14.461 Part II
Problem Set 2, Solutions

Fall 2009

1 Unemployment Insurance and Saving

This problem studies unemployment insurance in the presence of a moral hazard problem, when agents are allowed to privately save (based on Shimer and Werning).

There is a representative risk-averse worker who maximizes his expected utility from consumption

$$E\sum_{t=0}^{\infty}\beta^{t}u\left(c_{t}\right),$$

with u increasing and concave. At time 0 the worker is unemployed. Each period an unemployed worker draws a wage from the cumulative distribution function $F\left(\cdot\right)$ with support $[\underline{w},\infty)$ and $\underline{w}\geq 0$. The worker observes the wage and decides whether to accept it or reject it. If he accepts w, he will be employed forever at that wage; if he rejects it he stays unemployed and sample a new wage next period (there is no recall, meaning that the offer lasts only one period). In either case, the worker decides how much to consume at the end of the period.

1. Consider an autarky economy where agents cannot privately save. If a worker is unemployed he has 0 consumption. Let $V\left(w\right)$ denote the value of an agent who is employed at wage w and U the value of being unemployed. Derive an expression for $V\left(w\right)$ and a recursive expression for U. The strategy of the worker can be represented by the acceptance rule $a\left(w\right)\in\{0,1\}$. Determine the optimal acceptance rule

   ANSWER.

The value of being employed at wage w is

$$V(w) = \frac{u(w)}{1-\beta}$$

The value of being unemployed before taking the draw is

$$U = \int \max\{V(w), u(0) + \beta U\} \, dF(w)$$

Note that

$$\max\{V(w), u(0) + \beta U\} = \begin{cases} u(0) + \beta U & \text{if } w \leq \hat{w} \\ u(w)/(1-\beta) & \text{if } w > \hat{w} \end{cases}$$

where $\hat{w}$ is such that

$$\frac{u(\hat{w})}{1-\beta} = u(0) + \beta U$$

In other words,

$$a(w) = \begin{cases} 0 & \text{if } w \leq \hat{w} \\ 1 & \text{if } w > \hat{w} \end{cases}$$

2. Next, suppose there is a risk-neutral planner who can give each period an unemployment benefit to the worker if he is unemployed and get a tax from the worker if he is employed. His objective is to minimize the expected present value of the net costs of the insurance scheme, subject to giving expected utility U to the unemployed worker. Assume first that the planner can observe all the wages sampled and make benefits and taxes contingent on those. What is the optimal unemployment insurance scheme?

ANSWER

The idea here is that first best is attainable. The planner can tax employed workers 100%, and redistribute the proceeds as lump-sum transfer to every agent in the economy. Workers would be indifferent about taking a job. This achieves complete insurance. See Shimer and Werning.

Formally, suppose the planner chooses a constant unemployment benefit and a state contingent tax schedule for the employed:

$$C(U) = \min_{c, \tau(w)} \int \left\{ [c + \beta C(U')] [1 - a(w)] + \frac{\tau(w)}{1-\beta} a(w) \right\} dF(w)$$

subject to

$$\begin{aligned} U &= \int \max\left\{\frac{u\left(w-\tau\left(w\right)\right)}{1-\beta}, u\left(c\right)+\beta U'\right\} dF\left(w\right) \\ a\left(w\right) &= 1\left\{\frac{u\left(w-\tau\left(w\right)\right)}{1-\beta}, u\left(c\right)+\beta U'\right\} \end{aligned}$$

The assumption (embedded in the last two constraints) is that the planner can't force a worker to accept/reject an offer (this is the private acceptance rule for the worker).

