14.461 Part II
Problem Set 1, Solutions

Fall 2009

1 Wage Dispersion

This problem extends the search model with random matching and Nash bargaining seen in class to allow for match-specific productivity. This simple extension will be able to generate wage dispersion.

Time is discrete and horizon infinite. There is a continuum of risk-neutral ex-ante homogeneous workers of measure 1 and a continuum of larger measure of risk-neutral ex-ante homogeneous firms. They have common discount factor β. Workers can search freely, while firms have to pay a cost k to open a vacancy. At the beginning of the period firms post vacancies and workers search for a job. Then matching takes place according to a standard constant returns to scale matching function. Let $\mu(\theta_t)$ denote the probability a worker meets a firm and $\mu(\theta_t)/\theta_t$ the probability a firm meets a worker, where θ_t is the market tightness. Assume $\mu(\theta)$ is continuous and twice differentiable with $\mu'(\theta) > 0$ and $\mu''(\theta) < 0$ for all $\theta \in [0, \infty)$. Also, assume $\mu(\theta) \leq \min\{\theta, 1\}$. When a firm and a worker meet, they draw a match-specific productivity y from a distribution $F(\cdot)$ with full support on $Y \equiv [\underline{y}, \overline{y}]$, where y is observed by both and constant until separation, that happens with probability s. Assume that $F(.)$ is differentiable, with $f(.)$ denoting the associated density function. After observing y, the worker and the firm bargain the wage w_t according to generalized Nash bargaining. If a worker is unemployed (either because is not matched or because walk away from the match) he gets b (home production) and can search next period.

1. Write down the value functions for an unemployed worker (U_t), an employed worker in a match producing y ($V_t(y)$), a firm with an unfilled vacancy (W_t), and a firm with a filled vacancy and productivity y ($J_t(y)$). All the value functions are evaluated at the beginning of the period before matching and separation take place. Use $e_t(y) \in \{0,1\}$ to denote whether a job is created after observing y.

   ANSWER.

$$\begin{aligned}
U_t &= \mu(\theta_t)\left(\int [w_t(y) + \beta V_{t+1}(y)] e_t(y) + (1 - e_t(y))(b + \beta U_{t+1}) dF(y)\right) \\
&\quad + (1 - \mu(\theta_t))(b + \beta U_{t+1}) \\
V_t(y) &= s(b + \beta U_{t+1}) + (1 - s)(w_t(y) + \beta V_{t+1}(y)) \\
W_t &= -k + \frac{\mu(\theta_t)}{\theta_t}\left(\int e_t(y)[y - w_t(y) + \beta J_{t+1}(y)] dF(y)\right) \\
&\quad + \left(1 - \frac{\mu(\theta_t)}{\theta_t}\right)\beta \max\{W_{t+1}, 0\} \\
J_t(y) &= s\beta \max\{0, W_{t+1}\} + (1 - s)(y - w_t(y) + \beta J_{t+1}(y))
\end{aligned}$$

2. Focus on the steady state equilibrium. Write down the generalized Nash bargaining problem, assuming that the worker has bargaining power η, and show how the surplus is split among the worker and the firm. Do all matches lead to job creation?

   ANSWER.

$$U = \mu(\theta)\left(\int [w(y) + \beta V(y)] e(y) dF(y)\right) + \left(1 - \mu(\theta)\int e(y) dF(y)\right)(b + \beta U) \tag{1}$$

$$V(y) = \frac{s(b + \beta U) + (1 - s) w(y)}{1 - \beta(1 - s)} \tag{2}$$

$$k = \frac{\mu(\theta)}{\theta}\left(\int [y - w(y) + \beta J(y)] e(y) dF(y)\right) \tag{3}$$

$$J(y) = \frac{(1 - s)(y - w(y))}{1 - \beta(1 - s)} \tag{4}$$

Fix y. Then **Nash Bargaining** require

$$w(y) = \max_{\omega}\left[\omega + \beta\tilde{V}(\omega; y) - b - \beta U\right]^{\eta}\left[y - \omega + \beta\tilde{J}(\omega; y)\right]^{1-\eta}.$$

The relationship is going to be consummated if

$$\omega + \beta\tilde{V}(\omega; y) \geq b + \beta U, \tag{5}$$

and

$$y - \omega + \beta \tilde{J}(\omega; y) \geq 0, \tag{6}$$

for some ω so that there is something to bargain over.

