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Lamb's problem at its simplest

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Abstract

This article revisits the classical problem of horizontal and vertical point loads suddenly applied onto the surface of a homogeneous, elastic half-space, and provides a complete set of exact, explicit formulas which are cast in the most compact format and with the simplest possible structure. The formulas given are valid for the full range of Poisson's ratios from 0 to 0.5, and they treat real and complex poles alike, as a result of which a single set of formulas suffices and also exact formulas for dipoles can be given.

1. Introduction

Lamb's problem deals with the response elicited by a vertical or horizontal point load applied suddenly onto the surface of an elastic half-space. This classical problem harks back to the early 20th century, when Lamb (1904) enunciated this now famous and emblematic problem in seismology.

The first truly complete solutions to Lamb's problem were given by Pekeris (1955) and by Chao (1960), who provided closed form expressions for the components of motion elicited by a vertical and a horizontal load, respectively, but only when Poisson's ratio is $\frac{1}{4}$. This problem was taken up again by Mooney (1974), who extended the Pekeris solution to any arbitrary Poisson's ratio, but he did so only for the vertical component while ignoring the radial one. Then in 1979, Richards considered this problem once again and gave a complete set of exact formulas for both loading cases and for any Poisson's ratio in a paper that has remained largely unknown within the elastodynamics and wave propagation communities. Part of the reason may have been that Richards only presented the final formulas in the context of a note on spontaneous crack propagation without indicating where their rather complicated derivation could be found. Also, he did not summarize these in his book (Aki and Richards, 2002), perhaps because he judged these to be unimportant. This seems to be corroborated by two comments in his brief article, namely "*these formulas would be only a minor curiosity*", and "*Perhaps the main achievement of this paper* …", as if he were unsure of their true worth.

It is the purpose of this technical note to present a new rendition of Lamb's problem by means of a very compact set of exact formulas for all loading cases and for any Poisson's ratio which are fully equivalent to, but much simpler than Richards'. In a nutshell, the formulas are obtained by casting the equations of motion in the Laplace–radial wavenumber domain (s,k), carrying out an inverse Laplace transform into the time domain by a contour integration based on a generalization of the so-called Bateman-Pekeris theorem, followed by Hankel transforms evaluated with the aid of Mooney's integral together with an expansion of the integrand into partial fractions. Although the

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formulas herein are listed without proof, their complete derivation can be found online in Kausel, 2012.

The ensuing provides the complete set of formulas in Lamb's problem cast in a consistent, right-handed, upright coordinate system, and which is valid for the full range of Poisson's ratios from 0 to 0.5. We bring about a significant organization and simplification to the formulas by reducing the number as well as the form of the constants involved, and most importantly, by providing expressions which make no distinction between real or complex roots, a feature that provides for a seamless transition into the complex domain and greatly simplifies the task of taking derivatives. This allows us also to provide simple, explicit expressions for a subset of Lamb dipoles.

2. Notation and definitions

Consider a lower, homogeneous elastic half-space $z \le 0$ onto whose upper surface $z = 0$ we suddenly apply either an upright vertical or a horizontal-*x* (tangential) point force of unit intensity which remains constant thereafter. Thus, the force has the time dependence of a step function $\mathcal{H}(\tau)$ i.e. a Heaviside function. A right-handed, cylindrical coordinate system r, θ, z with origin at the location of the force on the surface is used throughout, i.e. the vertical force and vertical displacements are both positive up, while the horizontal force acts in the positive x direction (i.e. radial direction with $\theta = 0$), see Fig. 1.

The following symbols are used in the ensuing:

r	radial distance (range)
θ	azimuth
μ	shear modulus
ρ	Mass density
ν	Poisson's ratio
C_R	Rayleigh wave velocity
C_S	shear wave velocity
C_P	pressure wave velocity
t	time
a	$= C_S / C_P$ = ratio of S to P wave velocity
κ_j	$= C_S / C_j$ = three dimensionless solutions to the Rayleigh characteristic equation
γ	$\equiv \kappa_1 = C_S / C_R$ = true Rayleigh root
τ	$= tC_S / r$ = dimensionless time

In a nutshell, the application of the formulas requires the straightforward task of finding the three roots $\kappa_1^2, \kappa_2^2, \kappa_3^2$ of the rationalized, bi-cubic Rayleigh function

$$16\left(1-a^2\right)\kappa^6 - 8\left(3-2a^2\right)\kappa^4 + 8\kappa^2 - 1 = 0 \qquad (1)$$

of which the first one is the real, true Rayleigh root and the remaining two are the non-physical roots. At low Poisson's ratio the latter two are real-valued, but they turn complex when Poisson's ratio exceeds the threshold $\nu > 0.2631$. Associated with each of the three roots κ_j, $j = 1,2,3$, we define the three sets of coefficients:

$$\boxed{A_j = \frac{\left(\kappa_j^2 - \frac{1}{2}\right)^2 \sqrt{a^2 - \kappa_j^2}}{D_j}}, \boxed{B_j = \frac{\left(1 - 2\kappa_j^2\right)\left(1 - \kappa_j^2\right)}{D_j}}, \boxed{C_j = \frac{\left(1 - \kappa_j^2\right)\sqrt{a^2 - \kappa_j^2}}{D_j}} \tag{2a}$$

$$\boxed{D_j = \left(\kappa_j^2 - \kappa_i^2\right)\left(\kappa_j^2 - \kappa_k^2\right)}, \qquad i \neq j \neq k \tag{2b}$$

These coefficients, which can be real or complex, will be used in the response functions given in section 4. Observe that the two indices i,k in eq. 2b are inferred from j by cyclic order. For example, $j = 1$ implies $i = 2$ and $k = 3$ (or what is exactly the same, $i = 3$, $k = 2$), and so forth. Thus, allowable triplets are $(i, j, k) = (1,2,3), (2,3,1), (3,1,2)$.

3. Preliminary comment on square roots terms in the formulae

As can be seen, the coefficients A_j, C_j given by eq. 2 contain square root terms in the three roots of eq. 1. In a practical implementation of the formulas listed in the ensuing and for the true Rayleigh root $\kappa_1 \equiv \gamma > 1$ and for $\tau < \gamma$, it might seem preferable to define the real, transformed coefficients

$$\bar{A}_1 = \frac{\left(\gamma^2 - \frac{1}{2}\right)^2 \sqrt{\gamma^2 - a^2}}{\left(\gamma^2 - \kappa_2^2\right)\left(\gamma^2 - \kappa_3^2\right)}, \qquad \bar{C}_1 = \frac{\left(1 - \gamma^2\right)\sqrt{\gamma^2 - a^2}}{\left(\gamma^2 - \kappa_2^2\right)\left(\gamma^2 - \kappa_3^2\right)}$$

such that for $\tau < \gamma$

$$\frac{A_1}{\sqrt{\tau^2 - \gamma^2}} = \frac{\bar{A}_1}{\sqrt{\gamma^2 - \tau^2}}, \qquad \frac{C_1}{\sqrt{\tau^2 - \gamma^2}} = \frac{\bar{C}_1}{\sqrt{\gamma^2 - \tau^2}}, \qquad C_1\sqrt{\tau^2 - \gamma^2} = (-)\bar{C}_1\sqrt{\gamma^2 - \tau^2}$$

all of which follow readily from the transformations

$$\frac{\sqrt{a^2 - \kappa_1^2}}{\sqrt{\tau^2 - \kappa_1^2}} \equiv \frac{\sqrt{a^2 - \gamma^2}}{\sqrt{\tau^2 - \gamma^2}} = \frac{\mathrm{i}\sqrt{\gamma^2 - a^2}}{\mathrm{i}\sqrt{\gamma^2 - \tau^2}} = \frac{\sqrt{\gamma^2 - a^2}}{\sqrt{\gamma^2 - \tau^2}}, \qquad \text{and}$$

$$\begin{aligned}\sqrt{a^2 - \kappa_1^2}\sqrt{\tau^2 - \kappa_1^2} &\equiv \sqrt{a^2 - \gamma^2}\sqrt{\tau^2 - \gamma^2} = \left(\mathrm{i}\sqrt{\gamma^2 - a^2}\right)\left(\mathrm{i}\sqrt{\gamma^2 - \tau^2}\right) \\ &= (-)\sqrt{\gamma^2 - a^2}\sqrt{\gamma^2 - \tau^2}\end{aligned}$$

(Note: use of $-\mathrm{i}$ instead of $+\mathrm{i}$ leads to the same result, because the negative sign cancels in either the product or division).

Although we could indeed express our response functions in terms of a mixed set of coefficients $\bar{A}_1, A_2, A_3$ and $\bar{C}_1, C_2, C_3$, we prefer to use the definitions given by eq. 2 throughout for the following reasons:

- Fewer constants are needed for the response functions, which allows adding terms via summation signs. This facilitates and simplifies in turn the derivation and presentation of the formulas for the dipoles.
- The resulting formulas are then largely independent of the numbering sequence used for the roots, so these could be renumbered according to user preference or programming environment. Also, any leading negative signs, such as the one above, now arise naturally from the formulation itself, which thus avoids ad-hoc sign reversals that depend on the numbering sequence.
- The same spatio-temporal differentiation rules apply to all roots, see sections 8 and 10.
- Most importantly, our formulas are then *universal*, that is, they are valid no matter what Poisson's ratio may be, and whether or not the false roots are real or appear in complex conjugate pairs.

Of course, when implementing the formulas into a computer program, users can readily make use of the transformations above to avoid the complex algebra for the term contributed by the Rayleigh root. Similarly, when the roots appear in complex-conjugate pairs, one could also replace their contribution to a summation by doubling the real part of one of these. However, these are merely implementation issues that will be sidestepped herein and left to the readers to sort out.

4. Response functions

With reference to Fig. 1, the response functions are (their derivation can be found in Kausel, 2012):

Vertical displacement due to vertical load

$$u_{zz}(r,\tau)=\frac{(1-\nu)}{2\pi\mu r}\begin{cases}\frac{1}{2}\left(1-\sum_{j=1}^{3}\frac{A_j}{\sqrt{\tau^2-\kappa_j^2}}\right) & a<\tau<1\\ 1-\frac{A_1}{\sqrt{\tau^2-\gamma^2}} & 1\le\tau<\gamma\\ 1 & \tau\ge\gamma\end{cases} \tag{3}$$

Radial displacement due to horizontal load

$$u_{rx}=\frac{(\cos\theta)}{2\pi\mu r}\begin{cases}\frac{1}{2}(1-\nu)\tau^2\sum_{j=1}^{3}\frac{C_j}{\sqrt{\tau^2-\kappa_j^2}} & a<\tau<1\\ 1+(1-\nu)\tau^2\frac{C_1}{\sqrt{\tau^2-\gamma^2}} & 1\le\tau<\gamma\\ 1 & \tau\ge\gamma\end{cases} \tag{4}$$

Tangential displacement due to horizontal load

$$u_{\theta x}=\frac{(1-\nu)(-\sin\theta)}{2\pi\mu r}\begin{cases}\frac{1}{2}\left[1-\sum_{j=1}^{3}C_j\sqrt{\tau^2-\kappa_j^2}\right] & a<\tau<1\\ 1-C_1\sqrt{\tau^2-\gamma^2} & 1\le\tau<\gamma\\ 1 & \tau\ge\gamma\end{cases} \tag{5}$$

Radial displacement due to a vertical load:

$$u_{rz}(r,\tau)=\frac{\tau}{8\pi\mu r}\begin{cases}\frac{1}{\pi\left(1-a^2\right)^{3/2}}\left\{2K(n)-\sum_{j=1}^{3}B_j\,\Pi\left(n^2m_j,n\right)\right\}, & a<\tau<1\\ \frac{n^{-1}}{\pi\left(1-a^2\right)^{3/2}}\left\{2K\left(n^{-1}\right)-\sum_{j=1}^{3}B_j\,\Pi\left(m_j,n^{-1}\right)\right\}, & 1\le\tau<\gamma\\ \frac{n^{-1}}{\pi\left(1-a^2\right)^{3/2}}\left\{2K\left(n^{-1}\right)-\sum_{j=1}^{3}B_j\,\Pi\left(m_j,n^{-1}\right)\right\}+\frac{2}{\sqrt{\tau^2-\gamma^2}}D, & \tau\ge\gamma\end{cases} \tag{6}$$

where

$$n^2=\frac{\tau^2-a^2}{1-a^2},\quad m_j=\frac{1-a^2}{a^2-\kappa_j^2},\quad D=\frac{\left(2\gamma^2-1\right)^3}{8\left(1-a^2\right)\gamma^6-4\gamma^2+1} \tag{7}$$

In these expressions,

$$K(n)=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\sqrt{1-n^2\sin^2\theta}}\ ,\qquad \Pi(m,n)=\int_0^{\frac{1}{2}\pi}\frac{d\theta}{\left(1+m\sin^2\theta\right)\sqrt{1-n^2\sin^2\theta}} \tag{8}$$

are the complete elliptic functions of the first and third kind, respectively.
In the case of complex roots, the characteristic m turns complex, in which case the elliptic Π function satisfies the complex conjugate symmetry $\Pi\left(m^*,n\right)=\Pi^*(m,n)$. To the best of the author's knowledge and as of this writing, only Mathematica —but not Maple or Matlab— seems to provide the capability of complex characteristic. However, it is not difficult to implement effective numerical routines which allow for complex values of these parameters (one such routine is available online as a supplement to this article).

Vertical displacement due to horizontal load
Because of the reciprocity principle, the vertical displacement elicited by a horizontal load is numerically equal to the horizontal (radial) displacement due to a vertical load, except for a sign change and also the fact that it varies with the cosine of the azimuth:

$$u_{zx}(r,\theta,\tau)=-u_{rz}(r,\tau)\cos\theta$$

5. Displacements at depth along the epicentral axis

At $r=0$ and a depth $d=|z|$, the displacements at dimensionless time $\tau = tC_S / |z|$ are

$$\boxed{u_{xx} = \frac{1}{4\pi\mu|z|}\left[f_S(\tau)\mathcal{H}(\tau-1) - f_P(\tau)\mathcal{H}(\tau-a)\right]} \tag{9a}$$

$$\boxed{u_{zz} = \frac{1}{2\pi\mu|z|}\left[g_P(\tau)\mathcal{H}(\tau-a) - g_S(\tau)\mathcal{H}(\tau-1)\right]} \tag{9b}$$

where

$$f_P = \frac{2\tau(\tau^2-a^2)S_1}{(2\tau^2-2a^2+1)^2-4\tau(\tau^2-a^2)S_1}, \qquad f_S = 1+\frac{(2\tau^2-1)\tau^2}{(2\tau^2-1)^2-4\tau(\tau^2-1)S_2} \tag{10a}$$

$$g_P = \frac{\tau^2(2\tau^2-2a^2+1)}{(2\tau^2-2a^2+1)^2-4\tau(\tau^2-a^2)S_1}, \qquad g_S = \frac{2\tau(\tau^2-1)S_2}{(2\tau^2-1)^2-4\tau(\tau^2-1)S_2} \tag{10b}$$

and

$$S_1 = \sqrt{\tau^2+1-a^2}, \qquad S_2 = \sqrt{\tau^2-1+a^2} \tag{10c}$$

Although simple in appearance, at large times $\tau \gg 1$ the above representation suffers from severe cancellations. The reason is that although the sum of the two functions tends to a constant (static) value, individually each function grows without bound with time. This problem can be avoided by means of the following fully equivalent formulas for $\tau > 1$:

$$u_{xx} = \frac{1}{4\pi\mu|z|}\left\{1-(1-a^2)\left[\frac{(\tau^2-a^2)(2\tau^2-2a^2+1)^2}{\frac{1}{2}(1+S_1/\tau)D_1}+\frac{2\tau^2(2\tau^2-1)(\tau^2-1)}{\frac{1}{2}(1+S_2/\tau)D_2}\right]+\frac{1}{D_1D_2}\sum_{j=2,4,\ldots}^{12} a_j\tau^j\right\} \tag{11a}$$

$$u_{zz} = \frac{1}{2\pi\mu|z|}\left\{(1-a^2)\left[\frac{2\tau^2(2\tau^2-2a^2+1)(\tau^2-a^2)}{\frac{1}{2}(S_1/\tau+1)D_1}+\frac{(\tau^2-1)(2\tau^2-1)^2}{\frac{1}{2}(S_2/\tau+1)D_2}\right]+\frac{1}{D_1D_2}\sum_{j=2,4,\ldots}^{12} b_j\tau^j\right\} \tag{11b}$$

where

$$D_1 = 16(1-a^2)\tau^6+8(6a^4-8a^2+3)\tau^4-8(6a^6-10a^4+6a^2-1)\tau^2+(1-2a^2)^4 \tag{12a}$$

$$D_2 = 16(1-a^2)\tau^6-8(3-4a^2)\tau^4+8(1-2a^2)\tau^2+1 \tag{12b}$$

and the coefficients of the two summations (obtained with Matlab's symbolic tool) are

$$\left.\begin{aligned}
a_{12} &= 128(1-a^2)\\
a_{10} &= -64(1+4a^2-6a^4)\\
a_8 &= -16(3-15a^2-4a^4+24a^6)\\
a_6 &= 16a^2(4-17a^2+10a^4+8a^6)\\
a_4 &= 16a^2(1-3a^2+7a^4-6a^6)\\
a_2 &= -(1-10a^2+40a^4-48a^6+16a^8)
\end{aligned}\right\} \tag{13a}$$

$$\left.\begin{aligned}
b_{12} &= 128\left(1-a^2\right) \\
b_{10} &= 64\left(1-2a^2\right)\left(2-4a^2+a^4\right) \\
b_8 &= -16\left(21-37a^2+4a^4+36a^6-16a^8\right) \\
b_6 &= 16\left(3+26a^2-78a^4+70a^6-8a^8-8a^{10}\right) \\
b_4 &= 4\left(15-87a^2+116a^4+24a^6-136a^8+64a^{10}\right) \\
b_2 &= \left(11-28a^2+16a^4\right)\left(1-2a^2\right)^3
\end{aligned}\right\} \tag{13b}$$

At large times, the above converge to

$$D_1 = D_2 \to 16\left(1-a^2\right)\tau^6 + \cdots, \qquad \Sigma \to a_{12}\tau^{12} + \cdots \qquad \tfrac{1}{2}\left(1+S_j/\tau\right) \to 1$$

so

$$u_{xx} \to \frac{1}{4\pi\mu|z|}\left\{1 - \frac{\left(1-a^2\right)8}{16\left(1-a^2\right)} + \frac{128\left(1-a^2\right)}{16^2\left(1-a^2\right)^2}\right\} = \frac{1}{8\pi\mu|z|}\frac{2-a^2}{\left(1-a^2\right)} = \frac{3-2\nu}{8\pi\mu|z|} \tag{14a}$$

$$u_{zz} \to \frac{1}{4\pi\mu|z|}\left(\frac{2-a^2}{1-a^2}\right) = \frac{3-2\nu}{4\pi\mu|z|} \tag{14b}$$

which are the correct limits predicted by the Cerruti and Boussinesq theories for static tangential and vertical loads applied onto the surface of a half-space.

6. Discontinuities and singularities

a) Arrival of P wave, $\tau = a$:

At the arrival of the P wave, all response functions at the surface are continuous, even if their slopes are not. Still, all temporal derivatives are zero at $\tau = a^-$ while shortly thereafter at $\tau = a^+$, they are well defined.

On the other hand, of the two response functions at depth on the epicentral axis, the response due to a vertical load exhibits a jump of magnitude

$$\Delta u_{zz}\big|_{\tau=a} = \frac{a^2}{2\pi\mu|z|} = \frac{1-2\nu}{4(1-\nu)\pi\mu|z|} \tag{15}$$

which is properly accounted to by the Heaviside term $\mathcal{H}(\tau - a)$ in eq. (9b), so the singularity in the temporal derivative will arise naturally from the differentiation of that step function.

b) Arrival of S wave, $\tau = 1$:
All but one of the response functions —but not their slopes— are continuous at the surface. The notable exception is the tangential displacement due to a horizontal load (eq. 5), which is discontinuous at that location. Analyzing eq. 5 with Matlab's symbolic tool, this discontinuity can be shown to be given by:

$$\begin{aligned}\Delta u_{\theta x}\Big|_{\tau=1} &= \frac{(1-\nu)}{4\pi\mu r}(-\sin\theta)\left\{1 - C_1\sqrt{1-\kappa_1^2} + C_2\sqrt{1-\kappa_2^2} + C_3\sqrt{1-\kappa_3^2}\right\} \\ &= \frac{1}{2\pi\mu r}(-\sin\theta)\end{aligned} \tag{16}$$

Observe the remarkable fact that the jump is ultimately independent of Poisson's ratio, even though the function itself depends on that ratio at all other times. This shows that the discontinuity is caused by the arrival of an SH wave. Furthermore, the jump implies that the temporal derivative (i.e. with respect to τ) will exhibit a singularity

$$\frac{\delta(\tau-1)}{2\pi\mu r}(-\sin\theta) \tag{17}$$

which must be accounted for when the temporal derivatives are obtained by direct differentiation at $\tau = 1^-$ and $\tau = 1^+$, i.e. in the case of the dipoles considered later on.

At points below the surface along the epicentral axis, the horizontal displacement is discontinuous, and the jump is

$$\Delta u_{xx}\Big|_{\tau=1} = \frac{1}{2\pi\mu|z|} \tag{18}$$

but again this discontinuity is properly accounted for by the Heaviside term $\mathcal{H}(\tau-1)$ which will contribute properly to the singularity of the slope at this point in time.

c) Arrival of R wave, $\tau = \gamma$
The functions u_{zz}, u_{rx} in eqs. 3, 4 exhibit an integrable, negative-valued singularity in the neighborhood $\tau = \gamma^-$. Immediately thereafter, following the passage of the Rayleigh wave, these response functions jump to their final, positive, static values at $\tau = \gamma^+$, i.e.

$$\Delta u_{zz}\Big|_{\tau=\gamma} = \frac{1-\nu}{2\pi\mu r}, \qquad \Delta u_{rx}\Big|_{\tau=\gamma} = \frac{(\cos\theta)}{2\pi\mu r} \tag{19}$$

Thus, when differentiated with respect to time, these jumps will add a positive singularity immediately following the negative singularity, i.e.

$$\frac{1-\nu}{2\pi\mu r}\delta(\tau-\gamma) \qquad \text{and} \qquad \frac{(\cos\theta)}{2\pi\mu r}\delta(\tau-\gamma) \tag{20}$$

respectively. These must be accounted for when computing velocities, dipoles, or the response function due to impulsive loads.

On the other hand, the displacement functions u_{rz}, u_{zx} also exhibit an integrable singularity at $\tau=\gamma^+$ —due to the term in D in eq. 6— but are otherwise continuous.

