Holographic View of Non-relativistic Physics

by

Koushik Balasubramanian

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Abstract

Motivated by the AdS/CFT correspondence for relativistic CFTs, it seems natural to generalize it to non-relativistic CFTs. Such a dual description could provide insight into strong coupling phenomena observed in condensed matter systems. Scale invariance can be realized in non-relativistic theories in many ways. One freedom is the relative scale dimension of time and space, called the dynamical exponent z. In this thesis, we will mainly focus on the case where $z = 2$, however gravity duals for other values of z have also been found.

In the first part of the thesis, we study NRCFTs that are Galilean invariant. Discrete light cone quantization (DLCQ) of $\mathcal{N} = 4$ super Yang-Mills theory is an example of such a system with $z = 2$ scaling symmetry. A more realistic example of a system with the same set of symmetries is a system of cold fermions at unitarity. These non-relativistic systems respect a symmetry algebra known as the Schrödinger algebra. We propose a gravity dual that realizes the symmetries of the Schrödinger algebra as isometries. An unusual feature of this duality is that the bulk geometry has two extra dimensions than the CFT, instead of the usual one. The additional direction is a compact direction and shift symmetry along this direction corresponds to the particle number transformation.

This solution can be embedded into string theory by performing a set of operations (known as the Null-Melvin twist) on $AdS_5 \times S^5$ solution of type IIB supergravity. This method also provides a way of finding a black hole solution which has asymptotic Schrödinger symmetries. The field theory dual of these gravity solutions happens to be a modified version of DLCQ $\mathcal{N} = 4$ super Yang-Mills theory. The thermodynamics of these theories is very different from that of cold atoms. This happens to be a consequence of realizing the entire Schrödinger group as isometries of the spacetime. We give an example of a holographic realization in which the particle number symmetry is realized as a bulk gauge symmetry. In this proposal, the Schrödinger algebra is realized in the bulk without the introduction of an additional compact direction. Using this proposal, we find a confining solution that describes a non-relativistic system at finite density. We use the holographic dictionary to compute the conductivity of this system and it is found to exhibit somewhat unusual behavior.

In the second part of the thesis we study gravity duals of Lifshitz theories. These are non-relativistic scale invariant theories that are not boost invariant. These theories do not have a particle number symmetry unlike the boost invariant NRCFTs. We present solutions of 10D and 11D supergravity theories that are dual to Lifshitz theories. We present a black hole solution that is dual to a strongly interacting Lifshitz theory at finite temperature. We show that the finite temperature correlators in the interacting theories do not exhibit ultra-local behavior which was observed in free Lifshitz theories.

Thesis Supervisor: John McGreevy
Title: Associate Professor

Dedication

To My Loving Parents

for their

Limitless Love

and

Endless Encouragement.

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Chapter 1

Introduction

1.1 An alternate view of strongly interacting systems

The holographic principle relates a gravity theory to a quantum field theory (QFT) in one lesser dimension. A prototype example of this correspondence is the AdS/CFT duality, which is a conjectured equivalence between a gravity theory on Anti-de Sitter (AdS) space and a strongly coupled conformal field theory (CFT) in one lesser dimension [124, 69, 168]. In some sense, the correspondence suggests that the emergent degrees of freedom of the strongly coupled field theory are described by a gravitational theory. This can be taken as a "definition" of quantum gravity given the lack of alternate definitions.

The original suggestion for the correspondence was based on studies of D-branes in flat space. In the low energy limit, one obtains an equivalence between a supersymmetric gauge theory that describes the light open string modes describing the worldvolume theory on the branes and a gravity theory in AdS. The gravity theory becomes classical in the limit where the number of colors (in the gauge theory) becomes infinite. In this limit, it is possible to compute observables in the strongly coupled gauge theory by studying classical (or semi-classical) gravity. This feature is pleasing - the correspondence maps intractable computations in the field theory to simple computations in classical gravity. For example, the AdS/CFT correspondence (also known as gauge/gravity correspondence) maps the problem of computing transport coefficients to solving linear wave equations in a black hole background [113]. It is a highly non-trivial task to obtain transport quantities of a strongly interacting many body system using conventional field theory techniques. These techniques use perturbative methods to describe the transport properties which treats the plasma as a system of weakly-interacting particles (or quasi-particles). Viscosity is proportional to the mean free path, which diverges when the interaction is turned off. Hence, perturbative methods produce a large value for the shear viscosity-to-entropy ratio. This cannot explain the low values of viscosity of hot quark-gluon plasma observed in experiments. It is essential to incorporate interactions to obtain a finite mean free path, which seems to be naturally encoded in the gravity description. The holographic calculation predicts a small value for the shear viscosity-to-entropy ratio for a plasma made by putting the *dual QFT* at finite temperature. The finiteness of viscosity just turns out to be a consequence of the properties of black hole horizons. The absorption of energy by black hole horizon mimics the dissipative effects in the hydrodynamics of the high temperature plasma. Though this result does not directly apply to QCD at finite temperature, it seems to provide a good estimate of the viscosity of quark-gluon plasma measured in heavy ion collisions. More importantly, holography has provided conceptual insights that can augment our current understanding of the quark gluon plasma.

More generally, we would like to exploit the power of holographic duality to understand features of strongly interacting systems, such as high-T_c superconductors and QCD at finite density, that are not well understood using conventional techniques. It seems important to develop new ideas to understand the concepts underlying such many body phenomena. Holography seems to provide fertile grounds for harvesting new ideas about strongly interacting theories. This motivates a study of CFTs with holographic description (holographic CFTs). String theory has a plethora of AdS groundstates and each AdS solution is dual to some CFT. This landscape of CFTs resembles the landscape of CFTs that describe the dynamics near quantum critical points arising in quantum phase transitions. The prospect of studying many body physics using holography has made the existence of the string landscape a pleasant feature.

Recently, there have been many attempts to use holography to mimic condensed matter systems. A bulk of these studies take a phenomenological approach to AdS/CFT where the full framework of string theory is not required. Though a string embedding could be helpful in providing an explicit description of the microscopic dynamics of the dual QFT, such an embedding is not always available and might be non-trivial. The phenomenological approach focus on capturing some of the essential features of a real-world problem. A drawback of the phenomenological approach to holography, is that there is no explicit Lagrangian description of the QFT. However, the absence of a Lagrangian description is not an obstacle in using these holographic QFTs to learn about strongly interacting theories. In fact, there are examples of theories which do not have a Lagrangian description. Holography provides a prescription for obtaining correlation functions of all the gauge invariant observables. This can be treated as a definition of the dual field theory in the absence of a Lagrangian description.

Though the holographic models are somewhat far from describing realistic condensed matter systems, it has spurred a lot of studies on gravitational instabilities. As a result of these studies a numerous instabilities of charged black holes have been identified. At this point, the implications of these instabilities for quantum gravity is not clear. The precise comparison between the experimental results and holographic theories may not be possible now, but developments in the future might yield fruitful results. As mentioned earlier, these studies help in visualizing various non-perturbative features that could arise in many body systems. The hope is that some of these features might become more transparent from experimental or theoretical studies.

In the following sections we will a present a lightning review of the AdS/CFT correspondence.

1.2 A phenomenological approach to AdS/CFT

In this section, we will find a geometric description of a relativistic conformal field theory in $d+1$ dimensions. This serves as a starting point to find phenomenological holographic duals of quantum critical points. The goal is to find a metric whose isometry group matches with the symmetry group of the conformal field theory. First, let us list the symmetries of a relativistic CFT:

Translations: $P_\mu = -i\partial_\mu$, Dilations: $D = -ix^\mu\partial_\mu$

Lorentz transformations: $M_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) = -(x_\mu P_\nu - x_\nu P_\mu)$

Special Conformal Transformations: $K_\mu = i(2x_\mu x\cdot\partial - x^2\partial_\mu) = -2x_\mu D + x^2 P_\mu$

These set of generators respect the following algebra

$[M_{\mu\nu}, P_\rho] = i(g_{\nu\rho}P_\mu - g_{\mu\rho}P_\nu)$, $[D, K_\mu] = -iK_\mu$, $[M_{\mu\nu}, K_\rho] = i(g_{\nu\rho}K_\mu - g_{\mu\rho}K_\nu)$, $[D, P_\mu] = iP_\mu$

$[M_{\mu\nu}, M_{\rho\tau}] = i\left(g_{\mu\tau}M_{\nu\rho} + g_{\nu\rho}M_{\mu\tau} - g_{\mu\rho}M_{\nu\tau} - g_{\nu\tau}M_{\mu\rho}\right)$, $[D, K_\mu] = -iK_\mu$, $[P_\mu, K_\nu] = 2i(g_{\mu\nu}D + M_{\mu\nu})$

The $d+1$ dimensional conformal group is isomorphic to $SO(d+1,2)$ There is a unique geometry whose isometry group is $SO(d+1,2)$- AdS_{d+2} and it is a maximally symmetric spacetime. The line element of this $d+2-$dimensional spacetime is given by

$$ds^2 = L^2\left(\frac{-dt^2 + d\vec{x}^2 + dr^2}{r^2}\right) \tag{1.2.1}$$

The generators of the isometry group are the Killing vectors-
Translations: $\tilde{P}_\mu = -i\partial_\mu$, Dilations: $\tilde{D} = -ix^\mu\partial_\mu - ir\partial_r$
Lorentz transformations: $\tilde{M}_{\mu\nu} = i(x_\mu\partial_\nu - x_\nu\partial_\mu) = -(x_\mu\tilde{P}_\nu - x_\nu\tilde{P}_\mu)$
Special Conformal Transformations: $\tilde{K}_\mu = i(2x_\mu x\cdot\partial - x^2\partial_\mu) = -2x_\mu\tilde{D} + x^2\tilde{P}_\mu$
We have introduced a $\sim$ to distinguish the Killing vectors from the generators of the conformal group. The Killing vectors approach the generators of the conformal group when $r \longrightarrow 0$. Hence, the conformal symmetry group is the symmetry group of the theory living on the conformal boundary of AdS, which is at $r=0$. The interpretation of the radial direction (r) in the dual field theory will be discussed later in this chapter.

In order to completely specify the gravity theory, we need to identify the source that supports this spacetime. Anti-de Sitter spacetime is a space with constant negative curvature and it is maximally symmetric. The only source that respects the symmetries of AdS is a negative cosmological constant. In $d+2$ dimensions the value of cosmological constant is $\Lambda = -\frac{d(d+1)}{2L^2}$. The $(d+2)-$dimensional action can be written as

$$S = \frac{1}{2\kappa^2_{d+2}}\int d^{d+2}x\sqrt{g}\,(R-2\Lambda)$$

In the next section we will go through the original arguments in [124] which led to the discovery of AdS/CFT correspondence.

1.3 A braney picture of the correspondence

Let us list down some important facts about D-branes that will be helpful for our discussion. Dirichlet p-branes or Dp branes are $p+1-$dimensional topological defects on which strings can end. The open strings which end there have tension and hence their light states are localized on the brane. These provide a string theory description of worldvolume degrees of freedom.

The open string spectra contains massless vector degrees of freedom; these worldvolume degrees of freedom generally include a gauge field propagating in $p+1-$dimensions. The worldvolume gauge dynamics on a D p-branes is described by the Born-Infeld action. The bosonic part of the Born-Infeld action is given by

$$S_{BI} = -\tau_p^{(0)} \int d^{p+1}\xi \; e^{-\phi} \sqrt{-\det\left[G_{\alpha\beta} + B_{\alpha\beta} + 2\pi\alpha' F_{\alpha\beta}\right]} \tag{1.3.1}$$

$G_{\alpha\beta}$ is the pullback of the metric $G_{\mu\nu}$, $B_{\alpha\beta}$ is the pullback of the NS-NS two form potential $B_{\mu\nu}$ and $F_{\alpha\beta}$ is the gauge field living on the brane. This action comes from interactions involving worldsheets with one boundary. So the $Dp-$ brane tension is given by:

$$\tau_p \equiv \frac{\tau_p^{(0)}}{g_s} = \frac{(2\pi\sqrt{\alpha'})^{1-p}}{2\pi\alpha' g_s} \tag{1.3.2}$$

where g_s is the string coupling constant. It is also known that D-branes carry Ramond-Ramond(RR) charges. This can be seen by computing the amplitude for D-branes to emit RR gauge bosons.

Now, let us consider a stack of N D3 branes. The back reaction on the geometry from this stack of branes is determined by its contribution to the source term in Einstein's equations. The source term in Einstein's equation is proportional to the product of 10D Newton's constant G_N and the stress tensor of the N D3 branes. The stress tensor is proportional to the product of the number of branes and the D3 brane tension. The tension of the D3 brane is inversely proportional to the string coupling constant (g_s) and G_N is proportional to g_s^2. Hence the backreaction of the stack of N D3 branes is proprtional to $g_s N \equiv \lambda$, which is negligible when $\lambda \ll 1$. When $\lambda \gg 1$, the D3-branes collapse into an extremal black brane carrying the RR charge of the D3 branes.

The black 3-brane metric carrying N units of RR 5-form flux (α' is the string tension) is

$$\begin{aligned} ds^2 &= H^{-1/2}(\rho)\eta_{\mu\nu}dx^\mu dx^\nu + H^{1/2}(\rho)dx^m dx^m\,, \quad \mu,\nu \in 0,\ldots,3\,, \quad m,n \in 4,\ldots,9\,, \\ H &= 1 + \frac{L^4}{\rho^4}\,, \quad L^4 = 4\pi g_s N \alpha'^2\,, \quad \rho^2 = x^m x^m\,. \end{aligned} \tag{1.3.3}$$

The metric approaches flat space as $\rho/L \to \infty$. and in the "near-horizon" (small ρ/L)

limit the line element takes the following nice form,

$$ds^2 \rightarrow \frac{\rho^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{\rho^2}d\rho^2 + L^2 d\Omega_{S^5}^2\,. \tag{1.3.4}$$

The near horizon geometry is just $AdS_5 \times S^5$. The coordinate transformation $\rho = L^2/r$ makes the line element of AdS take the form in (1.2.1).

Note that the horizon is extremal, that is, it has no temperature. Let us consider a probe with energy E (as measured by an observer at infinity) and let us denote the energy measured by an observer located at ρ by $E(\rho)$. These two are related by a redshift, since $H(\rho)$ varies with the radial coordinate. This relation is given by $E = H^{-1/4}E(\rho)$. In this relation, $H^{-1/4}(\rho)$ is the redshift factor. Long wavelength excitations close to the horizon cannot get past the gravitational potential barrier. It has also been shown that the excitations at infinity can't get to the region close to the horizon due to the low absorption cross-section of the horizon[5]. This implies that the dynamics of the closed strings propagating in flat space at infinity decouples from the dynamics of the low energy dynamics of the strings near the horizon. This limit is equivalent to keeping the energy (in string units) measured by an observer at infinity fixed and taking the Regge slope α' to zero. The classical supergravity description of the near horizon geometry is good when $g_s \rightarrow 0, N \rightarrow \infty$.

Now let us examine the case when $\lambda \ll 1$. In this case, the backreaction is negligible and the D3-brane admits a description in terms of the worldvolume gauge theory. The world volume gauge theory is described some non-abelian generalization of the Born-Infled action. The precise form of this action is not required for the current discussion; we are just interested in the low energy dynamics. In the low energy limit (or $\alpha' \rightarrow 0$ keeping energy fixed), the world volume theory is described by $\mathcal{N} = 4$ $U(N)$ Yang-Mills theory. The $\mathcal{N} = 4$ SYM theory is a conformal field theory. The closed strings propagating in flat space decouple from the dynamics of the world volume theory in the limit where $\alpha' \rightarrow 0$ (keeping energy fixed).

We now have two two different descriptions of a stack of D3 branes. In both these descriptions the dynamics of closed strings in flat space decouple from the low energy descriptions. This observation led Maldacena [124] to conjecture that the string theory on $AdS_5 \times S^5$ provides a description of $\mathcal{N} = 4$ SYM theory at strong t'Hooft coupling .

In the large N limit, we have a duality between classical type IIB supergravity and the *large N limit of SU(N) $\mathcal{N}=4$ SYM theory!*

1.4 Holographic view of RG flows

We saw that the isometries of the bulk spacetime becomes the conformal generators of the field theory at the boundary. This fact suggests that the field theory can be thought of as living on the conformal boundary of the bulk spacetime, which is at $\rho = \infty$. Boundary value of the fields in the gravitational theory correspond to sources for gauge invariant operators in the field theory. Hence, small deformations of the dual field theory correspond to linear perturbations around the gravitational background. If the boundary theory is deformed by some operator $\mathcal{O}_\phi(\vec{x})$, this modifies the boundary conditions for the dual bulk field $\phi(x, \rho)$. The equation of motion for the fluctuations $\phi(x, \rho)$, is a linear wave equation. In order to study the response of the system to the small deformations, we evaluate correlation functions in the filed theory. For concreteness, let us evaluate the two-point correlation function $\langle \mathcal{O}_\phi(\vec{x}) \mathcal{O}_\phi(\vec{x'}) \rangle$. This correlation functions can be computed from the behavior of the fluctuations obtained by solving the wave equation, which has a unique solution for $\phi(x, \rho)$ that respects causality. The holographic dictionary provides an in-principle way of finding the generating functional for all the correlation functions using the gravity theory. The generating functional is related to the on-shell action of the bulk field ϕ. The correlation functions can be determined by taking derivatives of the generating functional with respect to the boundary value $\phi(\vec{x}, \infty)$. The on-shell action, and hence the generating functional can be computed from the boundary behavior of the field ϕ. In other words, the correlation function $\langle \mathcal{O}_\phi(\vec{x}) \mathcal{O}_\phi(\vec{x'}) \rangle$ can be computed from large ρ asymptotic behavior of ϕ at x and x'. When the points x and x' are separated by short distances, the asymptotics of ϕ at x and x' are determined mostly by the behavior of the solution near the boundary (at large ρ). When the separation is large, the asymptotic behavior of ϕ at x and x' probe regions far from the boundary. From this discussion, we can see that the short distance (high energy) physics in the field theory correspond to the *bulk physics* near the boundary, while the long distance (low energy) physics in the field theory correspond to bulk physics far away from the boundary. This allows us to interpret the radial coordinate ρ as the renormalization group scale in the field theory. We could have reached the same conclusion by analyzing the redshift factor.

There is a detailed recipe for computing all gauge invariant observables in the boundary field theory using classical computations in the gravity side [69, 168]. This recipe makes holography a powerful tool for studying strongly correlated systems.

1.5 Synopsis

Rest of thesis is organized as follows.

1. In the second chapter, we propose a generalization of the AdS/CFT correspondence to non-relativistic conformal field theories which are invariant under Galilean transformations. Such systems govern ultracold atoms at unitarity, nucleon scattering in some channels, and more generally, a family of universality classes of quantum critical behavior. We construct a family of metrics which realize these symmetries as isometries. They are solutions of gravity with negative cosmological constant coupled to pressureless dust. We discuss realizations of the dust, which include a bulk superconductor. We develop the holographic dictionary and find two-point correlators of the correct form. A strange aspect of the correspondence is that the bulk geometry has two extra dimensions.

2. We construct string theory duals of non-relativistic critical phenomena at finite temperature and density in the third chapter. Concretely, we find black hole solutions of type IIB supergravity whose asymptotic geometries realize the non-relativistic conformal group (also known as Schrödinger group)as isometries. We then identify the non-relativistic conformal field theories to which they are dual. We analyze the thermodynamics of these black holes, which turn out to describe the system at finite temperature and finite density. The strong-coupling result for the shear viscosity of the dual non-relativistic field theory saturates the KSS bound.

3. In the fourth chapter, we point out that the spectrum of the particle number operator in the earlier examples is not a necessary consequence of the existence of a gravity dual. We record some progress towards more realistic spectra. In particular, we construct bulk systems with asymptotic Schrödinger symmetry and only one extra dimension. In examples, we find solutions which describe these Schrödinger-symmetric systems at

finite density. A lift to M-theory is used to resolve a curvature singularity. As a happy byproduct of this analysis, we realize a state which could be called a holographic Mott insulator.

4. In the fifth chapter, We describe solutions of 10-dimensional supergravity comprising null deformations of $AdS_5 \times S^5$ with a scalar field, which have $z = 2$ Lifshitz symmetries. The bulk Lifshitz geometry in 3+1-dimensions arises by dimensional reduction of these solutions. The dual field theory in this case is a deformation of the $\mathcal{N}$=4 super Yang-Mills theory. We discuss the holographic 2-point function of operators dual to bulk scalars. We also discuss deformations of $AdS \times X$ in 11-dimensional supergravity, which are somewhat similar to the solutions above. In some cases here, we expect the field theory duals to be deformations of the Chern-Simons theories on M2-branes stacked at singularities.

5. In the sixth chapter, we propose candidate gravity duals for a class of non-Abelian $z = 2$ Lifshitz Chern-Simons (LCS) gauge theories studied by Mulligan, Kachru and Nayak. These are nonrelativistic gauge theories in 2+1 dimensions in which parity and time-reversal symmetries are explicitly broken by the presence of a Chern-Simons term. We show that these field theories can be realized as deformations of DLCQ $\mathcal{N} = 4$ super Yang-Mills theory. Using the holographic dictionary, we identify the bulk fields of type IIB supergravity that are dual to these deformations. The geometries describing the groundstates of the non-Abelian LCS gauge theories realized here exhibit a mass gap.

6. In the seventh chapter, we construct analytically a black hole which asymptotes to a vacuum Lifshitz solution. We study its thermodynamics and scalar response functions. The scalar wave equation turns out to be exactly solvable. Interestingly, the Green's functions do not exhibit the ultralocal behavior seen previously in the free Lifshitz scalar theory.

7. In the final chapter, we present some concluding remarks about the pros and cons of non-relativistic holography and the holographic principle in general.

Chapters 2,3 and 4 are edited and re-mixed versions of [16], [1] and [19]. Chapters 5, 6 and 7 are slightly modified versions of [20], [18] and [17].

Chapter 2

Gravity Duals of Non-relativistic Conformal Field Theories

A bulk of this chapter and the first item in the synopsis (Chapter 1) appeared in "Gravity duals for non-relativistic CFTs." with John McGreevy [16] and is reprinted with the permission of *Phys.Rev.Lett.* Copyright (2008) by The American Physical Society. Some portions of this chapter appeared in "Hot Spacetimes for Cold Atoms." with Allan Adams and John McGreevy [1] and is reprinted with the permission of *JHEP*.

2.1 Introduction

Many attempts have been made to use the AdS/CFT correspondence [124, 69, 168] to study systems realizable in a laboratory. One does not yet have a holographic dual matching the precise microscopic details of any such system and is therefore led to try to match the universality class of the system. In general, physics in the far infrared is described by a (sometimes trivial) fixed point of the renormalization group. It has been argued that the associated zero-temperature conformal field theory (CFT) controls a swath of the finite-temperature phase diagram, namely the region in which the temperature is the only important scale, and no dimensionful couplings have turned on (see *e.g.* [153]). In trying to use gauge/string duality to study such questions, it seems urgent, then, to match the symmetries of that theory to those of the sytem of interest.

The AdS/CFT correspondence so far gives an effective description of relativistic conformal field theories at strong coupling. Not many of these are accessible experimentally.

However, there are many *non-relativistic* conformal field theories which govern physical systems. Such examples arise in condensed matter physics [153], atomic physics [139, 147, 24, 173, 108], and nuclear physics [130]. In the first situation, these are called 'quantum critical points'; this term refers to a second-order phase transition that is reached not by varying the temperature, but by varying some coupling constants at zero temperature.

A particularly interesting example where exquisite experimental control is possible is the case of cold fermionic atoms at unitarity (*e.g.* [139] and references therein and thereto). These are scale invariant in the following sense. The interactions are tuned (by manipulating the energy of a two-body boundstate to threshold) to make the scattering length infinite (hence the system is strongly coupled). The material is dilute enough that the effective range of the potential can be treated as zero. While the density of material is a dimensionful quantity, the dependence of the energy and other physical quantities on the density is determined by the symmetry algebra. A recent paper studying the constraints from Galilean conformal symmetry on such systems is [137].

In this chapter, we will describe a bulk dual of non-relativistic CFTs, analogous to the AdS gravity description of relativistic CFTs, at strong coupling. We approach this question by considering the algebra of generators of the non-relativistic conformal group, which appears in [137] (related work includes [104, 77, 136]). We seek a solution of string theory (in its low-energy gravity approximation) whose asymptotic boundary symmetry group is not the Poincaré symmetry group (*i.e.* Lorentz and translations), but rather Galilean invariance and translations. We also demand a non-relativistic version of scale invariance and special conformal transformations.

Clearly we will have to introduce some background energy density to find such a solution (*i.e.* it will not be just a solution of gravity with a cosmological constant). Holographically, this is because the theories we are trying to describe arise in general (perhaps not always) from relativistic microscopic theories (not necessarily conformal) upon by adding in some background of Stuff (which fixes a preferred reference frame) and then taking some limit focussing on excitations with small velocities in this frame.

An analogy which may be useful is the following. The plane-wave symmetry group is a contraction of the AdS isometry group; the plane wave solution arises from AdS in considering a limit focussing on the worldline of a fast-moving particle. So it might be possible to take some limit of AdS analogous to the plane wave limit to find our solution. The limit involved would be like the Penrose limit, but replacing the single particle worldline with a uniform background number density. We leave such a derivation from a more microscopic system for the future and proceed to guess the end result.

Scale invariance can be realized in non-relativistic theories in a number of ways. One freedom is the relative scale dimension of time and space, called the 'dynamical exponent' z (see *e.g.* [91]). The familiar Schrödinger case has $z = 2$, and the conformal algebra for this case is described in [137]. Fixed points with $z \neq 2$ can be Galilean invariant, too.

Non-relativistic systems which enjoy conformal invariance in d spatial dimension are governed by a symmetry algebra known as the d-dimensional Schrödinger algebra. To be specific, the relevant algebra is generated by angular momenta M_{ij}, momentum P_i, Galilean boosts K_i, dilatations D and rest mass N. Here $i, j = 1..d$ label the spatial dimensions. The commutators which are neither zero nor determined by spin are:

$$
\begin{aligned}
&[K_i,\, P_j] = i\delta_{ij}N,\; [D,\, P_i] = iP_i,\; [D,\, K_i] = (1-z)iK_i \\
&[H,\, K_i] = -iP_i, \quad [D,\, H] = ziH.
\end{aligned}
\tag{2.1.1}
$$

where z, the "dynamical exponent", determines the relative scaling between the time-coordinate and the spatial coordinates, $[t]$=length^z. In the special case $z = 2$, the algebra may be extended by an additional "special conformal" generator, C, whose non-trivial commutation relations are

$$
[D, C] = -2iC \quad [H, C] = -iD.
$$

In this case $z = 2$, both D and N may be diagonalized, so representations of the Schrödinger algebra are in general labeled by two numbers, a dimension Δ and a "number" ℓ. For fermions at unitarity, this number is precisely the fermion number.

Motivated by the relativistic AdS/CFT correspondence, it is natural to wonder whether there exists gravitational duals for non-relativistic CFTs. By analogy to the relativitstic case, we expect such a gravitaional description to realize the symmetry group of the CFT as the isometry group of the spacetime. However, since there are now *two* symmetry generators which may be diagonalized and whose eigenvalues label inequivalent representations (in the AdS case, there is only one, the dimension), we may expect any spacetime which has the Schrödinger algebra linearly realized as its isometry group to be two dimensions higher than the CFT, as opposed to one-dimension higher as in the case of AdS.

In the next section we will explicitly construct such geometries. More precisely, we will construct a $d+3$-dimensional metric realizing the d-spatial-dimensional Schrödinger group as its isometry group, and such a system is holographically dual to non-relativistic CFTs at zero temperature and zero density. We will refer to these metrics as Sch^z_{d+3}, where z labels the dynamical exponent and d the number of spatial dimensions (note that d+3 = (d+1)+2); in the special case $z = 1$, $Sch^1_{d+3} = AdS_{d+3}$.

Importantly, this metric is not a vacuum solution of the Einstein equations. As a result, it is necessary to couple the system to additional background field strengths (a pressureless dust and a negative cosmological constant) whose stress tensors cancel the non-zero Einstein tensor of the spacetime metric[1]. As we shall see in the next chapter, this system – a metric with Schrödinger isometries supported by background field strengths for massive tensor fields – has a natural embedding into string theory.

We provide some basic checks for our conjecture by a comparison of Green functions for scalar operators as computed in the NRCFT and gravity. The n-point Green's function of an operator $\mathcal{O}$ in a NRCFT, is determined by the scaling dimension $\Delta_{\mathcal{O}}$ and the particle number $N_{\mathcal{O}}$ [137]. Indeed, as shown by Nishida and Son [137], this is the case in any nonrelativistic CFT, since that's what's required to specify a representation of the $z = 2$ Schrödinger algebra. We will show that the two-point Green's function calculated using the gravity theory has the same form as that of [137].

A basic entry in the gauge/gravity dictionary is the following. The global currents of the boundary gauge theory (such as the R-currents) couple to gauge fields in the bulk. In the simplest example of the $\mathcal{N} = 4$ theory, the R-currents are coupled to the Kaluza-Klein vector fields in AdS_5 associated with the isometries of the 5-sphere. So turning on a chemical potential for R-charge in some $U(1)$ Cartan direction inside $SO(6)$ means in the bulk AdS_5 turning on some background gauge potential A_0. Our solutions can be sourced by

[1]The recent papers [66, 21] find a solution of the vacuum einstein equations with only a cosmological constant – this is just the DLCQ of AdS, with the periodic identification breaking the AdS symmetry group to its Schrödinger subgroup. As we will discuss in considerably more detail in the next chapter, this corresponds to a degenerate limit of the backgrounds considered in [16, 160]

a bulk electric field, which should correspond to the conserved number density of particles comprising the Stuff. That the particle number can be conserved is a consequence of the possibility of absence of antiparticles in a non-relativistic theory [2].

In the next section we write down a metric with this isometry group [3]. It is sourced by a stress tensor describing a negative cosmological constant plus dust. After demonstrating the action of the isometry group, we provide answers to the important question: why doesn't the dust collapse due to the gravitational attraction? In section three, we generalize the AdS/CFT dictionary to extract information about the field theory dual, including anomalous dimensions and Green's functions. In section four, we highlight some of the many open questions raised by this construction.

2.2 Geometry

The metric we will study is

$$ds^2 = -\frac{2\beta^2 dt^2}{r^{2z}} + \frac{d\vec{x}^2 + 2d\xi dt}{r^2} + \frac{dr^2}{r^2}. \tag{2.2.1}$$

Here $\vec{x}$ is a d-vector, and z is the advertised dynamical exponent. The coordinate ξ is something new whose interpretation we will develop below[4]. The metric (2.2.1) is nonsingular. As is manifest in the chosen coordinates, it is conformal to a pp-wave spacetime. When $\beta = 0$, the metric in 2.2.1 is just AdS in lightcone coordinates. When one of the null directions is compactified, the isometriy group of AdS is broken down to Schrodinger group [66, 21]. However, the causal structure of AdS in lightcone coordinates is different from the causal structure of Galilean invariant theories which is naturally encoded when β is non-zero. For the rest of this chapter, we set β^2 to half. For this value of β^2 the metric in 2.2.1 is

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + 2d\xi dt}{r^2} + \frac{dr^2}{r^2}.$$

[2]That is to say, the dual field theories that we are describing must not have particle production. It is of course possible for non-relativistic theories to nevertheless contain both particles and antiparticles, as indeed do the systems under study in [106], which, unlike ours, are invariant under $t \to -t$.

[3]Earlier work on geometric realizations of the Schrödinger group include [54] and references therein, wherein the metrics we study are called Bargmann-Einstein structures.

[4]In the case $d = 2, z = 3$, this metric describes the near-horizon geometry of D3-branes in the presence of lightlike tensor fields [8]; its causal structure was studied in [99]. The sources present in those solutions did not preserve the rotations M_{ij}.

We shall show that the isometries of the above metric comprise the $d+1$-dimensional non-relativistic conformal group with algebra 2.1.1. It is clear that the above metric is invariant under translations in $\vec{x}, t$ and under rotations of $\vec{x}$. It is also invariant under the scale transformation

$$x' = \lambda x, \quad t' = \lambda^z t, \quad r' = \lambda r, \quad \xi' = \lambda^{2-z}\xi \quad .$$

It is invariant under the Galilean boost $\vec{x}' = \vec{x} - \vec{v}t$ if we assign the following transformation to the coordinate ξ: $\xi' = \xi + \frac{1}{2}(2\vec{v}\cdot\vec{x} - v^2 t)$. In the special case $z = 2$, special conformal transformations act as

$$x' = \frac{x}{1+ct}, \; t' = \frac{t}{1+ct}, \; r' = \frac{r}{1+ct}, \; \xi' = \xi + \frac{c}{2}(\vec{x}.\vec{x} + r^2).$$

One can check that the vector fields generating these isometries have Lie brackets satisfying the algebra (2.1.1) [5]. The only non-obvious identification is that the rest mass is identified with $N = \partial_\xi$. Perhaps not surprisingly, the spectrum of the number operator in the theories dual to geometries of the form (2.2.1) is the set of integers, since it arises from the tower of Kaluza-Klein momenta in the ξ direction. The name for the ξ-momentum conservation law in the n-point functions

$$l_1 + l_2 + ... = l_1' + l_2' + ...$$

is "Bargmann's superselection rule on the mass" [78]. The stress tensor is an operator which commutes with the particle number operator (this is a consequence of the Schrödinger algebra). The fluctuations of the bulk metric dual to the stress tensor therefore have zero ξ-momentum.

Time reversal is an antiunitary operation, which means that it complex conjugates the wavefunctions. In particular, say in a weakly coupled theory with field operator Ψ, it acts on an operator $\mathcal{O}_l$ by

$$T : \mathcal{O}_l = \Psi^l ... \Psi^{\dagger\; l-k} \mapsto \mathcal{O}_{-l}.$$

Note that although the metric is not invariant under simple time reversal, $t \to -t$, it is invariant under the combined operation $t \to -t, \xi \to -\xi$, which, given the interpretation of

[5] We believe that no $(d+2)$-dimensional metric can realize this isometry group.

the ξ-momentum as rest mass, can be interpreted as the compositionof charge conjugation and time-reversal.

The fact that the number operator in a non-relativistic conformal field theory is gapped (one Li atom, two Li atoms, three...) tells us that ξ must be periodic. But ξ is a null direction in the bulk geometry. As such, we appear to be forced into a discrete light cone quantization (DLCQ). This will be made more precise in the next chapter.

At first sight, compactifying ξ may look problematic. For example, this may appear to violate boost invariance. However, boost invariance remains unbroken precisely because the ξ direction is null; this follows from the commutator $[\hat{N}, \hat{K}_i] = 0$ in the Schrödinger algebra. Perhaps more troublingly, compactifying ξ would appear to introduce a null conical singularity at $r \to \infty$, which suggests that our metric should not be reliable in the strict IR. However, this singularity is unphysical. As we shall seein the next chapter, the singularity goes away as soon as we turn on any finite temperature – the would-be null singularity is lost behind a finite horizon which shrouds a garden-variety schwarzschild singularity. Meanwhile, physically, we always have some finite T in a realistic cold-atom system, and thus a natural IR regulator. Finally, and most sharply, even in the strict $T \to 0$ case, the dynamics will resolve this "singularity" in a fashion familiar from the study of null orbifolds of flat space [116, 120, 55, 95]: a pulse of stress-energy sent towards large r is steadily blue-shifted until its back-reaction is no longer negligible; analysis of the back-reaction then shows that the would-be null-singularity turns over into a spacelike singularity shrouded behind a (microscopic) horizon. All of which is to say, the strict $T \to 0$ limit of our NRCFT is unstable to thermalization upon the introduction of any energy, no matter how small. The challenge, in both the NRCFT and the dual spacetime, is not to crank up a finite temperature, but to drive the temperature low.

The stress tensor sourcing the metric in equation (2.2.1) consists of a negative cosmological constant [6] $\Lambda = -10$ plus a pressureless dust, with constant density $\mathcal{E}$:

$$T_{ab} = \Lambda g_{ab} + \mathcal{E}\delta_a^0\delta_b^0 g_{00}.$$

As mentioned in the introduction, it is natural to include a gauge field in the bulk. In a non-relativistic theory, the particle number can represent a conserved charge of the boundary

[6] We have set $8\pi G_N = 1$. The indices a, b run over all $d+3$ dimensions of the bulk.

theory, which will correspond to a bulk gauge field. The fact that we want to consider a finite density of matter suggests that we should consider configurations of this gauge field in the presence of nonzero charge density.

We observe that because of the non-stationary form of the metric, the stress tensor for an electric field in the radial direction F_{rt} has only a 00 component. Maxwell's equation in the bulk, coupled to a background current j^b is $\frac{1}{\sqrt{g}}\partial_a\left(\sqrt{g}F^{ab}\right) = j^b$. To produce an electric field in the r direction, which gives the correct behavior of T_{00} we consider a nonzero j^ξ of the form $j^\xi = \rho_0 r^{-z+2}$, which by Gauss' law generates an electrostatic potential $A_0 = \frac{\rho_0}{z(z+d)}r^{-z}$. The current and gauge field are therefore related by the London equation $j_a = m_A^2 A_a$, with $m_A^2 = z(z+d)$. [7].

Having gained this insight, we observe that our metric is sourced by the ground state of an Abelian Higgs model in its broken phase. The model

$$S = \int d^{d+3}x\sqrt{g}\left(F^2 + |D\Phi|^2 - V\left(|\Phi|^2\right)\right)$$

with $D_a\Phi \equiv (\partial_a + ieA_a)\Phi$, produces the correct dust stress tensor with a Mexican-hat potential

$$V\left(|\Phi|^2\right) = g\left(|\Phi|^2 - v^2\right)^2 + \Lambda$$

for arbitrary g, as long as $e^2v^2 = m_A^2 = z(z+d)$ as above [8]. It is tempting to relate the parameter φ to the phase of the boundary wavefunction. We hope in the future to use this theory with finite g to describe the vortex lattice formed by rotating the cold-atom superfluids.

2.3 Correspondence

Consider a scalar field in the background (2.2.1), with action $S = \frac{1}{2}\int d^{d+3}x\sqrt{g}\left(\partial_\mu\phi\partial_\nu\phi g^{\mu\nu} - m^2\phi^2\right)$. As in AdS, the transverse fluctuations of the metric $h_{xy}(k_z)$ satisfy the same equation of motion as a massless scalar, and so we can take the $m = 0$ case as a proxy for this component of the metric fluctuations.

[7]Recent holographic studies of spontaneously broken global symmetries of the boundary theory (at finite temperature) include [72].

[8]Our metric is therefore a solution of the same system studied in the first and last references in [72], with very different asymptotics.

Translation invariance allows us to Fourier decompose in the directions other than r, $\phi(r) = e^{i\omega t+i\vec{k}\cdot x+il\xi}f_{\omega,\vec{k},l}(r)$. Then the wave equation in the background (2.2.1) takes the form:

$$\begin{aligned}&\left(-r^{d+3}\partial_r\left(\tfrac{1}{r^{d+1}}\partial_r\right)+r^2(2l\omega+\vec{k}^2)\right.\\&\left.+r^{4-2z}l^2+m^2\right)f_{\omega,\vec{k},l}(r)=0.\end{aligned}\tag{2.3.1}$$

To find the generalization of the gauge/gravity dictionary to this case, we first study asymptotic solutions of this equation near the boundary. Writing $f \propto r^{\Delta}$, we find for $z \leq 2$:

$$\Delta_{\pm} = 1+\frac{d}{2}\pm\sqrt{\left(1+\frac{d}{2}\right)^2+m^2+\delta_{z,2}l^2}\ .\tag{2.3.2}$$

In the case $z > 2$, there is no scaling solution near the boundary; the ξ-momentum dominates the boundary behavior, and the solutions vary with r faster than any power of r. It would be interesting to understand a possible meaning of the asymptotic solutions which do arise, and to interpret the critical nature of $z = 2$ in terms of the boundary theory. As usual, these exponents $\Delta_{\pm}$ can be interpreted as scaling dimensions of spin-zero boundary operators in the boundary theory. We look forward to extending this analysis to the fluctuations of fields of other spin.

For definiteness, we focus on the familiar, critical $z = 2$ case with $d = 3$ spatial dimensions. In that case, the momentum in the ξ-direction simply adds to the mass of the bulk scalar, *i.e.* they appear in the combination $l^2 + m^2$. In this case the wave equation is solved by

$$f_{\omega,\vec{k},l}(r) = Ar^{5/2}K_{\nu}(\kappa r)$$

where K is a modified Bessel function, A is a normalization constant, and [9]

$$\nu = \sqrt{\left(\frac{5}{2}\right)^2+l^2+m^2},\quad \kappa^2 = 2l\omega+\vec{k}^2.$$

As usual in AdS/CFT, we choose K over I because it is well-behaved near the horizon,

[9] For general d, with $z \leq 2$, the solution is $f_{\omega,\vec{k},l}(r) = Ar^{1+\frac{d}{2}}K_{\nu}(\kappa r)$ with $\nu = \sqrt{(1+\frac{d}{2})^2+M^2}$, $\kappa^2 = 2l\omega+\vec{k}^2$, $M^2 \equiv m^2+\delta_{z,2}l^2$.

$K(x) \sim e^{-x}$ at large x.

To compute correlators, we introduce a cutoff near the boundary at $r = \epsilon$, and normalize $f_\kappa(\epsilon) = 1$ (so $A = \epsilon^{-5/2} K_\nu(\kappa\epsilon)^{-1}$). The on-shell bulk action is

$$S[\phi_0] = \left[\int d^{d+2}X \, \phi(X) \partial_r \phi(X) \right]_{r=\epsilon}$$

where $X \in \{\vec{x}, t, \xi\}$ are the coordinates on the boundary. This evaluates to

$$S[\phi_0] = \frac{1}{2} \int dp \phi_0(-p) \mathcal{F}(\kappa, \epsilon) \phi_0(p)$$

with the 'flux factor'

$$\mathcal{F}(\kappa, \epsilon) = \lim_{r \to \epsilon} \sqrt{g} g^{rr} f_\kappa(r) \partial_r f_\kappa(r) = \sqrt{g} g^{rr} \partial_r \ln K_\nu(\kappa r)|_{r=\epsilon}$$

Expanding near the boundary and keeping the leading singular term, this gives

$$\langle \mathcal{O}_1(\omega, \vec{k}) \mathcal{O}_2(\omega', \vec{k}') \rangle \propto \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \delta(k + k') \left(\frac{(2l\omega + \vec{k}^2)\epsilon^2}{4} \right)^\nu$$

(in the case $\phi = h_{xy}$, $\mathcal{O} = T_{xy}$). Fourier transforming to position space, this is

$$\langle \mathcal{O}_1(x, t) \mathcal{O}_2(0, 0) \rangle \propto \frac{\Gamma(1-\nu)}{\Gamma(\nu)} \delta_{\Delta_1, \Delta_2} \theta(t) \frac{1}{|t|^\Delta} e^{-lx^2/2|t|}$$

where we used translation invariance to put the second operator at $(\vec{x}, t) = (\vec{0}, 0)$, and $\Delta_{i,j}$ are the scaling dimensions of the operators that we found in (2.3.2). This is the form one expects for the correlators in a Galilean-invariant conformal field theory in d spatial dimensions with dynamical exponent 2.

2.4 Discussion

In discussions of AdS/CFT, one often hears that the CFT 'lives at the boundary' of the bulk spacetime. The spacetime (2.2.1) is conformal to a pp-wave spacetime, and hence has a boundary which is one dimensional – for $z > 1$, g_{tt} grows faster than the other components at small r. While one might be tempted to speculate that this means that the dual field theory is one dimensional, there is no clear evidence for this statement (in the usual case, only the UV of the field theory can really be said to live at the boundary of the AdS space), and indeed we have nevertheless used it to compute correlators which look like those of a d-dimensional nonrelativistic field theory.

Perhaps the most mysterious aspect of this new correspondence is not of too few dimensions, but of too many. The ξ direction is an additional dimension of the bulk geometry besides the usual radial direction, which seems to be associated to the conserved rest mass. We will discuss an approach to find a holographic description without this additional direction in chapter 4. We have computed correlators at fixed l; this is reasonable since the theory is non-relativistic and so a sector with definite l can be closed (unlike in a relativistic theory where these sectors unavoidably mix by production of back-to-back particles and antiparticles). The relation to the plane wave appears here, too: the plane wave describes a relativistic system in the infinite momentum frame, where there is also no particle production.

We would like to think of these solutions as scaling limits which zoom in on small fluctuations about particular states of relativistic theories (possibly CFTs), where a preferred rest frame is fixed by some density of Stuff. The non-stationary ($d\xi dt$) form of our metric suggests that it may arise as a limit of a higher-dimensional rotating spacetime. That the gauge field in our background could arise as the corresponding KK gauge field would not be surprising. It was shown in [102] that the Schrödinger metric can indeed arise as such a limit of an asymptotically AdS solution.

Chapter 3

Hot spacetime for cold atoms

The material in this chapter and the second item in the synopsis (Chapter 1) appeared in "Hot Spacetimes for Cold Atoms." with Allan Adams and John McGreevy [1] and is reprinted with the permission of *JHEP*.

3.1 Introduction

The hydrodynamics of cold atoms with interactions at the unitarity limit (for reviews, see [33]) is a subject crying out for an effective strong-coupling description. Experimentally, these systems are under extensive and detailed study, and rich data exist. For example, the shear viscosity has been extracted [157] from energy-loss during sloshing experiments [108], leading to an estimate for the ratio $\frac{\eta}{s}$ which approaches the bound conjectured by [113]. While this bound is universal in *relativistic* systems with classical gravity duals [35, 113], the system of cold fermionic atoms at unitarity is most certainly *not* a relativistic one, though it does have non-relativistic conformal symmetry (for a systematic discussion, see [137]). One could imagine that the nonrelativistic nature of the system has an important effect on this ratio. Indeed, the counterexamples to the $\frac{\eta}{s}$ bound proposed in [41, 159, 38, 42] arise in nonrelativistic systems. Theoretically, however, these systems are hard: perturbative techniques are inadequate, and lattice methods are difficult to apply to dynamical questions (though see [131, 132, 133]).

In the relativistic context, an effective strong-coupling description of a CFT can sometimes be found in terms of a gravitational theory in extra-dimensional spacetimes [124, 169] in which the conformal symmetry of the CFT arise as the isometries of the geometry. Via the Hawking phenomenon, the thermal ensemble of such systems is constructed by placing a black hole in the extra-dimensional geometry [169]. The rigid structure of black hole spacetimes, when combined with the finite-temperature gauge/gravity duality, has led to the observation of universal behavior of these strongly coupled gauge theories at finite temperature. For a nice review of this work, see [163].

In the non-relativistic case, a natural guess for a strong-coupling description is a dual geometry whose isometries reproduce the symmetries of the non-relativistic CFT (NRCFT). Such geometries were constructed in the previous chapter [160, 16], with the metric taking the form in 2.2.1[1] The dynamical exponent takes the value $z = 2$ for cold fermions at unitarity. We showed that this spacetime solve the equations of motion of Einstein gravity coupled to a gauge field of mass $m_A^2 = \frac{z(z+d)}{L^2}$ and a cosmological constant $\Lambda = \frac{(d+1)(d+2)}{L^2}$, and was argued to be dual to an NRCFT at zero temperature and zero density. To study via duality an NRCFT at finite temperature and density, then, we need to put a black hole inside this geometry.

In this chapter we will construct black holes with the asymptotics of (2.2.1), show that they arise as solutions of string theory, and identify the specific non-relativistic conformal field theories to which they are dual. An analysis of the thermodynamics of these black hole spacetimes shows that they describe the dual non-relativistic CFTs at finite temperature and finite density, with the previously-studied geometry of [16, 160] arising in the zero temperature, zero density limit. Along the way we identify a scaling limit which describes the system at zero temperature but nonzero density. To produce these solutions, we will use a solution-generating technique called the Null Melvin Twist [25, 6, 65, 100, 101], which we will review in detail below. This Melvinizing procedure is the sought-for analog of the plane wave limit described in the introduction and conclusion of [16].

We should emphasize at this point that the NRCFT describing Lithium atoms tuned to a Feschbach resonance probably does not literally have a weakly-coupled gravity dual. However, the Lithium system has closely related cousins which do have 't Hooft limits – indeed, the NRCFTs dual to our black hole spacetimes are precisely such creatures. Our hope is that these ideas will be useful for studying strongly-coupled cold atoms in at least

[1]For previous appearances of related spacetimes in the pre-gauge/gravity-duality literature, see [53, 54]. For studies of the supersymmetrization of the Schrödinger group please see [154]. See also [166].

the same sense in which the $\mathcal{N} = 4$ theory has been useful for studying universal properties of strongly-coupled relativistic liquids, including those made out of QCD.

This chapter is organized as follows. We will present a string embedding for Schrödinger spacetime in section 2. The solutions are constructed using the Null Melvin Twist, a machine which eats supergravity solutions and produces new ones. The machine has several dials, which we will tune to various ends. The input solution that produces the metric (2.2.1) is the extremal D3-brane in type IIB. When (in section 2.3) we feed to the Melvinizing machine the near-extremal D3-brane, we find that it produces black brane solutions which asymptote to the spacetimes (2.2.1). We then provide some rudimentary understanding of the identity of the theory at weak coupling in this realization. In section four we analyze the thermodynamics. In section five we compute the shear viscosity, and show that the strong-coupling universality found by [35, 113] extends beyond the class of relativistic systems. We conclude with a discussion of interesting open questions. The appendices contain the details of the Melvinization process, an argument for frame-independence of the viscosity calculation, and some progress towards a 5d effective action.

3.2 Embedding in String Theory

In this section we will show that the zero-temperature solutions described above have a natural embedding into solutions of Type II string theory. For simplicity, we will mostly focus on the case $z = 2$, $d = 2$, though many other cases may be equally straightforward. These solutions may be generated in a number of equivalent ways. One useful technique is the "Null Melvin Twist," (which was named in [65]) which will be described in detail momentarily. We will use this technique to construct solutions which embed the Sch_5^2 geometry discussed above into string theory, as well as solutions which describe the system at finite temperature and chemical potential. A second useful technique, which is in fact completely equivalent for the backgrounds we consider, is a simple modification of Discrete Light Cone Quantization; this presentation will make the structure of the dual field theory transparent. Let's begin with the twist.

3.2.1 The Null Melvin Twist

The Null Melvin Twist is a solution generating technique for IIB supergravity which eats known solutions and spits out new solutions with inequivalent asymptotics. Melvinization has largely been used to construct gravity duals of non-commutatitve and non-local field theories [25, 6, 65, 100, 101]; in the case at hand, the Null twisting produces an extremely mild form of non-locality, that of non-relativistic theories with instantaneous interactions. Importantly, these backgrounds have the property that all curvature scalars are identical to those of the original solution [6]; as a result, the constraints on when the supergravity is reliable (*e.g.* $\lambda \ll 1$) carry over directly. This will also be clear from our analysis of the boundary field theory below.

