset{*}{w}{}_{i}^{g} d_{ij}^{gh} \qquad (4.36) $$

that is, a detailed coefficient representing producing and purchasing industries as well as supplying and receiving regions. In contrast, the coefficients of the income-multiplier matrix Ω, using the block-diagonal matrix $\hat{\overset{*}{W}}$, has two components. The first is

$$ {}^{w}M_{j}^{gh} = \sum_{i=1}^{m} \overset{*}{w}{}_{i}^{g} d_{ij}^{gh} \qquad (4.37) $$

and the second is $\tilde{\psi}^{gh}$ (or $\tilde{\psi}^{gh}$, if $C_k = I$). The coefficients of Ω , which are a product of the two components, can be represented as

$$ {}^{w}\omega_{j}^{gh} = \sum_{r=1}^{n} \tilde{\psi}^{gr} ({}^{w}M_{ij}^{rh}) \qquad (4.38) $$

Thus, unlike the procedure described on p. for the open model, it is not possible to use the components of each row vector Ω^{g} in combination with the elements $\tilde{y}^{h}$ of the final demand vector to obtain 'detailed,' or 'industry-specific' regional income impacts. To implement the same procedure described on p , i.e. to obtain individual industry specific regional income coefficients it is necessary to use the vector of the column sums of the matrix Ω . In this case, however, it is possible only to determine the income earned in a region g and industry i as a consequence of a change in the non-consumption final demand for the output of that industry in that region. In other words

$$w_j^h = \sum_{g=1}^{n} \omega_j^{gh} \tilde{y}_j^h \qquad (4.39)$$

where $\sum_{g=1}^{n} \omega_j^{gh}$ is the scalar coefficient representing the column sum of the matrix Ω. Another procedure is required to obtain a matrix Ω of dimension (mn x mn) whose coefficients ω_{ij}^{gh} would show the impact on income in industry i and region g that is consistent with a changed non-consumption final demand for another commodity j in another region h.

Thus, all detailed income impacts (i.e. region and industry-specific income impacts) can be determined simultaneously using the partitioned matrix solution, but only at the cost of losing some of the computational advantages of the procedure developed in Chapter 4. This alternate procedure involves the following adjustments to the structure of the matrices used in the partitioned matrix solution.

The dimension of the matrix $\tilde{\psi}$ must be increased to (mn x mn). This requires that a correct procedure be found to expand the (n x n) matrix C_k to the same dimension. After a number of trials it was found that C_k must have the same form as the expanded interregional trade matrix C. Each element of $\hat{C}_k$, i.e. $C_{(k)}^{gh}$, must be replaced with an (m x m) diagonal submatrix $\hat{C}_{(k)}^{gh}$, all of whose elements along the principal diagonal are identical to the element of C_k which they replace. This transformation does not change the meaning of the matrix C_k. Of course, if C_k = I there is no problem in expanding its dimension from (n x n) to (nm x nm).

The other adjustments are more straight forward. The (n x nm) block diagonal matrix $\hat{\overset{*}{W}}$ must be replaced by an (nm x nm) diagonal matrix $\hat{\overset{*}{W}}$. Similarly the (nm x n) block-diagonal matrix $\hat{\overset{*}{C}}$ must be replaced by an (nm x nm) diagonal matrix $\hat{\overset{*}{C}}$. With these adjustments the matrix $\Phi = C_k \hat{\overset{*}{W}} D \hat{\overset{*}{C}}$ can be expanded to an (nm x nm) matrix whose coefficients will be the interregionally and interindustrially-specific ϕ_{ij}^{gh}, in contrast to the interregionally-specific coefficients ϕ^{gh} in the previous formulation. An (nm x nm) matrix $\overline{\Psi} = (I-\Phi)^{-1}$ can now be determined. Even though this procedure involves inverting a matrix of the same dimension as D, it is still computationally more efficient than the standard MRIO procedure which would require the inversion of an [(mn + mn) x (mn x mn)] matrix to obtain income-multipliers incorporating induced impacts which are region and industry-specific. (It is well known that the computations involved in inverting a matrix increase with the cube of the matrix's dimension, i.e. a doubling of the dimension from mn to 2mn involves an eightfold increase in computations).

For purposes of dimensional compatibility with the revised matrix $\tilde{\Psi}$, the diagonal matrix $\overset{*}{Z}$ must also be expanded to dimension (nm x nm). This requires that the coefficients $\overset{*}{Z}^{h}$ be replaced by the coefficients $\overset{*}{Z}_{j}^{h}$ all along the principal diagonal of the matrix. With this adjustment, an (nm x nm) matrix $\Lambda = C_k\hat{\overset{*}{Z}}$ can be determined. With Λ and $\overline{\Psi}$ it is then possible to determine the (nm x nm) matrices $\Psi = \tilde{\Psi}(I-\Lambda\tilde{\Psi})^{-1}$ and $\tilde{\Psi} = \Psi C_k$ which are required in order to determine detailed impacts.

The final step involves replacing the (n x 1) column vector ${}^{w}\tilde{Y}$ by the (nm x 1) column vector ${}^{w}\tilde{Y}$ whose elements are region and industry-specific.

The solution of the enlarged income vector remains unchanged in form. It is the same as in equation (4.29), except for the change in the dimension of the component vectors and matrices. Thus, instead of determining an (n x 1) region-specific vector of wage and salary income, it is now possible to determine an (nm x nm) region and industry-specific vector of wage and salary income. This solution is analogous to the open model solution in equation (2.45) except that now induced effects have been incorporated into the multiplier matrix.

With this version of the augmented model income-multiplier it is possible to obtain the same set of submatrix and scalar income-multipliers as can be obtained from the open model (which are presented in Chapter 3 and Annex B.4).

At this point, we can amend our discussion on p. , to note that by replacing the dimensionally revised matrix $\tilde{\Psi}$ into equation (3.19) and (4.19) it is possible to determine an (nm x nm) matrix $\tilde{\Xi}$ to represent an induced output multiplier matrix for the exogenously-determined wage and salary income. Then it is possible to obtain the same types and number of submatrix or scalar multipliers from $\tilde{\Xi}$ as can be obtained from D and Ξ. This replacement is predicated, however, on inverting a matrix $\tilde{\Psi}$ of the same dimension as D, rather than the much smaller matrix $\tilde{\Psi}$ which was assumed to have been used in the discussion on output multipliers earlier in this chapter.

The enlarged version of the matrix $\tilde{\Psi}$, however, is not required in determining the employment multipliers in the augmented model. As is noted in Annexes A.3 and A.4 the direct employment coefficient are estimated from 'employment-production functions' rather than the inter-industry transactions tables from which the direct technical production coefficients are estimated. Hence, the two sets of coefficients are not totally consistent with each other. Unlike the income coefficients, therefore, they cannot be incorporated directly into the augmented matrix. In fact the consumption coefficients of the northeast quadrant of the augmented matrix, which are necessary for 'closing' the model (see Annex A.3), are linked to a measure of income (such as wage and salary income) and not to a measure of employment. Therefore, the employment multiplier incorporating induced effects is not obtained directly from the augmented model as is the income-multiplier. The (mn x 1) region and industry-specific employment vector is obtained from an equation which is analogous to that of the open model,[5/] i.e.

$$E = \hat{\overset{*}{E}} X \qquad (4.40a)$$

or

$$E = \hat{\overset{*}{E}} \left[D(I + \hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D) \quad D\hat{\overset{*}{C}}\tilde{Y} \right] \begin{bmatrix} \tilde{Y} \\ \hline {}^{W}\tilde{Y} \end{bmatrix} \qquad (4.40b)$$

where $\hat{\overset{*}{E}}$ is a (nm x nm) diagonal matrix of direct employment to output coefficients, and the matrices in the expression in brackets are the same as those in the upper row of equation (5.1). Alternately, E can be expressed as:

$$E = \hat{\overset{*}{E}}D\tilde{Y} + \hat{\overset{*}{E}}\Xi\tilde{Y} + \hat{\overset{*}{E}}\tilde{\Xi}({}^{W}\tilde{Y}) \qquad (4.41)$$

5/ In the open MRIO model the calculation is done according to the procedure, now well established in the national employment estimations made by the U.S. Department of Labor, Bureau of Labor Statistics, Factbook for Estimating the Manpower Needs of Federal Programs, Bulletin No. 1832. Washington, D.C.: U.S. Government Printing Office, 1975 where each element in a row of the open model inverse matrix is multiplied by the employment-to-output ratio for the particular industry represented by the row. The same procedure is used at the multiregional level, with the industries now being differentiated by the region, as well as the industry, in which the output is produced.

where the total impact on region and industry-specific employment is the sum of the impacts transmitted by three separate multiplier matrices.

The first multiplier matrix $\hat{\overset{*}{E}}D$ shows the direct plus indirect employment impact of non-consumption final demand. The second multiplier matrix $\hat{\overset{*}{E}}\,\Xi$ shows the employment impact of induced output consequent upon a changed non-consumption final demand. The third multiplier matrix $\hat{\overset{*}{E}}\,\tilde{\Xi}$ shows the employment impact of induced output consequent upon a change in exogenously determined wage and salary income.

As in the case of the augmented model output multiplier matrices, the first and second multiplier matrices above are of the same dimension as D and the third multiplier matrix is of dimension (mn x n). However, if instead of the (n x n) interregional income multiplier matrix $\tilde{\Psi}$ the enlarged (mn x mn) version of that matrix is used then the third multiplier matrix $\hat{\overset{*}{E}}\,\tilde{\Xi}$ would also be of the same dimension as the other two. Then the same types and numbers of submatrix and scalar multipliers can be obtained from each of the three employment-multiplier matrices as were obtained from the open model output multiplier matrix D.

Equation (4.41) is computationally more efficient than equation (4.29), $\tilde{\Psi}$ is an (nm x nm) matrix rather than an (n x n) matrix. As a result, if an analyst is not interested in the structure of the larger interregional multipliter matrix $\tilde{\Psi}$ itself, it will be more cost-effective to adapt equation (4.41) to the determination of industry and region-specific income impacts. This involves pre-multiplying the matrices D, Ξ and $\tilde{\Xi}$ by the diagonal matrix $\overset{*}{W}$ of direct income coefficients and adding the results, i.e.,

$$W = \hat{\overset{*}{W}}DY + \tilde{\overset{*}{W}}\,\Xi\,\tilde{Y} + \hat{\overset{*}{W}}\,\tilde{\Xi}\,({}^{w}\tilde{Y}) \tag{4.42}$$

where W and $\tilde{Y}$ are (nm x 1) column vectors; ${}^{w}\tilde{Y}$ is an (n x 1) column vector; $\hat{\overset{*}{W}}$, D and Ξ are (nm x nm) matrices and $\tilde{\Xi}$ is an (nm x n) matrix.

CHAPTER 5

RELATIONSHIP BETWEEN THE MULTIPLIERS

In Chapter 3 a theoretical and an operational reason was given to justify the closure of the model with respect to consumption. The theoretical reason can be rephrased as follows: the level of total income depends on the regional and industrial composition of production which depends on the location-specific sectoral composition of consumption, which depends in turn on the income generated in each sector and region. Thus, the augmented model relaxes the more rigid assumption that the level of income is independent of the composition of production, or in other words, that disaggregation adds nothing to an analysis in more aggregated terms (see Annex A-3). The operational reason can be restated as follows: if all final demand components cannot be projected independently, it is desirable to use a less inclusive multiplicand and incorporate the induced effects into the multiplier.

What has not been demonstrated, is that for equivalent changes in the composition of exogenous demand, both the open and augmented multiplier matrices of the model will of necessity have to generate identical gross output and income levels. That is incorporating the feedback from income to production via consumption does not involve additional injections into the income formation process. This can be demonstrated formally as follows:

Relationship Between the Open and Augmented Model Multipliers

In the open model the $(n \times 1)$ vector of region-specific wage and salary income W is determined by the formula

$$W = \hat{\overset{*}{W}}X + {}^{w}Y \qquad (5.1)$$

where $\hat{\overset{*}{W}}$ is a $(n \times 1)$ block-diagonal matrix and ${}^{w}Y$ is of the same dimension as W. Substituting the open model solution for X = DY, into the above equation we can write W as a function only of the exogenously determined variables, i.e.

$$W = \hat{\overset{*}{W}}DY + {}^{w}Y \qquad (5.2)$$

where the $(n \times mn)$ matrix $\hat{\overset{*}{W}}D$ is the income multiplier matrix. The augmented model solution for income from equation (4.29) is

$$W = \tilde{\Psi}(\hat{\overset{*}{W}}D\tilde{Y} + {}^{w}\tilde{Y}) \qquad (5.3)$$

In form the augmented model expression $(\hat{\overset{*}{W}}D\tilde{Y} + {}^{w}\tilde{Y})$ in equation (5.3) is analogous to the expression on the right hand side of the open model equation (5.2). However, the coefficients of the open model exogenous vectors Y and ${}^{w}Y$ are by construction, larger than the corresponding coefficients of the augmented model exogenous vectors $\tilde{Y}$ and ${}^{w}\tilde{Y}$. These two equations will, however, give the same result. Let us assume that the interregional multiplier $\tilde{\Psi}$ ensures that the smaller base of autonomous injections $\tilde{Y}$ and ${}^{w}\tilde{Y}$ will result in the same vector of regional income W as the open model solution using

Y and ${}^{W}Y$. In other words, assuming for the moment, that ${}^{W}Y$ (and, therefore, ${}^{W}\tilde{Y}$) is zero, then the augmented income multiplier matrix $\tilde{\Psi}\hat{\overset{*}{W}}D$ is larger than the open multiplier matrix $\overset{*}{W}D$, and the induced effect incorporated in the former is strictly a consequence of using the smaller multiplicand $\tilde{Y}$ instead of Y.

That this assumption about the interregional income multiplier matrix W is correct can now be demonstrated by analyzing the augmented model solution of the gross output vector. In equation (3.20), the solution of the gross output vector in the augmented model is given as

$$X = D\ (I + \hat{\overset{*}{C}}\ \tilde{\Psi}\ \hat{\overset{*}{W}}D)\ Y + D\hat{\overset{*}{C}}\ \tilde{\Psi}({}^{W}\tilde{Y}) \tag{5.4}$$

Expanding the above equation we get

$$X = D\tilde{Y} + D\hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D\tilde{Y} + D\hat{\overset{*}{C}}\ \tilde{\Psi}({}^{W}\tilde{Y}) \tag{5.5}$$

rearranging, this equation can be written as

$$X = D\tilde{Y} + D\hat{\overset{*}{C}}\left[\tilde{\Psi}\ (\hat{\overset{*}{W}}D\tilde{Y} + {}^{W}\tilde{Y})\right] \tag{5.6}$$

from equation (5.3) it is clear that the expression in brackets is nothing other than the vector of regional incomes W, hence

$$X = D\tilde{Y} + D\hat{\overset{*}{C}}W \tag{5.7}$$

$$= D(\tilde{Y} + \hat{\overset{*}{C}}W) \tag{5.8}$$

The matrix $\hat{\overset{*}{C}}W$ is by definition the matrix of consumption demand $\hat{C}$ (see Annex C.2). Hence, the vector of consumption demand $\bar{C} = e\hat{C}$ and equation (5.8) can be written as

$$X = D(\tilde{Y} + \bar{C}) \tag{5.9}$$

The expression in parenthesis is the same as total final demand Y (see Annex C.2), therefore

$$X = DY$$

which is the desired open model solution for gross outputs. Thus, ignoring $^{w}\tilde{Y}$ for the moment[1/], the augmented output multiplier matrix $D(I + \hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D)$ is larger than the open model output multiplier matrix D by the amount $D\hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D$ to offset the reduction in the value of the multiplicand from Y to $\tilde{Y}$. This remains unchanged even if ^{w}Y is assumed positive as is evident in Annex E using actual 1963 data.

In other words, since $\overline{C}$ is a component of the multiplican Y in the open model, the augmented model multiplicand must always be

$$\tilde{Y} = Y - \overline{C} \qquad (5.10)$$

Hence, the notion of equivalent changes in the composition of exogenous demand, mentioned at the beginning of the chapter, refers to the fact that the exogenous demand vectors of the two versions of the model must always fulfill the condition expressed in (5.10). Thus, the total level of national income w_o will be different depending on the composition of demand in the vectors Y and $\tilde{Y}$, but the same solution will be obtained whether the open or augmented model is used.

A subsidiary point referred to in Chapter 4, which could not be demonstrated at the time, was that if the intra-household consumption

1/ Even though a positive vector ^{w}Y does not enter into the solution of gross outputs in the open version of the model, it must enter into the solution of gross outputs in the augmented version, because consumption induced output $D\overline{C}$ (from 5.9) is based on the assumption that consumption demand is a function of income W. And this variable can, even in the open model, include an exogenously determined component, namely ^{w}Y.

coefficients $\overset{*}{z}^h$ are assumed to be zero then the commodity-specific consumption coefficients $\overset{*}{c}_i^{gh}$ must be larger than if the coefficients $\overset{*}{z}^h$ are assumed to be positive. This can now be demonstrated as follows:

Assuming ${}^w\tilde{Y} = 0$, then based on (2.71) and (4.18) we can write

$$X = D\tilde{Y} + \Xi\tilde{Y} \tag{5.11}$$

where $\Xi = D\hat{\overset{*}{C}}\tilde{\Psi}\hat{\overset{*}{W}}D$

It has already been argued that the gross output vector X must be the same irrespective of the version of the MRIO model used for the estimation. This means that the matrix in (5.11) must be the same whether the matrix $\hat{\overset{*}{Z}}$ is a positive diagonal matrix or a null matrix. The value of $\hat{\overset{*}{Z}}$ will not effect the values of the open model matrices D or $\hat{\overset{*}{W}}$. However, if $\hat{\overset{*}{Z}} = 0$ then the matrix $\tilde{\Psi}(=\Psi C_k)$ reduces to $\tilde{\overline{\Psi}}$ $(=\overline{\Psi}C_k)$, where $\tilde{\Psi} > \tilde{\overline{\Psi}}$, since $\Psi > \overline{\Psi}$ according to (4.17). Therefore, for the value of Ξ to remain unchanged the coefficients of $\hat{\overset{*}{C}}'$, with $\hat{\overset{*}{Z}}=0$, must be larger than the coefficients of $\hat{\overset{*}{C}}$, with $\hat{\overset{*}{Z}} > 0$, i.e.,

$$\hat{\overset{*}{C}}'\tilde{\overline{\Psi}} = \hat{\overset{*}{C}}\tilde{\Psi} \tag{5.12}$$

$$\text{or} \quad \hat{\overset{*}{C}}' = \hat{\overset{*}{C}}\tilde{\Psi}\tilde{\overline{\Psi}}^{-1} \tag{5.13}$$

Thus, $\hat{\overset{*}{C}}' > \hat{\overset{*}{C}}$ by the amount $\tilde{\Psi}\tilde{\overline{\Psi}}^{-1}$. Hence, if $\hat{\overset{*}{Z}} > o$ in an empirically estimated model, and it is desired to treat $\hat{\overset{*}{Z}} = o$, then it is necessary to distribute the value of $\overset{*}{z}^h$ amongst the commodity-specific consumption coefficients $\overset{*}{c}_i^{gh}$ rather than to incorporate them in the coefficients $\overset{*}{u}^h$ of the secondary balance equation (C.2.24). Otherwise, X will be under-estimated by the amount $\tilde{\Psi}\tilde{\overline{\Psi}}^{-1}$.

Relationship Between MRIO and Keynesian Multipliers

As noted earlier the augmented MRIO model incorporates a type of income formation process. It has not been possible in the past with the standard formulation of the augmented MRIO model solution to demonstrate how this process differs from the more familiar Keynesian process and how the model could be respecified to incorporate the Keynesian process.

A simplified illustration of the difference between the aggregate Keynesian and Leontief multipliers in a closed economy, will be presented next to clarify the discussion. For convenience, traditional input-output notation will be used in lieu of the notation commonly used in national income models.

In a net accounting framework, we can write

$$v_o \equiv y_o \tag{5.14}$$

where v_o is gross national income and y_o is gross national product. In a closed economy v_o represents value added in a national income (N-I) model and y_o represents aggregate demand. Assuming no government sector

$$y_o = c_o + \tilde{y}_o \tag{5.15}$$

where c_o is the rate of consumption and $\tilde{y}_o$ the rate of gross investment. Setting the rate of consumption, c_o, as a linear function of income v_o we can write

$$c_o = \overset{*}{c}\, v_o \tag{5.16}$$

where $\overset{*}{c}$ is the marginal propensity to consume. Substituting (5.16) into (5.15) and then (5.15) into (5.14), we get the familiar Keynesian equations in I-0 notation

$$v_o = \overset{*}{c}v_o + \tilde{y} \tag{5.17}$$

$$= \frac{1}{1-\overset{*}{c}}\,\tilde{y}_o \tag{5.18}$$

where $\frac{1}{1-\overset{*}{c}}$ is the aggregate Keynesian multiplier. Using the same notation, but within a gross accounting framework the Leontief model can be represented as

$$x_o = r_o + y_o \tag{5.19}$$

where x_o is a scalar denoting total gross output, and r_o a scalar denoting total intermediate demand. In a closed economy, aggregate demand y_o is also final demand. Setting intermediate demand r_o as a linear function of output x_o we can write

$$r_o = ax_o \tag{5.20}$$

where a is a technical parameter.

Substituting (5.20) into (5.19), the aggregate Leontief multiplier can be written as

$$x_o = ax_o + y_o \tag{5.21}$$

$$= \frac{1}{1-a}\,y_o \tag{5.22}$$

where $\frac{1}{1-a}$ is the scalar representation of the inverse coefficient matrix $(I-A)^{-1}$.

From equations (5.18) and (5.22) it is clear that the Keynesian multiplier process originates in a net accounting framework and that the Leontief multiplier process is a consequence of expanding to a gross accounting framework.

In the Leontief model income is a linear function of output

$$v_o = \overset{*}{v} x_o \tag{5.23}$$

where value added v_o is also total primary supply in a closed economy and $\overset{*}{v}$ is the income-to-output ratio.

Substituting (5.22) into (5.23) we can write

$$v_o = \frac{\overset{*}{v}}{1-a}\, y_o \tag{5.24}$$

$$\text{or} \quad v_o = y_o \tag{5.24b}$$

because $\frac{\overset{*}{v}}{1-a} = 1$, since by definition in a closed economy $\overset{*}{v} = 1\text{-}a$.

On the basis of identity (5.14) and equation (5.18), it is also possible to write (5.22) as

$$x_o = \left(\frac{1}{1-a}\right)\left(\frac{1}{1-\overset{*}{c}}\right)\tilde{y}_o \tag{5.25}$$

The multiplier in this equation resembles a compound multiplier. Substituting (5.25) into (5.23), we get

$$v_o = \overset{*}{v}\left(\frac{1}{1-a}\right)\left(\frac{1}{1-c}\right)\tilde{y}_o \tag{5.26}$$

$$= \frac{1}{1-c}\,\tilde{y}_o \tag{5.27}$$

In other words, the open Leontief model in equation (5.22) does not contain an income formation process, whereas the income formation process contained in equation (5.25) is a consequence of incorporating the Keynesian multiplier process in an augmented model of the type

$$x_o = r_o + c_o + \tilde{y}_o \tag{5.28}$$

and

$$v_o = \overset{*}{v} x_o \tag{5.29}$$

which after introducing the functional relations for r_o and c_o, can be written in matrix form (but with all elements as scalars) as

$$\begin{bmatrix} x_o \\ \hline v_o \end{bmatrix} = \left[\begin{array}{c|c} a & \overset{*}{c} \\ \hline \overset{*}{v} & o \end{array}\right] \begin{bmatrix} x_o \\ \hline v_o \end{bmatrix} + \begin{bmatrix} \tilde{y}_o \\ \hline o \end{bmatrix} \tag{5.30}$$

This is in fact the form of the augmented input-output model in a closed economy (i.e. without any secondary balance equation). Thus, replacing all the scalars except v_o by vectors and matrices equation (5.30) can be written as

$$\begin{bmatrix} X \\ \hline v_o \end{bmatrix} = \left[\begin{array}{c|c} A & \overset{*}{C}_o \\ \hline \overset{*}{V}_o & 0 \end{array}\right] \begin{bmatrix} X \\ \hline 0 \end{bmatrix} + \begin{bmatrix} \tilde{Y} \\ \hline 0 \end{bmatrix} \tag{5.31}$$

where X, $\tilde{Y}$ and $\overset{*}{C}_o$ are (mxl) column vectors of gross outputs, non-consumption final demand and marginal consumption coefficients respectively, $\overset{*}{V}_o$ a $(1 \times m)$ vector of income coefficients and A an $(m \times m)$ technical coefficient matrix[2].

2/ In a totally closed economy, the column vectors of the A matrix approximate fixed proportion production functions, whereas in an open economy I-O supply model they do not approximate even this type of production function because imports are not distributed as industry-specific inputs.

As has been pointed out by Miyazawa (1963, p. 95) in a closed economy, despite the disaggregation of the industrial sector the value of total income v_o will not depend on the proportions of final demand unless an income distribution pattern is introduced[3]. That is, treating income as a scalar v_o instead of a vector V, it is possible to write the solution for v_o, in analogy to the MRIO partitioned matrix solution presented in Chapter 3, as

$$v_o = K_o \overset{*}{V}_o B \overset{*}{Y} \qquad (5.32)$$

where B is the (m × m) inverse matrix $(I-A)^{-1}$, and K_o the single economy counterpart of $\tilde{\Psi}$ from equation (4.29). From equation (5.31) it is clear that $\overset{*}{Z}$ and ^{w}Y are zero, which implies that in the open model there is no direct transactions between primary supply and final demand. With $\overset{*}{Z} = 0$, we can write

$$K_o = (I - L_o)^{-1} \qquad (5.33)$$

where L_o is the counterpart of Φ in equation (3.18), i.e.

$$L_o = \overset{*}{V}_o B \overset{*}{C}_o \qquad (5.34)$$

From equation (F.7) in Annex F, it is clear that in a closed economy

$$\overset{*}{V}_o = e\ (I-A) \qquad (5.35)$$

where e is a (1 × m) row vector, all of whose elements are unity.

Hence,

$$\overset{*}{V}_o B = e\ (I-A)(I-A)^{-1} = e \qquad (5.36)$$

3/ This requires that V_o be replaced by a (k × 1) column vector V, the (1 × m) row vector $\overset{*}{V}_o$ by a (k × m) matrix $\overset{*}{V}$, and the (m × 1) column vector $\overset{*}{C}_o$ by the (m × k) matrix $\overset{*}{C}$.

Substituting (5.35) into (5.33) we get

$$L_o = e\overset{*}{C}_o = \overset{*}{c} \qquad (5.37)$$

where $\overset{*}{c}$ is a scalar representing the aggregate marginal propensity to consume.

Hence,

$$K_o = \frac{1}{1 - \overset{*}{c}} \qquad (5.38)$$

is also a scalar.

Substituting (5.35) and (5.38) into (5.32), we can write

$$v_o = \frac{1}{1-\overset{*}{c}}\, e\tilde{Y} \qquad (5.39)$$

$$= \frac{1}{1-\overset{*}{c}}\, \tilde{y}_o \qquad (5.40)$$

Thus, the partitioned matrix I-O solution for total income reduces automatically to the standard aggregate Keynesian model solution in a closed economy with consumption a function of total value added. That is, total income is independent of the composition of final demand.

There are various ways in which, in a closed economy, income can be made to depend on the composition of final demand. One approach is to introduce income distribution into the model, as Miyazawa has done, without creating a secondary balance condition. Another is to introduce a secondary balance condition such as the equality between savings and investment, in the context of different sector-specific savings propensities, as has been done by Chenery and Clark (see Annex A.3).

This second approach is formally equivalent to an open economy assumption, in which imports are introduced as a row-vector of inputs in the I-0 model. A number of derivations are provided in Annex F to demonstrate how the scalar form of the open economy I-0 income multipliers differs from the two standard forms of the aggregate open economy Keynesian multipliers, i.e. in which imports are not a part of the multiplier and in which imports are a function of income. The I-0 income multipliers are bracketed by the two Keynesian multipliers. The difference, however, is due to the fact that in I-0 models imports are a function of gross outputs, which are in turn a function of final demand. Hence, eventhough a functional relation can be established between imports and final demand, this relation will differ from the relation between imports and income (i.e. the marginal propensity to import), because income and final demand are equal only in a closed economy.

The first approach, in which income distribution is introduced into the model, is, however, more germane to showing the relationship of the MRIO income multiplier matrix to the aggregate Keynesian multiplier.

Before this can be done, it is necessary to replace the standard MRIO assumptions with Keynesian assumptions. The current augmented MRIO model is based on a procedure which is very frequently used to augment I-0 models. In this procedure, the "household" sector is treated as a "fictitious industry". This requires that the traditional I-0 assumption that output supplied is equal to output demanded also be used for the household sector. In the MRIO model, in the case where $C_k = I$, this is equivalent to assuming an equality between the row sum and column sum of the household sector. Average consumption coefficients are then derived using this column sum.

These assumptions can be represented as

$$c^{*h}_i = \frac{\bar{c}^h_i}{w^h_o} \qquad (5.41)$$

$$\sum_{i=1}^{m} c^{*h}_i = c^{*h}_o \qquad (5.42)$$

$$(c^{*h}_o + z^{*h}_o + u^{*h}_o)w^h_o = \bar{c}^h_o \qquad (5.43)$$

and

$$w^h_o = \bar{c}^h_o \qquad (5.44)$$

All symbols are defined in Annex C.2. Clearly equations (5.42) and (5.43) are equal only if

$$c^{*h}_o + z^{*h}_o + u^{*h}_o = 1 \qquad (5.45)$$

Assuming a closed economy with no government sector (in which $u^{*h}_o = 0$), region-specific value added v^h_o can be used to replace region-specific wage and salary income w^h_o in equation (5.41). In this case, the equation in (5.44) will be replaced by the inequality

$$v^h_o > \bar{c}^h_o \qquad (5.46)$$

as a result of which the sum of coefficients in (5.45) would be

$$c^{*h}_o + z^{*h}_o < 1 \qquad (5.47)$$

In contrast to (5.45), the relation in (5.47) represents the Keynesian assumption that the region-specific MPC is less than one.

Assuming that both $^{w}\tilde{Y} = 0$ and $\hat{\overset{*}{Z}} = 0$, the MRIO partitioned matrix solution for regional income from equation (4.29) can be written as

$$V = \Psi C_k \hat{\overset{*}{V}} D\tilde{Y} \qquad (5.48)$$

or
$$V = \Psi TBC\tilde{Y} \qquad (5.49)$$

where
$$\Psi = (I - \Phi)^{-1} \qquad (5.50)$$

and
$$\Phi = TB\Gamma \qquad (5.51)$$

where
$$T = C_k \hat{\overset{*}{V}} \qquad (5.52)$$

All the vectors and matrices in the equations (5.48) through (5.52) have the same dimensions as their counterparts in Chapter 3.

The (n x n) matrix Φ does not automatically reduce to a scalar as does L_o in (5.34), because introducing regional differences in income, in the form of the vector V, is equivalent to Miyazawa's introduction of income distribution by income groups in a single economy. However, Φ can be aggregated to obtain the aggregate Keynesian MPC. This requires the transformation

$$e\Phi e^T = \overset{*}{c} \qquad (5.53)$$

where e is a (1 x mn) row-vector of unit elements and e^T is its transpose, i.e. an (mn x 1) column-vector. Only with this adjustment will the interregional income multiplier matrix Ψ in equation (5.50) reduce to the Keynesian multiplier $\left(\frac{1}{1-\overset{*}{c}}.\right)$

If the government sector is introduced, then the appropriate income concept to use in equation (5.41), is neither value added v_o^h, nor wage and salary income w_o^h, but disposable income, which is intermediate between the two.

If the closed economy assumption is dropped, and it is assumed that ${}^{w}\tilde{Y} > o$ and $\hat{\overset{*}{Z}} > o$, then expressing the relation between the MRIO income multiplier and the Keynesian multiplier becomes considerably more complex. It involves,combining the aggregation procedure used in (5.53) with the derivations in Annex F.

However, in summary, it can be argued that the "fictitious industry" approach to augmenting the MRIO model introduces an income-formation process that is quite distinct from the Keynesian multiplier process, and that even when Keynesian assumptions are introduced, the disaggregated MRIO interregional income multiplier does not automatically reduce to the aggregate Keynesian multiplier. That is, the solution of regional incomes is sensitive to the composition of final demand.

Conclusion

The conclusion is presented in two parts. In the first part, the potential of this dissertation's principal contribution, that is, the partitioned matrix solution of the reformulated augmented MRIO model, is outlined. In the second part, the capabilities and limitations of the MRIO multipliers and the contexts in which they can be meaningfully used, are discussed briefly. Both parts contain suggestions for future research.

Part I

In chapter 3, it was shown that the partitioned matrix solution of an augmented matrix can be adapted to the MRIO framework by reformulating the augmented MRIO model. This solution was then used in chapter 4 to rigorously demonstrate the induced effect incorporated in the augmented multiplier. It was also used to demonstrate that the open model and augmented model multipliers are systematically related, hence, it is unnecessary to invert the full augmented MRIO model. In chapter 5, it was shown with the aid of the partitioned matrix solution how Keynesian assumptions can be introduced, in a model closed with respect to consumption, in lieu of the standard I-O "fictitious" industry assumptions. Since there are many other ways, each with a different analytic purpose, in which the open MRIO model can be augmented, the partitioned matrix approach, presented in the previous chapters, can be used to increase the operational and theoretical flexibility of the MRIO model considerably.

As noted in chapter 4, computationally the partitioned matrix solution is much more efficient than the standard MRIO solution of the augmented model. In the partitioned matrix solution Φ is the only matrix that has to be inverted if the matrix $\hat{\overset{*}{Z}}$ of intra-household transactions is a null matrix. If $\hat{\overset{*}{Z}}$ is a positive diagonal matrix then both Φ and $\overline{\Psi}$ have to be inverted. As a result of the disaggregation of income by region, the matrices Φ and $\overline{\Psi}$ are only of dimension (n x n). Even if it is desired that the interregional income multiplier matrix $\tilde{\Psi}$ show industrial, as well as regional detail,[1/] the matrices Φ and $\overline{\Psi}$ will still be only of dimension (mn x mn). Both cases are more cost efficient than the corresponding MRIO solutions, which involve the inversion of (mn + n)x(mn + n) and (2mn x 2mn) matrices respectively.

This cost-efficiency can be realized each time the model is used at different levels of aggregation. The cost savings can also be realized in any of the other cases in which the model is closed with respect to different final-demand components because the interpretation of the matrices Φ and $\tilde{\Psi}$ can be generalized as follows:

$$\Phi = C_p \hat{\overset{*}{P}} D \hat{\overset{*}{F}} \tag{6.1}$$

where

$\hat{\overset{*}{P}}$ represents the (n x nm) diagonal block matrix of the coefficients of that component of primary supply with which the model is augmented

D represents the (nm x nm) open MRIO model multiplier matrix which functions to link the net and gross accounting frameworks

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[1/] If the analyst is not interested in the multiplier matrix $\tilde{\Psi}$ but only in the actual income impacts, then it is more efficient to use equation (4.42).

$\hat{\overset{*}{F}}$ represents the (nm x n) diagonal block matrix of the coefficients of that component of final demand with which the model is augmented

and C_p represents the interregional trade coefficient matrix for that component of primary supply with which the model is augmented

Similarly,

$$\bar{\Lambda} = (I - \Lambda\bar{\Psi})^{-1} \qquad (6.2)$$

where

$$\bar{\Psi} = (I-\Phi)^{-1}$$

and

$$\Lambda = C_p \hat{\overset{*}{Z}}$$

where

$\hat{\overset{*}{Z}}$ now represents the direct transactions between the two components respectively of final demand and primary supply with which the model is augmented

then

$$\tilde{\Psi} = \Psi C_p = \bar{\Psi}\bar{\Lambda}C_p \qquad (6.3)$$

where

$\tilde{\Psi}$ represents the interregional multiplier matrix of direct plus indirect coefficients resulting from the interaction of the internalized primary supply and final demand components via the Leontief-type MRIO inverse.

Thus, instead of closing the model with respect to personal consumption expenditures (PCE) only, it is possible to close it with respect to total consumption (that is, including state and local, as well as Federal, spending), or total aggregate demand, as in

Hansen and Tiebout (1963).[2/] The model can also be closed with respect to specific components of final-demand, for example, Bourque (1969) takes account specifically of the induced effects of state and local spending in a regional I-O model. It is, of course, necessary that these closures not be mechanical, but based on some reasonable theoretical relation between the components which are incorporated into the augmented model. For example, it is less plausible that investment is a fixed proportion of savings than it is that personal consumption expenditures are a fixed proportion of disposable personal incomes.

In addition to the computational savings, and more importantly, the partitioned matrix procedure also provides theoretical flexibility and clarity to the augmented MRIO model. As noted in chapter 5, the traditional I-O logic, where the augmented sectors are treated as "fictitious industries" requires that the row and column sums of the augmented sectors be equal. This results in an additional constraint on the conditions that must be satisfied before a sector can be incorporated into the augmented model. As a result it is necessary to assume that personal consumption expenditures equal disposable personal income, or government spending equals tax receipts, or imports equal exports, etc. (see Annex A.3, p.). This rules out the possibility, for example, of assuming a marginal propensity

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[2/] This latter is equivalent to translating the export base analysis from the regional to the national level. Thus, in this case the impact on the sub-economies of the system will be a function of exogenous developments in external trade only.

to consume (MPC) of less than unity, or budget or trade surpluses or deficits.

In the case of the MRIO model, this assumption of column sum and row sum equality is relaxed, in principle, for individual regional economies but retained for the national economy. In operational work, however, since it is assumed that $C_k = I$, the assumption is not even relaxed for regional economies. As a result, in the case of the MRIO model closed with respect to PCE it has been assumed that the average, as well as marginal propensity to consume out of wage and salary income is unity.

In contrast to the limiting assumptions of the "fictitious industry" approach it has been shown in chapter 5 that with the partitioned matrix approach the augmented model assumption of an average propensity to consume, (APC) = 1 can be replaced with the assumption of MPC < 1, that is, that the row sum of disposable personal incomes can be greater than the column sum of personal consumption expenditures. The resulting marginal consumption coefficients are conceptually different from the average technical coefficients of the processing sectors. Therefore, they should not be incorporated in the inverse of the technical coefficient matrix. This restriction is fulfilled in the partitioned matrix solution, where the effect of the marginal consumption coefficients on gross outputs enters via the inverse of matrix $(I - \Phi)$ and not the inverse of the Leontief matrix $(I - \Theta)$. This formulation has an additional advantage. If all of income were incorporated into the technical coefficient matrix, particularly in a closed economy, or in a model in which imports are not treated as inputs, then the model would be less stable because the stability of the model declines as the sum of the coefficients approaches unity. This problem is avoided in the partitioned matrix solution.[3/]

With the existing data base of the MRIO model it is possible to derive a different set of regional commodity-specific consumption coefficients using value-added as a proxy income measure. With this measure the inequality relation V > C can be substituted for the equality between "wage and salary" income and PCE. In fact, the existing data does not show such an equality between the last two variables and the difference between the two variables has had to be incorporated into the "other primary supply" category (which includes imports). This adjustment is not necessary in the partitioned matrix approach.

Even if a different income measure is used as the base for determining the PCE coefficients with the currently available MRIO data, the coefficients will still be average rather than marginal coefficients since the non-homogenous terms are implicitly assumed to be part of the multiplicand, i.e. the non-consumption final-demand elements. Additional data is required to estimate marginal rather than average consumption coefficients and to determine whether the MPC is greater than unity in some regions. This is a potential direction for future research.

[3/] The convergence conditions for the solution of an augmented model, using the partitioned matrix approach, is laid out in Miyazawa (1963, Part IV).

If such research is undertaken in the future, it will also be desirable that instead of value-added, and wage and salary income as proxy measures for disposable personal income, actual disposable personal income be determined. This is the more appropriate concept to use as the base for marginal PCE coefficients. At the national level some work has already been done in this direction by D. Belzer (1978).

In addition to the theoretical flexibility just discussed the partitioned matrix solution increases the clarity of the solution by distinguishing between the direct and indirect effect via the multiplier matrix D, and the induced effect via the multiplier matrices Ξ, Ω, $\tilde{\Xi}$, and $\tilde{\Psi}$. The latter two are particularly useful for determining the detailed impact on gross outputs, or the region-specific impact on disposable incomes, which are associated with exogenous changes in the regional distribution of government payments to employees ${}^{w}\tilde{Y}$.[4/] These changes can be initiated by local and state governments for a specific region, or by the federal government for all regions. In either case these changes can be independent of changes in commodity-specific government spending.

Another type of analysis, which has not yet been attempted in the MRIO research project, but for which the partitioned matrix solution could prove to be very valuable, is the analysis of income distribution within each region. Formally, the partitioned matrix approach would be very similar to what has already been presented,

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[4/] In the data base of the current MRIO model ${}^{w}\tilde{Y}$ represents only the government payments to employees. Income received from sales to the investment and export sectors are recorded under region and industry-specific payments to wage and salary recipients.

except that the matrices used would be somewhat different to reflect their expansion to k types of income recipients in each region. Then,

$$\left[\begin{array}{c} X \\ \hline W_k \end{array}\right] = \left[\begin{array}{c|c} C & O \\ \hline O & C_k \end{array}\right] \left[\begin{array}{c|c} \hat{A} & \hat{\overset{*}{C}}_k \\ \hline \hat{\overset{*}{W}}_k & \hat{\overset{*}{Z}}_k \end{array}\right] \left[\begin{array}{c} X \\ \hline W_k \end{array}\right] + \left[\begin{array}{c|c} C & O \\ \hline O & C_k \end{array}\right] \left[\begin{array}{c} \tilde{Y} \\ \hline w_{\tilde{Y}} \end{array}\right] \qquad (6.4)$$

where X, Y, C, and A are all as previously defined, and

$W_k = \left[w_k^g \right]$ is an (nk x 1) column vector of total income, where the element w_k^g represents the total income received by the kth income group in region g (see figure C.2b.1)

$w_{\tilde{Y}_k} = \left[w_{\tilde{Y}_k}^{g} \right]$ is also an (nk x 1) column vector of exogenously determined income where the element $w_{\tilde{Y}_k}^{g}$ represents the exogenously determined income for the kth income group in region g (see same figure)

$\hat{\overset{*}{C}}_k = \left[\overset{*}{c}{}^h_{i(k)} \right]$ is an (nm x nk) block diagonal matrix of consumption coefficients. For each region there is an (m x k) submatrix with commodity-specific consumption coefficients for each of the k income groups. The n regional blocks are arranged along the principal diagonal of the matrix $\hat{\overset{*}{C}}_k$ (see figure C.2b.2i)

$\hat{\overset{*}{W}}_k = \left[\overset{*}{w}{}^g_{j(k)} \right]$ is an (nk x nm) block diagonal matrix of income coefficients. For each region there is a (k x m) submatrix with industry-speicific coefficients for each of the k types of income recepients. The n regional blocks are arranged along the principal diagonal of the matrix $\hat{\overset{*}{W}}_k$ (see figure C.2b.2ii)

$\hat{\overset{*}{Z}}_k = \left[\overset{*}{z}{}^h_{(k)} \right]$ is an (nk x nk) block diagonal matrix of direct intra-household transactions. For each region there can be a (k x k) submatrix as in figure C.2b.2ii, or a (k x k) diagonal submatrix showing the direct intra-income group transactions. In eather case, the n regional blocks are arranged along the principal diagonal of the matrix $\hat{\overset{*}{Z}}_k$.

Finally,

$C_k = \left[c^{gh}_{(k)} \right]$ is now an (nk x nk) matrix of trade coefficients, (unless it is assumed that there are no interregional factor payments), which consists of n^2 diagonal submatrices of dimension (k x k) of "trade" coefficients for each income group between each pair of regions. The form of this matrix is identical to that of the interregional commodity trade matrix C (see figure C.2b.3)

The open model matrix $\Theta = C\hat{A}$ can now be augmented by the matrices $\Gamma = C\hat{C}^{*}_{k}$ (figure C.2b.4i), $T = C_k\hat{W}^{*}_{k}$ (figure C.2b.4ii) and $\Lambda = C_k\hat{Z}^{*}_{k}$ (figure C.2b.4iii). The coefficients $\gamma^{gh}_{i(k)}$, $\upsilon^{gh}_{j(k)}$, and $\lambda^{gh}_{(k)}$ are all now specified for each income group (for definitions see chapter 3). Hence, the coefficients of the matrices Φ and $\overline{\Psi}$ will represent respectively the direct, and the direct plus indirect interdependence between the k income groups in all regions. As a result, a "structural" analysis of the interdependencies between income groups via the matrices Φ and $\overline{\Psi}$ becomes, within the limits of the restrictive MRIO assumptions, almost as meaningful as a structural analysis of the interdependencies between industries via the matrices Θ and D.

Two points of caution are pertinent to any future research in the direction of introducing income distribution patterns within regions. First, the current MRIO data base provides a size distribution of income by region within the limitations of the U.S. national income account categories. As is pointed out in Annex A.3 these categories are only partially correlated with functional income distribution categories. Hence, the income concept used bypasses factor accounts, and treats income derived from factor services as direct payments from activities to institutions. This approach is different from the more general appraoch proposed by the U.N. in A System of National Accounts (1968), in which activities pay factors, which in turn transmit income to the institutions of the economy. The inclusion of factor accounts as distinct from institutional accounts is crucial in making the role of employment as a factor explicit, and in allowing the distribution of wealth to emerge as an important determinant of income distribution through the transfer of factor incomes to institutions. Research in this direction has already begun under the auspices of the International Labor Organization (ILO) and the International Bank for Reconstruction and Development (IBRD).[5/] It may be necessary to investigate the possibility of adapting this more general framework for use with the MRIO model, before an attempt is made to introduce a functional distribution of income into the augmented MRIO model.

The second point is related to the problem of increasing the level of disaggregation. As has been pointed out by Vining (1955), a particular classification of objects serves a purpose if operations upon it yield stabilities that assist decision-making. It can not be taken for granted that subclassifying these objects will be more effective in the establishment of the stabilities that are requisite for analytic purposes.

It is not possible to argue whether the assumption of the constancy of coefficients over a given period of time is more or less tenable as the level of disaggregation is increased. This is an empirical problem which will have to await the expected updating of the MRIO

[5/] See Pyatt and Roe (1977), and Pyatt and Round (1978, b and c).

model to a 1972, or even more recent (post oil crisis), data base. Only then will it be possible to determine how stable the MRIO coefficients are over time. Determining the stability of the existing level of disaggregation of the trade, technical, income and consumption coefficients may be a prerequisite of increasing the level of disaggregation of the income and consumption coefficients.

A less ambitious study which does not require determining the stability of the trade and technical coefficients is to analyse the stability only of the income and consumption coefficients, since one set of the latter can be obtained from the regional disaggregation of the 1972 national input-output model's value added and final-demand components (see, for example, Golladay and Haveman, 1977).

In this context another advantage of the partitioned matrix solution is that if the analyst wants to replace the income and consumption coefficients based on more recent data,[6/] it is not necessary to invert the whole augmented MRIO model. It is only necessary to determine a revised matrix $\tilde{\Psi}$. With a

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[6/] The issue of ensuring that the updated data for a portion of the I-O accounts is consistent with the overall framework is at the frontier of current research, see, for example, Lecomber (1971) Bachrach (1972), Malizia and Bond (1974), Mierynk (1975), and Hewings (1977). Other references can also be found in Taylor (1975). Most of these studies are directed at adjusting, updating or projecting the technical coefficient matrices which are too costly to revise totally and frequently. This type of analysis can be extended to matrices augmented with consumption and income coefficients.

revised interregional income multiplier matrix it is then possible to improve on the approximation of impacts on gross outputs and regional incomes even when it is not possible to update the basic MRIO multiplier matrix D. It is also possible to simulate the gross output and regional income impacts consequent upon hypothetical changes in the regional income distribution or consumption expenditure patterns, whether or not the existing MRIO income and consumption data are disaggregated further.

Part II

The MRIO model shares many of the strengths and limitations associated with the use of national and regional input-output techniques. Most of its limitations are a consequence of the simplifying assumptions that are explicitly made to facilitate the implementation and operationalization of a general equilibrium model at an affordable cost. For example, to estimate changes in gross output due to changes in final-demand, it is usually assumed that (1) each industry produces a homogenous commodity at the chosen level of aggregation (that is, there are no joint products), (2) there are constant returns to scale (that is, no externalities or synergistic effects), (3) marginal inputs are equal to average inputs, (4) the inputs required for a particular sub-component of an industry are the same as the average inputs required for the whole industry, (5) the averaged-industry technologies are constant over time, (6) there is no substitution of one input for another (such as labor for capital, or aluminum for steel, whether resulting from changing relative prices, or adoption of new technologies), (7) there are no capacity constraints or input bottlenecks, (8) there are no induced consumption or investment accelerator effects, and (9) there is no financial market impact on production. In addition, in using the multiregional input-output model, a tenth assumption is made: that the average interregional pattern of trade remains constant over time.

A number of these assumptions can be relaxed through different reformulations of the model. For example, supply constraints can be introduced in a linear programming version of the model, and the capability to analyse the path of change can be introduced in a dynamic version of the model. As yet, however, the operational MRIO model has not been reformulated in either the linear programming or dynamic version.

The basic MRIO model, like the basic I-O and Keynesian models, is a comparative-static demand model. As such the multipliers obtained from the model are "timeless" (see Annex A.1) and useful primarily in contexts where there are unemployed or under-employed resources. If the returns to scale are increasing in an industry, then it is likely that output will rise more rapidly than income, and the latter more rapidly than employment. The opposite will be the case if returns to scale are decreasing. This suggests that the MRIO income and employment multipliers are more likely to require adjustments than the MRIO output multipliers.

Despite these limitations, for purposes of planning, the current U.S. MRIO model is rich in interregional and inter-industrial detail. The dissertation has sought to show that the model contains a variety of submatrix and scalar multipliers which are useful in special cases, for example, where the multiplicand is in the form of a subvector or scalar.[7/]

For example, with a submatrix multiplier of the D_j^g type, it is possible to show by how much the output of all industries must change in region g (e.g., region 21 = Michigan) to be consistent with a

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[7/] Clearly the full model must be used if a policy analyst is interested in analyzing differential regional and industry-specific impacts consequent upon a changed final-demand expenditure pattern in each region and industry. However, it is also possible to focus only on a subset of changes and affected variables.

coefficients b^N_{ii}, (on the principal diagonal of B^N), which are invariably equal to or greater than unity, the coefficients of the MRIO submatrix multiplier D_{ii} (which is not located on the principal diagonal of D) are almost all less than unity even for the coefficients along the diagonal of the submatrix multiplier D_{ii}.

It was also shown in chapter 2 that the multiplier submatrices D^{gh}, with g = h (that is the submatrices located along the principal diagonal of the multiplier matrix D) are analogous to equivalently sectored regional I-O model matrices B^R. The main difference is that, unlike the submatrix multiplier D^{gg}, the multiplier matrix B^R does not take into account interregional feedbacks (If instead of a directly estimated regional I-O model, the national I-O model coefficients are used for the same region, then the multiplier matrix $B^{g(N)}$, unlike the submatrix multiplier D^{gg}, will not take into account the greater regional leakages as a result of interregional trade). Hence, the MRIO submatrix multiplier D^{gg} can be more useful than the regional I-O multiplier matrix B^R and can be used for any type of analysis, or planning purpose, for which the latter can be used with one exception.

The submatrix multiplier D^{gg} cannot be used for export base type analysis if it is assumed that the share of region g in the export of one or more commodities is likely to increase. This restriction is a direct consequence of using column coefficient trade matrices in which the share of each supplier is held constant. Thus, any final-demand subvector Y^g or $\tilde{Y}^g$ will have to exclude the non-externally-destined exports from region g to all regions h before it can be used changed subvector of final-demand in all regions for the output of the jth industry (e.g., industry 59 = motor vehicles and equipment).[8/] On the other hand, with a multiplier matrix of the D_{ij} type, it is possible to show by how much the adjustment of industry i (e.g., industry 43 = engines and turbines) in all regions must change to be consistent with a changed subvector of final-demand for the output of the same motor vehicles and equipment industry. Thus, an analyst interested only in the regionalized consequences of changes in the demand for automobiles does not have to use the whole matrix. If there is no change in the regional composition of the demand for automobiles, but only in the scale of the demand, then it is not even necessary to use the submatrix multipliers above, a subset of scalar multipliers would be sufficient. These possibilities of the model, which are explicitly incorporated in its structure, have seldom been used, for analytic or planning purposes, despite the fact that this information cannot be obtained from either national or regional I-O models.