7. Transformation into Cartesian coordinates

The displacement functions in all of the formulas in eqs. 3-6 can be cast in terms of cylindrical amplitudes $\tilde{u}_{\ell r}, \tilde{u}_{\ell\theta}, \tilde{u}_{\ell z}$ by writing the response functions in the form

$$u_r = \tilde{u}_{\ell r}(\cos\ell\theta), \qquad u_\theta = \tilde{u}_{\ell\theta}(-\sin\ell\theta), \qquad u_z = \tilde{u}_{\ell z}(\cos\ell\theta) \tag{21}$$

where $\ell=0$ for the vertical load, and $\ell=1$ for the horizontal load. These amplitudes are closely related to the integrals $I_1,\cdots I_4$ in Richards, 1979. The Cartesian components of the displacement vector are then given by

$$u_x = \tilde{u}_{\ell r}\cos\ell\theta\cos\theta + \tilde{u}_{\ell\theta}\sin\ell\theta\sin\theta \tag{22a}$$

$$u_y = \tilde{u}_{\ell r}\cos\ell\theta\sin\theta - \tilde{u}_{\ell\theta}\sin\ell\theta\cos\theta \tag{22b}$$

$$u_z = \tilde{u}_{\ell z}\cos\ell\theta \tag{22c}$$

8. Lamb dipoles

Of the nine possible point dipoles which may act within a continuous space, the explicit formulas for Lamb's problem allow obtaining a subset of these, namely the solutions for the six dipoles acting at the surface of the half-space which do not depend on derivatives of the displacement functions with respect to the vertical direction, see Fig. 2. When expressed in cylindrical coordinates, these six dipoles can be written as follows. Let u,v,w be a shorthand for the response functions due to a horizontal load (i.e. $u\equiv\tilde{u}_{1r}$, $v\equiv\tilde{u}_{1\theta}$, $w\equiv\tilde{u}_{1z}$ in eq. 21) and U,W are the response functions due to a vertical load (i.e. $U\equiv\tilde{u}_{0r}$, $W\equiv\tilde{u}_{0z}$). Then (Kausel, 2006):

$$\mathbf{G}_{xx} = -\frac{1}{2}\left\{\left(\frac{\partial u}{\partial r}+\frac{u-v}{r}\right)\hat{\mathbf{r}} + \left(\frac{\partial w}{\partial r}+\frac{w}{r}\right)\hat{\mathbf{k}} + \left(\frac{\partial u}{\partial r}-\frac{u-v}{r}\right)\cos 2\theta\,\hat{\mathbf{r}} - \left(\frac{\partial v}{\partial r}+\frac{u-v}{r}\right)\sin 2\theta\,\hat{\mathbf{t}} + \left(\frac{\partial w}{\partial r}-\frac{w}{r}\right)\cos 2\theta\,\hat{\mathbf{k}}\right\} \tag{23a}$$

$$\mathbf{G}_{yy} = -\frac{1}{2}\left\{\left(\frac{\partial u}{\partial r}+\frac{u-v}{r}\right)\hat{\mathbf{r}} + \left(\frac{\partial w}{\partial r}+\frac{w}{r}\right)\hat{\mathbf{k}} - \left(\frac{\partial u}{\partial r}-\frac{u-v}{r}\right)\cos 2\theta\,\hat{\mathbf{r}} + \left(\frac{\partial v}{\partial r}+\frac{u-v}{r}\right)\sin 2\theta\,\hat{\mathbf{t}} - \left(\frac{\partial w}{\partial r}-\frac{w}{r}\right)\cos 2\theta\,\hat{\mathbf{k}}\right\} \tag{23b}$$

$$\mathbf{G}_{xy} = -\frac{1}{2}\left\{ -\left(\frac{\partial v}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{t}} + \left(\frac{\partial u}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{r}}\sin 2\theta \right.$$
$$\left. + \left(\frac{\partial v}{\partial r} + \frac{u-v}{r}\right)\cos 2\theta\,\hat{\mathbf{t}} + \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\sin 2\theta\,\hat{\mathbf{k}} \right\} \tag{23c}$$

$$\mathbf{G}_{yx} = -\frac{1}{2}\left\{ \left(\frac{\partial v}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{t}} + \left(\frac{\partial u}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{r}}\sin 2\theta \right.$$
$$\left. + \left(\frac{\partial v}{\partial r} + \frac{u-v}{r}\right)\cos 2\theta\,\hat{\mathbf{t}} + \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\sin 2\theta\,\hat{\mathbf{k}} \right\} \tag{23d}$$

$$\mathbf{G}_{zx} = -\left\{ \frac{\partial U}{\partial r}\cos\theta\,\hat{\mathbf{r}} - \frac{U}{r}\sin\theta\,\hat{\mathbf{t}} + \frac{\partial W}{\partial r}\cos\theta\,\hat{\mathbf{k}} \right\} \tag{23e}$$

$$\mathbf{G}_{zy} = -\left\{ \frac{\partial U}{\partial r}\sin\theta\,\hat{\mathbf{r}} + \frac{U}{r}\cos\theta\,\hat{\mathbf{t}} + \frac{\partial W}{\partial r}\sin\theta\,\hat{\mathbf{k}} \right\} \tag{23f}$$

where $\hat{\mathbf{r}}, \hat{\mathbf{t}}, \hat{\mathbf{k}}$ are unit vectors in the radial, tangential and vertical directions, respectively. The $\mathbf{G}_{xx}, \mathbf{G}_{yy}$ are the response functions for cracks while the other functions are for single couples, with the first index identifying the direction of the forces in the couple and the second the direction of the moment arm, see Fig. 2. From the above we can readily infer also the expressions for a double couple lying in the horizontal plane on the surface (seismic moment) as well as for a torsional moment with vertical axis, namely

$$\begin{aligned}\mathbf{M}_z &= \mathbf{G}_{yx} + \mathbf{G}_{xy} \\ &= -\left\{ \left(\frac{\partial u}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{r}}\sin 2\theta + \left(\frac{\partial v}{\partial r} + \frac{u-v}{r}\right)\cos 2\theta\,\hat{\mathbf{t}} + \left(\frac{\partial w}{\partial r} - \frac{w}{r}\right)\sin 2\theta\,\hat{\mathbf{k}} \right\}\end{aligned} \tag{24}$$

and

$$\mathbf{T}_z = \tfrac{1}{2}\left(\mathbf{G}_{yx} - \mathbf{G}_{xy}\right) = -\frac{1}{2}\left(\frac{\partial v}{\partial r} - \frac{u-v}{r}\right)\hat{\mathbf{t}} \tag{25}$$

All of the above solutions for dipoles involve derivatives with respect to the range r, and all of Lamb's formulas contain terms of the form

$$F(r,\tau) = \tfrac{1}{r} f(\tau), \qquad \tau = \tau(t,r) = \frac{C_S t}{r}, \qquad \frac{\partial f(\tau)}{\partial r} = \frac{\partial f(\tau)}{\partial \tau}\frac{\partial \tau}{\partial r} = (-)\frac{\tau}{r}\frac{\partial f(\tau)}{\partial \tau}$$

which implies

$$\frac{\partial F}{\partial r} = \frac{\partial\left(\frac{1}{r} f\right)}{\partial r} = (-)\frac{1}{r^2}\left(f + \tau\frac{\partial f}{\partial \tau}\right) \tag{26}$$

This leads to the following derivatives of interest:

$$f = 1 \qquad\qquad \frac{\partial \frac{1}{r}}{\partial r} = (-)\frac{1}{r^2} \tag{27a}$$

$$f=\tau^2 \qquad \frac{\partial\left(\frac{1}{r}\tau^2\right)}{\partial r}=(-)\frac{3\tau^2}{r^2} \tag{27b}$$

$$f=\sqrt{\tau^2-\kappa_j^2}\,, \qquad \frac{\partial\frac{1}{r}\sqrt{\tau^2-\kappa_j^2}}{\partial r}=(-)\frac{1}{r^2}\frac{2\tau^2-\kappa_j^2}{\sqrt{\tau^2-\kappa_j^2}} \tag{27c}$$

$$f=\sqrt{\kappa_j^2-\tau^2} \qquad \frac{\partial\frac{1}{r}\sqrt{\kappa_j^2-\tau^2}}{\partial r}=\frac{1}{r^2}\frac{2\tau^2-\kappa_j^2}{\sqrt{\kappa_j^2-\tau^2}} \tag{27d}$$

$$f=\frac{1}{\sqrt{\tau^2-\kappa_j^2}}\,, \qquad \frac{\partial}{\partial r}\left(\frac{1}{r\sqrt{\tau^2-\kappa_j^2}}\right)=\frac{\kappa_j^2}{r^2\left(\tau^2-\kappa_j^2\right)^{\frac{3}{2}}} \tag{27e}$$

$$f=\frac{1}{\sqrt{\kappa_j^2-\tau^2}}\,, \qquad \frac{\partial}{\partial r}\left(\frac{1}{r\sqrt{\kappa_j^2-\tau^2}}\right)=(-)\frac{\kappa_j^2}{r^2\left(\kappa_j^2-\tau^2\right)^{\frac{3}{2}}} \tag{27f}$$

These derivatives are valid whether the roots κ_j are real or complex.

On the other hand, in section 6, we made reference to three discontinuities in the (dimensionless) time domain τ (eqs. 17, 19), which in the current notation are

$$\Delta u\big|_{\tau=\gamma}=\frac{1}{2\pi\mu r}\,, \qquad \Delta v\big|_{\tau=1}=\frac{1}{2\pi\mu r}\,, \qquad \Delta W\big|_{\tau=\gamma}=\frac{1-\nu}{2\pi\mu r} \tag{28}$$

Since the partial derivatives in the dipoles are with respect to r and not τ, then the above discontinuities lead to the singularities

$$\left.\frac{\partial u}{\partial r}\right|_{\tau=\gamma}=\left.\frac{\partial u}{\partial\tau}\frac{\partial\tau}{\partial r}\right|_{\tau=\gamma}=(-)\left.\frac{\tau}{r}\frac{\partial u}{\partial\tau}\right|_{\tau=\gamma}=(-)\frac{\gamma}{2\pi\mu r^2}\delta(\tau-\gamma) \tag{29b}$$

$$\left.\frac{\partial v}{\partial r}\right|_{\tau=1}=\left.\frac{\partial v}{\partial\tau}\frac{\partial\tau}{\partial r}\right|_{\tau=1}=(-)\left.\frac{\tau}{r}\frac{\partial v}{\partial\tau}\right|_{\tau=1}=(-)\frac{1}{2\pi\mu r^2}\delta(\tau-1) \tag{29a}$$

$$\left.\frac{\partial W}{\partial r}\right|_{\tau=\gamma}=\left.\frac{\partial W}{\partial\tau}\frac{\partial\tau}{\partial r}\right|_{\tau=\gamma}=(-)\left.\frac{\tau}{r}\frac{\partial W}{\partial\tau}\right|_{\tau=\gamma}=(-)(1-\nu)\frac{\gamma}{2\pi\mu r^2}\delta(\tau-\gamma) \tag{29c}$$

Taking these singularities into account, then the terms of interest which appear in the various dipoles are as follows:

[EDITOR: Please do not confuse the Roman, italic v on the left in 30a, 30c and the Greek ν on the right in 30a–d]

$$\frac{u-v}{r} = \frac{1}{2\pi\mu r^2}\begin{cases} \frac{1}{2}(1-\nu)\left[\sum_{j=1}^{3} C_j \frac{2\tau^2-\kappa_j^2}{\sqrt{\tau^2-\kappa_j^2}} - 1\right] & a<\tau<1 \\ \nu + (1-\nu) C_1 \frac{2\tau^2-\gamma^2}{\sqrt{\tau^2-\gamma^2}} & 1<\tau<\gamma \\ \nu & \tau>\gamma \end{cases} \tag{30a}$$

$$\frac{\partial u}{\partial r} = (-)\frac{1}{2\pi\mu r^2}\begin{cases} \frac{1}{2}(1-\nu)\tau^2 \sum_{j=1}^{3} C_j \frac{2\tau^2-3\kappa_j^2}{\left(\tau^2-\kappa_j^2\right)^{3/2}} & a<\tau<1 \\ 1+(1-\nu)\tau^2 C_1 \frac{\left(2\tau^2-3\gamma^2\right)}{\left(\tau^2-\gamma^2\right)^{3/2}} & 1\le\tau<\gamma \\ 1+\gamma\delta(\tau-\gamma) & \tau\ge\gamma \end{cases} \tag{30b}$$

$$\frac{\partial v}{\partial r} = (-)\frac{1}{2\pi\mu r^2}\begin{cases} \frac{1}{2}(1-\nu)\left[1-\sum_{j=1}^{3} C_j \frac{2\tau^2-\kappa_j^2}{\sqrt{\tau^2-\kappa_j^2}}\right] & a<\tau<1 \\ (1-\nu)\left[1-C_1 \frac{2\tau^2-\gamma^2}{\sqrt{\tau^2-\gamma^2}}\right] + \delta(\tau-1) & 1\le\tau<\gamma \\ 1-\nu & \tau\ge\gamma \end{cases} \tag{30c}$$

$$\frac{\partial W}{\partial r} = (-)\frac{(1-\nu)}{2\pi\mu r^2}\begin{cases} \frac{1}{2}\left(1+\sum_{j=1}^{3} \frac{A_j\,\kappa_j^2}{\left(\tau^2-\kappa_j^2\right)^{\frac{3}{2}}}\right) & a<\tau<1 \\ 1+\frac{A_1\gamma^2}{\left(\tau^2-\gamma^2\right)^{\frac{3}{2}}} & 1\le\tau<\gamma \\ 1+\gamma\,\delta(\tau-\gamma) & \tau\ge\gamma \end{cases} \tag{30d}$$

$$\frac{\partial w}{\partial r} = -\frac{\partial U}{\partial r} = \text{complicated} \tag{30e}$$

We omitted the last derivative in eq. 30e above because it is rather cumbersome, inasmuch as it involves derivatives of the elliptical functions in which τ appears as argument in the modulus n and n^{-1}, see eqs. 6,7. Nonetheless, these too could be written down in terms of explicit formulas without much problem, a task that is left to the readers to carry out.

To illustrate matters, consider the case of a torsional dipole. From eqs. 25, 30a, and 30c, we readily obtain

$$u_\theta = \frac{1}{4\pi\mu r^2}\left\{\mathcal{H}\left(\tau-1\right)+\delta\left(\tau-1\right)\right\} \tag{31}$$

which agrees perfectly with the exact solution obtained by the method of images (Kausel, 2006, pp. 83-85)

9. Graphic illustration

Figures 3-6 depict in their upper part a global view of the response functions given by equations 3-6 for the classical Pekeris-Chao case of $\nu = 0.25$, $\mu = C_S = r = 1$, and then underneath a blow-up of the corresponding response function for a full set of Poisson's ratios in the range from 0 to 0.5, which reveal the detail of the various wave arrivals together with their abrupt transitions. Thereafter, Fig. 7 shows the displacements along the vertical axis below the load. All of these agree perfectly with the known results. Observe that the plots for the response functions due to a horizontal load do *not* include the implicit factors $(\cos\theta)$ and $(-\sin\theta)$ factors, which must be added to obtain the actual variation with the azimuth.

Interested readers may wish to download an electronic supplement to this article containing the detailed mathematical proof of the formulas presented herein, and also a brief Matlab program which evaluates all of the components and for any Poisson's ratio.

10. Whither Q_1?

Cognoscenti will surely have noticed that the response functions given herein are free from cumbersome terms involving the complex poles Q_1, Q_2 which arise in contour integrals on the unit circle when the roots are complex (Mooney, 1974; Kausel 2006). Such integrals have the form

$$I = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{C\,d\theta}{C+D\sin^2\theta} = \frac{4Z}{\left(Q_2-Q_1\right)}, \qquad Z=\frac{C}{D}, \qquad Q_1Q_2=1, \qquad \left|Q_1\right|<1 \tag{32}$$

where C is complex and D is real and non-negative. The Q_1, Q_2 terms on the right hand side satisfy the equation

$$Q_{1,2} = 1+2Z \mp 2\sqrt{Z\left(Z+1\right)} \tag{33}$$

where the square root term $\sqrt{Z(Z+1)}$ corresponds to Richards' CROOT variable as given by his eq. 11. It can be shown (Kausel, 2012) that Q_1 can be obtained *explicitly* from the equation

$$Q_1 = 1 + 2Z - 2\sqrt{Z(Z+1)}\,\mathrm{sgn}\left[1 + 2\mathrm{Re}(Z)\right] \tag{34}$$

in which case

$$Q_2 - Q_1 = Q_1^{-1} - Q_1 = 4\sqrt{Z(Z+1)}\,\mathrm{sgn}\left[1 + 2\mathrm{Re}(Z)\right] \tag{35}$$

It can also be shown that

$$\frac{Z}{\sqrt{Z(Z+1)}} = \frac{Z}{\sqrt{Z}\sqrt{Z+1}}\mathrm{sgn}\left[1+2\mathrm{Re}(Z)\right] = \frac{\sqrt{Z}}{\sqrt{Z+1}}\mathrm{sgn}\left[1+2\mathrm{Re}(Z)\right] \tag{36}$$

Combination of both of the above yields finally

$$I = \frac{4Z}{(Q_2 - Q_1)} = \frac{\sqrt{Z}}{\sqrt{Z+1}} \tag{37}$$

which has *exactly the same form* as the solution for purely real coefficients C, D. Observe that the final ratio on the right is free from any numerical tests, such as the requirement for the absolute value of some quantity, say $|Q_1| < 1$, and indeed, *these quantities need not be evaluated in the first place*. Hence, exactly the same formulas apply to both real and complex roots. This is a significant improvement because it allows rendering all of the response functions in a much simpler, common form, and especially so concerning the variation with time. For example, compare below the classical treatment of complex roots on the left (Mooney, 1974) versus the current form on the right:

$$\frac{1}{(\tau^2 - a^2)\left[Q_2(\tau) - Q_1(\tau)\right]} \qquad \text{vs.} \qquad \frac{1}{\sqrt{\tau^2 - \kappa_j^2}} \tag{38}$$

Thus, only one set of functions is needed instead of two, and these are also readily amenable to further manipulations, such as taking derivatives to obtain explicit expressions for dipoles or for impulsive loads, as presented earlier. This is a significant advantage in our formulation which ultimately emanates from the lack of need to distinguish between real and complex roots.

11. Conclusions

This paper revisited Lamb's problem of a suddenly applied, horizontal and vertical point load applied at the surface of an elastic, homogeneous half-space of arbitrary Poisson's ratio. It presented a compact set of explicit space-time formulas for the following problems:

- All response functions for receivers placed at the surface of the half-space.

- All response functions for receivers placed at depth underneath the load, i.e. along the epicentral axis.
- Out of nine possible dipoles, explicit formulas are also given for the six dipoles which correspond to horizontal cracks and horizontally polarized single couples.

We also demonstrated that the integrals associated with complex roots of the Rayleigh function attain exactly the same form as those for the real roots. This allowed us to provide a unique set of formulas that is valid for any arbitrary Poisson's ratio, which simplified in turn the task of taking spatial and temporal derivatives needed for dipoles.

12. Acknowledgement

We wish to thank Dr. Paul G. Richards for openly reviewing an initial version of this paper and bringing to our attention his own, much earlier contribution to the subject matter. A careful cross-check ultimately demonstrated a complete agreement between our formulae for displacements, except for trivial changes in signs due to the different coordinate conventions employed by each of us. Readers should also be alerted to the fact that Richards' formula for a vertical load (his I_3) differs substantially from our eq. 3 as well as from the classical formulas found in either Mooney (1974) or Eringen-Suhubi (1975), which at first seemed to suggest to us that it might be incorrect. However, after carrying out some additional mathematical transformations with the aid of the Rayleigh function, we succeeded in demonstrating that our formulas were fully equivalent and thus constitute nothing but alternative mathematical realizations of one and the same response function.

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Lamb's problem revisited: Mathematical formulation

by **Eduardo Kausel**[1]

Abstract

This document supplements a concurrent, companion, main article by the writer containing a complete set of *exact* space-time formulas for Lamb's problem in a homogeneous half-space, for both vertical and horizontal (tangential) point loads which have been suddenly applied onto the half-space's surface, and for any Poisson's ratio. This second part provides the full mathematical derivation to the formulas reported in the summary article.

I. Introduction

Perhaps few problems in theoretical seismology enjoy a higher emblematic and iconic status than Lamb's problem. Indeed, it is the quintessential problem in seismology *par excellence*, enunciated first in the now famous 1904 paper by Sir Horace Lamb, Professor of Mathematics at the University of Adelaide in South Australia. In his paper, Lamb resorts to a precursor of what constitutes the modern integral transform method to obtain the response to either an impulsive (2-D) or a suddenly applied (3-D) vertical load acting on the surface of an elastic half-space. However, Lamb himself lacked in his time the advanced mathematical tools —not to mention computers— necessary to fully evaluate all of his integrals. Thus, Lamb assessed in some detail only the response in the far field at remote distances from the source. Still, to this day and in his honor, the problem of a dynamic source applied at the surface of an elastic half-space is referred to as *Lamb's problem.*

About four decades after Lamb, Cagniard (1939) finally managed to evaluate the requisite double integral transforms in Lamb's problem by means of a very ingenious yet arcane contour integration that few understood. Two decades later in turn, de Hoop (1960) succeeded in finding a substantial simplification to Cagniard's procedure in what is now referred to as the Cagniard–de Hoop method. This analytical strategy was also used by Pekeris (1955) and by Chao (1960) to obtain closed-form solutions —i.e. not requiring numerical integrations— for vertical and horizontal point loads suddenly applied onto an elastic half-space, but only when Poisson's ratio is $\nu = 0.25$.

We mention in passing that the terseness and obscurity of some of the mathematical details together with a few minor inconsistencies makes Pekeris' paper difficult to follow and requires a great deal of detective work on the part of the reader. For example, he does not indicate the direction of his displacements, even though he states that the load is positive downwards. By the time he presents plots of displacements versus time, they are displayed as either "up" and "down" or "inwards" and "outwards", but the signs in the plots are inconsistent with the formulas. Moreover —prefixing his equations with the letter P— formula P66 for the radial displacement exhibits a positive asymptote at long times, which is consistent with an inward radial displacement for a downward force. But in his plot, this quantity is negative. Also, Pekeris neglects twice a leading negative sign, first from P20 to P22, and then again from P32 and on. Of course, after this double sign

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reversal he obtains the correct final results. An examination of his developments seems to suggest that he at first may have worked out the formulation using a right-handed, Cartesian coordinate system, and only later adapted it in an *ad hoc* fashion to conform to Lamb's convention, namely an upward vertical and outward radial displacements for a downward force.