Interestingly, some solutions with Sch_5^z asymptotics have already appeared in the Melvinizing literature. For example, the $T = 0$ limit of one of the solutions in [6, 100] corresponds to $Sch_5^{z=3}$, though the form field backgrounds break part of the symmetry group. These were described as dual to "dipole theories" with a non-trivial star-algebra in the dual field theories. Here we will argue that these and some other backgrounds generated by the Null Melvin twist are in fact dual to NRCFTs.

The procedure itself is baroque but elementary. The first step is to choose a IIB background with two marked isometries, which we will call dy and $d\phi$. We then (1) boost along dy with boost parameter γ, which generates (in general) a new $dydt$ term in the metric, (2) T-dualize a la Buscher[2] along dy, which generates a new $dy \wedge dt$ term in the NS-NS B-field and a non-trivial dilaton profile, $g_{yy} \to \frac{1}{g_{yy}}$, $B_{ty} \to \frac{g_{ty}}{g_{yy}}$ and $\Phi \to \Phi - \frac{1}{2}\ln g_{yy}$, (3) re-diagonalize our isometry generators by shifting $d\phi \to d\phi + \alpha dy$, which generates a new term in the metric of the form $ds^2 \to ... + (d\phi + \alpha dy)^2$, then return to our original frame by (4) T-dualizing back along dy, which generates $dydt$ terms in the metric and $d\phi \wedge dy$ terms in B, and (5) boosting back along dy with boost parameter $-\gamma$.

All of this leaves us back in the original frame with a new metric, B-field and non-trivial dilaton, all of which are horrendously complicated functions of the two knobs, γ and α. The final step of the Null twist is to (6) simplify this morass by taking a scaling limit in which the boost becomes infinite, $\gamma \to \infty$, and the twisting goes to zero, $\alpha \to 0$, such that the product $\frac{1}{2}\alpha e^{\gamma} = \beta$ remains finite. The result is a new solution of the full IIB equations of

[2]The full Buscher rules, and our conventions for them, are given in an appendix.

motion with non-trivial background NS-NS 2-form and deformed metric with asymptotics inequivalent to the original solution.

3.2.2 Rampaging Melvin Eats Extremal D3-brane, Spits Out Sch_5^2

Let's apply this procedure to our canonical example, the extremal D3-brane, a solution of IIB supergravity with metric,

$$ds^2 = \frac{1}{h}\left(-d\tau^2 + d\vec{x}^2\right) + h\left(d\rho^2 + \rho^2 ds_{S^5}^2\right)$$

where $h^2 = 1 + \frac{R_A^4}{\rho^4}$ is the usual D3 harmonic function, and self-dual five-form flux

$$F^{(5)} = \frac{1}{r^5} d\tau \wedge dy \wedge dx_1 \wedge dx_2 \wedge dr + \Omega_5\, d\theta \wedge d\phi \wedge d\psi \wedge d\mu \wedge d\chi,$$

with $\Omega_5 = \frac{1}{8}\cos\theta\cos\mu\sin^3\mu$. We must first choose the two isometry directions, dy and $d\phi$, along which to Melvinize. Let's take dy to lie along the worldvolume and $d\phi$ along the S^5; without loss of generality, we can choose coordinates such that $y = x_3$. A particularly convenient (though by no means the only possible) choice for $d\phi$ is given by the Hopf fibration $S^1 \to S^5 \to \mathbb{P}^2$ with metric,

$$ds_{S^5}^2 = ds_{\mathbb{P}^2}^2 + (d\chi + \mathcal{A})^2$$

where χ is the local coordinate on the Hopf fibre and $\mathcal{A}$ is the 1-form potential[3] for the kahler form on $\mathbb{P}^2$, ie $J_{\mathbb{P}^2} = d\mathcal{A}$. We thus take $d\phi = d\chi$. Note that both dy and $d\phi$ act freely, which is important for our solution to remain non-singular.

Melvin being a very messy eater, we will hide the details of the procedure in the Appendix

[3]We can compute $\mathcal{A}$ from the Kähler potential on $\mathbb{P}^2$, $K = t\ln\sum|z_i|^2$, ie

$$\mathcal{A}_i = ir\frac{\bar{z^i}}{\sum|z^i|^2}.$$

For completeness, and because we had a pointlessly slow search for this data in the literature, we present an explicit set of conventions and coordinate systems in an appendix.

and simply write down the result, which is:

$$
\begin{aligned}
ds^2 &= \frac{1}{h}\left(-d\tau^2(1+\beta^2\rho^2)+dy^2(1-\beta^2\rho^2)+2d\tau dy(\beta^2\rho^2)\right)+h\rho^2(d\chi+\mathcal{A})^2 \\
B &= 2\beta\rho^2(d\chi+\mathcal{A})\wedge(d\tau+dy) \\
\Phi &= \Phi_0
\end{aligned}
$$

Note that nothing has happend to the five-form along the way, since T-duality takes $d\Omega^5$, the top form on the sphere, to $dy\wedge d\Omega^5$, so that the twist $d\phi \to d\phi+\beta dy$ of step (3) acts trivially. We thus have in our final solution the same five-form flux as in the beginning,

$$
F_5 = (1+*)\Omega_5 d\theta\wedge d\phi\wedge d\psi\wedge d\mu\wedge d\chi
$$

To locate the Sch_5^2 hiding inside this solution, a few more steps are helpful. Changing coordinates to $t=(y+\tau)/\sqrt{2}$ and $\xi=(y-\tau)/\sqrt{2}$, our background becomes,

$$
\begin{aligned}
ds^2 &= \frac{1}{h}\left(\beta^2\rho^2\,dt^2+2dtd\xi\right)+h\rho^2(d\chi+\mathcal{A})^2 \\
B &= 2\beta\rho^2(d\chi+\mathcal{A})\wedge dt \qquad \Phi=\Phi_0
\end{aligned}
$$

Adding back in all the terms we dropped in the first step then gives,

$$
\begin{aligned}
ds^2 &= \frac{1}{h}\left(-\beta^2\rho^2\,dt^2+2dtd\xi+d\vec{x}^2\right)+h\left(d\rho^2+\rho^2 ds^2_{S^5}\right) \\
B &= 2\beta\rho^2(d\chi+\mathcal{A})\wedge dt \qquad \Phi=\Phi_0
\end{aligned}
$$

Finally, taking the near-horizon limit, $h\to R_A^2/\rho^2$ and switching to the global radial coordinate $r=R_A^2/\rho$, in terms of which $h=\frac{r^2}{R_A^2}$, the solution becomes,

$$
\begin{aligned}
ds^2 &= \frac{R_A^2}{r^2}\left[-\frac{2\Delta^2}{r^2}dt^2+2dtd\xi+d\vec{x}^2+dr^2\right]+R_A^2 ds^2_{S^5} \\
B &= 2\sqrt{2}\,\Delta\frac{R_A^2}{r^2}(d\chi+\mathcal{A})\wedge dt \qquad \Phi=\Phi_0.
\end{aligned}
$$

Upon compactifying on the S^5, we precisely recover the Schrödinger geometry with $d=2$ and $z=2$, our sought-after Sch_5^2.

Note, too, the appearance of the parameter β, which was implicitly set to $1/\sqrt{2}$ in the earlier results of [160, 16], by a choice of units. The utility of this parameter is considerable in what follows. For now, note that retaining it allows a very revealing limit, *i.e.* $\beta \to 0$, in which the solution above reduces to the extremal D3-brane solution with which we began. This suggests that there should be a more intrinsic description of our solution as a garden-variety deformation of AdS; we will explore this relation later in this section.

A note on dimensions. As discussed in Section 2, ensuring the quantization of the spectrum of the Schrödinger number operator $\hat{N}$ requires compactifying the direction ξ, something not implemented in the Null Melvin Twist described above; this introduces a new dimensionful parameter to the game, the length scale L_ξ. Meanwhile, ϕ is an angular direction along a compact space and so $d\chi$ is dimensionless, which means α, and thus β, must have dimensions of 1/length. Our solutions would thus appear to have two dimensionful parameters, L_ξ and β. In the case above, however, the specific values of L_ξ and β can be rescaled by rescaling the coordinates as

$$t \to \beta t, \quad \xi \to \beta^{-1}\xi,$$

leaving only the product $\frac{\beta}{L_\xi}$ invariant. It is this ability to scale away[4] β which allowed [160, 16] to set $\beta = 1/\sqrt{2}$. Our extremal solutions are thus parameterized by a single dimensionless parameter, $\frac{\beta}{L_\xi}$. Holding L_ξ fixed while scaling β to zero gives a particularly trivial background which respects the Schrödinger group as its isometries; we shall return to this example below.

Relatedly, in the above we have set c to 1; it is easy to reintroduce c by taking $dt \to c\,dt$. Interestingly, the way we are getting a non-relativistic limit is *not* by taking $c \to \infty$; rather, the asymptotic geometry has an effective $c_{\text{eff}} \sim \frac{c\beta}{r^2}$ which goes to ∞ as we approach the boundary. In addition to L_ξ and β, then, our non-relativistic theory thus contains a finite velocity, c. It is natural to interpret this velocity as the speed of sound, v_s. We will test this interpretation in the future. For now, we suppress factors of c, which are easy to restore.

We have thus constructed a solution of Type IIB string theory of the form $\text{Sch}_5^2 \times S^5$ which is dual, according to the results of [16, 160], to some theory which respects the Schrödinger symmetry algebra, *i.e.* a non-relativistic conformal field theory with $d = 2$ and $z = 2$. We note that it is straightforward to repeat the analysis of this section for other IIB back-

[4]Scaling away the value of β will not be possible in the finite-temperature solutions described below.

grounds with the requisite isometries, including in particular other-dimensional branes and other choices of isometries along which to Melvinize. In the next section, we use the same techniques to construct a dual description of such an NRCFT at finite temperate and chemical potential.

3.2.3 Solutions with Finite Temperature and Chemical Potential

As we saw above, feeding the Melvinizing machine an extremal D3-brane produces a solution of IIB string theory with spacetime geometry $Sch_5^{z=2} \times S^5$ whose dual field theory is a NRCFT at zero temperature. Experience with AdS/CFT suggests that putting the NRCFT at finite temperature should correspond to the introduction of a Rindler horizon in the bulk spacetime, with modes of the boundary theory thermalized by the Hawking radiation of the black hole. So we need to figure out a way to embed a non-extremal black hole inside our Schrödinger spacetime. It is natural to guess that feeding Melvin a black D3-brane, which shares the asymptotic $AdS_5 \times S^5$ structure of the extremal D3-brane, should produce a black hole spacetime which is asymptotically $Sch_5^{z=2} \times S^5$. As we shall see, this is correct, with one important modification which will become clear after Melvin does his thing.

Thus motivated, let's Melvinize the black D3-brane solution of IIB string theory. The starting solution is,

$$ds^2 = \frac{1}{h}\left(-d\tau^2 f + dy^2 + d\vec{x}^2\right) + h\left(\frac{d\rho^2}{f} + \rho^2\left[ds^2_{\mathbb{P}^2} + (d\phi + \mathcal{A})^2\right]\right)$$

where $h = \frac{R_A^2}{\rho^2}$ is the near-horizon limit[5] of the usual D3 harmonic function and $f = 1 + g = 1 - \frac{\rho_H^4}{\rho^4}$ is the emblackening factor, together with the usual five-form fieldstrength supporting the S^5 and providing the 5d cc,

$$F^{(5)} = (1 + *)\Omega_5 \, d\theta \wedge d\phi \wedge d\psi \wedge d\mu \wedge d\chi$$

Melvinizing this solution along the Hopf fibre is a straightforward application of the recipe described above, so let's jump to the chase and simply write down the final result (for

[5]It is easy to keep the full geometry; we skip directly to the near horizon limit here for simplicity. Keeping the 1 leads to a simple modification of the above, a geometry which may be interpreted as a non-extremal IIB Fluxbrane; it would be interesting to understand the relation between the NRCFT dual to our geometry and the worldvolume theory of the full stringy Fluxbrane.

completeness, the full computation is presented in an appendix). In String Frame[6], using the coordinates $\{t, \xi, r\}$ introduced above, the result is

$$ds^2 = \frac{1}{r^2 K}\left(-2f\frac{\beta^2}{r^2}dt^2 + 2d\xi dt + \frac{1-f}{2}(dt-d\xi)^2\right) + \frac{1}{r^2}d\vec{x}^2 + \frac{dr^2}{r^2 f} + ds^2_{\mathbb{P}^2} + \frac{1}{K}\left(d\chi + \mathcal{A}\right)^2. \tag{3.2.1}$$

where $(d\chi + \mathcal{A})$ and $ds_{\mathbb{P}^2}$ are as above and

$$f = 1 - \frac{r^4}{r_H^4}, \qquad K = 1 + \frac{\beta^2 r^2}{r_H^4}.$$

In contrast to the $T = 0$ solution and to the AdS black hole, the dilaton now has a non-trivial profile,

$$\Phi = -\frac{1}{2}\ln K,$$

while the Neveu-Schwarz two form takes a slightly different form,

$$B = \frac{\sqrt{2}\beta}{r^2 K}\left((1+f)dt + (1-f)d\xi\right) \wedge (d\chi + \mathcal{A})$$

The five form field strength is again unmodified by the Melvinizing.

It is a simple but tedious exercise to verify that this is a solution to the full 10D IIB supergravity equations of motion. Explicitly, it solves

$$G_{\mu\nu} = \sum_{p=1,3,5} T^{(p)}_{\mu\nu} e^{-\delta^3_p \Phi},$$

where $T^{(p)}$ is the stress tensor for a minimally-coupled p-form field strength H,

$$T^{(H)}_{\mu\nu} = -\frac{2}{p(p+1)}\left(\frac{1}{4}g_{\mu\nu}H^2 - \frac{p}{2}H_{\mu\ldots}H_{\nu}^{\ \ldots}\right)$$

and $T^{(1)}$ is the dilaton stress tensor.

There are many things worth noting about this solution. We focus first on the region near the horizon. The component of the gauge field along the null killing vector normal to the horizon (*i.e.* $B_{\tau\phi} = A_\tau$) vanishes at the horizon. This is necessary to have a smooth euclidean continuation. The geometry contains a nice Rindler horizon with normal (and tangent) vector

[6]The transformation to 10d Einstein Frame multiplies the metric by $e^{-\frac{1}{2}\Phi} = K^{\frac{1}{4}}$.

∂_τ, just as in the pre-Melvin hole – in particular, the would-be null-singularity living near $r \to \infty$ arising from compactification of ξ is lost behind the Schwarzschild horizon at $r = r_H$. Near the horizon at $r = r_H$, it is useful to change coordinates to

$$r =: r_H - R^2;$$

up to an irrelevant constant scale factor the metric looks like

$$ds^2 = dR^2 - R^2\kappa^2 d\tau^2 + \frac{1}{r_H^2} d\vec{x}^2 + \ldots$$

with $\kappa \equiv \frac{2}{r_H}$. In order for this to produce a smooth euclidean cigar geometry, the euclidean time coordinate τ must be identified according to $i\kappa\tau \simeq i\kappa\tau + 2\pi$. This is the same result as for the pre-Melvin hole. Before Melvinizing, τ was also the asymptotic time coordinate, and this gave a Hawking temperature $T_H^{\text{pre-Melvin}} = \frac{\kappa}{2\pi} = \frac{1}{\pi r_H}$. With the Schrödinger asymptotics, however, $t = \frac{1}{\sqrt{2}}(\tau + y)$ is the natural time coordinate. Different Hamiltonians imply different temperatures. The euclidean continuation of our time coordinate t must be identified according to $i\kappa\sqrt{2}t \simeq i\kappa\sqrt{2}t + 2\pi$, and hence the Hawking temperature of our black hole is

$$T_H = \frac{\sqrt{2}}{\pi r_H}. \tag{3.2.2}$$

Note that the horizon is unmodified by the Melvin procedure [65]; only the asymptotics (and the relationship to the asymptotic coordinates and horizon coordinates) is changed.

Next note the relative factor of K between the $\mathbb{P}^2$ part of the metric and the $(d\chi + \mathcal{A})$ term: turning on a temperature has squashed the S^5 along the Hopf fibre (though, since $\mathcal{K}$ varies between 1 and $1 + \beta^2$ between boundary and horizon, the squashing is gentle). When we compactify to 5 dimensions, then, we should expect a non-trivial profile for the scalar field associated with this squashing mode. Note, too, that this squashing breaks the isometry group of the sphere from $SU(4)$ to a subgroup, and thus breaks the supersymmetry of the background accordingly[7]. Entertainingly, T-dualizing on this fibre still leaves us with a squashed sphere, but the dual dilaton is now constant. This gives a perhaps-simpler IIA description which may be convenient for various purposes.

[7]For work on the supersymmetric generalization of the Schrödinger group see [154].

This solution also admits a number of illuminating limits. As before, taking $\beta \to 0$ effectively un-does the Melvinization, returning us to the ordinary black D3-brane solution with an identification. Taking $T \to 0$ sends this solution to the zero temperature solution found above, ie to $Sch_5^2 \times S^5$.

All of which raises the obvious question, *what is β?* At $T = 0$, we saw that β could be scaled out of any physical question by a rescaling of coordinates, which can be thought of as a choice of units in the boundary theory. However, this is not the case at finite temperature. β thus represents a physical parameter of the finite-temperature black hole embedded in asymptotically Schrödinger spacetime. To anticipate what property of the spacetime/NRCFT this parameter represents[8], look back at the Melvinization procedure. In step (3), α turns on a mixing of the y direction (which will eventually become the ξ direction after boosting to the IMF) and angular momentum along the S^5. In finite temperature AdS/CFT, angular momentum along the sphere translates into finite chemical potential for the conserved R-charge dual to the angular momentum current. Combining the above, we should expect the CFT dual to this emblackened $Sch_5^{z=2}$ to have a finite ξ-momentum density, *aka* particle number density, which scales as some power of (β/r_H^m), where the factor of r_H^m with $m > 0$ is there to ensure that the density runs to zero as $T = \frac{\sqrt{2}}{\pi r_H} \to 0$ with β held fixed since, as we have seen, any finite β is unphysical at zero temperature so cannot determine any physical quantity like density in the zero temperature limit.

3.2.4 In Search of QCP: Finite Density at $T = 0$

The solutions found above have non-zero temperature and density. However, sending $T \to 0$ appears to send them *both* to zero. This is bizarre, particularly in a non-relativistic theory in which particle-antiparticle annihilation is absent so that the number density should stay fixed as we take the temperature to zero. We must be able to find a finite-density zero-temperature solution! And indeed it is useful to do so, since refrigeration techniques have reached the point that thermal effects on the cold atoms can be neglected for many purposes.

The answer was already implicit in the last paragraph of the previous subsection: the particle number density of our Schrödinger black holes scales as some power of β/r_H^m; the limit

[8] We will verify this through explicit computations of boundary thermodynamic quantities in the next section.

we should take, then, involves sending $r_H \to \infty$ (which removes the horizon and sends the temperature to zero) while taking $\beta \to \infty$ to hold the density fixed. A little experimentation suggests that the proper limit is to scale $r_H^2 \sim \beta^{-1} \to \infty$, keeping $\Omega \equiv \frac{\beta}{r_H^2}$ fixed.

To define the limit more precisely, we make the following replacements:

$$t = \sqrt{2}\tilde{\beta}\heartsuit\tilde{t}, \quad \xi = \frac{\tilde{\xi}}{\heartsuit\sqrt{2}\tilde{\beta}}, \quad r_H = \heartsuit^{1/2}\tilde{r}_H, \quad \beta = \heartsuit\tilde{\beta}. \tag{3.2.3}$$

We will show in Section 4 that the particle number density in these units is given by $\tilde{\rho} \propto \Omega^2 \equiv \frac{\tilde{\beta}^2}{\tilde{r}_H^4}$, while the rescaled temperature is $\tilde{T} = \frac{\sqrt{2}}{\pi\heartsuit^{3/2}\tilde{r}_H}$. To send $\tilde{T} \to 0$ keeping finite $\tilde{\rho}$, we should take $\heartsuit \to \infty$ holding objects with tildes fixed (we will drop the decorations at the end). Note that (3.2.3) includes the transformation which allowed us, at zero temperature, to scale away any finite β (notably, in the present limit, $\beta \to \infty$).

Ignoring the sphere directions for simplicity, the metric in the scaling limit $\heartsuit \to \infty$ takes a pleasingly simple form,

$$ds^2_{\Omega\neq 0} = \frac{1}{r^2\kappa}\left(-\frac{dt^2}{r^2} + 2dtd\xi + \Omega^2 r^4 d\xi^2\right) + \frac{d\vec{x}^2 + dr^2}{r^2} \tag{3.2.4}$$

while the B field takes the equally entertaining form,

$$B = \frac{1}{r^2\kappa}\left(dt + 2\Omega^2 r^4 d\xi\right) \wedge (d\chi + \mathcal{A})$$

and the dilaton remains non-trivial,

$$\Phi = -\frac{1}{2}\ln\kappa,$$

where $\kappa \equiv 1 + \Omega^2 r^2$. The five-form, as usual, goes along for the ride. Perhaps unsurprisingly given its pedigree, but surprisingly given its form (note that as $r \to \infty$, $r^2\kappa \sim r^4$, so the ξ direction asymptotes to a finite radius controlled by Ω), this background can be explicitly shown to solve the full equations of motion of IIB supergravity. We will study the thermodynamics of this solution alongside that of the finite-temperature case in the next section.

Note, that in our scaling limit, the horizon has run off to $r = \infty$. Happily, the null-singularity observed before near $r \to \infty$ is absent thanks to the $r^4 d\xi^2$ term – finite density has cut off this singularity. However, the dilaton still grows logarithmically. This means that the theory contains a region of strong coupling in the IR part of the geometry, somewhat similar to gravity duals of IR-strong gauge theories, such as Dp-branes with $p < 3$ [103]. It would be exciting to interpret this scale-dependence of the string coupling in terms of screening in the boundary theory at finite density.

If we can take the temperature to zero holding the chemical potential fixed, it stands to reason that we can take the chemical potential to zero holding the temperature fixed. Indeed we can! However, something rather stupid happens. To see this, let's run the scaling of the last section backward, holding T fixed but scaling $\Omega \to 0$. It is easy to identify the proper scaling here: we must take $\beta \to 0$. Sending the chemical potential to zero thus sends the solution back to the black D3-brane with which we began. Of course, we have compactified the light-like ξ direction, so what we really have is a DLCQ of AdS (and hence the breaking of the symmetry group from $SO(2,4)$ to its $Sch_5^{z=2}$ subgroup). However, as an NRCFT, it is rather disappointing – there is nothing in the trap.

As should by now be clear, the backgrounds we have been studying are intimately connected to DLCQs. Fleshing out this connection will shed light both on the spacetime solutions themselves and on the NRCFTs to which they are dual. The remainder of this section is thus devoted to an analysis of this connection.

3.2.5 The Null Melvin Twist with finite L_ξ as a modified DLCQ

The Null Melvin twist has the great advantage of being a concrete tool with which to generate new IIB solutions from our tired old examples, and as such has been studied rather extensively. However, at intermediate steps the solutions are far from simple, and the physical meaning of the procedure is rather opaque: what is the intrinsic relationship between the final and initial solutions?

Happily, there is another way of organizing the argument which is completely equivalent and which makes the connection between initial and final solutions manifest, following [25]. Let's start by studying the DLCQ of the original solution along the $\xi = (y-\tau)/\sqrt{2}$ direction by requiring all fields Φ to be invariant under translation along the light-cone ξ direction,

$$\Phi(\xi + L_\xi) = e^{L_\xi J_\xi}\Phi(\xi) \stackrel{!}{=} \Phi(\xi),$$

where $J_\xi = \partial_\xi$ is the momentum generator on the light-like direction. This produces the solution above at $\beta = 0$. So: how do we introduce β? The answer is suggested by step (3) of the Melvinization, in which we re-diagonalized our symmetry generators to mix the spatial translation current dy into the angular rotation current $d\phi \to d\phi + \alpha dy$; this replaces momentum along the (boosted) y-direction, J_y, with the sum of $J_y - \alpha J_\phi$, where $\alpha J_\phi = \partial_{\frac{1}{\alpha}\phi}$. Boosting back to our original frame and taking the boost large turns this into $J_\xi - \frac{1}{2}\alpha e^\gamma J_\phi \to J_\xi - \beta J_\phi = J_\beta$. To get the final solution with $\beta \neq 0$, then, we should perform a modified DLCQ of the original solution in which we orbifold not by a finite translation by $L_\xi J_\xi$, but by the modified current, $L_\xi J_\beta$, *i.e.* we should shift the light-cone momentum of every field by β times its charge under the $d\phi$ isometry. We'll refer to this as a DLCQ$_\beta$.

The physical meaning of the generated solution is thus relatively straightforward: our Null Melvin Twist, *aka* the DLCQ$_\beta$, is a restriction of the original solution to modes with fixed light-cone- and angular- momenta[9]. Note that the DLCQ$_\beta$ of zero-temperature $\mathcal{N} = 4$ SYM has a particularly trivial limit in which we hold L_ξ fixed and scale $\beta \to 0$ – this is just the usual DLCQ of AdS_5. The fact that this system realizes the Schrödinger group as isometries is a direct consequence of the compactification of ξ. Unfortunately, this limit ensures that the particle density of the groundstate is zero, so, while this does describe a NRCFT it describes the system at zero density only. It is thus extremely important to preserve the parameter β if we want to describe something like "fermions at unitarity" rather than "*no* fermions at unitarity."

3.2.6 The DLCQ$_\beta$ of $\mathcal{N} = 4$ SYM

The virtue of the DLCQ$_\beta$ prescription for our purposes is that it translates relatively easily into the dual field theory. To wit, the NRCFTs dual to our Schrödinger spacetimes arise as DLCQ$_\beta$'s of the boundary theories of the original solution. Importantly, the current by which we orbifold is $J_\beta = J_\xi - \beta J_R$, where J_R is the U(1) R-current dual to the isometry current on the S^5, J_ϕ, we used in Melvinization. For example, the NRCFT dual to $\mathrm{Sch}_5^{z=2}$ is simply the DLCQ$_\beta$ of the $\mathcal{N} = 4$ SU(N) SYM living on the boundary of the AdS$_5$ with which we started, with J_R the trace of the cartan of the $SO(6)$ R-symmetry (corresponding to the Hopf fibration we used in Melvinization).

[9]Note that for nonzero β, J_β is not actually light-like in the bulk. So this is in general a DLCQ only from the point of view of the boundary field theory.

This result may also be derived via direct application of the Null Melvin Twist to the field theory as follows. Start with the $\mathcal{N} = 4$ SYM theory and compactify it on a circle $y \simeq y + 2\pi L_y$ with all fields Φ required to satisfy the boundary conditions,

$$e^{L_y(\partial_y - i\alpha q_R)}\Phi(y) = \Phi(y),$$

where q_R is the charge of Φ under the specified $U(1)$ subgroup of the R-symmetry group. This is equivalent to inserting an R-symmetry-valued wilson line around the compact spatial direction [25]. For our special case, we chose the R-symmetry such that all three complex scalars in the **6** of $SO(6)$ carry the same charge; this corresponds to a shift on the Hopf circle. Now boost the y direction by $\gamma \to \infty$ to make it lightlike while scaling $\alpha \to 0$ such that $\beta = \frac{1}{2}\alpha e^{\gamma}$ remains fixed. At weak 't Hooft coupling, this has the following effect. On the potential terms in the Lagrangian it does nothing because the R-symmetry is a symmetry. On y-derivative terms it amounts to the replacement $\partial_\xi \Phi \to \partial_\xi \Phi - i\beta q_R \Phi$. In terms of ξ-momentum, the net effect is to shift the moding of each field by a constant piece proportional to β, ie

$$L_\xi(i\ell - i\beta q_R) = 2\pi i \qquad \Rightarrow \qquad \ell = \frac{2\pi N}{L_\xi} + \beta q_R,$$

where N is an integer. This is precisely the DLCQ$_\beta$ described above.

This theory seems remarkably simple, even moreso than the usual un-modified DLCQ of $\mathcal{N}$=4. To understand why, recall that the usual DLCQ tells us to expand every field in the theory in modes along the light-like ξ circle. This leaves a KK tower of massive modes, plus a single level of massless modes – the zero modes of ∂_ξ – which must be treated with, if not respect, at least care. The resulting theory is thus deliciously close to being non-relativistic, but the persistence of these zero modes reminds everyone that the theory is really Lorentz invariant. To get a truly non-relativistic theory, we would like to lift these zeromodes. But since the upshot of the DLCQ$_\beta$ is to shift the moding of all fields by β times their R-charge, that is precisely what the DLCQ$_\beta$ does. More precisely, the only zero-modes surviving the DLCQ$_\beta$ are those of R-scalars, ie of the vector bosons, and these must be dealt with carefully; among other things, they generate instantaneous interactions between the remaining non-zero modes of the matter fields. The result is a theory with only nonrelativistic excitations, with the spectrum gapped by two mass scales, the KK scale $1/L_\xi$ and the new scale β. In particular, something dramatic happens when $\beta \times (\frac{L_\xi}{2\pi}) \in \mathbb{Z}$: Φ picks back up a zero mode (this is just the fact that the wilson line along the DLCQ circle has phase βL_ξ). β is thus playing the role of an IR regulator for the DLCQ zero modes generated by a wilson line. Another curious feature of this scaling is that in order to excite a single KK mode, we need $\mathcal{O}(\frac{2\pi}{L_\xi \beta})$ would-be-zero-modes; this suggests that there is an interesting regime where $1 \ll \langle N \rangle \ll \frac{2\pi}{L_\xi \beta}$ where we can drop the KK modes but we still have a well-regulated theory. Understanding the interplay between these two scales in more detail, and especially to see it arise in the dual geometry, seems worthwhile; we leave such questions to future work.

3.3 Black Hole Thermodynamics in Schrödinger Spacetimes

3.3.1 Entropy

The Bekenstein-Hawking entropy density of the black hole (3.2.1) is

$$s \equiv \frac{S}{L_1 L_2} = \frac{1}{4G_{10}} L_y \frac{\pi^3}{r_H^3} R_A^8 = \frac{1}{4G_5} L_y \frac{R_A^3}{r_H^3}. \tag{3.3.1}$$

Note that the dependence on β cancels. To write this in terms of more physical variables, we need to relate L_y to L_ξ. What we mean by L_y in the formula (3.3.1) is the extent of the horizon in the y direction when ξ has period L_ξ. To figure out what this is, one need only look at the metric near the horizon, and plug in. Near the horizon, the metric takes the form:

$$ds^2 = dy^2 + ... = \frac{1}{2}(dt - d\xi)^2 + ...$$

where ... is terms which vanish when we ask about the invariant distance between two events separated only in the ξ direction by an amount $d\xi = L_\xi$. Therefore:

$$L_y = \frac{1}{\sqrt{2}} L_\xi.$$

Using the standard parameter map of AdS/CFT (which commutes with Melvinization)

$$\frac{R_A^8}{4G_{10}} = \frac{N^2}{2\pi^4},$$

and the temperature (3.2.2), we have

$$s = \frac{1}{8} N^2 \pi^2 L_\xi T^3.$$

For later comparison, it will be useful to note that in units where $16\pi G_5 = 1$, we have

$$s = L_\xi \pi^4 T^3. \tag{3.3.2}$$

3.3.2 A comment on the correspondence for the stress tensor

We begin with a comment about the mysterious-seeming equation (34) of [160], which built on ideas of [161]. The last section of [160] contains an assertion about which modes of the metric couple to which thermodynamic variables of the boundary theory, which is supported by matching to a weakly coupled Lagrangian. This expression can be understood more directly as follows. In the standard AdS/CFT examples, fluctuations of the metric which have nice equations of motion are the ones with one upper and one lower index. These are also the components which couple directly to the boundary stress tensor [121, 144], *i.e.*

$$I_{bdy} = \mathcal{O}\left(h^0\right) + \int_{\partial M} \sqrt{\gamma}\, 2\, (T_{\rm bdy})^\mu_\nu\, h^\nu_\mu + \mathcal{O}\left(h^2\right)$$

where γ is the metric on the boundary. Given h^μ_ν, to determine the perturbation of the metric $g_{\mu\nu} = g^{(0)}_{\mu\nu} + h_{\mu\nu}$, the right thing to do is to symmetrize:

$$h_{\mu\nu} = \frac{1}{2}\left(g_{\mu\rho} h^\rho_\nu + g_{\nu\rho} h^\rho_\mu\right).$$

Adding such a fluctuation to the zero-temperature "schrödinger metric", and setting

$$A_0 = h^\xi_t, \quad A_i = h^\xi_i, \quad \Phi = h^t_t, \quad B_i = h^t_i$$

gives Son's equation (34) to linear order in these fluctuations. If h^t_ξ is nonzero, there is a nonzero fluctuation of $h_{\xi\xi}$. For general z, and restoring factors of β, we find to linear order in the fluctuations

$$ds^2 = ds_0^2 + \left(\frac{A_0}{r^2} - 2\beta^2\frac{\Phi}{r^{2z}}\right)dt^2 + \frac{\Phi(r)}{r^2}d\xi dt + \left(\frac{A_i dx^i}{r^2} - \frac{\beta^2 B_i dx^i}{r^{2z}}\right)dt + \frac{B_i dx^i}{r^2}d\xi + h_{ij}dx^i dx^j + \ldots \tag{3.3.3}$$

So T_ξ^t is the number density of the field theory. An analogy which is useful for understanding this point is the following. In IIB on $AdS_5 \times S^5$, considered as a ten-dimensional theory with a nine-dimensional boundary, what is the meaning of T_μ^ϕ and T_ϕ^μ, components of the boundary stress tensor with indices in the sphere directions? The answer is that they give R-current densities. This is quite analogous to the statement that T_ξ^t gives the number density, since in our correspondence the particle number density is the density of ξ-momentum, just as the R-charge density is the density of momentum around the S^5 directions.

Note that the interpretation of $\mathcal{T}_t^\xi$ and $\mathcal{T}_\xi^\xi$ remains mysterious.

3.3.3 Expectation values of the stress tensor

Consider any bulk theory where the matter lagrangian doesn't involve derivatives of the metric. If the boundary metric is flat, the terms in the on-shell action which are linear in the metric fluctuations take the form [121][10]

$$I_{bdy} = \mathcal{O}(h^0) + \int_{\partial M} h_\mu^\nu \left(\Theta_\nu^\mu - \Theta\delta_\nu^\mu\right) + I_{bdy,ct} + \mathcal{O}(h^2)$$

where $I_{bdy,ct}$ contains counterterms involving the matter fields.

$$\Theta_{\mu\nu} = \frac{1}{2}\left(D_\mu n_\nu + D_\nu n_\mu\right)$$

is the extrinsic curvature of the boundary, and n^μ is an inward-pointing unit normal vector to the boundary. Taking $n^r = -\sqrt{g^{rr}}$, the formula to extract the expectation value of the field theory stress tensor from the bulk data is

$$(T_{\text{bdy}})_\nu^\mu = -2\sqrt{\gamma}\left(\Theta_\nu^\mu - \delta_\nu^\mu(\Theta + a) + \ldots\right)$$

[10]In what follows we studiously set the bulk coupling $\mathcal{K}_5 = \frac{1}{16\pi G_5}$ to one.

where γ is the metric on the boundary (*i.e.* the metric on a fixed-r subspace), and a is a counterterm coefficient, and .... is the contribution of other counterterms. This quantity should have a finite limit as $r \to 0$ (*i.e.* as it approaches the boundary).

In these nonrelativistic systems there is one further complication in the extraction of the expectation values of the field theory stress tensor. This is the fact that the description of nonrelativistic systems one finds here involves an extra dimension ξ, whose momenta are associated with the conserved particle number. Since the ordinary stress tensor of the nonrelativistic system, which we will denote $\mathcal{T}$, is an operator of particle number zero (*i.e.* it is of the form $\Psi^\dagger...\Psi$), it is related to the boundary stress tensor which depends on ξ by extracting the zeromode. This leads to an extra factor of L_ξ:

$$\mathcal{T}^\mu_\nu = -2 \lim_{r \to 0} L_\xi \sqrt{\gamma} \left(\Theta^\mu_\nu - \delta^\mu_\nu(\Theta + a) + ...\right)$$

We evaluate the stress tensor expectation values in terms of the five-dimensional description. It would be a useful check to redo this calculation in ten dimensions. The boundary counterterms we include are

$$I_{ct} = \int_{bdy} d^{d+2}X \sqrt{\gamma} \left(a_1 e^{\alpha_1 \Phi} + a_2 e^{\alpha_2 \Phi} A^2 + a_3 e^{\alpha_3 \Phi} A^4\right),$$

where $A^2 \equiv A_\alpha A_\beta \gamma^{\alpha\beta}$. Because of the asymptotic behavior of the solution, we can replace the factors of $e^{\alpha_i \Phi}$ with their Taylor expansion about the boundary $r = 0$; α_3 will not matter. Note that Φ here is a proxy for any 5d scalar quantity which behaves as $e^{-2\Phi} = K$.

We find that finite expectation values of the physical components of the stress tensor require the addition of some extrinsic terms:

$$I_{\text{ext}} = \int_{\text{bdy}} \sqrt{\gamma} n^r A^\mu F_{r\mu} a_4 e^{\alpha_4 \Phi}. \tag{3.3.4}$$

Note that to the order at which this term contributes to the stress tensor and free energy, we can rewrite

$$a_4 e^{\alpha_4 \Phi} = a_4 + a_4 \alpha_4 \Phi.$$

This term changes the boundary conditions on the massive gauge field away from purely Dirichlet [86, 111]. Using our result in appendix C that the coefficient of the F^2 term in the 5d lagrangian is $e^{-\frac{8}{3}\Phi}$, we see that the special choice $a_4 = 1, \alpha_4 = -\frac{8}{3}$ implies *Neumann* boundary conditions on A. Remarkably and mysteriously, it turns out that $a_4 = 1$ is required for finiteness of $\mathcal{T}_t^{\ t}$, and $\alpha_4 = -\frac{8}{3}$ is required for the first law of thermodynamics to be satisfied.

In a scale-invariant field theory with dynamical exponent z, the energy density and pressure in thermal equilibrium are related by[11]

$$z\mathcal{E} = d\mathcal{P}.$$

Just like tracelessness of the stress-energy tensor of a relativistic CFT (the special case $z = 1$), this relation arises as a Ward identity for conservation of the dilatation current. For our case with $d = z = 2$, this implies $\mathcal{E} = \mathcal{P}$. We constrain the counterterms to cancel the divergences at $r \to 0$ and so that the Ward identity is satisfied[12]. Identifying $\mathcal{E} = -\mathcal{T}_t^{\ t}$ and $\mathcal{P} = \mathcal{T}_i^{\ i}$, we find[13]

$$\mathcal{E} = \mathcal{P} = \frac{L_\xi}{4} (\pi T)^4 \left(1 + \aleph\delta^4\right). \tag{3.3.5}$$

The numerical factor $\aleph$ depends on counterterm coefficients which are not determined by finiteness of the energy, pressure, density, or by the Ward identity for scaling. We will fix $\aleph$ below by demanding the first law of thermodynamics. $\aleph$ will turn out to be zero, in agreement with [123, 90]. As a small check on our calculation, the action evaluated on the black hole solution satisfies

$$T\left(I_{\text{bulk}} + I_{\text{GH}} + I_{\text{ct}} + I_{\text{ext}}\right)\Big|_{\text{on-shell}} = \mathcal{P} L_1 L_2,$$

as expected for the free energy in the grand canonical ensemble. This equality is true of the regulated expressions for any choice of the counterterms.

[11] For the special cases $z = 1, 2$ the formula for general z was derived with Pavel Kovtun.

[12] The conditions on the counterterms we find are:

$$a_1 = -6, \alpha_1 = -\frac{1}{6}\cdot(2a_2 - 4a_4 - 2), a_4 = 1$$

for finiteness, and the Ward identity requires

$$0 = 17 - 18\alpha_1 - a_2(6 + 10\alpha_2) + 12a_3.$$

[13] In these expressions we have divided out a common factor of $\mathcal{K} = \frac{N^2}{16\pi^2}$ in all of the one-point functions.

Note that the thermodynamic potential densities $\mathcal{E}, \mathcal{P}$ in a system with dynamical exponent z should scale like T^{d+z} times some function of the dimensionless ratio $\frac{\mu}{T}$, in agreement with our expressions (3.3.5). In our $z = 2$ case, the factor of L_ξ makes up for the dimensions of the extra power of temperature.

As discussed in the previous subsection, the density is determined by $\langle \mathcal{T}_\xi^t \rangle$. This gives

$$\rho = 2\frac{L_\xi}{r_H^4} = \frac{1}{2} L_\xi (\pi T)^4. \tag{3.3.6}$$

Note that the still-mysterious T_μ^ξ components of the stress tensor are still divergent. That some components of the stress tensor would remain divergent in holographic calculations with degenerate boundaries was anticipated in [164]. The fact that the components which are hard to renormalize are precisely those whose physical interpretation is unclear is heartening.

3.3.4 Comments on chemical potential

Son [160] showed that the mode A_0 of the metric in (3.3.3) is the bulk field associated to the boundary number density current. The expansion of the finite-temperature metric (3.2.1) at the boundary gives

$$g_{tt} = -\frac{2\beta^2}{r^4} + \frac{4\beta^4}{3r_H^4}\frac{1}{r^2} + \ldots$$

Comparing with the parametrization of the fluctuations in (3.3.3), we see that $A_0 = \frac{4\beta^4}{3r_H^4} + \mathcal{O}(r^2)$ in our background. This suggests that $\frac{4\beta^4}{3r_H^4} = \frac{4}{3}\delta^4$ determines the chemical potential for the number density in this background. To extract more precisely the value of the chemical potential indicated by these falloffs of A_0 requires a better understanding of the couplings of these modes.

Following [123, 90], we can use a trick to determine the chemical potential which makes precise the comments at the end of section 3.3. The null killing vector at the horizon is $\mathfrak{v} \propto \partial_\tau = \frac{1}{\sqrt{2}}(\partial_t - \partial_\xi)$. If we normalize $\mathfrak{v}$ so that its component along the asymptotic time direction is unity,

$$\mathfrak{v} = \partial_t - \partial_\xi,$$

the temperature of the black hole is given by $T_H = \frac{\kappa}{2\pi}$; the surface gravity κ is defined as

$$\kappa^2 = -\frac{1}{2}\nabla_a \mathfrak{v}^b \nabla_c \mathfrak{v}^d g^{ab} g_{cd}.$$

This corroborates our earlier result that $T_H = \frac{\sqrt{2}}{\pi r_H}$. Now, the fact that the null killing vector at the horizon does not point only in the time direction says that the ensemble to which the black hole contributes has a density matrix $\hat{\rho} = e^{-\frac{1}{T}(\hat{H}-\mu\hat{N})}$; this is the translation operator by which the euclidean geometry is identified. In our t, ξ coordinates, this gives

$$\mu = -1. \tag{3.3.7}$$

3.3.5 Comments on the first law of thermodynamics

The first law of thermodynamics should read

$$\mathcal{E} + \mathcal{P} = Ts + \mu\rho.$$

Given the entropy density, thermodynamic relations determine $\mu\rho$ in a system with these symmetries. From the Bekenstein-Hawking formula, we have an entropy density of the form

$$s = c_1 L_\xi T^3,$$

where c_1 is a constant. But the thermodynamic relation $s = \frac{\partial \mathcal{P}}{\partial T}$ implies

$$\mathcal{P} = \frac{1}{4}c_1 L_\xi T^4 + p_0(\mu) = \frac{1}{4}Ts + p_0(\mu)$$

where the second term is temperature-independent but otherwise thus-far undetermined. The scale-invariance Ward identity, $z\mathcal{E} = d\mathcal{P}$, then implies

$$\mathcal{E} + \mathcal{P} = \left(\frac{d}{z} + 1\right)\mathcal{P}$$

so

$$\mu\rho = \mathcal{E} + \mathcal{P} - Ts = \left(\frac{1}{4}\left(\frac{d}{z}+1\right) - 1\right)Ts + \frac{1}{4}p_0$$

For our case $d = z = 2$, this gives

$$\mu\rho = \frac{1}{2}c_1 L_\xi T^4 + \frac{1}{4}p_0$$

Using the thermodynamic potentials $\mathcal{E}, \mathcal{P}, \rho$ extracted from $\mathcal{T}$, the enthalpy (the left hand side of the first law $\mathcal{E} + \mathcal{P} = Ts + \mu\rho$) is

$$\mathcal{E} + \mathcal{P} = \frac{1}{2}L_\xi(\pi T)^4(1 + \aleph\delta^4).$$

Using (3.3.6), (3.3.7) and (3.3.2), the right hand side is

$$Ts + \mu\rho = L_\xi(\pi T)^4 - \frac{1}{2}L_\xi(\pi T)^4.$$

Consistency of the first law therefore requires $\aleph = 0$, and determines the integration constant $p_0(\mu) = 0$.

3.3.6 Thermodynamics in physical variables

By rescaling $t \to t' = at, \xi \to \xi' = b\xi$, we can change the chemical potential to a value with respect to which it is possible to differentiate. In these new coordinates, we have

$$\mu' = -\frac{b}{a}, \quad T'_H = \frac{\sqrt{2}b}{\pi r_H}, \quad \mathcal{E}' = \mathcal{P}' = \frac{L_\xi}{r_H^4}\frac{1}{ab}, \quad \rho' = 2\frac{L_\xi}{r_H^4}\frac{1}{b^2}, \quad s' = \frac{L_\xi}{\sqrt{2}r_H^3}\frac{1}{b}.$$

The first law still checks. In order to preserve the dispersion relation $2l\omega + \vec{k}^2 = 0$ (*i.e.* to preserve the $g_{t\xi}$ metric coefficient), we should set $a = \frac{1}{b}$. In retrospect, the dispersion relation with positive mass should be $2l\omega = \vec{k}^2$; this can be accomplished by setting instead $a = -\frac{1}{b}$; this will also make the energy density positive.

Making the substitution $a = -\frac{1}{b}, b = \sqrt{\mu}, \frac{1}{r_H} = \frac{\pi T}{\sqrt{2\mu}}$, then, we have

$$\mathcal{E} = \mathcal{P} = \frac{1}{4}\frac{L_\xi(\pi T)^4}{\mu^2}, \qquad \rho = \frac{1}{2}\frac{L_\xi(\pi T)^4}{\mu^3}.$$

A small check on this result is the following. The free energy of a scalc-invariant theory at finite temperature and chemical potential can be written as

$$F = -VT^{\alpha} f\left(\frac{\mu}{T}\right) .$$

The power α is determined by dimensional analysis, and for general z turns out to be $\alpha = \frac{d+z}{z}$. Note that this value implies that $z\mathcal{E} = d\mathcal{P}$, in agreement with the scale-invariance Ward identity. Free nonrelativistic gases, both classical and quantum with either statistics, in the grand canonical ensemble give $\alpha = (d+2)/2$. For $z = 1$, $p = T^{d+1} f\left(\frac{\mu}{T}\right)$ is the familiar scaling (*e.g.* when $\mu \to 0$). The behavior of α at more general z can be argued as follows. With scaling exponent z, temperature (which is an energy), scales with z powers of inverse-length. Therefore $T^{1/z}$ scales with one power of inverse-length. The free energy density should scale with $d+z$ powers of inverse-length to make up for the scaling of $\int dt d^d x$. This gives $\alpha = \frac{d+z}{z}$, which agrees with the two familiar cases.

3.4 Viscosity

In this section we will study the shear viscosity η of the fluid described holographically by the metric (3.2.1). We will do this using the Kubo formula

$$\eta = -\lim_{\omega \to 0} \frac{1}{\omega} \mathrm{Im} G^R(\omega, \vec{k} = 0), \tag{3.4.1}$$

where G_R is the retarded two point function of the scalar mode of the stress tensor:

$$G^R(\omega, \vec{k} = 0) = -i \int d^d x dt e^{i\omega t} \theta(t) \langle [\mathcal{T}_{xy}(t, \vec{x}), \mathcal{T}_{xy}(0,0)] \rangle .$$

Here we emphasize that the stress tensor is an operator with particle-number zero:

$$\mathcal{T}_{\mu\nu}(t, \vec{x}) \equiv \int_0^{L_\xi} d\xi \; T_{\mu\nu}(t, \vec{x}, \xi) \tag{3.4.2}$$

It was argued in [115] that very generally the linearized Einstein equation for $\phi \equiv h^x_y(u) e^{-i\omega t}$ is the scalar wave equation in the same background. The argument uses only the $SO(2)$ symmetry of rotations in the xy-plane; this symmetry is preserved in our solution. We have also explicitly checked this statement using the ten-dimensional IIB supergravity equations of motion.

Note that unlike the familiar case of three spatial dimensions, in our $d = 2$ example there is no third dimension in which to give momentum to h^x_y. However, this momentum must be set to zero before taking the $\omega \to 0$ limit in the Kubo formula, and nothing is lost for the purposes of studying the viscosity.

We will show in the remainder of this section that the familiar relation

$$\frac{\eta}{s} = \frac{1}{4\pi}$$

also holds in these models. Note that the form of our metric violates the hypotheses of the general theorem [35]. It would be interesting to see how much further the assumptions made there can be relaxed.

3.4.1 Scalar wave equation in the finite-temperature solution

For convenience, we will discuss this problem in six-dimensional Einstein-frame (*i.e.* dimensionally reduce on the constant-volume $\mathbb{P}^2$). We show that the answer is frame-independent in the appendix.

The wave equation is

$$\Box\phi = -g^{\mu\nu}k_\mu k_\nu \phi + \frac{1}{\sqrt{g}}\partial_u\left(\sqrt{g}g^{uu}\partial_u\phi\right).$$

In this metric,

$$\sqrt{g} = \frac{\sqrt{K}}{2u^3}.$$

We will study Fourier modes of the form:

$$\phi(\tau, y, \vec{x}, u) = e^{i\frac{2}{r_H}\left(-\mathfrak{e}\tau + \mathfrak{q}_y y\right)} f_K(u) \quad ,$$

i.e. we have already set to zero the momentum in the spatial directions and the squashed-sphere directions. Note that $\mathfrak{e}, \mathfrak{q}_y$ are dimensionless variables, measured in units of the temperature (times $\frac{2\pi}{\sqrt{2}}$), *i.e.* they are the gothic variables of [162, 144, 145]. It will be crucial to distinguish $\mathfrak{e}$ from the variable ω conjugate to the asymptotic time coordinate t.