Simplifying,

$$C\left(U\right) = \min_{c,\tau(w)} \left[c+\beta C\left(U'\right)\right] F\left(\hat{w}\right) + \int_{\hat{w}} \frac{\tau\left(w\right)}{1-\beta} dF\left(w\right)$$

subject to

$$\begin{aligned} U &= \left[u\left(c\right)+\beta U'\right] F\left(\hat{w}\right) + \int_{\hat{w}} \frac{u\left(w-\tau\left(w\right)\right)}{1-\beta} dF\left(w\right) \\ u\left(\hat{w}-\tau\left(\hat{w}\right)\right) &= \left(u\left(c\right)+\beta U'\right)\left(1-\beta\right) \end{aligned}$$

(where we have assumed $w-\tau\left(w\right)$ is weakly increasing at the solution)

FOC wrt c, $\tau\left(w\right)$ and $U\prime$:

$$\begin{aligned} F\left(\hat{w}\right)\left(1+\lambda u'\left(c\right)\right) &= \eta u'\left(c\right)\left(1-\beta\right) \\ 1 &= \lambda u'\left(w-\tau\left(w\right)\right) \\ F\left(\hat{w}\right)\left(C'\left(U'\right)+\lambda\right) &= \eta\left(1-\beta\right) \end{aligned}$$

The first and the last equation imply

$$1 = C'\left(U'\right) u'\left(c\right)$$

which together with the EC implies

$$1 = \lambda u'\left(c\right)$$

Using the second FOC we get

$$w-\tau\left(w\right) = c$$

3. Next, assume that the planner can only observe if the worker is employed or unemployed and cannot observe the sampled wage. The planner will choose a sequence of unemployment benefits and employment taxes $\{c_t, \tau_t\}$ so that an unemployed worker at time t will enjoy consumption c_t and a worker who accepts a job at time t will pay τ_t in each period $t, t+1, \ldots$. Write down the recursive-form problem of a planner who wants to minimize the costs of this insurance scheme, subject to the private acceptance rule of the worker.

   ANSWER.

   $$C(U) = \min_{c,\tau,\hat{w},U'} \left\{ (c + \beta C(U')) F(\hat{w}) - \frac{\tau}{1-\beta} [1 - F(\hat{w})] \right\}$$

   subject to

   $$\begin{aligned} U &= \int \max \left\{ \frac{u(w-\tau)}{1-\beta}, u(c) + \beta U' \right\} dF(w) \\ u(\hat{w} - \tau) &= (1-\beta)(u(c) + \beta U') \end{aligned}$$

4. Guess that the cost function takes the form

   $$C(U) = A + \frac{u^{-1}((1-\beta)U)}{1-\beta}$$

   for a constant A. Verify the guess and argue that the optimal reservation wage is constant over time.

   ANSWER.

   Pluggin the conjecture into the bellman equation from the previous point:

   $$C(U) = \min_{c,\tau,\hat{w},U'} \left\{ \left(c + \beta \left[A + \frac{u^{-1}((1-\beta)U')}{1-\beta} \right] \right) F(\hat{w}) - \frac{\tau}{1-\beta} [1 - F(\hat{w})] \right\}$$

   subject to

   $$\begin{aligned} U &= \int \max \left\{ \frac{u(w-\tau)}{1-\beta}, u(c) + \beta U' \right\} dF(w) \\ \tau &= \hat{w} - u^{-1}((1-\beta)(u(c) + \beta U')) \end{aligned}$$

   Note the following property of CARA utility

   $$u(a+b) = -u(a)\,u(b)$$

Thus the problem becomes

$$C(U) = \min_{c,\hat{w},U'} \left(c + \beta \left[A + \frac{u^{-1}((1-\beta)U')}{1-\beta} \right] \right) F(\hat{w})$$
$$+ \frac{1-F(\hat{w})}{1-\beta} u^{-1}((1-\beta)u(c) + (1-\beta)\beta U') - \frac{1-F(\hat{w})}{1-\beta} \hat{w}$$

subjec to

$$U = -(u(c) + \beta U') \int \max\{u(w - \hat{w}), -1\} dF(w)$$

Note we can introduce a new variable $x \equiv u(c) + \beta U'$ so that the problem becomes

$$C(U) = \min_{c,\hat{w},U',x} \left(c + \beta A + \frac{\beta}{1-\beta} u^{-1}((1-\beta)U') \right) F(\hat{w})$$
$$+ \frac{1-F(\hat{w})}{1-\beta} u^{-1}((1-\beta)x) - \frac{1-F(\hat{w})}{1-\beta} \hat{w}$$

subjec to

$$\begin{aligned} U &= -x \int \max\{u(w - \hat{w}), -1\} dF(w) \\ x &= u(c) + \beta U' \end{aligned}$$