Using the equation for $\tilde{V}(\omega; y)$,the first inequality is equivalent to

$$\omega \geq (1 - \beta)(b + \beta U),$$

and using the equation for $\tilde{J}(\omega; y)$ the second inequality is equivalent to

$$y - \omega \geq 0 \text{ or } \omega \leq y,$$

A necessary and sufficient condition for the existence of a ω satisfying (5) and (6) is thus

$$y \geq (1 - \beta)(b + \beta U),$$

Then $e_t(y) = 1$ iff $y \geq \hat{y}$, where

$$\hat{y} = (1 - \beta)(b + \beta U).$$

An equivalent way to see this consists of computing the (steady state) surplus of the relationship as a function of y, which can be shown to be

$$S(y) = \frac{y - (1 - \beta)(b + \beta U)}{1 - (1 - s)\beta}$$

This has a simple interpretation: y is the per-period benefit of the match, while $(1 - \beta)(b + \beta U)$ is the per-period cost of the match (best alternative use). Only when the former exceeds the latter the match will create surplus.

3. Find the steady state equilibrium. In particular, characterize (as far as you can) the steady state job creation and wages.

ANSWER.

Using the cutoff rule for $e(y)$ found above, (3) and (4) imply

$$k = \frac{\mu(\theta)}{\theta} \frac{\int_{\hat{y}}^{\bar{y}} [y - w(y)]\, dF(y)}{1 - \beta(1 - s)}$$

From bargaining

$$w(y) = (1-\beta)(b+\beta U) + \eta[y - (1-\beta)(b+\beta U)]$$

Also, using the cutoff rule on (1) gives

$$(1-\beta)U = b + \mu(\theta)\left(\int_{\hat{y}}^{\bar{y}} [w(y) - b + \beta(V(y) - U)]\, dF(y)\right)$$

and then

$$(1-\beta)U = b + \frac{\eta}{1-\eta}\theta k,$$

so that

$$w(y) = \eta(y + \beta\theta k) + (1-\eta)b. \tag{7}$$

$$k = \frac{\mu(\theta)}{\theta}\frac{\int_{\hat{y}}^{\bar{y}} [(1-\eta)(y-b) - \eta\beta\theta k]\, dF(y)}{1-\beta(1-s)}$$

Then we can solve the following two equations in θ and $\hat{y}$

$$\hat{y} = b + \frac{\eta}{1-\eta}\beta\theta k.$$

and

$$k = \frac{\mu(\theta)}{\theta}\frac{\int_{\hat{y}}^{\bar{y}} [(1-\eta)(y-b) - \eta\beta\theta k]\, dF(y)}{1-\beta(1-s)}.$$

4. Recover the steady state wage distribution. How does it vary with an increase in b?

   ANSWER. The steady state wage distribution is computed from $F(y)$ and the equation that characterizes wages as a function of y (7), for $y > \hat{y}$ (otherwise, we observe no wages). An increase in b has three effects on this distribution: a direct effect (through an increase in the reservation wage) of increaseing $w(y)$ for every y, and two indirect effects, one through θ and another through the truncation point $\hat{y}$.

   The steady state level of θ is characterized by

$$k = \frac{\mu(\theta)}{\theta}\frac{\int_{b+\frac{\eta}{1-\eta}\beta\theta k}^{\bar{y}} [(1-\eta)(y-b) - \eta\beta\theta k]\, dF(y)}{1-\beta(1-s)}$$

   It can be shown using the implicit function th., that $\partial\theta/\partial b < 0$. Hence, both the effect on $\hat{y}$ and on $w(y)$ are ambigous - depend on parameters.