The Pekeris solution for a vertical load was later reported in the well-known book by Graff (1975). At about this same time, Mooney (1974) generalized Pekeris' results for vertical point loads acting on half-spaces with arbitrary Poisson's ratio, but considered only the vertical component while ignoring the radial displacement. The vertical load problem was also taken up by Eringen and Suhubi in their monumental book on *Elastodynamics* (1975), where they provided results for the radial component of displacement up to a Poisson's ratio $\nu < 0.2631$, namely the value at which the false roots of the equation for the speed of Rayleigh waves turn complex. In all cases, displacements were given in closed form only for points on the surface and on the axis of symmetry below the load, and not at other interior points or along the axis of symmetry.

Finally, in 1979, Richards closed the loop by providing a nearly complete set of formulae for both horizontal and vertical loads and for any Poisson's ratio, but without giving any proofs or derivations, and no indication as to where those proofs might be found. Perhaps because of that reason, and partly also because of the title that he chose for his paper in the context of crack propagation, that seminal contribution remained largely unnoticed and unknown, except among a handful of researchers in that field. Indeed, until very recently, this contribution was unknown to the writer and to his numerous and well-informed corresponding colleagues in soil dynamics.

It is likely that with the pair of the seminal papers by Pekeris and Chao, the geophysicists and mathematicians of the time may have declared victory and decided that Lamb's problem was essentially fully solved, only needing some details to be filled in here or there. Thus, when Mooney presented his partial extension to the vertical load case for arbitrary Poisson's ratios, he did so only as an appendix to his paper, as if were not particularly noteworthy in its own right. Similarly, when Richards gave his set of formulas, he may not have felt that they were important enough to require providing their complete proof —or that the proof was simply "obvious" or requiring anything more than basic training in calculus— and he himself appeared to have been almost apologetic when he begins with his brief article with the comment "these formulas would be only a minor curiosity", and then ends it with "Perhaps the main achievement of this paper …", as if he were unsure of its true worth. However, the vacuum left by the absence of proofs is regrettable because these formulas are most certainly not easy to come by or derive vis-à-vis the available solutions by Pekeris and Chao. Thus, considering that such proof is not elementary and that it has remained missing in publications as well as in books, it is felt that it should now be made public for the benefit of those with an interest in theoretical methods.

Now, the solutions to Lamb's problem given by both Mooney and by Richards consist of two rather different sets of formulas, namely one set for Poisson's ratios in the range

$0 \le \nu < 0.2631$, and another set when Poisson's ratios exceeds the threshold $\nu = 0.2631$, which marks the transition point from real to complex false roots in Rayleigh's characteristic equation. The solutions for the latter depend on a function of dimensionless time characterized by two alternative, complex-valued expressions $Q_1(\tau), Q_2(\tau)$, of which the correct one must satisfy the condition $|Q_1| < 1$, but which can change from one into the other as time goes by. The presence of this seemingly discontinuous function of time adds substantial complexity to the response functions, and causes difficulty when temporal derivatives are needed. In Appendix 2 we demonstrate how to dispose of these functions altogether and allow us to attain formulas which are identical in form whatever Poisson's ratio may be. However, inasmuch as we discovered this property only after completing this lengthy article, and taking advantage of that property would have required a substantial rewrite, we abstained from using that property herein, even if we did so in our main paper summarizing the formulas. This will allow the reader to compare both formulations and verify that they do indeed give the same results.

This article begins by presenting the known integral transforms from the frequency-wavenumber domain into the space-time domain. It then goes on to express the integrands in the Laplace domain —properly made dimensionless— integrates these into the time domain with aid of the so-called Bateman-Pekeris Theorem, and finally carries out an inverse Hankel transform from wavenumbers into the space domain by means of the Cagniard-De Hoop technique.

II. Integral transforms

Consider a homogeneous, elastic half-space with shear modulus μ and Poison's ratio ν which is subjected to either a horizontal or an upward vertical *point load* of unit amplitude. This load acts at the free surface at the origin of coordinates and is suddenly applied i.e. it is a step function in time. In the frequency-wavenumber (ω, k) domain, this problem is characterized by a load with amplitude $(\mathrm{i}\omega)^{-1}$ together with a set of Green's functions $g_{\alpha\beta}(\omega, k)$ which are known in closed form, as given later on. Displacements in the space-time domain at a radial distance *r*, depth $d = -z \ge 0$, azimuth θ and time t are then obtained —at least formally— by means of the usual inverse Fourier-Hankel transforms (e.g. Kausel, 2006, pp. 161-164)

$$u_{rx}(r,d,t) = \frac{(\cos\theta)}{2\pi\mu} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega} \mathrm{e}^{\mathrm{i}\omega t} \left\{ \int_0^\infty g_{xx} \frac{\partial}{\partial kr} J_1(kr) k\, dk + \int_0^\infty g_{yy} \frac{1}{kr} J_1(kr) k\, dk \right\} d\omega \tag{1a}$$

$$u_{\theta x}(r,d,t) = \frac{(-\sin\theta)}{2\pi\mu} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega} \mathrm{e}^{\mathrm{i}\omega t} \left\{ \int_0^\infty g_{xx} \frac{1}{kr} J_1(kr) k\, dk + \int_0^\infty g_{yy} \frac{\partial}{\partial kr} J_1(kr) k\, dk \right\} d\omega \tag{1b}$$

$$u_{zx}(r,d,t) = \frac{(\cos\theta)}{2\pi\mu} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega} \mathrm{e}^{\mathrm{i}\omega t} \left\{ \int_0^\infty g_{zx} J_1(kr)\, k\, dk \right\} d\omega \tag{1c}$$

$$u_{rz}(r,d,t) = -\frac{1}{2\pi\mu} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega} \mathrm{e}^{\mathrm{i}\omega t} \left\{ \int_0^\infty g_{xz} J_1(kr) k\, dk \right\} d\omega \tag{1d}$$

$$u_{zz}(r,d,t) = \frac{1}{2\pi\mu} \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{1}{\mathrm{i}\omega} \mathrm{e}^{\mathrm{i}\omega t} \left\{ \int_0^\infty g_{zz} J_0(kr) k\, dk \right\} d\omega \tag{1e}$$

with the vertical components being positive up. Making use of the recurrence relationships of the Bessel functions, the first two of the five integral transforms can also be written as

$$u_{rx} = \frac{1}{2\pi\mu}\left[\tfrac{1}{2}\left(u_{0x} - u_{2x}\right) + \tfrac{1}{2}\left(u_{0y} + u_{2y}\right)\right](\cos\theta) \tag{2a}$$

$$u_{\theta x} = \frac{1}{2\pi\mu}\left[\tfrac{1}{2}\left(u_{0x} + u_{2x}\right) + \tfrac{1}{2}\left(u_{0y} - u_{2y}\right)\right](-\sin\theta) \tag{2b}$$

where

$$u_{0x} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\left\{\int_0^\infty g_{xx}J_0\left(kr\right)k\,dk\right\}d\omega \tag{3a}$$

$$u_{2x} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\left\{\int_0^\infty g_{xx}J_2\left(kr\right)k\,dk\right\}d\omega \tag{3b}$$

$$u_{0y} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\left\{\int_0^\infty g_{yy}J_0\left(kr\right)k\,dk\right\}d\omega \tag{3c}$$

$$u_{2y} = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\left\{\int_0^\infty g_{yy}J_2\left(kr\right)k\,dk\right\}d\omega \tag{3d}$$

which are given in terms of "pure" Fourier-Hankel transforms i.e. not involving derivatives or ratios in the Bessel functions. Hence, if g_n denotes any arbitrary, generic Green's function, then *all* displacement components can be expressed in terms of integrals of the form

$$u_n\left(r,d,t\right) = \frac{1}{2\pi\mu}\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{1}{\mathrm{i}\omega}\mathrm{e}^{\mathrm{i}\omega t}\left\{\int_0^\infty g_n\left(k,d,\omega\right)J_n\left(kr\right)k\,dk\right\}d\omega, \qquad n = 0,1,2 \tag{4}$$

The generic Green's functions $g_n\left(k,\omega\right)$ include square root terms of the form

$$\sqrt{k^2 - k_P^2}, \qquad k_P = \omega / C_P, \qquad \sqrt{k^2 - k_S^2}, \qquad k_S = \omega / C_S \tag{5}$$

which for complex values of the radial wavenumber k are multi-valued, so appropriate branch cuts are needed for their unambiguous definition.

III. Integration method

To carry out the integral transforms, we proceed along the following steps:

1) Cast the frequency integrals into the Laplace domain:

Make $s = \mathrm{i}\omega$, which changes $g_n\left(k,d,\omega\right) \to f_n\left(k,d,s\right)$. The f_n functions are similar to the g_n functions, but they now involve square root terms of the form

$$\sqrt{k^2 + h_P^2}, \qquad h_P = s / C_P, \qquad \sqrt{k^2 + h_S^2}, \qquad h_S = s / C_S \tag{6}$$

Hence, these functions are fully real for any real pair k,s. This changes the generic integral transform into

$$u_n\left(r,d,t\right) = \frac{1}{2\pi\mu}\frac{1}{2\pi\mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\frac{1}{s}\mathrm{e}^{st}\left\{\int_0^\infty f_n\left(k,d,s\right)J_n\left(kr\right)k\,dk\right\}ds \tag{7}$$

2) Make the variables dimensionless

$$kr = \left(\frac{kC_S}{s}\right)\left(\frac{sr}{C_S}\right) = \xi\sigma\,, \qquad \xi = \frac{kC_S}{s}, \qquad \sigma = \frac{sr}{C_S}\,, \tag{8a}$$

$$\zeta = \frac{sd}{C_S} = \frac{d}{r}\sigma\,, \qquad \frac{dk\,ds}{s} = \frac{d\xi\,d\sigma}{r} \tag{8b}$$

in which case the functions change from $k\,f_n(k,d,s)$ to $\xi F_n(\xi,\zeta,\sigma)$ such that

$$\begin{aligned} U_n(r,d,t) &= 2\pi\,\mu r\,u_n(r,d,t) \\ &= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \mathrm{e}^{\sigma\tau}\left\{\int_0^\infty F_n(\xi,\zeta,\sigma)\,J_n(\xi\sigma)\,\xi\,d\xi\right\}d\sigma \\ &= \mathcal{L}^{-1}\left\{\int_0^\infty F_n(\xi,\zeta,\sigma)\,J_n(\xi\sigma)\,\xi\,d\xi\right\} \end{aligned} \tag{9}$$

with dimensionless displacement functions $U_n(r,d,t)$. The Green's functions $F_n(\xi,\zeta,\sigma)$ in the Laplace domain can readily be derived and are known in closed form. They are

$$F_{xx}(\xi,\zeta,\sigma) = \frac{\beta}{R}\left[2\xi^2(e_P - e_S) - e_S\right] \tag{10a}$$

$$F_{zx}(\xi,\zeta,\sigma) = \frac{\xi}{R}\left[2\alpha\beta\,e_P - \left(2\xi^2+1\right)e_S\right] \tag{10b}$$

$$F_{xz}(\xi,\zeta,\sigma) = \frac{\xi}{R}\left[2\alpha\beta\,e_S - \left(2\xi^2+1\right)e_P\right] \tag{10c}$$

$$F_{zz}(\xi,\zeta,\sigma) = \frac{\alpha}{R}\left[2\xi^2(e_S - e_P) - e_P\right] \tag{10d}$$

$$F_{yy}(\xi,\zeta,\sigma) = \frac{1}{\beta}e_S \tag{10e}$$

where

$$R(\xi^2) = 4\xi^2\alpha\beta - \left(1+2\xi^2\right)^2 \qquad = \text{Rayleigh function} \tag{11a}$$

$$\alpha = \sqrt{\xi^2 + a^2}\,, \qquad \beta = \sqrt{\xi^2+1}\,, \qquad a = \frac{C_S}{C_P} \tag{11b}$$

$$e_P = \exp(-\zeta\alpha)\,, \qquad e_S = \exp(-\zeta\beta) \qquad \zeta = \frac{sd}{C_S} \tag{11c}$$

In particular, at the free surface $d = 0$ i.e. $\zeta = 0$ we have $e_P = e_S = 1$, in which case

$$F_{xx} = -\frac{\beta}{R} \tag{12a}$$

$$F_{xz} = F_{zx} = -\frac{\xi}{R}\left[1 + 2\xi^2 - 2\alpha\beta\right] \tag{12b}$$

$$F_{zz} = -\frac{\alpha}{R} \tag{12c}$$

$$F_{yy} = \frac{1}{\beta} \tag{12d}$$

3) Define the branch cuts

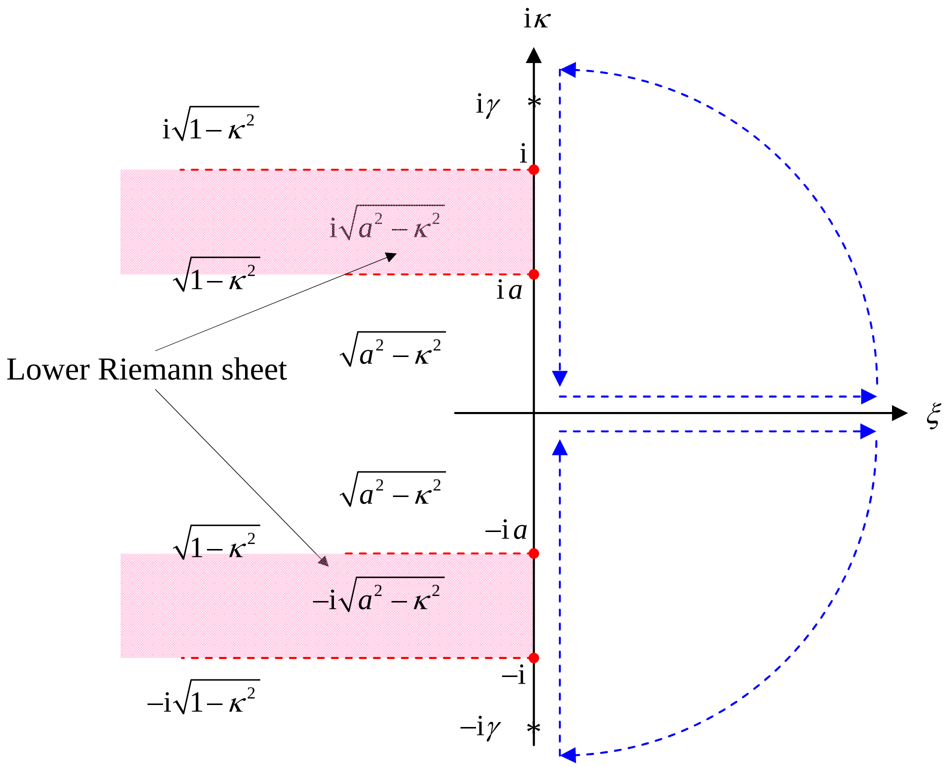

Figure 1: Branch points, branch cuts, poles, and values of α, β along imaginary axis, used for Green's functions $F(\mathrm{i}\kappa, \sigma)$ (after Pekeris)

For complex wavenumbers $\Xi = \xi + \mathrm{i}\kappa$, $F_n(\Xi)$ has branch points at $\pm\mathrm{i}a, \pm\mathrm{i}$ and poles at $\pm\mathrm{i}\gamma$, where $\gamma = C_S / C_R$ is the true root of the Rayleigh function as defined later on. Defining the branch cuts as in Pekeris, see Fig. 1, then for $\xi = \mathrm{Re}(\Xi) > 0$ the functions satisfy the complex-conjugate symmetry $F_n(\Xi^*) = F_n^*(\Xi)$; in particular, $F_n(-\mathrm{i}\kappa) = F_n^*(\mathrm{i}\kappa)$ and vice versa, $F_n(\mathrm{i}\kappa) = F_n^*(-\mathrm{i}\kappa)$. From Fig. 1 and for the branch cuts shown, we infer that the multi-valued functions $\alpha(\mathrm{i}\kappa), \beta(\mathrm{i}\kappa)$ attain the values given in Table 1.

Table 1

κ	α	β	$\alpha\beta$
$0 \le \kappa \le a$	$\alpha = \sqrt{a^2-\kappa^2}$	$\beta = \sqrt{1-\kappa^2}$	$\sqrt{a^2-\kappa^2}\sqrt{1-\kappa^2}$
$a \le \kappa \le 1$	$\alpha = \mathrm{i}\sqrt{\kappa^2-a^2}$	$\beta = \sqrt{1-\kappa^2}$	$\mathrm{i}\sqrt{\kappa^2-a^2}\sqrt{1-\kappa^2}$
$1 \le \kappa < \infty$	$\alpha = \mathrm{i}\sqrt{\kappa^2-a^2}$	$\beta = \mathrm{i}\sqrt{\kappa^2-1}$	$-\sqrt{\kappa^2-a^2}\sqrt{\kappa^2-1}$

4) Change the path of integration (Bateman-Pekeris Theorem)

Since $J_n(\xi\sigma) = \frac{1}{2}\left[H_n^{(1)}(\xi\sigma) + H_n^{(2)}(\xi\sigma)\right]$ then

$$U_n(r,d,t) = \mathcal{L}^{-1}\left\{\int_0^\infty F_n(\xi,\zeta,\sigma)\tfrac{1}{2}H_n^{(1)}(\xi\sigma)\xi\,d\xi + \int_0^\infty F_n(\xi,\zeta,\sigma)\tfrac{1}{2}H_n^{(2)}(\xi\sigma)\xi\,d\xi\right\} \tag{13}$$

The first Hankel function is bounded in the upper complex half-plane $\kappa > 0$, while the second Hankel function is bounded in the lower half-plane. In addition, for $\xi > 0$, there are no poles on the right complex half-plane and all functions $F_n(\xi,\zeta,\sigma)$ are regular there. Hence, following Pekeris, we proceed to carry out a contour integration in both the upper and lower quadrants, each of which yields zero, except that integrations along the imaginary axis pass through poles, and thus must be evaluated "principal value":

$$\int_0^\infty F_n(\xi,\zeta,\sigma)H_n^{(1)}(\xi\sigma)\xi\,d\xi + \int_\infty^0 F_n(\mathrm{i}\kappa,\zeta,\sigma)H_n^{(1)}(\mathrm{i}\kappa\sigma)(\mathrm{i}\kappa)\,d(\mathrm{i}\kappa) = 0 \qquad \text{Upper h.sp.}$$

and

$$\int_0^\infty F_n(\xi,\zeta,\sigma)H_n^{(2)}(\xi\sigma)\xi\,d\xi + \int_{-\infty}^0 F_n(\mathrm{i}\kappa,\zeta,\sigma)H_n^{(2)}(\mathrm{i}\kappa\sigma)(\mathrm{i}\kappa)\,d(\mathrm{i}\kappa) = 0 \qquad \text{Lower h.sp.}$$

so

$$\begin{aligned}
\int_0^\infty F_n(\xi,\zeta,\sigma)H_n^{(1)}(\xi\sigma)\xi\,d\xi &= -\int_0^\infty F_n(\mathrm{i}\kappa,\zeta,\sigma)H_n^{(1)}(\mathrm{i}\kappa\sigma)\kappa\,d\kappa \\
\int_0^\infty F_n(\xi,\zeta,\sigma)H_n^{(2)}(\xi\sigma)\xi\,d\xi &= -\int_0^{-\infty} F_n(\mathrm{i}\kappa,\zeta,\sigma)H_n^{(2)}(\mathrm{i}\kappa\sigma)\kappa\,d\kappa, \ \kappa \to -\kappa \\
&= -\int_0^\infty F_n(-\mathrm{i}\kappa,\zeta,\sigma)H_n^{(2)}(-\mathrm{i}\kappa\sigma)\kappa\,d\kappa
\end{aligned} \tag{14}$$

But

$$H_n^{(1)}(\mathrm{i}\kappa\sigma) = \mathrm{i}^{-(n+1)}\tfrac{2}{\pi}K_n(\kappa\sigma) \tag{15a}$$

$$H_n^{(2)}(-\mathrm{i}\kappa\sigma) = \mathrm{i}^{n+1}\tfrac{2}{\pi}K_n(\kappa\sigma) \tag{15b}$$

so

$$\int_0^\infty F_n(\xi,\zeta,\sigma)H_n^{(1)}(\xi\sigma)\xi\,d\xi = -\mathrm{i}^{-(n+1)}\tfrac{2}{\pi}\int_0^\infty F_n(\mathrm{i}\kappa,\zeta,\sigma)K_n(\kappa\sigma)\upsilon\,d\upsilon \tag{16a}$$

$$\int_0^\infty F_n(\xi,\zeta,\sigma)H_n^{(2)}(\xi\sigma)\xi\,d\xi = -\mathrm{i}^{(n+1)}\tfrac{2}{\pi}\int_0^\infty F_n(-\mathrm{i}\kappa,\zeta,\sigma)K_n(\kappa\sigma)\kappa\,d\kappa \tag{16b}$$

Hence

$$\begin{aligned}
\int_0^\infty F_n(\xi,\zeta,\sigma)J_n(\xi\sigma)\xi\,d\xi &= -\tfrac{1}{\pi}\int_0^\infty\left[\mathrm{i}^{-(n+1)}F_n(\mathrm{i}\kappa,\zeta,\sigma) + \mathrm{i}^{(n+1)}F_n(-\mathrm{i}\kappa,\zeta,\sigma)\right]K_n(\kappa\sigma)\kappa\,d\kappa \\
&= -\tfrac{1}{\pi}\tfrac{1}{\mathrm{i}^{(n+1)}}\int_0^\infty\left[F_n(\mathrm{i}\kappa,\zeta,\sigma) + (-1)^{(n+1)}F_n^*(\mathrm{i}\kappa,\zeta,\sigma)\right]K_n(\kappa\sigma)\kappa\,d\kappa \\
&= \frac{2}{\pi}\begin{cases} -(-1)^{\frac{1}{2}n}\,\mathrm{Im}\int_0^\infty\left[F_n(\mathrm{i}\kappa,\zeta,\sigma)\right]K_n(\kappa\sigma)\kappa\,d\kappa & n = 0,2,\cdots \\ -(-1)^{\frac{1}{2}(n+1)}\,\mathrm{Re}\int_0^\infty\left[F_n(\mathrm{i}\kappa,\zeta,\sigma)\right]K_n(\kappa\sigma)\kappa\,d\kappa & n = 1,3,\ldots \end{cases}
\end{aligned} \tag{17}$$

This equation constitutes a generalization of the Bateman-Pekeris theorem.