In fact, in chapter 2, it was shown that the submatrix D_{ij} is a regional expansion of the national I-O model's inverse coefficient b^N_{ij}. Hence, the MRIO multiplier matrix D can be used for any type of analysis, or planning purposes, for which an equivalently sectored national I-O multiplier matrix B^N can be used. It is even more useful, because of its regional detail. Therefore, unlike the national model

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[8/] Of course, the MRIO model's final-demand subvector for industry j is not sensitive to shifts in the internal composition of the auto industry, say from gas guzzlers to compacts.

as a multiplicand with the submatrix multiplier D^{gg}.9/

This exception is not very significant because one of the main criticisms levelled against export base theory is that it assumes exports are determined autonomously and that it neglects feedback effects. However, the issue of whether or not interregional feedback effects are significant empirically has not been demonstrated conclusively. This is another potential area for future research. R. Miller (1966 and 1969) has suggested that these feedback effects are not significant for a region as a whole (though they could be significant for specific sectors in a region) and that feedback effects

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9/ The interregional trade flow tables from which the trade coefficient matrices C are derived, include intra-regional, as well as interregional and international trade, hence

$$x_i^g = x_i^{gg} + \sum_{h=2}^{n} x_i^{gh} \qquad \text{(i)}$$

where x_i^g represents the total amount of industry i in the supplying region g, x_i^{gg} represents intra-regional trade and $\sum_{h=2}^{n} x_i^{gh}$ represents all regional outflows to other regions. Regional outflows to other regions which are not destined for external exports cannot be distinguished from those destined for external exports in the trade flow tables. They can, however, be implicitly defined as

$$\sum_{h=2}^{n} \bar{x}_i^{gh} = (x_i^{gg} + \sum_{h=2}^{n} x_i^{gh}) - {}^{E}y_i^g \qquad \text{(ii)}$$

where $\sum_{h=2}^{n} \bar{x}_i^{gh}$ represents internal interregional exports from region g, and ${}^{E}y_i^g$ represents the externally destined exports of region g.

vary systematically with a region's degree of self-sufficiency. More recent studies by D. Greytack (1970 and 1974) dispute these conclusions and suggest that interregional feedbacks for a region as a whole are significant and that there is no systematic relationship between the size of the feedback and the degree of regional self-sufficiency. The MRIO model appears to be well-suited to testing the above hypothèsis. Using equation (i) or (ii) [from footnote 9] for exports, and a similar equation for imports, it is possible to separate interregional exports and imports for each commodity in a region and add the results to the external imports and exports for that region in such a way that column and row sums for each commodity are equal. It should then be possible to compare the consequences of a changed final-demand for that region using the abstracted I-O model for the region and its counterpart embedded in the MRIO model.

Another area for future research, which emerges from the analysis in this dissertation, is to determine how the MRIO model can be used to analyse interregional balance of payments implications associated with the process of income formation.[10/] To do this it may be useful to show how the interregional income multiplier matrix $\tilde{\Psi}$ [11/] is related to the interregional income multipliers of Metzler's (1950) model of

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[10/] Of course, even if the MRIO model should prove capable of determining some useful balance of payments implications via interregional income multipliers, there is still the problem that at the regional level it is more difficult than it is at the national level, to implement policies designed to adjust balance of payments.

[11/] Its component matrix C_k of interregional factor payments is on a current account basis. Therefore, it can be used in balance of payments analysis to determine net factor income for each region, separate from the balance of trade in goods and non-factor services.

income determination in a multi-region system. This model is a strictly Keynesian one in which it is possible for the marginal propensity to spend in a region to be greater than one, as well as less than one, provided the system's overall marginal propensity to spend is less than one (which is required for purposes of the model's stability).[12] It is not difficult theoretically to incorporate the regional variation in the marginal propensity to spend if the partitioned matrix solution is used as in chapter 5. It will be more difficult, however, to incorporate the Keynesian assumption of imports as a function of domestic income (hence a marginal propensity to import) rather than the I-O assumption of imports as function of gross outputs or final-demand. A meaningful solution of this problem will be particularly useful for a MRIO model in which national economies are used as the regions of the model.

It should be clear from the discussion in this dissertation that, despite its limiting assumptions as a comparative-static, linear, fixed coefficient, demand model, there is still much potential in the MRIO model as a working tool, which has not yet been expolited. In this context, the partitioned matrix solution of the augmented model has been presented as a methodological contribution designed to increase the model's flexibility and utility.

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[12] A review of this model is also available in H. Richardson (1969, p. 254-270).

ANNEXES

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The Analysis of Multipliers in Single Economy

The Multiplier is the marginal effect of a change of one economic variable upon another economic variable, of which the first variable is a component; for instance, the marginal effect of a change in primary employment upon total employment, or of a change in investment upon national income. In recent years multipliers of various kinds have been applied as tools of analysis in a number of fields of economic inquiry such as the theory of employment, national income determination, and foreign trade. There have arisen, however, some misunderstanding and confusions. The present paper intends to clear up many of the difficulties involved by surveying briefly the main types of multiplier and their correct interpretation.

(O. Lange, 1943, p. 227)

If we extend [Keynes'] concept of a marginal propensity to consume of less than one, to all industries, we get a matrix multiplier with extraordinary formal analogies with the simple multiplier. To counterbalance the increased complexity, there is a much richer, more complete result. Even though a matrix multiplier should prove too difficult in practice, it yields considerable clarification of principle, for by taking a broader standpoint, it shows more clearly the meaning and limitations of the Keynesian multiplier.

(R. M. Goodwin, 1949, p. 537)

ANNEX A

REVIEW OF SELECTED KEYNESIAN AND INPUT-OUTPUT MULTIPLIERS

One of the objectives of instrumental knowledge is to specify, with the aid of simplified models abstracted from reality, the consequences of changes in internal or external events or actions.

Impact Analysis

In impact analysis policy analysts outline the ex-post, ex-ante or hypothetical consequences of changes in internal (or external) policies and programs on a society and its institutions.

The potentially multi-dimensional concept of social impact has been narrowed considerably in economic analysis. Economic impacts refer to the effects of changes in one set of economic variables on another set of economic variables, to the exclusion of non-economic stimuli and impacts. In addition, in this type of analysis the emphasis is on measurement which implies focusing on the quantitative as distinct from the qualitative, or intangible, aspects of an economic impact.

As a result of frequent use, the term impact has become synonymous with numerous others, such as 'effect', 'response', 'result', and 'incidence'. "As such it cannot be expected to be a precise and well-defined expression. Indeed, in the discipline of economics, ... where the term is used in a manner which suggests a precise and technical meaning, there exists ambiguity in interpretation of the concept" (Jensen, 1976, p. 44).

The technical meaning of impact is derived from analytic economic models. In these the economy is viewed as an interconnected network of abstract (i.e. aggregated) units involved in the production, distribution and consumption of goods and services within a defined geographic boundary. The models are specified in such a way that some variables (known as the endogenous variables) are a function of other variables (known as the exogenous variables). With the aid of these analytic models it is possible to identify and attribute the stimulus for change to specific (single or composite) exogenous variables, and thereby to determine real or hypothetical economic impacts.

Analytic models may be either descriptive or formal. Descriptive models are useful in providing general background information and an overview of the problems, as well as the types, and possibly the directions, of impacts that may be expected to occur from a given economic stimulus. Descriptive studies do not attempt to quantify the magnitude of the effect. In formal models, on the other hand, it is necessary to establish stable quantitative relationships between the stimulating variable and the affected variable. This then enables the analyst to make reasonably accurate, but still only approximate, ex-ante estimates of the likely magnitude of an impact. Formal operational models are often derived from simple theoretical models of the economy, such as the Keynesian and Leontief models. Where funds permit, other types of formal models are constructed, including optimizing and programming models, or the even more ambitious econometric models, which incorporate _ad hoc_ hypotheses to simulate the complexity of actual economies.

All formal models rely on the technical concept of the multiplier to estimate the total impact of a given stimulus on a variable. The stimulus can be associated with a change in the level and/or in the composition of almost any economic variable that can be quantified. Traditionally, at the national level analysts have focused on changes in the rate of investment, consumption, government expenditures, foreign exports, as well as on changes in technology, energy prices, etc. At the regional level they have focused on the regional equivalents of the above, with the difference that regional trade will include, in addition to external trade, internal trade with other regions in the nation. With formal models, regional analysts have been able to also study the effects of changes in industrial complexes, incorporating the expansion or contraction, as well as the introduction or loss of firms and industries.

An increase or decrease in the rate of spending will often have a multiplicative effect in the same direction, as a result of the inter-relationships between activities in the economy. The magnitude of the multiplier is inversely proportional to the diversions out of the economy in successive rounds of transactions. A change in the composition of expenditures will manifest itself as the net result of the multiplicative effect of the changes in the magnitude of its constituent components.

The indicator variables, in terms of which impact is measured by the multipliers, have generally been industry output, household incomes, and employment. Other types of traditional economic impacts have been measured or described, including (a) effects on an economy's trade and payments balances; (b) effects on an economy's growth potential, (c) effects on government revenues and fiscal balances, (d) effects on project benefits and costs, etc. More recently attempts have been made to go beyond the traditional types of economic impacts to investigate non-economic impacts resulting from economic stimuli. For example, attempts have been made to formally measure water requirements (Davis, 1969), environmental effects, (Isard, 1969; Leontief, 1970), quality of life (Hirsch, 1971).

The types of impacts identified and analysed have been direct, indirect, and induced. They have also included total (or gross), as well as net impacts (which incorporate compensated changes).[1/]

1/ That is, instead of measuring the total impact of a particular economic stimulus, the analyst measures only the differences which would remain in the economy after the consequences of hypothetical alternative expenditures triggered by the change, have been assessed. For example, a decline in private investment and employment may in part be offset by an increase in government expenditure: only the net result may be of consequence. On the question of whether to use total or net measures of impact, Stone's comments are relevant:

> there is no uniquely 'correct' concept of 'impact'. The analysis to be adopted in any case - whether to consider compensated or uncompensated change (or some variant of these two extremes)- should depend on what questions one seeks to answer and/or on what assumptions about the likelihood and nature of compensatory expenditures are reasonable. (Stone, 1973, p.6)

The utility of the multiplier concept for assessing a variety of impacts consequent upon different types of stimuli should be apparent from the preceding litany. The literature on multipliers is very extensive and it is not possible, or necessary, to provide an exhaustive review. As a result, many prominent types of multipliers, particularly those derived from price-responsive, or supply-constrained, or dynamic models, are excluded from consideration here. In addition, the many variants of fiscal, foreign trade and input-output multipliers (including export base, and basic service multipliers) are not reviewed. The extent of the review is a function of the clarity of context it provides.

In Annex A.1, the gensis of the concept of the multiplier and its comparative static form is reviewed because the concept has been around for a long time and there has been a drift in its meaning. In Annex A.2 and A.3, a selected set of aggregate Keynesian multipliers, and disaggregated Input-output (or Leontief) multipliers are reviewed respectively. In Annex A.4, multipliers in the context of single sub-national economies are reviewed, and the conceptual basis for a system of economies connected by trade is anticipated in the analysis of macro-economic internal and external multipliers for a single economy within a system of economies.

ANNEX A.1

CONCEPT OF THE MULTIPLIER

The concept of the multiplier was introduced into the mainstream of economic analysis in the period 1931-1936, during the Great Depression.[1/] Prior to that, particularly in business-cycle literature, the importance of the relationship of an increment of investment to an increment of income had been recognized widely, from Tugan-Baranowsky and Wicksell onwards. But these economists and their followers had been content merely to state a tendency. It was only after Keynes introduced the 'marginal propensity to consume' concept, inspired by Kahn's notion of 'leakages', that the necessary tools for analytically more precise thinking on this subject became available.

Leakages and the Multiplier

At the height of the Great Depression, it had been observed that increases in primary employment in construction work (and in the manufacture of materials entering into construction) tended to increase the demand for consumers' goods, and thereby cause an increase in secondary employment.

This immediately posed the question as to why the 'chain reaction' did not go on and on. Why did the employment of a thousand workers in

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1/ The presentation in this section follows in part the exposition by Hansen (1953, ch.4), though many of the ideas presented are now commonplace. Other sources for the articulation of specific points are referred to explicitly.

an economy suffering from acute unemployment not lead to the employment of another thousand, in an unbroken succession until 'full employment' was reached? Professional economists at the time were unable to show precisely what was wrong with this line of reasoning. Some even argued that the employment effects of public work expenditures would be limited entirely to the direct effect of the initial spending itself.

The first meaningful answer to the problem was given by Kahn (1931) in a landmark article, in which he sought to show how much secondary employment would be induced if the government, for example, increased employment in public works. He argued that the reemployment process would peter out because of 'leakages'. He pointed out that a part of the increment of income will be used to pay off debts; a part saved in the form of idle bank deposits; a part invested in securities purchased from others, who, in turn fail, to spend the proceeds; and a part spent on imports, which would not help domestic employment. In addition, a part of the purchases might be supplied from excess stocks of consumers' goods which, if not replaced, would also not contribute to new domestic employment. As a result of these leakages, the chain reation is gradually exhausted.[2/] In the process, however, the primary employment will have indeed induced a certain amount of secondary employment, contrary to the expectation of the

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2/ As is clear in the above-cited list of qualifying conditions, particularly the third one, Kahn was observing and analysing a dynamic temporal process, in contrast to the analytic process implicit in the comparative-static multipliers of the Keynesian model.

critics of the New Deal who saw the full effect limited to the initial spending only; but the amount so induced would be less than was expected by those who had hopes that increased government spending would set in motion a cumulative process which could of itself eventually lead to full employment.

The Marginal Propensity To Consume And The Multiplier

In 1936, Keynes (1936, p. 113-131) introduced the behavioral propositon that "when the real income of the community increases or decreases, its consumption will increase or decrease but not so fast", that is, formally $\Delta Y_w > \Delta C_w$, where ΔY_w and ΔC_w are the changes in real income and consumption, respectively, measured in wage units.[3/] He then defined dC_w/dY_w as the 'marginal propensity to consume' (MPC). With this concept he was able to show that the magnitude of the multiplier effect of investment on income would vary in direct proportion to the MPC. The MPC concept contained implicitly the complementary concept of the marginal propensity to save (MPS), which was in a sense a more precise summary of the net result of Kahn's numerous leakages that could limit the multiplier process.

With the introduction of the MPC concept it became possible to argue that if the MPC is zero, there would be no multiple expansion beyond the initial expenditure, as was implicitly assumed by the

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3/ It is now customary to use price deflators rather than wage-units to obtain real volume (as opposed to nominal volume which incorporates the inflation in the absolute price level).

critics of the New Deal. On the other hand, if the MPC is unity, then the multiplier would be infinite and the cumulative effect of any initial increment of the investment would continue indefinitely, as had been implicitly assumed by some proponents of public works programs.

Even though the idea of leakages was initially based on empirical observation, it can be shown theoretically that for a model to be stable and have a convergent solution it is necessary for the MPC to be less than one, that is, that MPS>0. On the other hand, it appears to be only an empirical observation rather than a theoretical necessity that the MPC is greater than zero. "No proof has yet been presented to show that the multiplier will be greater than one", (Samuelson, 1973, p. 229, emphasis added). Hence, even though in theory there is no reason to exclude the point zero from the range of the MPC, it is empirically most likely that the MPC will be contained in the closed interval $0 < \frac{dC}{dY} < 1$. It is also assumed that under normal circumstances the MPC will be non-negative, even though, as will be evident, in the form of the multiplier in the next section, the multiplier will be positive even if MPC < 0, so long as it is greater than minus one (Samuelson, 1941, p.119).

Despite his general functional formulation, Keynes was very cautious about placing any numerical value upon the multiplier because empirically establishing the magnitude of a multiplier, derived from consumption schedules, is quite complex. It involves determining not only the slope and position of the consumption function but also shifts in the function. Thus, the magnitude of a multiplier will vary between normal and crisis situations, as well as at different stages of development, and most importantly for different groups in the population. Similarly, the period over which a multiplier can be assumed to remain stable will vary with time and place. This is why multipliers, like other empirically estimated constants, should be used with caution, that is, only if there are strong grounds for assuming that the context in which they are used is similar to that in which they were estimated.

The Formal Multiplier

The general functional relationship between income Y and investment I can be stated as $Y = f(I)$. Many specific functional relationships are subsumed under this general formulation, the most common of which is $Y = kI$, where k is a constant.

An incremental change in income dY, can then be stated as a function of the incremental change in investment, dI, as follows:

$$dY = k(dI) \tag{A1.1}$$

or

$$k = \frac{dY}{dI} \tag{A1.2}$$

In a closed economy, national income is equal to aggregate demand, that is $Y = C + I$, where C is consumption. For the accounting identity to hold after an incremental change, it is necessary that

$$dY = dC + dI \tag{A1.3}$$

or

$$dI = dY - dC \tag{A1.4}$$

Substituting this expression for dI into equation (A1.2) we get

$$k = \frac{dY}{dY - dC} \tag{A1.5}$$

dividing both the numerator and the denominator by dY,

we get
$$k = \frac{1}{1 - \frac{dC}{dY}} = \frac{1}{1 - MPC} \tag{A1.6}$$

where $\frac{dC}{dY}$ is the marginal propensity to consume.

Alternately, noting that

$$\frac{dS}{dY} = 1 - \frac{dC}{dY}, \tag{A1.7}$$

where $\frac{dS}{dY}$ is the 'marginal propensity to save' (MPS)

we can write the multiplier as

$$k = \frac{1}{dS/dY} = \frac{1}{MPS} \tag{A1.8}$$

Thus, the multiplier can be treated formally either as the reciprocal of the MPS, or the reciprocal of the marginal reluctance to consume 1-MPC. We will see in part B that the two conditions are not always the same. When MPS = 1-MPC, the secondary effects of an initial increase in investment dI will vary in inverse proportion to the MPS. With this formulation it is possible to show that if the MPC is close to unity, small fluctuations in investment could cause large fluctuations in income and employment; while if the marginal propensity to consume is not much above zero, very large fluctuations of investment would be needed to produce any substantial fluctuations in income and employment.

Again these general conclusions would have to be qualified, based on the empirical relationship of the 'average propensity to consume' (APC) to the MPC.

From the relation in equation (A1.2) it is clear that the multiplier k is the marginal effect of a change of one variable on another, and from the relations in equations (A1.1) and (A1.3) that the effecting variable, that is, the multiplicand dI, is a component of the affected variable dY, or, simply put, the multiplier represents the impact of a part on the whole.

The multiplier can also be represented mathematically as a converging series of effects

$$k = \frac{1}{1 - \frac{dC}{dY}} = 1 + \left(\frac{dC}{dY}\right) + \left(\frac{dC}{dY}\right)^2 + \left(\frac{dC}{dY}\right)^3 + \ldots, \qquad \text{(A1.9)}$$

where $0 < \frac{dC}{dY} < 1$.

In this case the multiplier can be interpreted as the analytic (or undated) iteration of the expansionary effect of an incremental change in an exogenous variable.

Before proceeding to discuss the various interpretations of Keynesian multipliers, it will be useful to review briefly some early criticisms related to the meaningfulness and utility of multiplier analysis.

Meaningfulness of Multiplier Analysis

The multiplier concept has the potential for degenerating into a tautology, where the relationship is made inevitable by definition (Haberler, 1936). The algebraic derivation of the multiplier formula does not automatically confer on it the status of a meaningful concept. For that it is necessary to demonstrate the existence of a stable consumption function, that is, that the concept refers to a behavioral hypothesis. This is all too often overlooked in the use of multipliers in models with little or no behavioral content.

Hansen, in his defense of Keynes, accepted the thrust of Haberler's criticism and noted that Keynes' multiplier analysis was based on equations stated in terms of functional relations between variables, and therefore to be distinguished from identity equations, which explain nothing. He pointed out that the Keynesian consumption and saving schedules represent a type of behavior, that is, the propensity to consume and the propensity to save respectively at different levels of income, and in this sense were analogous to the Marshallian demand schedules which also represent a type of behavior, that is, the propensity to buy at different prices.

Therefore, he argued that only when using the Keynesian schedules relating the demand for investment to income, and the supply of saving to income, is it possible to mutually determine the level of income and the amount invested (or saved) at the point of intersection of the two schedules. In other words, it is at the intersection of the Aggregate Demand schedule (which is itself dependent on the Investment and Consumption functions), and the Aggregate Supply Schedule, that the amount invested simultaneously determines the level of income.

This analytic model of behavior underlies the determination of total income using multiplier analysis. It is a first approximation of the effect of an exogenously provided increment of investment, assuming a stable consumption function. Even this behavioral interpretation of the Keynesian multiplier concept, however, has been criticised by some commentators (such as A. G. Hart, quoted in Ackley (1970, p.309)) as a useless 'fifth wheel' that adds nothing to the ideas or results already implied in the use of the consumption function. In a sense this is true, though the multiplier is an operationally convenient summary of the results.

Different Analytic Interpretations of Multipliers

The interpretation of multipliers will often vary with the type of analytic model from which they are derived. Even though the analytic model in Keynes' General Theory is basically a comparative-static model, the bulk of his discussion on the multiplier often implicitly, and occasionally explicitly, assumed a dynamic framework. He distinguished between two dynamic interpretations of the multiplier: the logical and the periodic.

The Quasi-Dynamic or Logical Multiplier

The logical multiplier, which holds good continuously, without time-lags at all moments of time, is derived from a quasi-dynamic or moving-equilibrium analysis. It assumes that a change in aggregate investment "has been foreseen sufficiently in advance for the consumption-goods industries to advance <u>pari passu</u> with the capital-goods industries" (Keynes, 1936, p. 122), suggesting that there is no consumption-goods production lag. Similarly, assuming that the expansion has been foreseen, there will be no consumer expenditure lag, that is, desired consumption will always equal actual consumption. The relationship of consumption to income will be normal throughout, but the magnitude of the multiplier need not be constant. The desired ratio of consumption to income can gradually change as income changes, resulting in a gradual change in the normal magnitude of the multiplier. In this way, the system can be changing over time, but it will always be in equilibrium -- a moving equilibrium.

It was this concept that Keynes employed for the most part in his presentation of "The Marginal Propensity to Consume and the Multiplier" (Keynes, 1936, Chapter 10), and not the comparative-static multiplier,[4] even though his basic Consumption-Investment model was a comparative-static model.

The Fully Dynamic or Period Multiplier

In the period multiplier, in contrast to the logical multiplier concept, it is assumed that an expansion in the output of capital-goods industries in response to a change in investment demand is not fully foreseen. This is a fully-dynamic concept because the consequences of the expansion take effect gradually, subject to a time-lag, with the full effect emerging only after an interval.

4/ This has been pointed out by Hansen (1953, p.).

Keynes divided the lagged adjustment to an initial increment of investment into two parts, (1) a lag in the increase of investment in related industries, and (2) a consumption-expenditure lag. In the first case he suggested that it was possible to observe "a series of increments in aggregate investment occurring in successive periods over an interval of time" (Keynes, 1936, p. 123). Implicit in this statement is the suggestion that the multiplier (and therefore the marginal propensity to consume) holds at every instant, but that the multiplicand (the increment of investment) changes. In the second case, however, he argued that the consumption-expenditure lag causes the multiplier (dC/dY) at first to fall sharply and then in successive periods to return to a normal ratio. In other words, the lag in consumption-expenditure-increases relative to income-increases throws the relationship of actual current consumption to actual current income out of line, resulting in a drop in the marginal propensity to consume.[5/] Keynes refers in this case to a "temporary reduction of the marginal propensity to consume, that is, of the multiplier itself, ... As time goes on, ... the marginal propensity to consume rises temporarily above its normal level, to compensate for the extent to which it previously fell below it, and eventually returns to its normal level." (Keynes, 1936, p. 124)

5/ A simple illustration to clarify this point is provided in Hansen (1953, p. 111).

This idea of an oscillating MPC caused much confusion and debate. Many argued that the 'propensity to consume' should refer to a normal relationship, not to a temporary one which does not correspond to normal desires.

Keynes, however, argued that the changing marginal propensity, and therefore the changing multiplier, is in fact based on a behavior pattern, namely, a definite and predictable expenditure lag during the transition period from one equilibrium position to another.

> In every interval of time the theory of the multiplier holds good in the sense that the increment of aggregate demand is equal to the increment of aggregate investment multiplied by the marginal propensity to consume. (Keynes, 1936, p. 123)

This is potentially a verifiable behavior hypothesis and not a mere tautology.

This 'short-run-normal' MPC concept is not an issue either in the logical multiplier or in the comparative-static multiplier. In the former, the variables of the system remain continuously in a normal relation to each other as a result of instantaneous adjustments, while in the latter the analysis is restricted to the normal ratio which will eventually be reached when the community has settled down to a new steady level of aggregate investment.

The Comparative Static Multiplier

Unlike the two types of multipliers just discussed, the comparative-static multipliers have no time-dimension: they are timeless. The analytic framework in which they are used simply compares two equilibrium positions, that is, "the multiplier formulae give a comparison of values the dependent variable has in two equilibrium positions of the system, differing in the values of the independent variable" (Lange, 1943, p. 237).

Unfortunately, due to the current emphasis on the more complex dynamic models, it has become common place to contrast comparative-static multipliers to them and to refer to comparative-static multipliers as representing effects which take place instantaneously, if time is treated in continuous terms, or a given unit of time (such as one year) if time is treated in discrete terms. This erroneous imputation of a time dimension presumably stems from the observation that the exogenous variables used in multiplier analysis refer to annual rates of spending. Another reason may lie in the fact that Kahn also originally conceived of the multiplier process in a temporal context.

Thus, when the basically comparative-static Keynesian consumption-investment model is expressed in its multiplier form it has become difficult to keep a discussion of it in non-dynamic terms. Hence, instead of asking what will be the different equilibrium levels of income that correspond to different given levels of investment, there is a tendency to ask the related but separate question of how is it that one dollar of investment spending can possibly cause an increase of more than one dollar in total spending.

The typical answer provided to the latter question is in the form of a converging iterative expenditure sequence. Ackley disapproves of the following type of explanation:

> The original 'injection' of new investment spending causes the rate of production of investment goods to be increased in order to meet the enlarged demand. This leads to an equivalent increase in income in the capital goods industry. The recipients of this extra income may save some part of the increment, but will use most of it to enlarge their consumption spending. This will increase production and income payments in the consumer goods industries. The recipients of this further income increase will, in turn, save a part, but respend most of it, creating new income, new spending, new income, in an endless chain. When we consider the sum of all the successively smaller and smaller series of income increases, we find that the total increase in income (including the initial increase resulting from investment) is a multiple of the 'injection', a multiple the size of which clearly depends on the percentage of each 'round' of income which is respent. (Ackley, 1970, p. 313)

Incidentally, the same type of explanation is popularized in Samuelson's textbook (1973, p. 229). This type of explanation tends to subtly transform the elements of the iterative form of the multiplier solution from analytic rounds into <u>temporal</u> stages, thereby setting up what Ackley considers to be the impossible problem of trying to conceive that the whole sequence is compressed into a 'single instant of time', with all rounds occurring simultaneously (ibid., p. 314). A more fruitful interpretation of the analytic rounds will be provided in part C when interpreting I-O multipliers.

It cannot be overemphasized that in comparative-static analysis there is no single instant of time into which the converging rounds are compressed. As noted earlier, the comparative-static multiplier is timeless. This can be demonstrated by treating all the variables of the model as rates. Then the common time dimension in the ratio of the two rates cancel out and the multiplier becomes a <u>dimensionless constant</u>.

$$\frac{dC/dt}{dY/dt} = \frac{dC}{dY} \quad \text{(A1.10)}$$

When this dimensionless constant is multiplied by an exogenous variable expressed as a rate, then the product also has the dimensions of a rate. But there is no specification of when this rate will materialize, only that the resulting rate is consistent with the exogenously changed rate.

It is still possible and meaningful to identify the actual configuration of a dynamic economy in a base period with the first equilibrium position, and another real or hypothetical configuration of the same economy at another date with the second equilibrium position of a comparative-static model, as has been done by Hicks and others. However, this has nothing to do with outlining via multiplier analysis the time frame or structural path through which the whole adjusts to changes in its components.

In this type of analytic framework not only is the path and duration through which the impulses are transmitted not taken into account, but even the existence of a feasible path within some assumed time-period is not determined. Despite this, and its superficial resemblance to a non-explanatory and purely mathematical relation, the comparative-static macro-economic multiplier relation is still not tautological. The additional increment of new income at the later equilibrium position can still be interpreted to be equal to the additional increment of investment times the multiplier based on a normal propensity to consume, if that propensity can be assumed to be common to both equilibrium positions. During the transition (which is left out of the account), actual saving, using Keynesian terminology, is assumed to be equal to actual investment ex-post; but only at the new equilibrium income level is desired saving equal to desired investment ex-ante. In other words, whereas dynamic models will focus on the nature of the adjustment process and the type of production and expenditure lag that should be assumed, in a comparative-static framework any lag that might exist is assumed to have been overcome with the rate of consumption once again at the normal or desired ratio to income.[6/]

Thus, the time-less multiplier analysis disregards the transition and deals only with the new equilibrium income level that is consistent with a new steady level of aggregate investment. Of course, the stimulus for change need not arise only from investment expenditures, as will be shown in the next section.

6/ In fact dynamic model solutions will approximate comparative-static model solutions asymptotically provided the assumption of stationary state, or 'stable reproduction' (as opposed to expanding or contracting reproduction), is incorporated as a characteristic of the economy rather than of the adjustment process being analysed. The rate of approximation in calendar time will then be a function of the lag structures introduced.

ANNEX A.2

AGGREGATE MACRO-ECONOMIC MULTIPLIERS

The salient feature of macro-economic multipliers is that the magnitude of the impact is independent of the composition of the change in the exogenous variable and is undifferentiated in its results. These scalar multipliers are, however, amongst the most commonly used in theoretical and policy analysis.

We have already seen how the concept of the multiplier arose to explain the cumulative effect of government public works expenditures and/or investment expenditures. However, a matter which at an early stage of the development of the macro-economic multiplier concept gave rise to some misunderstanding, was the type of initial expenditure necessary to activate the multiplier process.

According to Hansen the initial expenditure did not have to be limited to investment outlays on capital goods. Keynes, in fact, had used not only the term investment, whether private or public, to describe the initial expenditure, but also the term "loan expenditure". The latter could involve funds paid out directly to consumers in the form of grants, credits, etc., or it could involve an increase in the take-home pay resulting from tax reduction, the deficit being financed by borrowing. Whatever the initial increase in expenditure, whether private or public investment, or simply an increase in private-consumption outlays resulting from a tax reduction, or perhaps from the spending of privately held liquid assets, the effect, as far as the multiplier process was concerned, would be the same.

It was this idea, which was contained in Keynes' original analysis, that the multiplier need not be limited to investment injections, that led to the explicit generalization of the investment multiplier concept to other types of multipliers such as consumption multipliers, spending multipliers, etc.

Simple and Compound Multipliers in a Closed Economy without a Government Sector

Simple multipliers refer to multipliers which involve only one marginal relationship. Compound multipliers refer to multipliers which involve more than one marginal relationship.

Simple Multipliers 1/

The investment multiplier which has already been defined in the previous section is a simple multiplier. In analogy to it we can treat consumption C as the exogenous variable to obtain the simple consumption multiplier which is equal to the reciprocal of the marginal reluctance to invest:

$$\frac{dY}{dC} = \frac{1}{1 - \frac{dI}{dY}} \tag{A2.1}$$

The interpretation of the consumption multiplier is similar to that of the investment multiplier, in that it shows the marginal effect upon national income of an increase in the rate of consumption.

1/ The following discussion of simple and compound multipliers is based, in part, on the analysis provided by Lange (1943).

This multiplier can also be obtained as the sum of an infinite geometric progression:

$$\frac{dY}{dC} = 1 + \frac{dI}{dY} + \left(\frac{dI}{dY}\right)^2 + \left(\frac{dI}{dY}\right)^3 + \dots , \text{ if } \left|\frac{dI}{dY}\right| < 1. \qquad \text{(A2.2)}$$

Both the investment and the consumption multipliers cannot, however, be derived from a model of the form

$$\frac{dC}{dY} + \frac{dI}{dY} = 1$$

even though each one can be derived separately from such a model. It is necessary instead for purposes of the model's stability to introduce $\frac{dS}{dY} > \frac{dI}{dY}$, that is, that the marginal propensity to save has to be greater than the marginal propensity to invest. If, $\frac{dS}{dY} < \frac{dI}{dY}$, then the system is unstable as the functions will not intersect.

When the condition $\frac{dS}{dY} > \frac{dI}{dY}$ is combined with the stability condition $\left|\frac{dC}{dY}\right| < 1$, which is necessary for the geometric series of the investment multiplier to be convergent (and is implied in the observed empirical fact $0 < \frac{dC}{dY} < 1$), we obtain

$$\frac{dC}{dY} + \frac{dS}{dY} = 1,$$

or

$$\frac{dC}{dY} + \frac{dI}{dY} < 1.$$

From this we get the implicit closed interval for the marginal propensity to invest $0 < \frac{dI}{dY} < 1$ to meet the stability condition $\left|\frac{dI}{dY}\right| < 1$ which is necessary for the consumption multiplier to be convergent.

Compound Multipliers

This concept was introduced in 1941 by Lange and Angell, both of whom originally used the term 'cumulative' multipliers. It was designed to overcome a problem implicit in the use of simple multipliers. For example, in the investment and consumption multipliers the multiplicands dI and dC refer to the total increments in the rate of investment and in the rate of consumption in the economy respectively. However, in general, this total is composed of two parts. In the case of the multiplicand dI, the first part is the initial autonomous increment in investment, which leads to the second part, that is, positive or negative additional investments. These are induced by the increase in national income consequent upon the increase in consumption generated by the initial investment. A similar analysis can be made for the multiplicand dC. Thus, the multiplicands dI and dC represent not the initial increment in I or C, but the total increment, which includes in addition to the initial increment, all induced increments.

The realization that the multiplicands contained induced effects imposed serious limitations upon the practical use of the simple investment and consumption multipliers. In this form they could not be applied to as straightforward a problem as the effect of a change in government investment, or consumption, expenditures upon national income, unless it was possible to estimate the private investment or consumption expenditures induced by the initial government expenditures (Samuelson, 1940 and 1942).

This was not an academic issue, because the uselessness of the investment multiplier formula had become particularly apparent during the period 1936-40, when many economists held the belief that

> because of its allegedly adverse effect upon business confidence, government investment causes a diminution of private investment to such an extent that it results in a fall of the national income. This argument was frequently expressed in the form of the statement that the multiplier is negative. This statement was a wrong formulation of a basically meaningful (though empirically unfounded) proposition. What critics of government spending meant to say was that the multiplicand dI, not the multiplier 1/(1 -dC/dY), is negative, because the (allegedly) negative induced private investments outweigh the positive initial increment made by the government. (Lange, 1943, p. 229)

To clarify this issue the initial autonomous increment in the rate of investment can be set as dI_o. Then national income will initially increase by an equal amount. This increase will, however, lead in turn to induced consumption equal to $(\frac{dC}{dY})\ dI_o$ and to induced investment equal to $(\frac{dI}{dY})dI_o$, and thus to an induced increase in income of $(\frac{dC}{dY}+\frac{dI}{dY})dI_o$. This induced income will, in turn, lead to a further induced increase in income of $(\frac{dC}{dY}+\frac{dI}{dY})(\frac{dC}{dY}+\frac{dI}{dY})dI_o$; etc. (Samuelson, 1942). The total increase in national income will then be:

$$dY = [\,1 + (\frac{dC}{dY} + \frac{dI}{dY}) + (\frac{dC}{dY} + \frac{dI}{dY})^2 + \ldots\,]\ dI_o \qquad \text{(A2.3)}$$

The 'compound investment multiplier' derived from the above relation is:

$$\frac{dY}{dI_o} = \frac{1}{1 - (\frac{dC}{dY} + \frac{dI}{dY})}, \qquad \text{if } \left|\frac{dC}{dY} + \frac{dI}{dY}\right| < 1. \qquad \text{(A2.4)}$$

An analogous argument was set out by Lange for consumption, in which an initial autonomous increment dC_o in the rate of consumption leads first to an equivalent increase in the national income and then through induced investment and consumption, to further increments in income $(\frac{dC}{dY} + \frac{dI}{dY})\, dC_o$, $(\frac{dC}{dY} + \frac{dI}{dY})(\frac{dC}{dY} + \frac{dI}{dY})\, dC_o$, etc.

The total increase in national income in this case will be:

$$dY = [1 + (\frac{dC}{dY} + \frac{dI}{dY}) + (\frac{dC}{dY} + \frac{dI}{dY})^2 + \ldots]\, dC_o \qquad (A2.5)$$

The 'compound consumption multiplier' derived from the above relation is:

$$\frac{dY}{dC_o} = \frac{1}{1 - (\frac{dC}{dY} + \frac{dI}{dY})}, \qquad \text{if } \left|\frac{dC}{dY} + \frac{dI}{dY}\right| < 1 \qquad (A2.6)$$

When using the compound multipliers, the multiplicands dI_o and dC_o refer only to the autonomous increments in investment or consumption. All induced changes in investment and consumption are taken care of directly by the compound multiplier formula itself. These multipliers can, therefore, be used for the problems for which the simple multiplier formulae proved inadequate.

The identity between the two compound multipliers led Lange and Angell to introduce the simple spending multiplier:

$$\frac{dY}{dA_o} = \frac{1}{1 - \frac{dA}{dY}} \qquad (A2.7)$$

in which dA_o is the autonomous increment in the rate of spending, $\frac{dA}{dY} = \frac{dC}{dY} + \frac{dI}{dY}$ is the marginal propensity to spend, and $1 - \frac{dA}{dY}$ the marginal reluctance to spend.

From $1 - \frac{dC}{dY} = \frac{dS}{dY}$, and $\frac{dA}{dY} = \frac{dC}{dY} + \frac{dI}{dY}$, we get $1 - \frac{dA}{dY} = \frac{dS}{dY} - \frac{dI}{dY}$,

which shows that the marginal reluctance to spend is the difference between the marginal propensity to save and the marginal propensity to invest. The marginal reluctance to spend can be referred to as the 'marginal propensity to hoard' (MPH), with hoarding being defined in this case as the difference between desired receipts and desired expenditures. The stability condition of the system remains the same as before but can be expressed in the form $1 - \frac{dA}{dY} > 0$, that is, that the reluctance to spend is an increasing function of national income.

The Relationship between Simple and Compound Multipliers

From the equations (A1.7), (A2.1) and (A2.7) it can be shown that

$$\frac{dY}{dI} = \frac{(dY/dA)}{(dI/dI_o)} \qquad (A2.8)$$

and

$$\frac{dY}{dC} = \frac{(dY/dA)}{(dC/dC_o)} \qquad (A2.9)$$

In this case the denominators are also multipliers, namely,

$$\frac{dI}{dI_o} = \frac{1 - (dC/dY)}{1 - (dA/dY)} \qquad (A2.10)$$

and

$$\frac{dC}{dC_o} = \frac{1 - (dI/dY)}{1 - (dA/dY)} \qquad (A2.11)$$

The ratio dI/dI_o states that the marginal effect of autonomous investment upon the rate of investment in the economy is equal to the ratio of the marginal reluctance to consume to the marginal reluctance to spend (or, in other words, the ratio of the marginal propensity to save to the marginal propensity to hoard). The ratio dC/dC_o states that the marginal effect of an autonomous change in consumption upon the rate of consumption in the economy is equal to the ratio of the marginal reluctance to invest to the marginal reluctance to spend.

The effect upon national income of any given autonomous change in spending, can be determined by using either the simple spending multiplier (A2.7), or any of the two simple investment and consumption multipliers (A1.7) or (A2.1) The use of the latter, however, requires a knowledge of the compound investment and consumption multipliers in (A2.10) or (A2.11) which presupposes the same data as the spending multiplier.

Government Spending Multipliers

In the exposition of the compound multiplier it was assumed that the autonomous component of expenditures originated in the government sector, rather than in the private sector. This is not necessary. The formal analysis can be conducted, in principle, for an economy without a government sector. On the other hand, if one wants to analyze the government spending multiplier specifically, an alternate approach to the problem is to define $Y = \overline{C} + \overline{I} + G$, where $\overline{C}$ and $\overline{I}$ are total private consumption and investment expenditures respectively, and G government expenditures. In this case it is also necessary to differentiate between disposable income Y_d, and taxes T (Allen, 1968, pp. 138-140), such that:

$$Y = Y_d + T \qquad \text{(A2.12)}$$

Setting $\overline{C} = C_1 + C_o$ as total private consumption expenditures, of which only a part C_o is autonomous

where $C_1 = \frac{dC_1}{dY_d}(Y_d)$ is private consumption as a linear function of disposable income rather than total income

and setting	$A_o = C_o + I_o$	as private autonomous expenditures
where	$I = I_o$	is total private investment expenditures, all of which are treated as autonomous

we can write the national income identity and equation (A2.12) as:

$$Y = C_1 + A_o + C \tag{A2.13}$$

and

$$Y_d = cY_d + A_o + G - T \tag{A2.14}$$

where $c = \frac{dC_1}{dY_d} = MPC$ 2/

In this case	$\overline{S} = S_1 + S_g$	is total public and private savings
where	$S_g = T - G$	is public savings as the difference between taxation and government spending

and $S_1 = Y_d - \frac{dC_1}{dY_d}(Y_d) = \frac{dS_1}{dY_d}(Y_d) = s(Y_d)$

where $s = \frac{dS_1}{dY_d} = MPS$

Instead of the equality of total savings to total investment of the simple investment multiplier, equilibrium now requires

$\overline{S} = A_o$, that is, public and private savings must be equal to private autonomous expenditures

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2/ Since transfers affect income distribution and hence the average aggregate marginal propensity to consume, its inclusion will only complicate the results further.

Setting the changes in autonomous expenditures and taxes equal to zero, that is, $dA_o = 0$, and $dT = 0$, and differentiating (A2.14) and (A2.12) with respect to dG,

we get

$$\frac{dY_d}{dG} = \frac{1}{1-c} = \frac{1}{s} \tag{A2.15}$$

and

$$\frac{dY}{dG} = \frac{dY_d}{dG} = \frac{1}{s} \tag{A2.16}$$

Similarly setting the changes in autonomous expenditures and government spending equal to zero, that is, $dA_o=0$, and $dG=0$, and differentiating (A2.14) and (A2.12) with respect to dT,

we get

$$\frac{dY_d}{dT} = \frac{-1}{1-c} = -\frac{1}{s} \tag{A2.17}$$

and

$$\frac{dY}{dT} = \frac{dY_d}{dT} + 1 \tag{A2.18}$$

or

$$\frac{dY}{dT} = \frac{1}{1-c} + 1 = -\frac{c}{1-c} = -\frac{1-s}{s} \tag{A2.19}$$

The results of equations (A2.16) and (A2.19) can be combined. That is, on the basis of equations (A2.12) and (A2.14) we can write

$$Y_d = \frac{1}{s} (A_o + G - T) \tag{A2.20}$$

and

$$Y = \frac{A_o + G - T}{s} + T$$

$$= \frac{1}{s} (A_o + G) - \frac{1-s}{s} T \tag{A2.21}$$

From which we get the equilibrium relation between changes in increments as

$$dY = \frac{1}{s} d(A_o + G) - \frac{1-s}{s} dT \tag{A2.22}$$

Hence the regular linear multiplier $\frac{1}{s}$ applies to autonomous expenditure (A_o = G alike)[3/], but only to a proportion (1-s) of taxes.[4/]

3/ For an analysis of the type above, but with changes in taxes a function of income rather than autonomous, see R.G.D. Allen (1968, pp. 145-147). The analysis can also be made more complex if there is a need to distinguish between the multiplier effect from Government deficit spending and the multiplier effect from Government expenditures out of taxes.

4/ Note that:

$$\frac{1}{s} - \frac{1-s}{s} = 1$$

and $$\frac{1}{s} = 1 + \frac{1-s}{s} > \frac{1-s}{s} = \frac{c}{s}$$

In the case of a balanced budget, that is, if government expenditures and receipts exactly balance (G = T), then

$$Y = \frac{1}{s} A_o + \frac{1}{s} G - \frac{1-s}{s} T \tag{A2.23}$$

$$= \frac{1}{s} A_o + \left(\frac{1}{s} - \frac{1-s}{s}\right) G$$

$$= \frac{1}{s} A_o + G$$

from which

$$dY = \frac{1}{s} dA_o + dG \tag{A2.24}$$

That is, the multiplier level of income $(\frac{1}{s}dA_o)$ is raised by the exact amount dG of a change in government spending.

Alternatively setting $dA_o = 0$, we get dY=dG in equation (A2.24) Thus, a simultaneous balanced increase in government expenditure and taxation (i.e., dG=dT) raises income by the amount of the change in government expenditures only, with no multiplier effect of its own.5/

It should be apparent by now that the condition for the stability of each model's equilibrium is different. In the case of the simple investment multiplier it is necessary for total savings to equal total investment, whereas, in the case of the fiscal multiplier just discussed, it is necessary for private and public saving to balance autonomous private expenditures and in the case of the simple spending multiplier (as well as the two compound multipliers) it is necessary for hoarding to balance autonomous spending. Thus, in equilibrium it

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5/ Additional references on the balanced budget and other fiscal multipliers can be found in Samuelson (1974).

is not always necessary for the flow of funds allocated to savings to be equal to the flow of funds allocated to investment. What is necessary is that the injections into the circular flow of income be balanced by leakages from it. This generalization will be presented more formally after discussing foreign trade multipliers in the next section.

Open Economy Multipliers

Up to this point we have assumed throughout that the economy is closed, that is, that it does not engage in trade with other economies. This close economy assumption, found in many theoretical models, is useful for analytic and heuristic purposes, and occasionally for policy purposes, as a first approximation of the multiplier effects, in a highly self-sufficient economy. However, it is generally not useful for policy purposes in economies with low degrees of self-sufficiency, that is, those which are highly dependent on trade, whether exports, imports, or both. This dependence on trade is particularly evident in most regional economies and many small national economies. With the increasing integration of the global economy, it is also becoming a feature of large national economies. Therefore, it is necessary to modify the above analysis when dealing with an open economy, with significant imports and exports.

Investment, consumption and spending multipliers can also be derived from an open national income model in a manner strictly analogous to those already developed in the context of a closed economy. Their interpretation will have to reflect the additional set of leakages via imports and injections via exports.

If we treat government spending as part of autonomous expenditures in a closed economy, then, as we saw in the case of the simple spending multiplier, the only type of leakage that could prevent an infinite expansionary effect in national (or regional) income consequent upon an autonomous change in the rate of spending is hoarding. In open economies, however, another type of leakage exists to prevent an infinite expansion (in addition to, or even in lieu of, hoarding): part of domestic (or local) spending does not flow back to income recipients in the economy under study. Expenditures on imported goods, or external spending, need to be distinguished from expenditures on domestically produced goods, or internal spending, which flow back to income-recipients within the economy. On the other hand, external spending may be offset by income received from selling domestically produced goods to other economies. The income received from the sale of exports, that is, external receipts, also needs to be distinguished from the income received from the sale of goods in the domestic (or local) markets, that is, internal receipts.

It should be recalled from the discussion on p. .... , that from the point of view of a country's circular flow of income, expenditures financed by borrowed funds also constitute an autonomous injection into the income stream. That is, firms, and hence households, earn income as a consequence of these expenditures just as they would, had the expenditures been financed from revenues rather than borrowed funds. Exports also constitute an injection into the domestic circular flow of income. Thus, an increase in exports would lead directly to an increase in the revenue of the firms producing the goods destined for export, and therefore, indirectly, via factor accounts, to an increase in the income of the income recipients in the economy. If these households spend part of their increased income on domestically produced goods, a multiplier effect similar to that resulting from an increase in domestic investment or consumption expenditures will be set in motion.

Thus, domestic income will vary directly with exports and inversely with imports, just as it varies directly with investment expenditures and inversely with savings. Therefore, the condition for equilibrium in the circular flow of income requires that the two sources of leakages, savings S, and imports M, balance the two sources of injections, investment I, and exports E:

$$S + M = I + E \tag{A2.25}$$

It is clear from this formulation that saving can exceed or fall short of investment, if it is offset in the former case by an excess of exports over imports, and in the latter case by an excess of imports over exports, because

$$S - I = E - M \tag{A2.26}$$

We can assume that investment and exports are exogenously determined and that, in analogy to savings, the import of goods varies solely with income, that is, there is an empirically demonstrable 'marginal propensity to import' (MPM) showing the fraction of an increment of income that households would wish to devote to the purchase of imported goods. Then if we start from a position of equilibrium and make a change in the volume of injections into the circular flow, income will change until, at the new equilibrium level, leakages again equal injections. That is

$$dS + dM = dI + dE \tag{A2.27}$$

Substituting the marginal propensity to save, $s = \frac{dS}{dY}$, and the marginal propensity to import, $m = \frac{dM}{dY}$, into equation (A2.27) we get

$$s(dY) + m(dY) = dI + dE \tag{A2.28}$$

from which we obtain the multiplier relationship.

$$dY = \frac{1}{s + m}(dI + dE) \tag{A2.29}$$

This way of explicitly incorporating gross imports into the multiplier via the marginal propensity to import, is not the only way to introduce the open economy assumptions into the analysis. We can instead use the concept of net imports, in one of two ways.

First, still regarding imports as a leakage, the term 'save' in the concept marginal propensity to save can be made to include the increment spent on net imports. The increment can be added or subtracted depending on whether the increment of gross imports is greater than or less than the increment of gross exports.

Alternately, treating imports as a component of the multiplicand rather than the multiplier, we may enter the excess of the increment in gross imports over the increment of gross exports as negative domestic investment, in which case the multiplicand dI is reduced by the amount of the induced increment of net imports. If the increment of exports exceeds the induced increment of gross imports, the excess can be regarded as positive investment and added to the domestic investment figure.

Other types of open economy multipliers are presented in Annex A.4.

General Form of the Multiplier

From this brief review (relative to the vast literature on the subject), it should be clear that the issue of what is to be included in the multiplier and what in the multiplicand cannot be decided in the abstract. It will depend on what can be demonstrated to be an empirical propensity, and not affected by the policy variables whose impact is being analysed.

However, one general statement about multipliers can be made, that the circular flow of income will only be in equilibrium when the total flow of all leakages is equal to the total flow of all injections (Lipsey, 1964, p. 489). This may be written, using the notation of L for leakages and A for autonomous injections, as

$$L = A \tag{A2.30}$$

Thus, for the flow of income to remain in equilibrium, any change in injections must be matched by an equal change in leakages:

$$dL = dA. \tag{A2.31}$$

If leakages or the propensity not to pass money on is assumed to depend on income, i.e.

$$L = f(Y), \tag{A2.32}$$

and the simplifying assumption is made that a constant fraction of any change in income will not be passed on, i.e.

$$dL = h(dY). \tag{A2.33}$$

then, substituting (A2.33) into (A2.31) we obtain

$$h(dY) = dA \tag{A2.34}$$

or

$$dY = \frac{1}{h}\,dA \tag{A2.35}$$

which is the general form of the multiplier as the reciprocal of h, the marginal propensity not to pass income on.

ANNEX A.3

DISAGGREGATED INPUT-OUTPUT MULTIPLIERS

Input-output analysis was introduced into economic literature by W. Leontief (1936) at the same time as J. M. Keynes introduced his General Theory. The input-output model on which the analysis is based is a simplification of the Walrasian General equilibrium analysis, though its forebears can be traced back to Quesnay's Tableau Economique (Phillips, 1955). The simplification consisted of assuming a fixed input to output ratio for the production of each commodity in the economy, thereby doing away with the problem of choice (or in the words of Cameron, 1955, "the problem of economizing").