The wave equation becomes

$$0 = u^3 \partial_u \left(\frac{4f}{u} \partial_u f_K \right) - \left(-\frac{u}{f} \mathfrak{e}^2 + \delta^2 (\mathfrak{e} - \mathfrak{q}_y)^2 + u \mathfrak{q}_y^2 \right) f_K.$$

The indicial equation near the horizon arises from setting $f_K = (1-u)^\alpha$ and demanding that the most singular terms at $u = 1$ cancel. This gives

$$0 = \alpha^2 + \frac{\mathfrak{e}^2}{4}.$$

The solution obeying incoming-wave boundary conditions at the horizon takes the form

$$f_K(u) = (1-u)^{-i\mathfrak{e}/2} F_K(u)$$

where F_K is analytic at $u = 1$. Next, to study the hydrodynamic limit, we can expand F_K in a small-frequency expansion:

$$f_K(u) = (1-u)^{-i\mathfrak{e}/2} \left(1 + \mathfrak{e} F_1(u) + \mathfrak{q}_y F_2(u) + \ldots \right);$$

here the ellipses denote terms of order $\mathfrak{e}^2, \mathfrak{e}\mathfrak{q}_y, \mathfrak{q}_y^2$ [14]. Plugging back into the wave equation, we find, just as in the AdS black hole [143, 162, 144],

$$F_1(u) = i \ln \frac{1+u}{2}, \quad F_2(u) = 0.$$

Using $g^{uu} = \frac{4u^2 f}{\sqrt{K}}$ and $\sqrt{-g} = \frac{\sqrt{K}}{2u^3 r_H^4}$, this produces a flux factor

$$-\mathcal{F} = \mathcal{K} \sqrt{-g} g^{uu} f_{-K}(u) \partial_u f_K(u) = \mathcal{K} \frac{2(1-u^2)}{u r_H^4} \left(\frac{i\mathfrak{e}}{4} \frac{1}{1-u} - \frac{1}{4} \frac{i\mathfrak{e}}{1+u} \right) + \mathcal{O}\left(\mathfrak{e}^2, \mathfrak{q}_y \mathfrak{e}, \mathfrak{q}_y^2\right)$$

where

$$\mathcal{K} = \frac{N^2}{16\pi^2}$$

is the normalization of the bulk action, written here in terms of field theory variables. It will cancel in η/s. We need the relationship between the momenta associated to the horizon

[14] Actually, the correct expansion treats $\mathfrak{e}$ as the same order as $\mathfrak{q}_y^2$; this will not affect the viscosity calculation.

coordinates and asymptotic coordinates:

$$\mathfrak{q}_y = \frac{1}{\sqrt{2}}(\omega + l)\, r_H, \quad \mathfrak{e} = \frac{1}{\sqrt{2}}(l - \omega)\, r_H.$$

Note that we have restored factors of $\frac{1}{\pi r_H}$ in the definition of the t, ξ momenta relative to the gothic momenta. At the boundary $u = \epsilon$, the flux factor is therefore

$$-\mathcal{F}|_{u=\epsilon} = \frac{\mathcal{K}}{r_H^3}\left(\frac{i}{2}\frac{\omega - l}{\sqrt{2}} + \mathcal{O}\left(\omega, l, k^2, \mathfrak{q}_y^2\right)\right).$$

We dropped contact terms in this expression. The real-time AdS/CFT prescription of [162] says that the retarded Green's function is obtained from the flux factor by

$$G^R(\omega, \vec{k} = 0) = -2\mathcal{F}|_{u=\epsilon}.$$

At this point, we pause to consider whose Green's functions we are studying. The momentum-space correlator in the Kubo formula (3.4.1) has had a factor of the volume of spacetime divided out by translation invariance:

$$G(\omega, \vec{k}) = \frac{1}{VT}\int d^{d+1}x_1 \int d^{d+1}x_2\; e^{ik_1\cdot x_1 + ik_2\cdot x_2} G(x_1, x_2);$$

the factor $VT = L_1 L_2 T$ is $\delta^{d+1}(0)$ in momentum space. As emphasized in equation (3.4.2), the field theory stress tensor is the zeromode in the ξ direction of the operator to which h^μ_ν couples. Therefore, when we relate the two-point function of $T(t, \vec{x}, \xi)$ to the momentum-space Green's function G^R, we should not divide out by the associated factor of L_ξ:

$$\langle [\mathcal{T}_{xy}(\omega, \vec{k}), \mathcal{T}_{xy}(-\omega, -\vec{k})] \rangle = \int d\xi_1 d\xi_2 \int d^d\vec{x}\, dt\, e^{i\omega t - i\vec{k}\cdot\vec{x}} \langle [T_{xy}(t, \vec{x}, \xi_1), T_{xy}(0, 0, \xi_2)] \rangle.$$

Putting this together, the Kubo formula for the viscosity then gives

$$\eta = -\lim_{\omega\to 0}\frac{1}{\omega}\mathrm{Im}\, G^R(\omega, \vec{k} = 0) = 2\mathcal{K} L_\xi \frac{1}{\sqrt{2} r_H^3} = \frac{\pi L_\xi T^3 N^2}{32}.$$

Note that in d spatial dimensions η indeed has mass dimension d. This is identical to the familiar $\mathcal{N} = 4$ answer except for *a)* the interpretation as the viscosity of a theory in two spatial dimensions, and *b)* the factors of $1/\sqrt{2}$ which come from the relation between the asymptotic time coordinate and the coordinate which becomes null at the horizon.

Taking the ratio $\frac{\eta}{s}$ reproduces the KSS value

$$\frac{\eta}{s} = \frac{1}{4\pi}.$$

3.5 Discussion

Our black hole lives in a space with very different asymptotics from AdS. There structure of the horizon, however, is the same as that of the AdS black hole; this is guaranteed by the manner in which it was constructed [65]. The calculation of the viscosity is not obviously determined only by the geometry near the horizon. However, the factors conspire mysteriously to preserve the viscosity ratio. Our result, then, is some further indication that the membrane paradigm should be taken seriously.

In this chapter we have focussed on an example with dynamical exponent $z = 2$ in $d = 2$ dimensions, which is related to the $\mathcal{N} = 4$ theory by a twisted version of discrete light cone quantization. Work on constructing string theory realizations for critical phenomena with other values of z, d is in progress.

It would be interesting to find the black hole solution which asymptotes to the NR metric with spherical spatial section, *i.e.* the analog of the black hole in global coordinates in AdS. The melvinization can't work quite the same if the starting point is AdS in global coordinates, because the analog of y is then an angular variable.

Having identified a zero-temperature background with nonzero density, and its likely weak-coupling description, we can calculate the Bertsch parameter (see *e.g.* [33]) for this theory. The Bertsch parameter is the cold-atoms analog of the famous $\frac{3}{4}$-ratio of strong and weak coupling free energies in the $\mathcal{N} = 4$ theory.

The boundary field theory we are studying clearly contains bosonic excitations, which carry charge under the number-density operator. There should be a chemical potential to temperature ratio above which they simply Bose condense. In this regard, it would be interesting to Melvinize the Sakai-Sugimoto model [155]; it has a better chance of describing a system conaining only fermionic atoms.

A nice check on our result for the viscosity and our understanding of the thermodynamics of the solution will be the location of the diffusion pole in the shear channel of the stress-tensor correlators.

3.A Details of Melvinization

In this appendix, we review the Null Melvin Twist, as formalized in a seven-step dance in [65].

3.A.1 Buscher Rules and Conventions

$$g'_{yy} = \frac{1}{g_{yy}} \qquad g'_{ay} = \frac{B_{ay}}{g_{yy}} \qquad g'_{ab} = g_{ab} - \frac{g_{ay}g_{yb} + B_{ay}B_{yb}}{g_{yy}} \tag{3.A.1}$$

$$\Phi' = \Phi - \frac{1}{2}\ln g_{yy} \qquad B'_{ay} = \frac{g_{ay}}{g_{yy}} \qquad B'_{ab} = B_{ab} - \frac{g_{ay}B_{yb} + B_{ay}g_{yb}}{g_{yy}} \tag{3.A.2}$$

3.A.2 The Hopf Vector on $\mathbb{P}^2$

In constructing our solutions we were forced to pick an isometry direction along S^5. A particularly convenient choice involved realizing S^5 as a Hopf fibration over $\mathbb{P}^2$, which we now review to make your life easier than ours was (if you don't already know this stuff).

The round metrics on $\mathbb{P}^n$ and S^{2n+1} may be elegantly expressed in terms of the left-invariant one-forms of SU(n). For SU(3), these can be written in coordinates as,

$$\sigma_1 = \frac{1}{2}(\mathrm{d}\theta\cos(\psi) + \mathrm{d}\phi\sin(\theta)\sin(\psi))$$

$$\sigma_2 = \frac{1}{2}(\mathrm{d}\theta\sin(\psi) - \mathrm{d}\phi\cos(\psi)\sin(\theta))$$

$$\sigma_3 = \frac{1}{2}(\mathrm{d}\psi + \mathrm{d}\phi\cos(\theta))$$

In terms of these 1-forms, the metrics on $\mathbb{P}^2$ and S^5 may be written,

$$ds^2_{\mathbb{P}^2} = \mathrm{d}\mu^2 + \sin^2(\mu)\left(\sigma_1^2 + \sigma_2^2 + \cos^2(\mu)\sigma_3^2\right)$$

$$ds^2_{S^5} = \mathrm{d}s^2_{\mathbb{P}^2} + \left(\mathrm{d}\chi + \sin^2(\mu)\sigma_3\right)^2$$

where χ is the local coordinate on the Hopf fibre and $\mathcal{A} = \sin^2(\mu)\sigma_3 = \frac{\sin^2(\mu)}{2}(\mathrm{d}\psi + \mathrm{d}\phi\cos(\theta))$ is the 1-form potential for the kahler form on $\mathbb{P}^2$ ($d\chi + \mathcal{A}$ is the vertical one-form along the Hopf fibration). This explicit coordinate presentation is necessary to verify that our various solutions in fact solve the full 10d IIB supergravity equations of motion, and to study the linearized equations of motion for the fluctuations.

3.A.3 Constructing the finite temperature solution

We now walk through the melvinization of the black D3-brane in all its majesty.

Step 1: We start with the black D3-brane solution,

$$ds^2 = \frac{1}{h}\left(-d\tau^2 f + dy^2 + d\vec{x}^2\right) + h\left(\frac{d\rho^2}{f} + \rho^2\left[ds^2_{\mathbb{P}^2} + (d\phi + \mathcal{A})^2\right]\right)$$

where $h^2 = 1 + \frac{R_A^4}{\rho^4}$ is the usual D3 harmonic function and $f = 1 + g = 1 - \frac{\rho_H^4}{\rho^4}$ is the emblackening factor. In what follows, nothing untoward will happen to the $d\vec{x}^2$, $d\rho^2$ or $ds^2_{\mathbb{P}^2}$ factors, so we'll drop those terms and reintroduce them after the dust settles. The truncated metric is thus,

$$ds^2 = \frac{1}{h}\left(-d\tau^2 f + dy^2\right) + h\rho^2(d\phi + \mathcal{A})^2$$

Step 2: Boost by γ, ie $\tau \to c\tau - sy$ with $c = \cosh(\gamma)$ and $c^2 - s^2 = 1$:

$$ds^2 = \frac{1}{h}\left(-d\tau^2(1 + gc^2) + dy^2(1 - gs^2) + 2d\tau dy(gcs)\right) + h\rho^2(d\chi + \mathcal{A})^2$$

Step 3: T-dualize along the dy isometry using the Buscher rules listed above:

$$ds^2 = -d\tau^2\frac{f}{h(1-gs^2)} + h\left(\rho^2(d\chi + \mathcal{A})^2 + dy^2\frac{1}{1-gs^2}\right) \tag{3.A.1}$$

$$B = 2dy \wedge d\tau\left[\frac{-gcs}{1-gs^2}\right] \qquad \Phi = \Phi_0 - \frac{1}{2}\ln\left[\frac{1-gs^2}{h}\right] \tag{3.A.2}$$

Step 4: Shift the local 1-form $d\chi$ to $d\chi + \alpha dy$ to give

$$ds^2 = -d\tau^2\frac{f}{h(1-gs^2)} + h\left(\rho^2(d\chi + \mathcal{A})^2 + dy^2\frac{1 + \rho^2\alpha^2(1-gs^2)}{1-gs^2} + 2dy(d\chi + \mathcal{A})(\alpha\rho^2)\right)$$

Note that α has dimensions of $\frac{1}{\text{length}}$.

Step 5: T-dualizing back along dy gives

$$\begin{aligned}
ds^2 &= -\frac{d\tau^2}{h(1-gs^2)}\left[f - \frac{g^2c^2s^2}{1+\rho^2\alpha^2(1-gs^2)}\right] + \frac{2dyd\tau}{h}\left[\frac{gcs}{1+\rho^2\alpha^2(1-gs^2)}\right] \\
&\quad + \frac{dy^2}{h}\left[\frac{1-gs^2}{1+\rho^2\alpha^2(1-gs^2)}\right] + h\rho^2(d\chi+\mathcal{A})^2\left[\frac{1}{1+\rho^2\alpha^2(1-gs^2)}\right] \\
B &= \frac{\alpha\rho^2}{1+\rho^2\alpha^2(1-gs^2)}(d\chi+\mathcal{A})\wedge\left[gcs\,d\tau + (1-gs^2)dy\right] \\
\Phi &= \Phi_0 - \frac{1}{2}\ln\left[\frac{1-gs^2}{h}\right]
\end{aligned}$$

Step 6&7: We now boost back by $-\gamma$ and take a double scaling limit $\alpha \to 0$ with $\alpha c = \beta$ held fixed. Since many terms do not survive this, it is easiest to do both steps at once and report only the result, adding back in all the terms we dropped in the first step,

$$\begin{aligned}
ds^2 &= \frac{1}{hK}\left[-d\tau^2(1+\beta^2\rho^2)f + dy^2(1-\beta^2\rho^2 f) + 2d\tau dy(\beta^2\rho^2 f)\right] \\
&\quad + \frac{1}{h}d\vec{x}^2 + h\left[\frac{d\rho^2}{f} + \rho^2 ds^2_{\mathbb{P}^2} + \frac{\rho^2}{K}(d\chi+\mathcal{A})^2\right] \\
B &= \frac{2\beta\rho^2}{K}(d\chi+\mathcal{A})\wedge(f\,d\tau + dy) \\
\Phi &= \Phi_0 - \frac{1}{2}\ln K
\end{aligned}$$

Note that β has dimensions of $\frac{1}{\text{length}}$.

Step 8: Finally, we take the near-horizon limit, $h \to R_A^2/\rho^2$. To compare with the solutions of [160, 16], it is convenient to switch variables to the radial radial coordinate

$$\frac{r}{R_A} = \frac{R_A}{\rho}$$

in terms of which the boundary is at $r = 0$ and the horizon at $r_H = R_A^2/R_H$. In terms of r and the parameter $\Delta = \beta R_A^2$ we have

$$\beta^2\rho^2 = \frac{\Delta^2}{r^2} \qquad h = \frac{r^2}{R_A^2} \qquad f = 1 - \frac{r^4}{r_H^4} \qquad K = 1 + \frac{\Delta^2 r^2}{r_H^4},$$

with the metric taking the form,

$$\begin{aligned}
ds^2 &= \frac{R_A^2}{r^2K}\Big[-d\tau^2(1+\frac{\Delta^2}{r^2})f + dy^2(1-\frac{\Delta^2}{r^2}f) + 2d\tau dy(\frac{\Delta^2}{r^2}f) \\
&\qquad + Kd\vec{x}^2 + K\frac{dr^2}{f} + r^2\left(Kds^2_{\mathbb{P}^2} + (d\chi+\mathcal{A})^2\right)\Big] \\
B &= 2\Delta\frac{R_A^2}{r^2K}(d\chi+\mathcal{A})\wedge(f\,d\tau+dy) \\
\Phi &= \Phi_0 - \frac{1}{2}\ln K.
\end{aligned}$$

Between the boundary and the horizon, K varies smoothly between 1 and $1+\frac{\Delta^2}{r_H^2}$. Importantly, the surface $r=r_H$, where $f\to 0$ and $B_t\to 0$, remains a non-singular null horizon. Near the horizon, ∂_τ is a timelike killing vector which is perpendicular to the null geodesics which span the horizon. We thus have a non-rotating black hole with $\Omega_H=0$. This might seem somewhat miraculous, since the geometry is not static but, like Kerr, only stationary, and so we might reasonably expect a Killing horizon outside the black hole. In fact, this construction, which preserved the near-horizon geometry at each step, had built into it that the horizon would be irrotational (and, in particular, have no additional killing horizon). We could introduce rotation by starting with a bifurcate killing horizon surrounding an ergosphere – *i.e.* by starting with a rotating black D3 – but, since we will exploit the unbroken rotational symmetry of our solution to compute the viscosity, we'll leave this generalization to later consideration.

The upshot is that we have a two-parameter family of finite-temperature solutions labeled by the r_H and Δ defined in units of R_A. This family has two simple and familiar limits, $\Delta\to 0$ and $r_H\to\infty$. Taking $\Delta\to 0$, which sends $K\to 1$, is easily seen to return us to the non-extremal black D3-brane solution with which we began.

Taking $r_H\to\infty$, by contrast, takes us to the globally non-singular Schrödinger geometry. To see this directly, it is useful to work in light-cone coordinates $t=(y-\tau)/\sqrt{2}$ and $\xi=(y+\tau)/\sqrt{2}$, in terms of which the solution becomes,

$$\begin{aligned}
ds^2 &= \frac{R_A^2}{r^2K}\left[-\frac{2\Delta^2}{r^2}f dt^2 + 2dtd\xi - \frac{g}{2}(dt-d\xi)^2 + Kd\vec{x}^2 + K\frac{dr^2}{f} + r^2\left(Kds^2_{\mathbb{P}^2}+(d\chi+\mathcal{A})^2\right)\right] \\
B &= \sqrt{2}\,\Delta\frac{R_A^2}{r^2K}(d\chi+\mathcal{A})\wedge([1+f]\,dt+[1-f]d\xi) \qquad \Phi=\Phi_0-\frac{1}{2}\ln K.
\end{aligned}$$

In the limit $r_H \to \infty$, which takes $f \to 1$ and $K \to 1$, the metric reduces to,

$$\begin{aligned} ds^2 &= \frac{R_A^2}{r^2}\left[-\frac{2\Delta^2}{r^2}dt^2 + 2dtd\xi + d\vec{x}^2 + dr^2\right] + R_A^2 ds^2_{S^5} \\ B &= 2\sqrt{2}\,\Delta\frac{R_A^2}{r^2}(d\chi + \mathcal{A}) \wedge dt \qquad\qquad \Phi = \Phi_0, \end{aligned}$$

which, upon compactifying on the S^5, is the Schrödinger geometry with $z = 2, d = 2$. Studying the finite-r_H solution near $r \ll r_H$ gives the same result. We have thus embedded a black hole in an asymptotically Schrödinger spacetime.

One final set of coordinates will be useful in the computations below. In terms of the dimensionless quantities $u = r^2/r_H^2 = R_H^2/\rho^2$, $\delta = \Delta/r_H = \beta R_H$ and $\mu = R_A/r_H = R_H/R_A$, the solution takes the form,

$$\begin{aligned} ds^2 &= \frac{\mu^2}{uK}\left[-\frac{2\delta^2}{u}f\,dt^2 + 2dtd\xi - \frac{g}{2}(dt - d\xi)^2 + K d\vec{x}^2\right. \\ &\qquad \left. + \frac{KR_A^2}{4\mu^2 u f}du^2 + \frac{uR_A^2}{\mu^2}\left(K ds^2_{\mathbb{P}^2} + (d\chi + \mathcal{A})^2\right)\right] \\ B &= \sqrt{2}\,\delta\frac{\mu R_A}{uK}(d\chi + \mathcal{A}) \wedge ((1+f)\,dt + (1-f)d\xi) \\ \Phi &= \Phi_0 - \frac{1}{2}\ln K. \end{aligned}$$

where

$$f = 1 - u^2 \qquad K = 1 + \delta^2 u,$$

These variables simplify many of the computations.

3.B Frame (in)dependence of the viscosity calculation

After compactifying to D dimensions, the string frame metric is related to the D-dimensional Einstein-frame metric by the Weyl rescaling

$$g_{\mu\nu}^{E,D} = e^{\frac{4\Phi}{D-2}} g_{\mu\nu}^{(\mathrm{str})}.$$

In our solution, the dilaton is

$$e^{2\Phi} = \frac{1}{K}$$

so we have

$$g_{\mu\nu}^{E,D} = K^{\frac{2}{2-D}} g_{\mu\nu}^{(\mathrm{str})}$$

In the special case $D = 10$, this says $g_{\mu\nu}^{E,10} = K^{\frac{1}{4}} g_{\mu\nu}^{(\mathrm{str})}$.

Now consider the wave equation in a conformal frame reached by an arbitrary power of K, where the metric is:

$$g_{\mu\nu}^{a} = K^{a} g_{\mu\nu}^{(\mathrm{str})}.$$

We have

$$\det g^{a} = K^{Da} \det g^{(\mathrm{str})}, \quad \sqrt{g^{a}} = \frac{K^{5a-1}}{2u^3 r_H^4} \mathrm{vol}_{10-\mathrm{D}}$$

where $\mathrm{vol}_{10-\mathrm{D}}$ is the constant volume of the compact dimensions, which will scale out of the wave equation.

The wave equation for a scalar in this background is

$$\begin{aligned} \Box\phi &= \frac{1}{K^{aD/2}} 2u^3 K \partial_u \left(K^{-a} 4u^2 f \frac{K^{aD/2-1}}{2u^3} \partial_u \phi \right) + \ldots \\ &= \frac{4u^3}{K^{aD/2-1}} \partial_u \left(\frac{K^{a\left(\frac{D}{2}-1\right)-1} f}{u} \partial_u \phi \right) + \ldots \end{aligned}$$

The einstein-frame condition above says that in D-dimensional Einstein frame, $K^{a} = e^{-\frac{4\Phi}{D-2}}$ which says

$$K^{a\left(\frac{D}{2}-1\right)-1} = 1.$$

So we see that in einstein frame, in whatever number of dimensions we want to live in, say 10 or 6, the factor K does not appear in the wave equation.

This in turn implies that the viscosity is independent of $\delta \equiv \frac{\beta}{r_H}$.

3.C Comments on reduction to five dimensions

Let $\Gamma = -\frac{1}{2}\ln K$; this is the profile for both the 10d dilaton and the KK scalar associated to the Hopf direction. The following two equations are true:

$$0 = -\partial_\mu \left(\sqrt{g} F^{\mu\nu} e^{(\nu-3)\Gamma} \right) + z(z+d)\sqrt{g} e^{(3\nu-1)\Gamma} A^{\nu}$$

$$0 = 16\partial_\mu \left(e^{(3\nu-1)\Gamma}\sqrt{g}g^{\mu\nu}\partial_\nu\Gamma\right) + \sqrt{g}\left(e^{(\nu-3)\Gamma}F^2 + 2z(z+d)e^{(3\nu-1)\Gamma}A^2\right)$$

where

$$e^{2\Gamma} \equiv \frac{1}{K}$$

$$A = \frac{2\beta}{r^2 K}\left(f d\tau + dy\right)$$

and

$$ds^2 = K^\nu \frac{1}{r^2 K}\left(-\left(1+\frac{\beta^2}{r^2}\right)f d\tau^2 - \frac{\beta^2 f}{r^2}2dyd\tau + \left(1-\frac{\beta^2}{r^2}f\right)dy^2 + Kd\vec{x}^2 + K\frac{dr^2}{fr^2}\right)$$

These are the respective equations of motion for A_ν and Φ for a five-dimensional action of the form

$$S_5 = \int d^5x\sqrt{g}\left(R - c_1 e^{a_1\Phi+b_1\sigma}\left(\partial\Phi\right)^2 - c_2\left(\frac{1}{4}e^{a_2\Phi+b_2\sigma}F^2 + \frac{m_A^2}{2}e^{a_3\Phi+b_2\sigma}A^2\right)\right) + \dots$$

with $m_A^2 = \frac{z(z-d)}{L^2}$ as usual, and $a_1 + b_1 = a_3 + b_3 = 3\nu - 1, a_2 = \nu - 3$ and $a_2 = a_3$. $\nu = \frac{1}{3}$ is 5d Einstein frame. Here σ is the other scalar arising from the KK reduction. The ... indicate terms that do not depend on Φ, A. We have not yet been able to determine the rest of the action.

Chapter 4

On the particle number in Galilean holography

The material in this chapter and the third item in the synopsis (Chapter 1) appeared in "The Particle number in Galilean holography. " with John McGreevy [19] and is reprinted with the permission of *JHEP*.

4.1 Introduction

Particle production is a dramatic, necessary consequence of relativistic field theory. There is no particle production in Galilean-invariant field theories, which therefore have an extra conserved quantity. This quantity is often thought of as the particle number, but (since we can and will work in units of the mass of one particle) it is equivalent to the total rest mass. In systems with multiple species, to be discussed more below, the latter definition is the more useful one.

In the holographic realizations of NRCFTs discussed in the previous chapters, the symmetry associated with the conserved rest mass is realized geometrically in the gravity dual as the isometry of a circle, whose coordinate we call ξ. Compactifying on a circle with circumference L_ξ produces a spectrum of possible values of the rest mass of states in the theory of the form

$$\{\text{masses}\} = \frac{1}{2\pi L_\xi}\,\mathbb{Z}_+ \quad . \tag{4.1.1}$$

The main purpose of this (somewhat polemical) article is to point out that this particular spectrum is not a necessary consequence of the existence of a gravity dual.

The form of the spectrum (4.1.1) seems to be responsible for the strange thermodynamics found in [123, 90, 1, 114]:

$$F \sim -\frac{T^4}{\mu^2}, \ \mu < 0. \tag{4.1.2}$$

This is quite different from the behavior of unitary fermions, where in particular the chemical potential is positive, and the free energy scales like a positive power of μ. These theories are closely related to relativistic field theories, via (modifications of) the discrete lightcone quantization procedure (DLCQ). This fact is made particularly vivid in the calculation of the free energy (4.1.2) from a free relativistic field theory in DLCQ by [22]. The modifications of DLCQ in [26, 7] (associated with "β-deformation") simplify the theory by removing most of the troublesome [89] lightcone zeromodes, but do not change the spectrum of the lightcone momentum operator, $i\partial_\xi$.

In this work, our goal is to learn how to construct gravity duals for NRCFTs with other (ideally, more realistic) spectra. We demonstrate that it is not necessary to realize the Schrödinger algebra in a gravity dual entirely via isometries of the bulk metric. It was natural to try to realize the full algebra by isometries, since the obviously-geometric momentum and boost generators commute to the particle number operator $\hat{N}$,

$$[K_i, P^j] = iN\delta^i_j. \tag{4.1.3}$$

However, here we show that this algebra can be realized without the introduction of a ξ dimension, if the boost generator acts by gauge transformations on fields charged under an additional abelian gauge symmetry in the bulk. The construction is quite similar to the way that these symmetries are realized on states of a quantum system: a Galilean boost by velocity $\vec{v}$ acts on the phase of the wavefunction (in the Schrödinger representation) of a particle of mass m by

$$\psi(x,t) \mapsto e^{im\left(\frac{1}{2}v^2 t + \vec{v}\cdot\vec{x}\right)}\psi(x - vt, t) \ ; \tag{4.1.4}$$

from this expression one can show that (4.1.3) is satisfied.

Using this idea we construct solutions describing $d = 2$, $z = 2$ NRCFT without the additional circle. For most of this chapter we will employ a practical approach to holography advocated in *e.g.* [79, 128]: we do not yet know the constraints that quantum gravity imposes on effective field theories of gravity coupled to matter (known examples [171, 9, 88] are not very forceful, and it is clear that our grasp on the space of less-supersymmetric string vacua is poor), and so we will employ the simplest gravity models with which we can approach the physics of interest. We will, however, find it useful in §4.4 to lift one of our solutions to 11-dimensional supergravity in order to resolve a curvature singularity. That solution describes a system (at finite density and at zero temperature) with a gap for the charged excitations; it appears to provide a holographic description of a Mott insulator. This is an improvement over a previous holographic realization of an insulating state [138], which had zero density. We also succeed in constructing some examples where there are several species of particles, so that the spectrum of the number operator is not just integer multiples of a single mass; this is described in §4.5. Finally, in §4.6, we discuss a conjugate issue, namely whether the NRCFTs described by the constructions of [160, 16] have superfluid groundstates (at low temperature and finite density), and what such a state would look like from the point of view of the gravity dual. We relegate to an appendix a curious black hole solution with the new realization of asymptotic Schrödinger symmetry.

4.2 Getting rid of the ξ direction

In this section, we study the dimensional reduction on the particle-number circle of the systems discussed previously in [160, 16]. We are doing this because it provides a proof of principle that there can be gravity theories with Schrödinger symmetry which don't have this annoying extra dimension. Our real goal is to find new solutions where the spectrum of the mass operator can be different (*i.e.* not the KK tower of momentum modes on a circle), and where the thermodynamics may therefore be more like that of unitary fermions. Our immediate goal is to understand how the symmetries are realized.

A concern which remains even after dimensionally reducing to replace the role of the ξ-dimension with a gauge field in a lower-dimensional description is charge-conjugation invariance. In a relativistic QFT with charge-conjugation invariance (like the one living in the bulk here), the spectrum of a $U(1)$ symmetry must include both positive and negative charges. Below, we explicitly break this symmetry by imposing boundary conditions which introduce a background electric field.

We studied a black hole solution asymptotic to the metric (2.2.1) in the previous chapter [123, 90, 1]; this describes the dual NRCFT at finite density and finite temperature. In the black hole solution, ξ is not null everywhere because $g_{\xi\xi}$ is not identically zero as in the vacuum solution. This implies that the radius of the circle is non-zero in the bulk and the supergravity approximation can be trusted in regions where the radius is large compared to the string length scale. Thus the nonzero $g_{\xi\xi}$ component acts like a regulator and this fact will be used here to construct alternate holographic descriptions of Schrödinger algebra.

Dimensional reduction of this solution along ξ yields a lower-dimensional system with asymptotic Schrödinger symmetries. The matter content of the lower-dimensional gravity theory consists of a massive vector field, $U(1)$ gauge field and two scalars (higher-dimensional dilaton and the radion). We would like to have a simpler system that can aid us in understanding the lower dimensional realization of Schrödinger symmetry.

In §3.2.4 of the previous chapter, we studied a scaling limit of this black hole solution (finite μ, T) which had zero temperature, but had a non-trivial $g_{\xi\xi}$ component (let's call this solution $Sch_{\Omega\neq 0,T=0}$). This solution is singular in the IR ($r \to \infty$) and should not be taken too seriously. We will use it here as a helpful device to learn about possible bulk realizations of the Schrödinger algebra.

4.2.1 Dimensional Reduction of $Sch_{\Omega\neq 0,T=0}$

The geometry of $Sch_{\Omega\neq 0,T=0}$ is described by the following line element

$$ds_5^2 = \frac{1}{r^2\kappa^{2/3}}\left(\frac{-dt^2}{r^2} + 2dtd\xi + (\kappa-1)r^2 d\xi^2\right) + \kappa^{1/3}\left(\frac{d\vec{x}^2 + dr^2}{r^2}\right) \tag{4.2.1}$$

where $\kappa = 1+\Omega^2 r^2$, for some constant Ω which determines the density. This can be obtained as a classical solution of the following action[1]:

$$S_5 = C_0 \int d^5x\sqrt{-g_5}\left[R_5 - \frac{4}{3}(\nabla\Phi)^2 + V(\Phi) - \frac{1}{4}e^{-8\Phi/3}F_5^2 - \frac{m^2}{2}A_5^2\right] \tag{4.2.2}$$

with $\Phi = -\frac{1}{2}\log\kappa$, $A_5 = \frac{1}{2}r^{-2}\kappa^{-1}\left(dt + (\kappa-1)r^2 d\xi\right)$ and $F_5 = dA_5$. In this subsection, we will perform a series of manipulations using this solution to identify a 4-dimensional

[1]The potential $V(\Phi)$ is [123, 90] $V(\Phi) = 4e^{2\Phi/3}\left(e^{2\Phi} - 4\right)$ but it will disappear soon.

system that admits asymptotic solutions which respect the Schrödinger group in two space dimensions.

Dimensional reduction of the above action along ξ direction yields a lower-dimensional system with Schrödinger symmetry. In this system, the particle number symmetry is realized as a bulk gauge symmetry. The $g_{\xi\xi}$ component of the higher-dimensional metric appears as a scalar field (the radion field $e^{2\sigma}$) in the lower-dimensional system. With the metric ansatz

$$ds_5^2 = ds_4^2 + e^{2\tilde{\sigma}} \left(d\xi + B\right)^2 \tag{4.2.3}$$

the lower-dimensional action can be written as

$$S_4 = C_0 L_\xi \int d^4x \, e^{\tilde{\sigma}} \sqrt{-G_D} \Big[R_D - \frac{4}{3} \left(\nabla\Phi\right)^2 + V\left(\Phi\right) - \frac{1}{4} e^{-8\Phi/3} F_D^2 - \frac{m^2}{2} A_D^2 +$$

$$4\nabla\Phi \cdot \nabla\tilde{\sigma} - \frac{e^{2\tilde{\sigma}}}{4} \left(dB\right)^2 - \frac{1}{2} e^{-8\Phi/3} \left(\nabla A_\xi\right)^2 - \frac{m^2}{2} A_\xi^2 + \mathcal{L}_{int}\left(A_\xi, B, A\right) \Big] \,. \tag{4.2.4}$$

Note that the line element in (4.2.1) can be written as

$$ds_5^2 = \frac{1}{r^2 \kappa^{2/3}} \left(\frac{-dt^2}{r^2} + \Omega^2 r^4 \left(d\xi + \frac{dt}{r^4 \Omega^2} \right)^2 - \frac{dt^2}{\Omega^2 r^4} \right) + \kappa^{1/3} \left(\frac{d\vec{x}^2 + dr^2}{r^2} \right) \,. \tag{4.2.5}$$

Scaling t by $\Omega Q^{1/2}$ and scaling ξ by $Q^{-1/2}/\Omega$ in the above expression we get

$$ds_5^2 = \frac{1}{r^2 \kappa^{2/3}} \left(\kappa Q \frac{-dt^2}{r^4} + r^4/Q \left(d\xi + Q \frac{dt}{r^4} \right)^2 \right) + \kappa^{1/3} \left(\frac{d\vec{x}^2 + dr^2}{r^2} \right) \,. \tag{4.2.6}$$

Under this rescaling $A_t \longrightarrow \Omega Q^{1/2} A_t$. Hence, the higher-dimensional line element can be written as

$$ds_5^2 = ds_D^2 + e^{2\tilde{\sigma}} \left(d\xi + B\right)^2 \tag{4.2.7}$$

$$e^{2\tilde{\sigma}} = \frac{r^2}{Q\kappa^{2/3}}, \qquad B = \frac{Q}{r^4} dt \,. \tag{4.2.8}$$

The $4-$dimensional line element in the above expression is

$$ds_D^2 \equiv (G_D)_{\mu\nu} dx^\mu dx^\nu = \kappa^{1/3} \left(-\frac{Q dt^2}{r^6} + \frac{d\vec{x}^2 + dr^2}{r^2} \right) \tag{4.2.9}$$

If we now define $e^{2\sigma} = \Omega^2 e^{2\tilde{\sigma}}$ and take the scaling limit, $\Omega \to 0$, holding σ fixed, we are left with an extremum of the much-simpler action

$$S_4 = C_0 L_\xi \int d^4x \; e^{\sigma} \sqrt{-G_D} \Big[R_D - 2\Lambda - \frac{e^{2\sigma}}{4} (dB)^2 \Big] \,, \tag{4.2.10}$$

where we have chosen units so that the cosmological constant is $\Lambda = -6$. In the above limit, $A = 0$, $\Psi = 0$, $\Phi = 0$ and $\kappa = 1$; the 4-dimensional solution is (4.2.9) with $\kappa = 1$. Note that Q is related to the chemical potential. When $Q \to \infty$, the Schrödinger symmetries become an exact symmetry of the above system; however, the metric becomes degenerate in the $Q \to \infty$ limit. Rewriting the 'string frame' action (4.2.10) in 4d Einstein frame (and throwing away the fields A, Ψ, Φ which vanish) we see that

$$ds_E^2 = e^{\sigma} \left(-Q \frac{dt^2}{r^6} + \frac{d\vec{x}^2 + dr^2}{r^2} \right), \quad B = Q \frac{dt}{r^4}, \quad e^{\sigma} = \frac{r}{\sqrt{Q}} \tag{4.2.11}$$

is a solution of the simple action

$$S_4^E = \mathfrak{K} \int d^4x \sqrt{-g_4} \Big[R_4 - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4} (dB)^2 - \frac{3}{2} (\partial\sigma)^2 \Big] \,, \tag{4.2.12}$$

where we have named $\mathfrak{K} \equiv C_0 L_\xi$ the effective 4d coupling. Note that the apparent strong coupling behavior of the action for the gauge field B at the boundary ($g_{\text{eff}}^{-2} \sim e^{3\sigma} \sim r^3 \to 0$) is an artifact of dimensional reduction.

4.2.2 Symmetry Generators

Let us try to understand how the Schrödinger symmetry group is realized by the above action and asymptotics. It is clear that the symmetries of the Schrödinger group are not realised as isometries in the lower-dimensional theory: the putative symmetry generators in the lower-dimensional theory do not solve the Killing equation. The metric is of the Lifshitz form [106] and seems to have scaling symmetry with dynamical exponent $z = 3$. What equation determines the symmetry generators of the lower-dimensional action? It is not hard to guess that the appropriate symmetry generators should solve the equation obtained by dimensional reduction of the higher-dimensional Killing equation.

So the symmetry generators of the lower-dimensional theory are:

- Particle Number: The $U(1)$ gauge charge associated with the massless gauge field B is the particle number. This acts by $B \to B + d\lambda$, and by phase rotations on charged fields in the bulk, of which we should include one or more. Let us introduce such a field $\Phi \equiv |\Phi|e^{i\varphi}$ of charge ℓ; we take Φ to vanish in the solution shown above. ℓ is the mass of the associated particle.

- Translations and rotations are realized as usual by isometries.

- Galilean boosts act as follows:

$$t \to t, \quad \vec{x} \to \vec{x} - \vec{v}t, \quad \varphi \to \varphi + \ell \left(\frac{1}{2} v^2 t + \vec{v} \cdot \vec{x} \right), \tag{4.2.13}$$

  where φ is the phase of a field of charge ℓ under the particle-number gauge symmetry. The role previously played by ξ in the Schrödinger geometry is now played by the phase φ of charged bulk fields. In summary, the boost generator is:

$$K^i = -t\partial_i + \text{gauge shift}$$

  where the gauge transformation parameter is $\lambda = \frac{1}{2}v^2 t + \vec{v} \cdot \vec{x}$. Note the similarity to the action in quantum mechanics given in Eqn. (4.1.4).

- Scale symmetry acts by

$$D = -2t\partial_t - x^i\partial_i - r\partial_r;$$

The generators of these symmetries satisfy the Schrödinger algebra.

The asymptotic profiles of the fields are *not* preserved by these transformations, but one can show, as follows, that they are nevertheless (asymptotic) symmetries of the system. The higher dimensional (5D) Killing equation can be written as

$$\delta_\eta \left(G_D\right)_{AB} = \mathcal{L}_\eta \left(G_D\right)_{AB} = 0 \ . \tag{4.2.14}$$

If the above equation is only true as $r \to 0$ (which we denote by ≈ 0), as is the case in the solutions described above, then the symmetry is realized only asymptotically. The lower dimensional (4D) metric (g) does not satisfy the 4d Killing equation, *i.e.* $\delta_\eta g_{\mu\nu} \neq 0$. The above equation (4.2.14), however, implies[2]

$$\delta_\eta\left(e^{-\sigma}g_{\mu\nu}+e^{2\sigma}B_\mu B_\nu\right)\approx 0 \tag{4.2.18}$$

$$\delta_\eta\left(e^{2\sigma}B_\mu\right)\approx 0 \tag{4.2.19}$$

$$\delta_\eta\left(e^{2\sigma}\right)\approx 0 \tag{4.2.20}$$

These quantities have the transformation properties of tensors. We also know (from its higher-dimensional origin) that the action can be written as a functional of these quantities, that is

$$S_D[g,B,\sigma]=S[e^{-\sigma}g_{\mu\nu}+e^{2\sigma}B_\mu B_\nu,e^{2\sigma}B_\mu,e^{2\sigma}]\ . \tag{4.2.21}$$

When the symmetries are realized as isometries, $\delta_\eta S$ vanishes as a consequence of $\delta_\eta G_{AB}$ vanishing. In the present case, $\delta_\eta S$ will vanish as a consequence of $\delta_\eta\left(e^{-\sigma}g_{\mu\nu}+e^{2\sigma}B_\mu B_\nu\right)$, $\delta_\eta\left(e^{2\sigma}B_\mu\right)$ and $\delta_\eta\left(e^{2\sigma}\right)$ vanishing.

In the solutions described above, these quantities only vanish asymptotically near the boundary. Note that we do not know a solution of the lower-dimensional system (4.2.12) which exactly preserves the Schrödinger symmetry. This is perhaps unsurprising given that such a solution would correspond to the vacuum of a Galilean-invariant system, a very boring state indeed. Rather, the surprising fact is that the previous holographic realizations [160, 16] did provide such a solution.

From the form of B in (4.2.11), we see that the solution (4.2.9) has non-zero chemical potential ($\mu\neq 0$), but charge density zero. This can happen for example if the chemical potential is smaller than the particle mass.

[2] The transformation rules for the lower-dimensional fields can be obtained from the transformation rule for the higher-dimensional metric:

$$\delta_\eta g_{\mu\nu}=[g_{\mu\rho}\partial_\nu\eta^\rho+g_{\rho\nu}\partial_\mu\eta^\rho+\eta^\rho\partial_\rho g_{\mu\nu}] \tag{4.2.15}$$

$$\delta_\eta B_\mu=[B_\rho\partial_\mu\eta^\rho+\eta^\rho\partial_\rho B_\mu]+\partial_\mu\eta^{D+1} \tag{4.2.16}$$

$$\delta_\eta e^{2\sigma}=\left[\eta^\rho\partial_\rho e^{2\sigma}\right]. \tag{4.2.17}$$

Note that the quantities within [ ] are the changes due to the coordinate transformations in the lower dimensions, while the transformation of x^{D+1} generates field transformations.

4.2.3 Wave equation

The wave equation for a probe scalar field with charge ℓ under the Kaluza Klein gauge field (and mass m^2) takes the following form

$$\left(-\omega^2 r^6 + m^2 + r^2(2\ell\omega + k^2)\right)\Phi - r^{d+3}\partial_r\left(r^{-d-1}\partial_r\Phi\right) = 0\ . \tag{4.2.22}$$

Notice that the first term in this equation is unimportant for the boundary behavior ($r \to 0$), but does spoil the Schrödinger invariance of the equation (and renders us and Mathematica unable to solve it analytically).

The D-dimensional (Einstein-frame) action that produces this equation of motion is perhaps surprising:

$$S_{\text{probe}}[\Phi] = \int \sqrt{g}\left[|(\partial - i\ell B)\Phi|^2 - \left(\ell^2 e^{-3\sigma} + m^2 e^{-\sigma}\right)|\Phi|^2\right] \quad . \tag{4.2.23}$$

This coupling to the background scalar σ is required in order that solutions for Φ represent the Schrödinger symmetry.

4.3 Black hole solution

The following is a black hole solution of (4.2.12) that asymptotes to the solution written in (4.2.11):

$$ds_E^2 = e^\sigma\left(-Qf\frac{dt^2}{r^6} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2 f}\right),\quad B = Q\frac{(1+f)dt}{2r^4},\quad e^{2\sigma} = \frac{r^2}{Q} \tag{4.3.1}$$

where $f = 1 - r^4/r_H^4$. The above solution can be obtained (by dimensional reduction) from the following five dimensional solution of Einstein's equation (with negative cosmological constant):

$$ds_5^2 = \frac{Q}{4r_H^8}r^2 dt^2 + \frac{d\vec{x}^2}{r^2} + \frac{(1+f)d\xi dt}{r^2} + \frac{dr^2}{fr^2} + \frac{r^2}{Q}d\xi^2 \tag{4.3.2}$$

The scaling symmetry of Sch algebra relates solutions with different values of Q and r_H. These solutions are also related to each other through the lightcone symmetry: $t \to \lambda t, \xi \to \lambda^{-1}\xi$. Even though compactification of ξ direction breaks this symmetry, this "symmetry" relates a system with chemical potential μ and temperature T to the system with chemical potential μ/λ^2 and temperature T/λ. This transformation along with the conformal Ward identity fixes the form (as a function of chemical potential and temperature) of the free energy [123].

A curious feature of the lower dimensional solution is the fact that the gauge field is non-vanishing at the horizon. This is not an indication that the solution is irregular. In fact, this solution was obtained from a regular solution in more dimensions. The gauge field obtained by dimensional reduction of the solution in [123, 90, 1] also has this feature.

Note also that t is not a time-like direction in (4.3.2). However, dimensional reduction of the higher-dimensional system results in t becoming a time-like direction. This feature can also be seen in rotating black hole solutions.

4.3.1 Thermodynamics

The temperature of the black hole in (4.3.1) is $T = \frac{\sqrt{Q}}{2\pi r_H^3}$ and the entropy density is

$$s = \frac{\mathfrak{K}}{\sqrt{Q} r_H} \,. \tag{4.3.3}$$

The energy density, pressure and free energy can be computed from the regularized action. It is possible to regularize the on-shell action and boundary stress tensor with the following boundary counterterms:

$$S_{ct} = \mathfrak{K} \int_{bdy} d^3x \sqrt{\gamma} \left(-4e^{-\frac{1}{2}\sigma} + e^{3\sigma} \frac{1}{2} n^r B^\mu F_{r\mu} \right) \,. \tag{4.3.4}$$

Note that the second term in (4.3.4) changes the boundary condition on the gauge field from Dirichlet to Neumann; this means that we are in the canonical ensemble (fixed ρ). In its origin as the dimensional reduction of the previous $d+3$-dimensional Schrödinger solution, this 'Neumannizing' term results from the dimensional reduction of the Gibbons-Hawking term.

In the higher-dimensional system, the number density is given by the momentum along the ξ direction. In the lower dimensional system this momentum appears as the charge density of the black hole which is given by

$$\rho = \frac{N}{L_x L_y} = \frac{\mathfrak{K}}{L_x L_y} \int_{bdy} d^2x \sqrt{\gamma} n^r e^{3\sigma} F_r^t \sim Q^{-1} \tag{4.3.5}$$

Using this we find

$$-\mathcal{F}=\mathcal{P}=\mathcal{E}=\frac{1}{2}\mathfrak{K}\rho^{2/3}T^{4/3}\sim\frac{T^4}{\mu^2} \tag{4.3.6}$$

where $\mathcal{C}$ is a numerical constant. The chemical potential, read off from $\frac{\partial\mathcal{F}}{\partial\rho}$, is

$$\mu\sim\frac{T^{4/3}}{\rho^{1/3}}\quad\text{or}\quad\rho\sim\frac{T^4}{\mu^3}\;. \tag{4.3.7}$$

The form of the thermodynamic quantities are the same as that in [123, 90, 1]. In the following section, we will present a solution describing a NRCFT with a finite density at zero temperature, which has a non-zero free energy (unlike the $T\to 0$ limit of (4.3.6)).

The solution (4.3.1) and the solution (4.2.11) with periodic imaginary time are saddle points of the same action. However, for any $T>0$, the black hole solution (4.3.1) has a smaller on-shell action and hence its contribution dominates. Like in planar AdS, the would-be Hawking-Page transition is at $T=0$ (where the two solutions coincide); unlike in AdS, here this does not follow from scale invariance. However, it has been brought to our attention (by Tom Faulkner) that the 5d uplift of the solution in this section is in fact isometric to the AdS_5 black brane solution; this explains the similarity in the phase diagram.

4.4 A Holographic Mott Insulator?

Let us now look at a marginal deformation of the above system; this will lead to an interesting new family of solutions. We will do this by adding a massless scalar field Ψ in the bulk. The corresponding operator will turn out to be marginally relevant in the presence of finite density. Consider the following action with two scalar fields:

$$S_4^E=\mathfrak{K}\int d^4x(-g_4)^{1/2}\Big[R_4-2\Lambda e^{-\sigma}-\frac{e^{3\sigma}}{4}(dB)^2-\frac{3}{2}(\partial\sigma)^2-\frac{1}{2}(\partial\Psi)^2\Big]\;. \tag{4.4.1}$$

The following background is a saddle point of the above action which has asymptotic Sch symmetry:

$$\begin{gathered}ds_E^2\left(\widehat{Sch}\right)=e^{\sigma}\left(-QK_x^2\frac{dt^2}{r^6}+K_x\frac{d\vec{x}^2}{r^2}+\frac{dr^2}{r^2}\right)\\ B=Q\frac{dt}{r^4},\quad e^{2\sigma}=\frac{r^2}{Q},\quad e^{2\Psi/\sqrt{5}}=\frac{1+\varsigma^2r^4/Q^2}{1-\varsigma^2r^4/Q^2}\end{gathered} \tag{4.4.2}$$

where $K_x^2 = 1 - \varsigma^4 r^8/Q^4$. The geometry is cut off at $r = r_0 = \sqrt{Q/\varsigma}$ where $K_x(r_0) = 0$. There is a curvature singularity at $r = r_0$, which we resolve below. ς is a dimensionless parameter describing the source for the operator dual to Ψ; this is a marginally relevant operator whose running produces the dimensional trasmutation scale r_0 in the solution. This solution has non-zero energy, presure, density and free energy[3], but has zero entropy. Note that the number density, identical to the calculation of (4.3.5), is Q^{-1} and the chemical potential is $\varsigma^2 Q^{-1}$.

The curvature singularity at $r = r_0$ can be resolved by dimensional oxidation. In general, dimensional reduction of a regular solution along a circle action with degenerate fibers can result in a curvature singularity in the lower dimensional metric [64]. In the next subsection we will show that such a resolution is available here.

The equations of motion have a symmetry which takes $\Psi \to -\Psi$. If we identify Ψ with the dilaton field, as we will in the next subsection, this transformation is an S-duality transformation. In the solution obtained by the action of this transformation (this reverses the sign of ς^2), the coupling dual to Ψ is marginally *irrelevant*. The phase diagram is thus similar to the BCS RG flow, where an attractive/repulsive coupling is relevant/irrelevant. We note, however, that even in our gravity solution for the marginally irrelevant case, the flow ends at a finite location in the bulk; perhaps this can be attributed to the strong coupling in the dual frame.