6) Add contribution of the Rayleigh pole

For purely imaginary wavenumbers $\xi \to \mathrm{i}\kappa$, the Rayleigh function (11a) appearing in the denominator of the integrands in eq. (17) is

$$
\begin{aligned}
R(\kappa^2) &= -\left[4\kappa^2\alpha\beta + (1-2\kappa^2)^2\right] \\
&= 4\kappa^2\sqrt{\kappa^2 - a^2}\,\sqrt{\kappa^2 - 1} - (1-2\kappa^2)^2
\end{aligned}
\tag{18}
$$

where in the second line we have assumed that both α, β in Table (1) lie on the same Riemann sheet. We also define the associated Rayleigh function $\tilde{R}(\kappa^2)$ as

$$
\begin{aligned}
\tilde{R}(\kappa^2) &= (1-2\kappa^2)^2 - 4\kappa^2\alpha\beta \\
&= (1-2\kappa^2)^2 + 4\kappa^2\sqrt{\kappa^2 - a^2}\,\sqrt{\kappa^2 - 1}
\end{aligned}
\tag{19}
$$

which can be interpreted as a Rayleigh function in which α, β lie on different Riemann sheets. The product of these two functions leads to a cubic equation in κ^2, namely

$$
\begin{aligned}
D = R\,\tilde{R} &= (4\kappa^2)^2(\kappa^2 - a^2)(\kappa^2 - 1) - (2\kappa^2 - 1)^4 \\
&= 16(1-a^2)\kappa^6 - 8(3-2a^2)\kappa^4 + 8\kappa^2 - 1 \\
&= 16(1-a^2)(\kappa^2 - \kappa_1^2)(\kappa^2 - \kappa_2^2)(\kappa^2 - \kappa_3^2), \qquad a = C_S / C_P
\end{aligned}
\tag{20}
$$

which constitutes the rationalized Rayleigh function $D(\kappa^2)$. It has three roots $\kappa_j^2,\ j = 1,2,3$, of which the first one $\kappa_1 = \gamma = C_S / C_R > 1$ is real and constitutes the "true" (and only) root of (18). The two "false" roots satisfy eq. 19, and can be either real or complex. The true root will lead to a pair of purely imaginary poles $\pm\mathrm{i}\gamma$ shown as stars in Fig. 1. Since the path of integration in eq. 17 passes straight though the two Rayleigh poles, this means that the integrals on the right hand side of eq. 17 must be understood in the principal value (PV) sense. Then again, as the path passes exactly "through" each of the Rayleigh poles, then each will contribute only half of its residue, or more precisely, either half of its imaginary part or half of its real part, as may be appropriate for a given n.

For Poisson's ratios $\nu < 0.2631$, the false roots will be real $\kappa_2 < \kappa_3 < a$ and they too will lead to pairs of poles lying on the imaginary axis, but inasmuch as these lie "below" and not on the path of integration, they do not affect the integral in (17). When $0.2631 < \nu < 0.317$, the false roots turn complex and appear in complex conjugate pairs yet they remain "invisible" by virtue of their location in the lower Riemann sheet. Nonetheless, when Poisson's ratio exceeds the threshold $\nu > 0.317$, one complex false root shows up from underneath each of the P-branch cuts and irrupts into the "open space" (shaded area) between the two branch cuts in the second and third quadrants, which exposes part of the lower Riemann sheet; hence, they constitute legitimate poles of the integrands. Still, since that region lies outside of the path of integration, then those poles do not affect the integrals in (17). Finally, complex roots have mirror (i.e. complex conjugate) poles lying below the right complex half-plane and once again they stay hidden in the lower sheet, so again they do not affect results. We conclude that none of the poles associated with the false roots contributes anything to the integrals and thus need not be considered. Hence, only the true Rayleigh poles must be accounted for.

Now, at the Rayleigh pole $\gamma > 1$, the Green's functions F_{xx}, F_{zz} given by 10a, 10d (or 12a, 12c) are purely imaginary, leading to residues which are purely real as a result of the πi factor in the contour integration around the pole. Hence, these poles do not contribute to either the $n = 0$ or $n = 2$ integrals, because their residues have no imaginary part. By contrast, the integral for F_{xz}, F_{zx} in 10b, 10c (or 12 b) which are associated with integrals $n = 1$ will produce non-vanishing residues. Still, since these are the only integrals affected, we can defer their evaluation until we consider the u_{rz} and u_{zx} displacement components.

7) Invert into the time domain

Abramowitz and Stegun 29.3.119 provide an explicit expression for the inverse Laplace transform of the modified Bessel function of order zero. Starting from that formula and making use of the recurrence relationships for these functions, we can readily infer also the Laplace transforms that yield the functions of order 1 and 2. These are:

$$K_0(\kappa\sigma) = \int_\kappa^\infty \frac{e^{-\sigma\tau} d\tau}{\sqrt{\tau - \kappa^2}}, \qquad \mathcal{L}^{-1}\left[K_0(\kappa\sigma)\right] = \frac{1}{\sqrt{\tau^2 - \kappa^2}} \mathcal{H}(\tau - \kappa) \tag{21a}$$

$$K_1(\kappa\sigma) = \int_\kappa^\infty \frac{\tau}{\kappa} \frac{e^{-\sigma\tau} d\tau}{\sqrt{\tau - \kappa^2}} \qquad \mathcal{L}^{-1}\left[K_1(\kappa\sigma)\right] = \frac{\tau / \kappa}{\sqrt{\tau^2 - \kappa^2}} \mathcal{H}(\tau - \kappa) \tag{21b}$$

$$K_2(\kappa\sigma) = \int_\kappa^\infty \left(2\frac{\tau^2}{\kappa^2} - 1\right) \frac{e^{-\sigma\tau} d\tau}{\sqrt{\tau - \kappa^2}} \qquad \mathcal{L}^{-1}\left[K_2(\kappa\sigma)\right] = \frac{2(\tau/\kappa)^2 - 1}{\sqrt{\tau^2 - \kappa^2}} \mathcal{H}(\tau - \kappa) \tag{21c}$$

8) Specialize to points on the surface

For points on the surface of the half-space $\zeta = 0$ for which the Green's functions are given by 12a-d, these functions no longer depend explicitly on the Laplace parameter, in which case $F_n(\mathrm{i}\kappa, 0, \sigma) = F_n(\mathrm{i}\kappa)$. Thus, these functions are "transparent" to the Laplace inversion, in which case

$$\mathcal{L}^{-1}\int_0^\infty F_n(\xi, 0, \sigma) J_n(\xi\sigma)\xi\, d\xi$$

$$= \frac{2}{\pi}\begin{cases} -(-1)^{\frac{1}{2}n} \int_0^\infty \mathrm{Im}\left[F_n(\mathrm{i}\kappa)\right]\mathcal{L}^{-1}\left[K_n(\kappa\sigma)\right]\kappa\, d\kappa & n = 0, 2, \cdots \\ -(-1)^{\frac{1}{2}(n+1)} \int_0^\infty \mathrm{Re}\left[F_n(\mathrm{i}\kappa)\right]\mathcal{L}^{-1}\left[K_n(\kappa\sigma)\right]\kappa\, d\kappa & n = 1, 3, \ldots \end{cases}$$

$$= \begin{cases} \frac{2}{\pi}\int_0^\tau \mathrm{Im}\left[-F_n(\mathrm{i}\kappa)\right]\frac{1}{\sqrt{\tau^2 - \kappa^2}}\kappa\, d\kappa, \quad n = 0 & \text{(22a)} \\ \frac{2}{\pi}\int_0^\tau \mathrm{Re}\left[F_n(\mathrm{i}\kappa)\right]\frac{\frac{\tau}{\kappa}}{\sqrt{\tau^2 - \kappa^2}}\kappa\, d\kappa, \quad n = 1 & \text{(22b)} \\ \frac{2}{\pi}\int_0^\tau \mathrm{Im}\left[-F_n(\mathrm{i}\kappa)\right]\frac{1 - 2\left(\frac{\tau}{\kappa}\right)^2}{\sqrt{\tau^2 - \kappa^2}}\kappa\, d\kappa, \quad n = 2 & \text{(22c)} \end{cases}$$

Observe that the negative sign of $\left[-F_n(\mathrm{i}\kappa)\right]$ in (22a,c) will cancel with the leading negative sign in the Green's functions in 12a,b,c. We have thus succeeded in casting the Green's function in the wavenumber-time domain.

9) Rationalization of the Green's functions

Before proceeding with the integration over wavenumbers κ, we eliminate the square root terms α,β in the denominator of the Green's functions by multiplication and division of the Green's function with the adjoined Rayleigh function. Multiplication of (12a,b,c) by (19), taking either the real or imaginary part as may be appropriate, and together with D given by (20) we obtain

$$\operatorname{Im}\left[-F_{xx}\right]=\frac{\operatorname{Im}\left[\beta\left(1-2\kappa^2\right)^2-4\kappa^2\alpha\beta^2\right]}{D}$$

$$=\frac{4}{D}\begin{cases}0 & 0<\kappa<a\\ \kappa^2\left(\kappa^2-1\right)\sqrt{\kappa^2-a^2} & a<\kappa<1\\ \sqrt{\kappa^2-1}\left(\kappa^2-\frac{1}{2}\right)^2+\kappa^2\left(\kappa^2-1\right)\sqrt{\kappa^2-a^2} & 1<\kappa\end{cases} \tag{23a}$$

$$\operatorname{Im}\left[-F_{zz}\right]=\frac{\operatorname{Im}\left[\alpha\left(1-2\kappa^2\right)^2-4\kappa^2\alpha^2\beta\right]}{D}$$

$$=\frac{4}{D}\begin{cases}0 & 0<\kappa<a\\ \sqrt{\kappa^2-a^2}\left(\kappa^2-\frac{1}{2}\right)^2 & a<\kappa<1\\ \sqrt{\kappa^2-a^2}\left(\kappa^2-\frac{1}{2}\right)^2+\kappa^2\left(\kappa^2-a^2\right)\sqrt{\kappa^2-1} & 1<\kappa\end{cases} \tag{23b}$$

$$\operatorname{Re}\left[F_{xz}\right]=-\frac{1}{D}\operatorname{Re}\left\{\mathrm{i}\kappa\left[\left(1-2\kappa^2\right)^3+8\alpha^2\beta^2-2\alpha\beta\left(1-2\kappa^2\right)\right]\right\}$$

$$=\frac{1}{D}\begin{cases}0 & 0<\kappa<a\\ -2\kappa\sqrt{\kappa^2-a^2}\sqrt{1-\kappa^2}\left(1-2\kappa^2\right) & a<\kappa<1\\ 0 & 1<\kappa\end{cases} \tag{23c}$$

$$\operatorname{Im}\left[-F_{yy}\right]=-\operatorname{Im}\frac{1}{\beta}$$

$$=\begin{cases}0 & 0<\kappa<1\\ \dfrac{1}{\sqrt{\kappa^2-1}} & 1<\kappa\end{cases} \tag{23d}$$

Observe from eq. 20 that $4/D$ in 23a,b contains a leading factor $\frac{1}{4}/\left(1-a^2\right)=\frac{1}{2}(1-\nu)$. We are now ready to proceed with the inversion into the spatial domain.

IV. Vertical displacement due to vertical load

The residue at the Rayleigh pole has no imaginary component so it contributes nothing to the integral. Thus, we can move on to the wavenumber integral, which is

$$\begin{aligned} U_{zz} &= \frac{2}{\pi}\int_0^\tau \frac{\mathrm{Im}\left[-F_{zz}\left(\mathrm{i}\kappa\right)\right]}{\sqrt{\tau^2-\kappa^2}}\kappa\, d\kappa \\ &= \frac{1}{4\left(1-a^2\right)}\left[f_1(\tau)+f_2(\tau)\right] \\ &= \tfrac{1}{2}(1-\nu)\left[f_1(\tau)+f_2(\tau)\right] \end{aligned} \tag{24}$$

where

$$f_1(\tau)=\frac{2}{\pi}\int_a^\tau \frac{\left(\kappa^2-\frac{1}{2}\right)^2\sqrt{\kappa^2-a^2}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\kappa\, d\kappa\,, \qquad a<\tau \tag{25a}$$

$$f_2(\tau)=\frac{2}{\pi}\int_1^\tau \frac{\kappa^2\left(\kappa^2-a^2\right)\sqrt{\kappa^2-1}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\kappa\, d\kappa\,, \qquad 1<\tau \tag{25b}$$

These two integrals together with their sum are evaluated in detail in Appendix IV. Finally, with $2\pi\mu r\, u_{zz}=U_{zz}$, we obtain for an upward vertical load of magnitude $V=1$ the upward displacement

$$u_{zz}(r,\tau)=\frac{1-\nu}{2\pi\mu r}\begin{cases}\dfrac{1}{2}\left(1-\displaystyle\sum_{j=1}^{3}\frac{A_j}{\sqrt{\left|\tau^2-\kappa_j^2\right|}}\right) & a<\tau<1\\[2ex] 1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma\\[2ex] 1 & \gamma<\tau\end{cases} \tag{26a}$$

$$A_j=\frac{\left(\kappa_j^2-\frac{1}{2}\right)^2\sqrt{\left|a^2-\kappa_j^2\right|}}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)}, \qquad i\neq j\neq k \tag{26b}$$

which agrees perfectly with the known solution in Eringen & Suhubi. For one real root and a pair of complex conjugate roots, the response function changes into

$$u_{zz}=\frac{1-\nu}{2\pi\mu r}\begin{cases}\dfrac{1}{2}\left(1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}}-2\,\mathrm{Re}\dfrac{\tilde{A}_2}{\sqrt{\tau^2-\kappa_2^2}}\right) & a<\tau<1\\[2ex] 1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma\\[2ex] 1 & \tau>\gamma\end{cases} \tag{27a}$$

$$\tilde{A}_2 = \frac{\left(\kappa_2^2 - \frac{1}{2}\right)^2 \sqrt{a^2 - \kappa_2^2}}{\left(\kappa_2^2 - \gamma^2\right)\left(\kappa_2^2 - \kappa_3^2\right)} \tag{27b}$$

which agrees fully with Mooney's solution. Observe that the square root term in $\tilde{A}_2$ does not carry an absolute sign.

V. Radial displacement due to vertical load

Pekeris solved this case only for Poisson's ratio $\nu = 0.25$, while Mooney did not consider it at all. Eringen and Suhubi, on the other hand, include expressions solely for real roots, i.e. for Poisson's ratios up to $\nu = 0.2631$. This section presents the full solution for any Poisson's ratio.

We consider first the residue at the Rayleigh pole. Evaluating (12b) for $\xi = \mathrm{i}\kappa$ and multiplying both the numerator and denominator by (19), we obtain after brief algebra

$$\begin{aligned} F_{xz}\left(\mathrm{i}\kappa\right) &= -\tfrac{1}{D}\mathrm{i}\kappa\left[4\left(1-2a^2\right)\kappa^4 - 2\left(3-4a^2\right)\kappa^2 + 1 - 2\left(1-2\kappa^2\right)\alpha\beta\right] \\ &= -\tfrac{1}{D}\mathrm{i}\kappa N\left(\kappa\right) \end{aligned} \tag{28}$$

where $N(\kappa)$ is the term in square brackets. At the Rayleigh pole and with α,β chosen from Table 1 with $\kappa = \gamma > 1$, $N(\gamma)$ evaluates to

$$N\left(\gamma\right) = 4\left(1-2a^2\right)\gamma^4 - 2\left(3-4a^2\right)\gamma^2 + 1 + 2\left(1-2\gamma^2\right)\sqrt{\gamma^2 - a^2}\sqrt{\gamma^2 - 1} \tag{29}$$

Since the integral in eq. 17 is in the positive κ direction, then the half of the pole which contributes to that integral lies immediately to the right of the path, which is thus clockwise and so it carries a leading negative sign. Hence, the residue at the Rayleigh pole is

$$\begin{aligned} P &= -\tfrac{2}{\pi}\,\mathrm{Re}\left\{\tfrac{1}{2}2\pi\mathrm{i}\lim_{\kappa\to\gamma}\left[\left(\kappa-\gamma\right)F\left(\mathrm{i}\kappa\right)\right]\kappa\,\mathcal{L}^{-1}\left[K_1\left(\kappa\sigma\right)\right]\right\} \\ &= -\gamma N\left(\gamma\right)\lim_{\kappa\to\gamma}\left[\frac{\kappa-\gamma}{D}\right]\frac{\tau}{\sqrt{\tau^2-\gamma^2}}\mathcal{H}\left(\tau-\gamma\right) \end{aligned} \tag{30}$$

But

$$\begin{aligned} \lim_{\kappa\to\gamma} D/\left(\kappa-\gamma\right) &= \lim_{\kappa\to\gamma}\left\{16\left(1-a^2\right)\left(\kappa+\gamma\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\right\} \\ &= 32\gamma\left(1-a^2\right)\left(\gamma^2-\kappa_2^2\right)\left(\gamma^2-\kappa_3^2\right) \end{aligned} \tag{31}$$

Now, the Rayleigh pole satisfies the characteristic equation, so

$$\sqrt{\gamma^2-a^2}\sqrt{\gamma^2-1}=\frac{\left(2\gamma^2-1\right)^2}{4\gamma^2} \tag{32}$$

and

$$\frac{\gamma N(\gamma)}{32\gamma\left(1-a^2\right)\left(\gamma^2-\kappa_2^2\right)\left(\gamma^2-\kappa_3^2\right)}=\frac{8\left(1-2a^2\right)\gamma^6-4\left(3-4a^2\right)\gamma^4+2\gamma^2-\left(2\gamma^2-1\right)^3}{32\left(1-a^2\right)\left(\gamma^2-\kappa_2^2\right)\left(\gamma^2-\kappa_3^2\right)\gamma^2} \tag{33}$$

Subtracting $D(\gamma)=16\gamma^6-24\gamma^4+8\gamma^2-1-16a^2\gamma^6+16a^2\gamma^4=0$ from the numerator, we obtain after simplifications

$$P=\frac{\left(2\gamma^2-1\right)^3}{16\left(1-a^2\right)\left(\gamma^2-\kappa_2^2\right)\left(\gamma^2-\kappa_3^2\right)\gamma^2}\frac{\tau}{\sqrt{\tau^2-\gamma^2}}\mathcal{H}(\tau-\gamma) \tag{34}$$

Also, from the expansion of $D(\kappa)$ in terms of the roots, we know that

$$\begin{aligned}&2\left(1-a^2\right)\left(\gamma^2+\kappa_2^2+\kappa_3^2\right)=\left(3-2a^2\right)\\&2\left(1-a^2\right)\left[\kappa_2^2\kappa_3^2+\gamma^2\left(\kappa_2^2+\kappa_3^2\right)\right]=1\\&16\left(1-a^2\right)\gamma^2\kappa_2^2\kappa_3^2=1\end{aligned}\quad,\qquad \text{so}\qquad \begin{aligned}&\kappa_2^2\kappa_3^2=\frac{1}{16\left(1-a^2\right)\gamma^2}\\&\kappa_2^2+\kappa_3^2=\frac{8\gamma^2-1}{16\left(1-a^2\right)\gamma^4}\end{aligned} \tag{35}$$

from which we can infer after brief algebra

$$\left(\gamma^2-\kappa_2^2\right)\left(\gamma^2-\kappa_3^2\right)=\gamma^4-\left(\kappa_2^2+\kappa_3^2\right)\gamma^2+\kappa_2^2\kappa_3^2=\frac{8\left(1-a^2\right)\gamma^6-4\gamma^2+1}{8\left(1-a^2\right)\gamma^2} \tag{36}$$

Hence, the contribution of the pole is

$$\begin{aligned}P&=\frac{\frac{1}{2}\left(2\gamma^2-1\right)^3}{\left(1-a^2\right)8\gamma^6-4\gamma^2+1}\frac{\tau}{\sqrt{\tau^2-\gamma^2}}\mathcal{H}(\tau-\gamma)\\&=\tfrac{1}{2}C\frac{\tau}{\sqrt{\tau^2-\gamma^2}}\mathcal{H}(\tau-\gamma)\end{aligned} \tag{37}$$

$$C=\frac{\left(2\gamma^2-1\right)^3}{8\left(1-a^2\right)\gamma^6-4\gamma^2+1} \tag{38}$$

For $\nu=0.25$, $a^2=\frac{1}{3}$ $\gamma^2=\frac{1}{4}\left(3+\sqrt{3}\right)$, in which case $\frac{1}{2}C=\frac{1}{4}$, which agrees with Pekeris. We now turn to the wavenumber integrals

$$U_{rz}=\frac{2}{\pi}\int_0^\tau \operatorname{Im}\left[G_{zx}(\kappa)\right]\frac{\frac{\tau}{\kappa}}{\sqrt{\tau-\kappa^2}}\kappa\,d\kappa=\begin{cases}f_1(\tau) & a<\tau<1\\ f_2(\tau) & 1<\tau\end{cases} \tag{39}$$

where for convenience we write $2\left(1-2\kappa^2\right)=(-4)\left(\kappa^2-\frac{1}{2}\right)$, then

$$f_1(\tau)=\tfrac{2\tau}{\pi}(-4)\int_a^\tau \frac{\left(\kappa^2-\frac{1}{2}\right)\sqrt{\kappa^2-a^2}\,\sqrt{1-\kappa^2}}{D\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa \tag{40a}$$

$$f_2(\tau)=\tfrac{2\tau}{\pi}(-4)\int_a^1 \frac{\left(\kappa^2-\frac{1}{2}\right)\sqrt{\kappa^2-a^2}\,\sqrt{1-\kappa^2}}{D\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa \tag{40b}$$

Observe that the integrals are identical, and differ only on the limits. From Appendix V, setting $P\left(\kappa^2\right)=\left(\kappa^2-\frac{1}{2}\right)\left(\kappa^2-a^2\right)$ and accounting for the factor $16\left(1-a^2\right)$ in D, we obtain the following results:

a) all roots are real

$$f_1(\tau)=\frac{\tau}{4\pi\left(1-a^2\right)}\left\{2K(n)-\sum_{k=1}^{3}B_k\,\Pi\left(n^2m_k,n\right)\right\} \tag{41a}$$

$$f_2(\tau)=\frac{\tau\,n^{-1}}{4\pi\left(1-a^2\right)}\left\{2K\left(n^{-1}\right)-\sum_{k=1}^{3}B_k\,\Pi\left(m_k,n^{-1}\right)\right\} \tag{41b}$$

Finally, accounting for the leading divisor $2\pi\mu r$, we obtain the radial displacement due to a vertical load $V=1$ as

$$\boxed{u_{rz}(r,\tau)=\frac{\tau}{8\pi\mu r}\begin{cases}\dfrac{1}{\pi\left(1-a^2\right)^{3/2}}\left\{2K(n)-\sum\limits_{k=1}^{3}B_k\,\Pi\left(n^2m_k,n\right)\right\}, & a<\tau<1\\[2ex] \dfrac{n^{-1}}{\pi\left(1-a^2\right)^{3/2}}\left\{2K\left(n^{-1}\right)-\sum\limits_{k=1}^{3}B_k\,\Pi\left(m_k,n^{-1}\right)\right\}, & 1<\tau<\gamma\\[2ex] \dfrac{n^{-1}}{\pi\left(1-a^2\right)^{3/2}}\left\{2K\left(n^{-1}\right)-\sum\limits_{k=1}^{3}B_k\,\Pi\left(m_k,n^{-1}\right)\right\}+\dfrac{2C}{\sqrt{\tau^2-\gamma^2}}, & \gamma<\tau\end{cases}} \tag{42a}$$

$$\boxed{B_k=\frac{\left(1-2\kappa_k^2\right)\left(1-\kappa_k^2\right)}{\left(\kappa_i^2-\kappa_k^2\right)\left(\kappa_j^2-\kappa_k^2\right)}},\quad \boxed{n^2=\frac{\tau^2-a^2}{1-a^2}},\quad \boxed{m_k=\frac{1-a^2}{a^2-\kappa_k^2}},\quad \boxed{C=\frac{\left(2\gamma^2-1\right)^3}{8\left(1-a^2\right)\gamma^6-4\gamma^2+1}} \tag{42b}$$

The above result for real roots agrees with the solutions given by Eringen and Suhubi.

b) On root is real, the other two are complex conjugates

In the case of complex roots, which develop when $\nu>0.2631$, the coefficients and indices are complex, in which case we must replace the two summations in (42a) by

$$\sum_{k=1}^{3} B_k\,\Pi\left(n^2 m_k, n\right) \quad\rightarrow\quad B_1\,\Pi\left(n^2 m_1, n\right) + 2\,\mathrm{Re}\left[B_2\,\Pi\left(n^2 m_2, n\right)\right] \tag{43a}$$

$$\sum_{k=1}^{3} B_k\,\Pi\left(m_k, n^{-1}\right) \quad\rightarrow\quad B_1\,\Pi\left(m_1, n^{-1}\right) + 2\,\mathrm{Re}\left[B_2\,\Pi\left(m_2, n^{-1}\right)\right] \tag{43b}$$

Although the characteristic of the elliptic function of the third kind is now complex —and thus so is also the function— it is not difficult to write a numerical routine for such case.