The Totally Closed Input-Output Model

In its original version the model was based on the assumption of a closed economy without a government sector. In addition, the model was fully closed, that is, all sectors of the economy, including households, were treated as intermediate sectors. The resulting homogenous set of equations had the form

$$X - AX = 0 \tag{A3.1}$$

where X was the vector of gross outputs and A the matrix of direct input to output ratios or coefficients. Each coefficient was defined as:

$$a_{ij} = X_{ij}/X_{oj}$$

such that

$$\sum_i a_{ij} = 1$$

where x_{ij} was the transaction representing the amount of industry i's output delivered as input to industry j, and x_{oj} the total output of industry j.

In the simplest form of the model Leontief assumed a stationary state economy without savings or investment (Leontief, 1941, p. 42-45). Later, this assumption was relaxed and savings incorporated in the model, through the introduction of a savings coefficient for each sector B_i. A proportionality factor was also introduced for total savings, β', and for sectoral technical productivity, A_i, so that in its most general form the input-output coefficient could be represented as

$$a_{ij} = \frac{A_i B_i \beta X_{ij}}{x_{oj}} \tag{A3.2}$$

This form of coefficient, however, has not been used in subsequent empirical and theoretical work since the model has been opened.

Before discussing the open model it should be pointed out that the solution of a homogenous set of equations as in formula (A3.1) cannot be obtained by inverting the matrix (I-A), because this matrix is singular. The solution requires finding the scalar parameter or eigen value λ, such that

$$\lambda X = AX \tag{A3.3}$$

for $X \neq 0$, and A a matrix of order m.

The procedure for determining eigen vectors and eigen values, and interpreting them can be found in Noble (1969, pp. 274-312) and Lancaster (1970, pp. 80-83) respectively.

The totally closed I-O model provides many insights despite its apparently circular form of argument (Brody, 1974, pp. 69 & 84).[1/] However, it was noticed soon after the input-output model was introduced that from the point of view of policy analysis the totally closed version of the model had two major drawbacks: (1) it contained no exogenous variables and hence no multipliers with which to estimate the impact of a policy change, and (2) it could not determine the absolute magnitude of gross outputs. It could only specify the relative ratios of gross outputs[2/], because in a system of n equations with n unknowns, only n-1 of the equations can be linearly independent, that is, the nth unknown will be determined automatically once the preceding n-1 unknowns have been solved.

Both drawbacks could be overcome simultaneously by 'opening' the model. Hence, during the Second World War, and thereafter, the open input-output model has been universally adopted for purposes of policy analysis.

The Open Input-Output Model

Once it was decided that the model had to be opened to be useful in policy analysis, the problem of what should be endogenous and what

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1/ This is particularly true when the totally closed model is transformed into its dynamic version, though its properties are difficult to interpret (Dorfman, Samuelson and Solow, 1958, p. 245-248 and Chapters 11 and 12).

2/ This is a feature it shared with all other general equilibrium models with n equations and n unknowns.

exogenous became an issue:

> The ultimate aim of analysis in terms of general interdependence is to achieve a system which is as nearly closed as possible.... [However] some elements of the system, for example, some aspects of investment decisions, must remain arbitrary in the sense that they are not understood and cannot be predicted from the knowledge of other economic variables. If a system is closed except for a few elements of the type just mentioned, these elements play the role of deus ex machina and control the system (Dusenberry and Kisten, 1953, p. 451).

Obviously, there is no correct answer to the question of what should be treated as autonomous and what as induced, or what should be included in the multiplier and what in the multiplicand. It depends on the issue under investigation, that is, what is assumed to change and what factual hypotheses underlie the empirical invariances. Ideally a model that is more flexible is to be preferred over one that is less flexible in terms of the degrees of closure that it can incorporate.

It has become common practice to link the I-O model which is based on a gross accounting framework to the national income accounts which are based on a net accounting framework. As a result, the exogenous variable has come to be identified with aggregate demand (C+I) in the context of a closed economy, and either aggregate demand plus gross exports (C+I+E), or aggregate demand plus net exports (C+I+E-M) in the context of an open economy, where M refers either to total gross imports, or, to competitive imports only.

Primal Version of the Open Model

The open input-output model, with an exogenous demand variable, shows that the supplies from each sector must satisfy all demands, including endogenously generated intermediate demands. This model has the form

$$X = AX+Y \tag{A3.4}$$

where Y represents the vector of exogenously determined final demand. In order to retain separately that portion of the model which is part of the national income accounts, all of the inputs are not treated as intermediate inputs. Therefore, in this case the intermediate direct coefficients a_{ij} do not add up to one as in the totally closed model. That is,

$$\sum_i a_{ij} < 1$$

or

$$1 - \Sigma\, a_{ij} = \overset{*}{v}_j > 0$$

where the $\overset{*}{v}_j$ represent the direct coefficients of the primary supply (or primary inputs) vector $\overset{*}{V}$. Hence the full open model contains a secondary balance equation:

$$v_o = \overset{*}{V}X \tag{A3.5}$$

where v_o is a scalar representing aggregate primary supply. This scalar can also be written in the form of a vector, V, if the coefficients $\overset{*}{v}_j$ are placed along the principal diagonal of a matrix $\hat{\overset{*}{V}}$, whose off-diagonal elements are all zero.

The solution of the full open model can then be written as

$$X = (I-A)^{-1} Y, \qquad \text{demand for gross outputs} \qquad (A3.6)$$

and

$$V = \hat{V}^{*}X = \hat{V}^{*}(I-A)^{-1} Y, \qquad \text{demand for factors} \qquad (A3.7)$$

in which I is the identity matrix and $(I-A)^{-1}$ the matrix multiplier which shows by how much X must change to be consistent with changes in Y.

The solution in equation (A3.6) shows how changes in X will depend on changes in Y only if the matrix A is assumed fixed because the total differential of equation (A3.3) is

$$dX = A(dX) + (dA)X + dY \qquad (A3.8)$$

$$= (I - A)^{-1} [(dA)X + dY] \qquad (A3.9)$$

For dX to depend only on dY it is necessary that (dA)X=0. Thus, the identity in (A3.9) is transformed into a type of 'behavioral' equation only by assuming that A is constant.[3/]

3/ The inverse $(I - A)^{-1}$ is important in models where expenditure allocations are independent of income levels. With respect to production activities, such independence usually requires that there should be no fixed factors of production since those elements of A which record the income shares of fixed factors will usually be sensitive to the level of production. However, if the production function is Cobb-Douglas, then income shares of factors are independent of output levels in all cases provided pricing policy is based on neo-classical marginal cost rules.

Alternatively, if product prices are given by average variable cost times some mark-up, returns to fixed factors will always be a constant share of total costs. It follows that fixed factors in production do not necessarily violate the conditions under which A can be assumed constant or given, once prices are known. (Pyatt and Round, 1978, p. 14)

Within this context, that is, where A is assumed fixed (and independent of prices), it is possible for a single inversion of (I-A) to be used repeatedly[4/] to determine (or test) the consistency of sectoral gross output levels (that is, including direct and indirect intermediate input requirements), with different, independently projected (or targeted) final demands.

Dual Version of the Open Model

Although the input-output system has been used principally for 'real' planning, it does contain an implicit set of prices (i.e., a dual), analogous to the shadow prices, that are obtained from programming models. If P_i represents the price of commodity i, and w the uniform wage-rate of the single factor L (labor), we can write

$$P_i = \sum_{j=1}^{m} p_j a_{ji} + wL_i \qquad \text{(A3.10)}$$

where L_i is the labor to output ratio in sector i.

This can also be written in matrix notation as

$$P = A^T P + wL \qquad \text{(A3.11)}$$

in which instead of the matrix A we use its transpose A^T. The solution of this system of equations is

$$P = (I-A^T)^{-1} V \qquad \text{(A3.12)}$$

with

$$V = wL.$$

4/ This can mean a considerable savings in computational effort compared with the elimination-of-variables method, especially if large equation systems are involved. However, if we use Cramer's rule to determine the inverse this advantage will be lost.

Since the transpose of I is I and the inverse of a transpose is the inverse (Noble, p. 23, Ex. 1.43a), we can write the solution as

$$P = \left[(I-A)^{-1}\right]^{T} V \qquad (A3.13)$$

From which it is possible to express the solution in a form in which it is not necessary to use the transpose of A, that is

$$P = V^{T}\left(I-A\right)^{-1} \qquad (A3.14)$$

This is analogous to the form used in Dorfman, Samuelson, and Solow (1958, p. 253):

$$\left(\frac{P}{W}^{T}\right) = L^{T}(I-A)^{-1} \qquad (A3.15)$$

Equation (A3.14) expresses the price of goods in terms of factor prices using the same multiplier matrix as the primal.

In the example used so far it has been assumed that the primary input vector represents only Labor with a uniform wage-rate w. The results can, however, be generalized to include more than one factor of production and non-uniform rates of remunerations for each factor (Yan, 1969, pp. 73-81).

The Iterative Solution of the Inverse Multiplier Matrix

The matrix $B=(I-A)^{-1}$ which enters both the primal and dual form of the model, is a multiplier matrix. Waugh (1950) showed that the matrix B can also be obtained from the converging series $\sum_{i=o} A^{i}$

$$(I - A)^{-1} = I + A^1 + A^2 + A^3 + \ldots \qquad (A3.16)$$

where $A^o = I$.

I and A both contain only non-negative numbers, hence, the sequence incorporating all the powers of A (i.e., the self-multiplication of A) has a non-negative solution. Thus, as is to be desired, the solution of equations (A3.6) and (A3.14) will also be non-negative.

If exact equality between $(I-A)^{-1}$ and $I + A + A^2 + A^3 + \ldots$ is not desired, then the end terms of this converging series can be dropped. The cut off point will depend on the degree of accuracy sought. This iterative approach can be used to bypass the costly inversion process and provide a more economical way of approximating the solution to the inverse of (I-A).

Output and Price Multipliers

Equation (A3.6) can be written on the basis of equation (A3.16) as a finite sequence with $i = n$,

$$X = (I-A)^{-1}Y = IY + AY + A^2Y + \ldots + A^nY \qquad (A3.17)$$

This is known as the Cornfield-Leontief multiplier chain according to which:

> We first compute the output requirements of the new Y itself; then we compute the first round direct requirements to produce the Y, which gives AY; then we compute the second-round direct requirements to produce the first-round items; etc. We thus build-up a growing total until the terms in the dwindling chain dwindle to negligible proportions. For Leontief's a_{ij}'s, the process can be shown to be convergent, and rather rapidly so. (Dorfman, Samuelson, Solow, loc. cit., with adjusted notation).

In analogy to the previous multiplier, equation (A3.14) can be written as:

$$P^T = V^T(I-A)^{-1} = V^TI + V^TA + V^TA^2 + \ldots + V^TA^n \qquad \text{(A3.18)}$$

This is known as the Gaitskell multiplier chain and it shows

> ... the total labor cost of a good as the sum of its initial direct labor cost, plus the direct-labor costs of the inputs it directly uses on the first round, plus the direct labor costs of the second round factors, needed to produce the first round factors, etc., until the terms of the infinite series become negligible. This series will be convergent; it is simply another way of looking at the previous multiplier series. (ibid.)

Interpretation of the Multiplier Matrix Coefficients

Needless to say, the rounds involved do not take place in calendar time, with the second round following the first, as in dynamic models.

> Artificial computational time is involved, and if we insist on giving a calendar-time interpretation we must think of the Gaitskell process as going 'backward' in time and the Cornfield process as showing how much production must be started many periods back if we are to meet the new consumption targets today. (ibid.)

It is best, however, not to insist on giving a calendar time interpretation, because the input-output flow model provides us only with comparative-static multipliers (see Chapter 5 and Annex A.1). Correctly interpreted, the rounds in the multiplier chain $\sum_{i=o}^{\infty} A^i$, represent successive approximations to the total change independent of time, that is, the kth round shows the effect of a transaction, k times removed from the direct change, but not reflecting readjustments to imbalances as might be found in dynamic models. This can be clarified with an example. Industry i may not sell directly to industry j, but only indirectly via industry g. This connection will only be picked up in the second round. The connection may be even more indirect, that is, industry i may sell to g who sells to j only via industry r. This connection will only be picked up in the third round of the iteration and so on.[5/] Thus, the rounds do not reflect any type of adjustment mechanism, but only the most indirect connections existing between a pair of industries in an economy. If the matrix A^n, with $n = 10$, is a null matrix, then that means the most indirect output necessary to meet the exogenous change in a variable is nine times removed from the direct change itself.

A last point in the interpretation of the inverse matrix. This point is pertinent to the relationship between the theoretical form and the operational form of the input-output model.

In principle the direct technical coefficients of the matrix A can refer to physical quantities measured in different units.[6/] Let us assume that $\bar{A}$ is the direct technical coefficient matrix whose elements are

$$\bar{a}_{ij} = \frac{q_{ij}}{q_{oj}} \qquad \text{(A3.19)}$$

where the q's refer to inputs and outputs measured in physical units.

Then, the direct technical coefficients measured in value units (i.e. in which $x_i = p_i q_i$) will take the form

5/ This type of explanation is normally used in the structural analysis of interindustry linkages (Yan, op. cit., p. 95).

6/ This will not affect the formal solution of the model (Brody, op. cit., p. 98). even though it does raise some interesting issues, e.g. on the definition of the physical units measuring the outout of service industries and differentiating them from labor input, etc.

$$a_{ij} = \frac{x_{ij}}{x_{oj}} = \frac{p_i q_{ij}}{p_j q_{oj}} = (\frac{p_i}{p_j})\bar{a}_{ij} \qquad \text{(A3.20)}$$

This formulation of value-derived coefficients in terms of quantity-derived coefficients times the price ratio of the quantities involved can be expressed in matrix notation as:

$$A = \hat{P}\,\bar{A}\,\hat{P}^{-1} \qquad \text{(A3.21)}$$

where $\hat{P}$ refers to a diagonal matrix of commodity prices.

Virtually all input-output models, with the exception of isolated cases in the Soviet Union, have been obtained from value transaction tables. Hence, the multiplier matrix B is based on the technical coefficient matrix A, rather than $\bar{A}$. This necessitates a change in the interpretation of the coefficients of B from the direct plus indirect requirements <u>per unit</u> of output for each sector, to the direct plus indirect requirements <u>per dollar of sales</u> of each commodity.

Income and Employment Multipliers

The output multiplier matrix B has stood the test of time (Bezdeck, 1974). Even when used in its dual version, given exogenously determined factor prices, its cost-push price forecasts appear to track observed prices fairly well (L. Taylor, 1975, p. 47).

Developments and variations in the model have more often than not been concerned with improving methods for the calculation of income and employment multipliers, which are in many cases more relevant for policy purposes than gross industry output multipliers.

In our discussion of the primal open input-output model we noted that, simultaneous with the creation of an exogenously determined final demand vector, a separate vector of primary inputs had to be introduced (equation A3.4). This convention is desirable (but not necessary) on two grounds. One, it ensures that the matrix A will fulfill the Hawkins-Simon conditions required for matrix inversion. Two, it is theoretically more satisfying to relax the assumption of fixed production coefficients for the individual sectors providing primary inputs, that is, the factors of production, even though the overall primary input coefficient vector will have to remain fixed as long as A is assumed constant, because

$$V = I - A \tag{A3.22}$$

by definition.

Provided the heroic, but often necessary, simplifying assumption is made that the use of each factor is proportional to sectoral production, then the primary input vector V, in equation (A3.7), can also be expressed as a full matrix whose coefficients v_{ij} show the amount of factor i used as input per unit of gross output in sector j. Alternately, each factor can be expressed separately as

$$L_o = \overset{*}{L}X = \overset{*}{L}\,(I-A)^{-1}\,Y = \bar{L}Y \tag{A3.23}$$

$$K_o = \overset{*}{K}X = \overset{*}{K}\,(I-A)^{-1}\,Y = \bar{K}Y \tag{A3.24}$$

$$M_o = \overset{*}{M}X = \overset{*}{M}\,(I-A)^{-1}\,Y = \bar{M}Y \tag{A3.25}$$

where L_o, K_o, M_o are scalars of labor, capital and import levels respectively that are consistent with the final demand vector Y, and $\overset{*}{L}$, $\overset{*}{K}$ and $\overset{*}{M}$ are the vectors of productivity ratios, and $\bar{L}$, $\bar{K}$ and $\bar{M}$ are the vectors whose coefficients represent direct plus indirect requirements per dollar of sales to final demand.

If each factor is disaggregated internally, that is, into different occupational or skill categories of labor, different types of capital goods, or end-use categories of imported goods, then

$$L = \overset{**}{L}X = \overset{**}{L}(I-A)^{-1} \quad Y = \overset{=}{L}Y \tag{A3.26}$$

$$K = \overset{**}{K}X = \overset{**}{K}(I-A)^{-1} \quad Y = \overset{=}{K}Y \tag{A3.27}$$

$$M = \overset{**}{M}X = \overset{**}{M}(I-A)^{-1} \quad Y = \overset{=}{M}Y \tag{A3.28}$$

where, X and Y are the same as before, and L, K and M are vectors, but $\overset{**}{L}$, $\overset{**}{K}$ and $\overset{**}{M}$ are now full matrices of factor input coefficients for each type of factor, and $\overset{=}{L}$, $\overset{=}{K}$, $\overset{=}{M}$ are now matrices of direct plus indirect factor requirements.

Employment Multipliers

In equation (A3.23) $\overline{L}$ represents the employment multiplier. It is smaller than the output multiplier B because the non-negative coefficients of $\overset{*}{L}$ are always less than one, that is

$$0 < \lambda_{ij} < 1$$

Therefore, using the summing vector e all of whose elements are unity

$$\overline{L} = \overset{*}{L}(I-A)^{-1} < e(I-A)^{-1} = eB \tag{A3.29}$$

The coefficients of $\overset{*}{L}$ measure the number of person years (or the number of workers) required per unit of output. They are derived from functions relating employment to gross production. These functions are estimated, using linear regression methods, with data from industrial surveys or manufacturing censuses. As such they are not totally consistent with the coefficients of A which are derived separately from the value transactions tables. Occasionally, as in Hansen and Tiebout

(1963), a transactions table is developed which is measured in units of employment rather than money values. This is not, however, a common practice.

Independently estimated productivity trends can sometimes be applied to update the coefficients of $\overset{*}{L}$. This must assume, in a sense, that the other factors are less productive, if $\overset{*}{L} + \overset{*}{K} + \overset{*}{M} = \overset{*}{V}$. But this is difficult to demonstrate because coefficients in physical units are not additive.

Income Multipliers

In order to use factor coefficients that are more consistent with the coefficients of matrix A, the factor coefficients are interpreted in value terms, as the income received by the factors. These can be obtained directly from the value added elements of the transactions table, provided they are aggregated by factor. Then, $\overline{L}$ and $\overline{K}$ would represent domestic income multipliers. $\overline{M}$ is also a type of income multiplier, but not for the income recipients of the economy under study.

The functional distribution of domestic income (i.e., of value added) need not be limited to the two factors, labor and capital. This is not the major problem encountered by analysts using operational input-output models. The main problem is the classification used in national accounts, which separate sectoral value added into wages, salaries, employee's contributions to social security, payments to entrepreneurs and property plus retained earnings, depreciation, and indirect taxes less subsidies. As can be observed, there is some, but not too much correspondence between these categories and the functional distribution of income to the primary factors of production (labor, land, capital, ...); nor is there any direct translation into the institutional categories which characterize the size distribution of personal income.

Thus, despite the obvious simplicity of the equations (A3.23), (A3.24), and (A3.26) and (A3.27), there are still many unresolved problems associated with attempts to graft income distribution and employment considerations onto the typical operational input-output model designed for use in economy-wide policy and planning exercises.

Partially Closing the Open Input-Output Model

The logic of partially closing the model can be illustrated by first solving outputs and resource requirements simultaneously (as in Dorfman, Samuelson and Solow, 1958, p. 260, and Clark, 1975, p. 133), rather than in the two-step procedure required by equations (A3.6) and (A3.7). This can be done by expressing the whole set of equations in partitioned matrix form as:

$$\left[\begin{array}{c|c} I - A & 0 \\ \hline -\overset{*}{V} & I \end{array}\right] \left[\begin{array}{c} X \\ \hline v_o \end{array}\right] = \left[\begin{array}{c} Y \\ \hline 0 \end{array}\right] \qquad \text{(A3.30)}$$

The upper row in the partitioned matrix of the left-hand side represents the basic input-output commodity demand equations (A3.6), while the lower part represents the factor demand equations (A3.7). A linear algebraic rule about the inversion of partitioned matrices[7/] shows that the inverse matrix will also have a north-east corner made up entirely of zeroes,

$$\left[\begin{array}{c|c} I - A & 0 \\ \hline -\overset{*}{V} & I \end{array}\right]^{-1} = \left[\begin{array}{c|c} (I-A)^{-1} & 0 \\ \hline \overset{*}{V}(I-A)^{-1} & I \end{array}\right] \tag{A3.31}$$

The solution of X and v_o are the same as those obtained in (A3.6) and (A3.7). In this form, however, it is easier to see that the vector $\overset{*}{V}$ of factor use does not feed back into the commodity balances in the upper part of the partitioned matrix. This is important because a closed-loop specification in an input-output model involves introducing a northeast quadrant of positive coefficients which is linked to the southwest quadrant of the partitioned matrix via an explicit set of relationships. The elements introduced into the northeast quadrant must have previously formed a part of exogenous demand. Hence, their introduction as an endogenous part of the model will reduce the magnitude of exogenous demand correspondingly, as well as the base from which injections can originate.

In general, whenever the northeast elements of a partitioned matrix are absent by remaining a part of exogenous demand in the commodity balance equations, then the unknowns X and v_o (or a component of v_o)

7/ See Intriligator (1971, p. 488).

can be calculated in sequence, since there will be no feedback linkages. Thus, a simultaneous solution is required to determine the interdependent output and factor use estimates only when an I-O model is 'closed' to complete one or more of the feedback links, because this solution cannot be decomposed into recursive steps. Hence, it is only then that the factor resource allocations can be said to be mutually consistent with the other endogenous variables in the model.

> One purpose of consistency modelling is to increase our knowledge and understanding of how an interdependent system operates. Variables from systems which are fully decomposable, which can be solved recursively, should not be thought of as interdependent. The output pattern is not dependent upon the factor availability or productivity; production may not be consistent with factor employment. Policies affecting resource use will have no impact on supply consistency because no feedback loop exists in the specification. (P. Clark, 1975, p. 134).

There is, however, no advantage to augmenting the open input-output matrix with the vector of factor productivity ratios as long as the northeast quadrant remains zero. In fact, from a computational point of view, there is a significant disadvantage because the iterative solution will converge more slowly. It is well known that the speed of convergence varies inversely with the column sum of the input coefficients.

The Need for the Partial Closure of the Input-Output Model with Respect to Consumption

There are many reasons for wanting to partially close the I-O model. Here we will discuss only a few of the reasons why it might be desirable to close the model with respect to the consumption component of final demand.

First, all the sectoral output, and factor use, forecasts are conditional on the ability to independently project all the national income components of final demand. Thus, it is possible to limit the analysis to the standard technical input-output relations, if the bill of goods can be estimated independently and used as a datum in the open input-output model.

Policy analysts, however, normally want to determine the variations in output, income or employment that are consistent with variations in selected instrumental variables. These instrumental variables tend to be investment rates, government spending rates, the rate of exports, etc. but not consumption rates. On the other hand, consumption usually makes up three quarters, or more, of final demand in most sectors. Thus, it is necessary that the policy analyst spend a considerable amount of effort on predicting sectoral consumption demand, in addition to estimating the sectoral values of the instrumental variables, before the open I-O model can be used. This can be an undesirable added burden.

Second, the procedure of retaining consumption demand as part of the exogenously determined final demand is justified, if the level of income and its use does not depend on the composition of production, that is, if the breakdown of income, generated by each sector, adds nothing to an analysis in more aggregated terms.[8/] However,

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8/ Cases illustrating the dependence or the independence of the level of income and its use on the composition of production can be found in Chenery and Clark (1959, pp. 63-64).

> if corporate savings and business taxes are not zero and are different in each sector, the result will no longer be independent of the composition of final demand ... The aggregate propensity to save will then be a weighted average of the propensities of the various sectors, as shown by Goodwin (1949) and Chipman (1950). If the open model is used in this case, it will usually be necessary to make several trial solutions before a level of income is found that is consistent with the savings-investment equality. (Chenery and Clark, 1959, p. 64).

As Chenery and Clark go on to note, the saving-investment equality of a closed economy without a government sector can be replaced by an equality between autonomous demand (government spending, investment and exports) and primary inputs (taxes, savings, and induced imports)[9/] in an open economy with a government sector.

Third, it is sometimes desirable to investigate the impact on total income, via the induced sectoral demand pattern, of simultaneous changes in income distribution patterns and exogenous non-consumption expenditures. Some studies, using the open model, have assumed changes in the distribution of income in order to alter the composition of the consumption component of final demand and thereby affect the pattern of gross outputs. This, however, is only one direction of causality. As is evident in equation (A3.7), the pattern of gross outputs is a determinant of the demand for factors and hence of the

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9/ Of course if primary inputs include only induced (i.e., non-competitive) imports, then the exports in autonomous demand must be defined as net of competitive imports. These definitions are usually used to maintain the appropriate account for foreign exchange demand. It is however possible, as noted earlier, for autonomous demands to include gross exports only, and for primary inputs to then include all of imports.

factor incomes which are distributed to the providers of factor services. Hence, the pattern of production will influence income distribution, via the factor accounts, when different income groups are identified in the model. Therefore, the pattern of income distribution is affected by the pattern of exogenous expenditures as much as it is by policy parameters which modify endogenous behavior. In this case, the pattern of production and the distribution of income will be determined simultaneously.

We have already noted that consumption demand plays a very important role in determining the level of activity in the economy. In spite of this, sectoral consumption projections cannot be estimated anew repeatedly in light of variations in income distribution patterns consequent upon changes in the pattern of gross outputs. It is therefore desirable, when possible, to take account of the relationships governing consumer behavior in the systematic analysis of general interdependence, rather than to take the outcome of these relations as given.

Augmented Matrix Approach to Closing the Input-Output Model with Respect to Consumption

To determine the level of income (which is dependent on the composition of production) within the model itself, it is necessary to assume aggregate linear consumption functions, in addition to fixed prices, and to tie total consumption demand to some measure of personal income.

In input-output analysis, it is common to equate the sum of sectoral wage payments (from the breakdown of value added) to the total personal consumption expenditure component of final demand, thereby treating the household sector as a "fictitious" industry. Despite some drawbacks, this framework gives explicit recognition to the circular flow of income within an economy. Alternate specifications, derived from a Keynesian framework, are discussed in Chapter 5. They utilize disposable personal income rather than wages, and marginal consumption coefficients rather than average consumption coefficients, as well as an aggregate marginal propensity to consume of less than one (MPC<1), rather than the equality between the 'fictitious' industry's supply and demand of gross output. Only under these conditions is the Leontief form of structural interdependence in production embedded into a broader Keynesian framework.

We will focus here on the formal approach in the context of the traditional input-output method. This consists of augmenting the open model's direct technical coefficient matrix A with a row vector of sectoral factor payment coefficients and a column vector of consumption coefficients based on the sum of these factor payments. In theoretical discussions this augmentation is presented in the following partitioned matrix format:

$$\left[\begin{array}{c} X \\ \hline \overset{*}{w}_o \end{array}\right] = \left[\begin{array}{c|c} A & \overset{*}{C} \\ \hline \overset{*}{W} & 0 \end{array}\right] \left[\begin{array}{c} X \\ \hline w_o \end{array}\right] + \left[\begin{array}{c} \overline{Y} \\ \hline 0 \end{array}\right] \qquad \text{(A3.32)}$$

in which W and A remain as previously defined, and

$\overset{*}{C} = \left[\overset{*}{c}_i = \frac{c_i}{w_o}\right]$ is the vector of average consumption coefficients

$\overline{Y} = Y - C$, is the vector of exogenous final demand less consumption demand

w_o is the scalar sum of sectoral wage payments

$\overset{*}{W} = \left[\overset{*}{w}_j = \frac{w_j}{x_j}\right]$ is the vector of sectoral wage coefficients

In operational input-output models, however, the southeast element of the augmented matrix is generally not zero due to intra-household transactions, and the last element of the exogenous vector of final demands is usually not zero due to wage payments in the government sector. Thus, the zero in the southeast quadrant is replaced by the element ${}^{w}\overset{*}{c}$, and the zero in the exogenous vector by the element ${}^{w}\tilde{y}$, which also necessitates replacing $\overset{*}{\overline{C}}$ by $\overset{*}{C} = \overset{*}{\overline{C}} - {}^{w}\overset{*}{c}$, and $\overline{Y}$ by $\tilde{Y} = \overline{Y} - {}^{w}\tilde{y}$.

The solution to this system of equations, including the primary inputs that are not incorporated into the augmented matrix, can be written as follows:

$$\begin{bmatrix} X \\ \hline w_o \end{bmatrix} = \left[\begin{array}{c|c} I-A & -\overset{*}{C} \\ \hline -\overset{*}{W} & I-{}^{w}\overset{*}{c} \end{array}\right]^{-1} \begin{bmatrix} \tilde{Y} \\ \hline {}^{w}\tilde{y} \end{bmatrix} \tag{A3.33}$$

$$\left(\overline{v}_o\right) = \left(\begin{array}{c|c} \overset{*}{\overline{V}} & \overline{v}\overset{*}{c} \end{array}\right) \begin{bmatrix} X \\ \hline w_o \end{bmatrix} + \left(\overline{v}\tilde{y}\right) \tag{A3.34}$$

Setting $^{a}X = \begin{bmatrix} X \\ \hline W_o \end{bmatrix}$, $^{a}B = \left[\begin{array}{c|c} I-A & -\overset{*}{C} \\ \hline -\overset{*}{W} & I-{}^{w}\overset{*}{c} \end{array}\right]^{-1}$ and $^{a}\tilde{Y} = \begin{bmatrix} \tilde{Y} \\ \hline {}^{w}\tilde{y} \end{bmatrix}$, it is possible to write equation (A3.33) in a more compact form as

$$ {}^{a}X = {}^{a}B \; {}^{a}\tilde{Y} \tag{A3.35} $$

Equation (A3.33) provides a solution to the levels of output and income which are mutually consistent because they are derived from the simultaneous solution of a fully determined system. The level of other primary inputs are calculated, as in equation (A3.34) only after the matrix inversion has been completed and in this sense are not fully consistent with the levels of output and income. The interactive consistency between outputs and income results from the closed loop system in which production leads directly to income determination (by-passing the translation of factor earnings into household income), which leads to expenditure patterns, which leads to production requirements needed both to supply the goods and to create the income flows. The endogenous expenditure system, that is, the non-zero elements in the northeast quadrant, are functionally related to the distribution of income, thereby providing the feedback, and an element of complexity, not appearing in the open input-output model.

Direct, Indirect and Induced Effects

The direct effects of changed exogenous demand are provided by the direct coefficient matrix A; the indirect effects by the difference between the inverse coefficient matrix B and the direct coefficient matrix A, that is, B-A; and the induced effects for sectoral production levels by the difference between the first m-order submatrix of the augmented inverse matrix ${}^{a}B$ and the unaugmented inverse matrix B, that is, ${}^{a}B^{m} - B$.

In addition, as a result of incorporating consumption, coefficients greater than unity are not confined to the on-diagonal elements of the augmented inverse matrix ${}^{a}B$, as they are in the non-augmented inverse matrix B.

Augmented Matrix Approach to Closing the Input-Output Model with Respect to Other Components of Final Demand

> The procedure followed here would apply equally to any other component of final demand which could be assumed to be equal to a certain portion of factor payments and to require inputs in given proportions. If we wish to assume balanced foreign trade and a given export composition, for example, we can construct a foreign trade sector with a row for imports and a column for exports in the same way. Another possibility is to make government expenditure equal to tax receipts. (Chenery and Clark 1959, p. 65)

It is more difficult to make investment endogenous in the input-output flow model. The induced investments in a flow model refer essentially only to replacement of production capacity (Carter and Leontief, 1966, p.) and not to the formation of new capacity which is often the *raison d'etre* of investment expenditures. It is also less plausible to make investment a fixed proportion of the rate of savings as would be required when both vectors are made endogenous in the augmented matrix. Most attempts to incorporate investment expenditures as an endogenous part of the model lead to the creation of dynamic versions of the model, where fixed capital formation responds to the gap created by the demand for productive capacity and its supply.

Iterative Approach to Closing the Input-Output Model with Respect to Consumption

Of the various ways of closing the input-output flow model, the most common is that in which consumption is made endogenous. This is analogous to Lange's approach with the macroeconomic compund multiplier, in which induced consumption is distinguished from autonomous changes in consumption. However, the augmented matrix approach just described presumes that the consumption of a commodity changes in direct proportion to income.[10/] In cases where this is deemed to be an unsuitable assumption, Engel elasticities can be used, with the induced income derived from an open input-output model using an iterative procedure. This approach can be summarized as follows:

> Initial change in final demand (that is, including only the autonomous component of consumption demand) → changes in output requirements → changes in income (direct and indirect income effect) → changes in the consumption of various goods and services (based on independent estimates) → changes in output requirements → changes in income (first-round induced income effect) → changes in the consumption of various goods and services (based on independent estimates) → changes in output requirements → changes in income (second-round induced income effect) → .... (until the desired accuracy is obtained).[11/]

It should be clear that this iterative procedure is separate from the iterative procedure used to approximate the inverse matrix B. It is important to note that this process uses the inverse matrix multiplier from the open input-output model, and the exogenous final demand vector (that is, multiplicand) from the augmented version of the input-output model.

10/ This involves the use of linear consumption functions based on budget shares.

11/ See Yan, (1969), p. 72).

ANNEX A.4

MULTIPLIERS IN SINGLE ECONOMIES

Macroeconomic and input-output multipliers have been used at both the national and the regional level. At the national level, the models from which the multipliers are derived refer to a single economy, without specifying how, and if, this single economy is composed of a system of sub-economies. At the regional level the models from which the multipliers are derived refer to a single economy, but without specifying the sub-economy's role as a component of a larger, national economy.

Formally, the models are identical. Quantitatively, they differ in the magnitude of the multipliers and, in the case of input-output models, in the size of the technical coefficients. The major difference between the two levels of analysis is qualitative. Generally, regional economies are much more open that the national economies in which they are embedded. This requires many adjustments in the interpretation of the concepts and gives rise to many more empirical estimation problems.

Regional Accounts

One of the more obvious and simpler conceptual differences between national and regional analysis is in the area of trade, where regional trade (both exports and imports) includes trade with other regions within the national boundary, as well as the region's share of trade beyond the national boundaries. A more fundamental difference is the problem of selecting an appropriate regional economic unit for which to establish regional accounts. This is both a conceptual problem, in terms of the purpose of analysis and the distinctiveness of a component economy's activities, and also an empirical problem, in terms of the availability of appropriate data. The distinctiveness of economic activity at the national level stems not only from more clearly defined physical and political boundaries but also from the greater sovereignty of the national government with respect to all types of economic policy, such as trade policy, monetary policy, etc. Despite some specialization on the basis of resource endowments, this distinctiveness of economic activity is often lacking at the regional level. The empirical problem arises from the fact that there is no systematic data base. Rather, the data are collected for different purposes by jurisdictions of varying boundaries such as counties, urban areas, watershed districts, political wards, census regions and so on, without any coordination.

Another difference between national and regional accounts, and therefore of the models based on these accounts, stems from the problem, particularly acute in small regional units, of determining where income is earned versus where it is received, and where a product is bought versus where the buyer lives. Each specification of personal income and product accounts will require a different data base and interpretation. Accounting problems occur not only in regional income and product accounts, but also in regional input-output accounts. For example,

> there is a problem in determining the regional location of consumption for an industrial firm or a government body, since the region where the good or service is used may not coincide with the region where it is recorded, as a purchase. The central office of a firm, for example, may record a capital purchase that then appears under the gross private capital formation sector of final demand for the specific region where the office is located, but this capital good may be used by a plant situated in any one of a number of regions in which the firm operates. Similarly with the public sector, the purchase of a good or service may be recorded in the accounts of a federal government agency in Washington, D.C., whereas the actual consumption may occur in one of the regions or even overseas. (K. Polenske, 1972, p. 12).

These few differences should be sufficient to show that the compilation of regional economic accounts,[1/] on which Keynesian and input-output models are based, requires both consideration of special problems not encountered in constructing national accounts, and detailed regional data that are not as readily available as national data.

Aggregate Regional Multipliers

Despite the problems enumerated above, aggregate Keynesian multipliers are not very different at the regional level from those already discussed at the national level in part B. The major difference between the models at the two levels, from which all the other differences flow, is that the closed economy assumption is less realistic at the regional level than it is at the national level.

A special type of aggregate multiplier is the economic base multiplier. It assumes that a stable functional relationship exists between the level of export activity and the level of total economic activity, and is defined as

$$\text{Base multiplier} = \frac{\text{Total employment (or income)}}{\text{Basic employment (or income)}}$$

It is, in a sense, a link between the aggregate Keynesian multipliers and the disaggregated I-O multipliers. Mathematically the economic base multiplier has been shown to be identical to the consolidated closed I-O model multiplier. This was demonstrated by

1/ For a detailed exposition of national and regional accounts see Stone (1961) and Czmanski (1973) respectively.

Billings (1969) and Garnick (1970, pp. 36-8). It was also shown in Chapter Five that the income multiplier of a consolidated closed I-O model in a closed economy is identical to the aggregate Keynesian multiplier.

It has been argued that for certain purposes the economic base multipliers, also known as basic service multipliers, can be used as "cost-effective alternatives to I-O multipliers for small regional impact studies. Indeed there are inexpensive means for augmenting the former such that differential multipliers are derived approximating most of the differential multipliers derived from I-O matrices" (Garnick, 1970, p. 36).

Regional Input-Output Models

There is no real substitute, however, for input-output analysis in larger regions, where exports can no longer be assumed to be the prime mover of a region's level of economic activity, and where changes in the sectoral distribution of autonomous expenditure are as important as changes in its overall magnitude.

The scarcity of regional data and the need to determine stable and reasonably accurate input-output multipliers has resulted in a debate on the merits of using non-survey methods to regionalize national I-O coefficients vs. using the more costly survey methods to estimate regional production coefficients.[2/] This is an important issue.

2/ (See Czamanski and Malizia, 1969; Schaffer and Chu, 1969; Hewings, 1971; Boster and Martin, 1972; Jones, et al. 1973; and McMenamin and Haring, 1974).

The estimate of likely impacts will vary with the magnitude of the multipliers, which will in turn vary with the size of input-output coefficients and the structure of the technical coefficient matrix. In integrated national economies, with few barriers to interregional trade, it can be expected that a certain degree of regional specialization will have occurred to reflect regional resource endowments. This will manifest itself in regional differences in technology and product-mix relative to the average national coefficients. Hence the survey based transactions tables are to be preferred to non-survey based tables but only if the added accuracy can justify the additional time and resources needed by the analyst in light of the problem at hand. The formal properties of the I-O multipliers at the regional level, however, remain the same as those at the national level.

Disaggregated Regional Income and Employment Multipliers

Of the many types of multipliers that can be obtained from the I-O models of single economies, income and employment multipliers tend to be more useful than output multipliers, even though the latter are an integral component of the former.[3/]

Income and employment multipliers can be derived at the regional level, as at the national level, from both the open version of the input-output model and the version closed with respect to consumption. If the latter are based on homogenous (or non-homogenous) consumption

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3/ Output multipliers, as we shall see later, are more important in multiregional input-output analysis in so far as the location of production is important to freight and transportation analysis.

functions derived from national data, then the resulting multipliers will not take account of leakages via regional imports.

In regional analysis it has, however, become common to define the multipliers in a different way from those already described at the national level. The two most frequently used multipliers are referred to as Type I and Type II income or employment multipliers.

The Type I income multiplier is defined as the ratio of the direct plus indirect income change to the direct income change resulting from a unit increase in final demand for any given sector. It can be written as

$$T(I)_j = \frac{\sum_i \overset{*}{w}_{oi} b_{ij}}{\overset{*}{w}_{oj}} \text{, for each industry } j \qquad \text{(A4.1)}$$

where $\overset{*}{w}_{oi}$ and $\overset{*}{w}_{oj}$ are the direct income coefficients and b_{ij} the direct plus indirect coefficient from the inverse matrix

$$B = (I - A)^{-1}.$$

In matrix notation all Type I income multipliers can be expressed in the form

$$T(I) = \overset{*}{W} B \hat{\overset{*}{W}}{}^{-1} \qquad \text{(A4.2)}$$

where $\overset{*}{W}$ is a (1 x m) row vector of direct income coefficients and $\hat{\overset{*}{W}}{}^{-1}$ is an (m x m) diagonal matrix of the coefficients $\left(\frac{1}{\overset{*}{w}_{oj}}\right)$.

The Type II income multiplier is defined as the ratio of the direct plus indirect plus induced income change to the direct income change resulting from an unit increase in the final demand for any given sector.

It can be written as

$$T(II)_j = \frac{{}^{a}b_{kj}}{\overset{*}{w}_{oj}} \quad , \qquad \text{for each industry } j \qquad (A4.3)$$

where ${}^{a}b_{kj}$ is the coefficient from the household row in the augmented matrix

$${}^{a}B = \left[\begin{array}{c|c} I-A & -\overset{*}{C} \\ \hline -\overset{*}{W} & I-{}^{w}\overset{*}{c} \end{array}\right]^{-1} = \left[\begin{array}{c|c} {}^{a}\bar{B} & {}^{a}B_c \\ \hline {}^{a}B_k & b \end{array}\right] \qquad (A4.4)$$

where $\overset{*}{C}$ is the (n x 1) vector of sectoral consumption coefficients and ${}^{w}\overset{*}{c}$ is the scalar for the intrahousehold consumption coefficient.

In matrix notation, all Type II income multipliers can be expressed in the form

$$T(II) = {}^{a}B_k \hat{\overset{*}{W}}^{-1} \qquad (A4.5)$$

where ${}^{a}B_k$ is a row vector of dimension [1x(n+1)], and there is an added element on the diagonal of $\hat{\overset{*}{W}}^{-1}$.

For each of the m industries in the model there is both a Type I and a Type II multiplier. If the consumption fuction used in the augmented I-O model is linear and homogenous, then it is unnecessary to carry out 2m separate calculations, because the relationship between the two types of multipliers is a constant one for a given input-output matrix regardless of the sector in question. The proof of this important finding has been demonstrated by Sandoval (1967), and Bradley and Gander (1969). Skipping the details of the proof in the latter source and using our own notation (for purposes of comparability with subsequent sections), the constant θ resulting from the ratio of the two multipliers can be expressed as

$$\Theta = \frac{T(II)}{T(I)} = \frac{{}^{a}B_{k}}{\overset{*}{W}B} = \frac{1}{1 - (\overset{*}{w}c + \overset{*}{W}\overset{*}{B}\overset{*}{C})} \tag{A4.6}$$

Thus the numerator of T(II) is Θ times the numerator of T(I), that is,

$$ {}^{a}B_{k} = \Theta\overset{*}{W}B \tag{A4.7}$$

Type I and Type II employment multipliers have also been defined for regional economies. The Type I employment multiplier is strictly analogous to the Type I income multiplier. It can be written as

$$\overset{*}{T}(I)_{j} = \frac{\sum_{i} \lambda_{oi} b_{ij}}{\lambda_{oj}}, \qquad \text{for each industry } j \tag{A4.8}$$

where λ_{oi} and λ_{ij} are the direct employment coefficients measured in physical units per unit of output in value terms.

In matrix notation all Type I employment multipliers can be expressed in the form

$$\overset{*}{T}(I) = \overset{*}{L}B\hat{L}^{-1} \tag{A4.9}$$

where $\overset{*}{L}$ is a (1 x m) row vector of direct employment coefficients and $\hat{L}^{-1}$ is an (m x m) diagonal matrix of the coefficients $(1/\lambda_{oj})$.

The Type II employment multiplier, however, is not analogous to the Type II income multiplier. As we noted in the discussion of employment multipliers in the open I-O model at the national level in Annex A-3, the employment coefficients are estimated independently of the technical interindustry production coefficients. As such they are not totally consistent with the latter and cannot be incorporated in the augmented matrix. More importantly, the consumption coefficients of the northeast quadrant of the augmented matrix ${}^{a}B$, which are necessary for 'closing' the model, are linked to a measure of income (wages, disposable income, etc.)

and not to a measure of employment. Hence, the Type II employment multiplier must be defined in analogy to the Type I multipliers using, however, the northwest quadrant of the augmented matrix (to include induced effects), rather than the open model inverse, that is,

$$\overset{*}{T}(II)_j = \frac{\sum_i \lambda_{oi}\, {}^{a}\bar{b}_{ij}}{\lambda_{oj}} \quad , \qquad \text{for each industry } j \tag{A4.10}$$

where ${}^{a}\bar{b}_{ij}$ is a coefficient from the northwest quadrant ${}^{a}\bar{B}$ in equation (A4.4).

In matrix notation the complete set of Type II employment multipliers can be written as

$$\overset{*}{T}(II) = \overset{*}{L}\,{}^{a}\bar{B}\;\hat{L}^{-1} \tag{A4.11}$$

In this case the constant relationship between the numerators of the Type I and Type II multipliers is based on their difference rather than their ratio. That is, if we set $\overset{*}{\theta} = \overset{*}{L}B\overset{*}{C}$, then

$$\overset{*}{L}\,({}^{a}B - B) = \overset{*}{\theta}({}^{a}B_k) \tag{A4.12}$$

or

$$\overset{*}{L}\,{}^{a}B = \overset{*}{L}B + \overset{*}{\theta}({}^{a}B_k) \tag{A4.13}$$

$$= \overset{*}{L}B + \overset{*}{\theta}\Theta\overset{*}{W}B \tag{A4.14}$$

The constants Θ and $\overset{*}{\theta}$ are scalars which are computationally easy to estimate. They can be used to obtain Type II multipliers, bypassing the need to invert the augmented matrix.

Internal & External Spending Multipliers

All the multipliers discussed so far do not clearly distinguish between domestic expenditures on locally produced goods and domestic expenditures on goods produced elsewhere. However, at both the national and regional levels, imports are a major source of leakages in open economies. Since imports are more significant at the regional level, the multipliers tend to be correspondingly smaller than multipliers at the national level. But these multipliers will also often underestimate the actual impact because they do not include feedback effects via trade.

One way to study this additional effect is to model the intereconomy trading pattern. This is often difficult and costly, because it requires domestic or local exports to be specified as a function of the other economies' income or gross output requirements. This would necessitate the inclusion of the production structure or income formation process of the other economies, which is not an easy task when there is more than one economy that trades with the economy under study.

For operational purposes, it is possible to specify exports as a function of local imports by assuming that the component of the income of other economies, which originates in the local spending of the economy under study, is an adequate proxy for the total income of the other economies. This may not be the best way to introduce feedback effects, but it does provide some interesting and useful analytic results.

This analysis, which has been presented systematically by Lange (1943, pp. 232-237), will be outlined here with a change in notation to conform to our other sections. In analogy with the other models already presented for single economies, this model can be interpreted for application to either the national or regional level.[4/] We will limit ourselves to the latter.

In the analysis to be presented, regional income (Y) is defined to include gross regional exports, in addition to regional aggregate demand. As such, it is to be distinguished from gross regional product (GRP), which adds to regional aggregate demand only the net balance of regional trade (which will be positive if gross regional export (GRE) exceeds gross regional imports (GRM), and negative if GRM exceeds GRE). This definition is, however, analogous to that used in input-output studies where all imports are incorporated in a row vector of input purchases.

4/ In fact, the essence of the open economy analysis used in this section is more general than is suggested in the text. It can apply to any open system. In the special case in which the system is an economy, it can apply to an analysis of the relations between _any_ exhaustive set of components that result from different ways of sectoring the economy. Thus, in addition to the traditional treatment in which a country or region has trade relations with other countries or regions, (i.e., international or interregional trade), an open system analysis can be applied to the relation between a country's private economy and the government's Treasury, or to the relation between the private sector and the socialized sector of a mixed economy, etc., with suitable adjustments in the interpretation of the variables.

Let us assume that regional aggregate demand, or the rate of internal spending (A_o), and regional imports, or the rate of external spending (M), are functions of regional income (Y), that is, $A_o = A_o(Y)$ and $M=M(Y)$ and that gross exports, or the rate of external receipts (E), are a function of gross imports, that is, $E=E(M)$. Then we can write the balance equations for total expenditures, or total spending (A), and total receipts, or total income (Y), as

$$A = A_o + M, \tag{A4.15}$$

and

$$Y^{5/} = A_o + E \tag{A4.16}$$

Instead of directly differentiating Y with respect to A_o in (A4.16), we can trace out step by step the consequences of an autonomous incremental change in internal spending (dA_o). At first aggregate income of the economy increases by the amount of the direct injection (dA_o). But of this, only a part $(\frac{dA_o}{dY})\, dA_o$,

based on the marginal propensity to spend for locally produced goods $(\frac{dA_o}{dY})$, is spent within the regional economy, leading to a further increase in the economy's income. Another part $(\frac{dM}{dY})\, dA_o$, based on the marginal propensity to spend for imported goods $(\frac{dM}{dY})$, is spent externally on goods produced elsewhere. From this external spending the amount

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5/ Only when A_o=A-M, from (A4.15) is substituted into (A4.16) do we obtain the GRP identity $\bar{Y} = A - M + E$.

$(\frac{dE}{dM})\ (\frac{dM}{dY})\ dA_o$, flows back to the regional economy, based on the other economies' marginal propensity to spend back $(\frac{dE}{dM})$. The product of the marginal propensities $(\frac{dE}{dM})$ and $(\frac{dM}{dY})$ can be interpreted as the marginal inducement to the other economies to spend back and to contribute thereby to an increase in the income of the regional economy under study. Thus, the initial increment of internal spending dAo leads to feedback from external spending, that is, $(\frac{dE}{dM})\ (\frac{dM}{dY})\ dA_o$, in addition to another round of increases in internal spending, that is, $(\frac{dA_o}{dY})\ dA_o$. The combined initial induced effects can be represented as

$$\left[\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})\right] dA_o$$

Out of this reduced incremental increase,

$$(\frac{dA_o}{dY})\left[\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})\right] dA_o$$

is spent internally, and

$$(\frac{dE}{dM})(\frac{dM}{dY})\left[\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})\right] dA_o$$

of the external spending flows back. Thus, the economy's income increases again by the amount

$$\left[\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})\right]^2 dA_o$$

This analytic process can be repeated until the cumulative increase in the economy's income can be represented by

$$dY = \{1 + [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})] + [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]^2 + \dots\}\ dA_o \qquad \text{(A4.17)}$$

The Internal Spending Multiplier

On the basis of (A4.17) the 'internal spending multiplier', whether the spending originates in regional consumption, investment or government expenditures, can be shown to be

$$\frac{dY}{dA_o} = \frac{1}{1 - [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]} \qquad \text{(A4.18)}$$

It is necessary for the stability of this model that

$$\left|\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})\right| < 1$$

This condition will generally be satisfied if

$$\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY}) > 0$$

is accepted as an empirically established property.

The internal spending multiplier can also be written in an alternate, analytically more useful, form.

On the basis of equation (A4.15) we can write:

$$\frac{dA_o}{dY} = \frac{dA}{dY} - \frac{dM}{dY} \qquad \text{(A4.19)}$$

Since A, Ao and M are all functions of Y, then

$$\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY}) = \frac{dA}{dY} - \frac{dM}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})$$

$$= \frac{dA}{dY} - (1 - \frac{dE}{dM})\frac{dM}{dY} \qquad \text{(A4.20)}$$

We can rewrite equation (A4.18) on the basis of (A4.20) as

$$\frac{dY}{dA_o} = \frac{1}{1 - \{\frac{dA}{dY} - [1 - (\frac{dE}{dM})]\ \frac{dM}{dY}\ \}} \qquad (A4.21)$$

This formulation shows clearly that the internal spending multiplier is smaller, if the marginal propensity to import is larger, and larger, if the other economies' marginal reluctance to spend back, that is, $[1 - (\frac{dE}{dM})]$ is smaller. Thus, it shows the income-reducing effects of imports, as well as the income-increasing effect of exports. Of course, imports will have this effect only if (dM/dY)>0, that is, if they are out of income. For imports, which are out of wealth (i.e., stocks of goods or of money), (dM/dY)=0, and there is no income-reducing effect.