4.4.1 Lift to eleven dimensions and mass gap

We begin our journey to a smooth uplift of the solution (4.4.2) by noting that the four dimensional action in (4.4.1) can be obtained as a consistent truncation of the following five dimensional action:

$$S_5^E = C_0 \int d^5x(-g_5)^{1/2} \Big[R_5 - 2\Lambda - \frac{1}{2} (\partial\Psi)^2 \Big] . \tag{4.4.3}$$

In particular, the equations of motion of S_5^E with ansatz

$$ds_5^2 = e^{-\sigma} ds_E^2 \left(\widehat{Sch}\right) + e^{2\sigma} (d\xi + B)^2 = ds_E^2 (Sch) + e^{2\sigma} (d\xi + B)^2 \tag{4.4.4}$$

[3] $\mathcal{E} = \mathcal{P} = -\mathcal{F} \sim \varsigma^2 Q^{-2}$

are the equations of motion of (4.4.1); we have defined

$$ds_E^2\left(Sch\right) \equiv e^{-\sigma} ds_E^2\left(\widehat{Sch}\right) = -QK_x^2\frac{dt^2}{r^6} + K_x\frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} \quad . \tag{4.4.5}$$

In fact, the action in (4.2.12) was obtained by dimensional reduction of a five dimensional system related to (4.4.3) by turning off Ψ.

We pause on our path to eleven dimensions to make some comments about the geometry (4.4.3). The asymptotics of the 5d metric are precisely AdS with a light-like identification:

$$ds^2 = \frac{2d\xi dt + d\vec{x}^2 + dr^2}{r^2}; \tag{4.4.6}$$

this is the realization of Schrödinger symmetry described in [66, 21]. Note that with the ansatz in (4.4.4), a gauge transformation of the B field which does not fall off at the boundary has a dramatic effect on the asymptotics. For example, the transformation $B \to B + \alpha dt$ is equivalent in the higher-dimensional description to a redefinition of the ξ coordinate by $\xi \to \xi + \alpha t$. This violates the periodicity of the ξ coordinate and is not an equivalence relation. Such a transformation is precisely what would be required in order to set the gauge field to zero at the IR boundary $B_t(r_0) = 0$.

Some evidence that this solution is not the result of Melvinization of a relativistic geometry is the fact that the free energy is finite at zero temperature and finite chemical potential; this is hard to get from a $T \to 0$ limit of $F \sim -T^4/\mu^2$.

The five-dimensional action (4.4.3) can in turn be obtained as a consistent truncation of type IIB supergravity [123]. Specifically, the action in (4.4.3) can be obtained from the consistent truncation of [123] by turning off the massive vector as well as the breathing and squashing modes u, v. This allows us to lift the solution in (4.4.2) to the following solution of type IIB supergravity:

$$ds_{10}^2 = ds_E^2\left(Sch\right) + e^{2\sigma}\left(d\xi + B\right)^2 + ds^2\left(S^5\right),$$

$$F_5 = 4\left(\Omega_5 + \star\Omega_5\right), \text{ and } \Phi = \Psi \tag{4.4.7}$$

where Φ is the IIB dilaton. However, there is a curvature singularity at $r = r_0$ even in this ten dimensional metric. The presence of this singularity is related to the non-trivial profile of the dilaton, consistent with our interpretation above in terms of dimensional transmutation. It is convenient to think of the dilaton as the radius of a compact direction in eleven dimensions [167]. This suggests that the singularity in the 10-D metric can be resolved by lifting it to M-theory. The details of the lift are described in Appendix B. After performing the lift, we get the following 11-dimensional solution, which is regular:

$$ds_{11}^2 = e^{-\Psi/6}\left[ds_E^2\left(Sch\right) + e^{2\sigma}\left(d\xi + B\right)^2 + ds^2\left(\mathbb{CP}^2\right) + d\chi_1^2\right] + e^{4\Psi/3}d\chi_2^2$$

$$F_4 = 2J \wedge J + 2J \wedge d\chi_1 \wedge d\chi_2 \tag{4.4.8}$$

where J is the Kähler form on $\mathbb{CP}^2$. We can now get two solutions of type IIA from this 11 dimensional solution - (a) by reducing along χ_1 and (b) by reducing along χ_2. The first reduction produces a regular metric (with a smoothly shrinking circle) and a constant dilaton, while the second system has a metric with a curvature singularity and non-trivial dilaton profile. The second system is related to the type IIB solution in (6.B.1) by T-duality. The two type-II solutions are related by S-duality.

Note that in the presence of fermion fields (as in eleven-dimensional supergravity), the regularity of the solution (6.B.4) requires antiperiodic boundary conditions around the χ_2 circle for the fermions, since in the neighborhood of r_0, χ_2 is merely an angle in polar coordinates in $\mathbb{R}^2$. This explicitly breaks any supersymmetries.

$\varsigma \equiv Q/r_0^2$ is a dimensionless parameter. It can be considered a perturbation of the non-normalizible falloff of Ψ, which from the IIB frame, is the string coupling. This encodes a marginally relevant deformation of the boundary theory. In vacuum, it is exactly marginal. It is driven marginally relevant by the finite density, and runs strong at $r = r_0$, producing this confining groundstate.

A finite temperature solution can be obtained simply by periodically identifying the Euclidean time direction in this solution. It is not clear that this is the thermodynamically favored solution[4]. If it is, then it implies that $e^{-E_{gap}/T}$ effects do not appear in observables in this state; this is consistent with an energy gap of order N^2. At $\varsigma \to 0$, a finite-temperature solution with a horizon is the one given in Section 4.3.

[4]We note that in contrast with the Hawking-Page transition [85, 169], in our case a double Wick rotation $t \to i\chi_2, \chi_2 \to -it$ does not provide a finite temperature deconfined solution with the same asymptotics, because of the $dtd\xi$ term in the metric. The ability to do this previously was a result of Lorentz invariance of the asymptotics.

4.4.2 What is a translation-invariant insulator?

The fact that the geometry ends smoothly in the IR (at r_0) strongly suggests that the excitations of this groundstate are gapped [169]. More precisely, regularity requires the boundary condition $\partial_r \varphi|_{r=r_0} = 0$ on any smooth 11-dimensional field φ. This *real* boundary condition in the IR implies the vanishing of the spectral density $\mathrm{Im}\,\langle \mathcal{O}\mathcal{O} \rangle$ of the dual operator $\mathcal{O}$, up to a discrete series of delta functions associated with normal modes. In particular, this applies to the bulk gauge field B which couples to the particle number current j, and implies a gap in the spectral density for j. This spectral density determines the conductivity.

Hence this solution is dual to a system at finite density with a gap for the charged excitations. We emphasize that the distinction between this solution and an ordinary confining groundstate of the dual gauge theory [169] is the presence of a nonzero charge density.

Such a thing can be called a Mott insulator. From the point of view of the dual field theory, it is the strong interactions that prevent the charge from moving. It is certainly not a band insulator or an Anderson insulator – indeed this system is translationally invariant. This raises a thorny point: translation invariance plus finite charge density implies that the center of mass of the system will accelerate in an external field, and hence $\mathrm{Re}\,\sigma(\omega) \propto \delta(\omega)$ – the DC conductivity is infinite. The system is actually a perfect conductor.

What we mean by calling the system an insulator is that we believe it would be an insulator if we pinned it down, for example by a boundary condition. We have not figured out how to show that the thing is actually an insulator in the above sense. The answer for the conductivity $\mathrm{Re}\,\sigma(\omega) \propto \delta(\omega)$ is not enough: the clean free Fermi gas also gives this answer, and obviously that is a metal. In that case, and quite generally [82] adding static impurities just turns the delta function into a transport peak. In real systems (*i.e.* with decent UV behavior) there is a sum rule that says that the spectral weight from the delta function has to be redistributed somehow upon adding a momentum sink. A possible concern is that the conclusion (*i.e.* whether the spectral weight gets redistributed away from $\omega = 0$) might depend on *how* the center of mass mode is frozen.

Even zero compressibility is not enough, at least in the presence of long-range forces (which presumably the dual field theory has): the 'jellium model' of a metal (in which the lattice of ions is approximated by a fixed uniform density of background charge) is incompressible if Coulomb interactions are included, but is also clearly a metal. Further analysis is required to test our conjecture. The application of an electric field of finite wavenumber may be the simplest approach.

There are several known examples of translation-invariant insulators. Quantum Hall states are insulators which preserve continuous translation invariance; the translation-invariance delta function in $\sigma(\omega)$ is shifted from zero to the cyclotron frequency because the (charged) center of mass mode is subjected to a magnetic field (this is known as Kohn's theorem). In contrast, the state discussed in this chapter is not subjected to an external magnetic field.

In strongly-correlated lattice models, the particles can fractionalize in such a way as to produce an integer number of fractionalized particles per unit cell, which can then realize an ordinary band insulator. The arguments of [156, 140] show that in a gapped system with a conserved particle number (not spontaneously broken) at incommensurate filling, either translation invariance is broken or the system exhibits groundstate degeneracy on a torus. All of the examples mentioned above realize the latter option. is

For realizing a finite-density insulating state which preserves continuous (non-magnetic) translations, it is crucial that the high-energy excitations of our system are not particles, but rather CFT excitations. If the system at the scale of the chemical potential were described in terms of charged particles, a state where the charges were localized would have to (spontaneously) break translation invariance, since the particles have to sit somewhere. It would be interesting to find a "slave unparticle" construction of such a state.

A state in which $\mathrm{Im}\,\langle\rho(\omega)\rho(-\omega)\rangle$ vanishes below some gap must be incompressible. Our system, as currently presented, does not have such a gap (there are zero-energy excitations, at least those associated with translation invariance), and naively the compressibility is finite. Indeed it seems to be a consequence of the scale invariance Ward identity that $\mu \propto \rho$, which is what we find with an appropriate choice of boundary counterterms. What this constraint on the compressibility has to say about our proposal for what would happen if one pinned this system down is not clear to us at present. One point to note is that we are forced to study the system at fixed particle number rather than fixed chemical potential (see §4.3.1). In general, an incompressible system ($d\rho/d\mu = 0$) does not have a homogeneous groundstate at fixed particle number. For example, consider a Mott insulator of repulsive bosons on a lattice. In general only certain values of the number density will admit homogenous groundstates states (*i.e.* those values with integer filling fraction) and at other values the system will

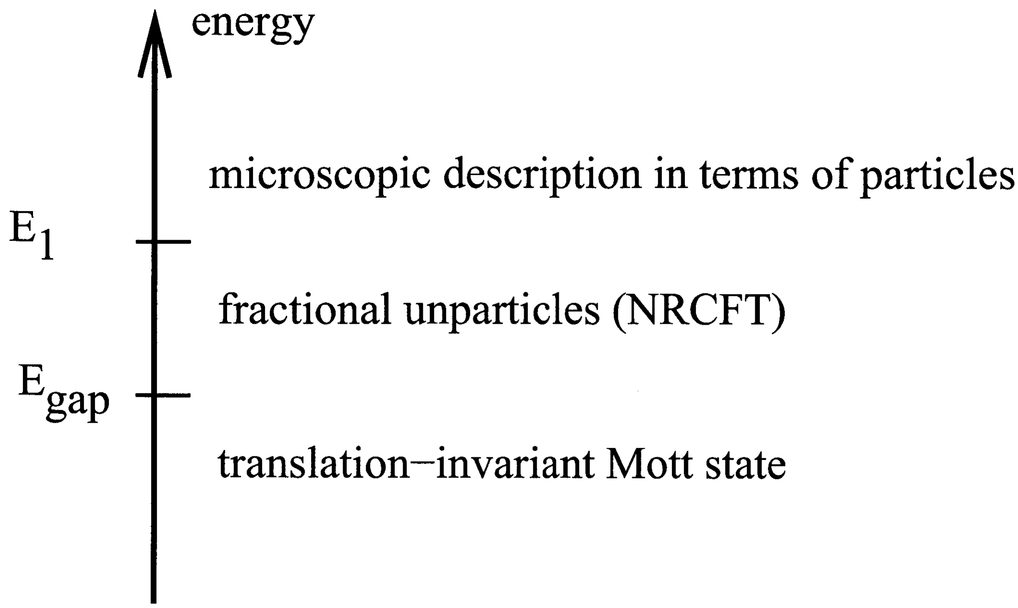

Figure 4-1: A scheme for realization a translation-invariant insulator. In the gravity dual, we are only addressing physics below the scale E_1.

phase separate. The phase separated solutions are compressible. It may be that the solution we have is describing such a mixed state. Another possible resolution is that the nonzero compressibility arises in our system via neutral modes – *i.e.* in our construction, it's not clear that all excitations have nonzero particle number.

A final disclaimer about our use of the name "Mott insulator" is that there is no local moment physics in our problem so far. It would be interesting to include spin degrees of freedom.

4.5 Examples with multiple species

In the Schrödinger metric of [160, 16], the ξ-momentum is dual to the particle number $\hat{N}$ of the dual NRCFT. Compactifying the ξ direction on a circle of radius L_ξ gives a spectrum of $\hat{N}$ which is just a tower of integer multiples of a fixed mass scale L_ξ^{-1}. In a system with multiple species of different mass (for example, in a pile of atoms consisting of several species) the mass operator will have a spectrum which is not just a tower of integer multiples of a fixed mass. Here we would like to find gravity duals with similar spectra. We can do this by adding more dimensions analogous to ξ.

We mention in passing that a usefully liberating perspective on the realization of the particle number was provided by [172]: in the analog of 'global coordinates' discussed there the particle-number circle is fibered non-trivially over the $t, \vec{x}$ directions.

We can solve a reasonable set of equations of motion with a nice metric with two ξ directions if we add a second gauge field. The Lagrangian is just

$$L = R + 2\Lambda - \frac{1}{4}F_1^2 - \frac{1}{2}m_1^2 A_1^2 - \frac{1}{4}F_2^2 - \frac{1}{2}m_2^2 A_2^2 \tag{4.5.1}$$

with (for $d = 2$)

$$m_1^2 = 4z, \qquad m_2^2 = -4(z-2) \tag{4.5.2}$$

$$\Lambda = \frac{1}{2}\left(26 - 7z + z^2\right) \tag{4.5.3}$$

(for $z = 2$, $\Lambda = 8$). The z-dependence of Λ is a novel development, compared to previous Schrödinger solutions.

The solution is

$$ds^2 = -r^{-2z}dt^2 + r^{-2}(-2d\xi_+ dt + d\vec{x}^2 + dr^2) + d\xi_-^2 r^{2z-4} \tag{4.5.4}$$

(the symmetries are discussed below) with

$$A_1 = \Omega_1 r^{-z} dt \tag{4.5.5}$$

$$A_2 = \Omega_2 r^{z-2} d\xi_- .$$

We believe that it is not possible to source the stress tensor for this metric with a single gauge field (which solves its own equations of motion).

Note that the mass-squared of the second gauge field is negative for many z's of interest ($z > 2$). According to (4.5.2), the second gauge field is massless for $z = 2$; we will comment below on some subtleties with this case.

4.5.1 Symmetries

This metric is invariant under galilean boosts just like the usual Schrodinger metric, with no action on ξ_-,

$$\xi_+ \to \xi_+ + \vec{v}\cdot\vec{x} - \frac{1}{2}v^2 t, \xi_- \to \xi_- . \tag{4.5.6}$$

It is scale invariant with $\xi_\pm$ both scaling like $length^{2-z}$:

$$\vec{x} \to \lambda\vec{x},\ t \to \lambda^z t,\ r \to \lambda r,\ \xi_\pm \to \xi_\pm \lambda^{2-z} \quad . \tag{4.5.7}$$

Interestingly, for $z = 2$, the $g_{\xi_-\xi_-}$ coefficient is 1. And, finally, $[K_i, P_j] = i\delta_{ij}\hat{N}$ with

$$\hat{N} = i\partial_{\xi_+} \ . \tag{4.5.8}$$

So, if we set $\xi_\pm = \xi^1 \pm \xi^2$ and compactify

$$\xi_1 \simeq \xi_1 + L_1, \quad \xi_2 \simeq \xi_2 + L_2 \tag{4.5.9}$$

then the spectrum of $\hat{N}$ is

$$\left\{ \frac{n_1}{L_1} + \frac{n_2}{L_2} | n_{1,2} \in \mathbb{Z} \right\}; \tag{4.5.10}$$

in particular $\frac{L_1}{L_2}$ needn't be rational. We can think of $i\partial_{\xi_1}$ and $i\partial_{\xi_2}$ as the conserved particle numbers of the individual particle species; only their sum appears in the Schrödinger algebra.

The isometries of this spacetime include $P_i = i\partial_{x^i}, K^j = ix^j\partial_{\xi_+} + it\partial_{x^j}$ and the Schrödinger algebra says: $[P_i, K^j] = i\delta_i^j\hat{N}$ so we have $\hat{N} = i\partial_{\xi_+}$. For $z = 2$, there is trivially a special conformal symmetry which acts on ξ_+ in the same way as on ξ in the usual Schrödinger spacetime, and does not act on ξ_-.

It would be interesting to realize a system with arbitrarily many ξ-directions (*i.e.* species). Note that the new realization of the Galilean algebra described in the rest of this chapter offers a simple possibility: one can just introduce a collection of gauge fields (perhaps coming from some p-form reduced on representatives of some rank $p-1$ cohomology group of a compactification space *e.g.* as described recently in [109]) and associate them with conserved particle numbers of various species. The Galilean boost will act by some linear combination of the gauge generators; this combination is the total mass appearing in the Galilean symmetry algebra.

The wave equation is qualitatively the same as in the one-species case [160, 16].

4.5.2 $z \to 2$

Note that the interesting case $z = 2$ is actually quite degenerate here. The Einstein equations determine the coefficients $\Omega_{1,2}$ in the solutions for the gauge fields to be (for $d = 2$!)

$$\Omega_1^2 = 2\frac{z-1}{z}\,, \qquad \Omega_2^2 = 2\frac{z-1}{z-2}\ . \tag{4.5.11}$$

Notice that Ω_2 has a pole at $z = 2$. But the stress tensor it produces is finite, because both the field strength and the mass go to zero as $z \to 2$! That is, we must take a scaling limit where $z \to 2, \Omega_2 \to \infty$ holding fixed $(z-2)\Omega_2^2$ to which the stress tensor of A_2 is proportional.

4.6 Comments on the superfluid state

The ground state of most assemblies of ultracold atoms, bosonic or fermionic, is a superfluid [33]. It is natural to ask whether the zero-temperature, finite-density solution found in [1] describes such a state. That it does not can be seen as follows. If shifts in the ξ-direction correspond to the particle-number symmetry, then the gravity dual of a superfluid ground state must somehow break translation invariance in the ξ direction in the IR region of the geometry. This is because the ground state wave function of a superfluid is localized in the space conjugate to the particle number.

A precedent for the required gravity description is the spontaneous breaking of the $U(1)_R$ symmetry in the Klebanov-Strassler [110] and Maldacena-Nunez [127] solutions, where it is indeed some isometry of the bulk geometry which is broken (to a discrete subgroup) by the exact solution in the IR region of the geometry. A possibility to keep in mind is that the symmetry may be broken by something other than the metric, *e.g.* some other field.

We note that it is not clear that the twisted DLCQ theories, to which the stringy embeddings of Schrödinger spaces found in [123, 90, 1] are dual, indeed have superfluid ground states. If not, how does the dual field theory avoid breaking the particle number symmetry at zero temperature? The fact that the only zero-temperature solution we know [1] is singular leaves open the likely possibility that there is a better, more correct solution with the same leading asymptotics which does describe a superfluid.

In the new realizations of the Schrödinger symmetry described in this chapter, the question of spontaneous breaking of the particle number symmetry becomes much more similar to the (well-developed) study of holographic superconductors in gravity duals of relativistic CFTs [80, 81, 94]. We note in particular that the system of §4.4 has a dimensionless parameter ς which controls the strength of the coupling. Upon the addition of a charged scalar to the bulk, we anticipate that varying ς will produce a quantum phase transition from the "Mott" "insulator" phase described here to a superfluid phase.

4.7 Conclusions

In this chapter we have introduced a new class of gravity duals of Galilean-invariant CFTs. This requires somewhat novel asymptotics. In particular, the bulk gauge field which represents the particle number symmetry becomes strongly coupled at the UV boundary. In the examples constructed by dimensional reduction, this strong coupling of the gauge field is resolved by the lift; this is the statement that the ξ direction becomes null at the boundary of an asymptotically-Schrödinger geometry. It is an interesting open problem to characterize the resolution independent of the lift.

In the most interesting new solution we found (described in Section 4.4), there was also a singularity at the IR end of the geometry. This curvature singularity was resolved by a lift to 11-dimensional supergravity; we emphasize that the shrinking circle in the IR is *not* the particle-number direction ξ. It would be interesting to characterize which singularities of this kind can be resolved (see *e.g.* [64, 71]). A necessary criterion for a resolution by oxidation is that the geometry be conformal to a regular metric. It would be most useful for our purposes to be able to describe the resolution without resorting to dimensional oxidation.

Solutions of related Einstein-Maxwell-dilaton systems have been studied recently in [74, 36, 76], mainly with AdS asymptotics in mind. It is possible that the near-infrared solutions studied in these papers can be integrated to the asymptotics described here.

Finally, we comment that although our goal in this chapter was to rid ourselves of the extra dimension ξ conjugate to the particle number, the 11-dimensional supergravity solution in Section 4.4 does indeed include such a dimension. Further, the regular 4d black hole solutions (without a ξ direction) which we found (in Section 4.3 and Appendix 4.A) all have equations of state similar to that following from DLCQ (4.1.2). It will be of great interest to find black hole solutions, with the asymptotics described here, which have other equations of state.

4.A Black hole solution in a system with two scalars

In this appendix we present another system in which we have found black holes with asymptotic Schrödinger symmetry. Its action is rather contrived.

Let us consider the following action

$$S_D^E = \int d^D x (-g_D)^{1/2} \Big[R_D - 2\Lambda e^{-\sigma} - \frac{e^{3\sigma}}{4} (dB)^2 - \frac{3}{2} (\partial\sigma)^2 - \frac{1}{2} (\partial\Psi)^2 + V_2(\sigma, \Psi) \Big] \quad (4.A.1)$$

where

$$V_2(\sigma,\Psi) = \left[12 e^{-\sigma} \left(\sinh\left(\Psi/\sqrt{5}\right)\right)^3 + 16\sqrt{Q}/r_0 \left(\tanh\left(\Psi/\sqrt{5}\right)\right)^{-9/4} \left(\sinh\left(\Psi/\sqrt{5}\right)\right)^5 \right] C_1^2 \quad (4.A.2)$$

The following background is a saddle point of this action

$$ds_E^2 = e^{\sigma} \left(-Q f K_x^2 \frac{dt^2}{r^6} + K_x \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{f r^2} \right)$$

$$B = Q \frac{(1 - r^4/r_H^4) dt}{r^4} + B(r_H), \quad e^{2\sigma} = \frac{r^2}{Q}, \quad \Psi = \sqrt{5} \tanh^{-1} \left(\frac{r^4}{r_0^4} \right) \quad (4.A.3)$$

where

$$K_x^2 = 1 - r^8/r_0^8 \quad \text{and} \quad f = 1 - C_1^2 \frac{r^4}{\sqrt{r_0^8 - r^8}}. \quad (4.A.4)$$

The above system has asymptotic Schrödinger symmetry, realized as in Section 4.2.2. The free energy of this system has the same form as the black hole in the system with one scalar *i.e.*,

$$\mathcal{F} \sim -\frac{T^4}{\mu^2}. \quad (4.A.5)$$

4.B Uplifting to M-theory

In this appendix we exhibit a useful sector of type IIA supergravity as a consistent truncation of eleven-dimensional supergravity[5]. Let us consider the following ansatz for the eleven-dimensional line element and the four-form flux:

$$ds^2 = g_{MN}dx^M dx^N = G_{\mu\nu}dx^\mu dx^\nu + e^{2\aleph}dz_{10}^2$$

$$\tilde{F}_4 = F_4 + H_3 \wedge dz_{10} \tag{4.B.1}$$

where g is the eleven-dimensional metric, G is the ten-dimensional metric, $\tilde{F}_4$ is the eleven-dimensional four-form flux, F_4 and H_3 are the ten-dimensional four-form and three-form flux. With this ansatz, the Bianchi identity becomes

$$d\tilde{F}_4 = 0 \quad \Leftrightarrow \quad dF_4 = 0 \quad \text{and} \quad dH_3 = 0. \tag{4.B.2}$$

The equation of motion for the eleven-dimensional four-form field strength can be written as

$$d \star \tilde{F}_4 = \frac{1}{4}\tilde{F}_4 \wedge \tilde{F}_4 \Leftrightarrow d\left(e^{\aleph} \star F_4\right) = \frac{1}{2}H_3 \wedge F_4 \quad \text{and} \quad d\left(e^{-\aleph} \star H_3\right) = \frac{1}{4}F_4 \wedge F_4\ . \tag{4.B.3}$$

The components of eleven-dimensional Ricci tensor ($\tilde{R}_{\mu\nu}$) are given by

$$\tilde{R}_{\mu\nu} = R_{\mu\nu} - \nabla_\mu\nabla_\nu\aleph - \nabla_\mu\aleph\nabla_\nu\aleph \tag{4.B.4}$$

$$\tilde{R}_{\mu 10} = 0 \tag{4.B.5}$$

$$\tilde{R}_{1010} = (\nabla_\mu\nabla^\mu\aleph + \nabla_\mu\aleph\nabla^\mu\aleph) \tag{4.B.6}$$

where $R_{\mu\nu}$ is the ten-dimensional Ricci scalar. After some algebra, the eleven-dimensional Einstein equations can be written as

$$\left(R_{\mu\nu} - \frac{1}{2}G_{\mu\nu}R\right) - \nabla_\mu\nabla_\nu\aleph - \nabla_\mu\aleph\nabla_\nu\aleph + (\nabla_\mu\nabla^\mu\aleph + \nabla_\mu\aleph\nabla^\mu\aleph)\,G_{\mu\nu} = T^F_{\mu\nu} + T^H_{\mu\nu} \tag{4.B.7}$$

[5]We will ignore fermions and assume that the Ramond-Ramond vector A_1 is turned off.

$$R - \frac{1}{48}F_4^2 + \frac{1}{12}e^{-2\aleph}H_3^2 = 0 \ . \tag{4.B.8}$$

The above equations can be obtained from the following action[6]

$$S_{10} = \int d^{10}x\sqrt{G}e^{\aleph}\left(R - \frac{1}{48}F_4^2 - \frac{e^{-2\aleph}}{12}H_3^2\right) + \frac{1}{2}\int B_2 \wedge F_4 \wedge F_4 \tag{4.B.10}$$

where $H_3 = dB_2$. Let us redefine $\tilde{G}_{\mu\nu} = e^{-\aleph}G_{\mu\nu}$ and $\aleph = 2\Phi/3$. In terms of the redefined variables, the action in (4.B.10) can be written as

$$S_{10} = \int d^{10}x\sqrt{\tilde{G}}e^{-2\Phi}\left(\tilde{R} + 4\left(\partial\Phi\right)^2 - \frac{e^{2\Phi}}{48}F_4^2 - \frac{1}{12}H_3^2\right) + \frac{1}{2}\int B_2 \wedge F_4 \wedge F_4 \ . \tag{4.B.11}$$

This action is the bosonic part of the type IIA supergravity action (with A_1 turned off) in string frame. Any solution of the ten dimensional action in (4.B.11) can be oxidized to give a solution of eleven-dimensional supergravity. The 10-D Einstein frame metric metric is related to the string frame metric $\tilde{G}$ through the following Weyl transformation $g_E = e^{\Phi/2}\tilde{G}$.

[6]We have made use of the following formula

$$\frac{1}{\sqrt{g}}\frac{\delta}{\delta g_{ab}}\int d^dx\sqrt{g}XR = X\left(R_{ab} - \frac{1}{2}Rg_{ab}\right) - \nabla_a\nabla_b X + g_{ab}\nabla_c\nabla^c X \ . \tag{4.B.9}$$

Chapter 5

Lifshitz solutions of 10D and 11D supergravity

A large portion of this chapter and the fourth item in the synopsis (Chapter 1) appeared in "Lifshitz spacetimes from AdS null and cosmological solutions " with K. Narayan [20] and is reprinted with the permission of *JHEP*. Some portions of this chapter appeared in "An analytic Lifshitz black hole" with John McGreevy [17] and is reprinted with the permission of *Phys. Rev.* **D**. Copyright (2009) by The American Physical Society.

5.1 Introduction

In this chapter, we discuss Lifshitz fixed points from a holographic perspective. These theories exhibit non-relativistic scale invariance but are not invariant under boosts. Various condensed matter systems admit descriptions in terms of Lifshitz fixed points, with dynamical exponent z given by the anisotropic scaling $t \to \lambda^z t,\ x^i \to \lambda x^i$. A Landau-Ginzburg description for such theories with $z = 2$ has the effective action $S = \int dt d^d x\ ((\partial_t \varphi)^2 - \kappa(\nabla^2 \varphi)^2)$. These theories, discussed early on in [93], arise in dimer models *e.g.* [11], representing universality classes of dimer statistical systems or as representing certain phases of systems with antiferromagnetic interactions as well as in models of liquid crystals. It was argued in [11] that the equal time correlation functions of a (2+1)-dim Lifshitz theory are identical to the correlators of an appropriate Euclidean 2-dim conformal field theory. Further it was discussed in [63] in the context of quantum critical points that finite temperature equal-time correlators of these theories exhibit ultra-locality in space.

Holographic duals of Lifshitz-like theories were studied in [106]. They found that the following metric provides a geometric realization of the symmetries of Lifshitz-like theories (with z as the dynamical exponent):

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{dx_i^2 + dr^2}{r^2} \, , \tag{5.1.1}$$

where $\vec{x} \equiv x_i$ denotes a $d-$dimensional spatial vector. In the case $d = 2$, this metric is a classical solution of the following action:

$$S = \frac{1}{2}\int d^4x \, (R - 2\Lambda) - \frac{1}{2}\int \left(F_{(2)} \wedge \star F_{(2)} + F_{(3)} \wedge \star F_{(3)}\right) - c\int B_{(2)} \wedge F_{(2)} \, , \tag{5.1.2}$$

where, $F_{(2)} = dA_{(1)}$, $F_{(3)} = dB_{(2)}$ and Λ is the 4-dimensional cosmological constant.

The tensor fields in [106] can be rewritten as one massive gauge field [165, 17]. The Chern-Simons-like coupling is responsible for the mass. A familiar example is that of a 2-form field strength F and a 3-form field strength H in five dimensions with $L = F\wedge\star F + H\wedge\star H + F\wedge H$: this gives the same equation of motion as $L = F\wedge\star F + A^2$. In the four dimensional case studied in [106], the dual of the 3-form field strength in four dimensions is a scalar field φ. Then

$$B_2 \wedge F_2 = -F_3 \wedge A_1 + \text{bdy terms} = -\star d\varphi \wedge A_1 = -\sqrt{g}\partial^\mu \varphi A_\mu . \tag{5.1.3}$$

The action then reduces to

$$F_2 \wedge \star F_2 + (\partial\varphi + A)^2, \tag{5.1.4}$$

and φ shifts under the A gauge symmetry, and we can fix it to zero, and this is just a massive gauge field[1]. Hence, the zero-temperature Lifshitz metric

$$ds^2 = -\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2 + dr^2}{r^2}, \tag{5.1.5}$$

is a solution of gravity in the presence of cosmological constant and a massive gauge field, and the gauge field mass is $m^2 = dz$. The bulk curvature radius has been set to one here and throughout the chapter; in these units, the cosmological constant is $\Lambda = -\frac{z^2+(d-1)z+d^2}{2}$. The gauge field profile is $A = \Omega r^{-z} dt$ (in the r coordinate with the boundary at $r = 0$), and

[1]This was also observed in [165].

the strength of the gauge field is (for $d = 2$)

$$\Omega^2 = 8\frac{z^2 + z - 2}{z(z+2)}.$$

We note in passing that the Schrödinger spacetime is a solution of the same action with a different mass for the gauge field and a different cosmological constant [160, 16]. Therefore we find the perhaps-unfamiliar situation where the same gravitational action has solutions with very different asymptopia. Another recent example where this happens is 'chiral gravity' in three dimensions, which has asymptotically AdS solutions as well as various squashed and smushed and wipfed solutions [118].

Given this fact, one might expect that the Lifshitz spacetime can be embedded into the same type IIB truncations as the Schrödinger spacetime (see [90, 1] and especially [123]). However, the scalar equation of motion is not satisfied by the Lifshitz background since F^2 is non-zero.

Recently, some obstacles in finding a string construction of such theories were pointed out in [117]. They showed, with reasonable ansatze for the fluxes, that it is not possible to have a classical solution of massive type IIA supergravity/M-theory of the form $Li_4 \times M_6$ (or M_7)[2]. This was shown to be true even when the product contains warp factors. To the best of our knowledge, solutions of 10- or 11-dimensional supergravity with Lifshitz symmetries have not yet been constructed. However, some ways of overcoming these obstacles were outlined in [83].

In this chapter we suggest alternative constructions, with explicit solutions of supergravities which have $z = 2$ Lifshitz symmetries. Lifshitz theories with dynamical exponent $z = 2$ are closely related to Galilean invariant CFTs (Schrödinger invariant theories). Note that Lifshitz theories have only non-relativistic scale invariance: these theories are not Galilean invariant. These theories do not have a conserved particle number unlike Galilean invariant theories. This suggests that Lifshitz invariant theories can be constructed by explicitly breaking Galilean invariance in Schrödinger invariant theories. We recall that holographic descriptions of Galilean invariant CFTs (with Schrödinger symmetry) were proposed in [160, 16]:

[2]Note however [15], which uses intersecting D3-D7 branes to construct $z = \frac{3}{2}$ Lifshitz spacetimes that are anistropic and in addition have a nontrivial dilaton that breaks this symmetry. Note also [164, 31], which construct Lifshitz-like solutions with a scalar having a radial profile. See also [112] which describes anisotropic Lifshitz-like solutions with anisotropic matter.

they can be embedded in string theory [123, 90, 1, 84, 51, 34]. In this description, the particle number symmetry is geometrically realized as an isometry of a circle (denoted by x^+). The geometry in this description has some resemblance to AdS in lightcone coordinates, with one of the lightcone directions compactified. In fact, AdS in light cone coordinates (with a compact lightcone direction) has the symmetries of the Schrödinger group [66, 23]. With a view to breaking the Schrodinger symmetry to a Lifshitz one, the shift symmetry along x^+ direction can be broken in many ways. For instance, adding backreacting branes (anti-branes) that are localized along this compact direction breaks this shift symmetry explicitly resulting in a geometry with Lifshitz symmetries.

Our construction in this chapter describes a possibly simpler way of breaking this shift symmetry by turning on a scalar field periodic in x^+ (with period determined by the radius of the x^+ direction). A scalar field with profile $\Phi(x^+)$ breaks the shift symmetry (asymptotic) along x^+ direction. Such solutions of supergravity have already been studied in the literature with a view to understanding cosmological singularities in AdS/CFT [47, 48, 14, 13]. We will refer to these solutions as null-deformed AdS solutions for convenience. We will review relevant aspects of these in the next section (sec. 2), but for now we describe some essential features of our proposed holographic system exhibiting $z = 2$ Lifshitz symmetry. The spacetimes we deal with are solutions of 10- or 11-dim supergravity comprising deformations of $AdS \times X$, alongwith a scalar $\Phi(x^+)$, the AdS-deformed metric being

$$ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \gamma(\Phi')^2 w^2 (dx^+)^2] + \frac{dw^2}{w^2} + d\Omega_S^2 \,, \tag{5.1.6}$$

with $\Phi' \equiv \frac{d\Phi}{dx^+}$. The constant γ is $\gamma = \frac{1}{4}$ for AdS_5 and $\gamma = \frac{1}{2}$ for AdS_4, with the x_i ranging over $1,2$ and 1 for AdS_5 and AdS_4 respectively (the $d\Omega_S^2$ is the metric for S^5 or X^7 respectively, with X^7 being some Sasaki-Einstein 7-manifold). We regard x^- as the time direction here, x^+ being a compact direction. We will discuss this metric in greater detail in the next section: there we will also describe a more general context that these solutions (and others discussed later) will naturally arise from.

It can be checked that these spacetimes (5.1.6) along with the scalar Φ and appropriate 5-form (or 4-form) field strength are solutions to the 10-dim (or 11-dim) supergravity equations. For instance, there is no S^5 or X^7 dependence and the resulting 5- or 4-dim system, with an effective cosmological constant from the flux, solves the equations $R_{MN} = -dg_{MN} +$

$\frac{1}{2}\partial_M\Phi\partial_N\Phi$, with $d = 4, 3$, for AdS_{d+1}, being the 5- or 4-dim effective cosmological constant. The spacetime (5.1.6) exhibits the following symmetries: translations in $x_i, x^- \equiv t$ (time), rotations in x_i and a $z = 2$ scaling $x^- \longrightarrow \lambda^2 x^-, x_i \longrightarrow \lambda x_i, w \longrightarrow \lambda w$ (x^+ being compact does not scale). Possible Galilean boosts $x_i \rightarrow x_i - v_i x^-,\ x^+ \rightarrow x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$, are broken by the $g_{++} \sim (\Phi')^2$ term. If $g_{++} = 0$, this is essentially AdS in lightcone coordinates and the system has a Schrodinger symmetry (as discussed in *e.g.* [66, 23, 123]): note however that these are not Schrodinger spacetimes of the sort discussed in [160, 16, 123, 90, 1, 84, 51, 34]. Similarly, in the present case with $g_{++} \neq 0$, there is no special conformal symmetry either. We discuss various aspects of this system in sec. 3 and sec. 4: this includes a discussion of the dimensional reduction of these systems and some aspects of the dual field theory (in part borrowing from [48]), which is the lightlike dimensional reduction, or DLCQ, along the x^+-direction of $\mathcal{N}$=4 super Yang-Mills theory with a nontrivial gauge coupling $g_{YM}^2 = e^{\Phi(x^+)}$. In particular we also discuss the holographic 2-point function of operators dual to bulk scalars. Our equal-time holographic 2-point function in particular recovers the spatial power-law dependence obtained in [106]. It is perhaps worth mentioning that the Lifshitz field theory here is an interacting strongly coupled dimensionally reduced limit of the $\mathcal{N}$=4 SYM theory, rather than a free Lifshitz theory.

Similarly we expect that the AdS_4-deformed solutions are dual to appropriate lightlike deformations of Chern-Simons theories arising on M2-branes stacked at appropriate supersymmetric singularities [4], dimensionally reduced along a compact direction.

In sec. 5, we describe a solution of 5-dimensional gravity with negative cosmological constant and a massless complex scalar, that are similar to the null solutions (5.1.6) above: these upon dimensional reduction give rise to $2 + 1$-dim Lifshitz spacetimes. This 5-dim solution can be uplifted to 11-dimensional supergravity.

Sec. 6 closes with a Discussion, while Appendix A provides some technical details for completeness.

5.2 Null deformations of AdS

The following solutions are discussed in [47, 48, 14, 13] as cosmological generalizations of AdS_5/CFT_4. The ten-dimensional Einstein frame metric, the scalar Φ, and 5-form flux are

$$ds^2 = \frac{R^2}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + R^2 d\Omega_5^2, \qquad \Phi = \Phi(x^\mu) \ ,$$
$$F_{(5)} = R^4(\omega_5 + *_{10}\omega_5) \ , \tag{5.2.1}$$

with $d\Omega_5^2$ being the volume element and ω_5 being the volume form of the unit five sphere S^5. This is a solution of the ten dimensional Type IIB supergravity equations of motion as long as the four-dimensional metric, $\tilde{g}_{\mu\nu}$, and the scalar Φ, are only dependent on the four coordinates, $x^\mu, \mu = 0, 1, 2, 3$, and satisfy the conditions,

$$\tilde{R}_{\mu\nu} = \frac{1}{2}\partial_\mu\Phi\partial_\nu\Phi, \qquad \partial_\mu(\sqrt{-\tilde{g}}\ \tilde{g}^{\mu\nu}\partial_\nu\Phi) = 0 \ , \tag{5.2.2}$$

where $\tilde{R}_{\mu\nu}$ is the Ricci curvature of the metric $\tilde{g}_{\mu\nu}$: these are equations governing 4-dim Einstein dilaton gravity.

The scalar Φ can be taken to be the dilaton with e^Φ then being the string coupling. As described in Appendix A, more general solutions exist where the S^5 is replaced by the base of any Ricci-flat 6-dim space: in these cases, Φ can be taken to be some other scalar, *e.g.* arising from the compactification.

Some details on these solutions that might be of relevance to the present context are reviewed in Appendix A.

We now specialise to null solutions where $\tilde{g}_{\mu\nu}$ and Φ are functions of only a lightlike variable x^+: if we further assume that $\tilde{g}_{\mu\nu}$ is conformally flat $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$, the metric and dilaton become (setting the AdS radius $R = 1$) for the AdS_5 case

$$ds^2 = \frac{1}{r^2}[e^{f(x^+)}(-2dx^+dy^- + dx_i^2) + dr^2] + d\Omega_5^2, \qquad \Phi = \Phi(x^+) \tag{5.2.3}$$

(see also [39, 119, 12] for related work). We use the variable y^- for convenience, reserving x^- for (5.1.6). We will refer to the coordinate system in (5.2.3) as conformal coordinates in what follows. The equations of motion in this case simplify drastically due to the lightlike nature of the solutions. The scalar equation of motion is automatically satisfied and the only nonzero Ricci component is R_{++}, giving $R_{++} = \frac{1}{2}(\partial_+\Phi)^2$, *i.e.*

$$R_{++} = \frac{1}{2}(f')^2 - f'' = \frac{1}{2}(\Phi')^2 \,, \tag{5.2.4}$$

with $\Phi' \equiv \frac{d\Phi}{dx^+}$, $f' = \frac{df}{dx^+}$. This is a single equation for two functions f, Φ, so that this is a fairly general class of solutions with a function-worth of parameters: choosing a generic Φ gives an e^f. One has to be careful though, since an arbitrary Φ does not necessarily give an e^f such that the pair is a sensible solution[3]. These solutions preserve half (lightcone) supersymmetry [47].

AdS_4 similarly admits generalizations of the solutions described above with the 11-dim metric and a scalar of the form $ds^2 = \frac{R^2}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + R^2 d\Omega^2_{X^7}$, $\Phi = \Phi(x^\mu)$. In this case, the scalar does not have any natural interpretation in the 11-dim theory directly: it arises instead from the 4-form flux after compactification on a 7-manifold X^7 as we discuss in Appendix A.1. The 11-dim supergravity equations are satisfied if the conditions in (5.2.2) hold, the $\tilde{R}_{\mu\nu}$ now being the Ricci tensor for the 3-dim metric $\tilde{g}_{\mu\nu}$. Pure 3-dim gravity has no dynamics but the scalar drives the system giLving rise to nontrivial dynamics. Consider now a 3-dim metric conformal to flat 3-dim spacetime: the 11-dim metric then in conformal coordinates is

$$ds^2 = \frac{1}{r^2}[e^{f(x^+)}(-2dx^+dy^- + dx_i^2) + dr^2] + d\Omega^2_{X^7} \,, \qquad \Phi = \Phi(x^+) \,. \tag{5.2.5}$$

The Einstein equation becomes

$$R_{++} = \frac{1}{4}(f')^2 - \frac{1}{2}f'' = \frac{1}{2}(\Phi')^2 \,, \tag{5.2.6}$$

of a form similar to the AdS_5 case.

Such a deformation, via a $\tilde{g}_{\mu\nu}$, could potentially lead to singularities on the Poincare horizon $r = 0$. For instance in the AdS_5 case, we have $R_{\mu\nu\alpha\beta} = \frac{1}{r^2}\tilde{R}_{\mu\nu\alpha\beta} - 2(g_{\mu\alpha}g_{\nu\beta} - g_{\mu\beta}g_{\nu\alpha})$, $R_{z\mu z\mu} = -\frac{2}{r^2}g_{\mu\nu}$, giving the curvature invariant $R_{ABCD}R^{ABCD} = r^4\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta} + O(r^0)$. Now for the null metrics in question here, $\tilde{R}_{\mu\nu\alpha\beta}\tilde{R}^{\mu\nu\alpha\beta}$ vanishes, since the lightlike

[3]For instance, in some related cosmological solutions and discussion in [122], certain regulated versions of singular solutions do not necessarily obey $R_{++} > 0$, which is essentially positivity of the energy density along null geodesics.

solutions admit no nonzero contraction. Thus the possible divergent r^4 term at the Poincare horizon $r \longrightarrow \infty$ is in fact absent. These null solutions are thus regular, except for possible singularities arising when e^f vanishes, as in the context of cosmological singularities [47, 48, 14, 13]: in that case, there were diverging tidal forces along null geodesics arising because the spacetime was essentially undergoing a crunch with e^f vanishing. For our purposes here, e^f and Φ will be regular functions of x^+, in which case we expect that these spacetimes are regular. It is however known that Lifshitz spacetimes have diverging tidal forces [106] (see also [32] which describes various geometric properties of the Schrodinger and Lifshitz spacetimes). It would seem that the singularities of the Lifshitz geometry then arise from the process of dimensional reduction of the above spacetimes (discussed in the next section).

In many cases, it is possible to find new coordinates such that boundary metric $ds_4^2 = \lim_{r\to 0} r^2 ds_5^2$ (AdS_5) or $ds_3^2 = \lim_{r\to 0} r^2 ds_4^2$ (AdS_4) is flat, at least as an expansion about the $r = 0$ boundary, if not exactly: this was studied for AdS_5 null cosmologies in [14]). These are *Penrose-Brown-Henneaux (PBH)* transformations, a subset of bulk diffeomorphisms leaving the metric invariant (in Fefferman-Graham form), and acting as a Weyl transformation on the boundary.

The coordinate transformation $w = re^{-f/2}$, $x^- = y^- - \frac{w^2 f'}{4}$, recasts these spacetimes (5.2.3), (5.2.5), in the form (5.1.6), reproduced here,

$$ds^2 = \frac{1}{w^2}[-2dx^+dx^- + dx_i^2 + \gamma(\Phi')^2 w^2 (dx^+)^2] + \frac{dw^2}{w^2} + d\Omega_S^2 \ , \tag{5.2.7}$$

using the equation of motion (5.2.4) or (5.2.6) for these solutions, with $\gamma = \frac{1}{4}$ for AdS_5 and $\gamma = \frac{1}{2}$ for AdS_4. Likewise, the x_i range over $1, 2$ and 1 for AdS_5 and AdS_4 respectively. We refer to this metric as written in PBH coordinates. In this lightlike case, this is an *exact* PBH transformation[4]. Now the boundary at $w = 0$ is manifestly flat 4D or 3D Minkowski spacetime, for the AdS_5 or AdS_4 cases respectively. With any infinitesimal regulator however, the regulated boundary $r = \epsilon$ is distinct from $w = \epsilon$, *i.e.* the holographic screens are distinct, although in the same conformal class.

Note that these are not normalizable deformations: *e.g.* in the AdS_5-deformed case, those would correspond to deformations where $w^2 g_{++} \sim w^4$.

[4]For t-dependent solutions, an exact PBH transformation is difficult to find in general, and one instead takes recourse to an expansion about the boundary [14].

In the next section, we study the dimensional reduction of these systems (5.1.6) (5.2.7) with a view to realizing spacetimes with Lifshitz symmetries [106] as a Kaluza-Klein reduction in one lower dimension.

5.3 Dimensional reduction to Lifshitz spacetimes

In the cosmological singularities context [47, 48, 14, 13], x^+ was regarded as a lightcone time coordinate, working in the conformal coordinate system (5.2.3): this introduces nontrivial lightcone time dependence into the system. From the dual gauge theory point of view, this makes the gauge coupling $g_{YM}^2 = e^{\Phi(x^+)}$ time dependent. Note that the boundary metric is either flat (in the PBH coordinates) or conformally flat (in the conformal coordinates): thus x^+ can be regarded equally well as a lightcone time or space variable in the boundary theory. In the bulk, although the worldsheet string is difficult to understand technically, it is natural to study string propagation on such spacetimes by fixing lightcone gauge as $\tau = x^+$, where τ is worldsheet time. In a sense, this has some parallels (and also some key differences) with the investigations of strings in plane wave spacetimes (see *e.g.* [98] for discussions of global properties and time-functions in plane wave spacetimes).
However, regarding x^+ as a time coordinate might appear problematic in the PBH coordinate system (5.1.6), (5.2.7), since $g_{++} = \gamma(\Phi')^2 > 0$, implying ∂_+ is a spacelike vector. Strictly speaking, the $x^+ = const$ surfaces are null surfaces since their normal dx^+ is null, noting that $g^{++} = 0$, while $x^- = const$ surfaces are spacelike, given that $g^{--} < 0$, suggesting again that x^- behaves like a time coordinate.

Now if x^+ represents a compact dimension, the discussion above needs to be qualified. Specifically the case $g_{++} < 0$ with x^+ treated as the time coordinate signals the presence of a closed timelike curve if x^+ is a compact dimension. In the present context, we have $g_{++} \sim \gamma(\Phi')^2 > 0$, and it is sensible to compactify x^+ on a spacelike circle. That is, we consider x^- to be the time coordinate. In this case, these are spacetimes with no x^- dependence, *i.e.* with time translation invariance. The scalar field must be a periodic nonsingular function $\Phi(x^+)$. A periodic Φ varying nontrivially over the compact x^+-direction has $\Phi' = 0$ at isolated x^+-values: this however is not problematic, since it will turn out that Φ' essentially disappears. At long wavelengths compared with the size of the x^+-circle, this gives an effective bulk 4-dim or 3-dim spacetime. The Kaluza-Klein compactification is natural and manifest in the PBH coordinates (5.1.6) (5.2.7): it can be performed in the standard way by writing $ds^2 = g_{mn}dx^m dx^n = G_{\mu\nu}dx^\mu dx^\nu + G_{dd}(x^d + A_\mu dx^\mu)^2$. Then the $\{g_{++}, g_{+-}\}$-terms can be rewritten as

$$\frac{1}{w^2}\left(\gamma w^2(\Phi')^2(dx^+)^2 - 2dx^+dx^-\right) = \gamma(\Phi')^2\left(dx^+ - \frac{dx^-}{\gamma w^2(\Phi')^2}\right)^2 - \frac{(dx^-)^2}{\gamma w^4(\Phi')^2} \, .$$

Thus the effective 4D or 3D metric, for AdS_5 or AdS_4 respectively, after compactifying on x^+ naively becomes

$$ds^2 = -\frac{(dx^-)^2}{\gamma w^4(\Phi')^2} + \frac{dx_i^2}{w^2} + \frac{dw^2}{w^2} \, , \tag{5.3.1}$$

where γ and the range of x_i have been defined after (5.1.6). Apart from the annoying factor of $(\Phi')^2$ which disappears as we will see below, these are thus spacetimes which exhibit a Lifshitz-like scaling with exponent $z = 2$, *i.e.*

$$x^- \equiv t \to \lambda^2 t, \qquad x_i \to \lambda x_i, \qquad w \to \lambda w \, . \tag{5.3.2}$$

The $z = 2$ Lifshitz scaling can also in fact be seen in the metric written in conformal coordinates (5.2.3): taking the compact coordinate x^+ to not scale, we see the scaling $y^- \sim w^2$. This is also consistent with the conformal-PBH coordinate transformation relation $y^- = x^- + \frac{w^2 f'}{4} \sim \lambda^2 y^-$. Likewise, the presence of the conformal factor $e^{f(x^+)}$ breaks boost invariance.