VI. Radial and tangential displacements due to horizontal load

This case was considered first by Chao, but only for a Poisson's ratio $\nu = 0.25$. We begin by defining the auxiliary function

$$h_n(\tau,\kappa) = \begin{cases} 1 & n = 0 \\ 1 - 2\left(\frac{\tau}{\kappa}\right)^2 & n = 2 \end{cases} \tag{44}$$

Now, from (22), (23d) we have for the yy term

$$U_{ny}(\tau) = \mathcal{L}^{-1}\int_0^\infty F_{yy}(\xi,0,\sigma)\,J_n(\xi\sigma)\,\xi\,d\xi = \tfrac{2}{\pi}\int_0^\tau \frac{\mathrm{Im}\left[-F_{yy}(\mathrm{i}\kappa)\right]}{\sqrt{\tau^2-\kappa^2}}\,h_n(\tau,\kappa)\,\kappa\,d\kappa \tag{45}$$

which from Appendix VI works out to

$$U_{0y}(\tau) = \tfrac{2}{\pi}\int_1^\tau \frac{1}{\sqrt{\kappa^2-1}\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa = \mathcal{H}(\tau-1) \tag{46a}$$

$$U_{2y}(\tau) = \tfrac{2}{\pi}\int_1^\tau \frac{1-2\left(\frac{\tau}{\kappa}\right)^2}{\sqrt{\kappa^2-1}\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa = (1-2\tau)\,\mathcal{H}(\tau-1) \tag{46b}$$

so

$$\tfrac{1}{2}\left(U_{0y} + U_{2y}\right) = (1-\tau)\,\mathcal{H}(\tau-1) \tag{47a}$$

$$\tfrac{1}{2}\left(U_{0y} - U_{2y}\right) = \tau\,\mathcal{H}(\tau-1) \tag{47b}$$

Also, from (22) and (23a), the relevant integrals for the xx component are

$$U_{nx}(\tau) = \mathcal{L}^{-1}\int_0^\infty F_{xx}(\xi,0,\sigma)\,J_n(\xi\sigma)\,\xi\,d\xi = \tfrac{2}{\pi}\int_0^\tau \frac{\mathrm{Im}\left[-F_{xx}(\mathrm{i}\kappa)\right]}{\sqrt{\tau^2-\kappa^2}}\,h_n(\tau,\kappa)\,\kappa\,d\kappa$$

$$= \frac{1}{4\left(1-a^2\right)}\begin{cases} \dfrac{2}{\pi}\displaystyle\int_a^\tau \frac{\kappa^2\left(\kappa^2-1\right)\sqrt{\kappa^2-a^2}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,h_n(\tau,\kappa)\,\kappa\,d\kappa & a<\tau<1 \\ \dfrac{2}{\pi}\displaystyle\int_a^\tau \frac{\kappa^2\left(\kappa^2-1\right)\sqrt{\kappa^2-a^2}+\left(\kappa^2-\frac{1}{2}\right)^2\sqrt{\kappa^2-1}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,h_n(\tau,\kappa)\,\kappa\,d\kappa & 1<\kappa \end{cases} \tag{48}$$

which can be written as

$$U_{0x} = \frac{1}{4\left(1-a^2\right)}\left[f_1(\tau) + f_2(\tau)\right] \qquad (49a)$$
$$= \tfrac{1}{2}(1-\nu)\left[f_1(\tau) + f_2(\tau)\right]$$

$$U_{2x} = U_{0x} - \frac{\tau^2}{2\left(1-a^2\right)}\left[f_3(\tau) + f_4(\tau)\right] \qquad (49b)$$
$$= U_{0x} - \tau^2(1-\nu)\left[f_3(\tau) + f_4(\tau)\right]$$

i.e.

$$\tfrac{1}{2}\left(U_{0x} - U_{2x}\right) = \tfrac{1}{2}\tau^2(1-\nu)\left[f_3(\tau) + f_4(\tau)\right] \qquad (50a)$$

$$\tfrac{1}{2}\left(U_{0x} + U_{2x}\right) = \tfrac{1}{2}(1-\nu)\left[f_1(\tau) + f_2(\tau) - \tau^2\left[f_3(\tau) + f_4(\tau)\right]\right] \qquad (50b)$$

where

$$f_1(\tau) = \frac{2}{\pi}\int_a^\tau \frac{\kappa^2\left(\kappa^2-1\right)\sqrt{\kappa^2-a^2}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa\, d\kappa \qquad (51a)$$

$$f_2(\tau) = \frac{2}{\pi}\int_1^\tau \frac{\left(\kappa^2-\tfrac{1}{2}\right)^2\sqrt{\kappa^2-1}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa\, d\kappa \qquad (51b)$$

$$f_3(\tau) = \frac{2}{\pi}\int_a^\tau \frac{\left(\kappa^2-1\right)\sqrt{\kappa^2-a^2}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa\, d\kappa \qquad (51c)$$

$$f_4(\tau) = \frac{2}{\pi}\int_1^\tau \frac{\left(\kappa^2-\tfrac{1}{2}\right)^2\sqrt{\kappa^2-1}}{\kappa^2\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa\, d\kappa \qquad (51d)$$

These integrals are evaluated in full in the Appendix VII, which also lists the sums $f_1 + f_2$ and $f_3 + f_4$ needed to form (50a,b). The result is

a) Real roots:

$$U_{0x} = \tfrac{1}{2}(1-\nu)\left[f_1(\tau) + f_2(\tau)\right]$$
$$= (1-\nu)\begin{cases} \frac{1}{2}\left[1 - \frac{A_1}{\sqrt{\gamma^2-\tau^2}} - \frac{A_2}{\sqrt{\tau^2-\kappa_2^2}} - \frac{A_3}{\sqrt{\tau^2-\kappa_3^2}}\right] & a < \tau < 1 \\ 1 - \frac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1 < \tau < \gamma \\ 1 & \tau > \gamma \end{cases} \qquad (52a)$$

$$A_j = \frac{\kappa_j^2\left(\kappa_j^2 - 1\right)\sqrt{\left|a^2 - \kappa_j^2\right|}}{\left(\kappa_j^2 - \kappa_i^2\right)\left(\kappa_j^2 - \kappa_k^2\right)}, \qquad i \neq j \neq k \tag{52b}$$

b) Complex roots:

$$U_{0x} = \tfrac{1}{2}(1-\nu)\left[f_1(\tau) + f_2(\tau)\right]$$

$$= (1-\nu)\begin{cases} \dfrac{1}{2}\left[1 - \dfrac{A_1}{\sqrt{\gamma^2 - \tau^2}} - 2\,\mathrm{Re}\left[\dfrac{\tilde{A}_2}{\sqrt{\tau^2 - \kappa_2^2}}\right]\right] & a < \tau < 1 \\ 1 - \dfrac{A_1}{\sqrt{\gamma^2 - \tau^2}} & 1 < \tau < \gamma \\ 1 & \tau > \gamma \end{cases} \tag{53a}$$

$$\tilde{A}_2 = \frac{\kappa_2^2\left(\kappa_2^2 - 1\right)\sqrt{a^2 - \kappa_2^2}}{\left(\kappa_2^2 - \kappa_1^2\right)\left(\kappa_2^2 - \kappa_3^2\right)} \tag{53b}$$

with Q_1 being given again by (27b). Also

a) Real roots:

$$\tfrac{1}{2}\left(U_{0x} - U_{2x}\right) = \tfrac{1}{2}\tau^2(1-\nu)\left[f_3(\tau) + f_4(\tau)\right]$$

$$= \tau\mathcal{H}(\tau - 1) + \tau^2(1-\nu)\begin{cases} \dfrac{1}{2}\left[\dfrac{C_1}{\sqrt{\gamma^2 - \tau^2}} + \dfrac{C_2}{\sqrt{\tau^2 - \kappa_2^2}} + \dfrac{C_3}{\sqrt{\tau^2 - \kappa_3^2}}\right] & a < \tau < 1 \\ \dfrac{C_1}{\sqrt{\gamma^2 - \tau^2}} & 1 < \tau < \gamma \\ 0 & \tau > \gamma \end{cases} \tag{54}$$

b) Complex roots

$$\tfrac{1}{2}\left(U_{0x} - U_{2x}\right) = \tfrac{1}{2}\tau^2(1-\nu)\left[f_3(\tau) + f_4(\tau)\right]$$

$$= \tau\mathcal{H}(\tau - 1) + \tau^2(1-\nu)\begin{cases} \dfrac{1}{2}\left[\dfrac{C_1}{\sqrt{\gamma^2 - \tau^2}} + 2\mathrm{Re}\dfrac{\tilde{C}_2}{\sqrt{\tau^2 - \kappa_2^2}}\right] & a < \tau < 1 \\ \dfrac{C_1}{\sqrt{\gamma^2 - \tau^2}} & 1 < \tau < \gamma \\ 0 & \tau > \gamma \end{cases} \tag{55}$$

where the C_j are as listed below, and Q_1 is given by (27b). Also, $\tilde{C}_2$ is like C_2, but without the absolute sign in the square root.. Combining the preceding equations with (47a) to form the expression $U_{rx} = \frac{1}{2}(U_{0x} - U_{2x}) + \frac{1}{2}(U_{0y} + U_{2y})$ and dividing that by $2\pi\mu r$, we obtain the radial displacement due to a horizontal force H

$$\boxed{u_{rx} = \frac{1}{2\pi\mu r}\begin{cases} \frac{1}{2}(1-\nu)\tau^2 \sum_{j=1}^{3} \frac{C_j}{\sqrt{|\tau^2 - \kappa_j^2|}} & a < \tau < 1 \\ 1 + (1-\nu)\tau^2 \frac{C_1}{\sqrt{\gamma^2 - \tau^2}} & 1 < \tau < \gamma \\ 1 & \tau > \gamma \end{cases}} \qquad \boxed{C_j = \frac{(1-\kappa_j^2)\sqrt{|a^2 - \kappa_j^2|}}{(\kappa_j^2 - \kappa_i^2)(\kappa_j^2 - \kappa_k^2)}}, \; i \neq j \neq k \tag{56}$$

When this equation is specialized to $\nu = 0.25$, we recover Chao’s solution. Also, the long term behavior $\tau > \gamma$, which is independent of Poisson’s ratio, agrees perfectly with the classical, static Cerruti solution.

In the case of complex roots, the summation on the first line of equation (56) for $a < \tau < 1$ must be replaced by

$$\tfrac{1}{2}(1-\nu)\tau^2 \left\{ \frac{C_1}{\sqrt{\gamma^2 - \tau^2}} + 2\,\mathrm{Re}\frac{\tilde{C}_2}{\sqrt{\tau^2 - \kappa_2^2}} \right\} \tag{57}$$

On the other hand, we can also form the expression

$$\begin{aligned} U_{\theta x} &= \tfrac{1}{2}(U_{0x} + U_{2x}) + \tfrac{1}{2}(U_{0y} - U_{2y}) \\ &= \left\{ \tfrac{1}{2}(1-\nu)\left[f_1(\tau) + f_2(\tau) \right] - \tau^2 \tfrac{1}{2}(1-\nu)\left[f_3(\tau) + f_4(\tau) \right] \right\} + \tau \mathcal{H}(\tau - 1) \end{aligned} \tag{58}$$

which after division by $2\pi\mu r$ leads us to the tangential displacement due to a horizontal load as

$$u_{\theta x} = \frac{1-\nu}{2\pi\mu r}\begin{cases} \frac{1}{2}\left[1 - \sum_{j=1}^{3} \frac{C_j \tau^2 + A_j}{\sqrt{|\tau^2 - \kappa_2^2|}} \right] & a < \tau < 1 \\ 1 - \frac{C_1 \tau^2 + A_1}{\sqrt{\gamma^2 - \tau^2}} & 1 < \tau < \gamma \\ 1 & \tau > \gamma \end{cases} \tag{59}$$

but

$$\frac{\tau^2 C_j + A_j}{\sqrt{|\tau^2-\kappa_j^2|}} = \frac{(\tau^2-\kappa_j^2)(1-\kappa_j^2)\sqrt{|a^2-\kappa_j^2|}}{\sqrt{|\tau^2-\kappa_j^2|}(\kappa_j^2-\kappa_i^2)(\kappa_j^2-\kappa_k^2)}$$

$$= \sqrt{|\tau^2-\kappa_j^2|}\frac{(1-\kappa_j^2)\sqrt{|a^2-\kappa_j^2|}}{(\kappa_j^2-\kappa_i^2)(\kappa_j^2-\kappa_k^2)}\operatorname{sgn}(\tau^2-\kappa_j^2) \tag{60}$$

$$= \sqrt{|\tau^2-\kappa_j^2|}\, C_j \operatorname{sgn}(\tau^2-\kappa_j^2)$$

so

$$\sum_{j=1}^{3}\frac{C_j\tau^2 + A_j}{\sqrt{|\tau^2-\kappa_2^2|}} = \sum_{j=1}^{3}\sqrt{|\tau^2-\kappa_j^2|}\, C_j \operatorname{sgn}(\tau^2-\kappa_j^2) \tag{61}$$

$$= -C_1\sqrt{\gamma^2-\tau^2} + C_2\sqrt{\tau^2-\kappa_2^2} + C_3\sqrt{\tau^2-\kappa_3^2}$$

Hence,

$$\boxed{u_{\theta x} = \frac{1-\nu}{2\pi\mu r}\begin{cases} \frac{1}{2}\left[1 + C_1\sqrt{\gamma^2-\tau^2} - C_2\sqrt{\tau^2-\kappa_2^2} - C_3\sqrt{\tau^2-\kappa_3^2}\right] & a<\tau<1 \\ 1 + C_1\sqrt{\gamma^2-\tau^2} & 1<\tau<\gamma \\ 1 & \tau>\gamma \end{cases}} \tag{62a}$$

This equation is in full agreement with Chao's solution when $\nu = 0.25$. In addition, the static behavior for $\tau \gg \gamma$ agrees with the classical Cerruti solution.

In the case of complex roots, the first line of equation (62) for $a<\tau<1$ must be replaced by

$$\tfrac{1}{2}\left(1 + C_1\sqrt{\gamma^2-\tau^2} - 2\operatorname{Re}\left[\tilde{C}_2\sqrt{\tau^2-\kappa_2^2}\right]\right) \tag{62b}$$

Displacements along the epicentral axis

In sharp contrast to the displacements on the surface, the displacements on the axis below the load can be evaluated readily and without much complication, as will be seen.

Horizontal load

Along the vertical axis, the radial and tangential displacements at depth $d = |z|$ due to a horizontal load at the surface are equal. Hence

$$u_x = \tfrac{1}{2}(u_{rx} + u_{\theta x})$$

$$= \tfrac{1}{2}\left\{\frac{1}{2\pi\mu}\left[\tfrac{1}{2}(u_{0x}-u_{2x}) + \tfrac{1}{2}(u_{0y}+u_{2y})\right] + \frac{1}{2\pi\mu}\left[\tfrac{1}{2}(u_{0x}+u_{2x}) + \tfrac{1}{2}(u_{0y}-u_{2y})\right]\right\} \tag{63}$$

$$= \frac{1}{4\pi\mu}(u_{0x}+u_{0y})$$

All integrals are now of order $n = 0$, for which $\lim_{r\to 0} J_0(kr) = 1$. Hence, the integral transforms no longer contain Bessel functions. In addition, to avoid confusion with derivatives, we denote in the ensuing the depth as $d \to |z|$. Also, we define

$$\tau = \frac{C_S t}{|z|}, \qquad \sigma = \frac{s|z|}{C_S}, \qquad \mathcal{L}^{-1}\left[F(\sigma)\right] = \frac{1}{2\pi \mathrm{i}}\int_{\bar{c}-\mathrm{i}\infty}^{\bar{c}+\mathrm{i}\infty} \exp(\sigma\tau) F(\sigma) d\sigma \tag{64}$$

which satisfies the inversion $\mathcal{L}^{-1}\left[\exp(-\sigma\tau_0)\right] = \delta(\tau - \tau_0)$.

a) Inversion of term in f_{xx}:

$$\begin{aligned}
u_{0x} &= \frac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{s}\exp(st)\int_0^\infty \frac{\sqrt{k^2+h_S^2}\left[2k^2(e_P - e_S) - h_S^2 e_S\right]}{4k^2\sqrt{k^2+h_P^2}\sqrt{k^2+h_S^2} - \left(h_S^2+2k^2\right)^2} k\,dk\,ds \\
&= \frac{1}{|z|}\mathcal{L}^{-1}\left\{\int_0^\infty \frac{2\xi^3\sqrt{\xi^2+1}\exp\left(-\sigma\sqrt{\xi^2+a^2}\right)}{4\xi^2\sqrt{\xi^2+a^2}\sqrt{\xi^2+1} - \left(1+2\xi^2\right)^2} d\xi - \int_0^\infty \frac{\xi\left(2\xi^2+1\right)\sqrt{\xi^2+1}\exp\left(-\sigma\sqrt{\xi^2+1}\right)}{4\xi^2\sqrt{\xi^2+a^2}\sqrt{\xi^2+1} - \left(1+2\xi^2\right)^2} d\xi\right\} \\
&= \frac{1}{|z|}\left\{\int_0^\infty \frac{\left(2\xi^2+1\right)\sqrt{\xi^2+1}\,\delta\left(\tau - \sqrt{\xi^2+1}\right)}{\left(1+2\xi^2\right)^2 - 4\xi^2\sqrt{\xi^2+a^2}\sqrt{\xi^2+1}}\xi d\xi - \int_0^\infty \frac{2\xi^2\sqrt{\xi^2+1}\,\delta\left(\tau - \sqrt{\xi^2+a^2}\right)}{\left(1+2\xi^2\right)^2 - 4\xi^2\sqrt{\xi^2+a^2}\sqrt{\xi^2+1}}\xi d\xi\right\} \\
&= \frac{1}{|z|}\left(I_S - I_P\right)
\end{aligned} \tag{65}$$

For the first integral we choose the substitution $y = \sqrt{\xi^2 + a^2}, y dy = \xi d\xi$, which leads us to

$$\begin{aligned}
I_P &= \int_a^\infty \frac{2y\left(y^2-a^2\right)\sqrt{y^2-a^2+1}\,\delta(\tau - y)}{\left(1+2y^2-2a^2\right)^2 - 4\left(y^2-a^2\right)y\sqrt{y^2-a^2+1}} dy \\
&= \frac{2\tau\left(\tau^2-a^2\right)\sqrt{\tau^2-a^2+1}}{4\tau\left(\tau^2-a^2\right)\sqrt{\tau^2-a^2+1} - \left(1+2\tau^2-2a^2\right)^2}\mathcal{H}(\tau - a)
\end{aligned} \tag{66}$$

Similarly, for the second integral we take $y = \sqrt{\xi^2+1}, y dy = \xi d\xi$, in which case

$$\begin{aligned}
I_S &= \int_1^\infty \frac{\left(2y^2-1\right)y^2\,\delta(\tau - y)}{\left(2y^2-1\right)^2 - 4\left(y^2-1\right)y\sqrt{y^2-(1-a^2)}} dy \\
&= \frac{\left(2\tau^2-1\right)\tau^2}{\left(2\tau^2-1\right)^2 - 4\tau\left(\tau^2-1\right)\sqrt{\tau^2-(1-a^2)}}\mathcal{H}(\tau - 1)
\end{aligned} \tag{67}$$

b) Inversion of term in f_{yy}

$$u_{0y} = \frac{1}{2\pi \mathrm{i}} \int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty} \frac{1}{s} \exp(st) \int_0^\infty \frac{\exp\left(-z\sqrt{k^2+h_S^2}\right)}{\sqrt{k^2+h_S^2}} k\,dk\,ds$$
$$= \mathcal{L}^{-1} \int_0^\infty \frac{\exp\left(-\sigma\sqrt{\xi^2+1}\right)}{\sqrt{\xi^2+1}} \xi\,d\xi = \int_0^\infty \frac{\delta\left(\tau - \sqrt{\xi^2+1}\right)}{\sqrt{\xi^2+1}} \xi\,d\xi = \int_1^\infty \frac{\delta(\tau-y)}{y} y\,dy \qquad (68)$$
$$= \int_1^\infty \delta(\tau-y)\,dy = \mathcal{H}(\tau-1)$$

c) Horizontal displacement of axis

Combination of the preceding results yields

$$u_x = \frac{1}{4\pi\mu|z|}\left[f_S(\tau)\mathcal{H}(\tau-1) - f_P(\tau)\mathcal{H}(\tau-a)\right] \qquad (69)$$

with

$$f_P(\tau) = \frac{2\tau\left(\tau^2-a^2\right)S_1}{\left(2\tau^2-2a^2+1\right)^2 - 4\tau\left(\tau^2-a^2\right)S_1} \qquad (70a)$$

$$f_S(\tau) = \frac{\left(2\tau^2-1\right)\tau^2}{\left(2\tau^2-1\right)^2 - 4\tau\left(\tau^2-1\right)S_2} + 1 \qquad (70b)$$

where

$$S_1 = \sqrt{\tau^2+1-a^2}, \qquad S_2 = \sqrt{\tau^2-\left(1-a^2\right)} \qquad (71)$$