Equation (A4.21) becomes identical to the simple spending multiplier of a closed economy if the other economies' marginal propensity to spend back $(\frac{dE}{dM}) = 1$.

If, on the other hand, $(\frac{dE}{dM}) = 0$, then equation (A4.21) becomes

$$\frac{dY}{dA_o} = \frac{1}{1 - (\frac{dA}{dY}) + (\frac{dM}{dY})} \qquad (A4.22)$$

This is the form commonly encountered in the analysis of foreign trade problems.6/

6/ Instead of treating imports as a function of income, we can treat them as a function of internal spending, that is, $M = M(A_o)$. In that case $(\frac{dM}{dY}) = (\frac{dM}{dA_o})\ (\frac{dAo}{dY})$ and equation (A4.22) must be rewritten in the form $\frac{dY}{dA_o} = \frac{1}{1 - \frac{dA}{dY}\ (\ 1 - \frac{dM}{dY}\ )}$.

The Total Spending Multiplier

If instead of an autonomous increment (dA_o) in the rate of internal spending, we start with an autonomous increment (dA) in the rate of total spending, and follow the same form of argument that led up to equation (A4.17), we get the cumulative or total increase in the economy's income as

$$dY = [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]\ \{1 + [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})] + [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]^2 + \ldots\}dA \quad (A4.22)$$

from which we get the 'total spending multiplier' of an open economy as

$$\frac{dY}{dA} = \frac{\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})}{1 - [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]} \quad (A4.23)$$

The stability condition remains the same. The total spending multiplier can also be written as

$$\frac{dY}{dA} = [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]\ \frac{dY}{dA_o} \quad (A4.24)$$

that is, as the internal spending multiplier times the sum of the marginal propensity to spend internally and the marginal inducement of other economies to spend back. Note that the total spending multiplier is smaller than the internal spending multiplier, that is, $(\frac{dY}{dA}) < (\frac{dY}{dA_o})$, since the stability condition requires that

$$|\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})| < 1.$$

Intersystem Multipliers

(The Effect of Internal Spending upon External Spending)

In this model an autonomous increase (dA_o) in the rate of internal spending raises the economy's income by the amount

$[\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]\ dA_o$, which induces additional external spending equivalent to $(\frac{dM}{dY})\ [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]\ dA_o$.

The economy's income increases again, now by the amount

$$[\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]^2\ dA_o$$

and external spending, in turn, increases by the amount

$$(\frac{dM}{dY})\ [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]^2\ dA_o$$

and so forth. The total increase in external spending is therefore:

$$dM = (\frac{dM}{dY})\{1 + [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})] + [\frac{dA_o}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]^2 + \ldots\}\ dA_o \quad \text{(A4.25)}$$

In light of equation (A4.17) we can rewrite the above relation to obtain the effect of internal spending (dA_o) on external spending (dM) as

$$\frac{dM}{dA_o} = (\frac{dM}{dY})(\frac{dY}{dA_o}) \quad \text{(A4.26)}$$

Analogously the relation

$$\frac{dM}{dA} = (\frac{dM}{dY})(\frac{dY}{dA}) \quad \text{(A4.27)}$$

shows the effect of total spending on external spending. The multipliers in equations (A4.26) and (A4.27) are, in a sense, <u>intersystem</u> internal spending and total spending multipliers respectively, because they measure the marginal effect of a change in a region's rate of internal or total spending upon its imports, and thus upon the external receipts of other economies or regions. Equations (A4.26) and (A4.27) show that the intersystem multiplier is equal to the product of the marginal propensity to spend externally (i.e., to import) and the internal or total spending multiplier.

The Export Multiplier

As noted earlier, both internal spending (A_o) and external receipts (E) are components of income and each can therefore have an expansionary effect on income.

"The effect of a change or an autonomous increment in the rate of external receipts (i.e., export receipts) upon the economy's income is the same as the effect of a change (or increment) of equal size in the rate of internal spending." (O. Lange, 1943, p. 236).

Interpreting dA_o as an autonomous increment in the rate of external receipts, equation (A4.25) can be used to evaluate the marginal increase in the economy's income. Thus, the internal spending multiplier can also be used as an external receipts multiplier, that is, as an export multiplier:

$$\frac{dY}{dE} = \frac{1}{1 - [\frac{dE}{dY} + (\frac{dE}{dM})(\frac{dM}{dY})]} \qquad \text{(A4.28)}$$

where $(\frac{dE}{dY})$ is the marginal propensity to export, suggesting that exports E are a function of Y , that is, E = E(Y).[7]

7/ $(\frac{dE}{dY})$ shows the direct effect of Y on E , while $(\frac{dE}{dM})(\frac{dM}{dY})$ shows the indirect effect of Y on E , via imports M .

With the same interpretation of dA_o, the <u>intersystem</u> multiplier (A4.26) indicates the marginal effect of an increase in the economy 's rate of external receipts (export receipts) upon its external spending (import expenditures). This multiplier may therefore be used to study the marginal effect of a change in the rate of exports upon the region's foreign balance. As can be seen from the formula, this effect is equal to the internal spending (now external receipts) multiplier times the marginal propensity to spend externally (i.e., times the marginal propensity to import). An autonomous increase dE in the rate of exports will increase, leave unchanged, or diminish the foreign balance, depending on whether $\frac{dM}{dE} \gtreqless 1$ (or in other words, depending on whether the reciprocal of the marginal propensity to import is greater than, equal to, or less than the internal spending multiplier).

A special case of equation (A4.26) is when a mechanism, or policy exists to equalize changes in external spending and in external receipts of the economy. This could occur in a context in which there was an absence of capital movements across regional boundaries. In this case, $\frac{dM}{dE} = 1$, that is,

$$\frac{dY}{dE} = \frac{dY}{dA_o} = \frac{1}{dM/dY} \qquad \text{(A4.29)}$$

This is an export multiplier which expresses the effect of a change in the rate of exports upon the region's income and which is equal to the reciprocal of the marginal propensity to import.

ANNEX B.1

DATA AND REFERENCES ON THE THEORETICAL AND OPERATIONAL UNDERPINNINGS OF THE U.S. MRIO MODEL

The MRIO Research Program and Model:

Leontief, W.W., and K. Polenske [1967]. "Multiregional Input-Output Research Program," EDA Report No. 1, July.

Polenske, K.R., [1970]. "A Multiregional Input-Output Model for the United States." EDA Report No. 21 (Harvard Econ. Res. Proj.), December.

Books:

Polenske, K.R. (ed.), [1972]. State Estimates of the Gross National Product, 1947, 1958, 1963, Vol. I of Multiregional Input-Output Analysis, Lexington Books, D.C. Heath and Co., Lexington, Mass.

Rodgers, J.M. [1972]. State Estimates of Outputs, Employment and Payrolls, 1947, 1958, 1963, Vol. II of Multiregional Input-Output Analysis (ed.) K.R. Polenske, Lexington Books, D.C. Heath and Co., Lexington, Mass.

Scheppach, R.C. Jr. [1973]. State Projections of the Gross National Product, 1970, 1980, Vol. III of Multiregional Input-Output Analysis, (ed.) K.R. Polenske, Lexington Books, D.C. Heath and Co., Lexington, Mass.

Rodgers, J.M. [1973]. State Estimates of Interregional Commodity Trade, 1963, Vol. V, Multiregional Input-Output Analysis (ed.) K.R. Polenske, Lexington Books, D.C. Heath and Co., Lexington, Mass.

Polenske, K.R. (ed.) [1974]. State Estimates of Technology, 1963, Vol. IV of Multiregional Input-Output Analysis, Lexington Books, D.C. Heath and Co., Lexington, Mass.

_____ (forthcoming). The United States Multiregional Input-Output Model.

A Users Guide:

Polenske, K.R., C.W. Anderson and M.M. Shirley [1972]. "A Guide for Users of the U.S. Multiregional Input-Output Model." DOT Report No. 2, (revised).

Additional Investigations into the Existence and Stability of the MRIO Models Based on the Three Approaches (Column Coeff., Row Coeff., and Gravity Coeff.) to Estimating the Trade Coefficient Matrix C:

Polenske, K.R. [1969]. "Empirical Implementation of a Multiregional Input-Output Gravity Trade Model," in Contributions to Input-Output Analysis, (ed.) A.P. Carter and A. Brody, Vol. I, North-Holland Publishing Co., Amsterdam, pp. 143-163.

_____ [1972]. "The Implementation of a Multiregional Input-Output Model for the United States" in Input-Output Techniques, (ed.) A. Brody and A.P. Carter, North-Holland Publishing Co., Amsterdam, pp. 171-189.

Fenel, Z. and N. Ng [1974]. "Comparison Tests of the Column Coefficient and the Gravity Coefficient Models." DOT Report No. 6, April.

Bon, R. [1975]. "Some Conditions of Macro-Economic Stability in Multiregional Models." Unpublished Ph.D. thesis, Massachusetts Institute of Technology, May.

Mohr, M. [1975]. "A Consistency Problem of Multiregional Input-Output (MRIO) and Existence Conditions of Constrained Biproportional Matrices." Unpublished Ph.D. thesis, M.I.T., May.

Additional Investigations into the Trade Flow Data:

Kaitz, G.M. [1974]. "Comparison of MRIO Value and Tonnage Trade Flows." MRIO Working Paper No. 1. University Research Program, U.S. Department of Transportation, September.

Mohr, M. [1974]. "Evaluation of the 1963 Interregional Commodity Trade Estimates." DOT Report No. 7, May.

Ng, N.K. [1975]. "Revisions of 1963 Commodity and Service Trade Flows." MRIO Working Paper No. 2, University Research Paper, U.S. Department of Transportation, Sept.

Investigations into Computational Procedures and Problems:

Luft, H.S. [1969]. "Computational Procedure for the Multiregional Model." EDA Report No. 16, Sept.

Cohen, C.P., P.W. Solenberger and G. Tucker [1970]. "Iterative and Inversion Techniques for Solving Large-Scale Multiregional Input-Output Models." EDA Report No. 17, June.

Fenel, Z. [1973]. "Computational Problems with Multiregional Input-Output Models." DOT Report No. 1 August.

TABLE 1. LISTING OF MULTIREGIONAL INPUT-OUTPUT DATA[*]
FOR THE UNITED STATES

Name of Matrix[**]	Matrix Dimension[†]	Years		
1. Final Demand (6 matrices for each year)	88x53	1947	1958	1963
2. Outputs, Employment, and Payrolls[≠]				
Outputs	88x53	1947	1958	1963
Employment	88x53	1947	1958	1963
Payrolls	88x53	1947	1958	1963
3. Projected Final Demands[≠] (6 matrices for each year)	88x53		1970	1980
4. Regional Input-Output Tables (51 matrices)	87x87			1963
5. Interregional Trade Flows (61 matrices)	52x52			1963
6. Projected Outputs[≢]	80x52		1970	1980
7. Projected Interregional Trade Flows (61 matrices for each year)	52x52		1970	1980
8. Regional Secondary Tables (51 matrices)	87x87			1963

[*]For 51 Regions (the 50 states and the District of Columbia) and 79 commodities based on the 87-industry classification scheme of the Bureau of Economic Analysis. Final demand data is available for each of the principal components: Private Consumption Expenditures, Gross Fixed Capital Formation, Government Expenditures, Change in Stocks and Exports, whereas Value Added data is available only for Payrolls/Employment and Other Value Added. Data on Imports and Secondary Transfers are also available.

[**]The data listed in this table are available on computer tapes from the National Technical Information Service, Springfield, Virginia 22151.

[†]The matrix dimensions include row and column sums.

[≠]These data were assembled for the study by Jack Faucett Associates, Inc., under a subcontract with the Harvard Economic Research Project.

[≢]These data are not available on the computer tapes submitted to the NTIS in 1970, but they are available on the revised tapes submitted to the Department of Transportation in 1973.

ANNEX B.2

FIGURES ILLUSTRATING THE COMPONENTS OF THE MRIO MATRIX θ

	Quadrant I Endogenous Purchasing Industries	Quadrant II Exogenous Final Demand	Row Sums
Endogenous Producing Industries	(square) (mxm)	Personal consumption expenditures (PCE) Other domestic aggregate demand Service industry residual (SIR) Exports (foreign demand) Secondary transfers-out (STRO)2/	Total regional consumption of endogenous sectors
	Quadrant III Endogenous Primary Supply	**Quadrant IV** Direct Supply From Endogenous Primary Sectors to Exogenous Final Demand	**Row Sums**
for Endogenous Producing Industries	Wage and salary income Other value added Directly allocated (non-competitive) imports Transferred (competitive) imports Secondary transfers-in (STRI)2/	(square) (kxk)	Total regional consumption of endogenous sectors
Column Sums	Total regional production of endogenous sectors	Total regional demand of exogenous sectors	

Figure B.2.1: Table of interindustry transactions for each region1/

Footnotes for Figure B.2.1:

1) In order to clarify the conceptual structure of the tables for the purposes of this dissertation the format of the presentation in Figure A.1.1. above differs from that used in Figure 4, page 19 of A Guide for Users of the U.S. Multiregional Input-Output Model, by Polenske, K., C. Anderson, and M. Shirley, 1973.

2) "Secondary products are double-counted in the national input-output tables to obtain a balance between total inputs and outputs. This double-counting is accomplished at the national level by estimating a transfer (secondary) matrix. For the multiregional study, the columns of the regional input-output table were first assembled on an establishment basis. The inputs purchased by a particular industry, therefore, consisted of the amounts required to produce its primary and secondary outputs. Secondary outputs were also double-counted in the state input-output tables to obtain an accounting balance, but with only a single column of transfers-out and a single row of transfers-in being added to each input-output table as an exogenous column and row, respectively...

The advantage of adding a single row and column of secondary products as exogenous components of each regional input-output table over alternative procedures is three-fold. First, the estimated input requirements for each region are not obscured by adding on elements from the transfer matrix. Although the primary and secondary matrices are added element by element in the national table, this is entirely an accounting device to assure a balanced table. Second, the production of secondary products by establishments may change significantly over time, and this method permits the researcher to alter the secondary product figures without recalculating the inverse matrix. Third, when industries are aggregated, some of the double-counting can be reduced. In the regional input-output tables constructed for this study, alterations to the tables that result from the reduction of the double-counting are isolated in the secondary row and column of each table. This makes it considerably easier to pinpoint any changes in the table that occur for other reasons.

There are at least two alternative ways in which secondary products could have been handled. First, the transfer row and column could have been placed as part of the inter-industry flow portion of each regional table and a column of coefficients calculated by dividing each flow in the column by the column sum. Second, an entire transfer matrix could have been used for each region, and the technical coefficients could have been calculated for the primary plus the secondary matrices. The advantage of these two methods is that the transfers are assumed to remain constant from year to year, and no special projections of the transfers are required." (Polenske, 1974:26-29).

[The last paragraph actually appears in the work cited as a footnote.]

$$
\begin{array}{cc}
 & \text{Purchasing Industries} \\
\text{Producing Industries} & \begin{bmatrix} a^g_{11} & a^g_{12} & \cdots & a^g_{1m} \\ a^g_{21} & a^g_{22} & \cdots & a^g_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^g_{m1} & a^g_{m2} & \cdots & a^g_{mm} \end{bmatrix}
\end{array}
= \underset{(m \times m)}{A^g}
$$

where $a^g_{ij} = \dfrac{\text{elements of quadrant I}}{\text{column sums}}$, and $\sum_{i=1}^{m} a^g_{ij} < 1$

Figure B.2.2: Matrix A^g of direct technical column coefficients for each region

$$\hat{A} = \begin{bmatrix} \begin{bmatrix} a_{11}^{1} & a_{12}^{1} & \cdots & a_{1m}^{1} \\ a_{21}^{1} & a_{22}^{1} & \cdots & a_{2m}^{1} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{1} & a_{m2}^{1} & \cdots & a_{mm}^{1} \end{bmatrix} & & & \\ & \begin{bmatrix} a_{11}^{2} & a_{12}^{2} & \cdots & a_{1m}^{2} \\ a_{21}^{2} & a_{22}^{2} & \cdots & a_{2m}^{2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{2} & a_{m2}^{2} & \cdots & a_{mm}^{2} \end{bmatrix} & & \\ & & \ddots & \\ & & & \begin{bmatrix} a_{11}^{n} & a_{12}^{n} & \cdots & a_{1m}^{n} \\ a_{21}^{n} & a_{22}^{n} & \cdots & a_{2m}^{n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1}^{n} & a_{m2}^{n} & \cdots & a_{mm}^{n} \end{bmatrix} \end{bmatrix}$$

Figure B.2.3: Expanded technical coefficient matrix $\hat{A}$ for all industries and regions

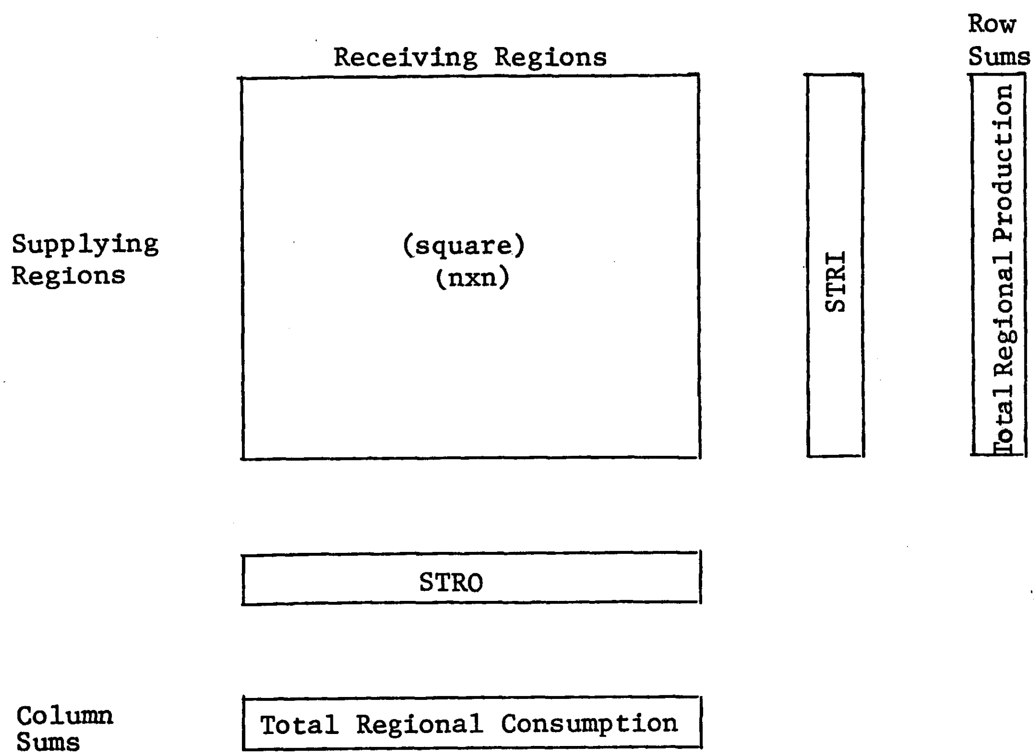

Figure B.2.4: Table of interregional trade flows for each commodity

$$\begin{array}{cc} & \text{Receiving Regions} \\ \begin{array}{c}\text{Supplying} \\ \text{Regions}\end{array} & \begin{bmatrix} c_i^{11} & c_i^{12} & \cdots & c_i^{1n} \\ c_i^{21} & c_i^{22} & \cdots & c_i^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_i^{n1} & c_i^{n2} & \cdots & c_i^{nn} \end{bmatrix} = \underset{(n \times n)}{C_i} \end{array}$$

$$\text{where} \quad c_i^{gh} = \frac{\text{elements of Quadrant I}}{\text{column sums}}, \text{ and} \qquad \sum_{r=1}^{n} c_i^{gh} = 1$$

Figure B.2.5: Matrix C_i of interregional trade flow column coefficients for each commodity

$$\begin{array}{cc} & \text{Receiving Region (h)} \\ \begin{array}{c}\text{Supplying} \\ \text{Region} \\ \text{(g)}\end{array} & \begin{bmatrix} c_1^{gh} & & & \\ & c_2^{gh} & & \\ & & \ddots & \\ & & & c_m^{gh} \end{bmatrix} = \underset{(m \times m)}{\hat{C}^{gh}} \end{array}$$

Figure B.2.6: Diagonal matrix $\hat{C}^{gh}$ of trade flow coefficients between each pair of regions for all commodities

$$
C = \begin{bmatrix}
\begin{bmatrix} c_1^{11} & & & \\ & c_2^{11} & & \\ & & \ddots & \\ & & & c_m^{11} \end{bmatrix} &
\begin{bmatrix} c_1^{12} & & & \\ & c_2^{12} & & \\ & & \ddots & \\ & & & c_m^{12} \end{bmatrix} &
\cdots &
\begin{bmatrix} c_1^{1n} & & & \\ & c_2^{1n} & & \\ & & \ddots & \\ & & & c_m^{1n} \end{bmatrix} \\
\begin{bmatrix} c_1^{21} & & & \\ & c_2^{21} & & \\ & & \ddots & \\ & & & c_m^{21} \end{bmatrix} &
\begin{bmatrix} c_1^{22} & & & \\ & c_2^{22} & & \\ & & \ddots & \\ & & & c_m^{22} \end{bmatrix} &
\cdots &
\begin{bmatrix} c_1^{2n} & & & \\ & c_2^{2n} & & \\ & & \ddots & \\ & & & c_m^{2n} \end{bmatrix} \\
\vdots & \vdots & \ddots & \vdots \\
\begin{bmatrix} c_1^{n1} & & & \\ & c_2^{n1} & & \\ & & \ddots & \\ & & & c_m^{n1} \end{bmatrix} &
\begin{bmatrix} c_1^{n2} & & & \\ & c_2^{n2} & & \\ & & \ddots & \\ & & & c_m^{n2} \end{bmatrix} &
 &
\begin{bmatrix} c_1^{nn} & & & \\ & c_2^{nn} & & \\ & & \ddots & \\ & & & c_m^{nn} \end{bmatrix}
\end{bmatrix}
$$

Figure B.2.7: Expanded interregional trade flow coefficient matrix C for all regions and commodities

$$
\theta = C\hat{A} = \left[\begin{array}{cccc|c|cccc|c|cccc}
c_1^{11}a_{11}^1 & c_1^{11}a_{12}^1 & \cdots & c_1^{11}a_{1m}^1 & c_1^{12}a_{11}^2 & c_1^{12}a_{12}^2 & \cdots & c_1^{12}a_{1m}^2 & \cdots & c_1^{1n}a_{11}^n & c_1^{1n}a_{12}^n & \cdots & c_1^{1n}a_{1m}^n \\
c_2^{11}a_{21}^1 & c_2^{11}a_{22}^1 & \cdots & c_2^{11}a_{2m}^1 & c_2^{12}a_{21}^2 & c_2^{12}a_{22}^2 & \cdots & c_2^{12}a_{2m}^2 & \cdots & c_2^{1n}a_{21}^n & c_2^{1n}a_{22}^n & \cdots & c_2^{1n}a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
c_m^{11}a_{m1}^1 & c_m^{11}a_{m2}^1 & \cdots & c_m^{11}a_{mm}^1 & c_m^{12}a_{m1}^2 & c_m^{12}a_{m2}^2 & \cdots & c_m^{12}a_{mm}^2 & \cdots & c_m^{1n}a_{m1}^n & c_m^{1n}a_{m2}^n & \cdots & c_m^{1n}a_{mm}^n \\
\hline
c_1^{21}a_{11}^1 & c_1^{21}a_{12}^1 & \cdots & c_1^{21}a_{1m}^1 & c_1^{22}a_{11}^2 & c_1^{22}a_{12}^2 & \cdots & c_1^{22}a_{1m}^2 & \cdots & c_1^{2n}a_{11}^n & c_1^{2n}a_{12}^n & \cdots & c_1^{2n}a_{1m}^n \\
c_2^{21}a_{21}^1 & c_2^{21}a_{22}^1 & \cdots & c_2^{21}a_{2m}^1 & c_2^{22}a_{21}^2 & c_2^{22}a_{22}^2 & \cdots & c_2^{22}a_{2m}^2 & \cdots & c_2^{2n}a_{21}^n & c_2^{2n}a_{22}^n & \cdots & c_2^{2n}a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
c_m^{21}a_{m1}^1 & c_m^{21}a_{m2}^1 & \cdots & c_m^{21}a_{mm}^1 & c_m^{22}a_{m1}^2 & c_m^{22}a_{m2}^2 & \cdots & c_m^{22}a_{mm}^2 & \cdots & c_m^{2n}a_{m1}^n & c_m^{2n}a_{m2}^n & \cdots & c_m^{2n}a_{mm}^n \\
\hline
\vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots & \ddots & \vdots & \vdots & & \vdots \\
\hline
c_1^{n1}a_{11}^1 & c_1^{n1}a_{12}^1 & \cdots & c_1^{n1}a_{1m}^1 & c_1^{n2}a_{11}^2 & c_1^{n2}a_{12}^2 & \cdots & c_1^{n2}a_{1m}^2 & \cdots & c_1^{nn}a_{11}^n & c_1^{nn}a_{12}^n & \cdots & c_1^{nn}a_{1m}^n \\
c_2^{n1}a_{21}^1 & c_2^{n1}a_{22}^1 & \cdots & c_2^{n1}a_{2m}^1 & c_2^{n2}a_{21}^2 & c_2^{n2}a_{22}^2 & \cdots & c_2^{n2}a_{2m}^2 & \cdots & c_2^{nn}a_{21}^n & c_2^{nn}a_{22}^n & \cdots & c_2^{nn}a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
c_m^{n1}a_{m1}^1 & c_m^{n1}a_{m2}^1 & \cdots & c_m^{n1}a_{mm}^1 & c_m^{n2}a_{m1}^2 & c_m^{n2}a_{m2}^2 & \cdots & c_m^{n2}a_{mm}^2 & \cdots & c_m^{nn}a_{m1}^n & c_m^{nn}a_{m2}^n & \cdots & c_m^{nn}a_{mm}^n
\end{array}\right]
$$

Figure B.2.8: **Matrix of trade adjusted production coefficients $\theta = C\hat{A}$**

$$
B = \left[\begin{array}{cccc|cccc|c|cccc}
1-c_1^{11}a_{11}^1 & -c_1^{11}a_{12}^1 & \cdots & -c_1^{11}a_{1m}^1 & -c_1^{12}a_{11}^2 & -c_1^{12}a_{12}^2 & \cdots & -c_1^{12}a_{1m}^2 & \cdots & -c_1^{1n}a_{11}^n & -c_1^{1n}a_{12}^n & \cdots & -c_1^{1n}a_{1m}^n \\
-c_2^{11}a_{21}^1 & 1-c_2^{11}a_{22}^1 & \cdots & -c_2^{11}a_{2m}^1 & -c_2^{12}a_{21}^2 & -c_2^{12}a_{22}^2 & \cdots & -c_2^{12}a_{2m}^2 & \cdots & -c_2^{1n}a_{21}^n & -c_2^{1n}a_{22}^n & \cdots & -c_2^{1n}a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
-c_m^{11}a_{m1}^1 & -c_m^{11}a_{m2}^1 & \cdots & 1-c_m^{11}a_{mm}^1 & -c_m^{12}a_{m1}^2 & -c_m^{12}a_{m2}^2 & \cdots & -c_m^{12}a_{mm}^2 & \cdots & -c_m^{1n}a_{m1}^n & -c_m^{1n}a_{m2}^n & \cdots & -c_m^{1n}a_{mm}^n \\
\hline
-c_1^{21}a_{11}^1 & -c_1^{21}a_{12}^1 & \cdots & -c_1^{21}a_{1m}^1 & 1-c_1^{22}a_{11}^2 & -c_1^{22}a_{12}^2 & \cdots & -c_1^{22}a_{1m}^2 & \cdots & -c_1^{2n}a_{11}^n & -c_1^{2n}a_{12}^n & \cdots & -c_1^{2n}a_{1m}^n \\
-c_2^{21}a_{21}^1 & -c_2^{21}a_{22}^1 & \cdots & -c_2^{21}a_{2m}^1 & -c_2^{22}a_{21}^2 & 1-c_2^{22}a_{22}^2 & \cdots & -c_2^{22}a_{2m}^2 & \cdots & -c_2^{2n}a_{21}^n & -c_2^{2n}a_{22}^n & \cdots & -c_2^{2n}a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
-c_m^{21}a_{m1}^1 & -c_m^{21}a_{m2}^1 & \cdots & -c_m^{21}a_{mm}^1 & -c_m^{22}a_{m1}^2 & -c_m^{22}a_{m2}^2 & \cdots & 1-c_m^{22}a_{mm}^2 & \cdots & -c_m^{2n}a_{m1}^n & -c_m^{2n}a_{m2}^n & \cdots & -c_m^{2n}a_{mm}^n \\
\hline
 & \vdots & & & & \vdots & & & \ddots & & \vdots & & \\
\hline
-c_1^{n1}a_{11}^1 & -c_1^{n1}a_{12}^1 & \cdots & -c_1^{n1}a_{1m}^1 & -c_1^{n2}a_{11}^2 & -c_1^{n2}a_{12}^2 & \cdots & -c_1^{n2}a_{1m}^2 & \cdots & 1-c_1^{nn}a_{11}^n & -c_1^{nn}a_{12}^n & \cdots & -c_1^{nn}a_{1m}^n \\
-c_2^{n1}a_{21}^1 & -c_2^{n1}a_{22}^1 & \cdots & -c_2^{n1}a_{2m}^1 & -c_2^{n2}a_{21}^2 & -c_2^{n2}a_{22}^2 & \cdots & -c_2^{n2}a_{2m}^2 & \cdots & -c_2^{nn}a_{21}^n & 1-c_2^{nn}a_{22}^n & \cdots & -c_2^{nn}a_{2m}^n \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
-c_m^{n1}a_{m1}^1 & -c_m^{n1}a_{m2}^1 & \cdots & -c_m^{n1}a_{mm}^1 & -c_m^{n2}a_{m1}^2 & -c_m^{n2}a_{m2}^2 & \cdots & -c_m^{n2}a_{mm}^2 & \cdots & -c_m^{nn}a_{m1}^n & -c_m^{nn}a_{m2}^n & \cdots & 1-c_m^{nn}a_{mm}^n
\end{array}\right]^{-1}
$$

Figure B.2.9: Matrix $B = (I-C\hat{A})^{-1}$

$$D = BC = \left[\begin{array}{cccc|c|cccc}
b^{11}_{11} & b^{11}_{12} & \cdots & b^{11}_{1m} & \cdots & b^{1n}_{11} & b^{1n}_{12} & \cdots & b^{1n}_{1m} \\
b^{11}_{21} & b^{11}_{22} & \cdots & b^{11}_{2m} & \cdots & b^{1n}_{21} & b^{1n}_{22} & \cdots & b^{1n}_{2m} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
b^{11}_{m1} & b^{11}_{m2} & \cdots & b^{11}_{mm} & \cdots & b^{1n}_{m1} & b^{1n}_{m2} & \cdots & b^{1n}_{mm} \\
\hline
 & \vdots & & & \ddots & & \vdots & & \\
\hline
b^{n1}_{11} & b^{n1}_{12} & \cdots & b^{n1}_{1m} & \cdots & b^{nn}_{11} & b^{nn}_{12} & \cdots & b^{nn}_{1m} \\
b^{n1}_{21} & b^{n1}_{22} & \cdots & b^{n1}_{2m} & \cdots & b^{nn}_{21} & b^{nn}_{22} & \cdots & b^{nn}_{2m} \\
\vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
b^{n1}_{m1} & b^{n1}_{m2} & \cdots & b^{n1}_{mm} & \cdots & b^{nn}_{m1} & b^{nn}_{m2} & \cdots & b^{nn}_{mm}
\end{array}\right]
\left[\begin{array}{cccc|c|cccc}
c^{11}_{1} & & & & & c^{1n}_{1} & & & \\
 & c^{11}_{2} & & & & & c^{1n}_{2} & & \\
 & & \ddots & & \cdots & & & \ddots & \\
 & & & c^{11}_{m} & & & & & c^{1n}_{m} \\
\hline
 & \vdots & & & \ddots & & \vdots & & \\
\hline
c^{n1}_{1} & & & & & c^{nn}_{1} & & & \\
 & c^{n1}_{2} & & & & & c^{nn}_{2} & & \\
 & & \ddots & & \cdots & & & \ddots & \\
 & & & c^{n1}_{m} & & & & & c^{nn}_{m}
\end{array}\right]$$

Figure B.2.10a: Matrix D = BC [with matrix B postmultiplied by matrix C, where $B = (I-CA)^{-1}$]

$$
D = \left[\begin{array}{cccc|c}
(b^{11}_{11}c^{11}_{1}+b^{12}_{11}c^{21}_{1}+\ldots+b^{1n}_{11}c^{n1}_{1}) & (b^{11}_{12}c^{11}_{2}+b^{12}_{12}c^{21}_{2}+\ldots+b^{1n}_{12}c^{n1}_{2}) & \ldots & (b^{11}_{1m}c^{11}_{m}+b^{12}_{1m}c^{21}_{m}+\ldots+b^{1n}_{1m}c^{n1}_{m}) & \ldots \\
(b^{11}_{21}c^{11}_{1}+b^{12}_{21}c^{21}_{1}+\ldots+b^{1n}_{21}c^{n1}_{1}) & (b^{11}_{22}c^{11}_{2}+b^{12}_{22}c^{21}_{2}+\ldots+b^{1n}_{22}c^{n1}_{2}) & \ldots & (b^{11}_{2m}c^{11}_{m}+b^{12}_{2m}c^{21}_{m}+\ldots+b^{1n}_{2m}c^{n1}_{m}) & \ldots \\
\vdots & \vdots & & \vdots & \\
(b^{11}_{m1}c^{11}_{1}+b^{12}_{n1}c^{21}_{1}+\ldots+b^{1n}_{m1}c^{n1}_{1}) & (b^{11}_{m2}c^{11}_{2}+b^{12}_{m2}c^{21}_{2}+\ldots+b^{1n}_{m2}c^{n1}_{2}) & \ldots & (b^{11}_{mm}c^{11}_{m}+b^{12}_{mm}c^{21}_{m}+\ldots+b^{1n}_{mm}c^{n1}_{m}) & \ldots \\
\hline
\vdots & \vdots & & \vdots & \\
\hline
(b^{n1}_{11}c^{11}_{1}+b^{n2}_{11}c^{21}_{1}+\ldots+b^{nn}_{11}c^{n1}_{1}) & (b^{n1}_{12}c^{11}_{2}+b^{n2}_{12}c^{21}_{2}+\ldots+b^{1n}_{12}c^{n1}_{2}) & \ldots & (b^{n1}_{1m}c^{11}_{m}+b^{12}_{1m}c^{21}_{m}+\ldots+b^{1n}_{1m}c^{n1}_{m}) & \ldots \\
(b^{n1}_{21}c^{11}_{1}+b^{n2}_{21}c^{21}_{1}+\ldots+b^{nn}_{21}c^{n1}_{1}) & (b^{n1}_{22}c^{11}_{2}+b^{n2}_{22}c^{21}_{2}+\ldots+b^{nn}_{22}c^{n1}_{2}) & \ldots & (b^{n1}_{2m}c^{11}_{m}+b^{n2}_{2m}c^{21}_{m}+\ldots+b^{nn}_{2m}c^{n1}_{m}) & \ldots \\
\vdots & \vdots & & \vdots & \\
(b^{n1}_{m1}c^{11}_{1}+b^{n2}_{m1}c^{21}_{1}+\ldots+b^{nn}_{m1}c^{n1}_{1}) & (b^{n1}_{m2}c^{11}_{2}+b^{n2}_{m2}c^{21}_{2}+\ldots+b^{nn}_{m2}c^{n1}_{2}) & \ldots & (b^{n1}_{mm}c^{11}_{m}+b^{n2}_{mm}c^{21}_{m}+\ldots+b^{nn}_{mm}c^{n1}_{m}) & \ldots
\end{array}\right]
$$

Figure B.2.10b: Matrix D = BC [as the product matrix, where B = $(I-\hat{CA})^{-1}$]
(continued on next page)

$$
\left.\begin{array}{c|ccccc}
\cdots & (b_{11}^{11}c_1^{1n}+b_{11}^{12}c_1^{2n}+\ldots+b_{11}^{1n}c_1^{nn}) & (b_{12}^{11}c_2^{1n}+b_{12}^{12}c_2^{2n}+\ldots+b_{12}^{1n}c_2^{nn}) & \ldots & (b_{1m}^{11}c_m^{1n}+b_{1m}^{12}c_m^{2n}+\ldots+b_{1m}^{1n}c_m^{nn}) \\
\cdots & (b_{21}^{11}c_1^{1n}+b_{21}^{12}c_1^{2n}+\ldots+b_{21}^{1n}c_1^{nn}) & (b_{22}^{11}c_2^{1n}+b_{22}^{12}c_2^{2n}+\ldots+b_{22}^{1n}c_2^{nn}) & \ldots & (b_{2m}^{11}c_m^{1n}+b_{2m}^{12}c_m^{2n}+\ldots+b_{2m}^{1n}c_m^{nn}) \\
 & \vdots & \vdots & & \vdots \\
\cdots & (b_{m1}^{11}c_1^{1n}+b_{m1}^{12}c_1^{2n}+\ldots+b_{m1}^{1n}c_1^{nn}) & (b_{m2}^{11}c_2^{1n}+b_{m2}^{12}c_2^{2n}+\ldots+b_{m2}^{1n}c_2^{nn}) & \ldots & (b_{mm}^{11}c_m^{1n}+b_{mm}^{12}c_m^{2n}+\ldots+b_{mm}^{1n}c_m^{nn}) \\
\hline
 & \vdots & \vdots & & \vdots \\
\hline
\cdots & (b_{11}^{n1}c_1^{1n}+b_{11}^{n2}c_1^{2n}+\ldots+b_{11}^{1n}c_1^{nn}) & (b_{12}^{n1}c_2^{1n}+b_{12}^{n2}c_2^{2n}+\ldots+b_{12}^{nn}c_2^{nn}) & \ldots & (b_{1m}^{n1}c_m^{1n}+b_{1m}^{n2}c_m^{2n}+\ldots+b_{1m}^{nn}c_m^{nn}) \\
\cdots & (b_{21}^{n1}c_1^{1n}+b_{21}^{n2}c_1^{2n}+\ldots+b_{21}^{nn}c_1^{nn}) & (b_{22}^{n1}c_2^{1n}+b_{22}^{n2}c_2^{2n}+\ldots+b_{22}^{nn}c_2^{nn}) & \ldots & (b_{2m}^{n1}c_m^{1n}+b_{2m}^{n2}c_m^{2n}+\ldots+b_{2m}^{nn}c_m^{nn}) \\
 & \vdots & \vdots & & \vdots \\
\cdots & (b_{m1}^{n1}c_1^{1n}+b_{m1}^{n2}c_1^{2n}+\ldots+b_{m1}^{nn}c_1^{nn}) & (b_{m2}^{n1}c_2^{1n}+b_{m2}^{n2}c_2^{2n}+\ldots+b_{m2}^{nn}c_2^{nn}) & \ldots & (b_{mm}^{n1}c_m^{1n}+b_{mm}^{n2}c_m^{2n}+\ldots+b_{mm}^{nn}c_m^{nn})
\end{array}\right]
$$

Figure B.2.10b: Matrix D = BC (Continued from previous page)

$$
\left[\begin{array}{c}
x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hline
x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \\ \hline
\vdots \\ \hline
x_1^n \\ x_2^n \\ \vdots \\ x_m^n
\end{array}\right]
=
\left[\begin{array}{cccc:cccc:c:cccc}
d^{11}_{11} & d^{11}_{12} & \cdots & d^{11}_{1m} & d^{12}_{11} & d^{12}_{12} & \cdots & d^{12}_{1m} & & d^{1n}_{11} & d^{1n}_{12} & \cdots & d^{1n}_{1m} \\
d^{11}_{21} & d^{11}_{22} & \cdots & d^{11}_{2m} & d^{12}_{21} & d^{12}_{22} & \cdots & d^{12}_{2m} & \cdots & d^{1n}_{21} & d^{1n}_{22} & \cdots & d^{1n}_{2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
d^{11}_{m1} & d^{11}_{m2} & \cdots & d^{11}_{mm} & d^{12}_{m1} & d^{12}_{m2} & \cdots & d^{12}_{mm} & & d^{1n}_{m1} & d^{1n}_{m2} & \cdots & d^{1n}_{mm} \\ \hdashline
d^{21}_{11} & d^{21}_{12} & \cdots & d^{21}_{1m} & d^{22}_{11} & d^{22}_{12} & \cdots & d^{22}_{1m} & & d^{1n}_{11} & d^{1n}_{12} & \cdots & d^{1n}_{1m} \\
d^{21}_{21} & d^{21}_{22} & \cdots & d^{21}_{2m} & d^{22}_{21} & d^{22}_{22} & \cdots & d^{22}_{2m} & \cdots & d^{1n}_{21} & d^{1n}_{22} & \cdots & d^{1n}_{1m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
d^{21}_{m1} & d^{21}_{m2} & \cdots & d^{21}_{mm} & d^{22}_{m1} & d^{22}_{m2} & \cdots & d^{22}_{mm} & & d^{1n}_{m1} & d^{1n}_{m2} & \cdots & d^{1n}_{mm} \\ \hdashline
 & \vdots & & & & \vdots & & & \ddots & & \vdots & & \\ \hdashline
d^{n1}_{11} & d^{n1}_{12} & \cdots & d^{n1}_{1m} & d^{n2}_{11} & d^{n2}_{12} & \cdots & d^{n2}_{1m} & & d^{nn}_{11} & d^{nn}_{12} & \cdots & d^{nn}_{1m} \\
d^{n1}_{21} & d^{n1}_{22} & \cdots & d^{n1}_{2m} & d^{n2}_{21} & d^{n2}_{22} & \cdots & d^{n2}_{2m} & \cdots & d^{nn}_{21} & d^{nn}_{22} & \cdots & d^{nn}_{2m} \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & & \vdots & \vdots & \ddots & \vdots \\
d^{n1}_{m1} & d^{n1}_{m2} & \cdots & d^{n1}_{mm} & d^{n2}_{m1} & d^{n2}_{m2} & \cdots & d^{n2}_{mm} & & d^{nn}_{m1} & d^{nn}_{m2} & \cdots & d^{nn}_{mm}
\end{array}\right]
\left[\begin{array}{c}
y_1^1 \\ y_2^1 \\ \vdots \\ y_m^1 \\ \hline
y_1^2 \\ y_2^2 \\ \vdots \\ y_m^2 \\ \hline
\vdots \\ \hline
y_1^n \\ y_2^n \\ \vdots \\ y_m^n
\end{array}\right]
$$

Matrix D is the product of post-multiplying B by C

Figure B.2.11: Solution of open MRIO Model X = DY

ANNEX B.3

SPECIFICATION OF THE OPEN MRIO MODEL SCALAR OUTPUT MULTIPLIERS

The MRIO scalar multipliers are presented in two parts. In the first part (Annex B.3) the MRIO scalar output multipliers are presented in detail, whereas in the second part (Annex B.4), the MRIO scalar income and employment multipliers are presented in outline only.

The MRIO output multipliers are presented sequentially, first in terms of the components of final demand in which the stimuli originate and then, within each of these categories, in terms of the components of gross output which are affected (i.e., to which the multiplier transmits the impact).[1/]

The first four multipliers, cases Ia through Id, and two other multipliers, cases IIb and IId, were initially described in the DiPasquale and Polenske study (1977). They are presented here for

---

[1/] For convenience of exposition and to avoid both the constant use of lengthy terms, such as "the demand for the output of industry j" on the demand side, and the image of animated commodities supplying products on the supply side, the word "commodity" will henceforth be used when referring to the output demanded and the word "industry" when referring to the supplier of that output. This convention is justified in light of the assumption made in all Input-Output models that there are no joint products, that is, an industry produces a homogeneous commodity. The interchangeable use of the terms industry and commodity in this annex has nothing to do with the data sources used in the construction of the model. That is, they do not in any way reflect the fact that commodity data based I-O models, incorporating secondary transfers in their accounts, are different from industry data based I-O models. For a discussion comparing the data structure and results of these two types of I-O models see Bozdogan (1974).

completeness, with greater elaboration and some adjustments in interpretation. The remaining multipliers are presented here for the first time.

Figures B.3.3 to B.3.18 are provided to illustrate each type of multiplier and to clarify the relationship between the scalar multipliers described in this annex and the vector/submatrix multipliers described in chapter 2. These illustrations are based on a three region, three commodity inverse matrix D presented in Figure B.3.1. Figure B.3.2 shows the relationship between the subvector form of the multiplicand and gross output vector, in which composition is important, and the scalar sum form of their respective subvector components, in which the variation in composition is ignored.

In Figures B.3.3-B. .18 the oval elements of the vector Y represent the components of the column subvector that are to be pre-multiplied by the oval elements of the appropriate row subvectors of the D matrix. The oval elements in the X vector will then represent the components of the column subvector resulting from the inner product form of matrix multiplication. Thus, the group of oval elements (or element as the case may be) in the indicated rows and columns of the D matrix represent together the vector or submatrix multiplier that shows by how much the oval elements of the output vector X must change in response to changes in the group of oval elements in the multiplicand final demand vector Y. Note that the other vector and submatrix multipliers within each case are highlighted by uncircled crosses.

Alternately, when weighting or composition effects can be ignored (as is often done when models based on scalar aggregates are used) the

$$
\begin{array}{ccc}
[x_i^g] & [d_{ij}^{gh}] & [y_j^h] \\
\left[\begin{array}{c}
x_{C1}^{R1} \\ x_{C2}^{R1} \\ x_{C3}^{R1} \\ \hline
x_{C1}^{R2} \\ x_{C2}^{R2} \\ x_{C3}^{R2} \\ \hline
x_{C1}^{R3} \\ x_{C2}^{R3} \\ x_{C3}^{R3}
\end{array}\right]
=
&
\left[\begin{array}{ccc:ccc:ccc}
d_{11}^{11} & d_{12}^{11} & d_{13}^{11} & d_{11}^{12} & d_{12}^{12} & d_{13}^{12} & d_{11}^{13} & d_{12}^{13} & d_{13}^{13} \\
d_{21}^{11} & d_{22}^{11} & d_{23}^{11} & d_{21}^{12} & d_{22}^{12} & d_{23}^{12} & d_{21}^{13} & d_{22}^{13} & d_{23}^{13} \\
d_{31}^{11} & d_{32}^{11} & d_{33}^{11} & d_{31}^{12} & d_{32}^{12} & d_{33}^{12} & d_{31}^{13} & d_{32}^{13} & d_{33}^{13} \\ \hdashline
d_{11}^{21} & d_{12}^{21} & d_{13}^{21} & d_{11}^{22} & d_{12}^{22} & d_{13}^{22} & d_{11}^{23} & d_{12}^{23} & d_{13}^{23} \\
d_{21}^{21} & d_{22}^{21} & d_{23}^{21} & d_{21}^{22} & d_{22}^{22} & d_{23}^{22} & d_{21}^{23} & d_{22}^{23} & d_{23}^{23} \\
d_{31}^{21} & d_{32}^{21} & d_{33}^{21} & d_{31}^{22} & d_{32}^{22} & d_{33}^{22} & d_{31}^{23} & d_{32}^{23} & d_{33}^{23} \\ \hdashline
d_{11}^{31} & d_{12}^{31} & d_{13}^{31} & d_{11}^{32} & d_{12}^{32} & d_{13}^{32} & d_{11}^{33} & d_{12}^{33} & d_{13}^{33} \\
d_{21}^{31} & d_{22}^{31} & d_{23}^{31} & d_{21}^{32} & d_{22}^{32} & d_{23}^{32} & d_{21}^{33} & d_{22}^{33} & d_{23}^{33} \\
d_{31}^{31} & d_{32}^{31} & d_{33}^{31} & d_{31}^{32} & d_{32}^{32} & d_{33}^{32} & d_{31}^{33} & d_{32}^{33} & d_{33}^{33}
\end{array}\right]
&
\left[\begin{array}{c}
y_{C1}^{R1} \\ y_{C2}^{R1} \\ y_{C3}^{R1} \\ \hline
y_{C1}^{R2} \\ y_{C2}^{R2} \\ y_{C3}^{R2} \\ \hline
y_{C1}^{R3} \\ y_{C2}^{R3} \\ y_{C3}^{R3}
\end{array}\right]
\end{array}
$$

Figure B. 3.1: X = DY for three Regions and three Commodities

oval elements of the D matrix and the vector Y can be added together (as the inverse coefficients are dimensionless) to obtain the corresponding scalar multipliers and multiplicands, respectively, for each type of vector or submatrix multiplier. The scalar sum of the corresponding elements of the vector X will then be the product obtained from multiplying the scalar multiplier by the scalar multiplicand.

I. Type of Multiplicand: A CHANGED FINAL DEMAND COMPONENT (fdc)

That is a changed final demand for each commodity in each region.

Types of Impacts:

Ia. DETAILED:

For some analyses, it is necessary to determine the effect of a changed final demand for the output of a single industry in a region on the gross output of the same industry, or another industry, in the same region, or in another region. This is the most disaggregated impact that can be determined by using the MRIO model and corresponds to the partial-derivatives $(\partial x_i^g / \partial y_j^h)$ described in chapter 2, that is to the coefficients (d_{ij}^{gh}) of the inverse matrix D. These coefficients will be referred to, henceforth, as "detailed" final-demand-component output-multipliers, $({}^{x}M_{ij}^{gh})$.[2/] The term "detailed" has been chosen to denote the fact that these multipliers are the least aggregated elements contained in the MRIO model since their coordinates specify industrial detail for both the supplying and demanding industry and regional detail for both the region in which the commodity originates and the region to which it is destined. In short, the MRIO "detailed" final-demand-component (fdc) output-multiplier is by definition identical to the MRIO

2/In a departure from the notation in the DiPasquale and Polenske study, the use of a small (x) as a superscript before the multiplier symbol has been adopted to denote an output-multiplier, e.g., ${}^{x}M_{ij}^{gh}$, instead of XM_{ij}^{gh}, to avoid the possibility of confusing a qualifying term used to designate the type of multiplier with the mathematical symbol for a vector (of gross output) which actually enters a mathematical operation. The same convention has been adopted to designate income and employment multipliers, i.e. ${}^{w}M_{ij}^{gh}$ instead of WM_{ij}^{gh}, and ${}^{e}M_{ij}^{gh}$ instead of EM_{ij}^{gh} respectively.

inverse coefficient:

$$x_{M}{}^{gh}_{ij} \equiv d^{gh}_{ij} \qquad \begin{array}{l} g,h = 1,2,\ldots,n \\ i,j = 1,2,\ldots,m \end{array} \qquad (B.3.1)$$

where g and h represent the industry and region, respectively, in which the final demand is located, and i and g the industry and region, respectively, in which the output is produced (see Fig. B.3.3).

In other words, the multiregional inverse coefficient for outputs, (d^{gh}_{ij}) shows the amount of output that has to be produced by industry i in region g to fulfill a unit of final demand for commodity j in region h. When this multiplier is multiplied by the amount of final demand for a commodity j, in region h, then the total amount of output that has to be produced to ensure the model's comparative-static equilibrium by an industry i located in region g is determined. The supplying industry i can be the same as or different from industry j. Similarly, the shipping region g can be the same as or different from the region h in which the demand is located. If ($i=j$) or ($g=h$) then the multiplier shows internal intra-industry or intra-regional effects respectively. For m-industries and n-regions, there are ($mn \times mn$) MRIO "detailed" final-demand-component output-multipliers.