5. Write down the Planner problem. Is the equilibrium constrained efficient?

ANSWER.

$$\begin{aligned}
P(u) &= \max_{u',\theta,e(y)} u\left(\mu(\theta)\int_{\underline{y}}^{\bar{y}} e(y)(y-b)\,dF(y)+b-\theta k\right)+(1-u)sb+ \\
&\quad +(1-u)(1-s)\frac{\int_{\underline{y}}^{\bar{y}} e(y)\,y\,dF}{\int_{\underline{y}}^{\bar{y}} e(y)\,dF}+\beta P(u') \\
&\quad \text{s.t.} \\
u' &= u\left(\mu(\theta)\int_{\underline{y}}^{\bar{y}}(1-e(y))\,dF(y)+1-\mu(\theta)\right)+(1-u)s
\end{aligned}$$

Taking pointwise derivative wrt $e(y)$

$$u\mu(\theta)(y-b)f(y)+(1-u)(1-s)\left[\frac{yf(y)\int_{\underline{y}}^{\bar{y}} e(y)\,dF-f(y)\int_{\underline{y}}^{\bar{y}} e(y)\,y\,dF}{\left(\int_{\underline{y}}^{\bar{y}} e(y)\,dF\right)^2}\right]-\lambda u\mu(\theta)f(y)$$

Even though this derivative does depend on $e(y)$, it is still true that $e(y)$ has a cutoff form. So $e(y)=1$ iff $y>\hat{y}$, where $\hat{y}$

$$u\mu(\theta)(\hat{y}-b)+(1-u)(1-s)\left[\frac{\hat{y}\int_{\underline{y}}^{\bar{y}} e(y)\,dF-\int_{\underline{y}}^{\bar{y}} e(y)\,y\,dF}{\left(\int_{\underline{y}}^{\bar{y}} e(y)\,dF\right)^2}\right]=\lambda u\mu(\theta)$$

The other FOC (already using the cutoff form for $e(y)$) :

$$\begin{aligned}
\beta P_u(u') &= \lambda \\
\mu'(\theta)\left[\int_{\hat{y}}^{\bar{y}}(y-b)\,dF(y)-(1-F(\hat{y}))\lambda\right] &= k
\end{aligned}$$

The Envelope Condition is

$$\begin{aligned}
P_u(u) &= \mu(\theta)\int_{\hat{y}}^{\bar{y}}(y-b)\,dF(y)+b-\theta k-sb-(1-s)\frac{\int_{\hat{y}}^{\bar{y}} y\,dF}{1-F(\hat{y})} \\
&\quad +\beta P_u(u')\left(\mu(\theta)F(\hat{y})+1-\mu(\theta)-s\right)
\end{aligned}$$

It is easy to see that these conditions are substantially different from the conditions characterizing the equilibrium.

2 Firms' Superior Information

This problem consider a competitive search model with asymmetric information and firms' limited liability similar to the one we have seen in class. The main difference is that now firms have private information.

Consider a competitive search model with the same preferences and technology described above, with the difference that match-specific productivity is now private information of the firm. Moreover, we assume that there is limited liability on the firm side, that is, a firm can always decide to go bankrupt, fire the worker and stop paying them, with no punishment. At the beginning of each period t, firms can open a vacancy at cost k which entitles them to post an employment contract C_t. Then workers observe all the posted contracts and decide where to apply. Let $\Theta_t(C_t)$ be the market tightness associated to contract C_t. Let $\mu(\Theta_t(C_t))$ denote the probability a worker applying for C_t meets a firm and $\mu(\Theta_t(C_t))/\Theta_t(C_t)$ the probability a firm posting C_t meets a worker. After a match is formed, y is drawn and observed by the firm. Assume also that $F(.)$ satisfies the monotone hazard rate condition $d(F(y)/f(y))/dy > 0$. Given firms' limited liability, firms will never make payments to workers they do not hire. If a worker is hired the match is productive until separation which happens with probability s. Finally, if a worker is not matched or not hired, he gets b and search next period.

1. Firms can post unrestricted contracts. Using the revelation principle, you can restrict attention to incentive-compatible and individually-rational direct mechanisms. How can you represent such a contract? Write down the form of the contract and the IC and IR constraints (for whom?) that such a contract has to satisfy.

   ANSWER. A contract is a pair of functions:

   $$C_t = \{e_t(y), \omega_t(y)\}$$

   The firm's expected utility conditional on matching, reporting $\tilde{y}$ and obtaining y is

   $$v_t(y, \tilde{y}) = e_t(\tilde{y})\, y - \omega_t(\tilde{y})$$

   IC is $v_t(y, y) \geq v_t(y, \tilde{y})$ for all $y, \tilde{y}$. **IR** is $v_t(y, y) \geq 0$ for all y.