Although simple in appearance, at large times $\tau \gg 1$ the above representation suffers from severe cancellations. This is because the sum of the two functions tends to a constant (static) value, but individually each function grows without bound with time. This problem can be avoided in various ways. For example, one can rationalize the denominators, then make the replacements in the numerators

$$S_1 \to (S_1-\tau)+\tau = \frac{1-a^2}{S_1+\tau}+\tau, \qquad S_2 \to (S_2-\tau)+\tau = -\frac{1-a^2}{S_2+\tau}+\tau \qquad (72)$$

and finally proceed to combine the various terms. This results in the following for $\tau > 1$:

$$u_{xx} = \frac{1}{4\pi\mu|z|}\left\{1-\left(1-a^2\right)\left[\frac{\left(\tau^2-a^2\right)\left(2\tau^2-2a^2+1\right)^2}{\frac{1}{2}\left(1+S_1/\tau\right)D_1} + \frac{2\tau^2\left(2\tau^2-1\right)\left(\tau^2-1\right)}{\frac{1}{2}\left(1+S_2/\tau\right)D_2}\right] + \frac{1}{D_1 D_2}\sum_{j=2,4,\ldots}^{12} a_j \tau^j\right\} \qquad (73)$$

where

$$
\begin{aligned}
D_1 &= \left(2\tau^2 - 2a^2 + 1\right)^4 - 16\tau^2\left(\tau^2 - a^2\right)^2\left(\tau^2 - a^2 + 1\right) \\
&= 16\left(1 - a^2\right)\tau^6 + 8\left(6a^4 - 8a^2 + 3\right)\tau^4 - 8\left(6a^6 - 10a^4 + 6a^2 - 1\right)\tau^2 + \left(1 - 2a^2\right)^4
\end{aligned}
\tag{74a}
$$

$$
\begin{aligned}
D_2 &= \left(2\tau^2 - 1\right)^4 - 16\tau^2\left(\tau^2 - 1\right)^2\left(\tau^2 - 1 + a^2\right) \\
&= 16\left(1 - a^2\right)\tau^6 - 8\left(3 - 4a^2\right)\tau^4 + 8\left(1 - 2a^2\right)\tau^2 + 1
\end{aligned}
\tag{74b}
$$

and

$$
\left.
\begin{aligned}
a_{12} &= 128\left(1 - a^2\right) & a_{10} &= -64\left(1 + 4a^2 - 6a^4\right) \\
a_8 &= -16\left(3 - 15a^2 - 4a^4 + 24a^6\right) & a_6 &= 16a^2\left(4 - 17a^2 + 10a^4 + 8a^6\right) \\
a_4 &= 16a^2\left(1 - 3a^2 + 7a^4 - 6a^6\right) & a_2 &= -\left(1 - 10a^2 + 40a^4 - 48a^6 + 16a^8\right)
\end{aligned}
\right\}
\tag{75}
$$

(to avoid errors, these coefficients were obtained with Matlab’s symbolic tool). At large times, the above converges to

$$
\begin{aligned}
&D_1 = D_2 \to 16\left(1 - a^2\right)\tau^6 + \cdots, \qquad \Sigma \to a_{12}\tau^{12} + \cdots \qquad \tfrac{1}{2}\left(1 + S_j / \tau\right) \to 1 \\
&u_{xx} \to \frac{1}{4\pi\mu|z|}\left\{1 - \frac{\left(1 - a^2\right)8}{16\left(1 - a^2\right)} + \frac{128\left(1 - a^2\right)}{16^2\left(1 - a^2\right)^2}\right\} = \frac{1}{8\pi\mu|z|}\frac{2 - a^2}{\left(1 - a^2\right)} = \frac{3 - 2\nu}{8\pi\mu|z|}
\end{aligned}
\tag{76}
$$

which is the correct limit predicted by Cerruti’s solution to a static tangential load.

Vertical load

Following closely the developments for the horizontal load, along the axis of symmetry the vertical displacement due to a vertical load is

$$
\begin{aligned}
u_{zz} &= \tfrac{1}{2\pi\mu}\tfrac{1}{2\pi \mathrm{i}}\int_{c-\mathrm{i}\infty}^{c+\mathrm{i}\infty}\tfrac{1}{s}\mathrm{e}^{st}\left\{\int_0^\infty f_{zz}\,k\,dk\right\}ds \\
&= \tfrac{1}{2\pi\mu z}\mathcal{L}^{-1}\left\{\int_0^\infty \frac{\sqrt{\xi^2 + a^2}\left[\left(1 + 2\xi^2\right)e_P - 2\xi^2 e_S\right]}{\left(1 + 2\xi^2\right)^2 - 4\xi^2\sqrt{\xi^2 + a^2}\sqrt{\xi^2 + 1}}\xi\, d\xi\right\} \\
&= \tfrac{1}{2\pi\mu z}\left\{\int_0^\infty \frac{\left(1 + 2\xi^2\right)\sqrt{\xi^2 + a^2}\,\delta\left(\tau - \sqrt{\xi^2 + a^2}\right)}{\left(1 + 2\xi^2\right)^2 - 4\xi^2\sqrt{\xi^2 + a^2}\sqrt{\xi^2 + 1}}\xi\, d\xi - \int_0^\infty \frac{2\xi^2\sqrt{\xi^2 + a^2}\,\delta\left(\tau - \sqrt{\xi^2 + 1}\right)}{\left(1 + 2\xi^2\right)^2 - 4\xi^2\sqrt{\xi^2 + a^2}\sqrt{\xi^2 + 1}}\xi\, d\xi\right\} \\
&= \tfrac{1}{2\pi\mu z}\left(I_P - I_S\right)
\end{aligned}
\tag{77}
$$

With appropriate substitutions as in the previous section, these integrals are

$$I_P = \int_a^\infty \frac{y^2\left(1+2y^2-2a^2\right)\delta(\tau-y)}{\left(1+2y^2-2a^2\right)^2-4\left(y^2-a^2\right)y\sqrt{y^2+1-a^2}}dy$$
$$= \frac{\tau^2\left(2\tau^2+1-2a^2\right)}{\left(2\tau^2+1-2a^2\right)^2-4\tau\left(\tau^2-a^2\right)\sqrt{\tau^2+1-a^2}}\mathcal{H}(\tau-a) \tag{78a}$$

$$I_S = \int_1^\infty \frac{2y\left(y^2-1\right)\sqrt{y^2+a^2-1}\,\delta(\tau-y)}{\left(2y^2-1\right)^2-4y\left(y^2-1\right)\sqrt{y^2-1+a^2}}dy$$
$$= \frac{2\tau\left(\tau^2-1\right)\sqrt{\tau^2-\left(1-a^2\right)}}{\left(2\tau^2-1\right)^2-4\tau\left(\tau^2-1\right)\sqrt{\tau^2-\left(1-a^2\right)}}\mathcal{H}(\tau-1) \tag{78b}$$

Hence

$$u_{zz} = \tfrac{1}{2\pi\mu z}\left[g_P(\tau)\mathcal{H}(\tau-a)-g_S(\tau)\mathcal{H}(\tau-1)\right] \tag{79a}$$

$$g_P = \frac{\tau^2\left(2\tau^2-2a^2+1\right)}{\left(2\tau^2-2a^2+1\right)^2-4\tau\left(\tau^2-a^2\right)S_1} \tag{79b}$$

$$g_S = \frac{2\tau\left(\tau^2-1\right)S_2}{\left(2\tau^2-1\right)^2-4\tau\left(\tau^2-1\right)S_2} \tag{79c}$$

As in the case of a horizontal load, each of these functions grows with time, even if in combination they do not. Thus, for times $t>1$, it is convenient to use an alternative formula:

$$u_{zz} = \frac{1}{2\pi\mu|z|}\left\{\left(1-a^2\right)\left[\frac{2\tau^2\left(2\tau^2-2a^2+1\right)\left(\tau^2-a^2\right)}{\frac{1}{2}\left(S_1/\tau+1\right)D_1}+\frac{\left(\tau^2-1\right)\left(2\tau^2-1\right)^2}{\frac{1}{2}\left(S_2/\tau+1\right)D_2}\right]+\frac{1}{D_1D_2}\sum_{j=2,4,\ldots}^{12} b_j\tau^j\right\} \tag{80}$$

where D_1, D_2 and S_1, S_2 are as before for the horizontal load. Also, the coefficients are now

$$\left.\begin{aligned}
b_{12} &= 128\left(1-a^2\right)\\
b_{10} &= 64\left(1-2a^2\right)\left(2-4a^2+a^4\right)\\
b_8 &= -16\left(21-37a^2+4a^4+36a^6-16a^8\right)\\
b_6 &= 16\left(3+26a^2-78a^4+70a^6-8a^8-8a^{10}\right)\\
b_4 &= 4\left(15-87a^2+116a^4+24a^6-136a^8+64a^{10}\right)\\
b_2 &= \left(11-28a^2+16a^4\right)\left(1-2a^2\right)^3
\end{aligned}\right\} \tag{81}$$

The long term limit is

$$u_{zz} \to \frac{1}{2\pi\mu|z|}\frac{1}{2}\left(\frac{2-a^2}{1-a^2}\right) = \frac{3-2\nu}{4\pi\mu|z|} \tag{82}$$

which is the correct result predicted by the Boussinesq static formula.

Appendix I: Expansion into partial fractions

We begin here with a mathematical formula needed in the ensuing sections. Consider the ratio of polynomials in κ^2

$$\frac{P_{n-1}\left(\kappa^2\right)}{Q_n\left(\kappa^2\right)}=\frac{a_0+a_1\kappa^2+\cdots a_{n-1}\kappa^{2(n-1)}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\cdots\left(\kappa^2-\kappa_n^2\right)} \tag{A1}$$

where the a_j, κ_j are either real or complex constants. The numerator is up to order $n-1$ in κ^2 (or less) while the denominator is of order n in κ^2. This can be expanded as

$$\frac{P_{n-1}\left(\kappa^2\right)}{Q_n\left(\kappa^2\right)}=\sum_{j=1}^{n}\frac{c_j}{\kappa^2-\kappa_j^2} \tag{A2}$$

with

$$c_j=\frac{P_{n-1}\left(\kappa_j^2\right)}{\prod_{k\neq j}^{n}\left(\kappa_j^2-\kappa_k^2\right)}=\frac{P_{n-1}\left(\kappa_j^2\right)}{(-1)^{n-1}\prod_{k\neq j}^{n}\left(\kappa_k^2-\kappa_j^2\right)}, \qquad \sum_{j=1}^{n}c_j=a_{n-1} \tag{A3}$$

Observe that the summation of the coefficients c_j equals the leading coefficient of the numerator, which is zero if the numerator is a polynomial of order smaller than $n-1$.

Appendix II: Mooney's integration formula

Using contour integration Mooney demonstrates a very useful integral that we shall generalize herein and make multiple uses later on. Let C be an arbitrary complex constant, and $D>0$ is real. Then consider the integral

$$J=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{C\,d\theta}{C+D\sin^2\theta}=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{Z\,d\theta}{Z+\sin^2\theta}, \qquad Z=\frac{C}{D}=X+\mathrm{i}Y$$

$$=Z\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{d\theta}{Z+\frac{1}{2}\left(1-\cos2\theta\right)}. \qquad \alpha=2\theta$$

$$=Z\frac{2}{\pi}\int_0^{\pi}\frac{d\alpha}{2Z+1-\cos\alpha}=Z\frac{1}{\pi}\int_0^{2\pi}\frac{d\alpha}{2Z+1-\cos\alpha}$$

Setting

$$z=e^{\mathrm{i}\alpha}, \qquad dz=\mathrm{i}z\,d\alpha, \qquad d\alpha=\frac{dz}{\mathrm{i}z}, \qquad \cos\alpha=\tfrac{1}{2}\left(z+z^{-1}\right), \qquad |z|=1$$

then by contour integration on the unit circle

$$J=\tfrac{1}{\pi}Z\oint_{|z|=1}\frac{dz}{\mathrm{i}z\left[2Z+1-\frac{1}{2}\left(z+z^{-1}\right)\right]}=\tfrac{1}{\pi\mathrm{i}}Z\oint_{|z|=1}\frac{dz}{\left[(2Z+1)z-\frac{1}{2}\left(z^2+1\right)\right]}$$

$$=\mathrm{i}\tfrac{2}{\pi}Z\oint_{|z|=1}\frac{dz}{z^2-2z(2Z+1)+1}=\mathrm{i}\tfrac{2}{\pi}Z\oint_{|z|=1}\frac{dz}{(z-z_1)(z-z_2)}$$

with

$$
\begin{aligned}
z_{1,2} &= 2Z+1 \mp \sqrt{(2Z+1)^2-1} \\
&= 2Z+1 \mp 2\sqrt{Z(1+Z)}
\end{aligned}
$$

which satisfies

$$z_1 z_2 = (2Z+1)^2 - \left[(2Z+1)^2 - 1\right] = 1$$

Thus, in every case $z_2 = z_1^{-1}$, which guarantees that one pole (z_1) is *within* the unit circle and the other (z_2) is *without*. The exception is when both are *on* the unit circle.

Case 1: C is real

a) Radicand is positive

If C is real then $Z \equiv X$ is real and $X(X+1) > 0$, i.e. $(2X+1)^2 - 1 > 0$. In that case the square root term is real, and

$$(2X+1)^2 > 1 \qquad \begin{cases} 2X+1 > 1 \\ 2X+1 < -1 \end{cases} \qquad \begin{cases} X > 0 \\ X < -1 < 0 \end{cases}$$

In the first case above $\quad z_1 = 2X+1-2\sqrt{X(X+1)}$

otherwise $\quad z_1 = 2X+1+2\sqrt{X(X+1)}$

so

$$z_2 - z_1 = 4\,\mathrm{sgn}(X)\sqrt{X(X+1)}$$

The residue is then

$$
\begin{aligned}
J &= 2\pi \mathrm{i} \tfrac{2}{\pi} X \frac{\mathrm{i}}{z_1 - z_2} = \frac{4X}{z_2 - z_1} = \frac{X\,\mathrm{sgn}(X)}{\sqrt{X(X+1)}} = \frac{|X|}{\sqrt{|X||X+1|}} = \frac{\sqrt{|X|}}{\sqrt{|X+1|}} \\
&= \frac{\sqrt{X}}{\sqrt{X+1}}
\end{aligned}
$$

because both X and $X+1$ are simultaneously either both positive or both negative, in which case any imaginary factors arising from the square roots would cancel in the ratio.

b) Radicand is negative

C is again real, so $Z = X$ is real, but now $X(X+1) < 0$, i.e. $(2X+1)^2 - 1 < 0$. Hence, the square root is purely imaginary. Since $0 < (2X+1)^2 < 1$, we can define $\cos\eta = 2X+1$ in which case we can write the square root term as

$$\sqrt{(2X+1)^2-1}=\mathrm{i}\sqrt{1-(2X+1)^2}=\mathrm{i}\sqrt{1-\cos^2\eta}=\mathrm{i}\sin\eta$$

then

$$z_{1,2}=2X+1\mp\mathrm{i}\sqrt{1-(2X+1)^2}=\cos\eta\mp\mathrm{i}\sin\eta=e^{\mp\mathrm{i}\eta}$$

Now both poles are on the unit circle. If so, there are two equal and opposite residues

$$J=\tfrac{1}{\pi}\frac{\mathrm{i}}{D}\oint_{|z|=1}\frac{dz}{(z-z_1)(z-z_2)}=\tfrac{1}{\pi}\frac{\mathrm{i}}{D}2\pi i\left(\frac{1}{z_1-z_2}+\frac{1}{z_2-z_1}\right)=0$$

in which case the integral vanishes. In summary

$$\boxed{\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{Cd\theta}{C+D\sin^2\theta}=\begin{cases}\dfrac{X}{\sqrt{X+1}} & X(X+1)>0\\ 0 & X(X+1)<0\end{cases}},\qquad X=\frac{C}{D}\qquad \text{(A4)}$$

Case 2: C is complex

Define the complex ratio

$$Z=\frac{C}{D}=X+\mathrm{i}Y \qquad \text{(A5)}$$

As in the real case, the contour integral will depend on two poles z_1, z_2, only one of which is within the unit circle, say z_1. These poles are now of the form

$$z_{1,2}=1+2Z\mp2\sqrt{Z(Z+1)}$$

It is easily shown that these two quantities satisfy the condition $z_1z_2=1$, i.e. they are reciprocals. The sign must be chosen so that $|z_1|<1$, after which the residue theorem gives

$$J=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{Cd\theta}{C+D\sin^2\theta}=\frac{4Z}{(z_2-z_1)}$$

This integral can be simplified further, as will be shown next. We begin by determining the sign needed to properly define z_1. For this purpose, we write the square root term as

$$\sqrt{Z(Z+1)}=\sqrt{(X+\mathrm{i}Y)(X+1+\mathrm{i}Y)}=\sqrt{(X^2+X-Y^2)+\mathrm{i}Y(2X+1)}$$
$$=\sqrt{R+\mathrm{i}I}=Ke^{\mathrm{i}\psi}$$

where

$$R=X^2+X-Y^2=\mathrm{Re}\left[Z(Z+1)\right],\qquad I=Y(2X+1)=\mathrm{Im}\left[Z(Z+1)\right]$$

$$K^4=(X^2+Y^2)\left[(X+1)^2+Y^2\right]\qquad \psi=\tfrac{1}{2}\arctan\frac{(1+2X)Y}{\left|X^2+X-Y^2\right|}$$

Hence

$$z_{1,2} = 1 + 2Z \mp 2K\mathrm{e}^{\mathrm{i}\psi} = (1 + 2X \mp 2K\cos\psi) + 2\mathrm{i}(Y \mp K\sin\psi)$$

and

$$\left|z_{1,2}\right|^2 = (1 + 2X \mp 2K\cos\psi)^2 + 4(Y \mp K\sin\psi)^2 > 0$$
$$= (1 + 2X)^2 + 4(Y^2 + K^2) \mp 4K\left[(1 + 2X)\cos\psi + 2Y\sin\psi\right]$$

Now, $-\frac{1}{2}\pi < \psi < \frac{1}{2}\pi$, so $\cos\psi > 0$. Also

$$\mathrm{sgn}(\psi) = \mathrm{sgn}(I) = \mathrm{sgn}\left[Y(2X + 1)\right] = \mathrm{sgn}(Y)\mathrm{sgn}(2X + 1)$$

so

$$\mathrm{sgn}(Y)\mathrm{sgn}(\sin\psi) = \left[\mathrm{sgn}(Y)\right]^2 \mathrm{sgn}(2X + 1) = \mathrm{sgn}(2X + 1)$$

and

$$(1 + 2X)\cos\psi = \mathrm{sgn}(1 + 2X)\left|(1 + 2X)\cos\psi\right|$$

so

$$\left|z_{1,2}\right|^2 = \left[(1 + 2X)^2 + 4(Y^2 + K^2)\right] \mp 4\mathrm{sgn}(2X + 1)\left[\left|(1 + 2X)K\cos\psi\right| + \left|2YK\sin\psi\right|\right]$$

The two terms in square brackets are always positive. Hence, to accomplish the largest cancellation between the first and the second term so as to obtain the smallest root, we see that if $\mathrm{sgn}(2X + 1) > 1$ we must use the negative sign to form z_1, and contrariwise if $\mathrm{sgn}(2X + 1) < 1$. Hence

$$z_1 = 1 + 2Z - 2\sqrt{Z(Z + 1)}\,\mathrm{sgn}[1 + 2X], \qquad |z_1| < 1 \tag{A6a}$$
$$z_2 = 1 + 2Z + 2\sqrt{Z(Z + 1)}\,\mathrm{sgn}[1 + 2X] \qquad |z_2| > 1 \tag{A6b}$$

so that

$$z_2 - z_1 = z_1^{-1} - z_1 = 4\sqrt{Z(Z + 1)}\,\mathrm{sgn}[1 + 2X], \qquad X = \mathrm{Re}\,Z$$

or

$$z_2 - z_1 = 4\sqrt{Z(Z + 1)}\,\mathrm{sgn}[1 + 2\mathrm{Re}Z]$$

On the other hand, since $Z = X + \mathrm{i}Y$ and $Z + 1 = X + 1 + \mathrm{i}Y$ share the same imaginary part, we see that their phases must have the same sign, namely $\mathrm{sgn}(Y)$. On the other hand, the sign of the phase of the product $Z(Z + 1)$ is controlled by the imaginary part of that product, i.e.

$$\mathrm{sgn}(I) = \mathrm{sgn}\left[Y(2X + 1)\right]$$

Thus, we see that

$$\sqrt{Z(Z+1)} = \left[\mathrm{sgn}(Y)\mathrm{sgn}(I)\right]\sqrt{Z}\sqrt{Z+1} = \left[\mathrm{sgn}(Y)\right]^2 \mathrm{sgn}[2X+1]\sqrt{Z}\sqrt{Z+1}$$
$$= \mathrm{sgn}[2X+1]\sqrt{Z}\sqrt{Z+1}$$
$$= \mathrm{sgn}[2\mathrm{Re}Z+1]\sqrt{Z}\sqrt{Z+1}$$

so

$$\frac{Z}{\sqrt{Z(Z+1)}} = \frac{Z}{\sqrt{Z}\sqrt{Z+1}}\mathrm{sgn}(2\mathrm{Re}Z+1) = \frac{\sqrt{Z}}{\sqrt{Z+1}}\mathrm{sgn}(2\mathrm{Re}Z+1)$$

Combining the preceding, we obtain

$$J = \frac{4Z}{(Q_2 - Q_1)} = \frac{Z}{\sqrt{Z(Z+1)}}\mathrm{sgn}(1+2X) = \frac{Z}{\sqrt{Z}\sqrt{Z+1}}\left[\mathrm{sgn}(1+2X)\right]^2$$
$$= \frac{\sqrt{Z}}{\sqrt{Z+1}}$$

Finally, if either $Z+1=0$ or $Z=0$, then the two roots coincide, their two contributions to the contour integral are equal and opposite, and the integral vanishes. We conclude then that

$$J = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{C\,d\theta}{C + D\sin^2\theta} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{Z\,d\theta}{Z + \sin^2\theta}$$

$$= \begin{cases} \dfrac{\sqrt{Z}}{\sqrt{Z+1}} & Z(Z+1) \neq 0 \\ 0 & X(X+1) \leq 0,\ Y = 0 \end{cases} \qquad \text{(A7)}$$

which is of exactly the same form as the earlier integral (A4) with real arguments, except that the real term X gets replaced by the complex term Z. We come to the important conclusion that *it is not necessary for us to distinguish between real and complex arguments in Mooney's integral*.