As is shown in Table in Annex E, for the open MRIO model the detailed final demand component output multipliers, including even some of those on the diagonal, are less than 1.0 (in principle it is possible for all of the diagonal coefficients to be less than one). Part of the importance for policy analyses in using the multiregional Input-Output multipliers, instead of the national Input-Output multipliers, now becomes evident. In the national inverse matrix for the open static

Input-Output model, a \$1 increase in the final demand for commodity i will always generate at least \$1 of output by the same industry, since \$1 of output must be sold to final users. As a result all elements on the diagonal of the national inverse matrix are always equal to or greater than 1.0. This national detailed final-demand-component output-multiplier does not, however, indicate in which region the output will be produced to fulfill the changed final demand requirement. It becomes obvious that if the national detailed intra-industry multiplier is larger than 1 but less than 2, as is usually the case, it is not possible for each of the n-regions, of the same economy to all produce more than \$1 of output for the same industry i ; therefore some, if not all, of the multiregional detailed multipliers for the open model will have to be less than 1.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c}
\times \\ 0 \\ \times \\ \hline \times \\ \times \\ \times \\ \hline \times \\ \times \\ \times
\end{array}\right]
\qquad
\left[\begin{array}{ccc|ccc|ccc}
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \hline
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\qquad
\left[\begin{array}{c}
\cdot \\ 0 \\ \cdot \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \cdot \\ \cdot \\ \cdot
\end{array}\right]
$$

$$x_i^g \qquad d_{ij}^{gh} = {}^{x}M_{ij}^{gh} \qquad y_j^h$$

Figure B.3.3: "Detailed" Final-Demand-Component (fdc) Output-Multiplier for $y_j^h = y_{C2}^{R1}$

Ib. INDUSTRY-SPECIFIC:

For some analyses, it is desirable to determine the nationwide impact on an industry which is generated by a changed final demand for one commodity in one region. This multiplier, to be called the MRIO "industry-specific" final-demand-component output-multiplier, is obtained by summing the "detailed" (fdc) output-multipliers for industry i in all the producing regions:

$$x_{M_{ij}}^{oh} = \sum_{g=1}^{n} d_{ij}^{gh} \qquad \begin{array}{r} h = 1,2,\ldots,n \\ i,j = 1,2,\ldots,m \end{array} \qquad \text{(B.3.2)}$$

where all the elements are defined as in equation (B.3.1), and o represents a summation over all regions, that is, a summation of all elements that refer to a specific producing industry in the column of the MRIO inverse that refers to the purchasing industry j in the consuming region h (see Fig. B.3.4). The impact of this multiplier is industry-specific because the dimension of the producing regions has been eliminated (as a result of combining the regions) but that of the producing industry retained. <u>For n-regions and m-industries there will be (m×mn) MRIO "industry-specific final-demand-component output-multipliers</u>.

When these multipliers are multiplied by the changed final demand for commodity j in region h, the total amount of output that has to be produced by industry i in all regions to fulfill the particular changed final demand is determined. These multipliers can be used when a policy analyst or planner needs to know how much output will have to be produced by each industry in the country as a result of a changed final demand for one commodity in one region.

For example, Region 1 will most likely not produce all of Industry 1's output required to fulfill a changed final demand for Commodity 1 in Region 1; instead some of the output required to fulfill the changed final demand will be produced by Industry 1 in Region 2, or Region 3, and then shipped to Region 1. The industry-specific (fdc) output-multiplier represents the total amount of Industry 1's output produced throughout the country to fulfill each dollar of the changed final demand for Commodity 1 in Region 1.

If the purchasing industry j in region h is the same as the supplying industry i then the multiplier ${}^{x}M_{ij}^{oh}$ for (i=j) will be larger than 1 for the same reason that the national detailed output multipliers on the diagonal of the national inverse matrix are larger than 1, namely, because \$1 of its output must be produced just to fulfill the \$1 of final demand (The rest of the amount above 1 represents the direct and indirect intermediate processing input requirements to produce that output.). However, if $i \neq j$ there is no reason for ${}^{x}M_{ij}^{oh}$ to be equal to or greater than one.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} 0 \\ \times \\ \times \\ \hdashline 0 \\ \times \\ \times \\ \hdashline 0 \\ \times \\ \times \end{array}\right]
\qquad
\left[\begin{array}{ccc:ccc:ccc}
\cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
\cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
\cdot & \cdot & \cdot & 0 & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\qquad
\left[\begin{array}{c} \cdot \\ \cdot \\ \cdot \\ \hdashline 0 \\ \cdot \\ \cdot \\ \hdashline \cdot \\ \cdot \\ \cdot \end{array}\right]
$$

$$x_i^o = \sum_g x_i^g \qquad \sum_g d_{ij}^{gh} = {}_xM_{ij}^{oh} \qquad y_j^h$$

Figure B.3.4: "Industry-Specific" Final-Demand-Component (fdc) Output-Multiplier for $y_j^h = y_{C1}^{R2}$

Ic. REGION-SPECIFIC:

A related summary scalar multiplier measure is the multiregional "region-specific" final-demand-component output-multiplier. This multiplier shows by how much all the producing industries in a region must change to maintain the model's comparative-static equilibrium consequent upon a changed final demand for one commodity in one region. It is obtained by summing the detailed (fdc) output-multipliers for all the industries i in the producing region g :

$$ {}^{x}M_{oj}^{gh} = \sum_{i=1}^{m} d_{ij}^{gh} \qquad \begin{array}{r} j = 1,2,\ldots,m \\ g,h = 1,2,\ldots,n \end{array} \qquad \text{(B.3.3)} $$

where all elements are defined as in equation 1 , and o represents a summation over all industries, that is, a summation of all the elements in a regional block of that column of the MRIO inverse matrix D which refers to the purchasing industry j in the consuming region h (see Fig. B. 3.5). The equilibrating impact of this multiplier is region-specific because the dimension of the producing industries has been eliminated (as a result of combining the industries) but that of the producing region retained. <u>For n-regions and m-industries there will be (n×mn) multiregional "region-specific" final-demand-component output-multipliers</u>.

When these multipliers are multiplied by the changed final demand for commodity j in region h , the total amount of output that has to be produced by all industries in each region g in order to fulfill the particular changed final demand is determined. These multipliers can be used when a policy analyst or planner needs to know how much output will have to be produced by each region in a country as a result of a changed final demand for the output of any one industry in any one region.

For example, to fulfill a changed final demand for commodity 1 in Region 2, it is necessary for Region 1 to increase not only its output of commodity 1 but also its output of commodities 2 and 3 if they are required in the production of commodity 1. Thus, the region-specific (fdc) output multiplier represents the total amount of Region 1's output of all commodities necessary to fulfill each dollar of the changed final demand for commodity 1, in Region 2. As for the industry-specific (fdc) multiplier, it is not theoretically necessary for the multiplier ${}^{x}M_{oj}^{gh}$ to be greater than 1 unless the region h in which the demand is located is the same as the region g in which the output is produced, (i.e., g=h).

The industry-specific and region-specific aggregate MRIO scalar output-multipliers may be specially useful to analysts and planners interested in the changes in industrial output nationwide for a specific industry, or in the total output for a specific region, that are consistent with the changed final demand for one commodity in one region. As summary measures, they are less detailed than the detailed output-multipliers, but more detailed than the total output-multipliers to be discussed next.

Output Vector		Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$		$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline \times \\ \times \\ \times \\ \hline \times \\ \times \\ \times \end{array}\right]
=
\left[\begin{array}{ccc:ccc:ccc}
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\qquad
\left[\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \hdashline \cdot \\ \cdot \\ \cdot \\ \hdashline \cdot \\ \cdot \\ \cdot \end{array}\right]
$$

$$x_o^g = \sum_g x_i^g \qquad \sum_i d_{ij}^{gh} = {}^{x}M_{oj}^{gh} \qquad y_j^h$$

Figure B. 3.5: "Region-Specific" Final-Demand-Component (fdc) Output-Multiplier for $y_j^h = y_{C1}^{R1}$

Id. TOTAL IMPACT:

An even more aggregate summary measure of (fdc) output multipliers can now be obtained. In this multiplier, both the regional and industrial origin of output are eliminated; only the region and industry in which the demand originates are retained. Hence, in terms of its equilibrating impact the multiplier is a "total" final-demand-component output-multiplier. It can be specified in a number of ways:

$$x_{M_{oj}}^{oh} = \sum_{g=1}^{n} x_{M_{oj}}^{gh} = \sum_{i=1}^{m} x_{M_{ij}}^{oh} = \sum_{g=1}^{n} \sum_{i=1}^{m} d_{ij}^{gh} \qquad \begin{array}{l} h = 1,2,\ldots,n \\ j = 1,2,\ldots,m \end{array} \qquad (B.3.4)$$

where all elements are as defined in equations (B.3.1),(B.3.2) and (B.3.3). That is, this multiplier can be obtained by adding either the m industry-specific output-multipliers, or the n region-specific output-multipliers, or the mn detailed output-multipliers for industry j in region h . It is, therefore, the sum of all the elements in that column of the inverse coefficient matrix D referring to the purchasing industry j in the consuming region h (see Fig. B.3.6). For m-industries and n-regions, there are (mn) MRIO "total" final-demand-component output-multipliers.

When this multiplier is multiplied by a changed final demand for the output of industry j in region h, the total amount of output that has to be provided nationally, by all industries in all regions, in order to fulfill the changed final demand for one commodity in one region is determined.

This multiplier is a considerably more summary measure than the detailed final-demand-component output-multipliers, as it does not specify how much each industry in each region will have to produce; still, it is important for planners as it sets out the aggregate national gross output implications that are consistent with a changed final demand for one commodity in one region.

<u>Output Vector</u>	<u>Multiplier Matrix of Inverse Coefficients</u>	<u>Multiplicand Vector of Final Demand</u>
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0 \end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & \cdot
\end{array}\right]
\left[\begin{array}{c} \cdot\\ \cdot\\ \cdot\\ \hline \cdot\\ \cdot\\ \cdot\\ \hline \cdot\\ 0\\ \cdot \end{array}\right]
$$

$$x_o^o = \sum\sum_{gi} x_i^g \qquad \sum\sum_{gi} d_{ij}^{gh} = x_{M\,oj}^{\;oh} \quad \cdot \quad y_j^h$$

Figure B.3.6: "Total" Final-Demand-Component (fdc) Output-Multiplier for $y_j^h = y_{C2}^{R3}$

In addition to the final-demand-component (fdc), or elements of the final demand vector, the stimulant for a changed gross output could originate in a changed industrial-demand (id), that is, a changed final demand for a specific commodity j in all regions, or in a changed regional demand (rd), that is, a changed final demand for all commodities in one region, or in a changed national demand, that is, a changed final demand for all commodities in all regions.

Thus, there are 12 other scalar output-multipliers that are analogous to the 4 that have just been described in considerable detail. They differ in terms of the exogenous stimulant, that is, the scalar multiplicand with which they can be combined, but are similar to those already discussed in terms of the impacted scalar sums of the gross output vector, that is, those scalar magnitudes which have to adjust for equilibrium purposes with the corresponding scalar magnitudes of a changed final demand.

These 12 scalar multipliers are presented in outline with limited elaboration on their derivation, interpretation, or use, which should be self-evident by now in light of their analogy to those already described.

<u>II. Type of Multiplicand</u>: A CHANGED "INDUSTRIAL DEMAND" (id)

That is a changed scalar sum of the final demands for one commodity in all regions.

<u>Types of Impacts</u>:

IIa. DETAILED:

This multiplier, to be called the MRIO "detailed" industrial-demand output-multiplier, refers to the equilibrating impact on each producing industry i in each producing region g resulting from a changed scalar final demand for commodity j in all regions. It is defined as:

$$ {}^{x}M^{go}_{ij} = \sum_{h=1}^{n} d^{gh}_{ij} \qquad \begin{array}{r} g = 1,2,\dots,n \\ i,j = 1,2,\dots,m \end{array} $$

that is, as the sum for all consuming regions h of the detailed (fdc) output-multipliers (${}^{x}M^{gh}_{ij}$) (see Fig. B.3.7). <u>For m-industries and n-regions, there are (m×mn) MRIO "detailed" industrial-demand (id) output-multipliers</u>.

These multipliers can be used when a policy analyst is not interested in the regional composition of the changed final demand for each commodity, but is interested in the regional composition of the adjustments that are necessary in the level of each industry's gross output to maintain the model's comparative-static equilibrium.

Output Vector		Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x^g_i]$		$D = (I-C\hat{A})^{-1}C = [d^{gh}_{ij}]$	$Y = [y^h_j]$

$$
\left[\begin{array}{c} \times \\ 0 \\ \times \\ \hline \times \\ \times \\ \times \\ \hline \times \\ \times \\ \times \end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot \\
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\ \hline
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\ \hline
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot \\
\times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot
\end{array}\right]
\left[\begin{array}{c} 0 \\ \cdot \\ \cdot \\ \hline 0 \\ \cdot \\ \cdot \\ \hline 0 \\ \cdot \\ \cdot \end{array}\right]
$$

$$ x^g_i \qquad\qquad \sum_h d^{gh}_{ij} = {}^{x}M^{go}_{ij} \qquad\qquad y^o_j = \sum_h y^h_j $$

Figure B.3.7: "Detailed" Industrial-Demand (id) Output-Multiplier for $y^h_j = y^{R1+R2+R3}_{C1}$

IIb. INDUSTRY-SPECIFIC:

This multiplier, to be called the MRIO "Industry-specific" industrial-demand output multiplier, refers to the equilibrating impact on the scalar sum of a specific producing industry i nationwide (that is, in all producing regions) consequent upon a changed scalar final demand for commodity j nationwide.

It is defined as:

$$x_{M_{ij}}^{oo} = \sum_{h=1}^{n} x_{M_{ij}}^{oh} = \sum_{g=1}^{n} \sum_{h=1}^{n} d_{ij}^{gh} \qquad i,j = 1,2,\ldots,m$$

that is, as the sum for all consuming regions h of the industry-specific (fdc) output-multipliers, or the sum for all producing and consuming regions (g and h, respectively) of the detailed (fdc) output-multipliers (see Fig. B.3.8). For m-industries and n-regions there are (m^2) MRIO "industry-specific" industrial demand output-multipliers.

These multipliers are in essence (though not necessarily in magnitude) equivalent to the detailed final-demand-component output-multipliers of an equivalently-sectored single economy national Input-Output model. This scalar multiplier and those in cases IId, IVb, and IVd, are the only ones that can be obtained from the national Input-Output model.

When ($i=j$) these multipliers show the effect of a scalar sum of a changed final demand for commodity j nationwide on a scalar sum of its own industry's gross output. When ($i \neq j$), these multipliers show the effect of a scalar sum of changed final demand for commodity j nationwide on the scalar sum of the gross outputs of other industries i nationwide.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} 0 \\ \times \\ \times \\ \hline 0 \\ \times \\ \times \\ \hline 0 \\ \times \\ \times \end{array}\right]
=
\left[\begin{array}{ccc:ccc:ccc}
\cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times \\
\cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times \\
\hdashline
\cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times \\
\cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times \\
\hdashline
\cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 \\
\cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times \\
\cdot & \cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times
\end{array}\right]
\left[\begin{array}{c} \cdot \\ \cdot \\ 0 \\ \hline \cdot \\ \cdot \\ 0 \\ \hline \cdot \\ \cdot \\ 0 \end{array}\right]
$$

$x_i^o = \sum_g x_i^g$ $\qquad$ $\sum_{hg}\sum d_{ij}^{gh} = {}^{x}M_{ij}^{oo}$ $\qquad$ $y_j^o = \sum_h y_j^h$

Figure B.3.8: "Industry-Specific" Industrial-Demand (id) Output-Multiplier for $y_j^h = y_{C3}^{R1+R2+R3}$

IIc. REGION-SPECIFIC:

This multiplier, to be called the MRIO "Region-specific" industrial-demand output-multiplier, refers to the equilibrating adjustment in the scalar sum of the gross output of all the producing industries in a specific producing region g, that is required in order to satisfy the changed scalar final demand for commodity j nationwide. It is defined as:

$$ {}^{x}M^{go}_{oj} = \sum_{i=1}^{m} {}^{x}M^{go}_{ij} = \sum_{h=1}^{n} {}^{x}M^{gh}_{oj} = \sum_{i=1}^{m} \sum_{h=1}^{n} d^{gh}_{ij} \qquad \begin{array}{l} g = 1,2,\ldots,n \\ j = 1,2,\ldots,m \end{array} $$

that is, as the sum of the detailed (id) output multipliers for all producing industries i, or the sum of the region-specific (fdc) output multipliers for all consuming regions h, or the sum of the detailed (fdc) output multipliers for all producing industries i and consuming regions h (see Fig. B.3.9). For m-industries and n-regions there are (n x m) MRIO "region-specific" industrial-demand output multipliers.

It can be used to determine the equilibrating change required in the scalar sum of regional output for each region due to a change in the scalar final demand for each commodity nationwide.

Output Vector		Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$		$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} \times \\ \times \\ \times \\ \hline 0 \\ 0 \\ 0 \\ \hline \times \\ \times \\ \times \end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
\cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot \\
\cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot \\
\cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot \\
\hline
\cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot \\
\cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot \\
\cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot \\
\hline
\cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot \\
\cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot \\
\cdot & \times & \cdot & \cdot & \times & \cdot & \cdot & \times & \cdot
\end{array}\right]
\left[\begin{array}{c} \cdot \\ 0 \\ \cdot \\ \hline \cdot \\ 0 \\ \cdot \\ \hline \cdot \\ 0 \\ \cdot \end{array}\right]
$$

$$x_o^g = \sum_i x_i^g \qquad \sum_{hi}\sum d_{ij}^{gh} = {}_{x}M_{oj}^{go} \qquad y_j^o = \sum_h y_j^h$$

Figure B.3.9: "Region-Specific" Industrial-Demand (id) Output-Multiplier for $y_j^h = y_{C2}^{R1+R2+R3}$

IId. TOTAL IMPACT:

This scalar multiplier, the MRIO "total" industrial-demand output-multiplier, refers to the equilibrating impact on the scalar sum of national gross output (that is, the aggregate gross output of all industries in all regions), consequent upon a changed scalar final demand for a single commodity j nationwide. It is defined as:

$$ {}^{x}M^{oo}_{oj} = \sum_{g=1}^{n} {}^{x}M^{go}_{oj} = \sum_{i=1}^{m} {}^{x}M^{oo}_{ij} = \sum_{h=1}^{n} {}^{x}M^{oh}_{oj} = \sum_{g=1}^{n} \sum_{i=1}^{m} \sum_{h=1}^{n} d^{gh}_{ij} \qquad j = 1,2,\ldots,m $$

that is, as the sum for all producing regions g of the "region-specific" (id) output-multipliers, or as the sum for all producing industries i of the industry-specific (rd) output-multipliers, or as the sum for all consuming regions h of the total (fdc) output multipliers (see Fig. B.3.10). <u>For m-industries and n-regions there are (m) MRIO "Total" industrial-demand output-multipliers.</u>

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c}
0\\ 0\\ 0\\ \hline 0\\ 0\\ 0\\ \hline 0\\ 0\\ 0
\end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\ \hline
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\ \hline
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot\\
0 & \cdot & \cdot & 0 & \cdot & \cdot & 0 & \cdot & \cdot
\end{array}\right]
\left[\begin{array}{c}
0\\ \cdot\\ \cdot\\ \hline 0\\ \cdot\\ \cdot\\ \hline 0\\ \cdot\\ \cdot
\end{array}\right]
$$

$$x_o^o = \sum_g\sum_i x_i^g \qquad \sum_h\sum_g\sum_i d_{ij}^{gh} = x_{M\,oj}^{oo} \qquad y_j^o = \sum_h y_j^h$$

Figure B.3.10: "Total" Industrial-Demand (id) Output-Multiplier for $y_j^h = y_{C1}^{R1+R2+R3}$

III. Type of Multiplicand: A CHANGED "REGIONAL DEMAND" (rd)

That is a changed scalar sum of the final demands for all commodities in one region.

Types of Impacts:

IIIa. DETAILED:

This multiplier, to be called the MRIO "detailed" regional-demand output-multiplier, refers to the separate equilibrating impacts on each producing industry i in each producing region g (that is, on a single component of the gross output vector), resulting from a changed scalar final demand for the output of all industries in one region h.

It is defined as:

$$ {}^{x}M_{io}^{gh} = \sum_{j=1}^{m} d_{ij}^{gh} \qquad \begin{array}{l} g,h = 1,2,\ldots,n \\ i = 1,2,\ldots,m \end{array} $$

that is, as the sum for all purchasing industries j of the detailed (fdc) output multipliers (see Fig. B.3.11). For m-industries and n-regions there are (mn×n) MRIO "detailed" regional-demand output-multipliers.

These multipliers can be used when a policy analyst is not interested in the industrial composition of the changed scalar final demand for all commodities in a region, but is interested in the industrial composition of the adjustments required in the gross output of each region. Only when (g=h) can this scalar multiplier and those in cases IIIc, Ia, and Ic, be derived from an equivalently sectored single economy regional Input-Output model.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c}
\times \\ 0 \\ \times \\ \hline \times \\ \times \\ \times \\ \hline \times \\ \times \\ \times
\end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & 0 & 0 & 0 & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\hline
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \times & \times & \times & \cdot & \cdot & \cdot
\end{array}\right]
\left[\begin{array}{c}
\cdot \\ \cdot \\ \cdot \\ \hline 0 \\ 0 \\ 0 \\ \hline \cdot \\ \cdot \\ \cdot
\end{array}\right]
$$

$$x_i^g \qquad \sum_j d_{ij}^{gh} = {}^{x}M_{io}^{gh} \qquad y_o^h = \sum_j y_j^h$$

Figure B.3.11: "Detailed" Regional-Demand (rd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R2}$

IIIb. INDUSTRY-SPECIFIC:

This multiplier, to be called the MRIO "Industry-specific" regional-demand output-multiplier, refers to the equilibrating impact of a changed scalar sum of the demands for all commodities in one region h on the scalar output of each producing industry i nationwide (that is in all regions).

It is defined as:

$$ {}^{x}M_{io}^{oh} = \sum_{g=1}^{n} {}^{x}M_{ij}^{gh} = \sum_{j=1}^{m} {}^{x}M_{ij}^{oh} = \sum_{g=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh} \qquad \begin{array}{l} h = 1,2,\ldots,n \\ i = 1,2,\ldots,m \end{array} $$

that is, as the sum for all producing regions g of the detailed (rd) output-multipliers, or the sum for all purchasing industries j of the industry-specific (dfc) output-multipliers, (see Fig. B.3.12). <u>For m-industries and n-regions there are (m×n) MRIO "industry-specific" regional demand output-multipliers</u>.

These multipliers can be used when the policy analyst is not interested in the industrial composition of the regional demand stimulus or in the regional composition of the gross output of the producing industries that are affected; therefore, it can be used to determine the nationwide effect on each industry of a changed scalar regional final demand.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c}
0 \\ \times \\ \times \\ \hline 0 \\ \times \\ \times \\ \hline 0 \\ \times \\ \times
\end{array}\right]
=
\left[\begin{array}{ccc:ccc:ccc}
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hdashline
0 & 0 & 0 & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\times & \times & \times & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot
\end{array}\right]
\left[\begin{array}{c}
0 \\ 0 \\ 0 \\ \hline \cdot \\ \cdot \\ \cdot \\ \hline \cdot \\ \cdot \\ \cdot
\end{array}\right]
$$

$$x_i^o = \sum_g x_i^g \qquad \sum_j \sum_g d_{ij}^{gh} = {}^{x}M_{io}^{oh} \qquad y_o^h = \sum_j y_j^h$$

Figure B.3.12: "Industry-Specific" Regional-Demand (rd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R1}$

IIIc. REGION-SPECIFIC:

This multiplier, called the MRIO "Region-specific" regional-demand output-multiplier, refers to the equilibrating impact of a changed scalar sum of final demands for all commodities in one region h on the scalar sum of the gross outputs of all industries in each producing region g .

It is defined as:

$$ {}^{x}M^{gh}_{oo} = \sum_{j=1}^{m} {}^{x}M^{gh}_{oj} = \sum_{i=1}^{m} {}^{x}M^{gh}_{io} = \sum_{j=1}^{m} \sum_{i=1}^{m} d^{gh}_{ij} \qquad g,h = 1,2,\ldots,n $$

that is, as the sum for all purchasing industries j of the region-specific (fdc) output multipliers, or as the sum for all producing industries i of the detailed (rd) output-multipliers, etc. (see Fig. B.3.13). <u>For m-industries and n-regions there are (n^2) MRIO "region-specific" regional-demand output-multipliers</u>.

When (g=h) this multiplier will show the effect on the region's own scalar gross output consequent upon a change in its scalar final demand. When (g≠h) the MRIO region-specific (rd) output-multiplier shows the effect on the scalar sum of the gross outputs of other regions g consequent upon a changed scalar regional final demand in a specific region h .

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c}
0\\ 0\\ 0\\ \hline \times\\ \times\\ \times\\ \hline \times\\ \times\\ \times
\end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 0 & 0 & 0\\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \times & \times\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \times & \times\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \times & \times\\ \hline
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \times & \times\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \times & \times\\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \times & \times & \times
\end{array}\right]
\left[\begin{array}{c}
\cdot\\ \cdot\\ \cdot\\ \hline \cdot\\ \cdot\\ \cdot\\ \hline 0\\ 0\\ 0
\end{array}\right]
$$

$x_o^g = \sum_i x_i^g$ $\qquad$ $\sum_j \sum_i d_{ij}^{gh} = {}^{x}M_{oo}^{gh}$ $\qquad$ $y_o^h = \sum_j y_j^h$

Figure B.3.13: "Region-Specific" Regional-Demand (rd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R3}$

IIId. TOTAL IMPACT:

This multiplier, called the MRIO "Total" regional-demand output-multiplier, refers to the equilibrating impact on the scalar sum, gross output nationwide (i.e., for all industries) necessitated by a changed scalar final demand in one region.

It is defined as:

$$ {}^{x}M_{oo}^{oh} = \sum_{g=1}^{n} M_{oo}^{gh} = \sum_{i=1}^{m} M_{io}^{oh} = \sum_{j=1}^{m} M_{oj}^{oh} = \sum_{g=1}^{n} \sum_{i=1}^{m} \sum_{j=1}^{m} d_{ij}^{gh} \qquad h = 1,2,\ldots,n $$

that is, as the sum for all producing regions g of the 'region-specific' (rd) output-multipliers, or the sum for all producing industries i of the industry-specific (rd) output-multipliers, or as the sum for all purchasing industries j of the 'total (fdc) output-multipliers, etc. (see Fig. B.3.14). For m-industries and n-regions there are (n) MRIO "total regional-demand output-multipliers.

This scalar multiplier is useful in determining the requisite change in a scalar of national gross output consequent upon a changed scalar of regional final demand in region h.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$\left[\begin{array}{c} 0\\0\\0\\ \hline 0\\0\\0\\ \hline 0\\0\\0 \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \hline \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \hline \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot\\ \cdot&\cdot&\cdot&0&0&0&\cdot&\cdot&\cdot \end{array}\right] \left[\begin{array}{c} \cdot\\\cdot\\\cdot\\ \hline 0\\0\\0\\ \hline \cdot\\\cdot\\\cdot \end{array}\right]$$

$x_o^o = \sum\limits_{g}\sum\limits_{i} x_i^g$ $\qquad$ $\sum\limits_{j}\sum\limits_{g}\sum\limits_{i} d_{ij}^{gh} = x_{M_{oo}}^{oh}$ $\qquad$ $y_o^h = \sum\limits_{j} y_j^h$

Figure B.3.14: "Total" Regional-Demand (rd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R2}$

IV. Type of Multiplicand: A CHANGED "NATIONAL DEMAND" (nd)

This is a changed scalar sum of the final demands for all commodities in all regions.

Type of Impact:

IVa. DETAILED:

This multiplier, called the "Detailed" national-demand output-multiplier, refers to the equilibrating impact on the gross output of each producing industry i in each producing region g necessitated by a changed scalar sum of national final demand, (that is, for all commodities in all regions).

It is defined as:

$$x_{M}{}^{go}_{io} = \sum_{h=1}^{n} x_{M}{}^{gh}_{io} = \sum_{j=1}^{m} x_{M}{}^{go}_{ij} = \sum_{h=1}^{n} \sum_{j=1}^{m} d^{gh}_{ij} \qquad \begin{array}{l} g = 1,2,\ldots,n \\ i = 1,2,\ldots,m \end{array}$$

that is, as the sum for all consuming regions h of the detailed (rd) output-multipliers, or as the sum for all purchasing industries j of the detailed (id) output-multipliers, or the sum for all purchasing industries j and consuming regions h of the detailed (fdc) output-multipliers (see Fig. B.3.15). For m-industries and n-regions there are (mn) MRIO "detailed" national demand output-multipliers.

This multiplier can be used by a policy analyst to estimate the impact on a single industry in a single region consequent upon a changed national demand which is not differentiated by the sector or location of demand.

Output Vector		Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$		$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} \times \\ 0 \\ \times \\ \hline \times \\ \times \\ \times \\ \hline \times \\ \times \\ \times \end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\hline
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\hline
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times
\end{array}\right]
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]
$$

$$x_i^g \qquad \sum_{hj}\sum d_{ij}^{gh} = {}^{x}M_{io}^{go} \qquad y_o^o = \sum_{hj}\sum y_j^h$$

Figure B.3.15: "Detailed" National-Demand (nd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R1+R2+R3}$

IVb. INDUSTRY-SPECIFIC:

This multiplier, to be called the MRIO "Industry-specific" national-demand output-multiplier, refers to the equilibrating impact on the scalar sum of the gross outputs nationwide of each producing industry i necessitated by a changed scalar final demand nationwide for all commodities.

It is defined as:

$$x_{M_{io}}^{oo} = \sum_{g=1}^{n} x_{M_{io}}^{go} = \sum_{h=1}^{n} x_{M_{io}}^{oh} = \sum_{j=1}^{m} x_{M_{ij}}^{oo} = \sum_{g=1}^{n} \sum_{h=1}^{n} \sum_{j=1}^{m} d_{ij}^{gh} \qquad i = 1,2,\ldots,m$$

that is, as the sum for all producing regions g of the detailed (nd) output-multipliers, or as the sum for all consuming regions h of the industry-specific (rd) output-multipliers, or as the sum for all purchasing industries j of the industry-specific (id) output-multipliers, etc. (see Fig. B.3.16). <u>For m-industries and n-regions there are (m) MRIO "Industry-specific" national-demand output-multipliers</u>.

This multiplier can be used by a policy analyst to estimate the impact on the total output of a single industry in all regions as a consequence of a change in national final demand.

$$\underset{\text{Output Vector}}{X = [x_i^g]} \qquad \underset{\text{Multiplier Matrix of Inverse Coefficients}}{D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]} \qquad \underset{\text{Multiplicand Vector of Final Demand}}{Y = [y_j^h]}$$

$$\left[\begin{array}{c} 0 \\ \times \\ \times \\ \hline 0 \\ \times \\ \times \\ \hline 0 \\ \times \\ \times \end{array}\right] = \left[\begin{array}{ccc|ccc|ccc} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \\ \times & \times & \times & \times & \times & \times & \times & \times & \times \end{array}\right] \left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \\ \hline 0 \\ 0 \\ 0 \end{array}\right]$$

$$x_i^o = \sum_g x_i^g \qquad \sum_{hjg}\sum\sum d_{ij}^{gh} = {}^{x}M_{io}^{oo} \qquad y_o^o = \sum_{hj}\sum y_j^h$$

Figure B.3.16: "Industry-Specific" National-Demand (nd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R1+R2+R3}$

IVc. REGION-SPECIFIC:

This multiplier, called the MRIO "Region-specific" national-demand output-multiplier, refers to the equilibrating impact on the scalar sum of the gross outputs for all industries of each producing region g necessitated by a changed scalar final demand nationwide for all commodities.

It is defined as:

$$x_{M}{}^{go}_{oo} = \sum_{i=1}^{m} x_{M}{}^{go}_{io} = \sum_{h=1}^{n} x_{M}{}^{gh}_{oo} = \sum_{j=1}^{m} x_{M}{}^{go}_{oj} = \sum_{i=1}^{m} \sum_{h=1}^{n} \sum_{j=1}^{m} d^{gh}_{ij} \qquad g = 1,2,\ldots,n$$

that is, as the sum for all producing industries i of the detailed (nd) output-multipliers, or as the sum for all consuming regions h of the region-specific (rd) output-multipliers, or as the sum for all purchasing industries j of the region-specific (id) output-multipliers, etc. (see Fig. B.3.17). For m-industries and n-regions there are (n) MRIO "Region-specific" national-demand output-multipliers.

This multiplier can be used by a policy analyst to estimate the impact on the total output of all industries in a single region as a consequence of a change in national final demand.

Output Vector		Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$		$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} \times \\ \times \\ \times \\ \hdashline \times \\ \times \\ \times \\ \hdashline 0 \\ 0 \\ 0 \end{array}\right]
=
\left[\begin{array}{ccc:ccc:ccc}
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\hdashline
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\times & \times & \times & \times & \times & \times & \times & \times & \times \\
\hdashline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right]
\quad
\left[\begin{array}{c} 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \\ \hdashline 0 \\ 0 \\ 0 \end{array}\right]
$$

$$x_o^g = \sum_i x_i^g \qquad \sum_h \sum_j \sum_i d_{ij}^{gh} = x_{M_{oo}^{go}} \qquad y_o^o = \sum_h \sum_j y_j^h$$

Figure B.3.17: "Region-Specific" National-Demand (nd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R1+R2+R3}$

IVd. TOTAL IMPACT:

This multiplier, called the MRIO "Total" national-demand output-multipliers, refers to the equilibrating impact on the scalar sum of national gross output necessitated by a changed scalar sum of national final demand. It is the most aggregate scalar multiplier that can be obtained from the MRIO model.

It is defined as:

$${}^{x}M^{oo}_{oo} = \sum_{g=1}^{n} {}^{x}M^{go}_{oo} = \sum_{i=1}^{m} {}^{x}M^{oo}_{io} = \sum_{h=1}^{n} {}^{x}M^{oh}_{oo} = \sum_{j=1}^{m} {}^{x}M^{oo}_{oj} = \sum_{g=1}^{n} \sum_{i=1}^{m} \sum_{h=1}^{n} \sum_{j=1}^{m} d^{gh}_{ij}$$

that is, it is the sum of all the coefficients of the MRIO inverse matrix D (see Fig. B.3.18). <u>For m-industries and n-regions there is only one MRIO "Total" national-demand output-multiplier</u>.

This multiplier can be used when the policy analyst is only interested in the magnitude of gross output (undifferentiated by industry or region in which the output is produced) that is consistent with a changed level of national final demand, (again undifferentiated by industry or region in which the demand is located).

From the tabular presentation of MRIO multipliers in table A, chapter 3 the following two figures based on the structure of summations of the coefficients can be abstracted. In Figure B.3.19 (|::) represents (d^{gh}_{ij}) and the position of the (o's) indicate the summations along those specified dimensions.

The information in parenthesis indicates the number of multipliers of that type in an n-region, m-industry MRIO model.

Output Vector	Multiplier Matrix of Inverse Coefficients	Multiplicand Vector of Final Demand
$X = [x_i^g]$	$D = (I-C\hat{A})^{-1}C = [d_{ij}^{gh}]$	$Y = [y_j^h]$

$$
\left[\begin{array}{c} 0\\0\\0\\\hline 0\\0\\0\\\hline 0\\0\\0 \end{array}\right]
=
\left[\begin{array}{ccc|ccc|ccc}
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\\hline
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\\hline
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0
\end{array}\right]
\left[\begin{array}{c} 0\\0\\0\\\hline 0\\0\\0\\\hline 0\\0\\0 \end{array}\right]
$$

$$x_o^o = \sum_{gi}\sum x_i^g \qquad \sum_{hjgi}\sum\sum\sum d_{ij}^{gh} = {}^{x}M_{oo}^{oo} \qquad y_o^o = \sum_{hj}\sum y_j^h$$

Figure B.3.18: "Total" National-Demand (nd) Output-Multiplier for $y_j^h = y_{C1+C2+C3}^{R1+R2+R3}$

	[mn]	[m]	[n]	[1]
[mn]	.. .. (mn×mn)	o. .. (m×mn)	.. o. (n×mn)	o. o. (mn)
[m]	.o .. (mn×n)	oo .. (m×m)	.o o. (n×m)	oo o. (m)
[n]	.. .o (mn×n)	o. .o (m×n)	.. oo (n×n)	o. oo (n)
[1]	.o .o (mn)	oo .o (m)	.o oo (n)	oo oo (1)

Figure B.3.19: An illustration of the structure of summations for the scalar multipliers and their numbers

The more abstract pattern of summations in Figure B.3.20 provides another logical basis for classifying the types of multiplier.

	o	o	oo
o	oo	oo	oo o
o	oo	oo	oo o
oo	oo o	oo o	oo oo

Figure B.3.20: The abstract pattern of summations

For example, to circumvent cumbersome notation problems when referring to a specific multiplier, one could refer to them as output multipliers a,b,c,d,... that is,

$${}_{a}x_{M} = x_{M}{}^{gh}_{ij} \qquad \text{output multiplier with no summations}$$

$${}^{x}_{b}M = {}^{x}M^{oh}_{ij}$$

$${}^{x}_{c}M = {}^{x}M^{go}_{ij}$$

$${}^{x}_{d}M = {}^{x}M^{gh}_{oj}$$

$${}^{x}_{e}M = {}^{x}M^{gh}_{io}$$

output multipliers with single summations

etc. This would be obtained by lettering Figure B.3.20 in sequence from bottom left to top right for each summation level on the "earliest" minor diagonal as in Figure B.3.21:

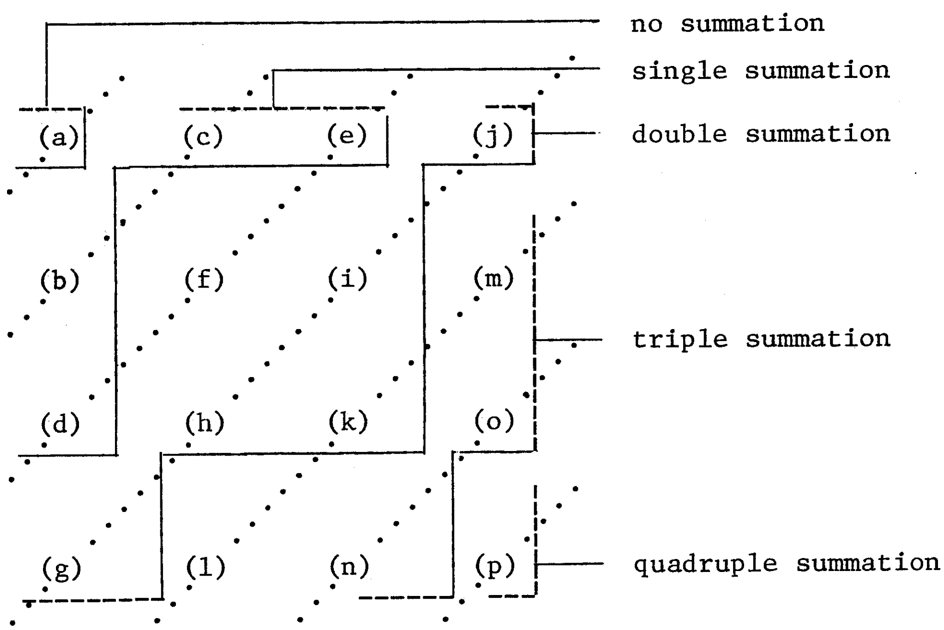

Figure B.3.21: An alternate notation for the different scalar multipliers

ANNEX B.4

SPECIFICATION OF SELECTED OPEN MRIO MODEL SCALAR INCOME AND EMPLOYMENT MULTIPLIERS

All MRIO scalar income and employment multipliers are based on multiplying the various MRIO scalar output multipliers by appropriate direct income or employment coefficients (or coefficient-sums to correspond to the dimension along which the output multiplier has been summed). There is little to be gained from a detailed discussion of these multipliers in light of the presentation in Annex B.3. Hence, only a selected number are defined in this annex with the notation explained at the end.

Selected MRIO Scalar Wage and Salary Income-Multipliers

I. Type of Multiplicand: A CHANGED "FINAL DEMAND COMPONENT (fdc)

Types of Impacts:

Ia. DETAILED:

$$ {}^{w}M^{gh}_{ij} = \left[\frac{x^{g}_{(w)i}}{x^{g}_{i}}\right] d^{gh}_{ij} = \overset{*}{w}{}^{g}_{i}\,({}^{x}M^{gh}_{ij}) \qquad \begin{matrix} i,j = 1,\dots,m \\ g,h = 1,\dots,n \end{matrix} \qquad \text{(B.4.1)} $$

shows the total amount of wage and salary income generated by industry i in region g in response to a unit change in the demand for commodity j in region h.

Ib. INDUSTRY-SPECIFIC:

$$ {}^{w}M^{oh}_{ij} = \sum_{g=1}^{n} {}^{w}M^{gh}_{ij} = \sum_{g=1}^{n} \left\{\left[\frac{x^{g}_{(w)i}}{x^{g}_{i}}\right]\right\} d^{gh}_{ij} = \overset{*}{w}{}^{o}_{i}\,({}^{x}M^{oh}_{ij}) \qquad \begin{matrix} i,j = 1,\dots,m \\ h = 1,\dots,n \end{matrix} \qquad \text{(B.4.2)} $$

shows the total amount of wage and salary income generated by industry i in all regions in response to a unit change in the demand for commodity j in region h.

Ic. REGION-SPECIFIC:

$$ {}^{w}M^{gh}_{oj} = \sum_{i=1}^{m} {}^{w}M^{gh}_{ij} = \sum_{i=1}^{m} \left\{ \left[\frac{x^{g}_{(w)i}}{x^{g}_{i}} \right] \right\} d^{gh}_{ij} = \overset{*}{w}{}^{g}_{o} \, ({}^{x}M^{gh}_{oj}) \qquad \begin{array}{l} j = 1,\ldots,m \\ h,g = 1,\ldots,n \end{array} \qquad (B.4.3) $$

shows the total amount of wage and salary income generated by all industries in region g in response to a unit change in the demand for commodity j in region h.

Id. TOTAL:

$$ {}^{w}M^{oh}_{oj} = \sum_{i=1}^{m} {}^{w}M^{oh}_{ij} = \sum_{g=1}^{n} {}^{w}M^{gh}_{oj} = \sum_{i=1}^{m} \sum_{g=1}^{n} {}^{w}M^{gh}_{ij} = \overset{*}{w}{}^{o}_{o} ({}^{x}M^{oh}_{oj}) \qquad (B.4.4) $$

$$ \begin{array}{l} j = 1,\ldots,m \\ h = 1,\ldots,n \end{array} $$

shows the total amount of wage and salary income generated by all industries in all regions in response to a unit change in the demand for commodity j in region h.

Selected MRIO Scalar Employment-Multipliers

I. Type of Multiplicand: A CHANGED 'FINAL DEMAND COMPONENT' (fdc)

Types of Impacts:

Ia. DETAILED:

$$ {}^{e}M^{gh}_{ij} = \left[\frac{x^{g}_{(e)i}}{x^{g}_{i}} \right] d^{gh}_{ij} = \overset{*g}{e_i} ({}^{x}M^{gh}_{ij}) \qquad \begin{array}{l} i,j = 1,\dots,m \\ g,h = 1,\dots,n \end{array} \quad (B.4.5) $$

shows the amount of employment generated by industry i in region g in response to a unit change in the demand for commodity j in region h.

Ib. INDUSTRY-SPECIFIC:

$$ {}^{e}M^{oh}_{ij} = \sum_{g=1}^{n} {}^{e}M^{gh}_{ij} = \sum_{g=1}^{n} \left\{ \left[\frac{x^{g}_{(e)i}}{x^{g}_{i}} \right] \right\} d^{gh}_{ij} = \overset{*o}{e} ({}^{x}M^{oh}_{ij}) \quad \begin{array}{l} i,j = 1,\dots,m \\ h = 1,\dots,n \end{array} \quad (B.4.6) $$

shows the amount of employment generated by industry i in all regions in response to a unit change in the demand for commodity j in region h.

II. Type of Multiplicand: A CHANGED 'INDUSTRIAL DEMAND' (id)

Types of Impacts

IIc. REGION-SPECIFIC

$$ {}^{e}M^{go}_{oj} = \sum_{h=1}^{n} \sum_{i=1}^{m} {}^{e}M^{gh}_{ij} = \sum_{h=1}^{n} \sum_{i=1}^{m} \left\{ \left[\frac{x^{g}_{(e)i}}{x^{g}_{1}} \right] \right\} d^{gh}_{ij} = \overset{*g}{e_o} ({}^{x}M^{go}_{oj}) \qquad (B.4.7) $$

$$ \begin{array}{r} j = 1,\dots,m \\ g,h = 1,\dots,n \end{array} $$

shows the amount of employment generated by all industries in region g in response to a unit change in the demand for commodity j in all regions.

III. <u>Type of Multiplicand</u>: A CHANGED 'REGIONAL DEMAND' (rd)

<u>Types of Impact</u>

IIId. TOTAL

$$^{e}M^{oh}_{oo} = \sum_{j=1}^{m} \sum_{g=1}^{m} {}^{e}M^{gh}_{oj} = \sum_{j=1}^{m} \sum_{i=1}^{m} {}^{e}M^{oh}_{ij} = \sum_{j=1}^{m} \sum_{g=1}^{n} \sum_{i=1}^{m} {}^{e}M^{gh}_{ij}$$

or

$$^{e}M^{oh}_{oo} = \overset{*o}{e_o} \left[{}^{x}M^{oh}_{oo} \right] \qquad h = 1,\ldots,n \qquad \text{(B.4.8)}$$

shows the amount of employment generated by all industries in all regions in response to a unit change in the demand for all commodities in region h.

NOTATION

The notation for the equation of the MRIO open model wage and salary income, and employment-multipliers are:

d^{gh}_{ij} a coefficient of the MRIO inverse technical coefficient matrix which shows the per unit amount of output that has to be generated by industry i in region g to fulfill final demand requirements in region h for the products of industry j.

$^{x}M^{gh}_{ij}$ the output-multiplier that shows by how much gross output in the producing industry i in the supplying region g must change to be consistent with a changed final demand in region h for commodity j.

$^{e}M^{gh}_{ij}$ the employment-multiplier that shows by how much employment in the producing industry i in the supplying region g must change to be consistent with a changed final demand in region h for commodity j.

${}^{w}M_{ij}^{gh}$ the 'wage and salary' income-multiplier that shows by how much 'wage and salary' income in the producing industry i in the supplying region g must change to be consistent with a changed final demand in region h for commodity j. It is a component of ${}^{v}M_{ij}^{gh}$ (the value added multiplier), whereas strictly speaking ${}^{e}M_{ij}^{gh}$ is not a component of ${}^{v}M_{ij}^{gh}$.

$\overset{*}{w}{}_{i}^{g}$ the wage and salary income coefficient $\left[\frac{x_{(w)i}^{g}}{x_{io}^{g}}\right] = \left[\frac{x_{(w)i}^{g}}{x_{oi}^{g}}\right]$, since $x_{io}^{g} = x_{oi}^{g}$

where $x_{(w)i}^{g}$ is total wage and salary income in the i^{th} industry in region g, and x_{io}^{g} the total output of that industry in that region.

$\overset{*}{e}{}_{i}^{g}$ the employment coefficient $\left[\frac{x_{(e)i}^{g}}{x_{io}^{g}}\right] = \left[\frac{x_{(e)i}^{g}}{x_{oi}^{g}}\right]$, since $x_{io}^{g} = x_{oi}^{g}$

where $x_{(e)i}^{g}$ is the total employment in the i^{th} industry in region g.

i,j the (m) producing and purchasing industries, respectively (used as subscripts).

g,h the (n) shipping and receiving regions, respectively (used as superscripts).

o the summation over all regions or industries.