These Lifshitz spacetimes are likely to not have any supersymmetry. However the null solutions described previously in fact do preserve some fraction of lightcone supersymmetry. Our belief is that the dimensional reduction along the x^+-direction breaks the lightcone supersymmetry completely.

Note that the nontrivial dependence on the x^+-direction through the $g_{++} \sim (\Phi')^2$ term breaks the Galilean boost invariance, $x_i \to x_i - v_i x^-$, $x^+ \to x^+ - \frac{1}{2}(2v_i x_i - v_i^2 x^-)$. If $\Phi' = 0$, then $g_{++} = 0$, boost invariance reappears, and the system has a larger Schrodinger symmetry.

If x^+ is noncompact, these systems admit a lightlike scaling symmetry $x^+ \to \lambda x^+$, $x^- \to \frac{1}{\lambda}x^-$, $Q \to \frac{Q}{\lambda}$, where the parameter Q appears in the combination Qx^+ in any function of x^+, *e.g.* $e^{f(x^+)} = e^{f(Qx^+)}$. This can be used to fix the parameter, say as $Q = 1$. The compactification of the x^+-dimension makes the system nonrelativistic, the compactification size becoming a physical (inverse) mass parameter. This lightlike scaling then is not a physical symmetry anymore, since it changes the physical parameters of the compactified nonrelativistic theory. The PBH coordinate system allows a natural interpretation to the compactification process: technically, this admits a natural Kaluza-Klein reduction by compactification on x^+.

5.3.1 Dimensional reduction, more rigorously

Consider a 5-dim metric of the form

$$ds^2 = -N^2(x^+)K^2(s^i)dt^2 + \frac{1}{N^2(x^+)}(dx^+ + N^2(x^+)A)^2 + \frac{1}{w^2}(ds^i)^2 \ , \tag{5.3.3}$$

where $N(x^+)$ governs the metric component g_{++}, with A being the Kaluza-Klein gauge field, and $s^i = x^i, w$ ($x^i \equiv x^1, x^2$). We have identified t as x^- earlier: the metric has no t-dependence. Define vielbeins[5]

$$\bar{e}^0 = Ne^0 = NKdt \ , \quad \bar{e}^+ = \frac{1}{N}(dx^+ + N^2 A_0 K dt) \ , \quad \bar{e}^i = e^i = \frac{1}{w}ds^i \ , \tag{5.3.4}$$

where $\bar{e}^\mu$ are vielbeins in the 5-dim metric, while e^μ are those of the lower dimensional metric: these satisfy $ds^2 = \eta_{MN}\bar{e}^M\bar{e}^N$. We take the Kaluza-Klein gauge field defined by the 1-form $A = A_0 e^0 = \frac{A_0}{N}\bar{e}^0$ to comprise purely a scalar potential with solely electric field strength, defined as $dA = \frac{1}{2}F_{0i}e^0 \wedge e^i$, in terms of the vielbeins e^μ of the lower dimensional spacetime. The field strength is related to the gauge field as $F_{0i} = -2w(\partial_i A_0 + A_0\frac{\partial_i K}{K})$. This is thus a "minimal" metric family that contains the AdS_5 null solution we have been discussing above.

Furthermore, we obtain a null-type metric of the form we have discussed earlier if we set $g_{tt} = -N^2K^2(1-A_0^2) = 0$, *i.e.* $A_0^2 = 1$: comparing with the earlier metric (5.1.6) (5.2.7), we see that $N = \frac{1}{\sqrt{\gamma\Phi'}}$, $K = \frac{1}{w^2}$. Dimensionally, we have $[N] = L, [K] = M^2, [A_0] = 0$, and $[e^A] = 0$, *i.e.* all vielbeins are dimensionless, consistent with the fact that the metric is dimensionless in units where $R_{AdS} = 1$ (the lhs is actually $\frac{ds^2}{R^2_{AdS}}$). With this simplified

[5]The metric in component form is

$$ds^2 = -N^2(x^+)K^2(s^i)(1 - A_0^2(s^i))dt^2 + \frac{(dx^+)^2}{N^2(x^+)} + 2A_0(s^i)K(s^i)dx^+dt + g_{ij}ds^i ds^j \ ,$$

ansatz however, it is difficult to separate the gauge field parts of the system from the lower dimensional metric per se: in other words, it is desirable to retain $K(s^i)$ and $A_0(s^i)$ separately towards understanding the lower dimensional effective action better.

We define the spin connection $\omega^a{}_b$ via the relations $d\bar{e}^a = -\omega^a{}_b \wedge \bar{e}^b$. We have (note *e.g.* $\omega^0{}_+ = -\omega_{0+} = \omega_{+0} = \omega^+{}_0$)

$$\begin{aligned} d\bar{e}^0 &= -\omega^0{}_+ \wedge \bar{e}^+ - \omega^0{}_i \wedge \bar{e}^i = \frac{w\partial_i K}{K}\bar{e}^i \wedge \bar{e}^0 + N'\bar{e}^+ \wedge \bar{e}^0 \ , \\ d\bar{e}^+ &= -\omega^+{}_0 \wedge \bar{e}^0 - \omega^+{}_i \wedge \bar{e}^i = \frac{1}{2}F_{0i}\bar{e}^0 \wedge \bar{e}^i + N'A_0\bar{e}^+ \wedge \bar{e}^0 \ , \\ d\bar{e}^i &= -\omega^i{}_0 \wedge \bar{e}^0 - \omega^i{}_+ \wedge \bar{e}^+ - \omega^i{}_j \wedge \bar{e}^j = -\bar{e}^w \wedge \bar{e}^i \ , \end{aligned} \tag{5.3.5}$$

and the spin connection becomes

$$\begin{aligned} \omega^0{}_+ = \omega^+{}_0 = N'\bar{e}^0 - N'A_0\bar{e}^+ + \frac{1}{4}F_{0i}\bar{e}^i \ , &\qquad \omega^i{}_+ = -\omega^+{}_i = \frac{1}{4}F_{0i}\bar{e}^0 \ , \\ \omega^0{}_i = \omega^i{}_0 = \frac{w\partial_i K}{K}\bar{e}^0 + \frac{1}{4}F_{0i}\bar{e}^+ \ , &\qquad \omega^i{}_w = -\omega^w{}_i = -\bar{e}^i \ . \end{aligned} \tag{5.3.6}$$

The curvature 2-forms are calculated using $R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b = R^a{}_{bcd}\bar{e}^c \wedge \bar{e}^d$. The relevant Riemann tensor components are

$$\begin{aligned} R^0{}_{+0+} = -(NN'' + N'^2)(1 - A_0^2) - \frac{1}{16}F_{0i}^2 \ , \quad R^i{}_{+i+} = -\frac{1}{16}F_{0i}^2 \ , \quad R^i{}_{jij} = -1 = R^i{}_{wiw} \ \ [i, j \neq w], \\ R^0{}_{i0i} = \frac{w\partial_w K}{K} - \frac{w^2\partial_i^2 K}{K} + \frac{1}{8}F_{0i}^2 \ \ [i \neq w] \ , \qquad R^0{}_{w0w} = -\frac{w\partial_w K}{K} - \frac{w^2\partial_w^2 K}{K} + \frac{1}{8}F_{0w}^2 \ . \end{aligned} \tag{5.3.7}$$

The metric determinant is $-g = -\frac{K^2}{w^6}$, and the Ricci scalar for this metric is

$$\begin{aligned} R^{(5)} &= \frac{1}{2K^2}\Bigg[-4(NN'' + (N')^2)K^2 - 12K^2 - 4w^2K\partial_i^2 K + 4wK\partial_w K \\ &\quad + \left[4(NN'' + (N')^2)K^2 + w^2(\partial_i K)^2\right]A_0^2 + 2w^2KA_0\partial_i A_0\partial_i K + w^2K^2(\partial_i A_0)^2\Bigg] \ , \\ &= -2(NN'' + (N')^2) - \left[\frac{2}{K}(w^2\partial_i^2 K - w\partial_w K + 3K)\right] + \frac{1}{8}F_{0i}^2 + 2(NN'' + (N')^2)A_0^2 \end{aligned} \tag{5.3.8}$$

(Numerical output corroborates this.) This higher dimensional Ricci scalar expanded in terms of the lower dimensional modes essentially gives the lower dimensional effective action on wavelengths long compared with the size of the compact dimension. Note that if there was no nontrivial x^+-dependence in this system, this would be the conventional Kaluza-Klein reduction with the lower dimensional fields (metric, massless gauge field and scalar) being independent of the compact dimension. The scalar $g_{++} = \frac{1}{N^2(x^+)}$ in this case is of a restrictive form, which therefore reflects in its lower dimensional kinetic term being a total derivative $\partial_+(NN')$.

The form of $R^{(4)}$ appearing here suggests that the lower dimensional spacetime is in fact of the form

$$ds^2 = -K^2(s^i)dt^2 + \frac{1}{w^2}ds^{i2} \qquad \Rightarrow \qquad R^{(4)} = -\frac{2}{K}\Big(w^2\partial_i^2 K - w\partial_w K + 3K\Big)\,. \tag{5.3.9}$$

Note that the $N(x^+)$, *i.e.* Φ', has disappeared from the effective metric. A closer look at the apparent gauge field mass term in (5.3.8) shows this to be $\int dx^+\partial_+(NN')$, which vanishes being the integral over a compact direction of a total derivative. On the other hand, the scalar kinetic terms do in fact contribute a mass term for the gauge field: we have the terms

$$-\frac{1}{2}g^{++}(\partial_+\Phi)^2 - g^{+t}\partial_+\Phi\partial_t\Phi \;\rightarrow\; -\frac{1}{2}N^2(1-A_0^2)(\Phi')^2 + \ldots \;\rightarrow\; \frac{1}{2}N^2(\Phi')^2A_0^2\,. \tag{5.3.10}$$

With $N^2 = \frac{1}{\gamma(\Phi')^2}$, the mass term becomes $\frac{m_A^2}{2} = \frac{1}{2\gamma}$, *i.e.* $m_A^2 = 4$ (AdS_5) or $m_A^2 = 2$ (AdS_4), agreeing with [17].

The 5-dim metric is a solution to the Einstein equations with a scalar depending only on the x^+-direction. Then the [00]-component equation of motion gives $\frac{1}{2}(-6 - \frac{1}{8}F_{0i}^2) = -4$, which gives $(\partial_i A_0 + A_0\frac{\partial_i K}{K})^2 = \frac{4}{w^2}$. admitting the solution $K = \frac{1}{w^2}$, $A_0 = -1$. These conditions are also satisfied by a massive ($m^2 = 4$) vector field with profile $A = A_0 e^0 = A_0 K dt = -\frac{dt}{w^2}$ in the $z = 2, d = 2$ Lifshitz (bulk) background metric ($Lif_{z=2}^{d=2}$):

$$ds^2 = -\frac{dt^2}{w^4} + \frac{dx^{i2}}{w^2} + \frac{dw^2}{w^2}\,. \tag{5.3.11}$$

As mentioned earlier, the fluxes that source $Lif_{z=2}^{d=2}$ are classically equivalent to a massive vector field with profile $A = -\frac{dt}{w^2}$. As a further check, assuming the scalar is a function of x^+ alone, the scalar equation of motion simplifies to $\partial_+(N^2(1-A_0^2)\frac{K}{w^3}\Phi') = 0$, verifying again the above solution. Note that time reversal invariance is broken in these solutions, by the gauge field in the lower dimensional system, and by the metric in the higher dimensional one.

What we have demonstrated here is that the on-shell Lifshitz spacetime with massive gauge field source is a solution to a 5-dim effective action corresponding to Einstein gravity with a massive gauge field and two scalars, one the remnant of the 10-dim dilaton and the other the Kaluza-Klein scalar corresponding to the radius of the compact dimension. The on-shell solution relates the two scalars and further fixes the gauge field mass in terms of the two scalars.

It is perhaps surprising that the naive dimensional reduction (5.3.1) involves $\Phi' \sim \frac{1}{N(x^+)}$ which however disappears in the metric (5.3.9) implied by (5.3.8): we do not have an intuitive way to understand this. The nontrivial dependence on the x^+-dimension might appear to complicate a Wilsonian separation-of-scales argument making it harder to justify why it is consistent for modes other than the ones here to be trivial. For instance, one could imagine turning on a lower dimensional vector potential $A_i dx^i$: this would arise from a Kaluza-Klein gauge field 1-form $A = A_0 e^0 + A_i e^i = \frac{A_0}{N}\bar{e}^0 + A_i \bar{e}^i$, with corresponding field strength $dA = \frac{1}{2}F_{\mu\nu}e^\mu \wedge e^\nu$. We do not have any conclusive result here for a consistent dimensional reduction: for instance, the 5-dim Ricci scalar has extraneous factors of $N(x^+)$ appearing in the analogous calculation, making it harder to interpret the lower dimensional system. However it is tempting to believe that some generalization of our "minimal" Kaluza-Klein ansatz (containing only A_0) will address these concerns and possibly also pave the way for more general Lifshitz spacetimes[6].

The calculation for the AdS_4-deformed solution is similar, resulting in a 2+1-dimensional bulk $z = 2$ Lifshitz theory. In sec. 6, we will find an alternative approach to uplift the $Lif_{z=2}^{d=2}$ background to 11-D supergravity.

5.3.2 Scalar probes and Lifshitz geometry

We would like to see how a bulk supergravity scalar sees the Lifshitz geometry at long wavelengths.

Consider the scalar action $S = \frac{1}{G_5}\int d^5x\ \sqrt{-g}\ g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi$: on restricting to modes with no

[6]It appears difficult however to find more general solutions in the higher dimensional AdS_5-deformed system within these ansatze or minor generalizations: in particular, attempts, in the cosmological context (S. Das, KN, S. Trivedi, unpublished), to find solutions with radial dependence for the dilaton (and metric) were not conclusive.

x^+-dependence (*i.e.* $\partial_+\varphi = 0$), this gives

$$
\begin{aligned}
S &= \frac{1}{G_5}\int \frac{d^4x dx^+}{w^5}\left[-\frac{w^4(\Phi')^2}{4}(\partial_-\varphi)^2 + w^2(\partial_i\varphi)^2 + w^2(\partial_w\varphi)^2\right] \\
&= \int \frac{d^4x}{w^5}\left[-w^4\left(\frac{\int dx^+(\Phi')^2}{4}\right)(\partial_-\varphi)^2 + w^2L(\partial_i\varphi)^2 + w^2L(\partial_w\varphi)^2\right] \\
&= \frac{1}{G_4}\int \frac{d^4x}{w^5}\left[-w^4(\partial'_-\varphi)^2 + w^2(\partial_i\varphi)^2 + w^2(\partial_w\varphi)^2\right] ,
\end{aligned}
\tag{5.3.12}
$$

where L is the size of the compact x^+-dimension, and $G_4 = \frac{G_5}{L}$ is the 4-dim Newton constant arising from dimensional reduction.

Thus we see that after the rescaling $x^- \to x^{-'} = \frac{L}{\int dx^+(\Phi')^2}x^-$, the scalar action at wavelengths long compared to the compactification size becomes that in the 4-dim $z = 2$ Lifshitz background (5.3.11).

A priori, this looks slightly different from a direct dimensional reduction of the equation of motion of the scalar, where it would seem that Φ' remains. The calculation here suggests that the Lifshitz geometry arises on scales large compared with the typical scale of variation (*i.e.* the compactification size), in other words effectively setting $\Phi' \sim const.$

5.4 The dual field theory

The field theory dual to the AdS_5 backgrounds is the $d = 4$ $\mathcal{N}=4$ super Yang-Mills theory with an appropriate lightlike deformation: taking the scalar to be the dilaton, the identification is essentially that given in [47, 48], *i.e.* the $\mathcal{N}=4$ SYM theory with the gauge coupling deformed to vary along the x^+-direction as $g^2_{YM}(x^+) = e^{\Phi(x^+)}$. Note that in the PBH coordinates (5.1.6), (5.2.7), the boundary metric $ds^2_4 = \lim_{r\to 0} r^2 ds^2_5$ on which the gauge theory lives is manifestly flat space. The lightlike deformation means that no nonzero contraction exists involving the metric and coupling alone, since only $\partial_+\Phi$ is nonvanishing with $g^{++} = 0$: thus various physical observables (in particular the trace anomaly) are unaffected by this deformation.

In the conformal coordinates (5.2.3), the base space on which the gauge theory lives is conformal to flat space with metric $\tilde{g}_{\mu\nu} = e^{f(x^+)}\eta_{\mu\nu}$. Various arguments were given in [48] discussing the role of the lightlike conformal factor in the gauge theory. The lightlike nature implies that various physical observables are in fact unaffected by the conformal factor since no nonzero contraction exists. However an important role played by the conformal factor is in providing dressing factors for operators and their correlators: specifically, conformally dressed operators in the conformally flat background behave like undressed operators in flat space, as we will discuss below in the context of the holographic 2-point function. The gauge coupling is again subject to the lightlike deformation alone as $g_{YM}^2(x^+) = e^{\Phi(x^+)}$.

In lightcone gauge $A_- = 0$ (compatible with Lorentz gauge $\partial_\mu A^\mu = 0$), the gauge kinetic terms reduce to those for the transverse modes A_i, the field A_+ being nondynamical: this is essentially similar to multiple copies of a massless scalar. Retaining modes of the form $A_i \equiv e^{ik_+x^+}A_i(x^-, x^i)$, with momentum k_+ along the x^+-direction, and approximating the coupling by its mean value say $g_{YM}^{(0)}$, this gives

$$\int d^3x dx^+ \frac{1}{g_{YM}^2(x^+)} \left[-2\partial_+A_i\partial_-A_i + (\partial_jA_i)^2\right] \rightarrow \int d^3x \, \frac{L}{(g_{YM}^{(0)})^2} \left[-iA_i\partial_tA_i + \frac{1}{k_+}(\partial_jA_i)^2\right] , \tag{5.4.1}$$

identifying $x^- \equiv t$, absorbing a k_+ into the definition of A_i, with L being the size of the compact x^+-direction. This heuristic argument shows the $z = 2$ Lifshitz scaling symmetry in the kinetic terms. In a sense, this is not surprising, since the $z = 2$ Lifshitz symmetry can be obtained by breaking Galilean (Schrodinger) symmetries: in the present case, the coupling varying along the compact x^+-direction breaks the x^+-shift symmetry. However, the field theory is really an interacting strongly coupled field theory with Lifshitz symmetries dual to the weakly coupled bulk Lifshitz geometry.

After the dimensional reduction along x^+, the theory becomes an interacting strongly coupled 3-dim gauge theory. The 3-dimensional gauge coupling is now naively $\frac{1}{g_3^2} = \int dx^+ \frac{1}{g_{YM}^2(x^+)} \sim \frac{L}{(g_{YM}^{(0)})^2}$, approximating the 4-dim coupling by its mean value. Then the theory is effectively 3-dimensional on length scales large compared with the compact direction.

In a sense, this sort of a DLCQ of $\mathcal{N}$=4 SYM with varying coupling is perhaps better defined than ordinary DLCQ. One would imagine the coupling variation causes the lightlike circle to "puff up", somewhat akin to momentum along the circle, so that the usual issue of strongly coupled zero modes stemming from DLCQ is perhaps less problematic here. This is of course not a rigorous treatment of the dimensional reduction of the $\mathcal{N}$=4 SYM theory, dual to *e.g.* the discussion of that of the bulk metric (5.3.3). It would be interesting to understand this better.

Similarly we expect that the field theory dual to the AdS_4 backgrounds is a lightlike deformation, dimensionally reduced, of the Chern-Simons theories on M2-branes at supersymmetric singularities [4], that have been found to be dual to $AdS_4 \times X^7$ backgrounds, with X^7 an appropriate Sasaki-Einstein 7-manifold. This is thus a $1+1$-dim field theory. It would also be interesting to explore this further.

5.4.1 The holographic 2-point function

The holographic 2-point function of operators $\mathcal{O}$ dual to massive bulk scalars φ in this deformed $\mathcal{N}$=4 SYM-Lifshitz theory can be obtained by the usual rules of AdS/CFT. Doing this calculation directly in the PBH coordinates (5.1.6), (5.2.7) is interesting. However an exact calculation is hindered by the fact that the wave equation for a massive scalar does not lend itself to separation of variables and solving for the exact mode functions appears difficult: possible mode functions $\varphi(x) = e^{ik_- x^- + ik_i x^i} e^{g(x^+)} \zeta(r)$ reduce the wave equation to

$$-2ik_- g' + \frac{r^3}{\zeta(r)} \partial_r \left(\frac{1}{r^3} \partial_r \zeta(r) \right) - k_i^2 - \frac{m^2}{r^2} + \gamma r^2 (\Phi')^2 k_-^2 = 0 \ , \tag{5.4.2}$$

the $r^2(\Phi')^2$ term being problematic. However, let us consider this equation near the boundary $r \to 0$, where this term is small and the metric asymptotes to the AdS_5 metric in lightcone coordinates. Then one finds the mode functions $e^{ik_- x^- + ik_i x^i} e^{i(k_i^2 - \omega^2)x^+/2k_-} (\omega r)^2 K_\nu(\omega r)$: not surprisingly, these are in fact the AdS_5 mode functions in lightcone coordinates. As we will see below, these also arise in the calculation in conformal coordinates (setting $e^f = 1$). This then gives the AdS_5 2-point function in lightcone coordinates $\langle O(x) O(x') \rangle \sim \frac{1}{[(\Delta \vec{x})^2]^\Delta}$, with $\Delta = 2 + \sqrt{4 + m^2}$. Note that the distance element arising from the calculation here is the 4-dimensional distance $(\Delta\vec{x})^2 = -2(\Delta x^+)(\Delta x^-) + \sum_{i=1,2} (\Delta x^i)^2$: this is the analytic continuation of the Euclidean 4-dim distance of the boundary theory in pure AdS_5. Now in the limit of a compactified x^+-dimension, with $\Delta x^+ \ll \Delta x^-, \Delta x^i$, this distance element reduces to $(\Delta \vec{x})^2 \sim \sum_{i=1,2} (\Delta x^i)^2$, so that

$$\langle O(x) O(x') \rangle \sim \frac{1}{[\sum_i (\Delta x^i)^2]^\Delta} \ . \tag{5.4.3}$$

For a massless bulk scalar, we have $\Delta = 4$, recovering the equal time 2-point function of the (2+1)-dim Lifshitz theory of [106]: it also corroborates the expectation [11] that the equal time correlators of this (2+1)-dim Lifshitz theory are identical to those of a 2-dim Euclidean conformal field theory.

We will now discuss the holographic 2-point function in conformal coordinates (5.2.3) where the conformal factor e^f appears explicitly: this calculation has been done in [48], noting the fact that the scalar wave equation in the lightlike deformed background can be solved exactly in these coordinates. We will not repeat this in detail here but will describe some essential points. Consider a minimally coupled scalar field of mass m propagating in the bulk 5-dim metric in (5.2.3), with action $S = -\int d^5x\sqrt{-g}\ (g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi + m^2\varphi^2)$, that is dual to an operator $O(x)$ in the boundary CFT with scaling dimension Δ. The wave equation following from the above action can be solved exactly for basis mode functions $e^{-f(x^+)/2}\ e^{i(k_i^2x^+ - \omega^2\int e^f dx^+)/2k_-}\ e^{ik_-x^- + ik_ix^i}(\omega r)^2K_\nu(\omega r)$, where $\nu = \sqrt{4+m^2}$.

The scalar action reduces, using the equation of motion, to a term at the (regulated) boundary $r=\epsilon$, given as $S = -\int d^4x\sqrt{-g}g^{rr}\ \varphi(\vec{x},r)\ \partial_r\varphi(\vec{x},r)|_{r=\epsilon}$: using the basis modes, this can be evaluated in momentum space giving (upto an overall ν-dependent constant)

$$S = \int d^2k_i dk_- dk_+\ \varphi(k_i,k_-,\omega^2)\varphi(-k_i,-k_-,\omega^2)\ \omega^{2\nu}\ , \tag{5.4.4}$$

where the integrals over all four variables, $k_i, i=1,2,\ k_-, k_+$ go from $[-\infty,\infty]$, and $\omega^2 = -2k_-k_+ + k_i^2$. This can be recast in position space as

$$S = C\int d^4x d^4x'\ e^{3f(x^+)/2}e^{3f(x'^+)/2}\ \varphi(\vec{x})\varphi(\vec{x}')\left(\frac{\Delta\lambda}{\Delta x^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}\ , \tag{5.4.5}$$

where C is a constant, $\Delta = 2+\nu$, and $\lambda = \int e^{f(x^+)}dx^+$ is the affine parameter along null geodesics stretched solely along x^+. The 4-dimensional distance element here is $(\Delta\vec{x})^2 = -2(\Delta x^+)(\Delta x^-) + \sum_{i=1,2}(\Delta x^i)^2$.

The boundary coupling between the (boundary value of the) scalar φ and the operator O is $S_{Boundary} = \int d^4x\sqrt{-\tilde{g}}\ O(x)\varphi(x)$, where $\tilde{g}_{\mu\nu} = e^f\eta_{\mu\nu}$ is the boundary metric and $\varphi(x) = \epsilon^{-\Delta_-}\varphi(x,\epsilon)$, with $\Delta_- = 2-\nu$.

By the usual prescriptions of AdS/CFT for calculating boundary correlation functions, equating the bulk action with the action of the boundary theory up to second order in the source $\varphi(x)$ gives

$$\begin{aligned}\sqrt{-\tilde{g}(x)}\sqrt{-\tilde{g}(x')}\langle O(x)O(x')\rangle &= \frac{\delta}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\varphi(\vec{x}')}\langle e^{\int d^4x\sqrt{-\tilde{g}}O(x)\varphi(x)}\rangle_{CFT} \\ &= \frac{\delta}{\delta\varphi(\vec{x})}\frac{\delta}{\delta\varphi(\vec{x}')}e^{-S_{Sugra}[\varphi(\vec{x})]}\ . \end{aligned} \tag{5.4.6}$$

From (5.4.5), we then get

$$\langle O(x)O(x')\rangle = Ce^{-f(x)/2}e^{-f(x')/2}\left(\frac{\Delta\lambda}{\Delta x^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}\ . \tag{5.4.7}$$

It is important to consider correlators of conformally dressed operators as emphasised in [48]. For instance, consider the operator $O(x)$ above with conformal dimension Δ in the SYM theory. Then a simple point to note is that the short distance limit of the correlator above gives $\langle O(x)O(x')\rangle \sim e^{-f(x^+)\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}$, by approximating $\frac{\Delta\lambda}{\Delta x^+} \sim \frac{d\lambda}{dx^+} = e^f$. Thus it is clear that the conformally dressed operators $e^{f(x^+)\Delta/2}O(x)$ have essentially a flat space 2-point function $\langle e^{f(x^+)\Delta/2}O(x)e^{f(x^+)\Delta/2}O(x')\rangle \sim \frac{1}{[(\Delta\vec{x})^2]^\Delta}$. In other words, the conformally dressed operators in the conformally flat background behave like undressed operators in the flat space background. More generally, the 2-point function for dressed operators at arbitrary points x, x', is

$$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x')\rangle = Ce^{\frac{f(x)(\Delta-1)}{2}}e^{\frac{f(x')(\Delta-1)}{2}}\left(\frac{\Delta\lambda}{\Delta x^+}\right)^{1-\Delta}\frac{1}{[(\Delta\vec{x})^2]^\Delta}\ . \tag{5.4.8}$$

In the compactified limit, we have $\Delta x^+ \ll \Delta x^-, \Delta x^i$. It is then consistent to approximate $\frac{\Delta\lambda}{\Delta x^+} \sim \frac{d\lambda}{dx^+} = e^f$. Furthermore, it is consistent to approximate $e^{f(x^+)} \sim 1$, essentially smearing the x^+ dependence relative to the uncompactified dimensions. This then simplifies the 2-point function for these operators which becomes

$$\langle e^{\frac{f(x)\Delta}{2}}O(x)e^{\frac{f(x')\Delta}{2}}O(x')\rangle \ \sim\ \langle O(x)O(x')\rangle \ \sim\ \frac{1}{[(\Delta\vec{x})^2]^\Delta} \ \sim_{\Delta x^+\ll\Delta x^-,\Delta x^i}\ \frac{1}{[\sum_i(\Delta x^i)^2]^\Delta}\ . \tag{5.4.9}$$

It is worth noting that the boundary hypersurfaces are different in the conformal and PBH coordinates: in the compactified system, they do not matter, *e.g.* in the 2-point function. Effectively we have smeared the conformal factor $e^f \rightarrow 1$. This does not mean that the metrics can also be similarly reduced by simply setting $e^f \rightarrow 1$: the radial coordinates mix

x^+-dependence.

5.5 Further Lifshitz-like solutions in 11-dim supergravity

Here we consider new solutions in 5-dim gravity with negative cosmological constant coupled to a massless complex scalar, which are similar to the null solutions discussed earlier. The $2+1$-dim Lifshitz spacetimes $Lif^{d=2}_{z=2}$ arise by dimensional reduction of these 5-dim solutions along one direction. These 5-dim solutions can be embedded in 11-dim supergravity.

First, we will study a solution in 5-dim with Lifshitz symmetries where the shift along x^+ is not broken by the metric, but only by a complex scalar field. The metric and the profile for the complex scalar field are:

$$ds^2 = R^2 \frac{-2dx^+dx^- + d\vec{x}^2 + dw^2}{w^2} + R^2 \left(dx^+\right)^2, \quad \varphi(x^+) = \sqrt{\frac{2}{\ell^2}} \frac{e^{i\ell x^+}}{R} \,. \tag{5.5.1}$$

Here, we have taken the periodicity of x^+ to be 2π. The normalization of the complex scalar field determines g_{++} and ℓ is an integer. The background in (5.5.1) is an extremum of the following action

$$S_5 = \kappa_5^2 \int d^5x \sqrt{g_5} \left(R_5 - 2\Lambda - \partial_\mu \bar{\varphi} \partial^\mu \varphi\right) \tag{5.5.2}$$

where $\Lambda = -6/R^2$ and $\bar{\varphi}$ denotes complex conjugate of φ. Note that the onshell value of $\partial_\mu \bar{\varphi} \partial^\mu \varphi$ is zero. This fact will be useful in finding an uplift of this solution to 11-D supergravity. It is not hard to dimensionally reduce along the x^+-direction now, as the metric is independent of x^+. We will use the following ansatz for the line element and the complex scalar to perform the KK reduction along x^+:

$$ds^2 = G_{\mu\nu} dx^\mu dx^\nu + R^2 (dx^+ + A)^2, \quad \varphi(x^+) = \sqrt{\frac{2}{\ell^2}} \frac{e^{i\ell x^+}}{R} \,. \tag{5.5.3}$$

This generalizes the metric in (5.5.1). The reduced action can be written as

$$S_4 = \kappa_4^2 \int d^4x \sqrt{G_4} \left(R_4 - 2\Lambda - \frac{1}{4} dA^2 + \frac{m^2}{2} A^2\right) \tag{5.5.4}$$

where $\Lambda = -6/R^2$ and $m^2 = 4/R^2$. Note that this action is the action obtained by dualizing the fluxes in [106] as mentioned earlier. Further, the equations of motion of the 5-dim action in vielbein indices can be written as[7]

$$R_{ab} - \frac{1}{2}R\eta_{ab} - \Lambda\eta_{ab} = (\partial_a\varphi\partial_b\bar{\varphi} + h.c) - \frac{1}{2}\eta_{ab}\left(\partial_c\varphi\partial^c\bar{\varphi} + \partial_5\varphi\partial^5\bar{\varphi}\right)$$

$$\Rightarrow R_{ab} = \left(F_{ac}F_b^c - \frac{1}{4}F^2\eta_{ab} + m^2 A_a A_b\right) - 2\Lambda\eta_{ab}$$

$$R_{a5} = (\partial_a\varphi\partial_5\bar{\varphi}\partial_a\varphi\partial_5\bar{\varphi}) \Rightarrow \nabla_a F^{ab} = m^2 A^b$$

$$R_{55} - \frac{1}{2}R\eta_{55} - \Lambda\eta_{55} = 2\partial_5\varphi\partial_5\bar{\varphi} - \frac{1}{2}\left(\partial_c\varphi\partial^c\bar{\varphi} + \partial_5\varphi\partial^5\bar{\varphi}\right) \quad \Rightarrow \quad \left(-\frac{1}{4}F^2 + \frac{m^2}{2}A^2\right) = -\Lambda.$$

Further, the scalar equation of motion is satisfied if and only if $A^2 = 1$. This condition also guarantees $\partial_\mu\varphi\partial^\mu\bar{\varphi} = 0$. Note that the 5-dim equations of motion are satisfied if the 4-dim equations of motion and constraints are satisfied. The $Lif^{d=2}_{z=2}$ metric and matter content solve the above equations of motion and also satisfies the constraints. Hence, $Lif^{d=2}_{z=2}$ background can be uplifted to a solution of 11-dim supergravity if the solution in 5-dim can be lifted to a solution of 11-dim supergravity. Note that the following eleven dimensional metric and 4-form flux

$$ds^2_{11} = g_{AB}dx^A dx^B + ds^2_{\mathbb{CP}^2} + d\chi_1^2 + d\chi_2^2\ ,$$

$$G_4 = 2J\wedge J + 2J\wedge d\chi_1\wedge d\chi_2 + \sqrt{3}d\varphi\wedge J\wedge(d\chi_1 - id\chi_2) + h.c. \tag{5.5.5}$$

is a solution of 11-dim supergravity if $g^{(5)}$ and φ satisfy the 5-dim equations of motion along with the constraint $d\varphi\wedge_5 \star_5 d\bar{\varphi} = 0$.[8] Here $\chi_{1,2}$ are coordinates in $S^1\times S^1$ and J is the

[7]The line element in terms of vielbeins can be written as $ds^2 = \eta_{ab}e^a e^b + e^5 e^5$, where $e^a = e^a_\mu dx^\mu$, $e^5 = dx^+ + A_\mu dx^\mu = dx^+ + e^5_\mu dx^\mu$. Further, $dx^\mu = E^\mu_a e^a$ and $dx^+ = e^5 - A_a e^a = E^+_a e^a + E^+_5 e^5$. Note that $\partial_5\varphi = (\partial_+ + E^\mu_5\partial_\mu)\varphi$ and $\partial_a\varphi = (E^+_a\partial_+ + E^\mu_a\partial_\mu)\varphi$.

[8]Using the properties of Kähler form and the constraint $d\varphi\wedge_5\star_5 d\bar{\varphi} = 0$, the equations of motion of 11-dim supergravity (in the background (5.5.5)) can be reduced to

$$R_{\mu\nu} = -4g_{\mu\nu} + \frac{1}{2}(\partial_\mu\varphi\partial_\nu\bar{\varphi} + h.c)\,, \quad \text{for } \mu,\nu = 0,1,2,3$$

$$R_{ij} = 6g_{ij}, \quad \text{for } i,j \text{ in } \mathbb{CP}^2$$

All other components of the Ricci tensor vanish. Note that the i,j components of Einstein's equations are trivially satisfied. Further, the Bianchi identity for the 4-form flux is also trivially satisfied. The flux equation is satisfied if $d\star_5 d\varphi = 0$. Hence, the conditions for (5.5.5) to be a solution of 11-dim supergravity

Kähler form on $\mathbb{CP}^2$. This is similar to some constructions in [61]. The $\mathbb{CP}^2$ space here can be generalized to any Kahler Einstein space.

Note that g_{++} does not vanish anywhere in this bulk solution. At this point, we are not clear about the interpretation of the dual field theory. One might guess that the dual field theory lives on $M5$ branes. Perhaps, it is convenient to study the type II theory on $D4$- or $D3$-branes obtained by dimensional reduction.

5.6 Discussion

We have discussed $z = 2$ Lifshitz geometries obtained by dimensional reduction along a compact direction of certain lightlike deformations of $AdS \times X$ solutions of 10- or 11-dimensional supergravity. We have also described some time-dependent (cosmological) solutions, with and without a nontrivial scalar (dilaton), and their anisotropic Lifshitz scaling.

Our discussion has been largely from the point of view of the bulk AdS-deformed theories. The duals in many of these cases are appropriate deformations of the $\mathcal{N}$=4 super Yang-Mills theory. In particular the constructions in this chapter can be taken to suggest precise field theories dual to AdS-Lifshitz spacetimes. In particular, the dual to the $z = 2$ AdS-Lifshitz theory is simply the dimensional reduction along the x^+-direction of the $\mathcal{N}$=4 SYM theory with gauge coupling $g_{YM}^2 = e^{\Phi(x^+)}$. Similarly we expect the $1 + 1$-dim duals in the AdS_4-deformed compactified cases are appropriate deformations of the Chern-Simons theories on M2-branes at supersymmetric singularities. It would be interesting to flesh these out further.

As we have discussed towards the end of sec. 3, the null solutions we have considered are of a particular type. Generalizing these solutions with more interesting ansatze, one might expect to find bulk spacetime solutions describing holographic renormalization group flows between *e.g.* AdS or Schrödinger and $z = 2$ Lifshitz spacetimes. These would correspond to higher dimensional analogs of *e.g.* similar RG flows discussed in [106]. It would be interesting to explore this further.

A solution that interpolates between the Schrödinger and Lifshitz background would break translation symmetry along the x^+-direction in the bulk but not asymptotically. As mentioned earlier, breaking of the translation symmetry along the x^+ direction corresponds

are the same as the conditions for extremizing the action in (5.5.2).

to breaking the particle number symmetry in the Schrödinger spacetime. A solution that breaks this symmetry only in the bulk (and not asymptotically), describes a state that breaks the particle number (and Galilean boost) symmetry spontaneously. In other words such a solution provides a holographic description of a superfluid ground state, in the sense that a scalar condensate spontaneously breaks a $U(1)$ global symmetry. It would be interesting to explore this further.

5.A The general setup for AdS_5 cosmological solutions

The solutions described in (5.2.1) are in fact part of a more general family of solutions of Type IIB supergravity or string theory, that are deformations of $AdS_5 \times X^5$, with X^5 being the base of a Ricci-flat 5-dim space. This can be seen by noting that a general metric of the form

$$ds^2 = Z^{-1/2}(x)\tilde{g}_{\mu\nu}dx^\mu dx^\nu + Z^{1/2}(x)\tilde{g}_{mn}dx^m dx^n \ , \tag{5.A.1}$$

is a solution of the equations of motion, as long as $Z(x)$ is a harmonic function on the flat, six dimensional tranverse space with coordinates x^m, $\tilde{g}_{mn}$ is Ricci-flat, depending only on x^m, and $\tilde{g}_{\mu\nu}$ and the scalar Φ are dependent only on the x^μ, satisfying the conditions (5.2.2). Taking the near horizon decoupling limit gives the solution in (5.2.1), the $d\Omega_5^2$ now being the metric on the base 5-space over which the transverse Ricci-flat space is a cone, with $\tilde{g}_{mn}dx^m dx^n = dr^2 + r^2 d\Omega_5^2$.

To see how this is obtained, note that the 10D IIB supergravity Einstein equations are

$$R_{MN} = \frac{1}{6}F_{MA_1A_2A_3A_4}F_N{}^{A_1A_2A_3A_4} + \frac{1}{2}\partial_M\phi\partial_N\phi \ , \tag{5.A.2}$$

the $F^2 = F_{ABCDE}F^{ABCDE}$ term vanishing because of the self-duality of the 5-form F. For the above backgrounds, it is clear that this equation with components along the S^5 directions is satisfied, since the scalar does not depend on the angular coordinates of the S^5: these equations are essentially the same as those for the $AdS_5 \times X^5$ solution. In the $\{\mu, r\}$-directions, the Ricci tensor is

$$R_{\mu\nu} = \tilde{R}_{\mu\nu} - \frac{4}{R^2}g_{\mu\nu} \ , \qquad R_{rr} = -\frac{4}{R^2}g_{rr} \ . \tag{5.A.3}$$

The term $-\frac{4}{R^2}g_{\mu\nu}$ in the first equation, as well as the R_{rr}-equation, are balanced by the 5-form contribution (which in effect provides a negative cosmological constant in 5-dimensions). This shows that the extra contribution $\tilde{R}_{\mu\nu}$ must balance the scalar kinetic energy for the Einstein equations with μ, ν-components to be satisfied. In effect the Einstein equation then becomes $R_{MN} = -4g_{MN} + \frac{1}{2}\partial_M\Phi\partial_N\Phi$: in fact it is easy to see that this equation is also valid when the scalar has radial r-dependence (as discussed below in the context of AdS_4 solutions). The scalar equation follows since it satisfies the massless free-field equation in 10 dimensions (with a trivial 3-form field strength) and is independent of r and the S^5 coordinates.

We expect similar solutions exist where the scalar is not the dilaton but arises from the 5-form flux through the compactification on a nontrivial 5-manifold, as in the $AdS_4 \times X^7$ case discussed below.

5.A.1 AdS_4 null and cosmological solutions

This is a straightforward generalization to AdS_4 of the cosmological solutions [47, 48, 14, 13] described above.

We are considering M-theory backgrounds with nontrivial metric and 3-form, that are generalizations of $AdS_4 \times X^7$, with X^7 being the 7-dim base space (possibly Sasaki-Einstein) of some Ricci-flat 8-dim space (say a CY 4-fold). With no other matter content, such backgrounds can be seen to arise by stacking M2-branes at a point on a Ricci-flat transverse space (which is a cone over the 7-dim space X^7) and taking the near horizon scaling limit, giving the $AdS_4 \times X^7$ background. The 11-dim supergravity equation of motion for the metric components are

$$R_{MN} = \frac{1}{12}G_{MB_1B_2B_3}G_N^{B_1B_2B_3} - \frac{1}{144}g_{MN}G_{B_1B_2B_3B_4}G^{B_1B_2B_3B_4} , \tag{5.A.4}$$

Consider now an ansatz for a deformation of $AdS_4 \times X^7$ of the form

$$ds^2 = \frac{1}{r^2}(\tilde{g}_{\mu\nu}dx^\mu dx^\nu + dr^2) + 4ds^2_{X^7} , \quad G_4 = 6\mathrm{vol}(M_4) + Cd\Phi(x^\mu) \wedge \Omega_3 , \tag{5.A.5}$$

with $\tilde{g}_{\mu\nu}$ being functions of x^μ alone, the scalar $\Phi = \Phi(x^\mu, r)$, C being a normalization constant, and Ω_3 is a harmonic 3-form on some Sasaki-Einstein 7-manifold X^7 with a non-trivial third Betti number (b_3). With a trivial scalar $\Phi = const$ and $\tilde{g}_{\mu\nu} = \eta_{\mu\nu}$, this is the $AdS_4 \times X^7$ solution (see *e.g.* [61] for the normalization). The condition $d\Omega_3 = 0$ ensures that the Bianchi identity is satisfied by the 4-form flux, while the flux equation $d\star G_4 + \frac{1}{2} G_4 \wedge G_4 = 0$ is satisfied if $d(\star d\Phi) = 0$ and $d \star \Omega_3 = 0$: these last two equations are the scalar equation of motion and the second condition for a harmonic form Ω_3. Further, the Einstein equations for the internal indices i, j, are satisfied if $d\Phi \wedge_4 \star_4 d\Phi = 0 \sim (\partial\Phi)^2$ (which is consistent with the null solutions described in the text). For instance, this kills the scalar terms in the second term in (5.A.4): further terms involving Φ in $G_{iB_1B_2B_3} G_j^{B_1B_2B_3}$ again necessarily force one of the B_i to be μ, thus involving the contraction $(\partial\Phi)^2$ which vanishes. A similar thing is true for the equation with μ, i-components, resulting in

$$R_{MN} = -3 g_{MN} + \frac{1}{2} \partial_M \Phi \partial_N \Phi \ , \qquad M, N = \mu, r.$$

In particular, note that this equation also holds for the case when the scalar Φ has radial r-dependence. The constant C can be used to normalize the coefficient of this scalar kinetic term to be $\frac{1}{2}$. The 4-form flux provides an effective negative cosmological constant in 4-dim. If Φ does not depend on r, the rr-component of this equation is simply $R_{rr} = -3 g_{rr}$, and the other equations with μ, ν-components simplify to

$$\tilde{R}_{\mu\nu} = \frac{1}{2} \partial_\mu \Phi \partial_\nu \Phi \ , \qquad \frac{1}{\sqrt{-\tilde{g}}}\, \partial_\mu (\sqrt{-\tilde{g}}\ \tilde{g}^{\mu\nu} \partial_\nu \Phi) = 0, \tag{5.A.6}$$

the second equation being the scalar equation of motion. In other words, a solution to the 3-dim Einstein-scalar system is automatically a solution to M-theory on AdS_4. It appears difficult to interpret the scalar Φ as the M-theory uplift of the IIA dilaton.

Chapter 6

String theory duals of Lifshitz-Chern-Simons gauge theories

The material in this chapter and the fifth item in the synopsis (Chapter 1) appeared in "String theory duals of Lifshitz-Chern-Simons gauge theories" with John McGreevy [18] and it is currently under consideration for publication in *Classical and Quantum Gravity.*

6.1 Introduction

Recently, Mulligan *et. al.* [135] studied an Abelian gauge theory in 2+1 dimensions with $z = 2$ Lifshitz scaling symmetry, $x \to \lambda x, t \to \lambda^2 t$. Parity symmetry is explicitly broken by the presence of a Chern-Simons term in this theory. Unlike the kinetic term in Maxwell-Chern Simons theory, the Lifshitz-type kinetic term is marginal. Hence, the Chern-Simons term and the Lifshitz kinetic term compete in the infrared (IR). The usual Maxwell kinetic term is a relevant operator in this non-relativistic theory and it tunes the system through a quantum phase transition between isotropic and anisotropic quantum Hall states. The critical point is reached by tuning the coupling to this operator to zero and the critical point is described by $z = 2$ Lifshitz-Chern-Simons (LCS) theory. A non-Abelian extension of this theory (so far without Chern-Simons term) is currently under investigation [134].

We would like to understand if this non-Abelian Lifshitz theory could be holographically related to a gravity theory in asymptotically $z = 2$ Lifshitz spacetime [106]. The string theory embeddings of Lifshitz solutions studied in [20, 52, 67], can be helpful in this regard[1]. Some of these solutions can be described as deformations of DLCQ (discrete light-cone quantization) of $\mathcal{N} = 4$ super Yang-Mills (SYM) theory[2]. We will argue below (in section 6.3) that the solutions studied in [52] are holographically dual to non-Abelian LCS theories with matter fields (which are organized into supermultiplets). In order to make a connection with the non-Abelian LCS theory discussed in [134], these additional matter fields must be lifted. We will now discuss an approach for finding gravity duals of LCS gauge theories without additional matter.

To obtain a 2+1 dimensional field theory, let us consider the theory on N D3-branes with one longitudinal direction compactified into a circle, whose coordinate we denote x_3, $x_3 \equiv x_3 + L_3$. Following [169], impose anti-periodic boundary conditions (APBCs) for the fermions and periodic boundary condition on the bosons. This boundary condition breaks supersymmetry and makes the fermions massive; the bosons then get mass through loop corrections. The masses of the bosons and fermions are of the order of inverse radius of the circle. For energies lower than the mass of the fermions the theory is effectively 2+1 dimensional and described by pure Yang-Mills theory. Type IIB supergravity in the AdS soliton solution is holographically dual to the confining groundstate of the theory described here [169].

Let us now deform this 2+1 dimensional theory by introducing a θ term that varies linearly along x_3, $\theta = kx_3/L_3$. This deformation produces a Chern-Simons term in the effective 2+1 dimensional theory:

$$\int \theta \,\mathrm{tr}\,(F \wedge F) = \int d\theta \wedge \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) = k \int \left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right) . \tag{6.1.1}$$

In the above equation we have integrated by parts to get the first equality, and neglected dependence on x_3 to get the second. In the string theory dual description, this deformation corresponds to turning on k units of RR-axion flux around the circle. We would like to know

[1] In this chapter we deal with the $z = 2$ case only. [67] found Lifshitz solutions of massive type IIA and type IIB supergravity for a general dynamical exponent z.

[2] The construction in [20] is a DLCQ (discrete light-cone quantization) of $\mathcal{N} = 4$ SYM theory, with a coupling that depends on the compact null direction. This non-trivial behavior of the coupling breaks the non-relativistic conformal symmetries of DLCQ $\mathcal{N} = 4$ theory to Lifshitz symmetries. Such a deformation of $\mathcal{N} = 4$ theory must result in a 2+1 dimensional non-Abelian Lifshitz theory with matter fields (in the adjoint representation). However, for our purposes, the non-trivial behavior of the coupling complicates the study of this effective 2+1 dimensional theory.

how the bulk geometry gets modified when this deformation is turned on. If we assume that the axion flux is small, then its backreaction on the metric can be neglected. However, the circle shrinks to zero size at the tip of the soliton, and hence there must be a source for the axion flux at the tip of the soliton. This suggests that the axion flux is sourced by D7 branes at the tip of the soliton which in principle resolves the conical singularity induced by the axion flux. This singularity is resolved when the number of D7 branes equals the axion flux. The presence of D7 branes makes the IR behavior different from that of the "undeformed" AdS soliton background. A related discussion appears in [58] as a holographic model of fractional quantum Hall systems. Note that this construction is similar to a holographic realization of $\mathcal{N} = 1$ super Yang-Mills-Chern-Simons theory found in [126].

We can now give an alternate interpretation of the low energy effective theory described earlier. Even though the identity of the flat-space brane system whose near-horizon limit we want is not completely clear, we can see that addition of D7 branes corresponds to addition of matter multiplets in the boundary theory. We also know that the matter multiplets transform in the fundamental representation of the $SU(N)$ gauge theory living on the D3 branes. The strings stretching between the D3 and D7 branes are massive. The 2+1 dimensional effective theory we get by integrating out these massive modes is 2+1 dimensional YMCS theory (see *e.g.* [27], [4]).

In the large k ($k \gg 1$) limit, we can utilize ideas of geometric transition to replace the D7 branes at the tip of the soliton by axion flux in the background of a "deformed soliton". Such a "deformed soliton" must be regular everywhere with x_3 being non-trivial in the IR. We might expect that the S^5 should become trivial in the IR in the deformed geometry. But, this need not happen if the non-compact part of the metric (deformed soliton) is multiplied by a warp factor that has a minimum in the IR. Note that the dilaton profile could also become singular in the IR. It appears that such a "deformed soliton" (if it exists) is dual to $\mathcal{N} = 0$ YMCS theory.