Subproblem. Given $(x, \hat{w})$, c and $U\prime$ have to solves the following sub-problem

$$\min_{c,U'} c + \frac{\beta}{1-\beta} u^{-1}((1-\beta)U')$$
$$s.t. \ u(c) + \beta U' = x$$

Foc

$$\min_{U'} u^{-1}(x - \beta U') + \frac{\beta}{1-\beta} u^{-1}((1-\beta)U')$$
$$\max \log(-x + \beta U') + \frac{\beta}{1-\beta} \log(-(1-\beta)U')$$

So that the solution to the subproblem is

$$U' = x$$

$$u(c) = (1-\beta)U' \Leftrightarrow c = u^{-1}((1-\beta)U')$$

Then, the objective function in the subproblem is

$$c + \frac{\beta}{1-\beta} u^{-1}((1-\beta)U') = \frac{1}{1-\beta} u^{-1}((1-\beta)x)$$

so that the bigger problem is

$$C(U) = \min_{\hat{w},x} \left\{ \left(\frac{u^{-1}((1-\beta)x)}{1-\beta} + \beta A \right) F(\hat{w}) - \frac{1-F(\hat{w})}{1-\beta} \left(\hat{w} - u^{-1}((1-\beta)x) \right) \right\}$$

subject to

$$U = -x \int \max\{u(w-\hat{w}), -1\} dF(w)$$

Or simply:

$$\min_{\hat{w},x} \beta A F(\hat{w}) - \frac{1-F(\hat{w})}{1-\beta}\hat{w} + \frac{u^{-1}((1-\beta)x)}{1-\beta}$$

subject to

$$U = -x \int \max\{u(w-\hat{w}), -1\} dF(w)$$

Eliminating x...(and using a property on $u^{-1}(\cdot)$ of CARA preferences)

$$x = \frac{-U}{\int \max\{u(w-\hat{w}), -1\} dF(w)}$$

$$\min_{\hat{w}} \beta A F(\hat{w}) - \frac{1-F(\hat{w})}{1-\beta}\hat{w} + \frac{u^{-1}((1-\beta)U) - u^{-1}\left(-\int \max\{u(w-\hat{w}), -1\} dF(w)\right)}{1-\beta}$$

$$\begin{aligned} C(U) &= \min_{\hat{w}} \left[\beta A F(\hat{w}) - \frac{1-F(\hat{w})}{1-\beta}\hat{w} + \frac{u^{-1}\left(-\int \max\{u(w-\hat{w}), -1\} dF(w)\right)}{1-\beta} \right] \\ &\quad + \frac{u^{-1}((1-\beta)U)}{1-\beta} \end{aligned}$$

which verifies the conjecture with

$$A = \min_{\hat{w}} \left[-\frac{1-F(\hat{w})}{1-\beta}\hat{w} + \frac{u^{-1}\left(-\int \max\{u(w-\hat{w}), -1\} dF(w)\right)}{1-\beta} + \beta A F(\hat{w}) \right]$$

5. OPTIONAL. Now, assume the worker can privately borrow and save at interest rate r. Consider the following policy: the planner give a constant benefit b to the unemployed in exchange for a constant employment tax τ. Write down the budget constraint of a worker employed at wage w with a assets. Then write down the value of being employed at wage w with assets a, $V(w, a)$, and the value of being unemployed with assets a, $U(a)$. Write down the problem of the planner who minimize the net present value of his net costs to deliver a unemployment benefit that is constant over time,

imposing a constant tax to the employed workers. Show that this plan achieves the same allocation than the optimal unemployment scheme considered in question 4.