ANNEX C.1

FIGURES ILLUSTRATING THE COMPONENTS OF THE AUGMENTED MRIO MATRIX ${}^{a}\theta$

$$
{}^{a}A^{g} = \left[\begin{array}{c:c} a^{g}_{ij} & \overset{*}{c}{}^{g}_{ik} \\ \hdashline \overset{*}{w}{}^{g}_{kj} & \overset{*}{z}{}^{g}_{kk} \end{array}\right] = \left[\begin{array}{cccc:c} a^{g}_{11} & a^{g}_{12} & \cdots & a^{g}_{1m} & \overset{*}{c}{}^{g}_{1k} \\ a^{g}_{21} & a^{g}_{22} & \cdots & a^{g}_{2m} & \overset{*}{c}{}^{g}_{2k} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a^{g}_{m1} & a^{g}_{m2} & \cdots & a^{g}_{mm} & \overset{*}{c}{}^{g}_{mk} \\ \hdashline w^{g}_{k1} & w^{g}_{k2} & \cdots & \overset{*}{w}{}^{g}_{km} & \overset{*}{w}{}^{g}_{kk} \end{array}\right]
$$

Figure C.1.1: Matrix ${}^{a}A^{g}$ of direct technical coefficients for each region augmented by a column of consumption coefficients and a row of wage and salary income coefficients

$$
{}^{a}\hat{A} = \left[\begin{array}{ccc} \left[\begin{array}{c:c} a^{1}_{ij} & \overset{*}{c}{}^{1}_{ik} \\ \hdashline \overset{*}{w}{}^{1}_{kj} & \overset{*}{z}{}^{1}_{kk} \end{array}\right] & & 0 \\ & \ddots & \\ 0 & & \left[\begin{array}{c:c} a^{n}_{ij} & \overset{*}{c}{}^{n}_{ik} \\ \hdashline \overset{*}{w}{}^{n}_{kj} & \overset{*}{z}{}^{n}_{kk} \end{array}\right] \end{array}\right]
$$

Figure C.1.2a: Expanded matrix ${}^{a}\hat{A}$ of augmented technical coefficient matrices for all regions (See also Fig. C.1.2b)

$$
{}^{a}\hat{A} = \left[\begin{array}{ccc}
\left[\begin{array}{cccc|c}
a^{1}_{11} & a^{1}_{12} & \cdots & a^{1}_{1m} & c^{*1}_{1} \\
a^{1}_{21} & a^{1}_{22} & \cdots & a^{1}_{2m} & c^{*1}_{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a^{1}_{m1} & a^{1}_{m2} & \cdots & a^{1}_{mm} & c^{*1}_{m} \\
\hline
w^{*1}_{1} & w^{*1}_{2} & \cdots & w^{*1}_{m} & z^{*1}
\end{array}\right] & & \\
& \left[\begin{array}{cccc|c}
a^{2}_{11} & a^{2}_{12} & \cdots & a^{2}_{1m} & c^{*2}_{1} \\
a^{2}_{21} & a^{2}_{22} & \cdots & a^{2}_{2m} & c^{*2}_{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a^{2}_{m1} & a^{2}_{m2} & \cdots & a^{2}_{mm} & c^{*2}_{m} \\
\hline
w^{*2}_{1} & w^{*2}_{2} & \cdots & w^{*2}_{m} & z^{*2}
\end{array}\right] & \\
& & \ddots \\
& & \left[\begin{array}{cccc|c}
a^{n}_{11} & a^{n}_{12} & \cdots & a^{n}_{1m} & c^{*n}_{1} \\
a^{n}_{21} & a^{n}_{22} & \cdots & a^{n}_{2m} & c^{*n}_{2} \\
\vdots & \vdots & \ddots & \vdots & \vdots \\
a^{n}_{m1} & a^{n}_{m2} & \cdots & a^{n}_{mm} & c^{*n}_{m} \\
\hline
w^{*n}_{1} & w^{*n}_{2} & \cdots & w^{*n}_{m} & z^{*n}
\end{array}\right]
\end{array}\right]
$$

Figure C.1.2b: Expanded matrix ${}^{a}\hat{A}$ of augmented technical coefficient matrices for all regions.

$$C_k = \begin{bmatrix} c_k^{rr} & \cdots & c_k^{rs} \\ \vdots & \ddots & \vdots \\ c_k^{sr} & \cdots & c_k^{ss} \end{bmatrix}$$

Figure C.1.3: Matrix C_k of interregional 'trade flow' coefficients for the 'industry' that is to be augmented.

$$^{a}\hat{C}^{gh} = \left[\begin{array}{c|c} \hat{c}_i^{gh} & 0 \\ \hline 0 & c_k^{gh} \end{array}\right] = \left[\begin{array}{ccc|c} c_1^{gh} & & 0 & \\ & \ddots & & 0 \\ 0 & & c_m^{gh} & \\ \hline & 0 & & c_k^{gh} \end{array}\right]$$

Figure C.1.4: Diagonal matrix $^{a}\hat{C}^{gh}$ of augmented trade flow coefficients between each pair of regions for all commodities $(1,2,\ldots,m,k)$

$$
{}^{a}C = \left[\begin{array}{c|c|c|c}
\begin{array}{cccc:c} c_1^{11} &&&& \\ & c_2^{11} &&& \\ && \ddots && \\ &&& c_m^{11} & \\ \hdashline &&&& c_k^{11} \end{array} &
\begin{array}{cccc:c} c_1^{12} &&&& \\ & c_2^{12} &&& \\ && \ddots && \\ &&& c_m^{12} & \\ \hdashline &&&& c_k^{21} \end{array} &
\cdots &
\begin{array}{cccc:c} c_1^{1n} &&&& \\ & c_2^{1n} &&& \\ && \ddots && \\ &&& c_m^{1n} & \\ \hdashline &&&& c_k^{1n} \end{array} \\ \hline
\begin{array}{cccc:c} c_1^{21} &&&& \\ & c_2^{21} &&& \\ && \ddots && \\ &&& c_m^{21} & \\ \hdashline &&&& c_k^{21} \end{array} &
\begin{array}{cccc:c} c_1^{22} &&&& \\ & c_2^{22} &&& \\ && \ddots && \\ &&& c_m^{22} & \\ \hdashline &&&& c_k^{22} \end{array} &
\cdots &
\begin{array}{cccc:c} c_1^{2n} &&&& \\ & c_2^{2n} &&& \\ && \ddots && \\ &&& c_m^{2n} & \\ \hdashline &&&& c_k^{2n} \end{array} \\ \hline
\vdots & \vdots & \ddots & \vdots \\ \hline
\begin{array}{cccc:c} c_1^{n1} &&&& \\ & c_2^{n1} &&& \\ && \ddots && \\ &&& c_m^{n1} & \\ \hdashline &&&& c_k^{n1} \end{array} &
\begin{array}{cccc:c} c_1^{n2} &&&& \\ & c_2^{n2} &&& \\ && \ddots && \\ &&& c_m^{n2} & \\ \hdashline &&&& c_k^{n2} \end{array} &
\cdots &
\begin{array}{cccc:c} c_1^{nn} &&&& \\ & c_2^{nn} &&& \\ && \ddots && \\ &&& c_m^{nn} & \\ \hdashline &&&& c_k^{nn} \end{array}
\end{array}\right]
$$

Figure C.1.5a: Expanded matrix ${}^{a}C$ of augmented interregional trade flow coefficient matrices for all regions with $C_k \neq I$.

$$
{}^{a}C = \left[\begin{array}{cc|ccc|cc}
\hat{c}_i^{11} & 0 & & \cdots & & \hat{c}_i^{1n} & 0 \\
0 & c_k^{11} & & \cdots & & 0 & c_k^{1n} \\
\hline
\vdots & \vdots & & \ddots & & \vdots & \vdots \\
\hline
\hat{c}_i^{n1} & 0 & & \cdots & & \hat{c}_i^{nn} & 0 \\
0 & c_k^{n1} & & \cdots & & 0 & c_k^{nn}
\end{array}\right]
$$

Figure C.1.5b: Expanded matrix ${}^{a}C$ of augmented interregional trade flow coefficient matrices for all regions, with $C_k \neq I$ (see also Fig. C.1.5a)

$$
{}^{a}C = \left[\begin{array}{cc|ccc|cc}
\hat{c}_i^{11} & 0 & & \cdots & & \hat{c}_i^{1n} & 0 \\
0 & I & & \cdots & & 0 & 0 \\
\hline
\vdots & \vdots & & \ddots & & \vdots & \vdots \\
\hline
\hat{c}_i^{n1} & 0 & & \cdots & & \hat{c}_i^{nn} & 0 \\
0 & 0 & & \cdots & & 0 & I
\end{array}\right]
$$

Figure C.1.6: Expanded matrix ${}^{a}C$ of augmented interregional trade flow coefficient matrices for all regions, with $C_k = I$

$$
{}^{a}C^{a}\hat{A} = \left[\begin{array}{ccc|c|ccc|c|c|ccc|c}
c^{11}_1a^1_{11} & c^{11}_1a^1_{12}\cdots & c^{11}_1a^1_{1m} & c^{11}_1c^{*1}_1 & c^{12}_1a^2_{11} & c^{12}_1a^2_{12}\cdots & c^{12}_1a^2_{1m} & c^{12}_1c^{*2}_1 & \cdots & c^{1n}_1a^n_{11} & c^{1n}_1a^n_{12}\cdots & c^{1n}_1a^n_{1m} & c^{1n}_1c^{*n}_1 \\
c^{11}_2a^1_{21} & c^{11}_2a^1_{22}\cdots & c^{11}_2a^1_{2m} & c^{11}_2c^{*1}_2 & c^{12}_2a^2_{21} & c^{12}_2a^2_{22}\cdots & c^{12}_2a^2_{2m} & c^{12}_2c^{*2}_2 & \cdots & c^{1n}_2a^n_{21} & c^{1n}_2a^n_{22}\cdots & c^{1n}_2a^n_{2m} & c^{1n}_2c^{*n}_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
c^{11}_ma^1_{m1} & c^{11}_ma^1_{m2}\cdots & c^{11}_ma^1_{mm} & c^{11}_mc^{*1}_m & c^{12}_ma^2_{m1} & c^{12}_ma^2_{m2}\cdots & c^{12}_ma^2_{mm} & c^{12}_mc^{*2}_m & \cdots & c^{1n}_ma^n_{m1} & c^{1n}_ma^n_{m2}\cdots & c^{1n}_ma^n_{mm} & c^{1n}_mc^{*n}_m \\
\hdashline
c^{11}_kw^{*1}_1 & c^{11}_kw^{*1}_2\cdots & c^{11}_kw^{*1}_m & c^{11}_kz^{*1} & c^{12}_kw^{*2}_1 & c^{12}_kw^{*2}_2\cdots & c^{12}_kw^{*2}_m & c^{12}_kz^{*2} & \cdots & c^{1n}_kw^{*n}_1 & c^{1n}_kw^{*n}_2\cdots & c^{1n}_kw^{*n}_m & c^{1n}_kz^{*n} \\
\hline
c^{21}_1a^1_{11} & c^{21}_1a^1_{12}\cdots & c^{21}_1a^1_{1m} & c^{21}_1c^{*1}_1 & c^{22}_1a^2_{11} & c^{22}_1a^2_{12}\cdots & c^{22}_1a^2_{1m} & c^{22}_1c^{*2}_1 & \cdots & c^{2n}_1a^n_{11} & c^{2n}_1a^n_{12}\cdots & c^{2n}_1a^n_{1m} & c^{2n}_1c^{*n}_1 \\
c^{21}_2a^1_{21} & c^{21}_2a^1_{22}\cdots & c^{21}_2a^1_{2m} & c^{21}_2c^{*1}_2 & c^{22}_2a^2_{21} & c^{22}_2a^2_{22}\cdots & c^{22}_2a^2_{2m} & c^{22}_2c^{*2}_2 & \cdots & c^{2n}_2a^n_{21} & c^{2n}_2a^n_{22}\cdots & c^{2n}_2a^n_{2m} & c^{2n}_2c^{*n}_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
c^{21}_ma^1_{m1} & c^{21}_ma^1_{m2}\cdots & c^{21}_ma^1_{mm} & c^{21}_mc^{*1}_m & c^{22}_ma^2_{m1} & c^{22}_ma^2_{m2}\cdots & c^{22}_ma^2_{mm} & c^{22}_mc^{*2}_m & \cdots & c^{2n}_ma^n_{m1} & c^{2n}_ma^n_{m2}\cdots & c^{2n}_ma^n_{mm} & c^{2n}_mc^{*n}_m \\
\hdashline
c^{21}_kw^{*1}_1 & c^{21}_kw^{*1}_2\cdots & c^{21}_kw^{*1}_m & c^{21}_kz^{*1} & c^{22}_kw^{*2}_1 & c^{22}_kw^{*2}_2\cdots & c^{22}_kw^{*2}_m & c^{22}_kz^{*2} & \cdots & c^{2n}_kw^{*n}_1 & c^{2n}_kw^{*n}_2\cdots & c^{2n}_kw^{*n}_m & c^{2n}_kz^{*n} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\
\hline
c^{n1}_1a^1_{11} & c^{n1}_1a^1_{12}\cdots & c^{n1}_1a^1_{1m} & c^{n1}_1c^{*1}_1 & c^{n2}_1a^2_{11} & c^{n2}_1a^2_{12}\cdots & c^{n2}_1a^2_{1m} & c^{n2}_1c^{*2}_1 & \cdots & c^{nn}_1a^n_{11} & c^{nn}_1a^n_{12}\cdots & c^{nn}_1a^n_{1m} & c^{nn}_1c^{*n}_1 \\
c^{n1}_2a^1_{21} & c^{n1}_2a^1_{22}\cdots & c^{n1}_2a^1_{2m} & c^{n1}_2c^{*1}_2 & c^{n2}_2a^2_{21} & c^{n2}_2a^2_{22}\cdots & c^{n2}_2a^2_{2m} & c^{n2}_2c^{*2}_2 & \cdots & c^{nn}_2a^n_{21} & c^{nn}_2a^n_{22}\cdots & c^{nn}_2a^n_{2m} & c^{nn}_2c^{*n}_2 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
c^{n1}_ma^1_{m1} & c^{n1}_ma^1_{m2}\cdots & c^{n1}_ma^1_{mm} & c^{n1}_mc^{*1}_m & c^{n2}_ma^2_{m1} & c^{n2}_ma^2_{m2}\cdots & c^{n2}_ma^2_{mm} & c^{n2}_mc^{*2}_m & \cdots & c^{nn}_ma^n_{m1} & c^{nn}_ma^n_{m2}\cdots & c^{nn}_ma^n_{mm} & c^{nn}_mc^{*n}_m \\
\hdashline
c^{n1}_kw^{*1}_1 & c^{n1}_kw^{*1}_2\cdots & c^{n1}_kw^{*1}_m & c^{n1}_kz^{*1} & c^{n2}_kw^{*2}_1 & c^{n2}_kw^{*2}_2\cdots & c^{n2}_kw^{*2}_m & c^{n2}_kz^{*2} & \cdots & c^{nn}_kw^{*n}_1 & c^{nn}_kw^{*n}_2\cdots & c^{nn}_kw^{*n}_m & c^{nn}_kz^{*n}
\end{array}\right]
$$

Figure C.1.7: The standard form of the MRIO augmented matrix ${}^{a}\left[C\hat{A}\right] = {}^{a}C^{a}\hat{A}$

$$
{}^{a}X = \left[\begin{array}{c}
x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hdashline \overset{*}{w}^1 \\ \hline
x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \\ \hdashline \overset{*}{w}^2 \\ \hline
\vdots \\ \hline
x_1^n \\ x_2^n \\ \vdots \\ x_m^n \\ \hdashline \overset{*}{w}^n
\end{array}\right]
\qquad
{}^{a}\tilde{Y} = \left[\begin{array}{c}
\tilde{y}_1^1 \\ \tilde{y}_2^1 \\ \vdots \\ \tilde{y}_m^1 \\ \hdashline w_{\tilde{y}}^1 \\ \hline
\tilde{y}_1^2 \\ \tilde{y}_2^2 \\ \vdots \\ \tilde{y}_m^2 \\ \hdashline w_{\tilde{y}}^2 \\ \hline
\vdots \\ \hline
\tilde{y}_1^n \\ \tilde{y}_2^n \\ \vdots \\ \tilde{y}_m^n \\ \hdashline w_{\tilde{y}}^n
\end{array}\right]
$$

Figure C.1.8: The augmented gross output vector ${}^{a}X$ and the augmented final demand vector excluding personal consumption expenditures ${}^{a}\tilde{Y}$

ANNEX C.2

CONSTRUCTION OF THE AUGMENTED MATRIX

$$\left[\begin{array}{c|c} C\hat{A} & C\hat{\overset{*}{C}} \\ \hline C_K\hat{\overset{*}{W}} & C_K\hat{\overset{*}{Z}} \end{array}\right]$$

The balance equations of the open MRIO are

For gross output $\quad X = C(\hat{A}X + Y), \quad$ (C.2.1)

and,

For the supply of primary inputs $\quad v_o = \overset{*}{V}_o X + {}^v y, \quad$ (C.2.2)

where X is an (nm x 1) column vector of gross outputs

C is an (nm x nm) expanded interregional trade coefficient matrix, composed of (n^2) diagonal submatrices $\hat{C}^{gh}$, each of dimension (m x m)

$\hat{A}$ is an (nm x nm) expanded block diagonal matrix of technical coefficients, composed of n submatrices A^g, each of dimension (m x m)

Y is an (nm x 1) column vector of final demands

and

v_o is a scalar of aggregate primary inputs (including imports)

$\overset{*}{V}_o$ is a (1 x nm) row vector of primary input coefficients

${}^v y$ is a scalar of total primary inputs supplied directly to final demand

To close the model with respect to consumption both the final demand column vector Y and the primary supply row vector $\overset{*}{V}$ have to be disaggregated in order to obtain:

For gross outputs $$X = C(\hat{A}X + \hat{\hat{C}} + \tilde{Y}) \qquad (C.2.3)$$

and

For the supply of primary factors $$W = \hat{\overset{*}{W}}X + {}^{w}C + {}^{w}\tilde{Y} \qquad (C.2.4)$$

and $$\tilde{v} = \overset{*}{\tilde{V}}X + {}^{\tilde{v}}c + {}^{\tilde{v}}\tilde{y} \qquad (C.2.5)$$

The column vector X and the matrices A and C remain unchanged. Two steps are necessary to decompose the final vector Y into

$$Y = \hat{\hat{C}} + \tilde{Y} \qquad (C.2.6)$$

where $\hat{\hat{C}}$ is an (mn x n) block-diagonal matrix of consumption demand in which the (m x 1) 'column subvector' of consumption demand for each region (h) is located in the h^{th} column position of the matrix as a block in descending stepwise fashion, (see Figure C.2a.2) and $\tilde{Y}$ is an (mn x 1) column vector of 'non-consumption final demand' (that is, including gross investments and exports, as well as Government expenditures if the matrix $\hat{\hat{C}}$ refers to 'personal consumption expenditures' only rather than to total consumption).

Step 1A: is to decompose the (nm x 1) column vector of final demand Y into the sum of two column vectors of the same dimension

$$Y = {}^{c}Y + {}^{nc}Y \qquad (C.2.7)$$

in which ${}^{c}Y$ is the column vector of consumption demand, and ${}^{nc}Y$ is the column vector of other final demand excluding consumption.

Step 2A: is to substitute the 'column block' diagonal matrix $\hat{\underline{C}}$ for the column vector ${}^{c}Y$ based on the observation that

$${}^{c}Y = \hat{\underline{C}}e^{T} \qquad (C.2.8)$$

where e^{T} is the transpose of the row vector e, all of whose elements are equal to one, and whose dimension in this context is (1 x n), i.e. e^{T} is (n x 1).

In addition, the notation for the second column vector is changed to $\tilde{Y}$:

$$\tilde{Y} = {}^{nc}Y \qquad (C.2.9)$$

To obtain equations (C.2.4) and (C.2.5) from equation (C.2.2) it is necessary to separately decompose $\overset{*}{V}$ and ${}^{v}y$ in two stages each: in the first stage two steps are necessary to decompose the (1 x mn) row vector of primary supply coefficients $\overset{*}{V}$, or its transpose $\overset{*}{V}{}^{T}$[1], into:

1/ It is more convenient to express the addition of row vectors in matrix notation using transposes.

$$\overset{*}{V}^{T} = \hat{\overset{*}{W}}^{T} + \overset{*}{\tilde{V}}^{T} \tag{C.2.10}$$

where $\hat{\overset{*}{W}}^{T}$ is the transpose of the (n x mn) block-diagonal matrix $\hat{\overset{*}{W}}$ of wage and salary income coefficients, in which the (1 x m) 'row sub-vector' of income coefficients for each region (g) is located in the $(g)^{th}$ row position of the matrix as a block in descending stepwise fashion (see figure C.2a.1), and $\overset{*}{\tilde{V}}^{T}$ is the transpose of the (1 x mn) row vector of primary supply coefficients $(\overset{*}{\tilde{V}})$ which excludes wage and salary income coefficients.

Step 1B: is to decompose the (1 x nm) row vector of primary supply coefficients $\overset{*}{V}$ into the sum of two row vectors of the same dimensions.

$$\overset{*}{V}^{T} = \overset{*}{W}^{T} + \overset{*}{\tilde{V}}^{T} \tag{C.2.11}$$

in which $\overset{*}{W}^{T}$ is the transpose of the row vector $\overset{*}{W}$ of wage and salary income coefficients, and $\overset{*}{\tilde{V}}^{T}$ is the transpose of the row vector $\overset{*}{\tilde{V}}$ of other primary supply coefficients.

Step 2B: is to substitute the block-diagonal matrix $\hat{\overset{*}{W}}$ for the row vector $\overset{*}{W}$ based on a similar observation to that cited earlier, that is

$$\underset{(1 \times nm)}{\overset{*}{W}} = \underset{(1 \times n)}{e}\,\underset{(n \times nm)}{\hat{\overset{*}{W}}} \tag{C.2.12}$$

in which e is the row vector all of whose elements are equal to one, and whose dimension in this context is (1 x n).

In the second stage, the scalar ${}^{v}y$ is decomposed into four parts

$$ {}^{v}y = e\left({}^{w}\overline{C}\right) + e\left({}^{w}\tilde{Y}\right) + {}^{\tilde{v}}\overline{c} + {}^{\tilde{v}}\tilde{y} \qquad \text{(C.2.13)} $$

where e is as defined before, **and**

${}^{w}\overline{C}$ is a column vector of dimension (n x 1) whose elements are the primary inputs directly supplied to consumption demand in each region, or alternately the wage and salary income in each region paid out directly from personal consumption expenditures in the same region.

${}^{w}\tilde{Y}$ is an (n x 1) column vector of primary inputs directly supplied to non-consumption final demand for each region, or more appropriately the exogenously determined part of wage and salary income, for each supplying region g.

${}^{\tilde{v}}\overline{c}$ is a (1 x 1) scalar of payments (excluding wage and salary income) for primary inputs supplied directly to consumption demand, by all regions.

${}^{\tilde{v}}\tilde{y}$ is a scalar of payments (excluding wage and salary income) for primary inputs supplied directly to non-consumption final demand by all regions.

Stage two is not completed until after the following two adjustments which are required for dimensional compatibility in the next set of operations.

Adjustment (1) consists of substituting the (n x n) diagonal matrix ${}^{w}\hat{\bar{C}}$ for the column vector ${}^{w}\bar{C}$, that is

$$(I)\,{}^{w}\bar{C} = {}^{w}\hat{\bar{C}} \qquad \text{(C.2.14)}$$

In the form of a diagonal matrix ${}^{w}\hat{\bar{C}}$, the interpretation given earlier for the column vector ${}^{w}\bar{C}$ becomes clearer.

Adjustment (2) consists of substituting a (1 x n) row vector ${}^{v}\bar{c}$ for the scalar ${}^{\tilde{v}}\bar{c}$, that is

$${}^{\tilde{v}}\bar{c} \text{ for the scalar } {}^{\tilde{v}}\bar{C}e^{T} \qquad \text{(C.2.15)}$$

The elements of the row vector ${}^{\tilde{v}}\bar{c}$ will now represent the payments (excluding wage and salary income) for primary inputs supplied directly to consumption demand in <u>each</u> purchasing region h [see Figure C.2a.3].

Before the matrices $\hat{\bar{C}}$ of consumption demand, and $\hat{\overset{*}{W}}$ of 'wage and salary' income coefficients can be made endogenous two more steps are required:

The first step (1C) is required by the interregional trade logic of the MRIO model. This requires the creation of an (n x n) factor payments matrix (C_k) whose elements (c_k^{gh}) show how much of the wage and salary income earned in region (g) originated in <u>each</u> of the other regions paying for (i.e., purchasing) its household services.

With the previous adjustments this allows us to rewrite the balancing equations (C.2.3) to (C.2.5) as:

for gross outputs $$X = C(\hat{A}X + \hat{\bar{C}} + \tilde{Y}) \qquad (C.2.16)$$

and,

for the supply of primary inputs $$W = C_k(\hat{\overset{*}{W}}X + {}^{w}\hat{\bar{C}} + {}^{w}\tilde{Y}) \qquad (C.2.17)$$

$$\tilde{v} = \overset{*}{\tilde{V}}X + {}^{\tilde{v}}\bar{C} + {}^{\tilde{v}}\tilde{y} \qquad (C.2.18)$$

The second step is to create analogue consumption functions linking the consumption expenditures represented by the matrices C, ${}^{w}\hat{C}$ and the vector ${}^{v}\bar{C}$ to the column vector of 'wage and salary' income W, which has the dimensions (n x 1) and whose elements w^g represent the wage and salary income earned in region g.

These functions are referred to as analogue consumption functions because in their construction they are based on the traditional input-output logic of total supply equals total demand for each industry, in this case the column sum of personal consumption expenditures for each region $\bar{c}_o^h$ is equal to the row sum of wage and salary income for the same region w^h. This traditional input-output assumption can be replaced by regular Keynesian consumption functions which is discussed in Chapter 5.

These linear analogue consumption functions can be written as

$$\hat{\bar{C}} = \hat{\overset{*}{C}}W \qquad (C.2.19)$$

$$^{w}\hat{\bar{C}} = \hat{\overset{*}{Z}}W \qquad (C.2.20)$$

and

$$\overset{v}{\overline{C}} = \overset{*}{U}W \qquad (C.2.21)$$

where

$\hat{\overset{*}{C}} = \left[\overset{*}{c}{}_j^h\right] = \left[\dfrac{\bar{c}_j^h}{\bar{c}_o^h}\right] = \left[\dfrac{\bar{w}_j^h}{w^h}\right]$ is an (mn x n) 'column-block' diagonal matrix of commodity-specific personal consumption expenditure coefficients, for each region h (see Figure C.2a.4). These coefficients represent essentially budget shares.

$\hat{\overset{*}{Z}} = \left[\overset{*}{z}{}^h\right] = \left[\dfrac{w\bar{c}^h}{\bar{c}_o^h}\right] = \left[\dfrac{w\bar{c}^h}{w^h}\right]$ is an (n x n) diagonal matrix of coefficients of personal consumption expenditures directly allocated for wage and salary payments (see Figure C.2a.5).

and

$\overset{*}{U} = \left[\overset{*}{u}{}^h\right] = \left[\dfrac{\tilde{v}\bar{c}^h}{\bar{c}^h}\right] = \left[\dfrac{\tilde{v}\bar{c}^h}{w^h}\right]$ is a (1xn) row vector of coefficients for payment to other primary suppliers directly allocated out of personal consumption expenditures.

These coefficients will represent marginal coefficients only if in their estimation the non-homogenous terms are included, otherwise if the non-homogenous terms have been incorporated instead in the corresponding multiplicand vectors $\tilde{Y}$ and ${}^{w}\tilde{Y}$, and the scalar ${}^{v}\tilde{y}$, then the coefficients $\overset{*}{c}{}^{h}_{j}$ will represent average coefficients only. This is commonly done but is not necessary.

Incorporating the coefficient matrices obtained in the second step into equations (2.16)-(2.18) we get:

$$X = C(\hat{A}X + \hat{\overset{*}{C}}W + \tilde{Y}) \qquad \text{(C.2.22)}$$

and

$$W = C_k(\hat{\overset{*}{W}}X + \hat{\overset{*}{Z}}W + {}^{w}\tilde{Y}) \qquad \text{(C.2.23)}$$

$$\tilde{v} = \overset{*}{\tilde{V}}X + \overset{*}{U}W + {}^{v}\tilde{y} \qquad \text{(C.2.24)}$$

which when expanded become:

$$X = C\hat{A}X + C\hat{\overset{*}{C}}W + C\tilde{Y} \qquad \text{(C.2.25)}$$

$$W = C_k\hat{\overset{*}{W}}X + C_k\hat{\overset{*}{Z}}W + C_k{}^{w}\tilde{Y} \qquad \text{(C.2.26)}$$

$$\tilde{v} = \overset{*}{\tilde{V}}X + \overset{*}{U}W + {}^{v}\tilde{y} \qquad \text{(C.2.27)}$$

Before this system of equations can be expressed in the form of a partitioned matrix it is necessary to show how the expanded trade flow matrix C and the expanded technical coefficient matrix A are augmented with the matrices just created. In the previous approach the expanded trade matrix C of the open MRIO model was augmented by separately augmenting each of its n^2 submatrices. This followed the traditional MRIO procedure of allocating each element of the (C_k) matrix to the principal diagonal of each (m x m) diagonal submatrix of trade flow coefficients C^{gh} between each pair of regions. In the formulation proposed here the (n x n) matrix C_k is appended directly as an undistributed block on the principal diagonal of a block diagonal interregional trade matrix, with the open MRIO model matrix C as the other block (see also Figure C.2a.7).

$$\left[\begin{array}{c|c} C & 0 \\ \hline 0 & C_k \end{array}\right]$$

Similarly, instead of augmenting each regional technical coefficient matrix separately as in the traditional MRIO approach, the diagonal coefficient matrices $\overset{*}{C}$, $\overset{*}{W}$ and $\hat{\overset{*}{Z}}$ are appended to the open MRIO model matrix $\hat{A}$ to create a differently augmented "technical" coefficient matrix of the same dimensions as in the standard approach

$$\left[\begin{array}{c|c} \hat{A} & \hat{\overset{*}{C}} \\ \hline \hat{\overset{*}{W}} & \hat{\overset{*}{Z}} \end{array}\right]$$

See Figure (C.2a.6).

The appropriate column vectors of endogenous and exogenous variables are presented in Figure (C.2a.9).

Now the system of equations in (C.2.25) through (C.2.27) can be written in partitioned matrix notation

$$\left[\begin{array}{c} X \\ \hline W \end{array}\right] = \left[\begin{array}{c:c} C & 0 \\ \hdashline 0 & C_k \end{array}\right] \left\{ \left[\begin{array}{c:c} \hat{A} & \hat{\overset{*}{C}} \\ \hdashline \overset{*}{W} & \overset{*}{Z} \end{array}\right] \left[\begin{array}{c} X \\ \hline W \end{array}\right] + \left[\begin{array}{c} \tilde{Y} \\ \hline w_{\tilde{y}} \end{array}\right] \right\} \tag{C.2.28}$$

and

$$(\tilde{v}) = (\overset{*}{\tilde{V}} \quad \overset{*}{\tilde{U}}) \begin{bmatrix} X \\ W \end{bmatrix} + \tilde{v}_{\tilde{y}} \tag{C.2.29}$$

ANNEX C.2a

FIGURES ILLUSTRATING THE REFORMULATED $^{a}\theta$, with $k = 1$

In the open model

$$v_o = \left[\overset{*}{v}{}^1_1\ \overset{*}{v}{}^1_2 \cdots \overset{*}{v}{}^1_m \,\middle|\, \overset{*}{v}{}^2_1\ \overset{*}{v}{}^2_2 \cdots \overset{*}{v}{}^2_m \,\middle|\, \cdots \,\middle|\, \overset{*}{v}{}^n_1\ \overset{*}{v}{}^n_2 \cdots \overset{*}{v}{}^n_m\right] [X]$$

This scalar v_o and coefficient vector $\overset{*}{V}$ are separated into two parts in the partitioned matrix approach: a vector W_k and a scalar $\tilde{v}_o$ with their associated coefficients in the form of matrix $\hat{\overset{*}{W}}$ and vector $\tilde{\overset{*}{V}}$ respectively.

(1)

$$W_k = \begin{bmatrix} w^1 \\ \hline w^2 \\ \hline \vdots \\ \hline w^n \end{bmatrix} = \left[\begin{array}{ccc} \overset{*}{w}{}^1_1\ \overset{*}{w}{}^1_2 \cdots \overset{*}{w}{}^1_m & & \\ \hline & \overset{*}{w}{}^2_1\ \overset{*}{w}{}^2_2 \cdots \overset{*}{w}{}^2_m & \\ \hline & & \ddots \\ \hline & & \overset{*}{w}{}^n_1\ \overset{*}{w}{}^n_2 \cdots \overset{*}{w}{}^n_m \end{array}\right] [X]$$

(2)

$$[\tilde{v}_o] = \left[\tilde{\overset{*}{v}}{}^1_1\ \tilde{\overset{*}{v}}{}^1_2 \cdots \tilde{\overset{*}{v}}{}^1_m \,\middle|\, \tilde{\overset{*}{v}}{}^2_1\ \tilde{\overset{*}{v}}{}^2_2 \cdots \tilde{\overset{*}{v}}{}^2_m \,\middle|\, \cdots \,\middle|\, \tilde{\overset{*}{v}}{}^n_1\ \tilde{\overset{*}{v}}{}^n_2 \cdots \tilde{\overset{*}{v}}{}^n_m\right] [X]$$

Figure C.2a.1: Partitioning of the row vector of primary supply coefficients $\overset{*}{V}$ into the diagonal block matrix $\hat{\overset{*}{W}}$ and the row vector $\tilde{\overset{*}{V}}$.

$$
\begin{bmatrix} y_1^1 \\ y_2^1 \\ \vdots \\ y_m^1 \\ \hline y_1^2 \\ y_2^2 \\ \vdots \\ y_m^2 \\ \hline \vdots \\ \hline y_1^n \\ y_2^n \\ \vdots \\ y_m^n \end{bmatrix}
=
\overset{(1)}{\begin{bmatrix} \begin{matrix} \bar{c}_1^1 \\ \bar{c}_2^1 \\ \vdots \\ \bar{c}_m^1 \end{matrix} & & & \\ & \begin{matrix} \bar{c}_1^2 \\ \bar{c}_2^2 \\ \vdots \\ \bar{c}_m^2 \end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix} \bar{c}_1^n \\ \bar{c}_2^n \\ \vdots \\ \bar{c}_m^n \end{matrix} \end{bmatrix}}
+
\overset{(2)}{\begin{bmatrix} \tilde{y}_1^1 \\ \tilde{y}_2^1 \\ \vdots \\ \tilde{y}_m^1 \\ \hline \tilde{y}_1^2 \\ \tilde{y}_2^2 \\ \vdots \\ \tilde{y}_m^2 \\ \hline \vdots \\ \hline \tilde{y}_1^n \\ \tilde{y}_2^n \\ \vdots \\ \tilde{y}_m^n \end{bmatrix}}
$$

$$
\underset{(mn \times 1)}{Y} = \underset{(mn \times n)}{\hat{\bar{C}}_k} \qquad \underset{(mn \times 1)}{\tilde{Y}}
$$

where Y = Final Demand (FD)

$\hat{\bar{C}}_k$ = Personal Consumption Expenditures (PCE)

$\tilde{Y}$ = FD - PCE

Figure C.2a.2: Separating the block diagonal matrix $\hat{\bar{C}}_k$, with k = 1, and the column vector $\tilde{Y}$ from the column vector of final demand Y.

$$({}^{v}y) = \left[\begin{array}{c|c} {}^{w}\hat{\bar{C}} & {}^{w}\hat{\tilde{Y}} \\ \hline {}^{\tilde{v}}\bar{C} & {}^{\tilde{v}}\tilde{y} \end{array}\right]_{k_1} = \overset{(1)}{\begin{bmatrix} {}^{w}\bar{c}^{1} & & & \\ & {}^{w}\bar{c}^{2} & & \\ & & \ddots & \\ & & & {}^{w}\bar{c}^{n} \end{bmatrix}} + \overset{(2)}{\begin{bmatrix} {}^{w}\tilde{y}^{1} \\ {}^{w}\tilde{y}^{2} \\ \vdots \\ {}^{w}\tilde{y}^{n} \end{bmatrix}}$$

$$\overset{(3)}{\begin{bmatrix} {}^{\tilde{v}}\bar{c}^{1} & {}^{\tilde{v}}\bar{c}^{2} & \dots & {}^{\tilde{v}}\bar{c}^{n} \end{bmatrix}} + \overset{(4)}{\begin{bmatrix} {}^{\tilde{v}}\tilde{y} \end{bmatrix}}$$

where
$$\begin{aligned} {}^{w}\hat{\bar{C}} &= [{}^{w}\bar{c}^{gg}] = (n \times n) \\ {}^{w}\tilde{Y} &= [{}^{w}\tilde{y}^{g}] = (n \times 1) \\ {}^{\tilde{v}}\bar{C} &= [{}^{\tilde{v}}\bar{c}^{h}] = (1 \times n) \\ {}^{\tilde{v}}\tilde{y} &= (1 \times 1) \end{aligned}$$

Figure C.2a.3: Separating the diagonal matrix ${}^{w}\hat{\bar{C}}$, with k = 1, the column vector ${}^{w}\tilde{y}$, the row vector ${}^{v}\tilde{\bar{C}}$ and the scalar ${}^{\tilde{v}}\tilde{y}$ from the scalar ${}^{v}y$.

$$
\begin{bmatrix}
\bar{c}^1_1 & & & 0 \\
\bar{c}^1_2 & & & \\
\vdots & & & \\
\bar{c}^1_m & & & \\
 & \bar{c}^2_1 & & \\
 & \bar{c}^2_2 & & \\
 & \vdots & & \\
 & \bar{c}^2_m & & \\
 & & \ddots & \\
 & & & \bar{c}^n_1 \\
 & & & \bar{c}^n_2 \\
 0 & & & \vdots \\
 & & & \bar{c}^n_m
\end{bmatrix}
\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}
=
\begin{bmatrix}
\overset{*}{c}{}^1_1 & & & 0 \\
\overset{*}{c}{}^1_2 & & & \\
\vdots & & & \\
\overset{*}{c}{}^1_m & & & \\
 & \overset{*}{c}{}^2_1 & & \\
 & \overset{*}{c}{}^2_2 & & \\
 & \vdots & & \\
 & \overset{*}{c}{}^2_m & & \\
 & & \ddots & \\
 & & & \overset{*}{c}{}^n_1 \\
 & & & \overset{*}{c}{}^n_2 \\
 0 & & & \vdots \\
 & & & \overset{*}{c}{}^n_m
\end{bmatrix}
\begin{bmatrix} w^1 \\ w^2 \\ \vdots \\ w^n \end{bmatrix}
$$

$$
\underset{(mn\times n)}{\hat{\bar{C}}_k}\ \underset{(n\times 1)}{e} = \underset{(mn\times n)}{\hat{\overset{*}{C}}_k}\ \underset{(n\times 1)}{W}\ , \text{ where } \overset{*}{c}{}^h_j = \frac{\bar{c}^h_j}{w^h} \text{ and } m > n
$$

Figure C.2a.4: The matrix $\hat{\overset{*}{C}}_k$ of personal consumption expenditure coefficients, in which $\hat{\bar{C}}_k e = \hat{\overset{*}{C}}_k W$

$$\begin{bmatrix} {}^{w}c^{1} & & & & 0 \\ & {}^{w}c^{2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & \cdot\ \cdot & \\ & & & & {}^{w}c^{n} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ \cdot \\ \cdot \\ \cdot \\ 1 \end{bmatrix} = \begin{bmatrix} \overset{*}{z}{}^{1} & & & & 0 \\ & \overset{*}{z}{}^{2} & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & \cdot \\ & & & & \overset{*}{z}{}^{n} \end{bmatrix} \begin{bmatrix} w^{1} \\ w^{2} \\ \cdot \\ \cdot \\ \cdot \\ w^{n} \end{bmatrix}$$

$$\underset{(n\times n)}{{}^{w}\hat{C}_{k}}\ \underset{(n\times 1)}{e} \;=\; \underset{(n\times n)}{\hat{\overset{*}{Z}}_{k}}\ \underset{(n\times 1)}{W}\ , \text{ where } \overset{*}{Z}{}^{h} = \frac{{}^{w}c^{h}}{w^{h}}$$

Figure C.2a.5: The coefficient matrix $\hat{\overset{*}{Z}}_{k}$ of personal consumption expenditures allocated directly as payments to the household sector, in which ${}^{w}\hat{C}_{k}e = \hat{\overset{*}{Z}}_{k}W$, with k = 1

$$\left[\begin{array}{c|c} \hat{A} & \overset{*}{\hat{C}} \\ \hline \overset{*}{\hat{W}} & \overset{*}{\hat{Z}} \end{array}\right] = \left[\begin{array}{cccc|cccc}
\begin{matrix} a^1_{11} & a^1_{12} & \cdots & a^1_{1m} \\ a^1_{21} & a^1_{22} & \cdots & a^1_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^1_{m1} & a^1_{m2} & \cdots & a^1_{mm} \end{matrix} & & & & \begin{matrix} \overset{*}{c}{}^1_1 \\ \overset{*}{c}{}^1_2 \\ \vdots \\ \overset{*}{c}{}^1_m \end{matrix} & & & \\
& \begin{matrix} a^2_{11} & a^2_{12} & \cdots & a^2_{1m} \\ a^2_{21} & a^2_{22} & \cdots & a^2_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^2_{m1} & a^2_{m2} & \cdots & a^2_{mm} \end{matrix} & & & & \begin{matrix} \overset{*}{c}{}^2_1 \\ \overset{*}{c}{}^2_2 \\ \vdots \\ \overset{*}{c}{}^2_m \end{matrix} & & \\
& & \ddots & & & & \ddots & \\
& & & \begin{matrix} a^n_{11} & a^n_{12} & \cdots & a^n_{1m} \\ a^n_{21} & a^n_{22} & \cdots & a^n_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a^n_{m1} & a^n_{m2} & \cdots & a^n_{mm} \end{matrix} & & & & \begin{matrix} \overset{*}{c}{}^n_1 \\ \overset{*}{c}{}^n_2 \\ \vdots \\ \overset{*}{c}{}^n_m \end{matrix} \\
\hline
\begin{matrix} \overset{*}{w}{}^1_1 & \overset{*}{w}{}^1_2 & \cdots & \overset{*}{w}{}^1_m \end{matrix} & & & & \overset{*}{z}{}^1 & & & \\
& \begin{matrix} \overset{*}{w}{}^2_1 & \overset{*}{w}{}^2_2 & \cdots & \overset{*}{w}{}^2_m \end{matrix} & & & & \overset{*}{z}{}^2 & & \\
& & \ddots & & & & \ddots & \\
& & & \begin{matrix} \overset{*}{w}{}^n_1 & \overset{*}{w}{}^n_2 & \cdots & \overset{*}{w}{}^n_m \end{matrix} & & & & \overset{*}{z}{}^n
\end{array}\right]$$

in which $\hat{A} = (mn \times mn)$;

$A^g = (m\ m)$;

$\overset{*}{\hat{C}} = (mn \times n)$;

$\overset{*}{\hat{W}} = (n \times mn)$;

$\overset{*}{\hat{Z}} = (n \times n)$

Figure C.2a.6: Partitioned matrix form of the expanded $\hat{A}$ matrix of technical coefficients augmented by the matrices of income coefficients $\overset{*}{\hat{W}}_k$, and consumption coefficients $\overset{*}{\hat{C}}_k$ and $\overset{*}{\hat{Z}}_k$: $\left\{ \left[\begin{array}{c|c} \hat{A} & \overset{*}{\hat{C}}_k \\ \hline \overset{*}{\hat{W}}_k & \overset{*}{\hat{Z}}_k \end{array}\right] \text{, with } k = 1 \right\}$

$$\left[\begin{array}{c:c} C & 0 \\ \hdashline 0 & C_k \end{array}\right] = \left[\begin{array}{c:c:c:c|c}
\begin{matrix} c_1^{11} & & & \\ & c_2^{11} & & \\ & & \ddots & \\ & & & c_m^{11} \end{matrix} & \begin{matrix} c_1^{12} & & & \\ & c_2^{12} & & \\ & & \ddots & \\ & & & c_m^{12} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} c_1^{1n} & & & \\ & c_2^{1n} & & \\ & & \ddots & \\ & & & c_m^{1n} \end{matrix} & \\
\hdashline
\begin{matrix} c_1^{21} & & & \\ & c_2^{21} & & \\ & & \ddots & \\ & & & c_m^{21} \end{matrix} & \begin{matrix} c_1^{22} & & & \\ & c_2^{22} & & \\ & & \ddots & \\ & & & c_m^{22} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} c_1^{2n} & & & \\ & c_2^{2n} & & \\ & & \ddots & \\ & & & c_m^{2n} \end{matrix} & \\
\hdashline
\vdots & \vdots & & \vdots & \\
\hdashline
\begin{matrix} c_1^{n1} & & & \\ & c_2^{n1} & & \\ & & \ddots & \\ & & & c_m^{n1} \end{matrix} & \begin{matrix} c_1^{n2} & & & \\ & c_2^{n2} & & \\ & & \ddots & \\ & & & c_m^{n2} \end{matrix} & \begin{matrix} \cdots \\ \cdots \\ \\ \cdots \end{matrix} & \begin{matrix} c_1^{nn} & & & \\ & c_2^{nn} & & \\ & & \ddots & \\ & & & c_m^{nn} \end{matrix} & \\
\hline
 & & & & \begin{matrix} c_k^{11} & c_k^{12} & \cdots & c_k^{1n} \\ c_k^{21} & c_k^{22} & \cdots & c_k^{2n} \\ \vdots & \vdots & \ddots & \vdots \\ c_k^{n1} & c_k^{n2} & \cdots & c_k^{nn} \end{matrix}
\end{array}\right]$$

in which

$C = (mn \times mn)$;

$\hat{C}^{gh} = (m \times m)$;

$C_k = (n \times n)$

Figure C.2a.7: Partitioned matrix form of the expanded matrix C of interregional trade flow coefficients augmented by the matrix C_k of interregional factor income (trade) flow coefficients for the household sector $\left[\begin{array}{c:c} C & 0 \\ \hdashline 0 & C_k \end{array}\right]$, with k = 1

$$
\left[\begin{array}{c|c}
C\hat{A} & C\hat{\overset{*}{C}} \\ \hline
C_k\hat{\overset{*}{W}} & C_k\hat{\overset{*}{Z}}
\end{array}\right]
=
\left[\begin{array}{ccc|ccc|c|ccc|ccc}
c_1^{11}a_{11}^1 & c_1^{11}a_{12}^1\cdots & c_1^{11}a_{1m}^1 & c_1^{12}a_{11}^2 & c_1^{12}a_{12}^2\cdots & c_1^{12}a_{1m}^2 & \cdots & c_1^{1n}a_{11}^n & c_1^{1n}a_{12}^n\cdots & c_1^{1n}a_{1m}^n & c_1^{11}c_1^{*1} & c_1^{12}c_1^{*2}\cdots & c_1^{1n}c_1^{*n} \\
c_2^{11}a_{21}^1 & c_2^{11}a_{22}^1\cdots & c_2^{11}a_{2m}^1 & c_2^{12}a_{21}^2 & c_2^{12}a_{22}^2\cdots & c_2^{12}a_{2m}^2 & \cdots & c_2^{1n}a_{21}^n & c_2^{1n}a_{22}^n\cdots & c_2^{1n}a_{2m}^n & c_2^{11}c_2^{*1} & c_2^{12}c_2^{*2}\cdots & c_2^{1n}c_2^{*n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_m^{11}a_{m1}^1 & c_m^{11}a_{m2}^1\cdots & c_m^{11}a_{mm}^1 & c_m^{12}a_{m1}^2 & c_m^{12}a_{m2}^2\cdots & c_m^{12}a_{mm}^2 & \cdots & c_m^{1n}a_{m1}^n & c_m^{1n}a_{m2}^n\cdots & c_m^{1n}a_{mm}^n & c_m^{11}c_m^{*1} & c_m^{12}c_m^{*2}\cdots & c_m^{1n}c_m^{*n} \\ \hline
c_1^{21}a_{11}^1 & c_1^{21}a_{12}^1\cdots & c_1^{21}a_{1m}^1 & c_1^{22}a_{11}^2 & c_1^{22}a_{12}^2\cdots & c_1^{22}a_{1m}^2 & \cdots & c_1^{2n}a_{11}^n & c_1^{2n}a_{12}^n\cdots & c_1^{2n}a_{1m}^n & c_1^{21}c_1^{*1} & c_1^{22}c_1^{*2}\cdots & c_1^{2n}c_1^{*n} \\
c_2^{21}a_{21}^1 & c_2^{21}a_{22}^1\cdots & c_2^{21}a_{2m}^1 & c_2^{22}a_{21}^2 & c_2^{22}a_{22}^2\cdots & c_2^{22}a_{2m}^2 & \cdots & c_2^{2n}a_{21}^n & c_2^{2n}a_{22}^n\cdots & c_2^{2n}a_{2m}^n & c_2^{21}c_2^{*1} & c_2^{22}c_2^{*2}\cdots & c_2^{2n}c_2^{*n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_m^{21}a_{m1}^1 & c_m^{21}a_{m2}^1\cdots & c_m^{21}a_{mm}^1 & c_m^{22}a_{m1}^2 & c_m^{22}a_{m2}^2\cdots & c_m^{22}a_{mm}^2 & \cdots & c_m^{2n}a_{m1}^n & c_m^{2n}a_{m2}^n\cdots & c_m^{2n}a_{mm}^n & c_m^{21}c_m^{*1} & c_m^{21}c_m^{*1} & c_m^{2n}c_m^{*n} \\ \hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \hline
c_1^{n1}a_{11}^1 & c_1^{n1}a_{12}^1\cdots & c_1^{n1}a_{1m}^1 & c_1^{n2}a_{11}^2 & c_1^{n2}a_{12}^2\cdots & c_1^{n2}a_{1m}^2 & \cdots & c_1^{nn}a_{11}^n & c_1^{nn}a_{12}^n\cdots & c_1^{nn}a_{1m}^n & c_1^{n1}c_1^{*1} & c_1^{n2}c_1^{*2}\cdots & c_1^{nn}c_1^{*n} \\
c_2^{n1}a_{21}^1 & c_2^{n1}a_{22}^1\cdots & c_2^{n1}a_{2m}^1 & c_2^{n2}a_{21}^2 & c_2^{n2}a_{22}^2\cdots & c_2^{n2}a_{2m}^2 & \cdots & c_2^{nn}a_{21}^n & c_2^{nn}a_{22}^n\cdots & c_2^{nn}a_{2m}^n & c_2^{n1}c_2^{*1} & c_2^{n2}c_2^{*2}\cdots & c_2^{nn}c_2^{*n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_m^{n1}a_{m1}^1 & c_m^{n1}a_{m2}^1\cdots & c_m^{n1}a_{mm}^1 & c_m^{n2}a_{m1}^2 & c_m^{n2}a_{m2}^2\cdots & c_m^{n2}a_{mm}^2 & \cdots & c_m^{nn}a_{m1}^n & c_m^{nn}a_{m2}^n\cdots & c_m^{nn}a_{mm}^n & c_m^{n1}c_m^{*1} & c_m^{n2}c_m^{*n}\cdots & c_m^{nn}c_m^{*n} \\ \hline
c_k^{11}w_1^{*1} & c_k^{11}w_2^{*1}\cdots & c_k^{11}w_m^{*1} & c_k^{12}w_1^{*2} & c_k^{12}w_2^{*2}\cdots & c_k^{12}w_m^{*2} & \cdots & c_k^{1n}w_1^{*n} & c_k^{1n}w_2^{*n}\cdots & c_k^{1n}w_m^{*n} & c_k^{11}z^{*1} & c_k^{12}z^{*2}\cdots & c_k^{1n}z^{*n} \\
c_k^{21}w_1^{*1} & c_k^{21}w_2^{*1}\cdots & c_k^{21}w_m^{*1} & c_k^{22}w_1^{*2} & c_k^{22}w_2^{*2}\cdots & c_k^{22}w_m^{*2} & \cdots & c_k^{2n}w_1^{*n} & c_k^{2n}w_2^{*n}\cdots & c_k^{2n}w_m^{*n} & c_k^{21}z^{*1} & c_k^{22}z^{*2}\cdots & c_k^{2n}z^{*n} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
c_k^{n1}w_1^{*1} & c_k^{n1}w_2^{*1}\cdots & c_k^{n1}w_m^{*1} & c_k^{n2}w_1^{*2} & c_k^{n2}w_2^{*2}\cdots & c_k^{n2}w_m^{*2} & \cdots & c_k^{nn}w_1^{*n} & c_k^{nn}w_2^{*n}\cdots & c_k^{nn}w_m^{*n} & c_k^{n1}z^{*1} & c_k^{n2}z^{*2}\cdots & c_k^{nn}z^{*n}
\end{array}\right]
$$

Figure C.2a.8: **Partitioned form of the augmented matrix** ${}^{a}[C\hat{A}] = \begin{bmatrix} C\hat{A} & C\hat{\overset{*}{C}} \\ C_k\hat{\overset{*}{W}} & C_k\hat{\overset{*}{Z}} \end{bmatrix}$**, with $k = 1$**

$$
\begin{bmatrix} X \\ \hdashline W \end{bmatrix} = \left[\begin{array}{c}
x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hline
x_1^2 \\ x_2^2 \\ \vdots \\ x_m^2 \\ \hline
\vdots \\ \hline
x_1^n \\ x_2^n \\ \vdots \\ x_m^n \\ \hdashline
w^1 \\ w^2 \\ \vdots \\ w^m
\end{array}\right]
\qquad
\begin{bmatrix} \tilde{Y} \\ \hdashline {}^{w}\tilde{Y} \end{bmatrix} = \left[\begin{array}{c}
\tilde{y}_1^1 \\ \tilde{y}_2^1 \\ \vdots \\ \tilde{y}_m^1 \\ \hline
\tilde{y}_1^2 \\ \tilde{y}_2^2 \\ \vdots \\ \tilde{y}_m^2 \\ \hline
\vdots \\ \hline
\tilde{y}_1^n \\ \tilde{y}_2^n \\ \vdots \\ \tilde{y}_m^n \\ \hdashline
w_{\tilde{y}}^1 \\ w_{\tilde{y}}^2 \\ \vdots \\ w_{\tilde{y}}^n
\end{array}\right]
$$

Figure C.2a.9: Partitioned vector form of the vectors X and $\tilde{Y}$ augmented by the vectors W_k and ${}^{w}\tilde{Y}_k$ respectively, with k = 1

ANNEX C.2b

FIGURES ILLUSTRATING THE REFORMULATED ${}^{a}\theta$, with $k > 1$

$$\begin{bmatrix} X \\ \hdashline W_k \end{bmatrix} = \left[\begin{array}{c} x_1^1 \\ x_2^1 \\ \vdots \\ x_m^1 \\ \hline x_1^2 \\ x_2^2 \\ \vdots \\ x_m^1 \\ \hline \vdots \\ \hline x_1^n \\ x_2^n \\ \vdots \\ x_m^n \\ \hline w_1^1 \\ w_2^1 \\ \vdots \\ w_k^1 \\ \hline w_1^2 \\ w_2^2 \\ \vdots \\ w_k^2 \\ \hline \vdots \\ \hline w_1^n \\ w_2^n \\ \vdots \\ w_k^n \end{array}\right] \quad \text{with } k > 1$$

$$\begin{bmatrix} \tilde{Y} \\ \hdashline {}^{w}\tilde{Y}_k \end{bmatrix} = \left[\begin{array}{c} \tilde{y}_1^1 \\ \tilde{y}_2^1 \\ \vdots \\ \tilde{y}_m^1 \\ \hline \tilde{y}_1^2 \\ \tilde{y}_2^2 \\ \vdots \\ \tilde{y}_m^2 \\ \hline \vdots \\ \hline \tilde{y}_1^n \\ \tilde{y}_2^n \\ \vdots \\ \tilde{y}_m^n \\ \hline {}^{w}\tilde{y}_1^1 \\ {}^{w}\tilde{y}_2^2 \\ \vdots \\ {}^{w}\tilde{y}_k^2 \\ \hline {}^{w}\tilde{y}_1^2 \\ {}^{w}\tilde{y}_2^2 \\ \vdots \\ {}^{w}\tilde{y}_k^2 \\ \hline \vdots \\ \hline {}^{w}\tilde{y}_1^n \\ {}^{w}\tilde{y}_2^n \\ \vdots \\ {}^{w}\tilde{y}_k^n \end{array}\right] \quad \text{with } k > 1$$