The analog of **Lemma 1** here is: Conditions IC and IR are equivalent to

$$
\begin{aligned}
v_t(y,y) &= v(\underline{y},\underline{y}) + \int_{\underline{y}}^{y} e(z)\,dz && (8)\\
v_t(\underline{y},\underline{y}) &\geq 0 && (9)\\
& e_t(y) \text{ non-decreasing} && (10)
\end{aligned}
$$

Proof (only one way). Take $y > \tilde{y}$

$$
\begin{aligned}
v(y,y) &\geq v(y,\tilde{y})\\
v(\tilde{y},\tilde{y}) &\geq v(\tilde{y},y)
\end{aligned}
$$

$$
\begin{aligned}
v(y,y) - v(\tilde{y},y) &\geq v(y,y) - v(\tilde{y},\tilde{y}) \geq v(y,\tilde{y}) - v(\tilde{y},\tilde{y})\\
e(y)(y-\tilde{y}) &\geq v(y,y) - v(\tilde{y},\tilde{y}) \geq e(\tilde{y})(y-\tilde{y})\\
e(y) &\geq \frac{v(y,y) - v(\tilde{y},\tilde{y})}{y-\tilde{y}} \geq e(\tilde{y})
\end{aligned}
$$

which means e must be non-decreasing. Write $y \equiv \tilde{y} + \Delta$, and let $g(y) \equiv v(y,y)$

$$
e(\tilde{y}+\Delta) \geq \frac{g(\tilde{y}+\Delta) - g(\tilde{y})}{\Delta} \geq e(\tilde{y})
$$

Taking $\Delta \to 0$

$$
g'(z) = e(z)
$$

Then

$$
\begin{aligned}
\int_{\underline{y}}^{y} g'(z)\,dz &= \int_{\underline{y}}^{y} e(z)\,dz\\
v(y,y) - v(\underline{y},\underline{y}) &= \int_{\underline{y}}^{y} e(z)\,dz
\end{aligned}
$$

2. Define a Competitive Search Equilibrium, restricting attention to the set of incentive-compatible and individually-rational contracts.

ANSWER.

Definition. A CSE is a sequence of IC and IR contracts $\{\mathbb{C}_t\}_{t=0}^{\infty}$, a sequence of functions $\{\Theta_t(\cdot)\}_{t=0}^{\infty}$ where $\Theta_t : \Omega_t \to \mathbb{R}_+$ and a bounded sequence of continuation utilities $\{U_t, V_t\}_{t=0}^{\infty}$ such that

(a) firm's profit maximization and free-entry

$$\frac{\mu\left(\Theta_t\left(C_t\right)\right)}{\Theta_t\left(C_t\right)}\int_{\underline{y}}^{\bar{y}}\left(e_t\left(y\right)y-w_t\left(y\right)\right)dF\left(y\right)-k\leq 0$$

with equality if $C_t \in \mathbb{C}_t$

(b) worker's optimal application at all t: for any $C_t \in \Omega_t$

$$\mu\left(\Theta_t\left(C_t\right)\right)\int_{\underline{y}}^{\bar{y}}\left(\omega_t\left(y\right)-e_t\left(y\right)\left(b+\beta\left(U_{t+1}-V_{t+1}\right)\right)\right)dF\left(y\right)+b+\beta U_{t+1}\leq U_t$$

with equality if $\Theta_t\left(C_t\right)<\infty$, where

$$U_t=\max_{C_t'\in\mathbb{C}_t}\mu\left(\Theta_t\left(C_t'\right)\right)\int_{\underline{y}}^{\bar{y}}\left(\omega_t'\left(y\right)-e_t'\left(y\right)\left(b+\beta\left(U_{t+1}-V_{t+1}\right)\right)\right)dF\left(y\right)+b+\beta U_{t+1}$$

or $U_t=b+\beta U_{t+1}$ if $\mathbb{C}_t$ is empty, and

$$V_t=s\left(b+\beta U_{t+1}\right)+\left(1-s\right)\beta V_{t+1}$$

3. Show if it is possible to characterize any competitive search equilibrium with a constrained optimization problem.

ANSWER.