Application of Mooney's integral:
In the context of this article, the constants C, D are of the form

$$C = \kappa_\alpha^2 - \kappa_j^2, \qquad D = \tau^2 - \kappa_\alpha^2 > 0 \qquad \text{(A8)}$$

where $\kappa_\alpha = a$ or $\kappa_\alpha = 1$, as the case may be, $\tau > \kappa_\alpha$, and κ_j is one of the three eigenvalues of the Rayleigh equation, the first of which is the true Rayleigh root $\kappa_1 \equiv \gamma > 1$, and the other two are either real or complex. Then

$$Z(Z+1)=\frac{\kappa_\alpha^2-\kappa_j^2}{\tau^2-\kappa_\alpha^2}\left[\frac{\kappa_\alpha^2-\kappa_j^2}{\tau^2-\kappa_\alpha^2}+1\right]=\frac{\left(\kappa_\alpha^2-\kappa_j^2\right)\left(\tau^2-\kappa_j^2\right)}{\left(\tau^2-\kappa_\alpha^2\right)^2}$$

At low Poisson's ratios all three roots κ_j^2 are real and they satisfy the condition $\kappa_1 \equiv \gamma > 1 > a > \kappa_2 > \kappa_3$, in which case $\kappa_\alpha^2-\kappa_1^2<0$ but $\kappa_\alpha^2-\kappa_{2,3}^2>0$. This implies that

$$\left(\kappa_\alpha^2-\kappa_1^2\right)\left(\tau^2-\kappa_1^2\right)>0 \quad \to \quad \kappa_1>\tau$$

and Mooney's integral is

$$I_1=\frac{\sqrt{Z_1}}{\sqrt{Z_1+1}}=\begin{cases}\dfrac{\sqrt{\kappa_1^2-\kappa_\alpha^2}}{\sqrt{\kappa_1^2-\tau^2}} & \kappa_\alpha<\tau<\kappa_1\\ 0 & \text{else}\end{cases} \tag{A9}$$

On the other hand, since

$$\frac{\sqrt{\kappa_1^2-\kappa_\alpha^2}}{\sqrt{\kappa_1^2-\tau^2}}=\frac{\mathrm{i}\sqrt{\kappa_\alpha^2-\kappa_1^2}}{\mathrm{i}\sqrt{\tau^2-\kappa_1^2}}=\frac{\sqrt{\kappa_\alpha^2-\kappa_1^2}}{\sqrt{\tau^2-\kappa_1^2}}$$

we see that we could optionally write the above in the complex form

$$I_1=\frac{\sqrt{Z_1}}{\sqrt{Z_1+1}}=\begin{cases}\dfrac{\sqrt{\kappa_\alpha^2-\kappa_1^2}}{\sqrt{\tau^2-\kappa_1^2}} & \kappa_\alpha<\tau<\kappa_1\\ 0 & \text{else}\end{cases} \tag{A10}$$

As for the other two roots,

$$\left(\kappa_\alpha^2-\kappa_{2,3}^2\right)\left(\tau^2-\kappa_{2,3}^2\right)>0 \quad \to \quad \tau>\kappa_\alpha>\kappa_{2,3}$$

so

$$I_{2,3}=\frac{\sqrt{Z_{2,3}}}{\sqrt{Z_{2,3}+1}}=\begin{cases}\dfrac{\sqrt{\kappa_\alpha^2-\kappa_{2,3}^2}}{\sqrt{\tau^2-\kappa_{2,3}^2}} & \kappa_\alpha<\tau\\ 0 & \text{else}\end{cases}$$

which is of the same form as (A10). Hence, we can write Mooney's integral for any of the three roots as

$$\boxed{I_j=\frac{\sqrt{Z_j}}{\sqrt{Z_j+1}}=\begin{cases}\dfrac{\sqrt{\kappa_\alpha^2-\kappa_j^2}}{\sqrt{\tau^2-\kappa_j^2}} & \kappa_\alpha<\tau\\ 0 & \text{else}\end{cases}}, \qquad j=1,2,3 \tag{A11}$$

Finally, since (A4), (A7) have exactly the same form other than the fact that Z is real in (A4) and complex in (A7), we could proceed to generalize (A11) to any Poisson's ratio and allow for complex roots κ_2, κ_3. These formulas will be used in the ensuing.

Appendix III: Generic wavenumber integral

Consider an integral of the form

$$I = \frac{2}{\pi}\int_{\kappa_\alpha}^{\tau} \frac{P_{n-1}\left(\kappa^2\right)\sqrt{\kappa^2-\kappa_\alpha^2}}{Q_n\left(\kappa^2\right)\sqrt{\tau^2-\kappa^2}}\kappa d\kappa \tag{A12}$$

where $\kappa_\alpha > 0$ appears both in the lower limit and in the square root term in the numerator, and

$$P_{n-1}\left(\kappa^2\right) = a_0 + a_1\kappa^2 + \cdots a_{n-1}\kappa^{2(n-1)} \tag{A13a}$$

$$Q_n\left(\kappa^2\right) = \left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\cdots\left(\kappa^2-\kappa_n^2\right) = \prod_{j=1}^{n}\left(\kappa^2-\kappa_j^2\right) \tag{A13b}$$

We define the transformed integration path

$$\kappa^2 = \kappa_\alpha^2 + \left(\tau^2-\kappa_\alpha^2\right)\sin^2\theta \tag{A14}$$

with

$$\left(\tau^2-\kappa_\alpha^2\right)\sin^2\theta = \kappa^2-\kappa_\alpha^2, \qquad \kappa_\alpha \le \kappa \le \tau, \qquad 0 \le \theta \le \tfrac{1}{2}\pi \tag{A15}$$

so that

$$\sin\theta = \sqrt{\frac{\kappa^2-\kappa_\alpha^2}{\tau^2-\kappa_\alpha^2}}, \qquad \cos\theta = \sqrt{1-\sin^2\theta} = \sqrt{\frac{\tau^2-\kappa^2}{\tau^2-\kappa_\alpha^2}} \tag{A16}$$

$$\begin{aligned} 2\kappa\, d\kappa &= 2\left(\tau^2-\kappa_\alpha^2\right)\sin\theta\cos\theta\, d\theta \\ &= 2\left(\tau^2-\kappa_\alpha^2\right)\sqrt{\frac{\kappa^2-\kappa_\alpha^2}{\tau^2-\kappa_\alpha^2}}\sqrt{\frac{\tau^2-\kappa^2}{\tau^2-\kappa_\alpha^2}}\, d\theta \\ &= 2\sqrt{\kappa^2-\kappa_\alpha^2}\sqrt{\tau^2-\kappa^2}\, d\theta \end{aligned} \tag{A17}$$

and

$$\kappa\, d\kappa = \sqrt{\kappa^2-\kappa_\alpha^2}\sqrt{\tau^2-\kappa^2}\, d\theta \tag{A18}$$

$$\frac{\sqrt{\kappa^2-\kappa_\alpha^2}}{\sqrt{\tau^2-\kappa^2}}\kappa\, d\kappa = \left(\kappa^2-\kappa_\alpha^2\right)d\theta \tag{A19}$$

Hence, with $\kappa^2 = \kappa^2(\tau,\theta)$ the integral changes into

$$I = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{P_{n-1}\left(\kappa^2\right)\left(\kappa^2-\kappa_\alpha^2\right)}{Q_n\left(\kappa^2\right)}d\theta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\left(a_0 + a_1\kappa^2 + \cdots a_{n-1}\kappa^{2(n-1)}\right)\left(\kappa^2-\kappa_\alpha^2\right)}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\cdots\left(\kappa^2-\kappa_n^2\right)}d\theta \tag{A20}$$

Using an expansion into partial fractions the above can be expressed as

$$\begin{aligned} I &= \frac{2}{\pi}\sum_{j=1}^{n} c_j \int_0^{\frac{1}{2}\pi} \frac{\kappa^2-\kappa_\alpha^2}{\kappa^2-\kappa_j^2} d\theta = \frac{2}{\pi}\sum_{j=1}^{n} c_j \int_0^{\frac{1}{2}\pi} \frac{\kappa^2-\kappa_j^2+\kappa_j^2-\kappa_\alpha^2}{\kappa^2-\kappa_j^2} d\theta \\ &= \frac{2}{\pi}\sum_{j=1}^{n} c_j \int_0^{\frac{1}{2}\pi} \frac{\kappa^2-\kappa_j^2+\kappa_j^2-\kappa_\alpha^2}{\kappa^2-\kappa_j^2} d\theta = \sum_{j=1}^{n} c_j \left\{ 1 - \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\kappa_\alpha^2-\kappa_j^2}{\kappa^2-\kappa_j^2} d\theta \right\} \\ &= \sum_{j=1}^{n} c_j - \sum_{j=1}^{n} c_j I_j = a_{n-1} - \sum_{j=1}^{n} c_j I_j \end{aligned} \tag{A21}$$

with

$$I_j = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\kappa_\alpha^2-\kappa_j^2}{\kappa^2-\kappa_j^2} d\theta = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\left(\kappa_\alpha^2-\kappa_j^2\right) d\theta}{\kappa_\alpha^2-\kappa_j^2+\left(\tau^2-\kappa_\alpha^2\right)\sin^2\theta} = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{C\,d\theta}{C+D\sin^2\theta} \tag{A22}$$

where $C=\kappa_\alpha^2-\kappa_j^2,\ D=\tau^2-\kappa_\alpha^2$

a) C is real

From Appendix II, the integrals above are

$$c_j I_j = \begin{cases} \dfrac{A_j}{\sqrt{\left|\tau^2-\kappa_j^2\right|}} & \left(\kappa_\alpha^2-\kappa_j^2\right)\left(\tau^2-\kappa_j^2\right) > 0 \\ 0 & \text{else} \end{cases} \tag{A23a}$$

with

$$A_j = c_j \sqrt{\left|\kappa_\alpha^2-\kappa_j^2\right|} = \frac{P_{n-1}\left(\kappa_j^2\right)}{\prod_{k\neq j}^{n}\left(\kappa_j^2-\kappa_k^2\right)} \sqrt{\left|\kappa_\alpha^2-\kappa_j^2\right|} \tag{A23b}$$

b) C is complex

Define $Z = C/D$ with C,D as above, then compute either one of

$$Q_{1,2} = 1 + 2Z \mp \sqrt{Z\left(Z+1\right)} \tag{A24}$$

The two expressions satisfy $Q_1 Q_2 = 1$. Of these, choose the one that satisfies $\left|Q_1\right| < 1$, in which case

$$I_j = \frac{4Z}{Q_1^{-1}-Q_1} = \frac{\sqrt{Z}}{\sqrt{Z+1}} = \frac{\sqrt{\dfrac{\kappa_\alpha^2-\kappa_j^2}{\tau^2-\kappa_\alpha^2}}}{\sqrt{\dfrac{\kappa_\alpha^2-\kappa_j^2}{\tau^2-\kappa_\alpha^2}+1}} = \frac{\sqrt{\kappa_\alpha^2-\kappa_j^2}}{\sqrt{\tau^2-\kappa_j^2}}\,, \tag{A25a}$$

$$c_j I_j = \frac{P_{n-1}\left(\kappa_j^2\right)}{\prod_{k\neq j}^{n}\left(\kappa_j^2-\kappa_k^2\right)}\frac{\sqrt{\kappa_\alpha^2-\kappa_j^2}}{\sqrt{\tau^2-\kappa_j^2}} \tag{A25b}$$

Since the complex roots $\kappa_2, \kappa_3 = \kappa_2^*$ appear in conjugate pairs, their net contribution is

$$2\,\mathrm{Re}\left[c_2 I_2\right] = 2\,\mathrm{Re}\left[\frac{P_{n-1}\left(\kappa_2^2\right)}{\prod_{k\neq j}^{n}\left(\kappa_2^2-\kappa_k^2\right)}\frac{\sqrt{\kappa_\alpha^2-\kappa_j^2}}{\sqrt{\tau^2-\kappa_j^2}}\right] \tag{A26}$$

Appendix IV: Wavenumber integrals in F_{zz} (vertical load)

a) First integral $f_1(\tau)$

Following Appendix III with $\kappa_\alpha = a$, $P_1\left(\kappa^2\right) = \left(\kappa^2 - \frac{1}{2}\right)^2$, $a_{n-1} = 1$, we can write

$$\begin{aligned} f_1(\tau) &= \frac{2}{\pi}\int_a^\tau \frac{\left(\kappa^2-\frac{1}{2}\right)^2\sqrt{\kappa^2-a^2}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa \\ &= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\left(\kappa^2-\frac{1}{2}\right)^2\left(\kappa^2-a^2\right)d\theta}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)} \\ &= 1-\sum_{j=1}^{3} c_j I_j \end{aligned} \tag{A27}$$

where

$$I_j = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\left(a^2-\kappa_j^2\right)d\theta}{a^2-\kappa_j^2+\left(\tau^2-a^2\right)\sin^2\theta}, \qquad c_j = \frac{\left(\kappa_j^2-\frac{1}{2}\right)^2}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)} \tag{A28}$$

For all real eigenvalues, the integrals I_j follow from Appendix II, which are

$$c_1 I_1 = \begin{cases} \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & a<\tau<\gamma \\ 0 & \text{else} \end{cases}, \qquad c_{2,3} I_{2,3} = \begin{cases} \dfrac{A_{2,3}}{\sqrt{\tau^2-\kappa_{2,3}^2}}, & a<\tau \\ 0 & \text{else} \end{cases} \tag{A29a}$$

where

$$A_j = c_j\sqrt{\left|a^2-\kappa_j^2\right|} = \frac{\left(\kappa_j^2-\frac{1}{2}\right)^2\sqrt{\left|a^2-\kappa_j^2\right|}}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)}, \qquad i\neq j\neq k \tag{A29b}$$

Hence

$$f_1(\tau)=\begin{cases}1-\sum_{k=1}^{3}\frac{A_k}{\sqrt{\left|\tau^2-\kappa_k^2\right|}} & a<\tau<\gamma\\ 1-\sum_{k=2}^{3}\frac{A_k}{\sqrt{\tau^2-\kappa_k^2}} & \tau>\gamma\end{cases}\tag{A30}$$

In the case of two complex conjugate roots, we obtain from eq. (A26), after accounting for the equivalence $\left(\kappa_2^2-\frac{1}{2}\right)^2=\frac{1}{4}\left(2\kappa_2^2-1\right)^2$

$$f_1=\begin{cases}1-\frac{A_1}{\sqrt{\gamma^2-\tau^2}}-2\mathrm{Re}\left[\frac{A_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & a<\tau<\gamma\\ 1-2\mathrm{Re}\left[\frac{A_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & \tau>\gamma\end{cases}\tag{A31a}$$

$$A_1=\frac{\left(\gamma^2-\frac{1}{2}\right)^2\sqrt{\gamma^2-a^2}}{\left|\kappa_2^2-\gamma^2\right|^2}\qquad A_2=\frac{\left(\kappa_2^2-\frac{1}{2}\right)^2\sqrt{a^2-\kappa_2^2}}{\left(\kappa_1^2-\kappa_2^2\right)\left(\kappa_3^2-\kappa_2^2\right)}\tag{A31b}$$

b) Second integral $f_2(\tau)$

In this case $\kappa_\alpha=1$, $P_1\left(\kappa^2\right)=\kappa^2\left(\kappa^2-a^2\right)^2$, $a_{n-1}=1$, and making use of Appendix III, we obtain

$$\begin{aligned}f_2(\tau)&=\frac{2}{\pi}\int_1^{\tau}\frac{\kappa^2\left(\kappa^2-a^2\right)\sqrt{\kappa^2-1}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa d\kappa\\&=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\kappa^2\left(\kappa^2-a^2\right)\left(\kappa^2-1\right)d\theta}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)}\\&=1-\sum_{j=1}^{3}c_jI_j\end{aligned}\tag{A33a}$$

$$I_j=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\left(1-\kappa_j^2\right)d\theta}{1-\kappa_j^2+\left(\tau^2-1\right)\sin^2\theta},\qquad c_j=\frac{\kappa_j^2\left(\kappa_j^2-a^2\right)}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)}\tag{A33b}$$

For all real eigenvalues, the integrals follow from Appendix II with $C=1-\kappa_j^2, D=\tau^2-1$, which yields

$$c_1I_1=\begin{cases}\frac{B_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma\\ 0 & \text{else}\end{cases},\qquad c_{2,3}I_{2,3}=\begin{cases}\frac{B_{2,3}}{\sqrt{\tau^2-\kappa_{2,3}^2}}, & 1<\tau\\ 0 & \text{else}\end{cases}\tag{A34a}$$

where

$$B_j = c_j\sqrt{\left|1-\kappa_j^2\right|} = \frac{\kappa_j^2\left(\kappa_j^2 - a^2\right)\sqrt{\left|1-\kappa_j^2\right|}}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)} \tag{A34b}$$

If we now take the ratio of the constants B_k to the constants A_k involved in the solution for $f_1(\tau)$ defined by eq. (A29b), we obtain

$$\frac{B_k}{A_k} = \frac{\kappa_k^2\left(\kappa_k^2 - a^2\right)\sqrt{\left|1-\kappa_k^2\right|}}{\frac{1}{4}\left(2\kappa_k^2-1\right)^2\sqrt{\left|a^2-\kappa_k^2\right|}} = \frac{4\kappa_k^2\sqrt{\left|a^2-\kappa_k^2\right|}\sqrt{\left|1-\kappa_k^2\right|}}{\left(2\kappa_k^2-1\right)^2}\operatorname{sgn}\left(\kappa_k^2-a^2\right) = \operatorname{sgn}\left(\kappa_k^2-a^2\right) \tag{A35}$$

which follows because the ratio obeys the eigenvalue problem for Rayleigh waves. Hence, $B_1 = A_1$ but $B_2 = -A_2$, $B_3 = -A_3$. It follows that

$$f_2 = \begin{cases} 1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} + \displaystyle\sum_{k=2}^{3}\frac{A_k}{\sqrt{\tau^2-\kappa_k^2}} & 1<\tau<\gamma \\ 1 + \displaystyle\sum_{k=2}^{3}\frac{A_k}{\sqrt{\tau^2-\kappa_k^2}} & \tau>\gamma \end{cases} \tag{A36}$$

We observe that the first two terms in f_1, f_2 have the same signs and contribute equally for $1<\tau<\gamma$, while the terms in κ_2, κ_3 are opposite and will cancel out when added up.

In the case of a pair of complex conjugate roots, and using again the method employed for eqs. (A35) and (A36), we infer from (A31a) that

$$f_2 = \begin{cases} 1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} + 2\operatorname{Re}\left[\dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & 1<\tau<\gamma \\ 1 + 2\operatorname{Re}\left[\dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & \tau>\gamma \end{cases} \tag{A37}$$

Finally, combining eqs. A30-31and A35-37, we obtain the sums $f_1 + f_2$ for real roots

$$f_1 + f_2 = 2\begin{cases} \dfrac{1}{2}\left[1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} - \displaystyle\sum_{k=2}^{3}\frac{A_k}{\sqrt{\tau^2-\kappa_k^2}}\right] & a<\tau<1 \\ 1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma \\ 1 & \tau>\gamma \end{cases} \tag{A38a}$$

with the A_j defined by (A29b); and for complex roots

$$f_1 + f_2 = 2\begin{cases} \dfrac{1}{2}\left[1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} - 2\operatorname{Re}\left[\dfrac{\tilde{A}_2}{\sqrt{\tau^2-\kappa_2^2}}\right]\right] & a < \tau < 1 \\ 1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1 < \tau < \gamma \\ 1 & \tau > \gamma \end{cases} \tag{A38b}$$

with A_j defined by (A31b).

Appendix V: Wavenumber integral in F_{rz} (vertical load)

Consider next an integral of the form

$$I = \int_{\kappa_\alpha}^{\tau} \frac{P_{n-2}\left(\kappa^2\right)\sqrt{\kappa^2-\kappa_\alpha^2}\sqrt{\kappa_\beta^2-\kappa^2}}{Q_n\left(\kappa^2\right)\sqrt{\tau^2-\kappa^2}}\kappa\, d\kappa\,, \qquad \kappa_\alpha < \tau < \kappa_\beta \tag{A39}$$

Then, choosing once more the modified path $\kappa^2 = \kappa_\alpha^2 + \left(\tau^2 - \kappa_\alpha^2\right)\sin^2\theta$ and proceeding as in Appendix III, we are lead to

$$\begin{aligned} I &= \int_0^{\frac{1}{2}\pi} \frac{P_{n-2}\left(\kappa^2\right)\left(\kappa^2-\kappa_\alpha^2\right)\sqrt{\kappa_\beta^2-\kappa^2}}{Q_n\left(\kappa^2\right)} d\theta \\ &= \int_0^{\frac{1}{2}\pi} \frac{P_{n-1}\left(\kappa^2\right)\sqrt{\kappa_\beta^2-\kappa^2}}{Q_n\left(\kappa^2\right)} d\theta, \qquad P_{n-1}\left(\kappa^2\right) = P_{n-2}\left(\kappa^2\right)\left(\kappa^2-\kappa_\alpha^2\right) \end{aligned} \tag{A40}$$

Relying once more on an expansion into partial fractions, we obtain

$$\begin{aligned} I &= \sum_{j=1}^{n} c_j \int_0^{\frac{1}{2}\pi} \frac{\sqrt{\kappa_\beta^2-\kappa^2}}{\kappa^2-\kappa_j^2} d\theta = \sum_{j=1}^{n} c_j \int_0^{\frac{1}{2}\pi} \frac{\kappa_\beta^2-\kappa_j^2+\kappa_j^2-\kappa^2}{\left(\kappa^2-\kappa_j^2\right)\sqrt{\kappa_\beta^2-\kappa^2}} d\theta \\ &= \sum_{j=1}^{n} c_j \left\{ \int_0^{\frac{1}{2}\pi} \frac{\kappa_\beta^2-\kappa_j^2}{\left(\kappa^2-\kappa_j^2\right)\sqrt{\kappa_\beta^2-\kappa^2}} d\theta - \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{\kappa_\beta^2-\kappa^2}} \right\} \\ &= \frac{1}{\sqrt{\kappa_\beta^2-\kappa_\alpha^2}} \sum_{j=1}^{n} c_j \left\{ \frac{\kappa_\beta^2-\kappa_j^2}{\kappa_\alpha^2-\kappa_j^2} \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\left(1+m_j\sin^2\theta\right)\sqrt{1-n^2\sin^2\theta}} - \int_0^{\frac{1}{2}\pi} \frac{d\theta}{\sqrt{1-n^2\sin^2\theta}} \right\} \\ &= \frac{1}{\sqrt{\kappa_\beta^2-\kappa_\alpha^2}} \sum_{j=1}^{n} c_j \left\{ \frac{\kappa_\beta^2-\kappa_j^2}{\kappa_\alpha^2-\kappa_j^2} \Pi\left(m_j, n\right) - K(n) \right\} \end{aligned} \tag{A41}$$

where

$$m_j = \frac{\tau^2 - \kappa_\alpha^2}{\kappa_\alpha^2 - \kappa_j^2}, \quad n^2 = \frac{\tau^2 - \kappa_\alpha^2}{\kappa_\beta^2 - \kappa_\alpha^2}, \qquad c_j = \frac{P_{n-1}\left(\kappa_j^2\right)}{\prod_{k \neq j}^{n}\left(\kappa_j^2 - \kappa_k^2\right)} = \frac{P_{n-2}\left(\kappa_j^2\right)\left(\kappa_j^2 - \kappa_\alpha^2\right)}{\prod_{k \neq j}^{n}\left(\kappa_j^2 - \kappa_k^2\right)} \tag{A42}$$

But

$$\sum_{j=1}^{n} c_j = a_{n-1} \qquad \text{and} \qquad B_j = c_j \frac{\kappa_\beta^2 - \kappa_j^2}{\kappa_\alpha^2 - \kappa_j^2} = \frac{P_{n-2}\left(\kappa_j^2\right)\left(\kappa_\beta^2 - \kappa_j^2\right)}{\prod_{k \neq j}^{n}\left(\kappa_j^2 - \kappa_k^2\right)} \tag{A43}$$

then

$$I = \frac{1}{\sqrt{\kappa_\beta^2 - \kappa_\alpha^2}}\left\{a_{n-1}K(n) - \sum_{j=1}^{n} B_j \Pi\left(m_j, n\right)\right\} \tag{A44}$$

Appendix VI: Wavenumber integral in F_{yy} (horizontal load)

$$f_1(\tau) = \tfrac{2}{\pi}\int_1^\tau \frac{\kappa\, d\kappa}{\sqrt{\kappa^2 - 1}\sqrt{\tau^2 - \kappa^2}}, \qquad 1 < \kappa < \tau \tag{A45}$$

Choose

$$\sin\phi = \frac{\sqrt{\kappa^2 - 1}}{\sqrt{\tau^2 - 1}}, \qquad \cos\phi = \sqrt{1 - \sin^2\phi} = \sqrt{\frac{\tau^2 - \kappa^2}{\tau^2 - 1}} \tag{A46}$$

so

$$\cos\phi\, d\phi = \frac{\kappa\, d\kappa}{\sqrt{\kappa^2 - 1}\sqrt{\tau^2 - 1}} \tag{A47}$$

and

$$d\phi = \frac{\kappa\, d\kappa}{\cos\phi\sqrt{\kappa^2 - 1}\sqrt{\tau^2 - 1}} = \frac{\kappa\, d\kappa}{\sqrt{\kappa^2 - 1}\sqrt{\tau^2 - \kappa^2}} \tag{A48}$$

Hence

$$f_1(\tau) = \tfrac{2}{\pi}\int_1^\tau \frac{\kappa\, d\kappa}{\sqrt{\kappa^2 - 1}\sqrt{\tau^2 - \kappa^2}} = \tfrac{2}{\pi}\int_0^{\frac{1}{2}\pi} d\phi = 1 \to \mathcal{H}(\tau - 1) \tag{A49}$$

Similarly

$$f_2(\tau) = \tfrac{2}{\pi}\int_1^\tau \frac{\kappa\, d\kappa}{\kappa^2\sqrt{\kappa^2 - 1}\sqrt{\tau^2 - \kappa^2}} = \tfrac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d\phi}{1 + \left(\tau^2 - 1\right)\sin^2\phi} = \frac{2}{\pi}\frac{\pi}{2}\frac{1}{\tau} = \frac{1}{\tau}\mathcal{H}(\tau - 1) \tag{A50}$$

and

$$2\tau^2 f_2(\tau) = 2\tau\mathcal{H}(\tau - 1) \tag{A51}$$

The last integral above is from Appendix III, with $C = 1, D = \tau^2 - 1$.