Figure C.2b.1: Partitioned vector form of the vectors X and $\tilde{Y}$ augmented by the vectors W_k and ${}^{w}\tilde{Y}_k$ respectively, with $k > 1$

$$\left[\begin{array}{cccc|cccc}
\begin{matrix} a^{1}_{11} & a^{1}_{12} & \cdots & a^{1}_{1m} \\ a^{1}_{21} & a^{1}_{22} & \cdots & a^{1}_{2m} \\ \vdots & \vdots & & \vdots \\ a^{1}_{m1} & a^{1}_{m2} & \cdots & a^{1}_{mm} \end{matrix} & & & & \begin{matrix} c^{*1}_{11} & c^{*1}_{12} & \cdots & c^{*1}_{1k} \\ c^{*1}_{21} & c^{*1}_{22} & \cdots & c^{*1}_{2k} \\ \vdots & \vdots & & \vdots \\ c^{*1}_{m1} & c^{*1}_{m2} & \cdots & c^{*1}_{mk} \end{matrix} & & & \\
 & \begin{matrix} a^{2}_{11} & a^{2}_{12} & \cdots & a^{2}_{1m} \\ a^{2}_{21} & a^{2}_{22} & \cdots & a^{2}_{2m} \\ \vdots & \vdots & & \vdots \\ a^{2}_{m1} & a^{2}_{m2} & \cdots & a^{2}_{mm} \end{matrix} & & & & \begin{matrix} c^{*2}_{11} & c^{*2}_{12} & \cdots & c^{*2}_{1k} \\ c^{*2}_{21} & c^{*2}_{22} & \cdots & c^{*2}_{2k} \\ \vdots & \vdots & & \vdots \\ c^{*2}_{m1} & c^{*2}_{m2} & \cdots & c^{*2}_{mk} \end{matrix} & & \\
 & & \ddots & & & & \ddots & \\
 & & & \begin{matrix} a^{n}_{11} & a^{n}_{12} & \cdots & a^{n}_{1m} \\ a^{n}_{21} & a^{n}_{22} & \cdots & a^{n}_{2m} \\ \vdots & \vdots & & \vdots \\ a^{n}_{m1} & a^{n}_{m2} & \cdots & a^{n}_{mm} \end{matrix} & & & & \begin{matrix} c^{*n}_{11} & c^{*n}_{12} & \cdots & c^{*n}_{1k} \\ c^{*n}_{21} & c^{*n}_{22} & \cdots & c^{*n}_{2k} \\ \vdots & \vdots & & \vdots \\ c^{*n}_{m1} & c^{*n}_{m2} & \cdots & c^{*n}_{mk} \end{matrix}
\end{array}\right]$$

Figure C.2b.2i: The northwest and northeast component matrices $\hat{A}$ and $\hat{\overset{*}{C}}_k$ of the partitioned matrix $\left[\begin{array}{c|c} \hat{A} & \hat{\overset{*}{C}} \\ \hline \hat{\overset{*}{W}}_k & \hat{\overset{*}{Z}}_k \end{array}\right]$, with $k > 1$

$$
\left[\begin{array}{cccccc}
\begin{matrix} w^{*1}_{11} & w^{*1}_{12} & \cdots & w^{*1}_{1m} \\ w^{*1}_{21} & w^{*1}_{22} & \cdots & w^{*1}_{2m} \\ \vdots & \vdots & & \vdots \\ w^{*1}_{k1} & w^{*1}_{k2} & \cdots & w^{*1}_{km} \end{matrix} & & & \begin{matrix} z^{*1}_{11} & z^{*1}_{12} & \cdots & z^{*1}_{1k} \\ z^{*1}_{21} & z^{*1}_{22} & \cdots & z^{*1}_{2k} \\ \vdots & \vdots & & \vdots \\ z^{*1}_{k1} & z^{*1}_{k2} & \cdots & z^{*1}_{kk} \end{matrix} & & \\
& \begin{matrix} w^{*2}_{11} & w^{*2}_{12} & \cdots & w^{*2}_{1m} \\ w^{*2}_{21} & w^{*2}_{22} & \cdots & w^{*2}_{2m} \\ \vdots & \vdots & & \vdots \\ w^{*2}_{k1} & w^{*2}_{k2} & \cdots & w^{*2}_{km} \end{matrix} & & & \begin{matrix} z^{*2}_{11} & z^{*2}_{12} & \cdots & z^{*2}_{1k} \\ z^{*2}_{21} & z^{*2}_{22} & \cdots & z^{*2}_{2k} \\ \vdots & \vdots & & \vdots \\ z^{*2}_{k1} & z^{*2}_{k2} & \cdots & z^{*2}_{kk} \end{matrix} & \\
& & \ddots & & & \ddots \\
& & \begin{matrix} w^{*n}_{11} & w^{*n}_{12} & \cdots & w^{*n}_{1m} \\ w^{*n}_{21} & w^{*n}_{22} & \cdots & w^{*n}_{2m} \\ \vdots & \vdots & & \vdots \\ w^{*n}_{k1} & w^{*n}_{k2} & \cdots & w^{*n}_{km} \end{matrix} & & & \begin{matrix} z^{*n}_{11} & z^{*n}_{12} & \cdots & z^{*n}_{1k} \\ z^{*n}_{21} & z^{*n}_{22} & \cdots & z^{*n}_{2k} \\ \vdots & \vdots & & \vdots \\ z^{*n}_{k1} & z^{*n}_{k2} & \cdots & z^{*n}_{kk} \end{matrix}
\end{array}\right]
$$

Figure C.2b.2ii: The southwest and southeast component matrices $\hat{\overset{*}{W}}_k$ and $\hat{\overset{*}{Z}}$ of the partitioned matrix $\left[\begin{array}{c|c} \hat{A} & \overset{*}{C}_k \\ \hline \hat{\overset{*}{W}}_k & \hat{\overset{*}{Z}}_k \end{array}\right]$, with $k > 1$

$$\left[\begin{array}{c:c:c:c}
\begin{matrix} c_1^{11} & & & \\ & c_2^{11} & & \\ & & \ddots & \\ & & & c_k^{11} \end{matrix} &
\begin{matrix} c_1^{12} & & & \\ & c_2^{12} & & \\ & & \ddots & \\ & & & c_k^{12} \end{matrix} &
\cdots &
\begin{matrix} c_1^{1n} & & & \\ & c_2^{1n} & & \\ & & \ddots & \\ & & & c_k^{1n} \end{matrix} \\ \hdashline
\begin{matrix} c_1^{21} & & & \\ & c_2^{21} & & \\ & & \ddots & \\ & & & c_k^{21} \end{matrix} &
\begin{matrix} c_1^{22} & & & \\ & c_2^{22} & & \\ & & \ddots & \\ & & & c_k^{22} \end{matrix} &
\cdots &
\begin{matrix} c_1^{2n} & & & \\ & c_2^{2n} & & \\ & & \ddots & \\ & & & c_k^{2n} \end{matrix} \\ \hdashline
\vdots & \vdots & & \vdots \\ \hdashline
\begin{matrix} c_1^{n1} & & & \\ & c_2^{n1} & & \\ & & \ddots & \\ & & & c_k^{n1} \end{matrix} &
\begin{matrix} c_1^{n2} & & & \\ & c_2^{n2} & & \\ & & \ddots & \\ & & & c_k^{n2} \end{matrix} &
\cdots &
\begin{matrix} c_1^{nn} & & & \\ & c_2^{nn} & & \\ & & \ddots & \\ & & & c_k^{nn} \end{matrix}
\end{array}\right]$$

Figure C.2b.3: The south-east component matrix C_k of the partitioned matrix

$$\left[\begin{array}{c:c} C & 0 \\ \hdashline 0 & C_k \end{array}\right], \text{ with } k > 1$$

$$
\hat{CC}^{*}_{k} = \left[\begin{array}{c:c:c:c}
c_1^{11*1}c_{11}\;\; c_1^{11*1}c_{12}\cdots c_1^{11*1}c_{1k} & c_1^{12*2}c_{11}\;\; c_1^{12*2}c_{12}\cdots c_1^{12*2}c_{1k} & \cdots & c_1^{1n*n}c_{11}\;\; c_1^{1n*n}c_{12}\cdots c_1^{1n*n}c_{1k} \\
c_2^{11*1}c_{21}\;\; c_2^{11*1}c_{22}\cdots c_2^{11*1}c_{2k} & c_2^{12*2}c_{21}\;\; c_2^{12*2}c_{22}\cdots c_2^{12*2}c_{2k} & \cdots & c_2^{1n*n}c_{21}\;\; c_2^{1n*n}c_{22}\cdots c_2^{1n*n}c_{2k} \\
\vdots \qquad\quad \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \qquad\quad \vdots & & \vdots \qquad\quad \vdots \qquad\quad \vdots \\
c_m^{11*1}c_{m1}\;\; c_m^{11*1}c_{m2}\cdots c_m^{11*1}c_{mk} & c_m^{12*2}c_{m1}\;\; c_m^{12*2}c_{m2}\cdots c_m^{12*2}c_{mk} & \cdots & c_m^{1n*n}c_{m1}\;\; c_m^{1n*n}c_{m2}\cdots c_m^{1n*n}c_{mk} \\
\hdashline
c_1^{21*1}c_{11}\;\; c_1^{21*1}c_{12}\cdots c_1^{21*1}c_{1k} & c_1^{22*2}c_{11}\;\; c_1^{22*2}c_{12}\cdots c_1^{22*2}c_{1k} & \cdots & c_1^{2n*n}c_{11}\;\; c_1^{2n*n}c_{12}\cdots c_1^{2n*n}c_{1k} \\
c_2^{21*1}c_{21}\;\; c_2^{21*1}c_{22}\cdots c_2^{21*1}c_{2k} & c_2^{22*2}c_{21}\;\; c_2^{22*2}c_{22}\cdots c_2^{22*2}c_{2k} & \cdots & c_2^{2n*n}c_{21}\;\; c_2^{2n*n}c_{22}\cdots c_2^{2n*n}c_{2k} \\
\vdots \qquad\quad \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \qquad\quad \vdots & & \vdots \qquad\quad \vdots \qquad\quad \vdots \\
c_m^{21*1}c_{m1}\;\; c_m^{21*1}c_{m2}\cdots c_m^{21*1}c_{mk} & c_m^{22*2}c_{m1}\;\; c_m^{22*2}c_{m2}\cdots c_m^{22*2}c_{mk} & \cdots & c_m^{2n*n}c_{m1}\;\; c_m^{2n*n}c_{m2}\cdots c_m^{2n*n}c_{mk} \\
\hdashline
\vdots \qquad\quad \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \qquad\quad \vdots & & \vdots \qquad\quad \vdots \qquad\quad \vdots \\
\hdashline
c_1^{n1*1}c_{11}\;\; c_1^{n1*1}c_{12}\cdots c_1^{n1*1}c_{1k} & c_1^{n1*1}c_{11}\;\; c_1^{n1*1}c_{12}\cdots c_1^{n1*1}c_{1k} & \cdots & c_1^{n1*1}c_{11}\;\; c_1^{n1*1}c_{12}\cdots c_1^{n1*1}c_{1k} \\
c_2^{n1*1}c_{21}\;\; c_2^{n1*1}c_{22}\cdots c_2^{n1*1}c_{2k} & c_2^{n1*1}c_{21}\;\; c_2^{n1*1}c_{22}\cdots c_2^{n1*1}c_{2k} & \cdots & c_2^{n1*1}c_{21}\;\; c_2^{n1*1}c_{22}\cdots c_2^{n1*1}c_{2k} \\
\vdots \qquad\quad \vdots \qquad\quad \vdots & \vdots \qquad\quad \vdots \qquad\quad \vdots & & \vdots \qquad\quad \vdots \qquad\quad \vdots \\
c_m^{n1*1}c_{m1}\;\; c_m^{n1*1}c_{m2}\cdots c_m^{n1*1}c_{mk} & c_m^{n1*1}c_{m1}\;\; c_m^{n1*1}c_{m2}\cdots c_m^{n1*1}c_{mk} & \cdots & c_m^{n1*1}c_{m1}\;\; c_m^{n1*1}c_{m2}\cdots c_m^{n1*1}c_{mk}
\end{array}\right]
$$

(with $k > 1$)

Figure C.2b.4i: The northeast component matrix $\hat{CC}^{*}_{k}$ of the partitioned matrix $\left[\begin{array}{c:c} \hat{CA} & \hat{CC}^{*}_{k} \\ \hdashline C_k\hat{W}^{*}_{k} & C_k\hat{Z}^{*}_{k} \end{array}\right]$ with $k > 1$

$$C_k\hat{\overset{*}{W}}_k = \left[\begin{array}{ccc|ccc|c|ccc}
c_1^{11*1} w_{11} & c_1^{11*1} w_{12} & \cdots c_1^{11*1} w_{1m} & c_1^{12*2} w_{11} & c_1^{12*2} w_{12} & \cdots c_1^{12*2} w_{1m} & \cdots & c_1^{1n*n} w_{11} & c_1^{1n*n} w_{12} & \cdots c_1^{1n*n} w_{1m} \\
c_2^{11*1} w_{21} & c_2^{11*1} w_{22} & \cdots c_2^{11*1} w_{2m} & c_2^{12*2} w_{21} & c_2^{12*2} w_{22} & \cdots c_2^{12*2} w_{2m} & \cdots & c_2^{1n*n} w_{21} & c_2^{1n*n} w_{22} & \cdots c_2^{1n*n} w_{2m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
c_k^{11*1} w_{k1} & c_k^{11*1} w_{k2} & \cdots c_k^{11*1} w_{km} & c_k^{12*2} w_{k1} & c_k^{12*2} w_{k2} & \cdots c_k^{12*2} w_{km} & \cdots & c_k^{1n*n} w_{k1} & c_k^{1n*n} w_{k2} & \cdots c_k^{1n*n} w_{km} \\
\hline
c_1^{21*1} w_{11} & c_1^{21*1} w_{12} & \cdots c_1^{21*1} w_{1m} & c_1^{22*2} w_{11} & c_1^{22*2} w_{12} & \cdots c_1^{22*2} w_{1m} & \cdots & c_1^{2n*n} w_{11} & c_1^{2n*n} w_{12} & \cdots c_1^{2n*n} w_{1m} \\
c_2^{21*1} w_{21} & c_2^{21*1} w_{22} & \cdots c_2^{21*1} w_{2m} & c_2^{22*2} w_{21} & c_2^{22*2} w_{22} & \cdots c_2^{22*2} w_{2m} & \cdots & c_2^{2n*n} w_{21} & c_2^{2n*n} w_{22} & \cdots c_2^{2n*n} w_{2m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
c_k^{21*1} w_{k1} & c_k^{21*1} w_{k2} & \cdots c_k^{21*1} w_{km} & c_k^{22*2} w_{k1} & c_k^{22*2} w_{k2} & \cdots c_k^{22*2} w_{km} & \cdots & c_k^{2n*n} w_{k1} & c_k^{2n*n} w_{k2} & \cdots c_k^{2n*n} w_{km} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\hline
c_1^{n1*1} w_{11} & c_1^{n1*1} w_{12} & \cdots c_1^{n1*1} w_{1m} & c_1^{n2*2} w_{11} & c_1^{n2*2} w_{12} & \cdots c_1^{n2*2} w_{1m} & \cdots & c_1^{nn*n} w_{11} & c_1^{nn*n} w_{12} & \cdots c_1^{nn*n} w_{1m} \\
c_2^{n1*1} w_{21} & c_2^{n1*1} w_{22} & \cdots c_2^{n1*1} w_{2m} & c_2^{n2*2} w_{21} & c_2^{n2*2} w_{22} & \cdots c_2^{n2*2} w_{2m} & \cdots & c_2^{nn*n} w_{21} & c_2^{nn*n} w_{22} & \cdots c_2^{nn*n} w_{2m} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
c_k^{n1*1} w_{k1} & c_k^{n1*1} w_{k2} & \cdots c_k^{n1*1} w_{km} & c_k^{n2*2} w_{k1} & c_k^{n2*2} w_{k2} & \cdots c_k^{n2*2} w_{km} & \cdots & c_k^{nn*n} w_{k1} & c_k^{nn*n} w_{k2} & \cdots c_k^{nn*n} w_{kn}
\end{array}\right]$$

(with $k > 1$)

Figure C.2b.4ii: The southwest component matrix $C_k\hat{\overset{*}{W}}_k$ of the partitioned matrix $\left[\begin{array}{c|c} C\hat{A} & C\hat{\overset{*}{C}}_k \\ \hline C_k\hat{\overset{*}{W}}_k & C_k\hat{\overset{*}{Z}}_k \end{array}\right]$, with $k > 1$

$$C_k\hat{\overset{*}{Z}}_k = \left[\begin{array}{ccc|ccc|c|ccc}
c_1^{11*1}z_{11} & c_1^{11*1}z_{12}\cdots & c_1^{11*1}z_{1k} & c_1^{12*2}z_{11} & c_1^{12*2}z_{12}\cdots & c_1^{12*2}z_{1k} & \cdots & c_1^{1n*n}z_{11} & c_1^{1n*n}z_{12}\cdots & c_1^{1n*n}z_{1k} \\
c_2^{11*1}z_{21} & c_2^{11*1}z_{22}\cdots & c_2^{11*1}z_{2k} & c_2^{12*2}z_{21} & c_2^{12*2}z_{22}\cdots & c_2^{12*2}z_{2k} & \cdots & c_2^{1n*n}z_{21} & c_2^{1n*n}z_{22}\cdots & c_2^{1n*n}z_{2k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
c_k^{11*1}z_{k1} & c_k^{11*1}z_{k2}\cdots & c_k^{11*1}z_{kk} & c_k^{12*2}z_{k1} & c_k^{12*2}z_{k2}\cdots & c_k^{12*2}z_{kk} & \cdots & c_k^{1n*n}z_{k1} & c_k^{1n*n}z_{k2}\cdots & c_k^{1n*n}z_{kk} \\
\hline
c_1^{21*1}z_{11} & c_1^{21*1}z_{12}\cdots & c_1^{21*1}z_{1k} & c_1^{22*2}z_{11} & c_1^{22*2}z_{12}\cdots & c_1^{22*2}z_{1k} & \cdots & c_1^{2n*n}z_{11} & c_1^{2n*n}z_{12}\cdots & c_1^{2n*n}z_{1k} \\
c_2^{21*1}z_{21} & c_2^{21*1}z_{22}\cdots & c_2^{21*1}z_{2k} & c_2^{22*2}z_{21} & c_2^{22*2}z_{22}\cdots & c_2^{22*2}z_{2k} & \cdots & c_2^{2n*n}z_{21} & c_2^{2n*n}z_{22}\cdots & c_2^{2n*n}z_{2k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
c_k^{21*1}z_{k1} & c_k^{21*1}z_{k2}\cdots & c_k^{21*1}z_{kk} & c_k^{22*2}z_{k1} & c_k^{22*2}z_{k2}\cdots & c_k^{22*2}z_{kk} & \cdots & c_k^{2n*n}z_{k1} & c_k^{2n*n}z_{k2}\cdots & c_k^{2n*n}z_{kk} \\
\hline
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
\hline
c_1^{n1*1}z_{11} & c_1^{n1*1}z_{12}\cdots & c_1^{n1*1}z_{1k} & c_1^{n2*2}z_{11} & c_1^{n2*2}z_{12}\cdots & c_1^{n2*2}z_{1k} & \cdots & c_1^{nn*n}z_{11} & c_1^{nn*n}z_{12}\cdots & c_1^{nn*n}z_{1k} \\
c_2^{n1*1}z_{21} & c_2^{n1*1}z_{22}\cdots & c_2^{n1*1}z_{2k} & c_2^{n2*2}z_{21} & c_2^{n2*2}z_{22}\cdots & c_2^{n2*2}z_{2k} & \cdots & c_2^{nn*n}z_{21} & c_2^{nn*n}z_{22}\cdots & c_2^{nn*n}z_{2k} \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\
c_k^{n1*1}z_{k1} & c_k^{n1*1}z_{k2}\cdots & c_k^{n1*1}z_{kk} & c_k^{n2*2}z_{k1} & c_k^{n2*2}z_{k2}\cdots & c_k^{n2*2}z_{kk} & \cdots & c_k^{nn*n}z_{k1} & c_k^{nn*n}z_{k2}\cdots & c_k^{nn*n}z_{kk}
\end{array}\right]$$

(with $k > 1$)

Figure C.2b.4iii: The southeast component matrix $C_k\hat{\overset{*}{Z}}_k$ of the partitioned matrix $\left[\begin{array}{c|c} C\hat{A} & C\hat{\overset{*}{C}}_k \\ \hline C_k\hat{\overset{*}{W}}_k & C_k\hat{\overset{*}{Z}}_k \end{array}\right]$, with $k > 1$

ANNEX D.1

THE PARTITIONED-MATRIX APPROACH TO DERIVING THE INVERSE OF A MATRIX:

Given an augmented matrix of the form $\left[\begin{array}{c|c} \alpha & \beta \\ \hline \alpha & \delta \end{array}\right]$ it is possible to find its inverse $\left[\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right] = \left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \beta \end{array}\right]^{-1}$ by pre-multiplying 1/ the former by the latter to obtain the identity matrix $\left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right]$, that is

$$\left[\begin{array}{c|c} E & F \\ \hline G & H \end{array}\right] \left[\begin{array}{c|c} \alpha & \beta \\ \hline \gamma & \delta \end{array}\right] = \left[\begin{array}{c|c} I & 0 \\ \hline 0 & I \end{array}\right] \qquad \text{(D.1.1)}$$

Using linear algebraic rules of matrix multiplication we obtain

$$E\alpha + F\gamma = I \qquad \text{(D.1.2)}$$

$$E\beta + F\delta = 0 \qquad \text{(D.1.3)}$$

$$G\alpha + H\gamma = 0 \qquad \text{(D.1.4)}$$

$$G\beta + H\delta = I \qquad \text{(D.1.5)}$$

Solving the elements of the identity matrix in sequence from quadrant I to IV[2/] we get:

1/ It is also possible to post-multiply the former by the latter to obtain a different, but equivalent solution since they hold for the same set of equations, see Hadley (1961, p. 109) and Miller (1969, p. 118).

2/ Quadrant I = northwest; Quadrant II = northeast; Quadrant III = southwest; and Quadrant IV = southeast of the partitioned matrix.

$$E\alpha = I - F\gamma$$ by subtracting $(F\gamma)$ from both sides

$$E = (I - F\gamma)\alpha^{-1}$$ by post-multiplying both sides by (α^{-1})

$$E = \alpha^{-1} - F\gamma\alpha^{-1} \qquad \text{(D.1.6)}$$

Substituting the solution for E into equation (D.1.3) and solving for F we get

$$(\alpha^{-1} - F\gamma\alpha^{-1})\beta + F\delta = 0$$

$$\alpha^{-1}\beta - F\gamma\alpha^{-1}\beta + F\delta = 0$$ by expansion

$$\alpha^{-1}\beta + F(\delta - \gamma\alpha^{-1}\beta) = 0$$ by rearranging terms

$$F(\delta - \gamma\alpha^{-1}\beta) = -\alpha^{-1}\beta$$ by subtracting $(\alpha^{-1}\beta)$ from both sides

$$F = -\alpha^{-1}\beta(\delta - \gamma\alpha^{-1}\beta)$$ by multiplying both sides by $(\delta - \gamma\alpha^{-1}\beta)$ (D.1.7)

The solution for G can be obtained from equation (D.1.4) as

$$G\alpha = -H\gamma$$ by subtracting $(H\gamma)$ from both sides

$$G = -H\gamma\alpha^{-1}$$ post-multiplying both sides by (α^{-1}) (D.1.8)

Substituting the solution for (g) into equation (D.1.5) and solving for (H) we get:

$$-H\gamma\alpha^{-1}\beta + H\delta = I$$

$$H(\delta - \gamma\alpha^{-1}\beta) = I$$ by rearranging terms

$$H = (\delta - \gamma\alpha^{-1}\beta)^{-1} \qquad \text{(D.1.9)}$$

Note that the solution for (G) is already expressed in terms of (H). It is also possible to transform (E) and (F) into expressions of (H) as follows:

substituting the solution for (H) into equation (D.1.7) we get

$$F = -\alpha^{-1}\beta H \qquad \text{(D.1.10)}$$

substituting this form of the solution for (F) into equation (D.1.6) we get

$$\begin{aligned} E &= \alpha^{-1} - (-\alpha^{-1}\beta H)\gamma\alpha^{-1} \\ &= \alpha^{-1} + \alpha^{-1}\beta H\gamma\alpha^{-1} \\ &= \alpha^{-1}\,(I + \beta H\gamma\alpha^{-1}) \end{aligned} \qquad \text{(D.1.11)}$$

The complete solution of the matrix $\left[\begin{array}{c|c}\alpha & \beta \\ \hline \gamma & \delta\end{array}\right]$ can now be stated in the form of another matrix by combining the results for each element obtained in equations (D.1.8) through (D.1.11).

$$\left[\begin{array}{c|c}\alpha & \beta \\ \hline \gamma & \delta\end{array}\right]^{-1} = \left[\begin{array}{c|c}E & F \\ \hline G & H\end{array}\right] = \left[\begin{array}{c|c}\alpha^{-1}(I + \beta H\gamma\delta^{-1}) & -\alpha^{-1}\beta H \\ \hline -H\gamma\alpha^{-1} & H\end{array}\right] \qquad \text{(D.1.12)}$$

where $H = (\delta - \gamma\alpha^{-1}\beta)^{-1}$

This partitioned approach to solving $\left[\begin{array}{c|c}\alpha & \beta \\ \hline \gamma & \delta\end{array}\right]^{-1}$ has two computational advantages over the direct inversion of $\left[\begin{array}{c|c}\alpha & \beta \\ \hline \gamma & \delta\end{array}\right]$:

First, if the matrix (α^{-1}) is already known, it is very easy to obtain the matrix (H), since the order of (H) is normally much less than the order of (α). As a result, the inversion of $(\delta - \gamma\alpha^{-1}\beta)^{-1}$ is computationally much simpler, and therefore less costly than the direct inversion of $\left[\begin{array}{c|c}\alpha & \beta \\ \hline \gamma & \delta\end{array}\right]$. Thus, it is possible for the matrices δ, γ, and β to be altered more frequently than α, with minimal additional cost.

Second, the solution for the matrices (E), (F), and (G) are a function of (H) and can be obtained simply by multiplying (H) with known matrices, α^{-1}, β, and γ.

ANNEX D.2

THE PARTITIONED MATRIX SOLUTION OF THE AUGMENTED MATRIX $\begin{bmatrix} I-\Theta & -\Gamma \\ -T & I-\Lambda \end{bmatrix}$:

Setting 1/

$$I - \Theta = \alpha$$
$$-\Gamma = \beta$$
$$-T = \gamma$$
$$I - \Lambda = \delta$$

we get:

$$\begin{bmatrix} I-\Theta & -\Gamma \\ -T & I-\Lambda \end{bmatrix}^{-1} = \begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix}^{-1} = \begin{bmatrix} E & F \\ G & H \end{bmatrix} \tag{D.2.1}$$

setting $(I-\Theta)^{-1} = B$ and using equations (D.1.8) to (D.1.11) we get:

$$\begin{aligned} E &= \alpha^{-1} (I + \beta H \gamma \alpha^{-1}) \\ &= (I - \Theta)^{-1} [I + (-\Gamma)H(-T)(I - \Theta)^{-1}] \\ &= B(I + \Gamma HTB) \end{aligned} \tag{D.2.2}$$

$$\begin{aligned} F &= -\alpha^{-1}\beta H \\ &= -(I - \Theta)^{-1}(-\Gamma)H \\ &= B\Gamma H \end{aligned} \tag{D.2.3}$$

1/ From equation (3.14).

$$G = -H\gamma\alpha^{-1}$$
$$= -H(-T)(I - \Theta)^{-1}$$
$$= HTB \qquad \text{(D.2.4)}$$

and

$$H = (\delta - \gamma\alpha^{-1}\beta)^{-1}$$
$$= [(I - \Lambda) - (-T)(I - \Theta)^{-1}(-\Gamma)]^{-1}$$
$$= [(I - \Lambda) - TB\Gamma] \qquad \text{(D.2.5)}$$

substituting

$$TB\Gamma = \Phi \qquad \text{(D.2.6)}$$

we can write (D.2.5) as:

$$H = [(I - \Lambda) - \Phi]^{-1} \qquad \text{(D.2.7)}$$

this matrix can be converted into the product of two matrices: first the non-inverted expression $[(I-\Lambda) - \Phi]$ can be transformed into the form $[I-\Lambda(I-\Phi)^{-1}](I-\Phi)$ as follows:

$$[I-\Phi-\Lambda] = (I-\Phi-\Lambda)[(I-\Phi)^{-1}(I-\Phi)]$$ by multiplying both sides by $I = (I-\phi)^{-1}(I-\phi)$

$$= (I-\Phi) - \Lambda(I-\Phi)^{-1}(I-\Phi)$$ by rearranging terms and expanding

$$= [I - \Lambda(I - \Phi)^{-1}](I - \Phi) \qquad \text{(D.2.8)}$$

substituting

$$\overline{\Psi} = (I-\Phi)^{-1} \qquad \text{(D.2.9)}$$

we can write (D.2.8) as:

$$(I - \Phi - \Lambda) = (I - \Lambda\overline{\Psi})\overline{\Psi}^{-1} \qquad \text{(D.2.10)}$$

Substituting (D.2.10) into equation (D.2.7) and using the procedure for inverting the product of two matrices (B. Noble, 1969, p. 15).

we get

$$
\begin{aligned}
H &= [(I - \Lambda) - \Phi]^{-1} \\
&= [(I - \Lambda\overline{\Psi})\overline{\Psi}^{-1}]^{-1} \\
&= \overline{\Psi}(I - \Lambda\overline{\Psi})^{-1} = \Psi
\end{aligned}
\tag{D.2.11}
$$

By combining the results obtained in equations (D.2.2) to (D.2.5) after substituting $H = \Psi$ from (D.2.11) into each, we get

$$
\left[\begin{array}{c|c} I - \Theta & -\Gamma \\ \hline -T & I-\Lambda \end{array}\right]^{-1} = \left[\begin{array}{c|c} B(I+\Gamma\Psi TB) & B\Gamma\Psi \\ \hline \Psi TB & \Psi \end{array}\right]
\tag{D.2.12}
$$

where $\Psi = \overline{\Psi}(I-\Lambda\overline{\Psi})^{-1}$

and $\overline{\Psi} = (I-\Phi)^{-1}$

and $\Phi = TB\Gamma = T(I-\Theta)^{-1}\Gamma$

or

$$
\Psi = [I - T(I-\Theta)^{-1}\Gamma]^{-1} \{I - \Lambda[I-T(I-\Theta)^{-1}\Gamma]^{-1}\}^{-1}
\tag{D.2.13}
$$

ANNEX D.3

THE SUBJOINED INVERSE METHOD FOR DERIVING THE SUBMATRICES D_{11} AND D_{21} OF THE PARTITIONED MATRIX SOLUTION

In Chapter 2 one of the reasons given for preferring the open MRIO model solution of

$$X = (I-C\hat{A})^{-1}CY \qquad \text{(D.3.1a)}$$

$$= BCY \qquad \text{(D.3.1b)}$$

over the more commonly used solution

$$X = (C^{-1}-\hat{A})^{-1}Y \qquad \text{(D.3.2)}$$

was that the component B of the former resembled the interregional model multiplier matrix $(I-A)^{-1}$. This fact can be used to adapt Miyazawa's subjoined inverse (Miyazawa, 1963) to the solution of the B_{11} component of the submatrix D_{11}.

In equation (C.2.16) it was noted that the open model MRIO balancing equations could be written as

$$X = C(\hat{A}X + \hat{\bar{C}} + \tilde{Y}) \qquad \text{(D.3.3)}$$

Substituting the analogue consumption function from (C.2.19) in the above equation we can write

$$X = C(\hat{A}X + \hat{\overset{*}{C}}W + \hat{Y}) \qquad \text{(D.3.4)}$$

In the open model the (n x 1) vector W is itself a function of X, i.e.

$$W = \hat{\overset{*}{W}}X \qquad \text{(D.3.5)}$$

where $\hat{\overset{*}{W}}$ is an (n x nm) block-diagonal matrix as described in equation (C.2.12). It is assumed in (D.3.5) that $C_k = I$. In the more general case when $C_k \neq I$, the vector W should be written as

$$W = C_k\hat{\overset{*}{W}}X \qquad \text{(D.3.6)}$$

where $C_k\hat{\overset{*}{W}}$ is a full matrix rather than a block-diagonal matrix. Substituting this form of W into equation (D.3.4) we get

$$X = C(\hat{A}X + \hat{\overset{*}{C}}C_k\hat{\overset{*}{W}}X + \tilde{Y}) \qquad \text{(D.3.7)}$$

Expanding this equation and substituting the notation introduced in Chapter 3, we can write

$$X = \theta X + \Gamma TX + C\tilde{Y} \qquad \text{(D.3.7)}$$

The solution of this system of equations is

$$X = (I-\theta-\Gamma T)^{-1}\, C\tilde{Y} \qquad \text{(D.3.8)}$$

The (mn x mn) 'enlarged multiplier matrix' $[(I-\theta-\Gamma T)^{-1}C]$ shows the total effect of non-consumption final demand $\tilde{Y}$ on gross outputs X via intermediate interindustry activity θ and induced consumption activity ΓT.

This enlarged multiplier matrix can also be written as the product of two separate multiplier matrices: the first being the traditional Leontief-type inverse matrix $B = (I-\theta)^{-1}$, and the second being what Miyazawa has called the 'subjoined inverse matrix' $(I-\Gamma TB)^{-1}$ which shows the induced effects of endogenous changes in the consumption expenditures of each region's 'household' sector.

This requires that the non-inverted expression $(I-\theta-\Gamma T)$ first be converted into the form $[I-\Gamma T(I-\theta)^{-1}]\,(I-\theta)$. Post-multiplying $(I-\theta-\Gamma T)$ by the identity matrix

$$I = [(I-\theta)^{-1}(I-\theta)]$$

we get

$$\begin{aligned}(I-\theta-\Gamma T) &= (I-\theta-\Gamma T)\ [(I-\theta)^{-1}(I-\theta)] \\ &= (I-\theta)-\Gamma T(I-\theta)^{-1}(I-\theta) \quad \text{by rearranging} \\ &= [I-\Gamma T(I-\theta)^{-1}](I-\theta) \quad \text{by factoring out the common expression } (I-\theta)\end{aligned}$$

Substituting $B = (I-\theta)^{-1}$ into the previous expression we can write

$$(I-\theta-\Gamma T) = (I-\Gamma TB)B^{-1} \qquad \text{(D.3.9)}$$

Substituting this expression into (D.3.8) and using the procedure for inverting the product of two matrices (Noble, 1969, p.15), we get the desired result

$$
\begin{aligned}
X &= (I-\theta-\Gamma T)^{-1}C\tilde{Y} \\
&= [(I-\Gamma TB)B^{-1}]^{-1}C\tilde{Y} \\
&= B(I-\Gamma TB)^{-1}C\tilde{Y} \qquad \text{(D.3.10)}
\end{aligned}
$$

where $B = (I-\theta)^{-1}$.

From the point of view of economic interpretation, the product of the two inverses is more useful than the enlarged inverse, because the former more clearly distinguishes between the inverse reflecting consumption activity (via the production process), and the inverse (of the same order matrix) reflecting production activity only, a distinction missing from the enlarged inverse.

The subjoined inverse can in turn be converted into a more practical form by introducing new matrices and showing the relationship between 'inter-income group' activities of the interrelated regions.

If the inverse $(I-\Gamma TB)^{-1}$ in equation (D.3.10) exists, it is possible to infer that the term $\sum_{\alpha=1}^{\infty}(\Gamma TB)^{\alpha}$ is convergent, in which case

$$
B(I-\Gamma TB)^{-1} = B[I + \sum_{\alpha=1}^{\infty} (\Gamma TB)^{\alpha}] \qquad \text{(D.3.11)}
$$

The second expression in brackets can also be written as

$$
\sum_{\alpha=1}^{\infty} (\Gamma TB)^{\alpha} = \sum_{\alpha=1}^{\infty} \Gamma(TB\Gamma)^{\alpha-1}TB \qquad \text{(D.3.12)}
$$

where the consumption expenditures Γ in the αth round is induced by income $TB\Gamma$ earned in the $(\alpha-1)^{th}$ round which was in turn induced by the income earned in the initial round TB.

Setting

$$\phi = TB\Gamma \tag{D.3.13}$$

into equation (D.3.12), we can write

$$\sum_{\alpha=1} \Gamma(TB\Gamma)^{\alpha-1}TB = \Gamma(\sum_{\alpha=0}^{\infty} \phi^{\alpha})TB \tag{D.3.14}$$

Setting

$$\overline{\Psi} = \sum_{\alpha=0}^{\infty} \phi^{\alpha} = (I-\phi)^{-1} \tag{D.3.15}$$

into the previous equation we can write

$$\Gamma(\sum_{\alpha=0}^{\infty} \phi^{\alpha})TB = \Gamma\,\overline{\Psi}TB \tag{D.3.16}$$

Equations (D.3.11) through (D.3.16) show that the convergence of $\sum_{\alpha}(\Gamma TB)^{\alpha}$ and that of $\sum_{\alpha}\phi^{\alpha}$ are equivalent. This is important because as pointed out by Miyazawa (1963, p. 100) the convergence condition of $\sum_{\alpha}\phi^{\alpha}$ is also that of $\sum_{\alpha}(\theta+\Gamma T)$, which is necessary for solving the enlarged inverse matrix $(I-\theta-\Gamma T)^{-1}$.

Substituting the result from (D.3.16) into equation (D.3.11) we get the desired result 1/

$$B(I-\Gamma TB)^{-1} = B(I+\Gamma\overline{\Psi}TB) \tag{D.3.17}$$

1/ The same result can be obtained (Miyazawa, 1968, p. 44), by setting $\overline{\Psi}=(I-\phi)^{-1}$, where $\phi=TB\Gamma$, and using the fact that by definition $\overline{\Psi}(I-TB\Gamma)=I$:

$$\Gamma TB = \Gamma(I)TB = \Gamma\overline{\Psi}(I-TB\Gamma)TB$$

or $$\Gamma TB = \Gamma\overline{\Psi}TB(I-\Gamma TB)$$

hence $$I-\Gamma TB = I-\Gamma\overline{\Psi}TB(I-\Gamma TB)$$

or $$(I-\Gamma TB)+[\Gamma\overline{\Psi}TB(I-\Gamma TB)] = I$$

$$[I+\Gamma\overline{\Psi}TB](I-\Gamma TB) = I$$

therefore $$(I-\Gamma TB)^{-1} = (I+\Gamma\overline{\Psi}TB)$$

where the identity matrices I in the first and second equations have the order of (n x n), and those in the third and following equations have the order of (mn x mn), respectively.

Substituting (D.3.17) into (D.3.10) we get

$$X = B(I + \Gamma\overline{\Psi}TB)C\tilde{Y} \tag{D.3.18a}$$

$$= B_{11}C\tilde{Y} \tag{D.3.18b}$$

$$= D_{11}\tilde{Y} \tag{D.3.18c}$$

as desired. Here X is only a function of exogenously determined non-consumption final demand $\tilde{Y}$. Both intra-household sector transactions Λ and exogenously determined income ${}^{w}\tilde{Y}$ are assumed to be zero. As such this solution of X is not as general as the partitioned matrix solution in Chapter 3.

Substituting (D.3.18a) into the open model solution for wage and salary income W in equation (D.3.6), we get

$$W = TB(I + \Gamma\overline{\Psi}TB)C\tilde{Y}$$

$$= (I + TB\Gamma\overline{\Psi})TD\tilde{Y}$$

$$= (I + \phi\overline{\Psi})TD\tilde{Y} \tag{D.3.19}$$

in which $I + \phi\overline{\Psi} = \overline{\Psi}$, because $(I-\phi)\overline{\Psi}\equiv I$.

Thus,

$$W = \overline{\Psi}TD\tilde{Y} \tag{D.3.20a}$$

$$= D_{21}\tilde{Y} \tag{D.3.20b}$$

This result also coincides with that obtained using the partitioned-matrix approach, if it is assumed that ${}^{w}\tilde{Y} = 0$.

In the subjoined matrix approach at partially closing the input-output model, the three matrices (T) with dimension (n x mn), (B) with dimension (mn x mn), and (Γ) with dimension (mn x n) can be multiplied in any of three sequences (1)$TB\Gamma$, (2)$B\Gamma T$, and (3)ΓTB (resulting in matrices of dimensions (n x n), (mn x mn) and (mn x mn) respectively, depending on which aspect of the interrelated propagation process is being emphasized. The first views the propagation process from the point of view of income formation, the second from the point of view of output production, and the third from the point of view of consumption expenditure.

Thus, the dependence of the endogenous vector (X) of gross outputs on the exogenous vector ($\tilde{Y}$) of non-consumption final demand which is expressed in equation (D.3.18a) can be written in three different ways (Miyazawa, 1963, p. 96):

(a) from the income-formation side ($T B \Gamma = \phi$)

$$
\begin{aligned}
X &= BC\tilde{Y} + B\Gamma(I-\phi)^{-1}TBC\tilde{Y} \\
&= BC\tilde{Y} + B\Gamma[I + TB\Gamma + (TB\Gamma)^2 + \ldots]TBC\tilde{Y}
\end{aligned}
\qquad \text{(D.3.21)}
$$

(b) from the consumption-expenditure side

$$
\begin{aligned}
X &= BC\tilde{Y} + B\Gamma(I-\phi)^{-1}TBC\tilde{Y} \\
&= B(I + \Gamma(I-\phi)^{-1}TB)C\tilde{Y} \\
&= B(I + \Gamma TB + (\Gamma TB)^2 + \ldots)C\tilde{Y}
\end{aligned}
\qquad \text{(D.3.22)}
$$

(c) from the production side

$$
\begin{aligned}
X &= BC\tilde{Y} + B\Gamma(I-\phi)^{-1}TBC\tilde{Y} \\
&= [I + B\Gamma(I-\phi)^{-1}T]BC\tilde{Y} \\
&= [I + B\Gamma T + (B\Gamma T)^2 + \ldots]BC\tilde{Y}
\end{aligned}
\qquad \text{(D.3.23)}
$$

Miyazawa points out that $B_{11} = B(I + \Gamma\overline{\Psi}TB)$ can be obtained in all these cases by projecting the propagation process into the income-formation side $\phi = TB\Gamma$. However, in deriving the sum of the geometrical progression from the consumption side (ΓTB), or the production side ($B\Gamma T$), the formula $[B(I+\Gamma\overline{\Psi}TB)]$ is not obtained directly, but rather indirectly via the equation $[B(I-\Gamma TB)^{-1}]$. From this he concludes "that <u>the income formation side has a homogenous character, which contrasts strikingly with the heterogenous character of production activity and consumption activity</u>." (<u>Ibid.</u>, p. 97).

In contrast to equation (D.3.8) where the propagation process occurs simultaneously on all three fronts, in equations (D.3.23) it is assumed that the entire propagation process is a succession of separate two-step movements; in the first step, the effect on production is direct and represented entirely by the effect of matrix D = BC, where in the second step, the effect is induced via the income-formation and consumption expenditure sides. Thus, "the two-step and the simultaneous processes have the same sum, even though obviously the truncated multiplier in the case of equations (D.3.21) - (D.3.22) has a larger value than the truncated multiplier in the case of equation (D.3.8)." 2/ (*Ibid.*)

2/ Actually it is the iterative form of (D.3.8), i.e.,

$$X = \tilde{Y} + (\phi+\Gamma T)C\tilde{Y} + (\phi+\Gamma T)^2 C\tilde{Y} + \ldots$$

that Miyazawa refers to in the quote above.

ANNEX E

Computer Results Using 1963 Data with C_k=I to Illustrate the Equivalence of the Standard and Partitioned Matrix Solution of the Augmented MRIO Model

E.1. Data base for 3 Regions and 3 Industries

E.2. Numerical illustration of the open MRIO model solution

E.3 Numerical illustration of the augmented MRIO model standard solution

E.4 Numerical illustration of the MRIO model partitioned matrix solution

TABLE.E1.1. INDUSTRIAL CLASSIFICATION SCHEME
(79 to 3 Industries)

C-1 Agriculture & Mining

IO- 1	Livestock & livestock prdts.
IO- 2	Other agricultural prdts.
IO- 5	Iron & ferro. ores mining
IO- 6	Nonferrous metal ores mining
IO- 7	Coal mining
IO- 8	Crude petro., natural gas
IO- 9	Stone & clay mining
IO-10	Chem. & fert. mineral mining

C-2 Construction & manufacturing

IO-11	New construction
IO-12	Maint. & repair construction
IO-13	Ordnance & accessories
IO-14	Food & kindred prdts.
IO-15	Tobacco manufactures
IO-16	Fabrics
IO-17	Textile prdts.
IO-18	Apparel
IO-19	Misc. textile prdts.
IO-20	Lumber & wood prdts.
IO-21	Wooden containers
IO-22	Household furniture
IO-23	Other furniture
IO-24	Paper & allied prdts.
IO-25	Paperboard containers
IO-26	Printing & publishing
IO-27	Chemicals, selected prdts.
IO-28	Plastics & synthetics
IO-29	Drugs & cosmetics
IO-30	Paint & allied prdts.
IO-31	Petroleum, related inds.
IO-32	Rubber, misc. plastics
IO-33	Leather tanning & prdts.
IO-34	Footwear, leather prdts.
IO-35	Glass & glass prdts.
IO-36	Stone & clay prdts.
IO-37	Primary iron, steel mfr.
IO-38	Primary nonferrous mfr.
IO-39	Metal containers
IO-40	Fabricated metal prdts.
IO-41	Screw mach. prdts., etc.
IO-42	Other fab. metal prdts.
IO-43	Engines & turbines
IO-44	Farm mach. & equip.
IO-45	Construction mach. & equip.
IO-46	Material hand. mach. & equip.
IO-47	Metalworking mach. & equip.
IO-48	Special mach. & equip.
IO-49	General mach. & equip.
IO-50	Machine shop prdts.
IO-51	Office, computing machines
IO-52	Service industry machines
IO-53	Elec. transmission equip.
IO-54	Household appliances
IO-55	Electric lighting equip.
IO-56	Radio, TV, etc., equip.
IO-57	Electronic components
IO-58	Misc. electrical mach.
IO-59	Motor vehicles, equip.
IO-60	Aircraft & parts
IO-61	Other transport. equip.
IO-62	Professional, scien. instru.
IO-63	Medical, photo. equip.
IO-64	Misc. manufacturing

C-3 Services*

IO- 3	Forestry & fishery prdts.
IO- 4	Ag., for., & fish. services
IO-65	Transportation & warehousing
IO-66	Communications, exc. brdcast.
IO-67	Radio & TV brdcasting
IO-68	Elec., gas, water, & san. serv.
IO-69	Wholesale & retail trade
IO-70	Finance & insurance
IO-71	Real estate & rental
IO-72	Hotels; repair serv., exc. auto
IO-73	Business services
IO-74	Research & development
IO-75	Automobile repair & services
IO-76	Amusements
IO-77	Med., ed. serv., nonprofit org.
IO-78	Federal gov't. enterprises
IO-79	State & local gov't. enterp.

*Non-traded commodities

TABLE E1. 2. **REGIONAL CLASSIFICATION SCHEME**
(51, 9, and 3 Regions)

R-1 NORTH			R-2 SOUTH			R-3 WEST		
Regional Classification			Regional Classification			Regional Classification		
9-Region	51-Region		9-Region	51-Region		9-Region	51-Region	
1 New England	6	Connecticut	5 South Atlantic	7	Delaware	4 West North Central	14	Iowa
	18	Maine		8	District of Columbia		15	Kansas
	20	Massachusetts		9	Florida		22	Minnesota
	28	New Hampshire		10	Georgia		24	Missouri
	38	Rhode Island		19	Maryland		26	Nebraska
	44	Vermont		32	North Carolina		33	North Dakota
				39	South Carolina		40	South Dakota
2 Middle Atlantic	29	New Jersey		45	Virginia			
	31	New York		47	West Virginia	8 Mountain	2	Arizona
	37	Pennsylvania					5	Colorado
			6 East South Central	1	Alabama		11	Idaho
3 East North Central	12	Illinois		16	Kentucky		25	Montana
	13	Indiana		23	Mississippi		27	Nevada
	21	Michigan		41	Tennessee		30	New Mexico
	34	Ohio					43	Utah
	48	Wisconsin	7 West South Central	3	Arkansas		49	Wyoming
				17	Louisiana			
				35	Oklahoma	9 Pacific	4	California
				42	Texas		36	Oregon
							46	Washington
							50	Alaska
							51	Hawaii

Table E1.3. Interregional Trade Flows for the Three Commodities C1, C2, and C3

1963 TRADE FLOW TABLE AGRIC/MINING

ROW	COLUMN	1 NORTH	2 SOUTH	3 WEST	4 RTRI	5 TOTAL SUPPLY
1	NORTH	13885510.	2504168.	864504.	1264535.	18518717.
2	SOUTH	5864761.	16529239.	2157973.	1963842.	26515815.
3	WEST	3664383.	2892995.	20750605.	2331973.	29639956.
4	RTRO	1264535.	1963842.	2331973.	0.	5560350.
5	TOTAL DEMAND	24679189.	23890244.	26105054.	5560350.	80234838.

1963 TRADE FLOW TABLE CONSTR/MANUF

ROW	COLUMN	1	2	3	4	5
1	NORTH	211817980.	32515533.	29878134.	19932190.	294143837.
2	SOUTH	26978459.	90580855.	10749477.	6684616.	134993407.
3	WEST	14501896.	8668978.	92431186.	6175578.	121777638.
4	RTRO	19932190.	6684616.	6175578.	0.	32792384.
5	TOTAL DEMAND	273230525.	138449982.	139234375.	32792384.	583707266.

1963 TRADE FLOW TABLE SERVICES

ROW	COLUMN	1	2	3	4	5
1	NORTH	190935733.	15094161.	9301656.	6615921.	221947471.
2	SOUTH	8782296.	88977907.	5954903.	3584371.	107299477.
3	WEST	8440907.	5805979.	95095110.	3804355.	113146352.
4	RTRO	6615921.	3584371.	3804355.	0.	14004647.
5	TOTAL DEMAND	214774856.	113462419.	114156024.	14004647.	456397946.

Table E1. 4. Transactions Tables for the Three Regions R1, R2, and R3

1963 TRANSACTIONS TABLE -NORTH- (1000$)

ROW	COLUMN	1 AGRIC/MINING	2 MANUF/CONSTR	3 SERVICES	4 PCE	5 OTHER FD	6 TOTAL CONSUMPTION
1	PRIMARY	4325541.	14463887.	1427624.	220297.	2250424.	24673773.
2	SECONDARY	2527554.	110897032.	15972874.	61068661.	70446878.	260912999.
3	TERTIARY	2906913.	36108883.	42874230.	114615604.	11695435.	208201065.
4	WAGE & SAL	1324324.	64607700.	60557788.	1484767.	53575653.	181550232.
5	OTHER PS[1]	7430320.	55774308.	94617759.	2174903.	-48852255.	111145035.
6	TOTAL PROD.	18514652.	281851810.	215450275.	181550232.	89116135.	786483104.
	1/of which IMPORTS	1572159.	6959708.	1593842.	0.	-10344802.	-219093.