Proposition. If $\{\mathbb{C}_t,\Theta_t\left(\cdot\right),U_t,V_t\}_{t=0}^{\infty}$ is a CSE, then any pair $\left(C_t,\theta\right)$ with $C_t\in\mathbb{C}_t$ and $\Theta_t\left(C_t\right)=\theta$ solves (call this problem **P1**)

$$U_t=\max_{e_t(\cdot),\omega_t(\cdot),\theta_t}\mu\left(\theta_t\right)\int_{\underline{y}}^{\bar{y}}\left[\omega_t\left(y\right)-e_t\left(y\right)\left(b+\beta\left(U_{t+1}-V_{t+1}\right)\right)\right]dF\left(y\right)+b+\beta U_{t+1}$$

subject to IC', IR', $e_t\left(y\right)\in[0,1]$ and non-decreasing for all $y\in Y$, and

$$\frac{\mu\left(\theta_t\right)}{\theta_t}\int_{\underline{y}}^{\bar{y}}\left(e_t\left(y\right)y-\omega_t\left(y\right)\right)dF\left(y\right)-k=0.$$

Proof. Analog to proposition 4 in the paper.

We can further simplify this problem by eliminating the wage schedule. Take expectations of IC'

$$\int_{\underline{y}}^{\bar{y}}v\left(y,y\right)dF-v\left(\underline{y},\underline{y}\right)=\int_{\underline{y}}^{\bar{y}}\int_{\underline{y}}^{y}e\left(z\right)dzdF$$

Using integration by parts

$$\begin{aligned}\int_{\underline{y}}^{\bar{y}} \left(\int_{\underline{y}}^{y} e(z)\, dz \right) f(y)\, dy &= F(y) \left(\int_{\underline{y}}^{y} e(z)\, dz \right) \Big|_{\underline{y}}^{\bar{y}} - \int_{\underline{y}}^{\bar{y}} e(y) F(y)\, dy \\ &= \int_{\underline{y}}^{\bar{y}} e(y) \left(\frac{1 - F(y)}{f(y)} \right) dF(y)\end{aligned}$$

Then

$$\begin{aligned}\int_{\underline{y}}^{\bar{y}} v(y, y)\, dF &= v(\underline{y}, \underline{y}) + \int_{\underline{y}}^{\bar{y}} e(y) \left(\frac{1 - F(y)}{f(y)} \right) dF(y) \\ &\geq \int_{\underline{y}}^{\bar{y}} e(y) \left(\frac{1 - F(y)}{f(y)} \right) dF(y)\end{aligned}$$

Thus,

$$\mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left(e(y) \left\{ y - \frac{1 - F(y)}{f(y)} \right\} - w(y) \right) dF \geq 0$$

From the zero profit condition

$$\mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \omega_t(y)\, dF(y) = \mu(\theta_t) \int_{\underline{y}}^{\bar{y}} e_t(y)\, y dF(y) - \theta_t k$$

Hence we can eliminate the wage schedule to reach the following maximization problem (call it **P2**):

$$U_t = \max_{e_t(y), \theta_t} \mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left[e_t(y) \left(y - b - \beta \left(U_{t+1} - V_{t+1} \right) \right) \right] dF(y) - \theta_t k + b + \beta U_{t+1}$$

subject to

$$\mu(\theta_t) \int_{\underline{y}}^{\bar{y}} e(y) \left\{ \frac{1 - F(y)}{f(y)} \right\} dF \leq \theta_t k$$

Claim. A solution to P1 has to solve P2. And given a solution to P2 we can construct a wage schedule that (together with the original solution to P2) constitutes a solution to P1.

Thus, to characterize the form of e($\cdot$) in a CSE we can directly work with P2.

Consider the derivative wrt $e_t(y)$

$$\mu(\theta_t) \left[y - b - \beta \left(U_{t+1} - V_{t+1} \right) \right] f(y) - \mu(\theta_t) \lambda \frac{1 - F(y)}{f(y)} f(y)$$

Note this is independent of $e(y)$ so that a corner solution is always chosen. Note also that assuming $\frac{1-F(y)}{f(y)}$ decreasing in y, this derivative is increasing in y. Thus, $e(\cdot)$ must take a cutoff form:

$$e_t(y) = \begin{cases} 0 & y < \hat{y} \\ 1 & y \geq \hat{y} \end{cases}$$

where

$$\hat{y} - b - \beta(U_{t+1} - V_{t+1}) - \lambda \frac{1 - F(\hat{y})}{f(\hat{y})} = 0$$