Now, let us consider a situation where the shrinking circle is non-trivially fibered over some four dimensional manifold. Such solutions are dual to 2+1 dimensional non-relativistic Chern-Simons theories. If the fiber does not degenerate in the IR, the circle can carry an axion flux even if the radius of the circle shrinks to zero size in the IR. If the geometry is non-singular then there are no additional sources (such as D7-branes) for the axion field.

In this chapter, we will study non-relativistic solutions with properties of the "deformed soliton" described above. These solutions arise via a null deformation[3] of type IIB on $AdS_5 \times S^5$ (with one of the lightcone coordinates compactified) and hence the dual field theory is a deformation of DLCQ of $\mathcal{N} = 4$ SYM theory. In contrast to the story in the AdS soliton, the IR scale (holographically, this is the point at which the x_3-circle shrinks) in this solution is not determined by the compactifcation size of the shrinking circle. Rather, there is a second mass scale in the problem, in addition to the inverse radius of the compact circle. A precedent for this situation is the confining solution studied in [70], where the confinement scale is determined by boundary conditions on the dilaton. The boundary condition on the dilaton introduces an an additional mass scale, for which Gubser [70] provided a field theory interpretation. We will use this idea to argue that the deformed $\mathcal{N} = 4$ SYM theory dual to our solution is described by non-Abelian $z = 2$ LCS theory at low energies (below the KK scale $1/L_3$, and above the dynamical scale – see Fig. 6-1). The low energy effective theory in this regime inherits the $z = 2$ scaling symmetry of DLCQ $\mathcal{N} = 4$ SYM theory, but it is not Galilean invariant.

The usual problem of DLCQ is the lightcone zeromodes [89]. They generally produce an infinitely-strongly-coupled static sector of the theory which must be solved first. Remarkably, here these zeromodes conspire to comprise exactly the auxiliary fields in the first-order description of the Lifshitz gauge theory [135]. These solutions seem to be dual to a pure glue theory in a wide range of energy scales. The freedom due to broken Lorentz invariance allows us to decouple the IR scale from the Kaluza-Klein scale.

The rest of the paper is organized as follows. In section 6.2, we will present a solution with RR axion flux which has asymptotic $z = 2$ Lifshitz scaling symmetry. In section 6.3, we will argue that the solution in section 6.2 is dual (in a window of energies) to large-N non-Abelian LCS theory. In particular, we will show that LCS theories can be realized as deformations of DLCQ $\mathcal{N} = 4$ SYM theory. The solution of §6.2 is not geodesically complete [43], for geodesics with sufficiently large momentum around the circle p^3; it is nevertheless useful for studying the physics of modes with $p^3 = 0$. In section 6.4, we study the dependence of the spectrum of glueballs with $p^3 = 0$ on the Chern-Simons level as a consistency check. In the final section 6.5, we provide two resolutions of the problems raised by [43]. One

[3]A null deformation is defined to be a deformation of a supergravity solution that preserves a null Killing vector.

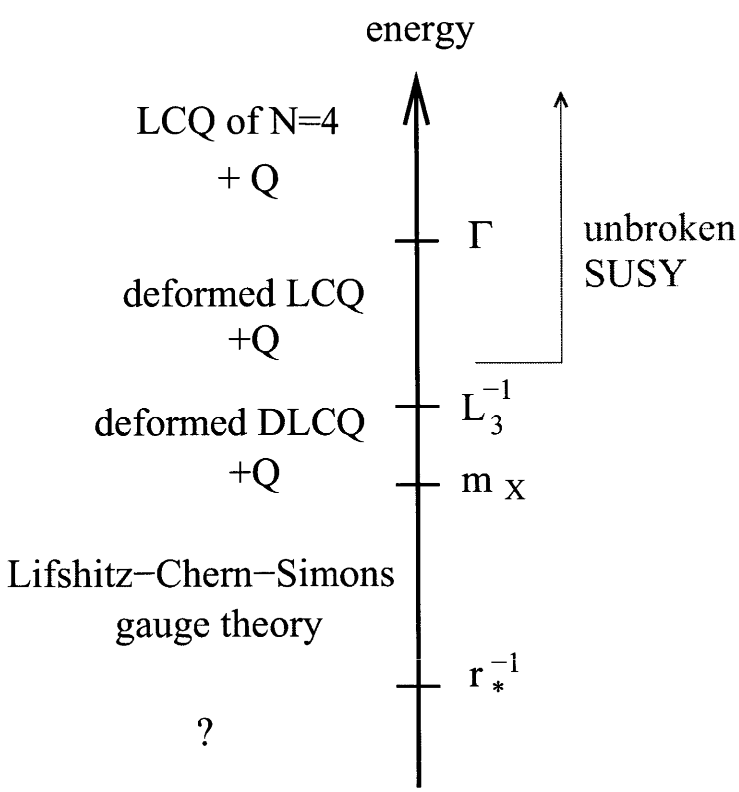

Figure 6-1: The hierarchy of energy scales considered in this chapter. The highest scale is Γ, which is the coefficient in the field theory action of certain protected-dimension operators which would be irrelevant as perturbations of the $\mathcal{N} = 4$ theory with the ordinary $z = 1$ scaling. In the bulk solution it appears via ordinary non-normalizable falloffs of the bulk metric. The next scale from the UV is the inverse-radius of the circle; at this scale the APBCs break supersymmetry. The scale m_X is the one we control the least. It is determined by a mass scale (called $M_{\rm BC}$ below) encoded in the metric boundary conditions; in the field theory, this is the coefficient of a certain operator whose dimension is not protected by supersymmetry. Below m_X, the physics is described by the Lifshitz-Chern-Simons gauge theory of interest. The scale $r_\star^{-1}$ is referred to in the paper as "the IR scale". The question mark is the subject of §6.5.

(in 6.5.1) is a realization of dilaton-driven confinement that follows upon perturbing the previously-described system by a certain (dangerously-irrelevant) operator. The other (in 6.5.2) looks like a Higgs vacuum of the theory, and is a candidate for the true groundstate. Several appendices sequester background information and technical details. In appendix 6.A, we will briefly review dilaton-driven confinement, primarily based on [70]. The remaining appendices give a family of confining solutions (6.B), an analysis of the supersymmetry which would be preserved if we used periodic boundary conditions (6.C), and a detailed analysis of the UV boundary data (6.D).

6.2 Null deformations of $AdS_5 \times S^5$

In this section we will study solutions of type IIB supergravity that can be obtained as null deformations of AdS. In order to study such solutions it is convenient to parametrize AdS by light cone coordinates. Compactifying one of the null directions is a simple example of a null deformation. This example does not alter the form of the metric or other supergravity fields but changes the boundary conditions. A compact null direction can be obtained from a compactified spatial direction by an infinite boost along the compact direction. In general, a null deformation modifies the supergravity fields in such a way that one of the null directions becomes spacelike.

In this chapter we will study a particular class of such deformations. The following is a solution of the type IIB equations of motion (see (6.B.3) for conventions):

$$ds^2_{\text{Lif}} = L^2\left(\frac{2dx_3dt + d\vec{x}^2 + dr^2}{r^2} + f(r)dx_3^2 + ds^2_{\Omega^5}\right), \quad f(r) = \frac{Q^2e^{2\Phi_0}}{4L_3^2} - \left(\frac{r^2}{r_0^4}\right) \tag{6.2.1}$$

$$F_5 = 2L^4(1+\star)\Omega_5, \quad C_0 = \frac{Qx_3}{L_3}, \quad \Phi = \Phi_0$$

In the above solution, x_3 is a compact direction ($x_3 \equiv x_3 + L_3$). Translation symmetry along x_3 is broken by the RR-axion profile $\chi(x_3)$. Similar solutions have been studied in the context of string embeddings of Lifshitz spacetime. In fact, the solution described above has asymptotic Lifshitz symmetries.[4] This geometry approaches $\text{Lif}^{d=2}_{z=2}$ geometry in the UV

[4]Note that the following scaling symmetry is an asymptotic isometry of (6.2.1)

$$t \to \lambda^2 t, \quad \vec{x} \to \vec{x}, \quad r \to \lambda r, \quad x_3 \to x_3.$$

($r \longrightarrow 0$). x_3 is a compact direction and hence cannot scale, as scaling will change the compactification radius. This solution is not invariant under time reversal symmetry and parity. The non-invariance of the solution under parity can not be seen in the geometry, but it can be inferred from the non-trivial profile for the RR-axion, which is a pseudo-scalar.

It is not possible to take $Q \longrightarrow 0$ in the above solution, while fixing r_0 and maintaining regularity. Further, when $Q = 0$ and $r_0 \longrightarrow \infty$, translation symmetry along x_3 is restored. Hence, we have asymptotic Lifshitz symmetries only when $Q \neq 0$.

The IR behavior of the geometry is more subtle. The x_3 circle shrinks in the IR when $f(r) = 0$, *i.e.*, when

$$r = r_\star = r_0^2 \left(\frac{Q e^{\Phi_0}}{2L_3} \right) . \tag{6.2.2}$$

There is no conical singularity at this locus. All curvature invariants of the metric in (6.2.1) are finite, as a consequence of g^{33} being zero. The geometry is free of curvature and conical singularities.

The metric in (6.2.1) is geodesically incomplete if the radial coordinate is restricted to lie between 0 and $r_\star$. In particular, it was shown that certain geodesics carrying non-zero momentum along x_3 do not lie entirely in the region $r < r_\star$. A straightforward way of extending these geodesics past $r_\star$ leads to closed timelike curves (CTCs) in this region. Clearly, it is important to understand the implications of this hidden singularity on the dual field theory. We discuss this issue and its resolutions in §6.5. Until then, we focus on questions pertaining to the field theory at energies above $r_\star^{-1}$, where the solution (6.2.1) is less problematic.

Geodesics carrying zero momentum along x_3 direction do not cross $r_\star$ and hence the physics of the zero modes may be insensitive to this region. The fact that modes with $p^3 = 0$ do *not* penetrate past $r = r_\star$, suggests that for the purposes of the 2+1-dimensional physics of interest to us in this chapter we may terminate the geometry at $r = r_\star$. This in turn suggests that the dual field theory (at least that of operators with $p_3 = 0$) is gapped. The energy scale associated with the gap $r_\star^{-1}$ (we will compute the energy gap for these modes in more detail below) is not determined by specifying L_3 alone. This is not unprecedented. In appendix 6.A, we review confining solutions where the confinement scale is not determined by the radius of the shrinking circle. By analyzing the boundary counterterms in appendix 6.D, we conclude that r_0 is determined by the boundary conditions on the metric (or vielbeins). The discussion in appendix 6.A indicates that the parameter r_0 corresponds to a mass deformation m_X in the dual field theory.

Before we proceed to analyze the dual field theory, let us make some more comments about the solution. When x_3 is non-compact, the solution with $r_0 \to \infty$ preserves $\mathcal{N} = 1$ supersymmetry (please see appendix 6.C). The on-shell action associated with this solution is non-zero when r_0 is finite and it is proportional to $r_\star^{-4}$. In order to regularize the action, we need to include boundary counterterms (please see appendix 6.D).

T-duality along x_3 produces a solution of massive type IIA supergravity. This equivalent description clarifies some aspects of the physics (though questions regarding the regularity of the solution are obscured). For instance, it is easier to check that the T-dualized solution has asymptotic Lifshitz symmetries. We can also see that r_0 is determined by boundary conditions on the dilaton (of massive type IIA). The T-dualized solution has a non-trivial flux associated with the NS-NS B- field. Further, this B field has a mass (determined by axion flux in the type IIB solution). This B field is related to the g^{t3} component of the IIB metric by T-duality. This suggests that the fluctuations of g^{t3} (in the presence of an axion flux) satisfies a massive wave equation. Further, we will see that this field is dual to a dimension 6 operator of $\mathcal{N} = 4$ SYM theory. The theory obtained by deforming $\mathcal{N} = 4$ SYM theory by this dimension 6 operator, is very similar to the non-commutative SYM theories studied in [60]. The S^5 factor remains unaffected by these deformations. These observations will be helpful in analyzing the dual field theory.

Let us now try to guess what the dual field theory (at low energies) could look like. At this point, we will not try to relate the parameters of the solution to the parameters of the field theory. We know that the solution has asymptotic Lifshitz symmetries. The presence of a shrinking circle suggests that the fermions must satisfy anti-periodic boundary conditions making them (and the scalars) massive. This suggests that the low energy theory is described by a Lifshitz-symmetric pure gauge theory. We can also infer (from the profile for RR-axion) that the dual field theory should contain a Chern-Simons term. So, it appears that the dual field theory is a non-Abelian version of Lifshitz Chern Simons theory at low energy. In the next section, we will present detailed arguments supporting this claim.

6.3 Identification of the dual field theory

In this section we will argue that the field theory dual to (6.2.1) is described by a non-Abelian LCS theory in a range of energies (above the IR scale, below the KK scale). Here is the strategy: first we study the UV asymptotics and conclude that the QFT is a deformation of the DLCQ of $\mathcal{N} = 4$ SYM. We organize the possible deformations to the QFT action by their scaling dimensions appropriate to the DLCQ theory, and identify the bulk fields to which they are dual following the extensive literature on AdS/CFT for $\mathcal{N} = 4$ SYM. This analysis can be done in the theory with x_3 noncompact (keeping the DLCQ scaling law in mind). We find a gauge theory coupled to fermions and scalars. Then we compactify x_3 to lift the fermions, and deform the boundary conditions on various supergravity modes to lift the scalars. As with any discussion of DLCQ, a tricky step in this analysis is the treatment of the zeromodes around the x_3 direction. By APBCs, the fermions have no such zeromodes. The scalar zeromodes are lifted by the mass deformation m_X. We are left with the zeromodes of the gauge field. We show that these organize themselves into a first-order description of the Lifshitz-Chern-Simons gauge theory.

To begin, let us note that the x_3-direction becomes null as we approach the boundary. The dual field theory lives on the conformal boundary which is $ds^2_{bdy} = 2dx_3dt + d\vec{x}^2$.[5] This is just Minkowski space in lightcone coordinates. Quantization of a field theory with x_3 compact, and t treated as the time variable is DLCQ. A scale transformation under which $\vec{x} \to \lambda\vec{x}$ requires $t \to \lambda^2 t$ to preserve the metric, and hence $z = 2$.

Since the solution (6.2.1) differs from (6.3.1) by non-normalizable field variations, the field theory dual is a deformation of the DLCQ of $\mathcal{N} = 4$ SYM theory. Operators that are irrelevant to the relativistic $\mathcal{N} = 4$ theory can be marginal or relevant in the deformed DLCQ theory. In order to study the dual field theory we must include irrelevant (with respect to $z = 1$ scaling) deformations of $\mathcal{N} = 4$ theory. We can ignore deformations that are irrelevant

[5]The notion of a conformal boundary need not be well-defined for non-relativistic backgrounds (for instance, Schrödinger spacetime [160, 16] is not conformally compact). Recently, a notion of anisotropic conformal infinity was introduced in [92, 150] for non-relativistic spacetimes that are not conformally compact in the conventional sense. We would like to point out that the metric in (6.2.1) is conformally compact unlike the Schrödinger spacetime. The conformal boundary is given by

$$ds^2_{bdy} = \lim_{r\to 0} \frac{r^2}{L^2} ds^2_{Lif} = 2dx_3dt + d\vec{x}^2.$$

with respect to $z = 2$ scaling symmetry as well; this means operators with dimension greater than 8 according the $z = 1$ counting.

In light-cone YM theory, the equation of motion involving $F^{ti} \equiv F_{3i}$ is a constraint equation (Gauss' law). Hence, in the 2+1 dimensional non-relativistic theory (with $z = 2$), $E^i \equiv F^{ti}$ appears as an auxiliary field with mass dimension $[E^i] = 1$. In the non-relativistic theory marginal operators have mass dimension 4. This suggests that we must include terms of the form tr $(F_{3i}F_{3j}F_{3i}F_{3j})$ in the 3+1 dimensional theory.

Now, let us consider the case where $r_0 \to \infty$ and x_3 is non-compact. In this case, the solution preserves $\mathcal{N} = 1$ supersymmetry. Hence, this solution is dual to a deformation of $\mathcal{N} = 4$ SYM theory that preserves $\mathcal{N} = 1$ supersymmetry[6]. The form of the metric suggests that this deformation breaks lightcone symmetry but preserves spatial rotation symmetry. The axion profile means that the deformation breaks parity. The deformation preserves the $SO(6)$ invariance of the undeformed theory. We will now identify the operators responsible for this deformation by studying the equations governing linear fluctuations of the supergravity fields.

When $r_0 \to \infty$, the line element (after reducing over the sphere) in (6.2.1) can be written as follows

$$ds_5^2 = \frac{2dx_3dt + d\vec{x}^2 + dr^2}{r^2} + \frac{Q^2e^{2\Phi_0}}{4L_3^2}dx_3^2 \,. \tag{6.3.1}$$

Note that when x_3 is non-compact, L_3 is a length scale that makes the metric non-dimensional, but it can be absorbed in a rescaling of the x_3 coordinate. The asymptotic solution (6.3.1) preserves four supercharges and is one studied in [52]. According to the previous discussion, we see that it is dual to a supersymmetric Lifshitz-Chern-Simons theory. The supergravity solution (6.3.1) implies that the extra matter which allows for supersymmetry produces a conformal fixed point. In the following, we deform this theory by relevant operators which lift the matter fields.

It is convenient to work with the vielbein formalism (see [75], [151] for a discussion on the utility of this formalism in non-relativistic holographic renormalization). In terms of

[6]The $\mathcal{N} = 4$ theory in the presence of a linear axion profile, and in particular the preservation of supersymmetry, have been studied recently in [59].

vielbeins, the five dimensional metric takes the following form[7]

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + \frac{Q^2 e^{2\Phi_0}}{4L_3^2}\left(dx^3 + \frac{4L_3^2}{Q^2 e^{2\Phi_0} r^2}dt\right)^2 = \eta_{ab}e^a e^b + e^y e^y \tag{6.3.2}$$

where

$$e^4 = dr/r,\ e^0 = 2L_3 \frac{dt}{Qe^{\Phi_0} r^2},\ e^y = \frac{Qe^{\Phi_0}}{2L_3}\left(dx_3 + \frac{4L_3^2}{Q^2 e^{2\Phi_0} r^2}dt\right),\ e^i = \frac{dx^i}{r}\ . \tag{6.3.3}$$

Let us also define $dx^\mu = \tilde{e}_a^\mu e^a$ and $dx^3 = \tilde{e}_0^3 e^0 + \tilde{e}_y^3 e^y$. [8] The operators we are interested in are dual to e_t^y and e_3^y. We will now determine the dimensions of these operators by studying the equation that governs linear fluctuations of e_t^y and e_3^y. Let us define $A' = \delta e_t^y dt$ and $\sigma = \delta e_3^y$. The equations of motion for δe_t^y and δe_3^y can be written as[9]

$$d\star_4 dA' = m^2 \star_4 A' \quad , \quad d\star_4 d\sigma = 0 \tag{6.3.4}$$

where $m^2L^2 = 16$ and $\star_4(e^{a_1}\wedge\cdots\wedge e^{a_k}) = e^{a_{k+1}}\wedge\cdots\wedge e^{a_4}$. Note that the above equations are true even when x^3 is non-compact. We can see that $A' = r^\Delta A_0'$ is a solution of the above equation if $(\Delta-2)^2 = m^2$ or $\Delta_\pm = 2\pm 4$. Hence, the dimcnsion of the operator that is dual to this mode is $\Delta_{\mathcal{O}} = 6$. The equation of motion for σ suggests that e_3^y is dual to a dimension 4 operator. Note that A' is massive due to the presence of axion flux. A similar observation was made in [49] where the fluctuations of NS-NS field becomes massive due to the presence of five form flux. They showed that these fluctuations correspond to a dimension 6 operator in $\mathcal{N}=4$ SYM theory.[10] The operator dual to the 2-form field is antisymmetric in Lorentz indices. In [57], it was shown that this dimension 6 operator lives in a short supermultiplet with tr $\left(W_\alpha^2 \bar{W}_{\dot{\alpha}}\right)$, where W_α denotes 10D $\mathcal{N}=1$ superfield strength.

[7]We will use μ,ν denote the spacetime indices $\{t, x_1, x_2, r\}$ and a, b to denote the vielbein indices $\{0,1,2,4\}$ throughout this chapter. Note that we will not use 3 to denote any vielbein index as this denotes the label of the compact direction. We will use the letter y to denote the fifth vielbein index.

[8]Note that when we reduce along x_3, e_μ^y shows up as a vector field and e_3^y as a scalar field in the lower dimensional theory. The non-trivial profiles for these fields are responsible for breaking Lorentz invariance in the lower dimensional theory. Some details about this reduction can be found in appendix 6.D.

[9]The following relations were used to derive (6.3.4):

$$\partial_y\varphi = \left(\partial_3 + \tilde{e}_y^\mu\partial_\mu\right)\varphi \quad \text{and} \quad \partial_a\varphi = \left(\tilde{e}_a^3\partial_3 + \tilde{e}_a^\mu\partial_\mu\right)\varphi.$$

Note: In the definition of $\star_4$, $a_i \neq y$.

[10]The authors identified this operator (anti-symmetric part) by expanding DBI and WZ action (for N D3 branes). Their analysis was restricted to the case where $SO(6)$ invariance is not broken.

The operator dual to e_t^y belongs to the same short multiplet and it can be written in terms of 10D $\mathcal{N}=1$ superfields as follows[11]

$$\tilde{\mathcal{O}}_6 = \int d^4\theta E^\alpha \mathrm{tr}\left(W_{\dot{\alpha}}^2 \bar{W}_\alpha\right) + h.c$$

where the boundary value of $e_t^y r^{-\Delta_-}$ has been promoted to a superfield E^α. When this operator is written in terms of the component fields, it must take the form $\mathcal{O}^t_\mu \underline{\tilde{e}}^\mu_y + \mathcal{O}^t_3 \underline{\tilde{e}}^3_y$, where $\underline{\tilde{e}}^\mu_y$ and $\underline{\tilde{e}}^3_y$ denote the coefficients of non-normalizable fall-offs of $\tilde{e}^\mu_y$ and $\tilde{e}^3_y$ respectively. Further, we know that $\tilde{e}^\mu_y = 0, g_{33}^{bdy} = g_{tt}^{bdy} = 0$. Using these facts we can see that the operator that is dual to e_t^y is of the form $\mathcal{O}_{x_3 x_3}$. Following [49, 57], we can write down this dimension 6 operator that is dual to e_t^y

$$\mathcal{O}_6 = i\mathrm{tr}\left([F_{3k}, F_{l3}]F^{kl} + F_{3k}\partial_3 X^I \partial^k X^I\right) + \text{terms involving fermions.}$$

Similarly, we can show that the operator dual to e_3^y is

$$\mathcal{O}_4 = T_{3t} = \mathrm{tr}\ \left(F_{3i}F_{ti} - \frac{1}{4}F^2\right) + \text{terms involving fermions and scalars.}$$

This operator belongs to the short multiplet tr $\left(W_\alpha \bar{W}_{\dot{\alpha}}\right)$. Note that tr $(F_{3i}F_{ti})$ appears as a kinetic term in the lower dimensional non-relativistic theory. Now, the only other $SO(6)$ invariant operator with dimension $\Delta \leq 8$ is the operator dual to the volume form of S^5. This operator has dimension 8 and it has been identified in [73, 57] to be

$$\tilde{\mathcal{O}}_8 = \mathrm{tr}\left(F_{IJ}F_{KJ}F_{IL}F_{KL} + \frac{1}{2}F_{IJ}F_{KJ}F_{KL}F_{IL} - \frac{1}{4}(F^2)^2\right)$$
$$+ \quad \text{terms involving fermions and scalars.}$$

This operator lies in tr $\left(W_\alpha^2 \bar{W}_{\dot{\alpha}}^2\right)$ and hence its dimension is protected. It was conjectured in [73] that moving away from the near-horizon geometry of D3 branes corresponds to deforming the $\mathcal{N}=4$ theory by the dimension 8 operator $\tilde{\mathcal{O}}_8$. This operator is irrelevant with respect to $z=1$ scaling and its effects disappear from the dual field theory when we take the strict near-horizon limit.

[11] We can write this in terms of 4D fields after reducing on a T^6.

Equation (6.3.1) describes the IR geometry of plane-wave-deformed D3 brane geometry [37]. Including deviations away from the IR region of a D3 brane geometry (which would ultimately glue it to the asymptotically flat $\mathbb{R}^{9,1}$) should correspond to adding the dimension 8 operator $\tilde{\mathcal{O}}_8$ [73]. When x_3 is compact and therefore does not scale, terms of the form $\mathcal{O}_8 \equiv \text{tr}\,(F_{3j}F_{3j}F_{3i}F_{3i})$ are not suppressed in the strict low-energy limit. Such terms in $\tilde{\mathcal{O}}_8$ cannot be ignored in the low energy effective theory that is dual to (6.3.1), with compact x_3.

When x_3 is non-compact, the theory that is dual to (6.3.1) is therefore described by the following action

$$S_1 = S_{\mathcal{N}=4}(\theta = Qx^3/L_3) + \int dtd^2xdx_3\,[\kappa_6\mathcal{O}_6 + \kappa_8\mathcal{O}_8 + \ldots] \tag{6.3.5}$$

where $\mathcal{O}_8 = \text{tr}\,([F_{3i}, F_{3j}]^2) +$ terms involving scalars and fermions, and θ is the theta-angle of $\mathcal{N} = 4$ theory. We will not worry about the operators denoted by ... as these are irrelevant with respect to both $z = 1$ and $z = 2$ scaling. Note that κ_6 has mass dimension -2. The non-normalizable fall-off of e_t^y suggests that the coupling is proportional to L_3^2/Q^2. Before we compactify x_3 let us define :

$$F^{ti} = F_{3i} = \sqrt{\frac{\kappa}{\kappa_6}}E_i, \quad F^{3t} = F_{t3} = E_3, \quad g'^2_1 = g_1L_3 = g^2\sqrt{\frac{\kappa_6}{\kappa}}, \quad \lambda_1 = \kappa_8\frac{\kappa^2}{\kappa_6^2}, \quad g_2 = g_3 = g$$

The action S_1 when written in terms of the new variables reads as follows

$$S_{3+1} \equiv \int dtd^2xdx_3\left[\frac{1}{2g'^2_1}\text{tr}\,(E_iD_tA_i + A_tD_iE_i) + \frac{1}{4g_2^2}\text{tr}\,\left(F_{ij}F^{ij}\right) + \frac{1}{2g_3^2}\text{tr}\,\left(E_3^2\right) + \right.$$

$$\left. \lambda_1\text{tr}\,\left([E_i, E_j]^2\right) + i\kappa\text{tr}[E_i, E_j]F^{ij} + \text{terms involving scalars and fermions}\right] \tag{6.3.6}$$

$$+\frac{Q}{L_3}\int dx_3 \wedge \text{tr}\,(A \wedge F)$$

We can see that this action resembles a gauge theory action written in first order formalism. Also, note that g'^2_1 has mass dimension -1.

At last we consider the theory with compact x_3. The field theory dual of (6.2.1) is a deformation of (6.3.6). Compactifying x_3 with anti-periodic boundary conditions on the fermions makes them massive. Kaluza-Klein reduction along x_3 of the last term in (6.3.6) induces a Chern-Simons term in the effective 2+1 dimensional theory. We can absorb the overall factor of L_3 by rescaling t.

The IR scale in the geometry r_0 is determined by non-trivial boundary behavior of e_3^y and e_t^y (see appendix 6.D). The discussion in appendix 6.A suggests that the non-trivial boundary conditions on $e_{3,t}^y$ are induced by some excited string state. The end result is a mass term m_X for the scalars, presumably by operator mixing. The heuristic calculation in appendix 6.A suggests that $M_{\rm BC} \sim \frac{m_X}{\Delta_K(\lambda)}$ which gives $M_{\rm BC} \ll m_X$ at large 't Hooft coupling. The placement of m_X in Fig. 6-1 is based on this estimate; unfortunately we do not know the precise relationship between m_X and the other scales in the problem.

We are interested in the low energy effective description for energies less than m_X, L_3^{-1}. We see that the modes with non-zero Kaluza-Klein momentum are massive (with mass $\geq L_3^{-1}$), and can be integrated out. Hence, the low energy dynamics is described by the dynamics of the modes with no dependence on x_3. Because of the APBCs, there are no modes of the fermion fields with this property. We will now show that the scalar zero modes can also be integrated out with impunity for energies less than m_X. To see this, let us study the behavior of the scalar zero mode propagator when the proper radius of x^3 in the boundary theory is $L_{prop} \sim \epsilon_3 L_3$. The propagator of the DLCQ theory is obtained by taking $\epsilon_3 \to 0$. When ϵ_3 is small but non-zero, the zero modes are dynamical and the momentum space propagator of the zero mode is given by

$$D(\omega, \vec{k}) \sim \left(\epsilon_3^2 \omega^2 - \vec{k}^2 - m_X^2\right)^{-1}$$

Now, for $\omega < m_X$, we can integrate out the zero mode without introducing divergence in the Feynman graphs containing zero mode propagators. However, when m_X is zero, the zero modes introduce divergences at $\vec{k} = 0$. In other words, the scalar zero mode gets strongly coupled with other zero modes and non-zero modes. Note that the zero modes are problematic when we try to quantize the 3+1 dimensional theory using DLCQ. This is because, to quantize the 3+1 dimensional theory, we need to quantize all non-zero modes. However, only the modes with mass less than m_X/ϵ_3 can be decoupled from the zero mode. Modes with mass greater than m_X/ϵ_3 get strongly coupled with the zero mode. However in our case, we are interested in finding the effective description for energies less than m_X. Hence, the scalar zero modes can be decoupled without introducing divergences.

So the only degrees of freedom for energies less than m_X are the zero modes of the gauge field. We can choose $A^t_\ell = A_{3\ell} = 0$ gauge for the non-zero ($\ell \neq 0$) modes, but we cannot choose this gauge for the zero modes of A^t. Usually the zero modes associated with A^t (or A_3) can be studied by choosing an alternate gauge. Here, we use the first order formalism to treat the zero modes of the gauge field. In the first order formalism, the zero modes of A_i, A_t, E_3 and E_i are the degrees of freedom. We will call the zero modes as A_i, A_t, E_3 and E_i instead of introducing new symbols. Not all of these are dynamical degrees of freedom. After integrating out all the massive modes and after dimensional reduction, (6.3.6) simplifies to

$$S \equiv \int dtd^2x \left[\frac{1}{2g_1'}\mathrm{tr}\left(E_iD_tA_i + A_tD_iE_i\right) + \frac{1}{4g'^2}\mathrm{tr}\left(F_{ij}F^{ij}\right) + \lambda_1'\mathrm{tr}\left([E_i,E_j]^2\right) + i\kappa'\mathrm{tr}[E_i,E_j]F^{ij}\right.$$

$$\left. + \frac{1}{2g_3'^2}\int d^2xdt\mathrm{tr}\left(E_3^2\right) + \frac{1}{2\alpha^2}\int d^2xdt\mathrm{tr}\left((D_iE_j)^2\right) + Q\int \mathrm{tr}\left(A\wedge F\right) + \text{irrelevant terms} \right. \quad (6.3.7)$$

Note that the couplings get corrected after the massive modes are integrated out. Further, we can see that E_3 is not dynamical and the equation of motion for E_3 is $E_3 = 0$! Hence, we can eliminate E_3 from the action. After eliminating E_3 we see that the action is same as the action for non-Abelian LCS theory. Note that this theory enjoys $z = 2$ classical scaling symmetry when x_3 is compact,

$$[t] = -2, [x_i] = -1, [E_i] = 1, [A_i] = 1, [A_t] = 2.$$

Further Galilean invariance is broken even when $r_0 \longrightarrow \infty$. This is due to the presence of dimension 6 operator. When $Q \neq 0$, it is not possible to scale A to make $g_1 = g_2$. However, this is possible when $Q = 0$. Note that α is a function of m_X ,L_3, κ and κ_6 *i.e.*

$$\frac{1}{\alpha^2} \sim \left(\frac{\kappa}{\kappa_6\mathcal{M}^2(m_X, L_3)}\right)$$

where $\mathcal{M}^2$ is a mass scale that appears in the action after integrating out the massive modes. Hence, $\mathcal{M}$ is a function of m_X and L_3.

6.4 The dependence of the gap on the CS level

We have argued that the field theory dual of (6.2.1) is a non-Abelian Lifshitz-Chern-Simons theory. Our gravity description ceases to exist when the Chern-Simons level is turned off. Reference [134] shows (perturbatively) that the weakly coupled theory without CS term flows to a free theory in the IR. Though this need not be true if we start the flow at strong coupling, this suggests that a classical supergravity description of the groundstate should not exist when $Q = 0$.

When $Q \neq 0$, our gravity solution (terminated at $r_\star$) has a minimum value of the warp factor, indicating that the mass gap has a non-trivial dependence on the Chern-Simons level. We will now show this more explicitly by computing masses of scalar glueballs. Note that parity P is not a good quantum number as it is explicitly broken by the CS term. The fluctuations of dilaton, axion and g_{33} mix as they are dual to gauge invariant operators with the same quantum numbers. The mixing between δg_α^α and the dilaton (or the other two modes) is suppressed in the large N-limit (see [45]). These modes cannot mix with any other fluctuation as other modes have different quantum numbers (J and C).

We can compute the 0^+ glueball spectrum by solving for the linearized fluctuations of the dilaton, axion and g_{33} subject to regular boundary conditions at $r = r_\star$ and the UV normalizability condition. At first glance, this might seem unreasonable as the metric is geodesically incomplete if r is restricted to lie within $r_\star$. However, the geodesics carrying zero momentum along x_3 do not penetrate the region past $r_\star$. [12] The solution we find in the section 6.5.2 will reveal that the calculation of this section is a good approximation in the regime $r_\star^{-1} \ll L_3^{-1}$.

The frequency of the 2+1 dimensional theory is obtained by scaling the frequency of 3+1 dimensional theory by L_3.[13] We will only consider the modes with zero spatial momentum, since we are only interested in the masses of the glueballs. Note that the metric has no explicit depence on x_3 and only derivatives of the axion field can appear in the equations of motion of the scalar field (dilaton, axion and g_{33}) fluctuations. Hence, the equations of motion for these fluctuations cannot have explicit dependence on x_3. This implies that the

[12]This is clear from the computation in [43].

[13]Recall that we had scaled by a factor of L_3 in the dual field theory to absorb an overall factor of L_3 after dimensional reduction.

fluctuations with zero momentum along x_3 get decoupled from the non-zero modes.[14] Let us now choose the following ansatz for the fluctuations of dilaton, axion and g_{33}

$$\delta\Phi = \varphi(r)e^{i\omega t/L_3}, \quad \delta\chi = \chi_1(r)e^{i\omega t/L_3}, \quad \delta g_{33} = \frac{h_{33}(r)}{r^2}e^{i\omega t/L_3} \,. \tag{6.4.1}$$

Here ω is the frequency in the $2+1$ dimensional theory. We define

$$\Gamma = \frac{Qe^{2\Phi_0}}{\sqrt{2}L_3}. \tag{6.4.2}$$

The contribution of the factors in the metric to the mass gap can only be functions of Γ and $r_\star$. This will allow us to study contribution of the axion to the dependence of the mass gap by just studying the dependence of the mass gap on Q. The equations of motion for the linearized fluctuations (φ, χ_1, and h_{33}) are

$$e^{2\Phi_0}Q\chi_1 = \omega h_{33} \tag{6.4.3}$$

$$e^{-2\Phi_0}r\omega^2\left(-8r_\star^2 h_{33} + r^2\left(r^2 - r_\star^2\right)\Gamma^2\varphi\right) + \frac{4Q^2}{\Gamma^2}r^2(3\varphi_1' - r\varphi'') = 0 \tag{6.4.4}$$

$$e^{-2\Phi_0}r^3\left(r_\star^2 - r^2\right)\omega^2 h_{33}(r) + 4\frac{r_\star^2 Q^2}{\Gamma^4}\left(2r\alpha^2\varphi(r) - 3h_{33}'(r) + rh_{33}''(r)\right) = 0 \,. \tag{6.4.5}$$

Note that the first equation was used to eliminate χ_1 from the other two equations of motion. The masses of the glueballs are eigenvalues of the above equations subject to regularity condition at $r = r_\star$. Further, the glueballs correspond to the normalizable modes of these fluctuations and hence the modes must satisfy normalizability condition. We can see that for fixed Γ and $r_\star$, the mass gap will have non-trivial dependence on Q.[15]

The above equations can be solved numerically by shooting. Here, we integrate from the boundary to the infrared by specifying normalizable boundary conditions for the fluctuations and using the shooting method to satisfy the regularity condition: $||\zeta'(r_\star)|| = 0$ where $\zeta = [h_{33}, \varphi, \chi_1]$. In order to specify the boundary condition we assume a power series expansion around $r = 0$ for the fluctuations and determine the coefficients (up to four terms) for which the modes are normalizable and the equations of motion is satisfied approximately

[14] The equation is in the variable separable form and hence the modes with non-zero momentum along x_3 can be separated from the zero mode.

[15] Fixing $r_\star$ is analogous to specifying the confinement scale in 3+1 dimensional YM theory. This scale is generated from a dimensionless coupling by dimensional transmutation.

near the boundary. We see that, χ_1 must fall of as r^6 near the boundary for all modes to be normalizable. With these boundary conditions, we integrate the system of equations numerically and determine the values of ω for which the regularity condition is satisfied. Figure 6-2 shows a plot of $||\zeta'(r_\star)||$ as a function of $\Omega = \omega e^{-\Phi_0}/Q$ for $\Gamma = 10$ and $r_\star = 1$. The points at which the graph touches the $\omega-$axis are points at which the regularity condition is satisfied. We can see from the figure that for $\omega \approx e^{\Phi_0}Q\{4.5,\ 6.5,\ 8.5,\ 10.5, ..\}$ we get normalizable solutions that satisfy regularity boundary condition. These are the values of the glueball masses measured in units where $r_\star = 1$. Figure 6-3 shows the radial profile of the solution corresponding to the lowest eigenvalue ($\Omega \approx 4.5$).

We emphasize that the identification of parameters between bulk and boundary described above is subject to renormalization. Further, the overall normalization of the couplings is difficult to obtain without further microscopic information. The dependence of the mass gap in the gauge theory found above is obtained by fixing Γ and $r_\star$. The logic is that Γ determines the coefficient κ (as explained in section 6.3), while $r_\star$ is analogous to Λ_{QCD} in QCD, *i.e.* a scale which determines the gauge coupling by dimensional transmutation. The CS coefficient then maps directly to the axion slope.

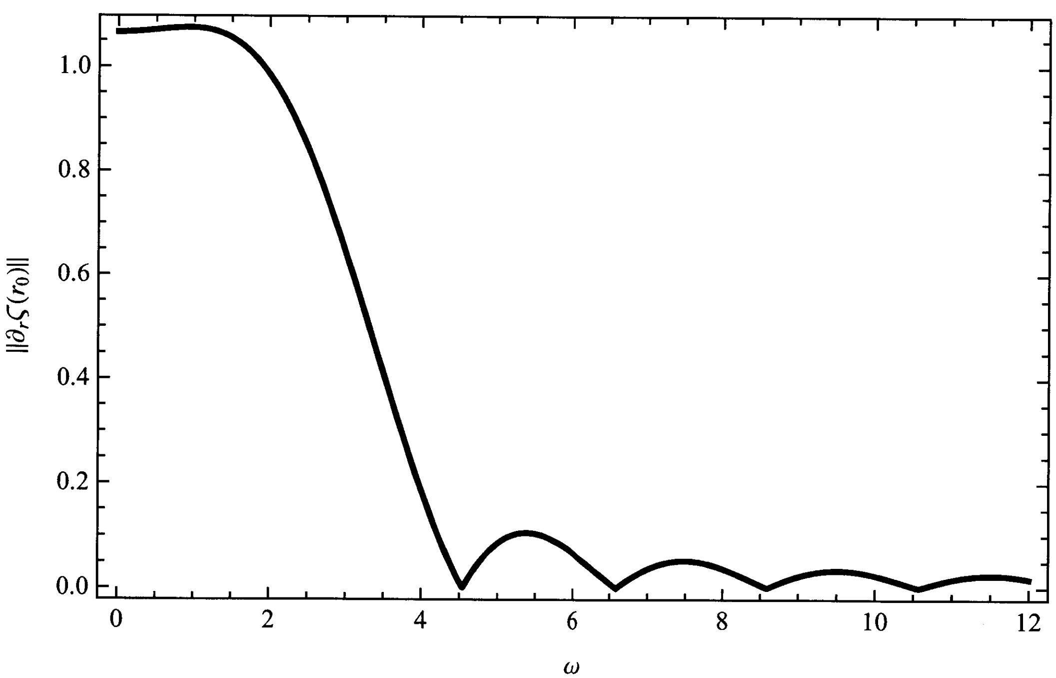

Figure 6-2: A graphical approach to find the glueball masses. We have chosen $r_\star = 1$ and $\Gamma = 10$.

We close this section with some comments.

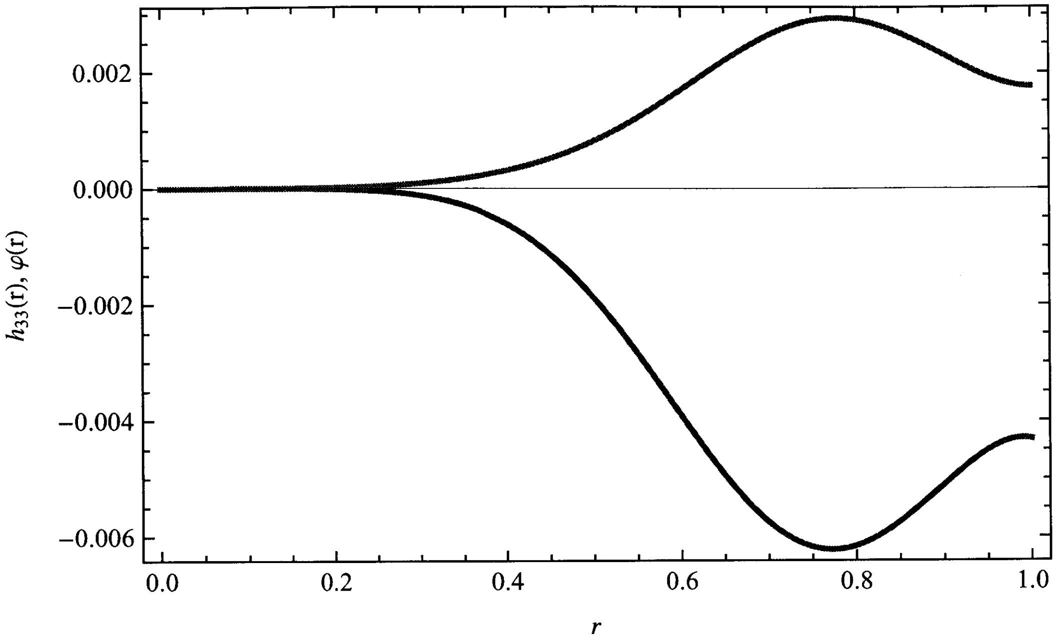

Figure 6-3: Plot of the normalizable mode of g^{33}, χ for the lowest eigenvalue. Note that the fluctuations of the axion is proportional to h_{33} and hence it is not plotted here.

1. Our results suggest that turning on a Chern-Simons term in Lifshitz gauge theory changes the sign of the beta functions computed in [134], and leads to a gapped state. This is a counterintuitive claim. Adding a CS term to an ordinary ($z = 1$) gauge theory in 2+1 dimensions, Abelian or non-Abelian, weakens the long-range gauge dynamics. This is simplest to see in the (gaussian) Abelian Maxwell-Chern-Simons theory (with noncompact gauge group) where the CS term produces a mass for the gauge boson [50]. In the non-Abelian theories or in the compact U(1) gauge theory, the story is more subtle, but the conclusion is the same [3, 148, 68]. A recent paper which relies crucially on this effect is [129].

   Recent studies of Abelian Lifshitz-Chern-Simons [135] make it clear that intuitions from $z = 1$ gauge theories do not always apply to Lifshitz gauge theories. It appears that, the long-range gauge dynamics in the model (6.3.7) is a complex interplay between the parameters that we call as g_1', λ' and the Chern-Simons level. The λ' term is irrelevant in the $z = 1$ case. Another crucial difference is that a $z = 1$ theory contains a E_i^2 term while it is absent in LCS theory. The perturbative dynamics of this model should be analyzed.

2. It would be interesting to calculate on both sides of the duality proposed in this chapter

observables which are sensitive to the Chern-Simons level Q. An interesting class of examples is given by Wilson-'t Hooft loops. In perturbation theory around the gaussian model, the CS coupling has an immediate effect on Wilson loops via its influence on the gluon propagator. At strong coupling, the effect of the axion profile is more subtle. In the bulk, the area of a fundamental string worldsheet ending on the quark trajectory computes the expectation value of a Wilson loop [125, 149]. But a fundamental string does not couple to the axion profile, and its action only sees the axion slope through the (weak) metric dependence on Q. In contrast, a D-string does couple to the axion, via the worldvolume Chern-Simons term

$$S_{\text{D-string}} \ni \int_{\text{D-string}} \chi F \, , \tag{6.4.6}$$

where F is the worldvolume gauge field on the D-string. A D-string configuration with p units of worldvolume flux carries F-string charge p, and therefore computes a mixed Wilson-'t Hooft loop describing the holonomy for a $(p, 1)$ dyon. This apparent tension is another counterintuitive manifestation of the effects of the CS level in the Lifshitz gauge theory.

3. We should comment on what happens to our theory when the hierarchy in Fig. 6-1 is re-ordered. If the radius L_3 is smaller than the deformation scale Γ, our argumentation in section 6.3 breaks down, because it relied on supersymmetry to make the identifications of the deformations. The gravity solution remains regular, however. Given the definition (6.4.2) of Γ, we note that $\Gamma \gg L_3^{-1}$ (at weak string coupling) requires $Q \gg 1$.

   If the radius L_3 is taken larger than the IR scale $r_\star$, then the model describes a 3+1 dimensional field theory with explicitly broken translation invariance (by the axion profile); the fact that the x_3 circle becomes timelike for $r > r_\star$ now represents a more serious problem, and the reader is referred to §6.5.

4. Since it does not rely on the structure of the S^5, the null deformations of $AdS_5 \times S^5$ described above have a generalization to many known AdS vacua of supergravity.

5. From the gravity solution, we see that the IR scale $r_\star$ and the KK scale L_3 may be made arbitrarily different while maintaining control over the solution. Why does this solution

allow for such a parametric separation? A simple QFT answer to this question would be progress toward a solution of confinement, but we offer the following observations. The reduced symmetry of the problem – P and T violation as well as Lorentz violation – allows for new ingredients, which are apparently helpful for this purpose. On the one hand, the P- and T-violating axion gradient along the circle provides an energetic incentive for the radius of the circle not to shrink. On the other hand, the Lorentz-violating couplings of the gauge fields are dual to the exotic boundary conditions on the vielbein; these boundary conditions play an important role in determining the IR scale.

6. Recall that tr (E_i^2) is a relevant operator in LCS theory. At least in the Abelian case, when the LCS theory is deformed by this operator (with a positive coefficient), it flows to a theory with $z = 1$ scaling symmetry. When the coupling to this relevant operator is negative, rotational symmetry is spontaneously broken [135]. Can such a nematic phase be seen in the gravity dual? At present, we do not have a concrete answe, but we give some preliminary ideas for understanding the relevant deformation in the gravity dual. We proceed by noting that the action of the transformation $t \to t + sx_3$ on DLCQ of (6.3.6) describes a deformation of (6.3.6) by tr (E_i^2) term (with a coefficient proportional to s). The theory obtained by compactifying x_3 (after the transformation) is no longer invariant under $z = 2$ scaling symmetry. The theory obtained by reducing along x_3 also contains the terms tr (E_i^4), tr $(E_i^2)^2$ and other terms that are irrelevant with respect to both $z = 1$ and $z = 2$ scaling. On the gravity side, the transformation $t \to t + sx_3$ generates a new solution with the metric given by

$$ds^2_{new} = L^2 \left(\frac{2dx_3 dt + d\vec{x}^2 + dr^2}{r^2} + \left(\frac{s}{r^2} + \frac{Q^2 e^{2\Phi_0}}{4L_3^2} - \frac{r^2}{r_0^4} \right) dx_3^2 \right)$$

A nematic phase would be encouraged when the coefficient of tr (E_i^2) is negative. This happens when $s < 0$ in which case x_3 is a compact time-like direction. It seems that this particular holographic realization of Lifshitz Chern-Simons theory does not admit a description of the nematic phase within the gravity regime.

6.5 Resolution of the "hidden singularity"

If the geometry (6.2.1) is extended past $r = r_\star$, the x_3 circle becomes timelike [43]. The presence of CTCs in the bulk need not be related to violation of unitarity in the UV description of the dual field theory.[16] Note that the asymptotic geometry of (6.2.1) is free of CTCs and the matter supporting this metric does not violate null energy condition (even in the region with CTCs). Hence, the CTC region is not linked to violation of unitarity in the UV description. Rather, the existence of the CTC region is an indication of IR instability in the state of the dual field theory. The local mass^2 of KK modes become tachyonic: $m^2(r)_{p^3} = g^{33}p_3^2 < 0$. Further, wound strings may become tachyonic when the radius is of order of the string length $\sqrt{\alpha'}$; their condensation would excise the CTC region [44].

We will now present two different resolutions of this singularity and discuss their implications for the IR behavior of the dual field theory.

6.5.1 Running Dilaton

In this section we show that it is possible to resolve the singularity by imposing non-trivial boundary conditions on the dilaton. As discussed earlier, such non-trivial boundary conditions on the dilaton corresponds to deformation of the dual field theory by a dangerous irrelevant operator (see appendix A). We will see that for one sign for the dangerous irrelevant coupling, the IR singularity is resolved. Note that this is the groundstate of a *different* theory from that determined by the asymptotics of (6.2.1) – the dilaton boundary conditions indicate a perturbation of the dual QFT by a (dangerously-irrelevant) operator.

The solution with a running dilaton is given by,

$$ds^2 = L^2\left(\frac{2H_d(r)dx_3dt + d\vec{x}^2}{r^2} + \frac{dr^2}{r^2H_d(r)^2} + f_d(r)dx_3^2 + ds^2_{\Omega^5}\right),\ H_d(r) = \sqrt{1-\frac{r^4}{r_1^4}}$$

$$F_5 = 2L^4(1+\star)\Omega_5,\ C_0 = \frac{Qx_3}{L_3},\ \Phi = \Phi_0+\log\left(H_d(r)\right),\ f_d(r) = \frac{H_d(r)}{r^2}\frac{Q^2e^{2\Phi_0}}{4L_3^2}\mathcal{F}\left(\sin^{-1}\left(\frac{r^2}{r_1^2}\right)\right) \tag{6.5.1}$$

[16] In some examples of rotating black holes in 2+1 D[146], the presence of CTC region was attributed to the violation of unitarity bound in the dual field theory. In these examples, either null-energy condition is violated or the asymptotic geometry contains region with CTCs.

where,

$$\mathcal{F}(x) = r_1^2 x + r_1^2 \int_0^x \xi \tan\xi d\xi - \frac{r_1^4}{r_0^4}\log\left(\cos^2 x\right)$$

The integral in $\mathcal{F}(x)$ can be evaluated in terms of Polylogarithms. It can be checked that this solution approaches (6.2.1) when $r_1 \to \infty$ and has the same asymptotic behavior as (6.2.1), except for the dilaton profile. In this solution, the geometry ends at $r = r_1$ and $f_d(r)$ remains positive throughout the geometry.