Appendix VII: Wavenumber integrals in F_{xx} (horizontal load)

We apply the previous method to obtain in detail the integrals $f_1(\tau), f_2(\tau), f_3(\tau), f_4(\tau)$ needed for the horizontal load case:

a) First integral

Here, $P(\kappa^2) = \kappa^2(\kappa^2 - 1)$, $a_{n-1} = 1$, $\kappa_\alpha = a$, so from Appendix III,

$$f_1(\tau) = \tfrac{2}{\pi}\int_a^\tau \frac{\kappa^2(\kappa^2-1)\sqrt{\kappa^2-a^2}}{(\kappa^2-\kappa_1^2)(\kappa^2-\kappa_2^2)(\kappa^2-\kappa_3^2)\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa$$

$$= \tfrac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\kappa^2(\kappa^2-1)(\kappa^2-a^2)\,d\theta}{(\kappa^2-\kappa_1^2)(\kappa^2-\kappa_2^2)(\kappa^2-\kappa_3^2)} \qquad \text{(A52a)}$$

$$= 1 - \sum_{j=1}^{3} c_j I_j$$

$$I_j = \tfrac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{(a^2-\kappa_j^2)\,d\theta}{a^2-\kappa_j^2+(\tau^2-a^2)\sin^2\theta}, \qquad c_j = \frac{\kappa_j^2(\kappa_j^2-1)}{(\kappa_j^2-\kappa_i^2)(\kappa_j^2-\kappa_k^2)},\ i \neq j \neq k \qquad \text{(A52b)}$$

For all real eigenvalues κ_j, the integrals follow from Appendix II and are

$$c_1 I_1 = \begin{cases} \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & a < \tau < \gamma \\ 0 & \text{else} \end{cases}, \qquad c_{2,3} I_{2,3} = \begin{cases} \dfrac{A_{2,3}}{\sqrt{\tau^2-\kappa_{2,3}^2}}, & a < \tau \\ 0 & \text{else} \end{cases} \qquad \text{(A53a)}$$

where

$$A_j = c_j\sqrt{\left|a^2-\kappa_j^2\right|} = \frac{\kappa_j^2(\kappa_j^2-1)\sqrt{\left|a^2-\kappa_j^2\right|}}{(\kappa_j^2-\kappa_i^2)(\kappa_j^2-\kappa_k^2)}, \qquad i \neq j \neq k \qquad \text{(A53b)}$$

Hence

$$f_1(\tau) = \begin{cases} 1 - \dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} - \dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}} - \dfrac{A_3}{\sqrt{\tau^2-\kappa_3^2}} & a < \tau < \gamma \\ 1 - \dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}} - \dfrac{A_3}{\sqrt{\tau^2-\kappa_3^2}} & \tau > \gamma \end{cases} \qquad \text{(A54)}$$

For complex roots, and in the light of (A26) and (A53), we obtain

$$f_1(\tau)=\begin{cases}1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}}-2\,\mathrm{Re}\left[\dfrac{\tilde{A}_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & a<\tau<\gamma\\[2ex] 1-2\,\mathrm{Re}\left[\dfrac{\tilde{A}_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & \tau>\gamma\end{cases} \tag{A55a}$$

with

$$\tilde{A}_2=\frac{\kappa_2^2\left(\kappa_2^2-1\right)\sqrt{a^2-\kappa_2^2}}{\left(\kappa_2^2-\gamma^2\right)\left(\kappa_2^2-\kappa_3^2\right)} \tag{A55b}$$

b) Second integral

Here, $P(\kappa^2)=\left(\kappa^2-\frac{1}{2}\right)^2$, $a_{n-1}=1$, $\kappa_\alpha=1$, so from Appendix III,

$$\begin{aligned}f_2(\tau)&=\frac{2}{\pi}\int_1^\tau\frac{\left(\kappa^2-\frac{1}{2}\right)^2\sqrt{\kappa^2-1}}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\kappa\,d\kappa\\ &=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\left(\kappa^2-\frac{1}{2}\right)^2\left(\kappa^2-1\right)d\theta}{\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)}\\ &=1-\sum_{j=1}^{3}c_jI_j\end{aligned} \tag{A56a}$$

$$I_j=\frac{2}{\pi}\int_0^{\frac{1}{2}\pi}\frac{\left(1-\kappa_j^2\right)d\theta}{1-\kappa_j^2+\left(\tau^2-1\right)\sin^2\theta}\,,\qquad c_j=\frac{\left(\kappa_j^2-\frac{1}{2}\right)^2}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)},\ i\neq j\neq k \tag{A56b}$$

For all real eigenvalues, the integrals I_j follow from Appendix II, which in this case result in

$$c_1I_1=\begin{cases}\dfrac{B_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma\\ 0 & \text{else}\end{cases},\qquad c_{2,3}I_{2,3}=\begin{cases}\dfrac{B_{2,3}}{\sqrt{\tau^2-\kappa_{2,3}^2}}, & 1<\tau\\ 0 & \text{else}\end{cases} \tag{A57a}$$

$$B_j=c_j\sqrt{\left|1-\kappa_j^2\right|}=\frac{\left(\kappa_j^2-\frac{1}{2}\right)^2\sqrt{\left|1-\kappa_j^2\right|}}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)},\qquad i\neq j\neq k \tag{A57b}$$

But

$$\frac{B_j}{A_j}=\frac{\left(\kappa_j^2-\frac{1}{2}\right)^2\sqrt{\left|1-\kappa_j^2\right|}}{\kappa_j^2\left(\kappa_j^2-1\right)\sqrt{\left|a^2-\kappa_j^2\right|}}=\frac{\left(2\kappa_j^2-1\right)^2}{4\kappa_j^2\sqrt{\left|1-\kappa_j^2\right|}\sqrt{\left|a^2-\kappa_j^2\right|}}\mathrm{sgn}\left(\kappa_j^2-1\right)=\mathrm{sgn}\left(\kappa_j^2-1\right) \tag{A58}$$

because the ratio satisfies the Rayleigh eigenvalue equation. Hence, $B_1=A_1$ but $B_2=-A_2$, $B_3=-A_3$. It follows that

$$f_2(\tau)=\begin{cases}1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}}+\dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}}+\dfrac{A_3}{\sqrt{\tau^2-\kappa_3^2}} & 1<\tau<\gamma\\ 1+\dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}}+\dfrac{A_3}{\sqrt{\tau^2-\kappa_3^2}} & \tau>\gamma\end{cases} \tag{A59}$$

For complex roots, and in analogy to eqs. (A55) we obtain

$$f_2(\tau)=\begin{cases}1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}}+2\,\mathrm{Re}\left[\dfrac{\tilde{A}_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & a<\tau<\gamma\\ 1+2\,\mathrm{Re}\left[\dfrac{\tilde{A}_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & \tau>\gamma\end{cases} \tag{A60}$$

It follows that for both real and complex roots, the sum is

$$f_1(\tau)+f_2(\tau)=2\begin{cases}\dfrac{1}{2}\left[1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}}-\dfrac{A_2}{\sqrt{\tau^2-\kappa_2^2}}-\dfrac{A_3}{\sqrt{\tau^2-\kappa_3^2}}\right] & a<\tau<1\\ 1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma\\ 1 & \tau>\gamma\end{cases} \tag{A61a}$$

and

$$f_1(\tau)+f_2(\tau)=2\begin{cases}\dfrac{1}{2}\left(1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}}-2\,\mathrm{Re}\dfrac{\tilde{A}_2}{\tau^2-\kappa_2^2}\right) & a<\tau<1\\ 1-\dfrac{A_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma\\ 1 & \tau>\gamma\end{cases} \tag{A61b}$$

with the A_j defined by eq. (A53b).

c) Third integral

Here, $P(\kappa^2)=\kappa^2-1$, $a_{n-1}=0$ (namely the coefficient of κ^4 in $P(\kappa^2)$, which does not exist), $\kappa_\alpha=a$ and

$$f_3(\tau) = \frac{2}{\pi}\int_a^\tau \frac{\sqrt{\kappa^2 - a^2}}{(\kappa^2 - \kappa_1^2)(\kappa^2 - \kappa_2^2)(\kappa^2 - \kappa_3^2)\sqrt{\tau^2 - \kappa^2}}\kappa d\kappa$$

$$= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{(\kappa^2 - 1)(\kappa^2 - a^2)}{(\kappa^2 - \kappa_1^2)(\kappa^2 - \kappa_2^2)(\kappa^2 - \kappa_3^2)} d\theta \tag{A62a}$$

$$= a_{n-1} - \sum_{j=1}^{3} c_j I_j = -\sum_{j=1}^{3} c_j I_j$$

$$I_j = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{(a^2 - \kappa_j^2) d\theta}{a^2 - \kappa_j^2 + (\tau^2 - a^2)\sin^2\theta}, \qquad c_j = \frac{\kappa_j^2 - 1}{(\kappa_j^2 - \kappa_i^2)(\kappa_j^2 - \kappa_k^2)},\ i \neq j \neq k \tag{A62b}$$

For all real eigenvalues, the integrals I_j follow from Appendix III, which are

$$c_1 I_1 = \begin{cases} \dfrac{-C_1}{\sqrt{\gamma^2 - \tau^2}} & a < \tau < \gamma \\ 0 & \text{else} \end{cases}, \qquad c_{2,3} I_{2,3} = \begin{cases} \dfrac{-C_{2,3}}{\sqrt{\tau^2 - \kappa_{2,3}^2}}, & a < \tau \\ 0 & \text{else} \end{cases} \tag{A63a}$$

where

$$C_j = \frac{(1 - \kappa_j^2)\sqrt{|a^2 - \kappa_j^2|}}{(\kappa_j^2 - \kappa_i^2)(\kappa_j^2 - \kappa_k^2)}, \qquad i \neq j \neq k \tag{A63b}$$

It follows that

$$f_3(\tau) = \begin{cases} \dfrac{C_1}{\sqrt{\gamma^2 - \tau^2}} + \dfrac{C_2}{\sqrt{\tau^2 - \kappa_2^2}} + \dfrac{C_3}{\sqrt{\tau^2 - \kappa_3^2}} & a < \tau < \gamma \\ \dfrac{C_2}{\sqrt{\tau^2 - \kappa_2^2}} + \dfrac{C_3}{\sqrt{\tau^2 - \kappa_3^2}} & \tau > \gamma \end{cases} \tag{A64}$$

In the case of complex roots, and again with reference to Appendix III, this is replaced by

$$f_3(\tau) = \begin{cases} \dfrac{C_1}{\sqrt{\gamma^2 - \tau^2}} + 2\mathrm{Re}\dfrac{\tilde{C}_2}{\sqrt{\tau^2 - \kappa_2^2}} & a < \tau < \gamma \\ 2\mathrm{Re}\dfrac{\tilde{C}_2}{\sqrt{\tau^2 - \kappa_2^2}} & \tau > \gamma \end{cases} \tag{A65a}$$

where

$$\tilde{C}_2 = \frac{(1 - \kappa_2^2)\sqrt{a^2 - \kappa_2^2}}{(\kappa_2^2 - \gamma^2)(\kappa_2^2 - \kappa_3^2)} \tag{A65b}$$

d) Fourth integral

Here $\kappa_\alpha = 1$, and added to $\kappa_j, j = 1,2,3$ we also have $\kappa_0 = 0$, $P = \left(\kappa^2 - \frac{1}{2}\right)^2$, $a_{n-1} = 0$. Then

$$
\begin{aligned}
f_4(\tau) &= \frac{2}{\pi}\int_1^{\tau} \frac{\left(\kappa^2-\tfrac{1}{2}\right)^2\sqrt{\kappa^2-1}}{\kappa^2\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)\sqrt{\tau^2-\kappa^2}}\,\kappa\,d\kappa \\
&= \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\left(\kappa^2-\tfrac{1}{2}\right)^2\left(\kappa^2-1\right)d\theta}{\kappa^2\left(\kappa^2-\kappa_1^2\right)\left(\kappa^2-\kappa_2^2\right)\left(\kappa^2-\kappa_3^2\right)} \\
&= \alpha_{n-1} - \sum_{m=0}^{3} c_m I_m = -\sum_{m=0}^{3} c_m I_m
\end{aligned}
\qquad \text{(A66)}
$$

where

$$
I_m = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\left(1-\kappa_m^2\right)d\theta}{1-\kappa_m^2+\left(\tau^2-1\right)\sin^2\theta} \qquad \text{(A67)}
$$

$$
c_m = \frac{\left(\kappa_m^2-\tfrac{1}{2}\right)^2}{\left(\kappa_m^2-\kappa_i^2\right)\left(\kappa_m^2-\kappa_j^2\right)\left(\kappa_m^2-\kappa_k^2\right)}, \qquad m = 0,1,2,3 \qquad i \neq j \neq k \neq m \qquad \text{(A68)}
$$

Taking into account the fact that $\kappa_0 = 0$, the coefficients are found to be

$$
c_0 = \frac{-1}{4\kappa_1^2\kappa_2^2\kappa_3^2} \qquad \text{(A69a)}
$$

$$
c_j = \frac{\left(\kappa_j^2-\tfrac{1}{2}\right)^2}{\kappa_j^2\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)}, \qquad i,j,k = 1,2,3, \qquad i \neq j \neq k \qquad \text{(A69b)}
$$

But from the Rayleigh equation (20) we know that $4\kappa_1^2\kappa_2^2\kappa_3^2 = \left[4\left(1-a^2\right)\right]^{-1} = \tfrac{1}{2}\left(1-\nu\right)$, so

$$
f_4(\tau) = -\sum_{m=0}^{3} c_m I_m = \frac{2}{1-\nu} I_0 - \sum_{j=1}^{3} c_j I_j \qquad \text{(A70)}
$$

Also, from Appendix II, with $\kappa_0 = 0$, $C = 1$, $D = \tau^2 - 1$, we infer

$$
I_0 = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{d\theta}{1+\left(\tau^2-1\right)\sin^2\theta} = \frac{1}{\tau}\mathcal{H}(\tau-1) = \begin{cases} \dfrac{1}{\tau}, & \tau > 1 \\ 0 & \text{else} \end{cases} \qquad \text{(A71)}
$$

For real values of the eigenvalues, the integrals again follow from Appendix II, and are

$$
I_j = \frac{2}{\pi}\int_0^{\frac{1}{2}\pi} \frac{\left(1-\kappa_j^2\right)d\theta}{1-\kappa_j^2+\left(\tau^2-1\right)\sin^2\theta} \qquad \text{(A72)}
$$

$$
c_1 I_1 = \begin{cases} \dfrac{D_1}{\sqrt{\gamma^2-\tau^2}} & 1 < \tau < \gamma \\ 0 & \text{else} \end{cases}, \qquad c_{2,3} I_{2,3} = \begin{cases} \dfrac{D_{2,3}}{\sqrt{\tau^2-\kappa_{2,3}^2}}, & 1 < \tau \\ 0 & \text{else} \end{cases} \qquad \text{(A73)}
$$

where

$$D_j = c_j \sqrt{\left|1-\kappa_j^2\right|} = \frac{\left(2\kappa_j^2 - 1\right)^2 \sqrt{\left|1-\kappa_j^2\right|}}{4\kappa_j^2\left(\kappa_j^2 - \kappa_i^2\right)\left(\kappa_j^2 - \kappa_k^2\right)} \tag{A74}$$

If we take the ratio of the constants D_j herein to the constants C_j obtained earlier for $f_3(\tau)$, we obtain

$$\frac{D_j}{C_j} = \frac{\left(2\kappa_j^2 - 1\right)^2 \sqrt{\left|1-\kappa_j^2\right|}}{4\kappa_j^2\left(1-\kappa_j^2\right)\sqrt{\left|a^2-\kappa_j^2\right|}} = \operatorname{sgn}\left(1-\kappa_j^2\right)\left[\frac{\left(2\kappa_j^2 - 1\right)^2}{4\kappa_j^2\sqrt{\left|1-\kappa_j^2\right|}\sqrt{\left|a^2-\kappa_j^2\right|}}\right] = \operatorname{sgn}\left(1-\kappa_j^2\right) \tag{A75}$$

This is because the term in brackets satisfies the Rayleigh equation. Since $\kappa_1 \equiv \gamma > 1$ while $\kappa_{2,3} < 1$, it follows that $D_1 = -C_1$ but $D_2 = C_2$, $D_3 = C_3$. Hence

$$f_4(\tau) = \frac{2}{1-\nu}\frac{\mathcal{H}(\tau-1)}{\tau} - \begin{cases} \dfrac{-C_1}{\sqrt{\gamma^2-\tau^2}} + \dfrac{C_2}{\sqrt{\tau^2-\kappa_2^2}} + \dfrac{C_3}{\sqrt{\tau^2-\kappa_3^2}} & 1 < \tau < \gamma \\ \dfrac{C_2}{\sqrt{\tau^2-\kappa_2^2}} + \dfrac{C_3}{\sqrt{\tau^2-\kappa_3^2}} & \tau > \gamma \end{cases} \tag{A76}$$

In the case of complex roots, it is found again that $B_1 = A_1$ while the contribution of the complex roots changes sign. Hence

$$f_4(\tau) = \frac{2}{1-\nu}\frac{\mathcal{H}(\tau-1)}{\tau} - \begin{cases} \dfrac{-C_1}{\sqrt{\gamma^2-\tau^2}} + 2\operatorname{Re}\dfrac{\tilde{C}_2}{\sqrt{\tau^2-\kappa_2^2}} & 1 < \tau < \gamma \\ 2\operatorname{Re}\dfrac{\tilde{C}_2}{\sqrt{\tau^2-\kappa_2^2}} & \tau > \gamma \end{cases} \tag{A77}$$

with C given again by eq. (A65b). Finally, the sum of the third and fourth integrals is

$$f_3 + f_4 = \frac{2}{1-\nu}\frac{\mathcal{H}(\tau-1)}{\tau} + 2\begin{cases} \dfrac{1}{2}\left[\dfrac{C_1}{\sqrt{\gamma^2-\tau^2}} + \dfrac{C_2}{\sqrt{\tau^2-\kappa_2^2}} + \dfrac{C_3}{\sqrt{\tau^2-\kappa_3^2}}\right] & a < \tau < 1 \\ \dfrac{C_1}{\sqrt{\gamma^2-\tau^2}} & 1 < \tau < \gamma \\ 0 & \tau > \gamma \end{cases} \tag{A78a}$$

$$f_3 + f_4 = \frac{2}{1-\nu}\frac{\mathcal{H}(\tau-1)}{\tau} + 2\begin{cases} \frac{1}{2}\left[\frac{C_1}{\sqrt{\gamma^2-\tau^2}} + 2\mathrm{Re}\frac{C_2}{\sqrt{\tau^2-\kappa_2^2}}\right] & a<\tau<1 \\ \frac{C_1}{\sqrt{\gamma^2-\tau^2}} & 1<\tau<\gamma \\ 0 & \tau>\gamma \end{cases} \tag{A78b}$$

with

$$C_j = \frac{\left(1-\kappa_j^2\right)\sqrt{\left|a^2-\kappa_j^2\right|}}{\left(\kappa_j^2-\kappa_i^2\right)\left(\kappa_j^2-\kappa_k^2\right)},\ i\neq j\neq k\ \text{ (real } \kappa_j\text{)} \qquad \tilde{C}_2 = \frac{\left(1-\kappa_2^2\right)\sqrt{a^2-\kappa_2^2}}{\left(\kappa_2^2-\kappa_1^2\right)\left(\kappa_2^2-\kappa_3^2\right)}\ \text{, (complex } \kappa_2\text{)} \tag{A79}$$

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