Plugging this cutoff form for $e(\cdot)$ into P2, we obtain the following maximization problem (call it **P3**)

$$U_t = \max_{\hat{y},\theta_t} \mu(\theta_t) \int_{\hat{y}}^{\overline{y}} [y - b - \beta(U_{t+1} - V_{t+1})] \, dF(y) - \theta_t k + b + \beta U_{t+1}$$

subject to

$$\mu(\theta_t) \int_{\hat{y}}^{\overline{y}} \frac{1 - F(y)}{f(y)} dF \leq \theta_t k$$

4. Show that there exists a CSE. Is this equilibrium unique?

   ANSWER. Start from $P3$, substract $V_t \ (= s(b + \beta U_{t+1}) + (1-s)\beta V_{t+1})$, and define $D_t \equiv U_t - V_t$.

$$D_t = \max_{\hat{y},\theta_t} \mu(\theta_t) \int_{\hat{y}}^{\overline{y}} [y - b - \beta D_{t+1}] \, dF(y) - \theta_t k + (1-s)(b + \beta D_{t+1})$$

   subject to

$$\mu(\theta_t) \int_{\hat{y}}^{\overline{y}} \frac{1 - F(y)}{f(y)} dF \leq \theta_t k$$

   Thus, the **CSE** can be characterized as follows:

   (a) For given $D_{t+1} \equiv U_{t+1} - V_{t+1}$, $(\hat{y}, \theta_t)$ must solve (call this **P4**)

$$\begin{aligned} \Phi(D_{t+1}) &\equiv \max_{\hat{y},\theta_t} \mu(\theta_t) \int_{\hat{y}}^{\overline{y}} [y - b - \beta D_{t+1}] \, dF(y) - \theta_t k \\ \mu(\theta_t) \int_{\hat{y}}^{\overline{y}} \frac{1 - F(y)}{f(y)} dF &\leq \theta_t k \end{aligned}$$

   (b) For given $\{\hat{y}, \theta_t\}_{t=0}^{\infty}$, the sequence $\{D_t\}_{t=1}^{\infty}$ must satisfy:

$$D_t = \Phi(D_{t+1}) + (1-s)(b + \beta D_{t+1})$$

Define $H(D) \equiv \Phi(D) + (1-s)(b+\beta D)$.

To show existence it is sufficient to show there exists a fixed point $H(D^*) = D^*$. As in the paper we use the intermediate value theorem. We need to find a point in which $H(D) - D$ is positive, and point in which is negative. The first point is zero

$$H(0) = \Phi(0) + (1-s)\, b > 0$$

As for the second point, consider the upper bound on D given by $(\bar{y} - b)/\beta$ (this comes from taking the upper bound for U and substracting a lower bound for V equal to zero). Then

$$\begin{aligned} H\left(\frac{\bar{y}-b}{\beta}\right) - \frac{\bar{y}-b}{\beta} &= \Phi\left(\frac{\bar{y}-b}{\beta}\right) + (1-s)\,\bar{y} - \frac{\bar{y}-b}{\beta} \\ &= (1-s)\,\bar{y} - \frac{\bar{y}-b}{\beta} < 0 \end{aligned}$$

since $\bar{y}(1-\beta(1-s)) > b$ (an assumption to make the problem interesting), and $\Phi\left(\frac{\bar{y}-b}{\beta}\right) = 0$ (since the solution of the problem for such a high D is $\theta = 0$ and $\hat{y} = \bar{y}$). This proves existence.

For uniqueness we require that H is a contraction. A sufficient condition for H to be a contraction with modulus β is that $|H'(D)| \leq \beta$.

$$H'(D) = \Phi'(D) + (1-s)\,\beta$$

From P4 we have

$$\Phi'(D) = -\mu(\theta_t)\,\beta\,(1 - F(\hat{y}))$$

which implies $H'(D) < \beta$. As for $H'(D) > -\beta$, we need

$$\Phi'(D) + (1-s)\,\beta > -\beta$$

5. Consider a social planner who does not observe y and cannot force firms to make transfers to the workers if they do not hire them (that is, the planner faces also a limited liability constraint on the firms' side). Write down the sequence problem that defines the region of the Pareto frontier where the firms receive zero utility.