1963 TRANSACTIONS TABLE -SOUTH- (1000$)

ROW	COLUMN	1	2	3	4	5	6
1	PRIMARY	3919132.	13077034.	882206.	1341437.	4662591.	23882400.
2	SECONDARY	3707227.	47837774.	7989131.	33156699.	41238838.	133929669.
3	TERTIARY	4962016.	17227194.	20304374.	59712278.	7699427.	109905289.
4	WAGE & SAL	2442191.	22903067.	28268133.	1536961.	41777769.	96928121.
5	OTHER PS[2]	11476089.	29455322.	46210494.	1180746.	-30417204.	57905447.
6	TOTAL PROD.	26506655.	130500391.	103654338.	96928121.	64961421.	422550926.
	2/of which IMPORTS	531158.	2480051.	734301.	0.	-3816491.	-70981.

1963 TRANSACTIONS TABLE -WEST- (1000$)

ROW	COLUMN	1	2	3	4	5	6
1	PRIMARY	8217698.	11728360.	750634.	1264160.	4114286.	26075138.
2	SECONDARY	3428865.	40901167.	8419743.	33168485.	49496716.	135414976.
3	TERTIARY	5249571.	16074814.	21522948.	60810153.	6718169.	110375655.
4	WAGE & SAL	2120227.	24556378.	32086879.	802272.	37496177.	97061933.
5	OTHER PS[3]	10595676.	24647179.	46560901.	1016863.	-25236420.	56584199.
6	TOTAL PROD.	29612037.	117907898.	109341105.	97061933.	71568928.	425511901.
	3/of which IMPORTS	391325.	1570559.	692345.	0.	-2699889.	-45660.

TABLE E2.1. 1963 3RX3C TECH COEFF MATRIX $\hat{A}$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C2	8 R3 C2	9 R3 C3
1	R1 C1	0.2336	0.0513	0.0066	0.0	0.0	0.0	0.0	0.0	0.0
2	R1 C2	0.1365	0.3935	0.0741	0.0	0.0	0.0	0.0	0.0	0.0
3	R1 C3	0.1570	0.1281	0.1990	0.0	0.0	0.0	0.0	0.0	0.0
4	R2 C1	0.0	0.0	0.0	0.1479	0.1002	0.0085	0.0	0.0	0.0
5	R2 C2	0.0	0.0	0.0	0.1399	0.3666	0.0771	0.0	0.0	0.0
6	R2 C3	0.0	0.0	0.0	0.1872	0.1320	0.1959	0.0	0.0	0.0
7	R3 C1	0.0	0.0	0.0	0.0	0.0	0.0	0.2775	0.0995	0.0069
8	R3 C2	0.0	0.0	0.0	0.0	0.0	0.0	0.1158	0.3469	0.0770
9	R3 C3	0.0	0.0	0.0	0.0	0.0	0.0	0.1773	0.1363	0.1968

Table E2.2. 1963 3RX3C TRADE FLOW COEFF MATRIX C

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1 C1	0.6138	0.0	0.0	0.1049	0.0	0.0	0.0332	0.0	0.0
2	R1 C2	0.0	0.8410	0.0	0.0	0.2428	0.0	0.0	0.2206	0.0
3	R1 C3	0.0	0.0	0.9173	0.0	0.0	0.1373	0.0	0.0	0.0843
4	R2 C1	0.2377	0.0	0.0	0.7740	0.0	0.0	0.0828	0.0	0.0
5	R2 C2	0.0	0.1034	0.0	0.0	0.6925	0.0	0.0	0.0794	0.0
6	R2 C3	0.0	0.0	0.0422	0.0	0.0	0.8098	0.0	0.0	0.0540
7	R3 C1	0.1485	0.0	0.0	0.1211	0.0	0.0	0.8841	0.0	0.0
8	R3 C2	0.0	0.0556	0.0	0.0	0.0647	0.0	0.0	0.7000	0.0
9	R3 C3	0.0	0.0	0.0405	0.0	0.0	0.0528	0.0	0.0	0.8618

Table E2.3. 1963 3RX3C MATRIX $\Theta = C\hat{A}$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1 C1	0.1434	0.0315	0.0041	0.0155	0.0105	0.0009	0.0092	0.0033	0.0002
2	R1 C2	0.1148	0.3309	0.0624	0.0340	0.0890	0.0187	0.0255	0.0765	0.0170
3	R1 C3	0.1440	0.1175	0.1825	0.0257	0.0181	0.0269	0.0149	0.0115	0.0166
4	R2 C1	0.0555	0.0122	0.0016	0.1144	0.0776	0.0066	0.0230	0.0082	0.0006
5	R2 C2	0.0141	0.0407	0.0077	0.0969	0.2538	0.0534	0.0092	0.0275	0.0061
6	R2 C3	0.0066	0.0054	0.0084	0.1516	0.1069	0.1586	0.0096	0.0074	0.0106
7	R3 C1	0.0347	0.0076	0.0010	0.0179	0.0121	0.0010	0.2453	0.0879	0.0061
8	R3 C2	0.0076	0.0219	0.0041	0.0091	0.0237	0.0050	0.0811	0.2428	0.0539
9	R3 C3	0.0064	0.0052	0.0081	0.0099	0.0070	0.0103	0.1528	0.1175	0.1696

Table E2.4. 1963 3RX3C MATRIX $B = (I - \widehat{CA})^{-1}$

COLUMN		1	2	3	4	5	6	7	8	9
		R1 C1	R1 C2	R1 C3	R2 C1	R2 C2	R2 C3	R3 C1	R3 C2	R3 C3
ROW										
1	R1 C1	1.1807	0.0606	0.0110	0.0279	0.0286	0.0051	0.0203	0.0157	0.0032
2	R1 C2	0.2454	1.5511	0.1239	0.1051	0.2183	0.0552	0.0947	0.1881	0.0496
3	R1 C3	0.2506	0.2399	1.2450	0.0695	0.0802	0.0519	0.0532	0.0601	0.0354
4	R2 C1	0.0849	0.0357	0.0070	1.1508	0.1295	0.0186	0.0431	0.0273	0.0050
5	R2 C2	0.0542	0.0978	0.0226	0.1748	1.3878	0.0931	0.0393	0.0712	0.0189
6	R2 C3	0.0370	0.0328	0.0178	0.2322	0.2035	1.2051	0.0344	0.0325	0.0205
7	R3 C1	0.0640	0.0279	0.0056	0.0367	0.0365	0.0064	1.3507	0.1652	0.0217
8	R3 C2	0.0323	0.0553	0.0131	0.0306	0.0597	0.0151	0.1694	1.3632	0.0918
9	R3 C3	0.0313	0.0271	0.0163	0.0307	0.0333	0.0202	0.2750	0.2265	1.2224

Table E2.5.1963 3RX3C MATRIX $D = BC = (I - C\hat{A})^{-1}C$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1 C1	0.7344	0.0548	0.0104	0.1479	0.0355	0.0058	0.0594	0.0266	0.0040
2	R1 C2	0.1897	1.3375	0.1180	0.1186	0.5399	0.0644	0.1006	0.4912	0.0561
3	R1 C3	0.1783	0.2134	1.1456	0.0865	0.1176	0.2149	0.0611	0.1013	0.1383
4	R2 C1	0.3320	0.0449	0.0074	0.9049	0.1001	0.0163	0.1362	0.0372	0.0059
5	R2 C2	0.0806	0.2297	0.0254	0.1457	0.9894	0.0795	0.0510	0.1816	0.0232
6	R2 C3	0.0830	0.0504	0.0680	0.1878	0.1510	0.9795	0.0508	0.0462	0.0842
7	R3 C1	0.2486	0.0364	0.0063	0.1987	0.0428	0.0071	1.1993	0.1247	0.0195
8	R3 C2	0.0523	0.1285	0.0164	0.0476	0.1430	0.0189	0.1533	0.9711	0.0810
9	R3 C3	0.0674	0.0388	0.0654	0.0603	0.0443	0.0832	0.2467	0.1672	1.0559

ENDOGENOUS VECTOR OF
1963 3RX3C EST OUTPUT X
(1000$)

ROW		
1	R1 C1	18511476.
2	R1 C2	281811540.
3	R1 C3	215354856.
4	R2 C1	26506021.
5	R2 C2	130470755.
6	R2 C3	103755036.
7	R3 C1	29616440.
8	R3 C2	117976031.
9	R3 C3	109381109.
10	TOTAL	1033383265.

EXOGENOUS VECTOR OF
1963 3RX3C FINAL DEMAND Y
(1000$)

ROW		
1	R1 C1	4456721.
2	R1 C2	131515539.
3	R1 C3	126311039.
4	R2 C1	6004028.
5	R2 C2	74395537.
6	R2 C3	67411705.
7	R3 C1	5378446.
8	R3 C2	82665201.
9	R3 C3	67528322.
10	TOTAL	565666538.

TABLE E3.1. 1963 3RX4C TECH COEFF MATRIX $^{a}\hat{A}$

COLUMN	1	2	3	4	5	6	7	8	9	10	11	12
ROW	R1 C1	R1 C2	R1 C3	R1 C4	R2 C1	R2 C2	R2 C3	R2 C4	R3 C1	R3 C2	R3 C3	R3 C4
1 R1 C1	0.2336	0.0513	0.0066	0.0122	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
2 R1 C2	0.1365	0.3935	0.0741	0.3364	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
3 R1 C3	0.1570	0.1281	0.1990	0.6313	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
4 R1 C4	0.0715	0.2292	0.2811	0.0082	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5 R2 C1	0.0	0.0	0.0	0.0	0.1479	0.1002	0.0085	0.0138	0.0	0.0	0.0	0.0
6 R2 C2	0.0	0.0	0.0	0.0	0.1399	0.3666	0.0771	0.3421	0.0	0.0	0.0	0.0
7 R2 C3	0.0	0.0	0.0	0.0	0.1672	0.1320	0.1959	0.6160	0.0	0.0	0.0	0.0
8 R2 C4	0.0	0.0	0.0	0.0	0.0921	0.1755	0.2727	0.0159	0.0	0.0	0.0	0.0
9 R3 C1	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.2775	0.0995	0.0069	0.0130
10 R3 C2	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1158	0.3469	0.0770	0.3417
11 R3 C3	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.1773	0.1363	0.1968	0.6265
12 R3 C4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0716	0.2083	0.2935	0.0083

TABLE E3.2. 1963 3RX4C TRADE FLOW COEFF MATRIX a_C

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R1 C4	5 R2 C1	6 R2 C2	7 R2 C3	8 R2 C4	9 R3 C1	10 R3 C2	11 R3 C3	12 R3 C4
1	R1 C1	0.6139	0.0	0.0	0.0	0.1048	0.0	0.0	0.0	0.0331	0.0	0.0	0.0
2	R1 C2	0.0	0.8482	0.0	0.0	0.0	0.2349	0.0	0.0	0.0	0.2146	0.0	0.0
3	R1 C3	0.0	0.0	0.9198	0.0	0.0	0.0	0.1330	0.0	0.0	0.0	0.0815	0.0
4	R1 C4	0.0	0.0	0.0	1.0000	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	R2 C1	0.2376	0.0	0.0	0.0	0.7741	0.0	0.0	0.0	0.0827	0.0	0.0	0.0
6	R2 C2	0.0	0.0987	0.0	0.0	0.0	0.7025	0.0	0.0	0.0	0.0772	0.0	0.0
7	R2 C3	0.0	0.0	0.0409	0.0	0.0	0.0	0.8158	0.0	0.0	0.0	0.0522	0.0
8	R2 C4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000	0.0	0.0	0.0	0.0
9	R3 C1	0.1485	0.0	0.0	0.0	0.1211	0.0	0.0	0.0	0.8842	0.0	0.0	0.0
10	R3 C2	0.0	0.0531	0.0	0.0	0.0	0.0626	0.0	0.0	0.0	0.7082	0.0	0.0
11	R3 C3	0.0	0.0	0.0393	0.0	0.0	0.0	0.0512	0.0	0.0	0.0	0.8664	0.0
12	R3 C4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	1.0000

TABLE E3.3. 1963 3RX4C MATRIX $^{a}\cancel{\theta} = {}^{a}C\,{}^{a}\hat{A}$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R1 C4	5 R2 C1	6 R2 C2	7 R2 C3	8 R2 C4	9 R3 C1	10 R3 C2	11 R3 C3	12 R3 C4
1	R1 C1	0.1434	0.0315	0.0041	0.0075	0.0155	0.0105	0.0009	0.0015	0.0092	0.0033	0.0002	0.0004
2	R1 C2	0.1158	0.3337	0.0629	0.2853	0.0328	0.0861	0.0181	0.0803	0.0248	0.0744	0.0165	0.0733
3	R1 C3	0.1444	0.1178	0.1830	0.5807	0.0249	0.0176	0.0261	0.0820	0.0144	0.0111	0.0160	0.0510
4	R1 C4	0.0715	0.2292	0.2811	0.0082	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0
5	R2 C1	0.0555	0.0122	0.0016	0.0029	0.1145	0.0776	0.0066	0.0107	0.0229	0.0082	0.0006	0.0011
6	R2 C2	0.0135	0.0388	0.0073	0.0332	0.0983	0.2575	0.0541	0.2403	0.0089	0.0268	0.0059	0.0264
7	R2 C3	0.0064	0.0052	0.0081	0.0258	0.1527	0.1077	0.1598	0.5026	0.0092	0.0071	0.0103	0.0327
8	R2 C4	0.0	0.0	0.0	0.0	0.0921	0.1755	0.2727	0.0159	0.0	0.0	0.0	0.0
9	R3 C1	0.0347	0.0076	0.0010	0.0018	0.0179	0.0121	0.0010	0.0017	0.2454	0.0880	0.0061	0.0115
10	R3 C2	0.0072	0.0209	0.0039	0.0179	0.0088	0.0230	0.0048	0.0214	0.0820	0.2457	0.0545	0.2420
11	R3 C3	0.0062	0.0050	0.0078	0.0248	0.0096	0.0068	0.0100	0.0315	0.1536	0.1181	0.1705	0.5428
12	R3 C4	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0	0.0716	0.2083	0.2935	0.0083

TABLE E3.4. 1963 3RX4C MATRIX $^{a}B = (^{a}I - {}^{a}C\,{}^{a}\hat{A})^{-1}$

COLUMN		1	2	3	4	5	6	7	8	9	10	11	12
ROW		R1 C1	R1 C2	R1 C3	R1 C4	R2 C1	R2 C2	R2 C3	R2 C4	R3 C1	R3 C2	R3 C3	R3 C4
1	R1 C1	1.1940	0.0862	0.0328	0.0560	0.0375	0.0436	0.0184	0.0331	0.0284	0.0290	0.0145	0.0257
2	R1 C2	0.4521	1.9509	0.4625	0.8696	0.2375	0.4244	0.2343	0.4406	0.2220	0.3961	0.2271	0.4111
3	R1 C3	0.5417	0.8107	1.7395	1.2830	0.2162	0.3125	0.2352	0.4230	0.1809	0.2692	0.1947	0.3404
4	R1 C4	0.3441	0.6868	0.6022	1.5768	0.1188	0.1898	0.1221	0.2241	0.1046	0.1699	0.1087	0.1933
5	R2 C1	0.0969	0.0570	0.0243	0.0428	1.1710	0.1610	0.0502	0.0862	0.0539	0.0444	0.0199	0.0351
6	R2 C2	0.1097	0.1911	0.0965	0.1794	0.3044	1.5913	0.3010	0.5757	0.0904	0.1511	0.0873	0.1575
7	R2 C3	0.0969	0.1264	0.0840	0.1495	0.4331	0.5147	1.5391	0.9395	0.0922	0.1209	0.0923	0.1597
8	R2 C4	0.0555	0.0745	0.0428	0.0774	0.2839	0.4415	0.4849	1.3872	0.0467	0.0646	0.0430	0.0756
9	R3 C1	0.0749	0.0467	0.0206	0.0364	0.0481	0.0541	0.0221	0.0396	1.3765	0.2085	0.0643	0.1088
10	R3 C2	0.0707	0.1177	0.0614	0.1134	0.0702	0.1193	0.0663	0.1239	0.3032	1.5904	0.3184	0.5832
11	R3 C3	0.0862	0.1129	0.0783	0.1387	0.0870	0.1182	0.0883	0.1578	0.5151	0.6317	1.6366	1.0745
12	R3 C4	0.0458	0.0615	0.0375	0.0675	0.0440	0.0639	0.0416	0.0756	0.3155	0.5360	0.5559	1.4566

TABLE E3.5. 1963 3RX4C MATRIX $^{a}D = {}^{a}B\,{}^{a}C = ({}^{a}I - {}^{a}C\,{}^{a}\hat{A})^{-1}\,{}^{a}C$

COLUMN		1	2	3	4	5	6	7	8	9	10	11	12
ROW		R1 C1	R1 C2	R1 C3	R1 C4	R2 C1	R2 C2	R2 C3	R2 C4	R3 C1	R3 C2	R3 C3	R3 C4
1	R1 C1	0.7461	0.0790	0.0315	0.0560	0.1576	0.0527	0.0201	0.0331	0.0678	0.0424	0.0162	0.0257
2	R1 C2	0.3669	1.7176	0.4440	0.8696	0.2581	0.7811	0.2643	0.4406	0.2309	0.7319	0.2467	0.4111
3	R1 C3	0.4108	0.7327	1.6173	1.2830	0.2460	0.4268	0.4333	0.4230	0.1958	0.3887	0.3227	0.3404
4	R1 C4	0.2550	0.6103	0.5632	1.5768	0.1407	0.3053	0.1853	0.2241	0.1137	0.2824	0.1496	0.1933
5	R2 C1	0.3457	0.0666	0.0252	0.0428	0.9231	0.1293	0.0452	0.0862	0.1476	0.0561	0.0219	0.0351
6	R2 C2	0.1531	0.3273	0.1045	0.1794	0.2580	1.1723	0.2629	0.5757	0.1087	0.2709	0.0992	0.1575
7	R2 C3	0.1761	0.1645	0.1438	0.1495	0.3566	0.3988	1.2715	0.9395	0.1205	0.1525	0.1671	0.1597
8	R2 C4	0.1085	0.1102	0.0608	0.0774	0.2312	0.3317	0.4034	1.3872	0.0666	0.0958	0.0661	0.0756
9	R3 C1	0.2618	0.0560	0.0224	0.0364	0.2118	0.0620	0.0240	0.0396	1.2236	0.1619	0.0586	0.1088
10	R3 C2	0.1051	0.1960	0.0717	0.1134	0.0985	0.2111	0.0785	0.1239	0.2763	1.1608	0.2843	0.5832
11	R3 C3	0.1501	0.1409	0.1399	0.1387	0.1388	0.1491	0.1662	0.1578	0.4655	0.4807	1.4291	1.0745
12	R3 C4	0.0854	0.0869	0.0581	0.0675	0.0770	0.0929	0.0674	0.0756	0.2841	0.3977	0.4868	1.4566

ENDOGENOUS VECTOR OF
1963 3RX4C EST OUTPUT X
(1000$)

ROW		
1	R1 C1	18503888.
2	R1 C2	281719929.
3	R1 C3	214928023.
4	R1 C4	181370973.
5	R2 C1	26490276.
6	R2 C2	130278568.
7	R2 C3	103984329.
8	R2 C4	96978474.
9	R3 C1	29651971.
10	R3 C2	118269525.
11	R3 C3	109603254.
12	R3 C4	97218329.
13	TOTAL	1408997540.

EXOGENOUS VECTOR OF
1963 3RX4C FINAL DEMAND Y
(1000$)

ROW		
1	R1 C1	2250424.
2	R1 C2	70446878.
3	R1 C3	11695435.
4	R1 C4	53575653.
5	R2 C1	4662591.
6	R2 C2	41238838.
7	R2 C3	7699427.
8	R2 C4	41777769.
9	R3 C1	4114286.
10	R3 C2	49496716.
11	R3 C3	6718169.
12	R3 C4	37496177.
13	TOTAL	331172363.

Table E4.1. Partitioned Matrices $\hat{\overset{*}{C}}$, $\hat{\overset{*}{Z}}$, and $\hat{\overset{*}{W}}$

1963 3RX3C CONS COEFF MATRIX $\hat{\overset{*}{C}}$

COLUMN		1	2	3
		R1	R2	R3
ROW				
1	R1 C1	0.0122	0.0	0.0
2	R1 C2	0.3364	0.0	0.0
3	R1 C3	0.6313	0.0	0.0
4	R2 C1	0.0	0.0138	0.0
5	R2 C2	0.0	0.3421	0.0
6	R2 C3	0.0	0.6160	0.0
7	R3 C1	0.0	0.0	0.0130
8	R3 C2	0.0	0.0	0.3417
9	R3 C3	0.0	0.0	0.6265

1963 3RX3C CONS ALLOC TO WAGES COEFF MATRIX $\hat{\overset{*}{Z}}$

COLUMN		1	2	3
		R1	R2	R3
ROW				
1	R1	0.0082	0.0	0.0
2	R2	0.0	0.0159	0.0
3	R3	0.0	0.0	0.0083

1963 3RX3C WAGE/SALARY COEFF MATRIX $\hat{\overset{*}{W}}$

COLUMN		1	2	3	4	5	6	7	8	9
		R1 C1	R1 C2	R1 C3	R2 C1	R2 C2	R2 C3	R3 C1	R3 C2	R3 C3
ROW										
1	R1	0.0715	0.2292	0.2811	0.0	0.0	0.0	0.0	0.0	0.0
2	R2	0.0	0.0	0.0	0.0921	0.1755	0.2727	0.0	0.0	0.0
3	R3	0.0	0.0	0.0	0.0	0.0	0.0	0.0716	0.2083	0.2935

Table E4.2. 1963 3RX3C TRADE ADJ CONS CCEFF MATRIX $C\hat{C}$

	COLUMN	1	2	3
		R1	R2	R3
ROW				
1	R1 C1	0.0075	0.0015	0.0004
2	R1 C2	0.2829	0.0830	0.0754
3	R1 C3	0.5791	0.0846	0.0528
4	R2 C1	0.0029	0.0107	0.0011
5	R2 C2	0.0348	0.2369	0.0271
6	R2 C3	0.0266	0.4989	0.0338
7	R3 C1	0.0018	0.0017	0.0115
8	R3 C2	0.0187	0.0221	0.2392
9	R3 C3	0.0256	0.0325	0.5399

Table E4.3. 1963 3RX3C Matrices $\hat{\overset{*}{\mathrm{DC}}}$ and $\hat{\overset{*}{\mathrm{WD}}}$ showing consumption-induced output coefficients and income generated by output coefficients respectively

1963 3RX3C MATRIX $\hat{\overset{*}{\mathrm{DC}}}$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1 C1	0.0339	0.0178	0.0124
2	R1 C2	0.5267	0.2260	0.2043
3	R1 C3	0.7972	0.1738	0.1220
4	R2 C1	0.0238	0.0568	0.0182
5	R2 C2	0.0943	0.3894	0.0773
6	R2 C3	0.0609	0.6576	0.0692
7	R3 C1	0.0192	0.0217	0.0704
8	R3 C2	0.0542	0.0612	0.3846
9	R3 C3	0.0552	0.0672	0.7219

1963 3RX3C MATRIX $\hat{\overset{*}{\mathrm{WD}}}$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1	0.1461	0.3705	0.3498	0.0621	0.1594	0.0756	0.0445	0.1430	0.0520
2	R2	0.0674	0.0582	0.0237	0.1602	0.2240	0.2826	0.0354	0.0479	0.0276
3	R3	0.0484	0.0408	0.0231	0.0418	0.0458	0.0289	0.1902	0.2602	0.3281

Table E4.4. 1963 Direct, and direct plus indirect interregional income coefficient matrices Φ and Ψ

1963 3RX3C MATRIX $\Phi =$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1	0.3472	0.1019	0.0820
2	R2	0.0354	0.2529	0.0341
3	R3	0.0289	0.0340	0.2970

1963 3RX3C $\overline{\Psi} = (I - \Phi)^{-1}$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1	1.5524	0.2205	0.1918
2	R2	0.0765	1.3524	0.0745
3	R3	0.0674	0.0745	1.4339

1963 3RX3C $\hat{\overset{*}{Z}}\overline{\Psi}$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1	0.0127	0.0018	0.0016
2	R2	0.0012	0.0214	0.0012
3	R3	0.0006	0.0006	0.0119

1963 3RX3C $\overline{\Lambda} = (I - \hat{\overset{*}{Z}}\overline{\Psi})^{-1}$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1	1.0129	0.0019	0.0016
2	R2	0.0013	1.0219	0.0012
3	R3	0.0006	0.0006	1.0120

1963 3RX3C $\Psi = \overline{\Psi}\overline{\Lambda}$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1	1.5727	0.2284	0.1969
2	R2	0.0793	1.3822	0.0772
3	R3	0.0692	0.0772	1.4513

Table E4.5: 1963 3RX3C MATRIX D

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1 C1	0.7344	0.0548	0.0104	0.1479	0.0355	0.0058	0.0594	0.0266	0.0040
2	R1 C2	0.1897	1.3375	0.1180	0.1186	0.5399	0.0644	0.1006	0.4912	0.0561
3	R1 C3	0.1783	0.2134	1.1456	0.0865	0.1176	0.2149	0.0611	0.1013	0.1383
4	R2 C1	0.3320	0.0449	0.0074	0.9049	0.1001	0.0163	0.1362	0.0372	0.0059
5	R2 C2	0.0806	0.2297	0.0254	0.1457	0.9894	0.0795	0.0510	0.1816	0.0232
6	R2 C3	0.0830	0.0504	0.0680	0.1878	0.1510	0.9795	0.0508	0.0462	0.0842
7	R3 C1	0.2486	0.0364	0.0063	0.1987	0.0428	0.0071	1.1993	0.1247	0.0195
8	R3 C2	0.0523	0.1285	0.0164	0.0476	0.1430	0.0189	0.1533	0.9711	0.0810
9	R3 C3	0.0674	0.0388	0.0654	0.0603	0.0443	0.0832	0.2467	0.1672	1.0559

ENDOGENOUS VECTOR OF
1963 3RX3C EST OUTPUT X
(1000$)

ROW		
1	R1 C1	9422143.
2	R1 C2	144450399.
3	R1 C3	41934414.
4	R2 C1	14912440.
5	R2 C2	68105304.
6	R2 C3	22238890.
7	R3 C1	17179959.
8	R3 C2	64868184.
9	R3 C3	22782105.
10	TOTAL	405893839.

EXOGENOUS VECTOR OF
1963 3RX3C NON-CONSUMPTION FD $\tilde{Y}$
~~(1000$)~~

ROW		
1	R1 C1	2250424.
2	R1 C2	70446878.
3	R1 C3	11695435.
4	R2 C1	4662591.
5	R2 C2	41238838.
6	R2 C3	7699427.
7	R3 C1	4114286.
8	R3 C2	49496716.
9	R3 C3	6718169.
10	TOTAL	198322764.

Table E4.6: 1963 3RX3C MATRIX $\Xi = D\hat{C}Y\hat{W}D$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1 C1	0.0116	0.0236	0.0209	0.0099	0.0175	0.0144	0.0086	0.0163	0.0124
2	R1 C2	0.1762	0.3619	0.3212	0.1427	0.2567	0.2038	0.1336	0.2532	0.1942
3	R1 C3	0.2324	0.5120	0.4646	0.1629	0.3160	0.2284	0.1381	0.2936	0.1923
4	R2 C1	0.0138	0.0224	0.0180	0.0178	0.0276	0.0284	0.0117	0.0195	0.0162
5	R2 C2	0.0729	0.1078	0.0817	0.1088	0.1635	0.1784	0.0588	0.0953	0.0779
6	R2 C3	0.0928	0.1173	0.0792	0.1650	0.2398	0.2785	0.0707	0.1088	0.0871
7	R3 C1	0.0133	0.0204	0.0163	0.0132	0.0197	0.0172	0.0235	0.0352	0.0384
8	R3 C2	0.0534	0.0740	0.0571	0.0515	0.0733	0.0612	0.1188	0.1720	0.1978
9	R3 C3	0.0832	0.1054	0.0780	0.0792	0.1075	0.0870	0.2143	0.3049	0.3609

ENDOGENOUS VECTOR OF
1963 3RX3C EST OUTPUT X2
(1000$)

ROW		
1	R1 C1	3739666.
2	R1 C2	56856611.
3	R1 C3	73965810.
4	R2 C1	4384157.
5	R2 C2	22824879.
6	R2 C3	28455595.
7	R3 C1	4760227.
8	R3 C2	20066365.
9	R3 C3	32391785.
10	TOTAL	247445094.

EXOGENOUS VECTOR OF
1963 3RX3C NON-CONSUMPTION FD $\tilde{Y}$
(1000$)

ROW		
1	R1 C1	2250424.
2	R1 C2	70446878.
3	R1 C3	11695435.
4	R2 C1	4662591.
5	R2 C2	41238838.
6	R2 C3	7699427.
7	R3 C1	4114286.
8	R3 C2	49496716.
9	R3 C3	6718169.
10	TOTAL	198322764.

Table E4.7: 1963 3RX3C MATRIX $\tilde{\Xi} = DC\overset{\hat{*}}{Y}$

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1 C1	0.0556	0.0333	0.0260
2	R1 C2	0.8604	0.4484	0.4177
3	R1 C3	1.2760	0.4317	0.3475
4	R2 C1	0.0432	0.0854	0.0354
5	R2 C2	0.1845	0.5658	0.1608
6	R2 C3	0.1527	0.9283	0.1631
7	R3 C1	0.0369	0.0399	0.1077
8	R3 C2	0.1167	0.1267	0.5736
9	R3 C3	0.1420	0.1613	1.0637

ENDOGENOUS VECTOR OF
1963 3RX3C EST OUTPUT X3
(1000$)

ROW		
1	R1 C1	5349071.
2	R1 C2	80494235.
3	R1 C3	99427049.
4	R2 C1	7210682.
5	R2 C2	39550658.
6	R2 C3	53079903.
7	R3 C1	7678629.
8	R3 C2	33055115.
9	R3 C3	54233563.
10	TOTAL	380078903.

EXOGENOUS VECTOR OF
1963 REGIONAL INCOME ${}^{w}\tilde{Y}$
(1000$)

ROW		
1	R1	53575653.
2	R2	41777769.
3	R3	37496177.
4	TOTAL	132849599.

Table E4.8: Endogenous vector of gross outputs X as the sum of the three vectors X1, X2, and X3 (1000$)

ROW		X	=	X1	+	X2	+	X3
1	R1 C1	18510880.		9422143.		5349071.		3739666.
2	R1 C2	281801245.		144450399.		80494235.		56856611.
3	R1 C3	215327272.		41934414.		99427049.		73965810.
4	R2 C1	26507279.		14912440.		7210682.		4384157.
5	R2 C2	130480841.	=	68105304.	+	39550658.	+	22824879.
6	R2 C3	103774387.		22238890.		53079903.		28455595.
7	R3 C1	29618815.		17179959.		7678629.		4760227.
8	R3 C2	117989663.		64868184.		33055115.		20066365.
9	R3 C3	109407453.		22782105.		54233563.		32391785.
10	TOTAL	1033417836.		405893839.		380078903.		247445094.

Table E4.9: 1963 3RX3C MATRIX $\Omega = Y\overset{*}{W}D$

ROW	COLUMN	1 R1 C1	2 R1 C2	3 R1 C3	4 R2 C1	5 R2 C2	6 R2 C3	7 R3 C1	8 R3 C2	9 R3 C3
1	R1	0.2547	0.6040	0.5601	0.1424	0.3108	0.1891	0.1155	0.2870	0.1527
2	R2	0.1085	0.1130	0.0623	0.2295	0.3258	0.3988	0.0671	0.0976	0.0676
3	R3	0.0856	0.0893	0.0595	0.0774	0.0948	0.0689	0.2819	0.3913	0.4820

ENDOGENOUS VECTOR W1 OF
1963 WAGE & SAL INCOME
(1000$)

ROW		
1	R1	80319787.
2	R2	32070003.
3	R3	35746637.
4	TOTAL	148136427.

EXOGENOUS VECTOR OF
1963 NON-CONSUMPTION FD Y
(1000$)

ROW		
1	R1 C1	2250424.
2	R1 C2	70446878.
3	R1 C3	11695435.
4	R2 C1	4662591.
5	R2 C2	41238838.
6	R2 C3	7699427.
7	R3 C1	4114286.
8	R3 C2	49496716.
9	R3 C3	6718169.
10	TOTAL	198322764.

Table E4.10: 1963 3RX3C MATRIX Ψ

ROW	COLUMN	1 R1	2 R2	3 R3
1	R1	1.5727	0.2284	0.1969
2	R2	0.0793	1.3822	0.0772
3	R3	0.0692	0.0772	1.4513

ENDOGENOUS VECTOR W2 OF
1963 WAGE & SAL INCOME
(1000$)

ROW		
1	R1	101183629.
2	R2	64887957.
3	R3	61352589.
4	TOTAL	227424175.

EXOGENOUS VECTOR ^{w}Y OF
1963 REGIONAL INCOME
(1000$)

ROW		
1	R1	53575653.
2	R2	41777769.
3	R3	37496177.
4	TOTAL	132849599.

Table E4.11: Endogenous vector of wage and salary income W as the sum of the two vectors W1 and W2 (1000$)

ROW		W	=	W1	+	W2
1	R1	181503415.		80319787.		101183629.
2	R2	96957960.	=	32070003.	+	64887957.
3	R3	97099226.		35746637.		61352589.
4	TOTAL	375560601.		148136427.		227424175.

ANNEX F

RELATIONSHIP BETWEEN AGGREGATE KEYNESIAN AND DISAGGREGATED INPUT-OUTPUT MULTIPLIERS

In the text the basic relationship between the simplest Keynesian macroeconomic multiplier and the scalar version of the single economy Leontief multiplier is presented for a closed economy. In this annex a number of the simplifying assumptions of the scalar version of the I-O model will be progressively relaxed in order to demonstrate how it effects the formal relationship between the I-O and Keynesian multipliers. The MRIO notation developed in the text will be used throughout this presentation with suitable adjustments in the dimensions of the vectors and matrices to reflect the different I-O context.

A. In a closed economy:

For convenience direct transactions between expenditures (or final demand) Y, and income (or primary supply) v_o will be assumed to be zero. That is, in the open I-0 model

$$ {}^{v}Y = 0 $$

which implies that in the augmented I-0 model

$$ \overset{*}{z} = 0 $$

and

$$ {}^{v}\tilde{Y} = 0 $$

The basic I-0 equations for gross outputs are:

$$X = AX + Y \tag{F.1}$$

where Y represents aggregate demand, i.e.

$$Y = C + I \tag{F.2a}$$

or

$$Y = C + I + G \tag{F.2b}$$

In equation (F.2a) the variable C refers to total consumption expenditures, whereas in equation (F.2b) it refers only to personal consumption expenditures. The difference is government spending G. In either case, the solution of this system of equations is:

$$X = (I-A)^{-1}Y \tag{F.3}$$

However, for the purposes of demonstrating the relationship between the Keynesian and I-0 multipliers based on different specifications of the I-0 model it is the secondary balance equation for total income v_o in the I-0 model which is important.

In the open model this equation is:

$$v_o = \overset{*}{V}_o X \tag{F.4}$$

$$= \overset{*}{V}_o (I-A)^{-1} Y \tag{F.5}$$

where v_o is a scalar and $\overset{*}{V}_o$ a (1 x m) row vector of direct income coefficients. In a closed economy

$$\sum_{i=1}^{m} a_{ij} + \overset{*}{v}_j = 1 \tag{F.6}$$

therefore

$$\overset{*}{V}_o = e(I-A) \tag{F.7}$$

where e is a summing vector of dimension (1 x m) all of whose elements are unity.

Substituting (F.7) into (F.4), we get

$$v_o = e(I-A)(I-A)^{-1}Y \tag{F.8}$$

$$= eY$$

$$= y_o \tag{F.9}$$

where v_o and y_o are scalars, and e is a summing row-vector of dimension (1 x m).

In this formulation v_o is total value added (or gross national income in a closed economy) and y_o is total aggregate demand (or gross national product in a closed economy). Thus, equation (F.8) is equivalent to the national income identity in a closed model:

$$VA \equiv AD \tag{F.10}$$

It is clear from equation (F.9) therefore, that the open version of the I-0 model is only embedded in the national income account framework and not in the framework of a Keynesian model. The most important point in this formulation is that in a closed economy national income cannot be different from domestic expenditures.

To incorporate the Keynesian propagation process, consumption must be made a part of the multiplier. One way to incorporate consumption into the model's multipliers is to rewrite the I-0 balance equations as

$$X = AX + C + \tilde{Y} \tag{F.11}$$

where

$$\tilde{Y} = I$$

or

$$\tilde{Y} = I + G$$

if the government sector is distinguished in the model. If consumption demand is assumed to be a linear function of total income, then

$$C = \overset{*}{\bar{C}}_o v_o \qquad \text{(F.12)}$$

where $\overset{*}{\bar{C}}_o$ is an (mxi) column vector of marginal (or average) consumption coefficients. Substituting (F.4) into (F.12) and the latter into (F.11) we get

$$X = AX + \overset{*}{\bar{C}}_o \overset{*}{V}_o X + \tilde{Y} \qquad \text{(F.13)}$$

The solution of this system of equations is

$$X = (I-A-\overset{*}{\bar{C}}_o \overset{*}{V}_o)^{-1} \tilde{Y} \qquad \text{(F.14)}$$

Substituting (F.14) into (F.4), we get

$$v_o = \overset{*}{V}_o \, (I-A-\overset{*}{\bar{C}}_o \overset{*}{V}_o)^{-1} \tilde{Y} \qquad \text{(F.15)}$$

Unfortunately, in this formulation the Keynesian propagation process is not evident. However, using the equivalence of the solution in (F.15) and the partitioned matrix solution from Annexes D.2 and D.3 we can write[1/] v_o as

$$v_o = \bar{\psi} \, \overset{*}{V}_o (I-A)^{-1} \tilde{Y} \qquad \text{(F.16)}$$

where $\bar{\psi}$ is a scalar, because

$$\bar{\psi} = (I-\phi)^{-1} \qquad \text{(F.17)}$$

and

$$\phi = \overset{*}{V}_o (I-A)^{-1} \overset{*}{\bar{C}}_o \qquad \text{(F.18)}$$

where ϕ is a scalar resulting from the product of the row-vector $\overset{*}{V}_o$, the matrix $(I-A)^{-1}$, and the column-vector $\overset{*}{\bar{C}}_o$.

1/ In the I-0 model the complicating factor introduced by the inter-regional trade matrices C and C_k are absent, hence the use of $B = (I-A)^{-1}$ instead of $D = (I-\theta)^{-1}C$, and the use of $\phi = \overset{*}{V}_o B \overset{*}{C}_o$ instead of $\Phi = C_k \overset{*}{W} D \overset{*}{C}$ in the equations in this annex.

Substituting (F.7) into (F.18), we get

$$\phi = [e(I-A)](I-A)^{-1}\bar{C}_o^*$$
$$= e\bar{C}_o^* = c = MPC \qquad (F.19)$$

That is, ϕ represents the aggregate marginal propensity to consume.[2/]
Hence,

$$\psi = (I-\phi)^{-1} = \frac{1}{1-c} \qquad (F.20)$$

Substituting (F.20) and (F.6) into equation (F.16) the aggregate Keynesian multiplier in a closed economy is obtained

$$v_o = \frac{1}{1-c}\ e\tilde{Y} \qquad (F.21)$$

$$= \frac{1}{1-c}\ \tilde{y}_o \qquad (F.22)$$

where y_o is a scalar sum of autonomous non-consumption expenditures. Thus, augmenting the open I-O model is equivalent to introducing the Keynesian multiplier process only if all of income,[3/] i.e., v_o, is incorporated. This procedure is different from the traditional interindustry method for augmenting the open model. There the household sector is treated as a 'fictitious' industry that satisfies the accounting identity of an equality between the supply and demand of an industry's output. Given that $c_o = eC$, it is incorrect to use

$$v_o = c_o \qquad (F.23)$$

2/ c will represent the aggregate average propensity to consume only if the coefficients of $\bar{C}^*$ are estimated as average consumption coefficients, i.e. excluding the non-homogenous terms.

3/ Assuming no taxes or corporate savings (see Annex A.3).

because c_o is a component of y_o which is the variable that is equal to v_o. Hence, in the traditional I-0 approach it is necessary to use a component of income in lieu of total income in the augmented I-0 income formation process. In contrast, in the derivation above, total income is used such that

$$v_o > c_o \tag{F.24}$$

The simplifying assumptions of the model above (e.g., a closed economy without taxes, etc.) is necessary to demonstrate the difference in the 'income formation' component of an I-0 model augmented with a 'fictitious industry' and the assumption of average 'technical' coefficients , and an I-0 model augmented with 'decision-making' units, and the assumption of behavioral 'marginal propensities to consume' coefficients.

B. <u>In an Open Economy</u>

Now the simplifying assumptions of a closed economy, without taxes, etc., can be relaxed in stages to clarify the logic of how the value of the multiplier changes with the different assumptions. Again for convenience direct transactions between final demand and primary supply are assumed to be zero. In addition to aggregate demand, final demand is now assumed to include gross exports (but no imports). The solution of gorss outputs will remain unchanged in form. However, the primary supply equation can be represented as two equations, one for value added v_o and the other for imports m_o, i.e.,

$$v_o = \overset{*}{V}_o X \tag{F.26a}$$

$$= \overset{*}{V}_o (I-A)^{-1} Y \tag{F.26b}$$

and $$m_o = \overset{*}{M}_o X \tag{F.27a}$$

$$= \overset{*}{M}_o (I-A)^{-1} Y \tag{F.27b}$$

Therefore, in contrast to (F.7), the row vector of value added coefficients must now be written as

$$\overset{*}{V}_o = e(I-A) - \overset{*}{M}_o \tag{F.28}$$

substituting (F.28) into (F.26a) we get the national income identity in the open economy

$$v_o = [e(I-A) - \overset{*}{M}_o](I-A)^{-1} Y \tag{F.29}$$

$$= eY - \overset{*}{M}_o (I-A)^{-1} Y \tag{F.30}$$

$$= y_o - m_o \tag{F.31}$$

where $y_o - m_o$ represents GNP. In this formulation, in contrast to that in the closed economy, it is possible for national income to be different from domestic expenditures. In fact

$$v_o < y_o, \text{ if } m_o > 0 \tag{F.32}$$

However, from equation (F.31) it is clear again that the open I-O model does not incorporate the Keynesian multiplier.

To incorporate the Keynesian process it is again necessary to augment the open I-O model. In an open economy context the form of the partioned matrix solution for v_o remains the same as in equation (F.16).

However, in this case the component ϕ of ψ is

$$\phi = \overset{*}{V}_o(I-A)^{-1}\overset{*}{\bar{C}}_o$$

$$= [e(I-A)-\overset{*}{M}_o](I-A)^{-1}\overset{*}{\bar{C}}_o \qquad (F.33)$$

$$= e\overset{*}{\bar{C}}_o-\overset{*}{M}_o(I-A)^{-1}\overset{*}{\bar{C}}_o$$

$$= c-m_c \qquad (F.34)$$

where m_c is a scalar coefficient representing the 'aggregate propensity to import out of marginal consumption expenditures'. Thus,

$$\bar{\psi} = (I-\phi)^{-1}$$

$$= \frac{1}{1-(c-m_c)}$$

$$= \frac{1}{1-c+m_c} = \frac{1}{s+m_c} \qquad (F.35)$$

where s = 1-c = MPS (the aggregate marginal propensity to save).

Substituting (F.35) and (F.28) into (F.16) we get

$$v_o = \frac{1}{s+m_c} \; [e(I-A)-\overset{*}{M}_o](I-A)^{-1}\tilde{Y} \qquad (F.36)$$

$$= \frac{1}{s+m_c} \; [e\tilde{Y}-\overset{*}{M}_o(I-A)^{-1}\tilde{Y}]$$

$$= \frac{1}{s+m_c} \; (\tilde{y}_o-\tilde{m}_o) \qquad (F.37)$$

where the scalar $\tilde{m}_o$ represents the total level of non-consumption imports in analogy to total imports m_o in (F.27b).

This result is slightly different from the traditional formulations of the macroeconomic multiplier in an open economy. In one case where imports m_o are treated as part of the multiplicand the macroeconomic multiplier is formulated as

$$v_o = \frac{1}{s}(y_o - m_o) \qquad (F.38)$$

In the other case where imports are assumed to be a function of income v_o, and therefore part of the multiplier, the macroeconomic multiplier is formulated as

$$v_o = \frac{1}{s+m} y_o \qquad (F.39)$$

where m is the propensity to import out of total income.

The difference between the macroeconomic formulations in (F.38) and (F.39), and the scalar I-O formulation in (F.37) is that in the latter imports are initially assumed to be a function of output as in (F.27a). Alternately, as in (F.27b), they are specified as a function of domestic expenditures Y rather than national income v_o.4/ Hence each component of domestic expenditures will give rise to a component-specific import propensity, as for example, $\overset{*}{M}_o(I-A)^{-1}\overset{*}{\bar{C}}_o$ in (F.34).

4/ Note that by assuming no direct transactions between primary supply and final demand the coefficient at the intersection of the column vector of marginal consumption coefficients $\overset{*}{C}$, and the row-vector of import-to-output coefficients $\overset{*}{M}$, is zero. If it were positive, it would represent the aggregate marginal propensity to import directly for consumption out of income, v_o.

The term $\overset{*}{M}_o(I-A)^{-1}\overset{*}{\bar{C}}_o$, represents consumption induced imports via the interindustry production process. As a result of this functional specification only those imports give rise to leakages in the multiplier formula which are associated with the sector that is incorporated directly into the augmented model. Thus, the value of the I-O multiplier in equation (F.37) is intermediate between the value of the multipliers in equations (F.38) and (F.39), i.e.

$$\frac{1}{s} > \frac{1}{s+m_c} > \frac{1}{s+m}, \text{ if } m > m_c > o \qquad \text{(F.40)}$$

This approach to determining the relationship between the scalar aggregate I-O multipliers and the corresponding Keynesian multipliers can be extended, for example, to the case where consumption demand is a function of a component of national income (whether or not the two are equal). Thus, if

$$v_o = w_o + v'_o \qquad \text{(F.41)}$$

and consumption demand is specified as

$$C = \overset{*}{C} w_o \qquad \text{(F.42)}$$

instead of $C = \overset{*}{\bar{C}} v_o$ as in (F.12), which implies that

$$\overset{*}{C} > \overset{*}{\bar{C}} \qquad \text{(F.43)}$$

then the augmented I-O model will be based on an open I-O model, in which the secondary balance equation can be expressed as three separate equations:

$$w_o = \overset{*}{W}_o(I-A)^{-1}Y \tag{F.44}$$

$$v'_o = \overset{*}{V}'_o(I-A)^{-1}Y \tag{F.45}$$

and

$$m_o = \overset{*}{M}_o(I-A)^{-1}Y \tag{F.46}$$

where the sum of the two components w_o and v'_o is equivalent to total value added v_o.

In this case, in contrast to (F.7) and (F.28), the vector of the 'component of income'-to-output ratios can be represented as

$$\overset{*}{W}_o = e(I-A)-\overset{*}{M}_o-\overset{*}{V}'_o \tag{F.47}$$

hence, substituting (F.47) into (F.44)

$$w_o = [e(I-A)-\overset{*}{M}_o-\overset{*}{V}'_o](I-A)^{-1}Y \tag{F.48}$$

$$= eY-m_o-\overset{*}{V}'_o(I-A)^{-1}Y$$

$$= y_o-m_o-v'_o \tag{F.49}$$

where y_o-m_o = GNP and the scalar v'_o represents that portion of national income which is excluded from the loop linking consumption expenditures to a measure of income in the augmented model. Thus, if w_o represents 'wage and salary income,' then v'_o represents, in addition to taxes, other income categories such as profit, interest, etc.[1/] If w_o represents all personal disposable income, including dividends and other personal income from the ownership of assets, then v'_o represents only taxes and corporate savings.

From the MRIO equation (5.3), with ${}^{w}\tilde{Y}=0$, and the I-O equation (F.16), the augmented model solution for w_o will be:

$$w_o = \overline{\psi}\overset{*}{W}_o(I-A)^{-1}\tilde{Y}$$

where, using equation (F.47)

$$\phi = \overset{*}{W}_o(I-A)^{-1}\overset{*}{C}_o$$

$$= [e(I-A)-\overset{*}{M}_o-\overset{*}{V}'_o](I-A)^{-1}\overset{*}{C}_o \qquad \text{(F.51)}$$

$$= e\overset{*}{C}_o-\overset{*}{M}_o(I-A)^{-1}\overset{*}{C}_o-\overset{*}{V}'_o(I-A)^{-1}\overset{*}{C}_o \qquad \text{(F.52)}$$

$$= c'-m'_c-v'_c \qquad \text{(F.53)}$$

and

$$\psi = (I-\phi)^{-1}$$

$$= \frac{1}{1-c'+m'_c+v'_c} \qquad \text{(F.54)}$$

Therefore, substituting equations (F.54) and (F.47) into (F.50),

$$w_o = \frac{1}{1-c'+m'_c+v'_c}\;(\tilde{y}_o-\tilde{m}_o-\tilde{v}'_o) \qquad \text{(F.55)}$$

1/ This latter assumption is used in the present form of the augmented MRIO model.

As a result of the different specification of the consumption coefficients $\overset{*}{C}$ in equation (F.42), the scalar c' in equations (5.53) to (5.55) is not the marginal propensity to consume out of total income as in (F.34). Similarly, $m_c' \neq m_c$ for the same reason. However, from (F.43) it is clear that

$$c' > c \qquad \text{(F.56)}$$

and

$$m_c' > m_c \qquad \text{(F.57)}$$

hence, the I-O multiplier in this formulation is smaller than that in (F.35),

$$\frac{1}{1-c+m_c} > \frac{1}{1-c'+m_c'+v_c'} \qquad \text{(F.58)}$$

and

$$\frac{1}{1-c} > \frac{1}{1-c+m_c} > \frac{1}{1-c'+m_c'+v_c'} > \frac{1}{1-c+m} \qquad \text{(F.59)}$$

i.e. the two Keynesian formulations bracket the two I-O formulations of the aggregate multiplier.

This approach can be extended readily to an analysis of the case where direct transactions between primary supply and final demand are assumed to be positive. If, ${}^{v}y > o$ in the open version of an I-O model in a closed economy, then in the augmented version it is possible that

$$(1) \quad \overset{*}{z} = 0 \qquad \text{and} \quad {}^{v}\tilde{y} > o \quad ,$$

or, $$(2) \quad \overset{*}{z} > 0 \qquad \text{and} \quad {}^{v}\tilde{y} = o \quad ,$$

or, $$(3) \quad \overset{*}{z} > 0 \qquad \text{and} \quad {}^{v}\tilde{y} > o.$$

In the open version of the I-O model in an open economy, it is possible for ${}^{m}Y > o$, in addition to, or, in lieu of $\overset{v}{Y} > o$ with corresponding adjustments in the augmented version. For illustrative purposes we will show the implication for ψ in case (2) of the augmented version of the I-O model in a closed economy.

Using equation (D.2.11) in the context of an I-O rather than an MRIO model, if $\overset{*}{z} > 0$, ψ can be expressed as

$$\psi = \bar{\psi}(I-\overset{*}{z}\bar{\psi})^{-1} \tag{F.60}$$

where

$$\bar{\psi} = (I-\phi)^{-1}$$

and

$$\phi = \overset{*}{V}_o(I-A)^{-1}\overset{*}{C}_o$$

Then, from (F.20)

$$\bar{\psi} = \frac{1}{1-c} \tag{F.61}$$

substituting (F.61) into (F.60)

$$\psi = \frac{1}{1-c}\,[1-\overset{*}{z}(\frac{1}{1-c})]^{-1}$$

$$= \frac{1}{1-c}\,[1-\frac{\overset{*}{z}}{1-c}]^{-1}$$

$$= \frac{1}{1-c}\,(\frac{1-c-\overset{*}{z}}{1-c})^{-1}$$

$$= \frac{1}{1-c}\,(\frac{1-c}{1-c-\overset{*}{z}})$$

$$= \frac{1}{1-c-\overset{*}{z}}$$

where $\overset{*}{z}$ is a scalar representing the marginal propensity to directly consume 'household' services.

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