There is a curvature singularity at $r = r_1$ which can be resolved by uplifting the solution to 11D supergravity.The details of the uplifting procedure is discussed in appendix B In the following, we will show that this geometry is geodesically complete. It is sufficient to focus on the case of null geodesics. The null geodesic equation is given by $\dot{r}^2 + V_{eff}(r) = 0$ where

$$V_{eff}(r) = r^2 H_d^2 \left(-E^2 \frac{Q^2 e^{2\Phi_0}}{4L_3^2} \mathcal{F}\left(\sin^{-1}\left(\frac{r^2}{r_1^2}\right)\right) \frac{r^4}{H_d(r)} - r^2 \frac{2p_3 E}{H_d(r)} + r^2\left(p_1^2 + p_2^2\right)\right)$$

In the above expression E, p_3 and p_i are the conserved quantities associated with ∂_t, ∂_3 and ∂_i. Note that $V_{eff}(r_1)$ is zero and hence the maximum possible value of r for radially ingoing geodesics is r_1. Hence, no geodesic can penetrate into the region past r_1. Hence, the 11D solution obtained by uplifting the solution in (6.5.1) to 11D supergravity is regular and provides a resolution of the "hidden singularity". There is also another solution with running dilaton obtained reversing the sign of the dilaton gradient near the boundary *i.e.*, $\Phi = \Phi_0 - \log H_d(r)$. In this case, g_{33} vanishes before $H_d(r)$ vanishes and the geometry is geodesically incomplete (or contains regions with CTCs). Hence, we did not present this solution here.

The dual field theory interpretation of the resolution discussed in this subsection is the following. Turning on a dangerous irrelevant deformation causes the gauge coupling to run. The gauge coupling can become strong or weak in the IR depending on the sign of the dangerous irrelevant deformation (dilaton gradient at the boundary). When the coupling becomes weak in the IR, the dynamics is controlled by the CS term which leads to an IR instability. When the gauge coupling becomes strong in the IR, the theory confines and overrides the effect of the CS term. The confinement scale is not related to the axion flux or Chern level in this solution. In the next section we will provide an alternate resolution of the singularity.

6.5.2 Excision of the CTC region

In this subsection, we provide an alternate resolution of the singularity which is similar to the enhançon mechanism [105]. Unlike the enhançon, there is no enhancement of gauge symmetry. Closed time-like curves can be prevented by placing localized sources at $r = r_\star$. This will preserve the asymptotic form of the metric but modifies the region beyond $r_\star$. We will show that the domain wall is described by smeared D3 branes located at $r = r_\star$. In a supersymmetric theory, the location of the brane specfies the vacuum expectation value (VEV) for some scalar field in the dual gauge theory. The moduli space of the theory describes all possible locations of the branes. In the system described here, the location of the branes is uniquely specified by the asymptotic boundary condition. This is natural in a theory with broken supersymmetry – the moduli space is lifted, leaving a unique groundstate.

Unlike in the previous subsection, the solutions described here are dual to states of the same QFT as (6.2.1), and hence represent a possible endpoint of the localized instabilities associated with the CTC region.

Before we describe this resolution, let us remind ourselves about the gravity dual of a spherically symmetric shell of D3 branes (smeared over S^5) that describes a special point in the Coulomb branch of $\mathcal{N} = 4$ SYM theory. The relevant solution of IIB supergravity is

$$ds^2 = \frac{L^2}{r^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dr^2 + r^2 ds^2_{\Omega_5}\right), \quad F_5 = 2L^4(1+\star)\Omega, \qquad \text{for } r < r_\star$$

$$ds^2 = \frac{L^2}{r_\star^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + \frac{L^2 r_\star^2}{r^4}\left(dr^2 + r^2 ds^2_{\Omega_5}\right), \qquad F_5 = 0, \qquad \text{for } r > r_\star \tag{6.5.2}$$

where Ω is the volume form on the 5-sphere. The interior geometry ($r > r_\star$) is just flat space[17] and the exterior geometry ($r < r_\star$) is $AdS_5 \times S^5$. The D3 branes are localized around $r = r_\star$ and acts as the source for Israel stress tensor ($S_{\mu\nu}$). Note that the metric is continuous at $r = r_\star$ and we will now show that it also satisfies the Israel junction condition.

[17]The solution looks more familiar in the coordinate system where boundary is at infinity ($r = L^2\rho^{-1}$, $r_\star = L^2\rho_\star^{-1}$). In this coordinate system, the solution is given by

$$ds^2 = \frac{\rho^2}{L^2}\left(\eta_{\mu\nu}dx^\mu dx^\nu\right) + \frac{L^2}{\rho^2}\left(d\rho^2 + \rho^2 ds^2_{\Omega_5}\right), \quad \text{for } \rho > \rho_\star$$

$$ds^2 = \frac{\rho_\star^2}{L^2}\eta_{\mu\nu}dx^\mu dx^\nu + \frac{L^2}{\rho_\star^2}\left(d\rho^2 + \rho^2 ds^2_{\Omega_5}\right), \quad \text{for } \rho < \rho_\star$$

In this coordinate system the boundary is at $\rho = \infty$.

The junction stress tensor is given by

$$K^{+}_{\mu\nu} + K^{-}_{\mu\nu} - G^{jun}_{\mu\nu}\left(K^{+} + K^{-}\right) = -\frac{2}{L}G^{jun}_{\mu\nu} = S_{\mu\nu}, \quad \text{for } \mu,\nu \in 0,1,2,3 \tag{6.5.3}$$

$$K^{+}_{ij} + K^{-}_{ij} - G^{jun}_{ij}\left(K^{+} + K^{-}\right) = -\frac{2}{L}G^{jun}_{ij} = S_{ij}, \quad \text{for } i,j \in S^5 \tag{6.5.4}$$

where $K^{\pm}_{AB} = \mp\frac{1}{2}n^r\partial_r G^{jun}_{AB}$ and G^{jun}_{AB} is the induced metric at the junction. The integrated Einstein equation tells us that the the Israel stress tensor is sourced by the D3 branes *i.e.*

$$S_{AB} = \frac{N}{\mathrm{Vol}(S^5)}\frac{\delta S_{D3}}{\delta G^{jun}_{AB}} \tag{6.5.5}$$

The right hand side of the above equation is the stress tensor of N D3 branes smeared over S^5 located at $r_\star$ and S_{D3} is the world volume action of $D3$ branes. The worldvolume action for Dp branes (with world volume gauge fields set to zero) is

$$S_{Dp} = -T_{Dp}\int d^{p+1}\xi e^{\frac{p-3}{4}\Phi}\sqrt{G^{\mathrm{PB}}_{Einstein}} + S_{WZW}$$

where $G^{PB}_{Einstein}$ is the Dp brane metric in Einstein frame. Note that when $p = 3$ there is no source term for the dilaton and the dilaton remains constant. After taking the derivative of the worldvolume action with respect to the metric, we can see that the Israel junction conditions in (6.5.5) are satisfied.

When one of the space directions of AdS is compactified with APBC on the fermions around this compact direction Horowitz and Silverstein [96] argued that the above solution is unstable and decays to an AdS soliton. Since the interior geometry is flat, perturbative techniques can be employed to show the existence of closed string tachyons from strings winding around the compact direction. Tachyon condensation excises the IR region leaving behind a cigar shaped geometry reflecting the confining nature of 3D Yang-Mills theory.

We may also expect such tachyons to develop in the region surrounding a region of CTCs. Let us make use of the intuition we get from Maxwell-Chern-Simons theory to guess the correct IR behavior of the solution in (6.2.1). Turning on a CS interaction weakens the gauge dynamics in the IR and can prevent confinement. Confinement in the dual gauge theory is prevented if the D3 brane shell system is stable in the presence of a linear axion profile. In the following, we will show that region with CTCs in (6.2.1) can be removed by placing a shell of D3 branes at $r = r_\star$.

The solution describing the shell of D3 branes in the presence of axion flux is

$$ds^2_{\text{ext}} = L^2\left(\frac{2dx_3dt + d\vec{x}^2 + dr^2}{r^2} + f_{ext}(r)dx_3^2 + ds^2_{\Omega^5}\right), \quad f_{ext}(r) = \frac{Q^2e^{2\Phi_0}}{4L_3^2}\left(1 - \left(\frac{r^2}{r_\star^2}\right)\right)$$

$$F_5 = 2L^4(1+\star)\Omega_5, \quad C_0 = \frac{Qx_3}{L_3}, \quad \Phi = \Phi_0, \quad \text{for } r < r_\star \tag{6.5.6}$$

$$ds^2_{\text{int}} = \frac{L^2}{r_\star^2}\left(2dx_3dt + d\vec{x}^2 + f_{int}(r)dx_3^2\right) + \frac{L^2r_\star^2}{r^4}\left(dr^2 + r^2ds^2_{\Omega_5}\right) \quad f_{int}(r) = \frac{Q^2e^{2\Phi_0}r_\star^2}{12L_3^2}\left(\frac{r^4}{r_\star^4} - \frac{r_\star^2}{r^2}\right)$$

$$F_5 = 0, \quad C_0 = \frac{Qx_3}{L_3}, \quad \Phi = \Phi_0, \quad \text{for } r < r_\star$$

There is no jump in axion flux and hence D7 brane sources are absent in this solution. The jump in 5-form flux is sourced by the shell of D3 branes. Note that the metric is continuous at $r = r_\star$. We will now show that this solution also satisfies Israel jump conditions, if $r_\star$ has the same relation to the UV variables as previously (6.2.2). The junction stress tensor is

$$K^+_{\mu\nu} + K^-_{\mu\nu} - G^{jun}_{\mu\nu}\left(K^+ + K^-\right) = -\frac{2}{L}G^{jun}_{\mu\nu} = S_{\mu\nu}, \quad \text{for } \mu,\nu \in 1,2$$

$$K^+_{ij} + K^-_{ij} - G^{jun}_{ij}\left(K^+ + K^-\right) = -\frac{2}{L}G^{jun}_{ij} = S_{ij}, \quad \text{for } i,j \in S^5$$

$$K^+_{33} + K^-_{33} - G^{jun}_{33}\left(K^+ + K^-\right) = 0 = S_{33}$$

$$K^+_{t3} + K^-_{t3} - G^{jun}_{t3}\left(K^+ + K^-\right) = -\frac{2}{L}G^{jun}_{t3} = S_{t3}$$

Since $G^{jun}_{33} = 0$ at $r = r_\star$, we can write S_{33} as $-(2/L)G^{jun}_{33}$. Hence, the form of the Israel stress tensor is same as (6.5.3) and (6.5.4). We already saw that a shell of D3 branes can provide this stress tensor. Hence, this solution provides a consistent way of removing the region with closed time like curves.

In the solution (6.5.6), the IR geometry is a planewave. Tidal forces become large as $r \to \infty$. This sort of mild singularity is familiar from the Lifshitz solution and we regard it as physically acceptable.

Evaluating the regulated on-shell action of the solution (6.5.6), we find that it compares favorably to that of (6.2.1). We conclude that the solution (6.5.6) is a truer groundstate. We cannot exclude the possibility of more favorable solution, such as a smooth solution which terminates at a finite value of r.

Just as the D3-brane shell solution (6.5.2) exhibits a mass gap in the spectrum of single-trace operators, so will (6.5.6). The reason in both cases is that the 5-sphere shrinks at $r = \infty$ – the geometry for $r > r_\star$ is roughly a (compact) ball. The analysis of §6.4 becomes a good approximation when this ball is small: $r_\star \gg L_3$. We must leave an analysis of the spectrum in the general case for the future.

6.A Dilaton-driven confinement

Here we review a holographic model for confinement studied in [70].[18] As in [169], there is a circle with APBCs. However, the confinement scale is not determined by the UV radius of the shrinking circle but is determined from the boundary conditions on the supergravity fields. The contents of this section is somewhat disconnected from the rest of the paper. The results will be helpful in section 4.

Gubser [70] found (numerically)[19] an asymptotically $AdS_5 \times S^5$ solution solution of type IIB supergravity with unusual boundary conditions for the dilaton field. The resulting non-trivial profile for the dilaton leads to confinement. The following is the solution that was studied in [70]

$$ds^2 = L^2 \left(1 - \frac{r^8}{r_0^8}\right)^{1/2} \left(\frac{-d\tau^2 + dy^2 + d\vec{x}^2 + dr^2}{r^2}\right) + L^2 ds^2_{S^5}$$

$$F_5 = L^4 \left(1 + \star\Omega_5\right), \Phi = \frac{\sqrt{6}}{2} \log\left(\frac{r_0^4 - r^4}{r^4 + r_0^4}\right) \tag{6.A.1}$$

where F_5 is the RR five-form flux of Type IIB supergravity and Φ is the dilaton. Here, we will assume that y is a compact direction with period L_y. The equations of motion and other details about the solution can be found in appendix 6.B. This appendix also contains a family of solutions of which the above solution is a special member distinguished by the fact that it preserves Lorentz invariance in the UV. The dilaton becomes singular at $r = r_0$. There is also a curvature singularity. However, the metric is conformal to a regular metric.

[18]This section contains some new results, some remarks benefitting from a decade of hindsight, and some minor differences in the style of presentation.

[19]The analytical solution has appeared previously in [107].

In fact, it is possible to resolve the singularity by "uplifting" the solution to a regular solution of 11D supergravity (see appendix 6.B).

The fact that the geometry ends smoothly in the IR signals a mass gap in the dual field theory[20]. In particular, this indicates that the matter fields have been made massive. Further, the presence of S^5 factor in the solution implies that this mass is $SO(6)$ invariant. In the AdS soliton case masses for all the fields are generated by the boundary conditions on fermions. In the present case, there must be two different mass scales.

Note that we can give the scalars an $SO(6)$-invariant mass by adding

$$L_m = m_X^2 \mathrm{tr} \sum_{I=1}^{6} X_I^2 \tag{6.A.2}$$

to the $\mathcal{N} = 4$ SYM Lagrangian. The fermions get mass through anti-periodic boundary condition. It was suggested that the mass term of the scalar (m_X) is responsible for the non-trivial behavior of the dilaton in the bulk. In particular, it was argued that the non-trivial boundary condition on the dilaton is induced by a "string field" that is dual to $\mathcal{O}_K = m_X^2 \mathrm{tr}\left(X^I X^I\right)$. Although $\mathcal{O}_K$ is a relevant operator at weak coupling, it acquires a large anomalous dimension at strong 't Hooft coupling and hence it is not visible in supergravity. Rather, it is dual to an excited mode of the IIB string in $AdS_5 \times S^5$ of mass of order $\sqrt{\lambda}$. Because of its large mass, this "string field" has a profile that decays extremely rapidly near the UV boundary of AdS. Hence, the effect of the "string field" on supergravity fields is felt just near the boundary. This effect appears as a non-trivial boundary condition on the dilaton. Such a boundary condition on the dilaton can in turn be described as a large-dimension multi-trace deformation of the QFT action[170, 28]. From the fact that such an irrelevant operator has an important effect on the IR physics, we are forced to call it 'dangerously irrelevant'.

It is difficult to justify the previous statements rigorously. In [70], the following heuristic calculation was presented to justify this picture and to estimate the mass gap in terms of m_X. Our purposes in redoing this calculation here are twofold:

1. to make explicit the dependence of the IR scale r_0^{-1} on non-normalizable deformations near the UV boundary,

[20]This solution is only relevant if fermions satisfy anti-periodic boundary conditions around y.

2. to interpret the holographic renormalization for the dilaton field Φ in this context in terms of a boundary potential.

Assume that adding the mass term (6.A.2) for scalars to the $\mathcal{N}=4$ Lagrangian (with UV cut-off $\mathcal{M}_1 > m_X$) corresponds to turning on a source for an excited string field ϕ_K in the bulk (with cut-off $r=\varepsilon_1$). In this calculation ϕ_K is treated as a linear perturbation in the bulk with mass m_K. Let us assume that ϕ_K interacts with the dilaton through an interaction term (in the bulk Lagrangian) of the form $W(\Phi)\phi_K^2$. (The choice of this coupling is made for convenience and should not be taken too literally.) The equations of motion can be written as

$$\nabla^2\phi_k + m_K^2\phi_K = 0 \implies \phi_K \overset{r\to 0}{\simeq} \mu^2\left(\frac{r}{\varepsilon_1}\right)^{4-\Delta_K} \tag{6.A.3}$$

$$\nabla^2\Phi = W'(\Phi)\phi_K^2\ . \tag{6.A.4}$$

Note that $W(\Phi)$ does not affect the ϕ_K equation of motion since m_K is very large. We must further specify boundary conditions for the dilaton at the UV cutoff ϵ_1: $\Phi(\varepsilon_1) = \Phi_0$ and $\partial_r\Phi|_{r=\varepsilon_1} = 0$ and $\phi_K(\varepsilon_1) = \mu^2$.

Now, let us integrate out all modes with mass greater than $\mathcal{M}_2$ (with $\mathcal{M}_2 < m_X, L_y^{-1}$) in the dual field theory. This corresponds to integrating along the radial direction from ε_1 to $\varepsilon_2(>\varepsilon_1)$ in the bulk. Integrating the dilaton equation of motion once we get

$$\frac{1}{\varepsilon_2^3}\partial_r\Phi|_{r=\varepsilon_2} = \int_{\varepsilon_1}^{\varepsilon_2}\frac{dr}{r^5}\lambda W'(\Phi)\phi_K^2 \approx W'(\Phi_0)\mu^4\int_{\varepsilon_1}^{\infty}\frac{dr}{r^5}\left(\frac{r}{\varepsilon_1}\right)^{8-2\Delta_K} \tag{6.A.5}$$

$$\implies \frac{1}{\varepsilon_2^3}\partial_r\Phi|_{r=\varepsilon_2} = \frac{W'(\Phi_0)}{2\Delta_K-4}\frac{\mu^4}{\varepsilon_1^4} \approx \frac{W'(\Phi(\varepsilon_2))}{2\Delta_K-4}\frac{\mu^4}{\varepsilon_1^4} \tag{6.A.6}$$

This implies that the boundary condition on Φ at $r=\varepsilon_2$ is determined by μ/ε_1 (assuming W and Δ are known). Note that $m_X \sim \mu/\varepsilon_1$. We have to choose $\mu < 1$ for the dual field theory to make sense. Further since $\Delta_K \gg 1$, we can see that the confinement scale $m_{\text{confine}} \sim 1/r_0 < m_X$. The separation between m_{confine} and m_X depends on the precise form of $W(\Phi)$. When W' is non-zero the boundary condition on the dilaton differs from that in the $\mathcal{N}=4$ theory. As we describe in appendix 6.D, a well-defined variational principle requires a boundary potential for the dilaton (this serves as the counterterm).

We emphasize that pure AdS does not satisfy this non-trivial boundary condition. The boundary potential for the dilaton represents a deformation of the $\mathcal{N}=4$ theory, and cannot be interpreted as a parameter specifying a state of the $\mathcal{N} = 4$ theory. A more modern perspective on the effects of such a boundary potential for the dilaton was given in [170, 28]: $W(\Phi)$ encodes a deformation of the dual field theory action by a combination of multitrace operators. From this point of view, the calculation [70] that we have just done is a nice example of holographic Wilsonian RG described in [87, 56]. The precise relationship between the SO(6)-invariant scalar mass and the multitrace operator encoded by $W(\Phi)$ is not clear, and the dangerousness of the irrelevance of this operator remains mysterious to us.

6.B A family of examples of dilaton-driven confinement

The solution in (6.A.1) is a particular member of a more general family of solutions of Type IIB supergravity. In this appendix we will present this family of solution. The following is a saddle point of Type IIB SUGRA action (with $B_2^{NS} = C_2^{RR} = 0$)

$$ds^2 = L^2\left(-\frac{d\tau^2\mathcal{K}_x}{r^2} + \frac{d\vec{x}^2\mathcal{K}_x}{r^2} + \frac{dy^2\mathcal{K}_x\mathcal{J}}{r^2} + \frac{dr^2}{r^2}\right) + L^2 ds^2_{S^5}$$

$$F_5 = 2L^4(1+\star)\Omega_5, \quad \Phi = \Psi = \frac{\mho}{2}\log\left(\frac{1+r^4/r_0^4}{1-r^4/r_0^4}\right), \quad \chi = 0 \tag{6.B.1}$$

when

$$\mathcal{K}_x = \sqrt{1-r^8/r_0^8} \quad \text{and} \quad \mathcal{J} = \frac{1+\frac{r^4}{r_0^4}}{\left(1-\frac{r^4}{r_0^4}\right)^{\sqrt{6-\mho^2}}}\,. \tag{6.B.2}$$

That is, the above field configuration satisfies the following equations (we choose units with $8\pi G_{10} = 1$):

$$\begin{aligned}
\nabla_M\nabla^M\Phi &= e^{2\phi}(\nabla_M\chi)^2 \\
\nabla_M\left(e^{2\Phi}\nabla^M\right)\chi &= 0 \\
R_{MN} &= \frac{1}{2}\partial_M\Phi\partial_N\Phi + \frac{1}{2}e^{2\phi}\partial_M\chi\partial_N\chi + \frac{1}{6}F_{MP_1\ldots P_4}F_N{}^{P_1\ldots P_4} \\
F_5 &= \star F_5, \quad dF_5 = 0
\end{aligned} \tag{6.B.3}$$

$$\int_{S^5} F_5 = N T_{D3} = N\sqrt{\pi} \implies L^8 = \frac{N^2}{4\pi^5}$$

Let us first consider the case when y is non compact; in this case, the solution preserves 3+1 dimensional Lorentz invariance in $t, \vec{x}, y$ when $\mho = \sqrt{6}$. All solutions preserve 2+1 dimensional Lorentz invariance. We studied the solution at $\mho = \sqrt{6}$ previously [19]; it realizes Schrödinger symmetry asymptotically.

All solutions with $\mho \neq 0$ are singular at $r = r_\star$. For any value of $\mho$ $(0 < \mho \leq \sqrt{6})$, the metric is conformal to a regular metric. Let us pick any solution from this family of solutions. The singularity in this solution can be resolved by "uplifting" the solution to a regular solution of 11 D supergravity. This can be done in more than one way. One way is to T-dualize along y to get a solution of type IIA. This T-duality will modify the profile of the dilaton, but it does not remove the singularity. The singularity in the type IIA solution can be removed by oxidizing this type IIA solution to a regular solution of 11D supergravity. The dilaton field becomes the "radion" associated with the 11-dimensional circle [167].

An alternate way of "uplifting" the solution was used in [19] to resolve a similar singularity. Here, the S^5 part of the metric is written as a Hopf fiber over $\mathbb{CP}^2$. Then we can obtain a solution of type IIA supergravity by T-dualizing along the Hopf circle (say χ_1). This solution of Type IIA supergravity can then be uplifted to 11D supergravity (see [19]). The uplifted solution is

$$ds_{11}^2 = e^{-\Psi/6} L^2 \left[\left(-\frac{d\tau^2 \mathcal{K}_x}{r^2} + \frac{d\vec{x}^2 \mathcal{K}_x}{r^2} + \frac{dy^2 \mathcal{K}_x \mathcal{J}}{r^2} + \frac{dr^2}{r^2} \right) + ds^2\left(\mathbb{CP}^2\right) + d\chi_1^2 \right] + L^2 e^{4\Psi/3} d\chi_2^2$$

$$F_4 = L^4 \left(\frac{1}{2} J \wedge J + 2J \wedge d\chi_1 \wedge d\chi_2 \right) \qquad (6.B.4)$$

where J is the Kähler form on $\mathbb{CP}^2$. The uplifted solution is regular. We can now get two solutions of type IIA from this 11 dimensional solution - (a) by reducing along χ_1 and (b) by reducing along χ_2. The first reduction produces a regular metric (with a smoothly shrinking circle) and a constant dilaton, while the second system has a metric with a curvature singularity and non-trivial dilaton profile. The second system is related to the type IIB solution in (6.B.1) by T-duality. The two type-II solutions are related by S-duality.

6.C Supersymmetry analysis

In this section we analyze the supersymmetry of the background in (6.3.1). We will assume x^3 to be non-compact in this section. In the following we will use use M, M_i, N to denote spacetime (10D) indices and a, b to denote vielbein indices. The conditions for a bosonic background (with $B_2^{NS} = C_2^{RR} = 0$) to preserve some supersymmetry are [62]:

Dilatino (λ) variation:

$$\delta\lambda = i\gamma^N \mathcal{P}_N \epsilon = 0 \tag{6.C.1}$$

Gravitino (ψ_M) variation:

$$\delta\psi_M = \left(\partial_M + \frac{1}{4}\omega_M^{ab}\Gamma_{ab} - \frac{i}{2}\mathcal{Q}_M\right)\epsilon + \frac{i}{192}\gamma^{M_1M_2M_3M_4}F_{MM_1M_2M_3M_4}\epsilon = 0 \tag{6.C.2}$$

where we have combined the Majorana-Weyl fermions ($\epsilon^{1,2}$) of type IIB supergravity into a single complex Weyl spinor $\epsilon = \epsilon^1 + i\epsilon^2$ (following [62]). The variables $\mathcal{Q}$ and $\mathcal{P}$ are defined as follows

$$\mathcal{Q} = -\frac{1}{2}e^{\Phi}d\chi, \quad \mathcal{P} = \frac{i}{2}e^{\Phi}d\chi + \frac{1}{2}d\phi.$$

Note that only $\mathcal{P}_3$ and$\mathcal{Q}_3$ are non-zero in our case. All other components of $\mathcal{Q}$ and $\mathcal{P}$ are zero[21]. Let us define $\tilde{Q} = QLe^{\Phi_0}$ for convenience. Γ denote "flat space" gamma matrices (*i.e.* $\{\Gamma^a, \Gamma^b\} = 2\eta^{ab}$) and $\gamma_M = e_M^a\Gamma_a$ are curved space gamma matrices. Note that only the non-compact part of the background in (6.3.1) is different from the undeformed $AdS_5 \times S^5$ background. Hence, we will suppress the S^5 part in the rest of the analysis. We have defined e_M^a in section 6.2. The spin connections associated with this choice of orthonormal basis are

$$\omega^{04} = 2e^0, \quad \omega^{y4} = 2e^0, \quad \omega^{i4} = e^i$$

Now, for the dilatino variation (6.C.1) to vanish, we must have $\gamma_t\epsilon = 0 = (\Gamma_0 + \Gamma_y)\epsilon$. [22]Using this constraint and the expressions for the spin connections, we can write the gravitino equations as follows

$$\partial_t\epsilon = 0, \quad \left(\partial_3 - \frac{i}{2}\mathcal{Q}_3 - \frac{1}{2}e_3^y\Gamma_y\right)\epsilon = 0,$$

[21] Supersymmetry of somewhat different null backgrounds with $\mathcal{Q} = 0, \mathcal{P} \neq 0$ were studied in [47, 48, 39, 40].

[22] Note that $\gamma_t^2 = (\Gamma_0 + \Gamma_y)^2 = 0$.

$$\left(\partial_j - \frac{L}{2r}\Gamma_j\left(1-\Gamma_4\right)\right)\epsilon = 0, \partial_r\epsilon - \frac{L}{2r}\Gamma_4\epsilon = 0$$

Note that the last two equations are the same as the equations in undeformed AdS_5. Note that ϵ must satisfy $(\Gamma_0 + \Gamma_y)\,\epsilon = 0$ and $\Gamma_4\epsilon = \epsilon$. The latter condition is the constraint we get from the last two equations (same as the undeformed case). We can see that the following satisfies all the equations.

$$\epsilon = e^{\frac{i\tilde{Q}x_3}{4L_3}}\left(\Gamma_0 + \Gamma_y\right)\left[\left(\frac{1+\Gamma_y}{2}\right) + e^{\frac{\tilde{Q}x_3}{4L_3}}\left(\frac{1-\Gamma_y}{2}\right)\right]\left(\frac{1+\Gamma_4}{2}\right)\frac{L}{r^{1/2}}\eta$$

where η is independent of t, x_i, x_3 and r. When $\mathcal{Q} = 0, \mathcal{P} \neq 0$ this result agrees with the results of [47, 48, 39, 40]. The above analysis shows that the background in (6.3.1) preserves 1/4 of the original supersymmetry. Hence, the operators dual to the null deformations in the bulk preserve 4 supercharges.

6.D Boundary terms

In this section, we will show that in the geometry (6.2.1), r_0 (and hence the IR cutoff scale $r_\star$) is determined by non-trivial boundary conditions on e^y. We will do this by finding the boundary terms that are required to have a well-defined variational principle. As discussed in appendix 6.A, the parameter that determines the boundary behavior of e^y corresponds to a mass-deformation in the dual field theory. Note that when x^3 is compact, the one point functions of the operators dual to e^y_3 and e^y_t must be finite to have a well-defined variation principle in addition to finiteness of stress tensor and one point function of other supergravity fields such as the dilaton and axion. In the case when x_3 is non-compact, the five dimensional stress tensor should be finite to have a well-defined variational principle. Finiteness of the stress tensor (and other one point functions) in the non-compact case does not guarantee the finiteness of the one point functions in the case where x_3 is compact. In fact when r_0 is finite, the geometry ends in the IR; while in the non-compact case the geometry does not end in the interior and r_0 is just a parameter associated with the plane wave. We will be exploiting this crucial difference in the asymptotic behavior in this section to interpret the parameter r_0 as a mass deformation of the dual field theory (when x_3 is compact).

When x_3 is compact, it is convenient to work with the reduced theory to find the boundary terms. The lower dimensional boundary terms need not oxidize to a local, intrinsic boundary term in the higher dimensional theory in general. It appears that in the case of gravity duals of dipole theories, the higher dimensional action contains boundary terms that are not local. In these cases, it should be possible to obtain the ten-dimenional counterterms by performing a Melvin twist of the AdS boundary terms. The action of the Melvin twist on the ten-dimensional Gibbons-Hawking term would produce unfamiliar extrinsic terms involving the B_{NS}, metric and the dilaton in addition to non-local intrinsic boundary terms. In fact application of holographic renormalization of a ten-dimensional solution seems to be less understood when the internal manifold is squashed.

In the present case, we will show that the lower dimensional boundary terms can be oxidized to local boundary terms in the higher dimensional theory when $r_0 \to \infty$. When r_0 is finite, the lower dimensional counterterms do not seem to uplift to a local counterterm of the higher dimensional theory. Further, the boundary conditions on the higher dimensional metric are complicated due to the compactness of x_3. It appears that a generalization of GH boundary term is required to include other non-trivial boundary behavior of the metric. Note that the Gibbons-Hawking boundary term imposes Dirichlet boundary condition on all the components of the metric. It is possible to impose other boundary conditions (without modifying the Gibbons-Hawking term) by introducing a heavy field (proxy for "string field") that induces the boundary condition on the metric components (as discussed in appendix 6.A). The fact that the higher dimensional counterterm needs to be modified when r_0 is finite is an indication that r_0 is determined by some boundary condition induced by an excited string state (or some heavy field).

We must understand the boundary conditions on e^y. We will do this by fixing the boundary terms in the reduced theory. First we present the action of the reduced theory without the boundary terms:

$$S = \frac{L_3}{2\kappa_4^2}\int d^4x\sqrt{g_4}\Bigg(R_4 - \frac{1}{2}\left(\partial\tilde{\Phi}\right)^2 - \frac{3}{2}\left(\partial\sigma\right)^2 - \left(\frac{Q^2}{2L_3^2}e^{2\tilde{\Phi}-3\sigma} - \frac{12}{L^2}e^{-\sigma}\right) \\ -\frac{1}{4}e^{3\sigma}F^2 - \frac{Q^2}{2L_3^2}e^{2\tilde{\Phi}}A^2\Bigg) \tag{6.D.1}$$

where $\tilde{\Phi}$ is the scalar field obtained from dimensional reduction of the type IIB dilaton, $e^{2\sigma}$

is the radion field associated with x_3. Some details of the reduction can be found in [52]. The dimensional reduction of (6.2.1) produces the following solution which is a saddle point of the reduced action (6.D.1)[23]:

$$ds^2 = e^{\sigma} L^2 \left(-e^{-2\sigma} \frac{dt^2}{r^4} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{r^2} \right), \quad A = \frac{L^2 e^{-2\sigma} dt}{r^2}, \quad \tilde{\Phi} = \Phi_0, \quad e^{2\sigma} = f(r) \quad (6.D.2)$$

Note that the term quadratic in the vector field A depends on $\tilde{\Phi}$. The linearized fluctuations of A and σ satisfy the equations in (6.3.4).

To determine the requisite boundary terms, we employ the following logic. We demand that in the limit $r_\star \to \infty$, the boundary conditions and boundary terms are the natural ones in 5d. Then we add intrinsic 4d counterterms which make the stress tensor finite with the same boundary conditions. This will give physics consistent with the desired scheme in Fig. 6-1.

The 5d boundary terms (when $r_\star \to \infty$) are:

$$S^{(5d)}_{bdy} = \int d^4x \sqrt{\gamma'} \left(K' - \frac{3}{L} - \frac{e^{2\Phi}}{4} \chi \Box_{\gamma'} \chi \right) \quad (6.D.3)$$

where $\Box_{\gamma'}$ is the Laplacian on the boundary metric γ' (see *e.g.* [158]). In particular, the 5d Gibbons-Hawking term K' imposes Neumann boundary conditions on σ.

For the action in (6.D.1) to be well-defined on (6.D.2) we need to introduce the following boundary terms

$$S_{bdy} = \int d^3x \sqrt{\gamma} \left(K + e^{-3\sigma} n^\mu A^\nu F_{\mu\nu} + n^r \partial_r \sigma W_1 \left(\sigma, A^2\right) - \frac{3}{L} e^{-\sigma/2} + W_2 \left(\sigma, A^2, \Phi\right) + W_3 \left(\sigma, A^2, \Phi\right) \right)$$

where the functions W_1 , W_2 and W_3 are defined as follows

$$W_1 \left(\sigma, A^2\right) = -\frac{2}{L} e^{-\sigma/2} - \frac{2}{L} e^{5\sigma/2} A^2$$

$$W_2 \left(\sigma, A^2, \Phi\right) = \frac{Q^2 L}{8 L_3^2} e^{2\Phi} \left(e^{\sigma/2} A^2 + e^{-5\sigma/2} \right)$$

$$W_3 \left(\sigma, A^2, \Phi\right) = \frac{1}{M^2_{\rm BC}} \left(\left(\frac{Q^2 L^2}{4 L_3^2} e^{2\Phi} e^{-2\sigma} - 1 \right) \left(L W_1 \left(\sigma, A^2\right) + \left(\frac{Q^3 L^4}{8 L_3^3} e^{3\Phi} \right) A^4 \right) \right)$$

[23]We note that this reduction was used in [10] to embed Lifshitz black holes in string theory.

$$+\frac{1}{2}\left(\frac{Q^2L^2}{4L_3^2}e^{2\Phi}e^{-2\sigma}-1\right)^2 e^{-\sigma/2}\right) .$$

The first three terms of the lower dimensional boundary term come from the reduction of the higher dimensional Gibbons-Hawking term. The higher dimensional "boundary cosmological constant" reduces to the fourth term ($e^{-\sigma/2}$) in the lower dimensional boundary integral, and the last term of (6.D.3) (the axion kinetic energy) reduces to W_2 term in the lower dimensional boundary action. This action without W_3 makes the stress tensor finite even when r_0 is finite. However, this does not make the variation with respect to e_3^y and e_t^y finite. This variation can be cancelled by adding W_3, with the specific coefficient

$$M_{\rm BC} = r_\star/4 = r_0^2 e^{\Phi_0} Q/2L_3. \tag{6.D.4}$$

Note that W_3 cannot be lifted to an intrinsic local 5d counterterm. We interpret W_3 as a boundary term that is induced by a "string field" which is not directly visible in supergravity. Further, when $M_{\rm BC}$ or r_0 is infinite, the lower dimensional boundary terms can be uplifted to the 5D boundary term in (6.D.3).

We emphasize the distinction between $\Gamma \sim \kappa_6 \sim \kappa_8$, which determines the coefficient in the perturbed action of operators whose dimensions are protected by supersymmetry above the KK scale (see Fig. 6-1), and $M_{\rm BC}$ which cannot be interpreted in this way and sources a "string field".

Chapter 7

Lifshitz Black Hole

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7.1 Introduction

In this chapter, we are interested in studying the thermal aspects of a Lifshitz field theory using holography. It was pointed out in [63] that the equal-time correlators of a free Lifshitz theory exhibit *ultra-local* behavior in the infinite volume limit. One is led to wonder whether this property should is shared by interacting Lifshitz theories, and whether the Lifshitz scaling is sufficient to produce this behavior. We use the AdS/CFt correspondence to show that Lifshitz field theories with gravity duals do not exhibit this behavior. Before we proceed to discuss the holographic approach, let us present a brief review of the results in [63]. Let us consider the following Gaussian model for a free Lifshitz theory in d space dimensions:

$$S[\chi] = \int d^d x dt \left[(\partial_t \chi)^2 - K \left(\nabla^2 \chi \right)^2 \right] \tag{7.1.1}$$

This action describes a fixed line parametrized by K, and the dynamical exponent is $z = 2$. This theory describes the critical behavior of *e.g.* quantum dimer models [11]. In many ways, the $d = 2, z = 2$ version of the theory (7.1.1) is like a relativistic boson in 1+1 dimensions[1]. The scaling behavior of the ground-state entanglement entropy for this class of theories was studied recently in [97, 11]. This analysis also supports the similarity with 2d CFT, in that a universal leading singular behavior is found.

In the free theory, the boson has logarithmic correlators

$$\langle \chi(x)\chi(0)\rangle \sim \int d\omega d^2k \frac{1}{\omega^2 - k^4} e^{i\vec{k}\cdot\vec{x} - i\omega t} \sim \ln x. \tag{7.1.2}$$

As in the familiar $d = z = 1$ case, the operators of definite scaling dimension are not the canonical bose field itself, but rather its exponentials and derivatives. In the connection with quantum dimer models, the bose field is a height variable constructed from the dimer configuration, and the exponentials of the bose field are order parameters for various dimer-solid orderings [11]. At zero temperature, the logarithmic behavior of the correlator of the bose fields implies that the two-point function of the order parameter decays as a power law. However, the equal-time correlators at finite temperature are *ultra-local* in the infinite-volume limit [63]: they vanish at any nonzero spatial separation. In [63], it was suggested that this might be a mechanism for the kind of local criticality (scaling in frequency, but not momentum) seen in the strange metal phase of the cuprates and in heavy fermion materials. One is led to wonder whether this property should is shared by interacting Lifshitz theories, and whether the Lifshitz scaling is sufficient to produce this behavior. In [63] the addition of perturbative interactions was shown to lead to a finite correlation length; these perturbations violate the Lifshitz scaling. Below we will show that interactions which preserve the Lifshitz scaling need not give ultralocal behavior.

They computed the two-point function for the case when $z = 2$ and showed that it exhibits power law decay. They also studied the holographic renormalization group flow for this case and found that AdS_4 is the only other fixed point of the flow. Lifshitz vacuum solutions were shown to be stable under perturbations of the bulk action in [2].

In this chapter we shall study a black hole solution which asymptotes to the Lifshitz spacetime with $d = 2, z = 2$. In section 2, an analytical solution for a black hole that asymptotes to the planar Lifshitz spacetime is written down. We present several actions

[1]Similar statements apply whenever $z = d$. However, constructing a rotation-invariant, local spatial kinetic operator that scales like p^{2d} is tricky for $d \neq 2^k$ for integer k. We note in passing that the existence of such theories seem to be suggested by the calculations of [141].

whose equations of motion it solves; they all involve some matter sector additional to (5.1.2). Section 3 presents an analysis of the thermodynamics of this black hole. In section 4, we solve the wave equation for a massive scalar field in this background; surprisingly, this equation is exactly solvable. We use this solution to calculate the two-point functions of boundary operators in section 5.

There have been numerous works related to black holes in Lifshitz spacetime including the author's work. [165] constructs a black hole solution in a related background with slightly different asymptotics. Danielsson and Thorlacius [46] found numerical solutions of black holes in global Lifshitz spacetime. Interestingly, these are solutions to precisely the system studied by [106], with no additional fields. Related solutions were found by [29]. [15] found solutions of type IIB supergravity that are dual to Lifshitz-like theories with spatial anisotropy and $z = 3/2$; these solutions have a scalar field which breaks the scaling symmetry.

7.2 Black hole solution

We shall now study a black hole in four dimensions that asymptotically approaches the Lifshitz spacetime in 5.1.1 with $z = 2$. We first observe that there is such a black hole in a system with a strongly-coupled scalar (*i.e.* a scalar without kinetic terms). The action is

$$S_1 = \frac{1}{2}\int d^4x\,(R - 2\Lambda) - \int d^4x \left(\frac{e^{-2\Phi}}{4}F^2 + \frac{m^2}{2}A^2 + \left(e^{-2\Phi} - 1\right)\right). \tag{7.2.1}$$

A solution of this system is

$$\Phi = -\frac{1}{2}\log\left(1 + r^2/r_H^2\right), \quad A = f/r^2 dt$$

$$ds^2 = -f\frac{dt^2}{r^{2z}} + \frac{d\vec{x}^2}{r^2} + \frac{dr^2}{fr^2}, \tag{7.2.2}$$

with

$$f = 1 - \frac{r^2}{r_H^2}.$$

Note that the metric has the same simple form as in the RG flow solution (eqn (4.1)) of [106].

We can get the same contributions to the stress tensor as from the scalar without kinetic terms from several more-reasonable systems. One such system is obtained by adding a second massive gauge field B which will provide the same stress-energy as the scalar. It has a slightly unfamiliar action:

$$S_2 = \frac{1}{2}\int d^4x\,(R-2\Lambda) - \int d^4x\left(\frac{1}{4}B^2 dA^2 + \frac{m_A^2}{2}A^2 + \frac{1}{4}dB^2 - \frac{m_B^2}{2}(1-B^2)\right) \qquad (7.2.3)$$

where A, B are one-forms, and $m_A^2 = 4$ and $m_B^2 = 2$. The solution looks like $B = B(r)dr, A = A(r)dt$ and the metric is same as (7.2.2). In the solution, the scalar functions take the form

$$B(r) = \sqrt{g_{rr}\left(1+\frac{r^2}{r_H^2}\right)}, \quad A(r) = \Omega f r^{-z} dt$$

Note that $B(r)$ isn't gauge-trivial (even though its field strength vanishes) because of the mass term. Since $B(r)$ asymptotes to 1, the effective gauge coupling of the field A is not large at the boundary.

The system with a strongly-coupled scalar in (7.2.1) is not equivalent to the system (7.2.3) with two gauge fields. For example, there are solutions of (7.2.1) where the scalar has a profile that depends both on r and x; such configurations do not correspond to solutions of (7.2.3).

It is not clear whether the solution written above is stable. We leave the analysis of the stability of such solutions to small perturbations to future work. As weak evidence for this stability, we show in the next section that these black holes are *thermodynamically* stable.

Another action with this Lifshitz black hole (7.2.2) as a solution is

$$S_3 = \frac{1}{2}\int d^4x\left(R - 2\Lambda - \frac{1}{2}dB^2 - (\partial\Phi - B)^2 - m_A A^2 - \frac{1}{2}e^{-2\Phi}F^2 - V(\Phi)\right) \qquad (7.2.4)$$

where $V(\Phi) = 2e^{-2\Phi} - 2$. In the solution, the metric and gauge field A take the same form as in (7.2.2). The other fields are

$$e^{-2\Phi} = 1 + \frac{r^2}{r_H^2}, \qquad B = d\Phi.$$

Note that the action (7.2.4) is *not* invariant under the would-be gauge transformation

$$B \to B + d\Lambda, \qquad \Phi \to \Phi + \Lambda,$$

because of the coupling to $F^2 - 4$ (the sum of the gauge kinetic term and the potential term)[2]. We are not bothered by this: it means that in quantizing the model, mass terms for the fluctuations B will be generated; however, such a mass term is already present.

We would also like to point out that in the three systems $S_{1,2,3}$ described above, the stress-energy tensor of the fields with local propagating degrees of freedom satisfy the dominant energy condition[3], *i.e.* $T^{(\Phi,A,B)}_{\mu\nu}$ $\left(= R_{\mu\nu} - (\frac{1}{2}R + \Lambda)g_{\mu\nu}\right)$ satisfies the following

$$\frac{T_{tt}}{T_{xx}} = \frac{T_{tt}}{T_{yy}} > -1 \text{ and } \frac{T_{tt}}{T_{rr}} > -1.$$

Hence, there are no superluminal effects in the bulk. This is basically a consequence of the fact that the squared-masses of the gauge fields are positive.

7.3 Lifshitz black hole thermodynamics

The Hawking temperature and entropy can be calculated using the near horizon geometry. The Hawking temperature is the periodicity of the Euclidean time direction in the near horizon metric (proportional to the surface gravity) *i.e.*, $T = \frac{\kappa}{2\pi}|_{r=r_H}$, with

$$\kappa^2 = -\frac{1}{2}\nabla^a v^b \nabla_b v^a$$

where $v = \partial_t$. Hence,

$$T = \frac{1}{2\pi r_H^2}. \tag{7.3.1}$$

The entropy of the black hole is

$$\mathcal{S} = \frac{\text{Area of Horizon}}{4G_N} = \frac{L_x L_y}{4G_4 r_H^2}. \tag{7.3.2}$$

[2]We note that this quantity does vanish on the solution of interest.

[3]We would like to thank Allan Adams, Alex Maloney and Omid Saremi for bringing this criterion to our attention.

We shall now evaluate the free energy, internal energy and pressure by calculating the on-shell action and boundary stress tensor. In order to renormalize the action, it is essential to add counterterms which are intrinsic invariants of the boundary.

Consider the following gravitational action:

$$
\begin{aligned}
S = & \frac{1}{2}\int_M d^4x\sqrt{g}\left(R - 2\Lambda - \frac{e^{-2\Phi}}{4}F^2 - \frac{m_A^2}{2}A^2 - V(\Phi)\right) \\
& - \int_{\partial M} d^3x\sqrt{\gamma}\left(K + c_N e^{-2\Phi} n^\mu A^\nu F_{\mu\nu}\right) \\
& + \frac{1}{2}\int_{\partial M} d^3x\sqrt{\gamma}\left(2c_0 - c_1\Phi - c_2\Phi^2\right) + \frac{1}{2}\int_{\partial M} d^3x\sqrt{\gamma}\left(\left(c_3 + c_4\Phi\right)A^2 + c_5 A^4\right) .
\end{aligned}
\tag{7.3.3}
$$

The second line of (7.3.3) contains extrinsic boundary terms: the Gibbons-Hawking term, and a 'Neumannizing term' which changes the boundary conditions on the gauge field. The last line of (7.3.3) describes the intrinsic boundary counterterms[4]. In the above expression, we have set $8\pi G = 1$. We have written the analysis in terms of S_1 (7.2.1); the analysis can be adapted for S_2 (7.2.3) by simply replacing Φ in (7.3.3) by $-\frac{1}{2}\log B^2$. If Neumann boundary conditions are imposed on the gauge field, then $c_N = 1$ and $c_i = 0$ for $i \geq 3$. Similarly, $c_N = 0$, if Dirichlet boundary condition is imposed on the gauge field.

The boundary stress tensor resulting from (7.3.3) is

$$
\begin{aligned}
T_{\mu\nu} = & K_{\mu\nu} - \left(K - c_0 + \frac{1}{2}c_1\Phi + \frac{1}{2}c_2\Phi^2\right)\gamma_{\mu\nu} + \frac{e^{-2\Phi}}{2}\left(n^r A_\mu \partial_r A_\nu + n^r A_\nu \partial_r A_\mu - n^r A_\alpha \partial_r A^\alpha \gamma_{\mu\nu}\right) \\
& + \left(c_3 + c_4\Phi + 2c_5 A^2\right) A_\mu A_\nu - \frac{1}{2}\left(c_3 + c_4\Phi + c_5 A^2\right) A^2 \gamma_{\mu\nu}
\end{aligned}
\tag{7.3.4}
$$

The values of c_i are determined by demanding that the action is "well-behaved". The action is well-behaved if the variation of the action vanishes on-shell and if the residual gauge symmetries of the metric are not broken. The values of c_i which makes the action well-defined also render finite the action and boundary stress tensor (please see appendix

[4] The most general combination of counterterms, which do not vanish at the boundary, is

$$
\frac{1}{2}\int_{\partial M} d^3x\sqrt{\gamma}\left(2c_0' + c_1'\Phi + c_2'\Phi^2\right) + \frac{1}{2}\int_{\partial M} d^3x\sqrt{\gamma}\left(\left(c_3' + c_4'\Phi\right)\left(A^2 - 1\right) + c_5'\left(A^2 - 1\right)^2\right)
$$

which has the same form as (7.3.3).

A). Implementing this procedure, we find for the energy density, pressure and free energy

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{1}{2}T\mathcal{S} = \frac{L_x L_y}{2r_H^4} \tag{7.3.5}$$

Satisfying the first law of thermodynamics (in the Gibbs-Duhem form $\mathcal{E} + \mathcal{P} = T\mathcal{S}$) is a nice check on the sensibility of our solution, since it is a relation between near-horizon $(T, \mathcal{S})$ and near-boundary $(\mathcal{E}, \mathcal{P})$ quantities.

Recently, [152] have described an alternative set of boundary terms for asymptotically Lifshitz theories. They do not include the Neumannizing term, but instead include an intrinsic but nonanalytic $\sqrt{A^\mu A_\mu}$ term.

7.4 Scalar response

In this section, we study a probe scalar in the black hole background (7.2.2). The scalar can be considered a proxy for the mode of the metric coupling to T^x_y.

7.4.1 Exact solution of scalar wave equation

Consider a scalar field ϕ of mass m in the black hole background (7.2.2)[5].

Let $u \equiv \frac{r^2}{r_H^2}$. Fourier expand:

$$\phi = \sum_k \phi_k(u) e^{-i\omega t + i\vec{k}\cdot\vec{x}}$$

The wave equation takes the form:

$$0 = \frac{u\left(-fk^2 + u\omega^2\right) + m^2 f}{4f^2u^2}\phi_k(u) - \frac{1}{fu}\phi_k'(u) + \phi_k''(u)$$

where $k^2 \equiv \vec{k}^2$. Near the horizon, the incoming $(-)$ and outgoing $(+)$ waves are

$$\phi_k \sim (1-u)^{\pm i\omega/2}.$$

[5]In the following we have set both the bulk radius of curvature and the horizon radius to one. This means that frequencies and momenta are 'gothic' [162], *i.e.* measured in units of r_H. Note that since $z = 2$, ω needs two factors of r_H to make a dimensionless quantity.