ANSWER. An allocation is $\left\{e_t(y), c_t(y), c_t^U, \theta_t\right\}_{t=0}^{\infty}$. Redefining $v_t(y, \tilde{y})$ as

$$v_t(y, \tilde{y}) = e_t(\tilde{y})\, y - c_t(\tilde{y})$$

the previous expressions for IC' and IR' still hold.

Given initial values (U_0, V_0) and an allocation, the following difference equations characterize the evolution of the values of being employed and unemployed (U_t, V_t)

$$U_t = \mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left(c_t(y) - e_t(y)\left(b + \beta\left(U_{t+1} - V_{t+1}\right)\right)\right) dF(y) + b + \beta U_{t+1} \quad (11)$$

$$V_t = s\left(b + \beta U_{t+1}\right) + (1-s)\beta V_{t+1} \quad (12)$$

The law of motion for unemployments is

$$u_{t+1} = u_t \left(1 - \mu(\theta_t) \int_{\underline{y}}^{\bar{y}} e_t(y)\, dF(y)\right) + (1 - u_t)\, s \quad (13)$$

The intertemporal budget constraint (**IBC**) is

$$0 \le \sum_{t=0}^{\infty} \beta^t \Bigg\{ u_t \left[\mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left(e_t(y)\, y + (1 - e_t(y))\left(b - c_t^U\right) - c_t(y)\right) dF(y) + (1 - \mu(\theta_t))\left(b - c_t^U\right)\right]$$
$$-u_t \theta_t k + (1 - \mu_t)\, s\left(b - c_t^U\right)\Bigg\}$$

or

$$0 \le \sum_{t=0}^{\infty} \beta^t \Bigg\{ u_t \left[\mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left(e_t(y)\left(y - b + c_t^U\right) - c_t(y)\right) dF(y) + b - c_t^U\right]$$
$$-u_t \theta_t k + (1 - \mu_t)\, s\left(b - c_t^U\right)\Bigg\}$$

An allocation is **feasible** if it satisfies the law of motion for u_t (13) , the **IBC** and IC', IR' and monotonicity:

$$v_t(y, y) = v\left(\underline{y}, \underline{y}\right) + \int_{\underline{y}}^{y} e(z)\, dz$$

$$v_t\left(\underline{y}, \underline{y}\right) = 0$$

$$e_t(y) \text{ non-decreasing}$$

Given u_0, an allocation $\left\{e_t(y), c_t(y), c_t^U, \theta_t\right\}_{t=0}^{\infty}$ is constrained efficientt if it solves the following sequence problem

$$\begin{aligned} & \max_{\left\{e_t(y),c_t(y),c_t^U,\theta_t\right\}_{t=0}^{\infty}} U_0 \\ \text{subject to } V_0 \geq \; & \bar{V} \text{ , Feasibility and} \\ & (11) \text{ and } (12) \end{aligned}$$

6. **OPTIONAL.** Write down a recursive version of the planner problem. Is the equilibrium constrained efficient?

ANSWER.

$$\begin{aligned} P(U_t, V_t, u_t) = & \max_{e_t(y),c_t(y),c_t^U,\theta_t} u_t \left[\mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left(e_t(y)\left(y - b + c_t^U\right) - c_t(y)\right) dF(y) + b - c_t^U - \theta_t k\right] \\ & + (1 - \mu_t) s \left(b - c_t^U\right) + \beta P(U_{t+1}, V_{t+1}, u_{t+1}) \end{aligned}$$

subject to

$$\begin{aligned} u_{t+1} &= u_t \left(1 - \mu(\theta_t) \int_{\underline{y}}^{\bar{y}} e_t(y) \, dF(y)\right) + (1 - u_t) s \\ U_t &= \mu(\theta_t) \int_{\underline{y}}^{\bar{y}} \left(c_t(y) - e_t(y)\left(b + \beta\left(U_{t+1} - V_{t+1}\right)\right)\right) dF(y) + b + \beta U_{t+1} \\ V_t &= s\left(b + \beta U_{t+1}\right) + (1 - s) \beta V_{t+1} \\ v_t(y, y) &= v\left(\underline{y}, \underline{y}\right) + \int_{\underline{y}}^{y} e(z) \, dz, \; v_t\left(\underline{y}, \underline{y}\right) = 0 \text{ and } e_t(y) \text{ non-decreasing} \end{aligned}$$

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