The solutions near the boundary at $u = 0$ are

$$\phi_k \sim u^{1 \pm \frac{1}{2}\sqrt{4+m^2}}$$

The *exact* solution to the wave equation is $\phi_k(u) = f^{-i\omega/2} u^{1-\frac{1}{2}\sqrt{m^2+4}} G_k(u)$ with

$$G_k(u) = A_1 \; {}_2F_1(a_+, b_+; c_+, u) u^{\sqrt{m^2+4}} + A_2 \; {}_2F_1(a_-, b_-; c_-, u) \tag{7.4.1}$$

and

$$(a_\pm, b_\pm; c_\pm) \equiv$$
$$\left(-\tfrac{i\omega}{2} \pm \tfrac{\sqrt{m^2+4}}{2} - \tfrac{1}{2}\sqrt{-k^2-\omega^2+1} + \tfrac{1}{2}, -\tfrac{i\omega}{2} \pm \tfrac{\sqrt{m^2+4}}{2} + \tfrac{1}{2}\sqrt{-k^2-\omega^2+1} + \tfrac{1}{2}; 1 \pm \sqrt{m^2+4}; u\right)$$

We emphasize that this is the exact solution to the scalar wave equation in this black hole; such a solution is unavailable for the $\mathrm{AdS}_{d>3}$ black hole. The difference is that the equation here has only three regular singular points, whereas the AdS_5 black hole wave equation has four. This is because in the AdS_5 black hole, the emblackening factor is $f = 1 - u^2$ which has two roots, whereas ours is just $f = 1 - u$.

The other example of a black hole with a solvable scalar wave equation is the BTZ black hole in AdS_3 [30][6]. The origin of the solvability in that case is the fact that BTZ is an orbifold of the zero-temperature solution. This is *not* the origin of the solvability in our case – this black hole is not an orbifold of the zero-temperature solution. This may be seen by comparing curvature invariants: they are not locally diffeomorphic. More simply, if the black hole were an orbifold, it would solve the same equations of motion as the vacuum solution. The fact that we were forced to add an additional matter sector (such as Φ or B_μ) to find the black hole solution immediately shows that they are not locally diffeomorphic.

Now we ask for the linear combination of (7.4.1) which is ingoing at the horizon. In terms of $\nu \equiv \sqrt{4+m^2}$, $\gamma \equiv \sqrt{1-\omega^2-k^2}$, this is the combination with

$$\frac{A_1}{A_2} = -(-1)^\nu \frac{\Gamma(\nu)}{\Gamma(-\nu)} \frac{\Gamma\left(\frac{1}{2}(1-i\omega-\nu-\gamma)\right)}{\Gamma\left(\frac{1}{2}(1-i\omega+\nu-\gamma)\right)} \frac{\Gamma\left(\frac{1}{2}(1-i\omega-\nu+\gamma)\right)}{\Gamma\left(\frac{1}{2}(1-i\omega+\nu+\gamma)\right)}. \tag{7.4.2}$$

In the massless case, one of the hypergeometric functions in (7.4.1) specializes to a Meijer

[6] Another example, in two dimensions, is [142].

G-function, and the solution is $\phi_k = u^2 f^{-i\omega/2} G_k(u)$ with

$$G_k(u) =$$
$$c_2\, {}_2F_1\left(-\tfrac{i\omega}{2} - \tfrac{1}{2}\sqrt{-k^2-\omega^2+1} + \tfrac{3}{2}, -\tfrac{i\omega}{2} + \tfrac{1}{2}\sqrt{-k^2-\omega^2+1} + \tfrac{3}{2}; 3; u\right) +$$
$$c_1 G_{2,2}^{2,0}\left(u \left| \begin{array}{c} \frac{1}{2}\left(i\omega - \sqrt{-k^2-\omega^2+1} - 1\right), \frac{1}{2}\left(i\omega + \sqrt{-k^2-\omega^2+1} - 1\right) \\ -2, 0 \end{array} \right. \right)$$

In this solution, the coefficient of c_1 (the Meier-G function) is purely ingoing at the horizon.

7.4.2 Correlators of scalar operators

In the previous section we wrote the solution for the wave equation in this black hole for a scalar field with an arbitrary mass. As mentioned earlier, the BTZ black hole also shares this property of having a scalar wave equation whose solutions are hypergeometric. Hence, one might expect that the two-point function of scalar operators in a Lifshitz-like theory to have a form that is similar to that of 2D CFTs.

The momentum space correlator for a scalar operator of dimension $\Delta = \Delta_-$ is determined from the ratio of the non-normalizable and normalizable parts of the solution. The asymptotic behavior of the solution in (7.4.1) is

$$\phi \sim u^{\frac{\Delta_+}{2}} \left(A_1 + \mathcal{O}(u)\right) + u^{\frac{\Delta_-}{2}} \left(A_2 + \mathcal{O}(u)\right) \tag{7.4.3}$$

Hence, the retarded Green's function (two-point function) is

$$G_{\text{ret}}(\omega, \vec{k}) = -\frac{A_1}{A_2} = (-1)^\nu \frac{\Gamma(\nu)}{\Gamma(-\nu)} \frac{\Gamma\left(\frac{1}{2}(1 - i\omega - \nu - \gamma)\right)\Gamma\left(\frac{1}{2}(1 - i\omega - \nu + \gamma)\right)}{\Gamma\left(\frac{1}{2}(1 - i\omega + \nu - \gamma)\right)\Gamma\left(\frac{1}{2}(1 - i\omega + \nu + \gamma)\right)} \tag{7.4.4}$$

with ν and γ defined above equation (7.4.2). Note that the correlator has a form very similar to that of a 2D CFT. It would be nice to know the precise connection between $z = 2$ Lifshitz-like theories in $2+1$ D with 2D CFTs that is responsible for this similarity. Note that the poles of the retarded Green's function do not lie on a straight line in the complex frequency plane, as they do for 2D CFTs.

Next, we would like to see whether the correlators exhibit ultra local behavior at finite temperature as observed in the free scalar Lifshitz theory [63]. We find that the Green's function is not ultra-local – this removes the possibility that Lifshitz-symmetric interactions require ultralocal behavior.

We will now calculate the two-point function of a scalar operator of dimension $\Delta = 4$ at finite temperature. In this case, the correlator is given by the coefficient of r^4 in the asymptotic expansion of the solution near $r = 0$. Kachru *et. al.* [106] showed that the correlator exhibits a power law decay at zero temperature.

We can evaluate this correlator by extracting the coefficient of the u^2 term (note that $u \propto r^2$) in the asymptotic expansion of the solution of the massless scalar wave equation. The behavior of the solution near $u = 0$ is

$$\phi(u,\vec{k},\omega) = 1-\frac{u}{4}\left(\vec{k}^2+2i\omega\right)-\frac{u^2}{64}\left(\left(\vec{k}^2\right)^2+4\omega^2\right)\left[-3+2\psi\left(\frac{1}{2}\left(-1+i\omega-\sqrt{1-\vec{k}^2-\omega^2}\right)\right)\right.$$

$$\left.+2\psi\left(\frac{1}{2}\left(-1+i\omega+\sqrt{1-\vec{k}^2-\omega^2}\right)\right)+2\gamma_E+2\ln u\right]+\mathcal{O}\left(u^3\right) \qquad (7.4.5)$$

where γ_E is Euler's constant, ψ is the digamma function. The behavior of the solution in the Euclidean black hole can be obtained by replacing ω by $-i|\omega|$. The choice of the negative sign gives the solution which is ingoing at the horizon, as appropriate to the retarded correlator [162]. Henceforth, we shall work with the solution for the Euclidean case. The correlator is the sum of the two digamma functions. All other terms in the coefficient of u^2 are contact terms. Hence, the correlator in momentum space is

$$\langle\mathcal{O}(-\omega,-\vec{k})\mathcal{O}(\omega,\vec{k})\rangle \propto \left(\left(\vec{k}^2\right)^2-4\omega^2\right)\left[\psi\left(\frac{1}{2}\left(-1+|\omega|-\sqrt{1-\vec{k}^2+\omega^2}\right)\right)+\right.$$

$$\left.\psi\left(\frac{1}{2}\left(-1+|\omega|+\sqrt{1-\vec{k}^2+\omega^2}\right)\right)\right] \qquad (7.4.6)$$

After dropping the contact terms, the above expression can be written as follows

$$\langle\mathcal{O}(-\omega,-\vec{k})\mathcal{O}(\omega,\vec{k})\rangle \propto \sum_{n=1}^{\infty}\mathcal{A}_n$$

where

$$\mathcal{A}_n = \frac{(2n-3)+|\omega|}{(2n-3)^2+2|\omega|(2n-3)+k^2-1} = \frac{a_n+\mid\omega\mid}{a_n^2+2a_n\mid\omega\mid+k^2-1}$$

We can now calculate the correlators in coordinate space by performing the Fourier transform of the above expression[7]. This is given by

$$D(|\vec{x}|,t) = \left[(4\partial_t^2-(\nabla^2)^2)\right]\sum_n \mathcal{F}_n \tag{7.4.7}$$

where, $D(|\vec{x}|,t)$ is the two-point function and

$$\sum_n \mathcal{F}_n = \sum_n \int kdkd\omega d\theta \frac{a_n+\mid\omega\mid}{a_n^2+2a_n\mid\omega\mid+k^2-1/4} e^{ik|\vec{x}|\cos\theta+i\omega t}$$

The short distance ($r \ll r_H$) behavior of the equal time correlator is

$$D(|\vec{x}|\ll r_H,0) = \left[(4\partial_t^2-(\nabla^2)^2)\mathcal{F}\right]_{t=0,|\vec{x}|\to 0} \propto \frac{1}{|\vec{x}|^8}. \tag{7.4.8}$$

As a check, we note that, the short distance behavior of this expression reproduces the zero-temperature answer $|\vec{x}|^{-8}$ found in [106].

The long distance ($|\vec{x}| \gg r_H$) behavior is

$$D(|\vec{x}|\gg r_H,0) = \left[(4\partial_t^2-(\nabla^2)^2)\mathcal{F}\right]_{t=0,|\vec{x}|\to\infty} \propto \frac{e^{-\sqrt{2}|\vec{x}|/r_H}}{|\vec{x}|^{3/2}}. \tag{7.4.9}$$

The correlator is not ultra-local, unlike the thermal correlator in free scalar Lifshitz theory.

7.5 Discussion

An important defect of this work which cannot have avoided the reader's attention is the fact that the matter content which produces the stress-energy tensor for this black hole is unfamiliar and contrived. There is no physical reason why terms such as A^2B^2 should not be added. In our defense, a perturbation analysis in the coefficient of such terms indicates that a corrected solution can be constructed. It is not clear how to embed such solutions

[7]We would like to thank Shamit Kachru and Mike Mulligan for sharing the Mathematica file that computes the spatial two-point function derived in [106].

in a UV-complete gravity theory. A numerical black hole solution of consistent truncations of type IIB supergravity were found in [10]. However, the lower dimensional effective action seems to be very different from those considered here.

7.A Regularizing the action and boundary stress tensor

In this appendix, we will show that the on-shell action and boundary stress tensor can be rendered finite by making the action well-behaved, *i.e.* the action is stationary on-shell under an arbitrary normalizable variation of the bulk fields, and the boundary terms in the action must not break the residual gauge symmetries of the metric.

We will first find the constraints imposed by finiteness of the free energy, internal energy and pressure on c_i.

The free energy of the boundary theory is

$$-\mathcal{F} = \frac{S_{\text{onshell}}}{\beta} = \frac{1}{2}L_x L_y \left[\frac{64c_N - 8c_0 + 16c_1 + 8c_2 + 6c_3 + 16c_4 - 15c_5}{32r_H^4} \right.$$

$$\left. - \frac{32 + 4c_1 + 8c_0 + 6c_3 + 2c_4 - 5c_5}{16\epsilon^2 r_H^2} + \frac{24 + 2c_3 - c_5 - 8c_N + 8c_0}{8\epsilon^4} \right] \tag{7.A.1}$$

where β is inverse temperature. We must set $-c_5 + 24 - 8c_N + 8c_0 + 2c_3 = 0$ and $-c_1 - 8 - 2c_0 - 3/2c_3 - 1/2c_3 + 5/4c_5 = 0$ to get rid of the divergences in the on-shell action. Further, finiteness of the boundary stress tensor and conformal ward identities impose more constraints on the counterterms.

The internal energy of the boundary theory is

$$\mathcal{E} = -L_x L_y \sqrt{\gamma} T_t^t = -L_x L_y \left(\frac{16 + 8c_0 - 2c_3 + 3c_5 + 8c_N}{8\epsilon^4} \right. \tag{7.A.2}$$

$$\left. - \frac{32 + 8c_0 + 4c_1 - 6c_3 - 2c_4 + 15c_5}{16r_H^2\epsilon^2} - \frac{8c_0 - 16c_1 - 8c_2 + 6c_3 + 16c_4 - 45c_5 + 64c_N}{64r_H^4} \right) \tag{7.A.3}$$

Similarly, the expression for pressure is

$$\mathcal{P} = \frac{1}{2} L_x L_y \sqrt{\gamma} T^i_i = L_x L_y \sqrt{\gamma} T^x_x = \frac{1}{2} L_x L_y \left[\frac{64 c_N - 8c_0 + 16c_1 + 8c_2 + 6c_3 + 16c_4 - 15c_5}{32 r_H^4} \right.$$

$$\left. - \frac{32 + 4c_1 + 8c_0 + 6c_3 + 2c_4 - 5c_5}{16\epsilon^2 r_H^2} + \frac{24 + 2c_3 - c_5 - 8c_N + 8c_0}{8\epsilon^4} \right] \tag{7.A.4}$$

Note that $\mathcal{F} = -\mathcal{P}$, as expected in the grand canonical ensemble. Hence, the condition for the divergences in pressure to cancel is same as the condition for divergences in the on-shell action to cancel. However, finiteness of energy imposes additional constraints on the counterterms. In the case of Schrödinger black hole, it is not possible to get rid of the divergence in the energy without the Neumannizing term [1].

The conformal Ward identity for conservation of the dilatation current requires $z\mathcal{E} = d\mathcal{P}$, and in our discussion $d = z = 2$. The residual gauge freedom of the metric is broken if this condition is not satisfied (see [160]). Note that making the boundary stress tensor finite does not ensure this condition. We must set $c_2 = 7/2$ for the conformal Ward identity to hold. After imposing these conditions, we find

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = L_x L_y \frac{15 - 2c_1 - 26c_N}{16 r_H^4} \tag{7.A.5}$$

In order to have a well-defined variational principle, we must ensure that $\delta S = 0$ onshell. We shall now determine the value of c_1 using this condition[8]. The variation of the action is

$$\delta S = \int_{\text{bulk}} \text{EOM} + \frac{1}{2} \int_{\text{bdy}} d^3x \sqrt{\gamma} \left[T^\mu_\nu \delta\gamma^\nu_\mu + \left((c_N - 1) e^{-2\Phi} n_\nu F^{\nu\mu} + \left(c_3 + c_4 \Phi + 2c_5 A^2\right) A^\mu \right) \delta A_\mu + \right.$$

$$\left. c_N A^\mu \delta\left(n^\nu e^{-2\Phi} F_{\nu\mu}\right) - \frac{1}{2}\left(c_1 + 2c_2\Phi - c_4 A^2 - 4c_N A^\mu n^\nu F_{\nu\mu} e^{-2\Phi}\right) \delta\Phi \right] \tag{7.A.6}$$

The first term vanishes onshell. Therefore, the boundary terms must also vanish onshell. Let us assume, for convenience that Dirichlet boundary condition is imposed on the gauge field ($c_N = 0$). Prescribing boundary conditions is equivalent to prescribing the coefficient of the non-normalizable mode of the solution. The allowed variations at the boundary fall

[8] We have determined the value of c_1 for the case where Dirichlet boundary condition is imposed on the gauge field. However, the method is general and can be used for other boundary conditions as well.

faster than the non-normalizable part of the solution, *i.e.*,

$$\begin{aligned}
\delta\gamma^{\mu}_{\nu} &= \delta\gamma^{\mu}_{\nu(1)} r^2 + \delta\gamma^{\mu}_{\nu(2)} r^4 + \ldots \\
\delta A_\mu &= r^{-2}\left(\delta A_{\mu(1)} r^2 + \delta A_{\mu(2)} r^4 + \ldots\right) \\
\delta\Phi &= r^2 \delta\Phi_{(1)} + r^4 \delta\Phi_{(2)} + \ldots
\end{aligned} \tag{7.A.7}$$

Substituting these expressions in (7.A.6) and using the conditions on c_i for energy and pressure to be finite[9], we find

$$\delta S = \int d^3x \left(\sqrt{\gamma} T^{\mu}_{\nu} r^2 \delta\gamma^{\nu}_{\mu(1)} + \mathcal{O}\left(r^2\right) \delta A_{\mu(1)} + \left(\frac{c_2 - c_1}{r_H^2}\right)\left(\delta\Phi_1 + \mathcal{O}\left(r^2\right)\right)\right) \tag{7.A.8}$$

Since $\mathcal{E}$ and $\mathcal{P}$ are finite, the first term in the integrand vanishes at the boundary. Hence, $c_1 = c_2 = 7/2$ for the variation of the action to vanish on-shell. Using the values of c_i found above in (7.A.5) we get

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{L_x L_y}{2 r_H^4}$$

After restoring factors of $8\pi G$,

$$\mathcal{E} = \mathcal{P} = -\mathcal{F} = \frac{L_x L_y}{16\pi G r_H^4} = -\frac{1}{2} T \frac{\partial \mathcal{F}}{\partial T} = \frac{1}{2} T \mathcal{S}$$

We have shown that the stress tensor and on-shell action can be regularized by making the action well-behaved, *i.e.* δS must vanish on-shell and the counterterms should not break any residual gauge symmetry.

[9] $c_0 = -(17 - c_1)/8$, $c_3 = -5 - c_1$, $c_4 = -2c_1$ and $c_5 = -3 - c_1$, when $c_N = 0$.

Chapter 8

Conclusions and outlook

We began by presenting an alternate view of strongly interacting systems. This alternate view led to interesting phenomenological techniques that can be useful in describing quantum critical points arising in condensed matter systems. We saw how the holographic correspondence can be extended to systems with more exotic symmetries such as non-relativistic conformal symmetries. However, the simplest embeddings of these holographic models in string theory do not resemble any realistic systems such as cold fermions at unitarity. It might be worth improving the existing tools in non-relativistic holography to build new phenomenological models that can have closer resemblance with realistic systems. In particular, the asymptotic behavior of the fields in a spacetime with no well-defined conformal boundary, seems to present a lot of strange features that is not seen the original holographic descriptions of relativistic CFTs. In particular, it would be nice to have a well-defined prescription for studying holographic renormalization in such spacetimes.

Non-relativistic holography might be useful as a conceptual tool for studying RG flows in a non-relativistic system. Currently, theorems such as "$a-$theorem" which constrain the RG flow is applicable only to relativistic systems. Non-relativistic holography might be helpful in understanding the behavior of RG flows in a non-relativistic theory. For instance, one can check if there is an analog of the holographic c-theorem for non-relativistic domain wall flows. Such studies might shed light on the behavior of RG flows when a finite chemical potential is turned for some conserved current.

We also stumbled upon solutions which seems to exhibit interesting but strange features - a translationally system at finite density which is confining. At present it is not clear if this is a feature of the large N limit or if it is a feature of strong interactions. Familiar ideas in condensed matter theory seems to suggest that this *cannot* be described as an insulator. However, it might be interesting to see the consequences of breaking translation invariance in these systems.

Another interesting feature of some of the holographic models encountered in thesis is the appearance of dangerous irrelevant couplings as boundary parameters. It is a rather non-trivial task to understand the effects of dangerous irrelevant operators in a field theory. Though the prescription for identifying such operators from the gravity side is not clear, it is possible to use heuristic reasons along with familiar holographic prescriptions to identify the effects of such operators. It would be nice to get a better understanding of such operators using the holography.

As mentioned in the introduction, the studies on applications of AdS/CFT to condensed matter systems have led to the "discovery" of numerous gravitational instabilities. It would be interesting to see if similar instabilities exist in asymptotically flat (or deSitter) spacetime. FInding such instabilities can enhance our understanding of classical gravity atleast, if not quantum gravity.

Before the era of holography, physicists searched for new phenomena and answers to old puzzles under one light of perturbative QFT-this was the only lamp post known till the development of the holographic principle. We have an additional lampost now - the holographic lamp post! We might find answers to some of the old puzzles or find new interesting phenomena by searching under this new lamppost. Such a search can at least get rid of the boredom of searching under one single lamppost!

Bibliography

[1] Allan Adams, Koushik Balasubramanian, and John McGreevy. Hot Spacetimes for Cold Atoms. *JHEP*, 11:059, 2008.

[2] Allan Adams, Alexander Maloney, Aninda Sinha, and Samuel E. Vazquez. 1/N Effects in Non-Relativistic Gauge-Gravity Duality. *JHEP*, 03:097, 2009.

[3] Ian Affleck, Jeffrey A. Harvey, Laszlo Palla, and Gordon W. Semenoff. THE CHERN-SIMONS TERM VERSUS THE MONOPOLE. *Nucl.Phys.*, B328:575, 1989.

[4] Ofer Aharony, Oren Bergman, Daniel Louis Jafferis, and Juan Maldacena. $N = 6$ superconformal Chern-Simons-matter theories, M2-branes and their gravity duals. *JHEP*, 10:091, 2008.

[5] Ofer Aharony, Steven S. Gubser, Juan Martin Maldacena, Hirosi Ooguri, and Yaron Oz. Large N field theories, string theory and gravity. *Phys.Rept.*, 323:183–386, 2000.

[6] Mohsen Alishahiha and Ori J. Ganor. Twisted backgrounds, pp-waves and nonlocal field theories. *JHEP*, 03:006, 2003.

[7] Mohsen Alishahiha and Ori J. Ganor. Twisted backgrounds, pp-waves and nonlocal field theories. *JHEP*, 03:006, 2003.

[8] Mohsen Alishahiha, Yaron Oz, and Jorge G. Russo. Supergravity and light-like noncommutativity. *JHEP*, 09:002, 2000.

[9] Luis Alvarez-Gaume and Edward Witten. Gravitational Anomalies. *Nucl. Phys.*, B234:269, 1984.

[10] Irene Amado and Anton F. Faedo. Lifshitz black holes in string theory. *JHEP*, 1107:004, 2011.

[11] Eddy Ardonne, Paul Fendley, and Eduardo Fradkin. Topological order and conformal quantum critical points. *Annals Phys.*, 310:493–551, 2004.

[12] Adel Awad, Sumit R. Das, Archisman Ghosh, Jae-Hyuk Oh, and Sandip P. Trivedi. Slowly Varying Dilaton Cosmologies and their Field Theory Duals. *Phys. Rev.*, D80:126011, 2009.

[13] Adel Awad, Sumit R. Das, Suresh Nampuri, K. Narayan, and Sandip P. Trivedi. Gauge Theories with Time Dependent Couplings and their Cosmological Duals. *Phys. Rev.*, D79:046004, 2009.

[14] Adel Awad, Sumit R. Das, K. Narayan, and Sandip P. Trivedi. Gauge Theory Duals of Cosmological Backgrounds and their Energy Momentum Tensors. *Phys. Rev.*, D77:046008, 2008.

[15] Tatsuo Azeyanagi, Wei Li, and Tadashi Takayanagi. On String Theory Duals of Lifshitz-like Fixed Points. *JHEP*, 06:084, 2009.

[16] Koushik Balasubramanian and John McGreevy. Gravity duals for non-relativistic CFTs. *Phys. Rev. Lett.*, 101:061601, 2008.

[17] Koushik Balasubramanian and John McGreevy. An analytic Lifshitz black hole. *Phys. Rev.*, D80:104039, 2009.

[18] Koushik Balasubramanian and John McGreevy. String theory duals of Lifshitz-Chern-Simons gauge theories. 2011. arXiv:1111.0634[hep-th].

[19] Koushik Balasubramanian and John McGreevy. The Particle number in Galilean holography. *JHEP*, 1101:137, 2011. arXiv:1007.2184[hep-th].

[20] Koushik Balasubramanian and K. Narayan. Lifshitz spacetimes from AdS null and cosmological solutions. *JHEP*, 1008:014, 2010.

[21] Jose L. F. Barbon and Carlos A. Fuertes. On the spectrum of nonrelativistic AdS/CFT. *JHEP*, 09:030, 2008.

[22] Jose L. F. Barbon and Carlos A. Fuertes. On the spectrum of nonrelativistic AdS/CFT. *JHEP*, 09:030, 2008.

[23] Jose L. F. Barbon and Carlos A. Fuertes. Ideal gas matching for thermal Galilean holography. *Phys. Rev.*, D80:026006, 2009.

[24] M. Bartenstein et al. Crossover from a Molecular Bose-Einstein Condensate to a Degenerate Fermi Gas. *Phys. Rev. Lett.*, 92:120401, 2004.

[25] Aaron Bergman, Keshav Dasgupta, Ori J. Ganor, Joanna L. Karczmarek, and Govindan Rajesh. Nonlocal field theories and their gravity duals. *Phys. Rev.*, D65:066005, 2002.

[26] Aaron Bergman, Keshav Dasgupta, Ori J. Ganor, Joanna L. Karczmarek, and Govindan Rajesh. Nonlocal field theories and their gravity duals. *Phys. Rev.*, D65:066005, 2002.

[27] Oren Bergman, Amihay Hanany, Andreas Karch, and Barak Kol. Branes and supersymmetry breaking in three-dimensional gauge theories. *JHEP*, 9910:036, 1999.

[28] Micha Berkooz, Amit Sever, and Assaf Shomer. Double-trace deformations, boundary conditions and spacetime singularities. *JHEP*, 05:034, 2002.

[29] Gaetano Bertoldi, Benjamin A. Burrington, and Amanda Peet. Black Holes in asymptotically Lifshitz spacetimes with arbitrary critical exponent. *Phys. Rev.*, D80:126003, 2009.

[30] Danny Birmingham, Ivo Sachs, and Siddhartha Sen. Exact results for the BTZ black hole. *Int. J. Mod. Phys.*, D10:833–858, 2001.

[31] Johan Blaaback, Ulf H. Danielsson, and Thomas Van Riet. Lifshitz backgrounds from 10d supergravity. *JHEP*, 02:095, 2010.

[32] Matthias Blau, Jelle Hartong, and Blaise Rollier. Geometry of Schroedinger Space-Times, Global Coordinates, and Harmonic Trapping. *JHEP*, 07:027, 2009.

[33] Immanuel Bloch, Jean Dalibard, and Wilhelm Zwerger. Many-Body Physics with Ultracold Gases. *Rev. Mod. Phys.*, 80:885–964, 2008.

[34] Nikolay Bobev, Arnab Kundu, and Krzysztof Pilch. Supersymmetric IIB Solutions with Schrődinger Symmetry. *JHEP*, 07:107, 2009.

[35] Alex Buchel and James T. Liu. Universality of the shear viscosity in supergravity. *Phys. Rev. Lett.*, 93:090602, 2004.

[36] C. Charmousis, B. Gouteraux, B. S. Kim, E. Kiritsis, and R. Meyer. Effective Holographic Theories for low-temperature condensed matter systems. *JHEP*, 11:151, 2010.

[37] Wissam Chemissany and Jelle Hartong. From D3-Branes to Lifshitz Space-Times. *Class.Quant.Grav.*, 28:195011, 2011.

[38] Aleksey Cherman, Thomas D. Cohen, and Paul M. Hohler. A sticky business: the status of the cojectured viscosity/entropy density bound. *JHEP*, 02:026, 2008.

[39] Chong-Sun Chu and Pei-Ming Ho. Time-dependent AdS/CFT duality and null singularity. *JHEP*, 04:013, 2006.

[40] Chong-Sun Chu and Pei-Ming Ho. Time-dependent AdS/CFT Duality II: Holographic Reconstruction of Bulk Metric and Possible Resolution of Singularity. *JHEP*, 02:058, 2008.

[41] Thomas D. Cohen. Is there a 'most perfect fluid' consistent with quantum field theory? *Phys. Rev. Lett.*, 99:021602, 2007.

[42] Thomas D. Cohen. Response to D.T. Son's comment on "Is there a 'most perfect fluid' consistent with quantum field theory?". 2007.

[43] Keith Copsey and Robert B. Mann. Hidden Singularities and Closed Timelike Curves in A Proposed Dual for Lifshitz-Chern-Simons Gauge Theories. 2011. 3 pages, letter style, 1 figure.

[44] Miguel S. Costa, Carlos A.R. Herdeiro, J. Penedones, and N. Sousa. Hagedorn transition and chronology protection in string theory. *Nucl.Phys.*, B728:148–178, 2005.

[45] Csaba Csaki, Hirosi Ooguri, Yaron Oz, and John Terning. Glueball mass spectrum from supergravity. *JHEP*, 9901:017, 1999.

[46] Ulf H. Danielsson and Larus Thorlacius. Black holes in asymptotically Lifshitz spacetime. *JHEP*, 03:070, 2009.

[47] Sumit R. Das, Jeremy Michelson, K. Narayan, and Sandip P. Trivedi. Time dependent cosmologies and their duals. *Phys. Rev.*, D74:026002, 2006.

[48] Sumit R. Das, Jeremy Michelson, K. Narayan, and Sandip P. Trivedi. Cosmologies with Null Singularities and their Gauge Theory Duals. *Phys. Rev.*, D75:026002, 2007.

[49] Sumit R. Das and Sandip P. Trivedi. Three-brane action and the correspondence between N=4 Yang-Mills theory and anti-De Sitter space. *Phys.Lett.*, B445:142–149, 1998.

[50] Stanley Deser, R. Jackiw, and S. Templeton. Three-Dimensional Massive Gauge Theories. *Phys.Rev.Lett.*, 48:975–978, 1982.

[51] Aristomenis Donos and Jerome P. Gauntlett. Solutions of type IIB and D=11 supergravity with Schrodinger(z) symmetry. *JHEP*, 07:042, 2009.

[52] Aristomenis Donos and Jerome P. Gauntlett. Lifshitz Solutions of D=10 and D=11 supergravity. *JHEP*, 1012:002, 2010.

[53] C. Duval, G. Burdet, H. P. Kunzle, and M. Perrin. BARGMANN STRUCTURES AND NEWTON-CARTAN THEORY. *Phys. Rev.*, D31:1841, 1985.

[54] C. Duval, Gary W. Gibbons, and P. Horvathy. Celestial Mechanics, Conformal Structures, and Gravitational Waves. *Phys. Rev.*, D43:3907–3922, 1991.

[55] Michal Fabinger and John McGreevy. On smooth time-dependent orbifolds and null singularities. *JHEP*, 06:042, 2003.

[56] Thomas Faulkner, Hong Liu, and Mukund Rangamani. Integrating out geometry: Holographic Wilsonian RG and the membrane paradigm. *JHEP*, 1108:051, 2011. 21 pages, 2 figures/ references added, minor changes.

[57] S. Ferrara, M.A. Lledo, and A. Zaffaroni. Born-Infeld corrections to D3-brane action in AdS(5) x S(5) and N=4, d = 4 primary superfields. *Phys.Rev.*, D58:105029, 1998. Latex, 16 pages Report-no: CERN-TH/98-147, UCLA/98/TEP/13.

[58] Mitsutoshi Fujita, Wei Li, Shinsei Ryu, and Tadashi Takayanagi. Fractional Quantum Hall Effect via Holography: Chern-Simons, Edge States, and Hierarchy. *JHEP*, 0906:066, 2009.

[59] Davide Gaiotto and Edward Witten. Janus Configurations, Chern-Simons Couplings, And The theta-Angle in N=4 Super Yang-Mills Theory. *JHEP*, 1006:097, 2010.

[60] Ori J. Ganor, Akikazu Hashimoto, Sharon Jue, Bom Soo Kim, and Anthony Ndirango. Aspects of Puff Field Theory. *JHEP*, 0708:035, 2007.

[61] Jerome P. Gauntlett, Seok Kim, Oscar Varela, and Daniel Waldram. Consistent supersymmetric Kaluza–Klein truncations with massive modes. *JHEP*, 04:102, 2009.

[62] Jerome P. Gauntlett, Dario Martelli, James Sparks, and Daniel Waldram. Supersymmetric AdS(5) solutions of type IIB supergravity. *Class.Quant.Grav.*, 23:4693–4718, 2006.

[63] Pouyan Ghaemi, Ashvin Vishwanath, and T. Senthil. Finite-temperature properties of quantum lifshitz transitions between valence-bond solid phases: An example of local quantum criticality. *Phys. Rev. B*, 72:024420, Jul 2005.

[64] G. W. Gibbons, Gary T. Horowitz, and P. K. Townsend. Higher dimensional resolution of dilatonic black hole singularities. *Class. Quant. Grav.*, 12:297–318, 1995.

[65] Eric G. Gimon, Akikazu Hashimoto, Veronika E. Hubeny, Oleg Lunin, and Mukund Rangamani. Black strings in asymptotically plane wave geometries. *JHEP*, 08:035, 2003.

[66] Walter D. Goldberger. AdS/CFT duality for non-relativistic field theory. *JHEP*, 03:069, 2009.

[67] Ruth Gregory, Susha L. Parameswaran, Gianmassimo Tasinato, and Ivonne Zavala. Lifshitz solutions in supergravity and string theory. *JHEP*, 1012:047, 2010.

[68] G. Grignani, G.W. Semenoff, P. Sodano, and O. Tirkkonen. G/G models as the strong coupling limit of topologically massive gauge theory. *Nucl.Phys.*, B489:360–386, 1997.

[69] S. S. Gubser, Igor R. Klebanov, and Alexander M. Polyakov. Gauge theory correlators from non-critical string theory. *Phys. Lett.*, B428:105–114, 1998.

[70] Steven S. Gubser. Dilaton driven confinement. 1999.

[71] Steven S. Gubser. Curvature singularities: The good, the bad, and the naked. *Adv. Theor. Math. Phys.*, 4:679–745, 2000.

[72] Steven S. Gubser. Breaking an Abelian gauge symmetry near a black hole horizon. *Phys. Rev.*, D78:065034, 2008.

[73] Steven S. Gubser and Akikazu Hashimoto. Exact absorption probabilities for the D3-brane. *Commun.Math.Phys.*, 203:325–340, 1999.

[74] Steven S. Gubser and Fabio D. Rocha. Peculiar properties of a charged dilatonic black hole in AdS_5. *Phys. Rev.*, D81:046001, 2010.

[75] Monica Guica, Kostas Skenderis, Marika Taylor, and Balt C. van Rees. Holography for Schrodinger backgrounds. *JHEP*, 1102:056, 2011.

[76] Umut Gursoy. Continuous Hawking-Page transitions in Einstein-scalar gravity. *JHEP*, 01:086, 2011.

[77] C. R. Hagen. Scale and conformal transformations in galilean-covariant field theory. *Phys. Rev.*, D5:377–388, 1972.

[78] C. R. Hagen. Scale and conformal transformations in galilean-covariant field theory. *Phys. Rev.*, D5:377–388, 1972.

[79] Sean A. Hartnoll. Quantum Critical Dynamics from Black Holes. 2009.

[80] Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz. Building a Holographic Superconductor. *Phys. Rev. Lett.*, 101:031601, 2008.

[81] Sean A. Hartnoll, Christopher P. Herzog, and Gary T. Horowitz. Holographic Superconductors. *JHEP*, 12:015, 2008.

[82] Sean A. Hartnoll, Pavel K. Kovtun, Markus Muller, and Subir Sachdev. Theory of the Nernst effect near quantum phase transitions in condensed matter, and in dyonic black holes. *Phys. Rev.*, B76:144502, 2007.

[83] Sean A. Hartnoll, Joseph Polchinski, Eva Silverstein, and David Tong. Towards strange metallic holography. *JHEP*, 04:120, 2010.

[84] Sean A. Hartnoll and Kentaroh Yoshida. Families of IIB duals for nonrelativistic CFTs. *JHEP*, 12:071, 2008.

[85] S. W. Hawking and Don N. Page. Thermodynamics of Black Holes in anti-De Sitter Space. *Commun. Math. Phys.*, 87:577, 1983.

[86] S. W. Hawking and Simon F. Ross. Duality between Electric and Magnetic Black Holes. *Phys. Rev.*, D52:5865–5876, 1995.

[87] Idse Heemskerk and Joseph Polchinski. Holographic and Wilsonian Renormalization Groups. *JHEP*, 1106:031, 2011.

[88] Simeon Hellerman. A Universal Inequality for CFT and Quantum Gravity. *JHEP*, 08:130, 2011.

[89] Simeon Hellerman and Joseph Polchinski. Compactification in the lightlike limit. *Phys. Rev.*, D59:125002, 1999.

[90] Christopher P. Herzog, Mukund Rangamani, and Simon F. Ross. Heating up Galilean holography. *JHEP*, 11:080, 2008.

[91] P. C. Hohenberg and B. I. Halperin. Theory of Dynamic Critical Phenomena. *Rev. Mod. Phys.*, 49:435–479, 1977.

[92] Petr Horava and Charles M. Melby-Thompson. Anisotropic Conformal Infinity. *Gen.Rel.Grav.*, 43:1391–1400, 2011. 11 pages.

[93] R. M. Hornreich, Marshall Luban, and S. Shtrikman. Critical Behavior at the Onset of k $\rightarrow$ -Space Instability on the lamda Line. *Phys. Rev. Lett.*, 35:1678–1681, 1975.

[94] Gary T. Horowitz. Introduction to Holographic Superconductors. 2010.

[95] Gary T. Horowitz and Joseph Polchinski. Instability of spacelike and null orbifold singularities. *Phys. Rev.*, D66:103512, 2002.

[96] Gary T. Horowitz and Eva Silverstein. The Inside story: Quasilocal tachyons and black holes. *Phys.Rev.*, D73:064016, 2006.

[97] Benjamin Hsu, Michael Mulligan, Eduardo Fradkin, and Eun-Ah Kim. Universal entanglement entropy in 2D conformal quantum critical points. 2008.

[98] Veronika E. Hubeny and Mukund Rangamani. Causal structures of pp-waves. *JHEP*, 12:043, 2002.

[99] Veronika E. Hubeny, Mukund Rangamani, and Simon F. Ross. Causal structures and holography. *JHEP*, 07:037, 2005.

[100] Veronika E. Hubeny, Mukund Rangamani, and Simon F. Ross. Causal structures and holography. *JHEP*, 07:037, 2005.

[101] Veronika E. Hubeny, Mukund Rangamani, and Simon F. Ross. Causally pathological spacetimes are physically relevant. *Int. J. Mod. Phys.*, D14:2227–2232, 2005.

[102] Takaaki Ishii and Tatsuma Nishioka. Flows to Schrodinger Geometries. *Phys.Rev.*, D84:125007, 2011. 19 pages, 5 figures, v2: typos corrected, references added, published version in PRD.

[103] Nissan Itzhaki, Juan Martin Maldacena, Jacob Sonnenschein, and Shimon Yankielowicz. Supergravity and the large N limit of theories with sixteen supercharges. *Phys. Rev.*, D58:046004, 1998.

[104] R. Jackiw and So-Young Pi. Finite and infinite symmetries in (2+1)-dimensional field theory. 1992.

[105] Clifford V. Johnson, Amanda W. Peet, and Joseph Polchinski. Gauge theory and the excision of repulson singularities. *Phys.Rev.*, D61:086001, 2000.

[106] Shamit Kachru, Xiao Liu, and Michael Mulligan. Gravity Duals of Lifshitz-like Fixed Points. *Phys. Rev.*, D78:106005, 2008.

[107] A. Kehagias and K. Sfetsos. On Running couplings in gauge theories from type IIB supergravity. *Phys.Lett.*, B454:270–276, 1999.

[108] J. Kinast, S. L. Hemmer, M. E. Gehm, A. Turlapov, and J. E. Thomas. Evidence for Superfluidity in a Resonantly Interacting Fermi Gas. *Phys. Rev. Lett.*, 92:150402, 2004.

[109] Igor R. Klebanov, Silviu S. Pufu, and Tiberiu Tesileanu. Membranes with Topological Charge and AdS4/CFT3 Correspondence. *Phys. Rev.*, D81:125011, 2010.

[110] Igor R. Klebanov and Matthew J. Strassler. Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities. *JHEP*, 08:052, 2000.

[111] Igor R. Klebanov and Edward Witten. AdS/CFT correspondence and symmetry breaking. *Nucl. Phys.*, B556:89–114, 1999.

[112] Peter Koroteev and Maxim Libanov. On Existence of Self-Tuning Solutions in Static Braneworlds without Singularities. *JHEP*, 02:104, 2008.

[113] P. Kovtun, D. T. Son, and A. O. Starinets. Viscosity in strongly interacting quantum field theories from black hole physics. *Phys. Rev. Lett.*, 94:111601, 2005.

[114] Pavel Kovtun and Dominik Nickel. Black holes and non-relativistic quantum systems. *Phys. Rev. Lett.*, 102:011602, 2009.

[115] Pavel Kovtun, Dam T. Son, and Andrei O. Starinets. Holography and hydrodynamics: Diffusion on stretched horizons. *JHEP*, 10:064, 2003.

[116] Albion Lawrence. On the instability of 3D null singularities. *JHEP*, 11:019, 2002.

[117] Wei Li, Tatsuma Nishioka, and Tadashi Takayanagi. Some No-go Theorems for String Duals of Non-relativistic Lifshitz-like Theories. *JHEP*, 10:015, 2009.

[118] Wei Li, Wei Song, and Andrew Strominger. Chiral Gravity in Three Dimensions. *JHEP*, 04:082, 2008.

[119] Feng-Li Lin and Wen-Yu Wen. Supersymmetric null-like holographic cosmologies. *JHEP*, 05:013, 2006.

[120] Hong Liu, Gregory W. Moore, and Nathan Seiberg. Strings in time-dependent orbifolds. *JHEP*, 10:031, 2002.

[121] Hong Liu and Arkady A. Tseytlin. D = 4 super Yang-Mills, D = 5 gauged supergravity, and D = 4 conformal supergravity. *Nucl. Phys.*, B533:88–108, 1998.

[122] Kallingalthodi Madhu and K. Narayan. String spectra near some null cosmological singularities. *Phys. Rev.*, D79:126009, 2009.

[123] Juan Maldacena, Dario Martelli, and Yuji Tachikawa. Comments on string theory backgrounds with non- relativistic conformal symmetry. *JHEP*, 10:072, 2008.

[124] Juan Martin Maldacena. The large N limit of superconformal field theories and supergravity. *Adv. Theor. Math. Phys.*, 2:231–252, 1998.

[125] Juan Martin Maldacena. Wilson loops in large N field theories. *Phys.Rev.Lett.*, 80:4859–4862, 1998.

[126] Juan Martin Maldacena and Horatiu Stefan Nastase. The Supergravity dual of a theory with dynamical supersymmetry breaking. *JHEP*, 0109:024, 2001.

[127] Juan Martin Maldacena and Carlos Nunez. Towards the large N limit of pure N = 1 super Yang Mills. *Phys. Rev. Lett.*, 86:588–591, 2001.

[128] John McGreevy. Holographic duality with a view toward many-body physics. *Adv.High Energy Phys.*, 2010:723105, 2010.

[129] John McGreevy, Brian Swingle, and Ky-Anh Tran. Fractional Chern Insulators from the nth Root of Bandstructure. *Phys.Rev.*, B85:125105, 2012. 9+3 pages, 6 figures/ v2: added refs, amplified discussion of deconfinement, improved discussion of translation invariance.

[130] Thomas Mehen, Iain W. Stewart, and Mark B. Wise. Conformal invariance for non-relativistic field theory. *Phys. Lett.*, B474:145–152, 2000.

[131] Harvey B. Meyer. A calculation of the shear viscosity in SU(3) gluodynamics. *Phys. Rev.*, D76:101701, 2007.

[132] Harvey B. Meyer. A calculation of the bulk viscosity in SU(3) gluodynamics. *Phys. Rev. Lett.*, 100:162001, 2008.

[133] Harvey B. Meyer. Computing the viscosity of the QGP on the lattice. *Prog. Theor. Phys. Suppl.*, 174:220–227, 2008.

[134] Michael Mulligan, Chetan Nayak, and Shamit Kachru. An Infrared Free Non-Abelian Gauge Theory. *to appear soon.*

[135] Michael Mulligan, Chetan Nayak, and Shamit Kachru. An Isotropic to Anisotropic Transition in a Fractional Quantum Hall State. *Phys.Rev.*, B82:085102, 2010. 29 pages, 1 figure.

[136] U. Niederer. The maximal kinematical invariance group of the free Schrodinger equation. *Helv. Phys. Acta*, 45:802–810, 1972.

[137] Yusuke Nishida and Dam T. Son. Nonrelativistic conformal field theories. *Phys. Rev.*, D76:086004, 2007.

[138] Tatsuma Nishioka, Shinsei Ryu, and Tadashi Takayanagi. Holographic Superconductor/Insulator Transition at Zero Temperature. *JHEP*, 03:131, 2010.

[139] K. M. O'Hara, S. L. Hemmer, M. E. Gehm, S. R. Granade, and J. E. Thomas. Observation of a Strongly-Interacting Degenerate Fermi Gas of Atoms. *Science*, 298:2179–2182, 2002.

[140] Masaki Oshikawa. Insulator, conductor, and commensurability: A topological approach. *Phys. Rev. Lett.*, 90:236401, Jun 2003.

[141] Daniel S. Park. Graviton and Scalar Two-Point Functions in a CDL Background for General Dimensions. *JHEP*, 06:023, 2009.

[142] Amanda W. Peet, Leonard Susskind, and Larus Thorlacius. Tachyon hair on two-dimensional black holes. *Phys. Rev.*, D48:2415–2420, 1993.

[143] G. Policastro, D. T. Son, and A. O. Starinets. The shear viscosity of strongly coupled N = 4 supersymmetric Yang-Mills plasma. *Phys. Rev. Lett.*, 87:081601, 2001.

[144] Giuseppe Policastro, Dam T. Son, and Andrei O. Starinets. From AdS/CFT correspondence to hydrodynamics. *JHEP*, 09:043, 2002.

[145] Giuseppe Policastro, Dam T. Son, and Andrei O. Starinets. From AdS/CFT correspondence to hydrodynamics. II: Sound waves. *JHEP*, 12:054, 2002.

[146] Joris Raeymaekers. Chronology protection in stationary three-dimensional spacetimes. *JHEP*, 1111:024, 2011.

[147] C. A. Regal, M. Greiner, and D. S. Jin. Observation of Resonance Condensation of Fermionic Atom Pairs. *Phys. Rev. Lett.*, 92:040403, 2004.

[148] S.J. Rey and A. Zee. SELFDUALITY OF THREE-DIMENSIONAL CHERN-SIMONS THEORY. *Nucl.Phys.*, B352:897–921, 1991.

[149] Soo-Jong Rey and Jung-Tay Yee. Macroscopic strings as heavy quarks in large N gauge theory and anti-de Sitter supergravity. *Eur.Phys.J.*, C22:379–394, 2001.

[150] Simon F. Ross. Holography for asymptotically locally Lifshitz spacetimes. *Class.Quant.Grav.*, 28:215019, 2011.

[151] Simon F. Ross and Omid Saremi. Holographic stress tensor for non-relativistic theories. *JHEP*, 0909:009, 2009.

[152] Simon F. Ross and Omid Saremi. Holographic stress tensor for non-relativistic theories. *JHEP*, 09:009, 2009.

[153] Subir Sachdev. *Quantum Phase Transitions.* Cambridge University Press, Cambridge, England, 1999.

[154] Makoto Sakaguchi and Kentaroh Yoshida. Super Schroedinger algebra in AdS/CFT. *J. Math. Phys.*, 49:102302, 2008.

[155] Tadakatsu Sakai and Shigeki Sugimoto. Low energy hadron physics in holographic QCD. *Prog. Theor. Phys.*, 113:843–882, 2005.

[156] D. J. Scalapino, S. R. White, and S. C. Zhang. Superfluid density and the drude weight of the hubbard model. *Phys. Rev. Lett.*, 68:2830–2833, May 1992.

[157] Thomas Schafer. The Shear Viscosity to Entropy Density Ratio of Trapped Fermions in the Unitarity Limit. *Phys. Rev.*, A76:063618, 2007.

[158] Kostas Skenderis. Lecture notes on holographic renormalization. *Class.Quant.Grav.*, 19:5849–5876, 2002.

[159] D T. Son. Comment on 'Is There a 'Most Perfect Fluid' Consistent with Quantum Field Theory?'. *Phys. Rev. Lett.*, 100:029101, 2008.

[160] D. T. Son. Toward an AdS/cold atoms correspondence: a geometric realization of the Schroedinger symmetry. *Phys. Rev.*, D78:046003, 2008.

[161] D. T. Son and M. Wingate. General coordinate invariance and conformal invariance in nonrelativistic physics: Unitary Fermi gas. *Annals Phys.*, 321:197–224, 2006.

[162] Dam T. Son and Andrei O. Starinets. Minkowski-space correlators in AdS/CFT correspondence: Recipe and applications. *JHEP*, 09:042, 2002.

[163] Dam T. Son and Andrei O. Starinets. Viscosity, Black Holes, and Quantum Field Theory. *Ann. Rev. Nucl. Part. Sci.*, 57:95–118, 2007.

[164] Marika Taylor. Holography for degenerate boundaries. 2000.

[165] Marika Taylor. Non-relativistic holography. 2008.

[166] Wen-Yu Wen. AdS/NRCFT for the (super) Calogero model. 2008.

[167] Edward Witten. String theory dynamics in various dimensions. *Nucl. Phys.*, B443:85–126, 1995.

[168] Edward Witten. Anti-de Sitter space and holography. *Adv. Theor. Math. Phys.*, 2:253–291, 1998.

[169] Edward Witten. Anti-de Sitter space, thermal phase transition, and confinement in gauge theories. *Adv. Theor. Math. Phys.*, 2:505–532, 1998.

[170] Edward Witten. Multi-trace operators, boundary conditions, and AdS/CFT correspondence. 2001.

[171] Edward Witten and Jonathan Bagger. Quantization of Newton's Constant in Certain Supergravity Theories. *Phys. Lett.*, B115:202, 1982.

[172] Daiske Yamada. Thermodynamics of Black Holes in Schroedinger Space. *Class. Quant. Grav.*, 26:075006, 2009.

[173] M. W. Zwierlein et al. Condensation of Pairs of Fermionic Atoms near a Feshbach Resonance. *Phys. Rev. Lett.*, 92:120403, 2004.