ANSWER.

$$a' = (1+r)\,a + w - c - \tau$$

$$V(w,a) = \frac{u(ra + w - \tau)}{1-\beta}$$

$$U(a) = \int \max \left\{ \frac{u(ra+w-\tau)}{1-\beta}, \begin{array}{c} \max_c u(c) + \beta U(a') \\ \text{s.t. } a' = (1+r)\,a + b - c. \end{array} \right\} dF(w)$$

The solution to this bellman eq. defines the worker's unemployment consumption $c_u(a)$, a reservation wage $\hat{w}(a)$, and next period's assets $a'(a)$.

The cost of unemployment insurance is defined as

$$C(a) = F(\hat{w}(a))\,(b + \beta C(a'(a))) - [1 - F(\hat{w}(a))]\,\frac{\tau}{1-\beta}$$

The optimal constant benefit policy solves

$$C(U_0, a) = \min_{b,\tau} C(a)$$

subject to

$$U(a) = U_0$$

It can be shown that $C(U_0, a)$ is independent of a.

2 Money and Competitive Search

This problem considers a money search model with divisible money a' la Lagos and Wright with a different market structure in the decentralized market. Instead of random matching and Nash bargaining, here we assume that sellers post a price and buyers direct their search towards the most convenient price.

Time is discrete and runs forever. The economy is populated by a continuum of measure 1 of infinitely lived agents with common discount factor β. There is a single, perishable consumption good. Each period is divided in two subperiods: day and night. During the night, all the agents have access to a linear technology to produce and consume and they trade in a centralized market. During the day, each agent becomes a buyer with probability

α and a seller (and producer) with probability $1-\alpha$. Buyers and sellers meet bilaterally in a decentralized market. The way we model the market structure during the day is competitive search.

During the night, the agents decide how much to produce H (hours spent working), how much to consume X, and how much money to keep in their pocket for the next day. At the beginning of the day, sellers decide whether to produce and if so they post the terms of trade, that is, a price and a quantity (q,d). Then buyers observe the posted terms of trade and direct their search towards the most attractive ones. Matching happens according to a constant returns to scale technology. Let $\Theta(q,d)$ be the market tightness associated to terms of trade (q,d), then $\mu(\Theta(q,d))$ denotes the probability a buyer looking for (q,d) meets a seller and $\mu(\Theta(q,d))/\Theta(q,d)$ the probability a seller posting (q,d) meets a buyer. After a match is formed, trade happens according to the posted terms of trade. Next, agents go to the next night market.

Agents' preferences at night are $U(X)-H$, while during the day they get $\mu(\Theta(q,d))\,u(q)$ if they become buyers and $\mu(\Theta(q,d))\,c(q)/\Theta(q,d)$ if they become sellers, where q is the quantity of the good traded during the day, $\Theta(q,d)$ the market tightness associated to the terms of trade (q,d), X is the quantity of the good consumed and H the hours of work during the night. The function U, u, and c satisfy standard assumptions.

Agents are anonymous in the decentralized market and the only medium of exchange is fiat money, a perfectly storable and divisible object in supply M_t at time t. Assume that γ is the money growth rate, that is, $M_{t+1}=(1+\gamma)M_t$.

1. Call $W_t(m)$ the value of an agent who enters the centralized market at time t with money balances equal to m and $V_t^B(m)$ and $V_t^S(m)$ the value of an agent who enters the decentralized market at time t with money balances m and realizes to be respectively a buyer or a seller. Write down the respective Bellman equations.

   ANSWER.

   $$W_t(m)=\max_{m'\geq 0, X\geq 0, H\in[0,\bar{H}]}\left\{U(X)-H+\beta\left[\alpha V_{t+1}^B(m')+(1-\alpha)V_{t+1}^S(m')\right]\right\},$$

   subject to

   $$X=H+\phi_t m-\phi_t m'$$

$$\begin{aligned} V_t^B(m) &= \max_{(d,q)} \mu(\Theta(d,q)) \left[u(q) + W_t^B(m-d) - W_t^B(m)\right] + W_t^B(m) \\ s.t.\ d &\leq m \end{aligned}$$

and

$$V_t^S(m) = \max_{(d,q)} \frac{\mu(\Theta(d,q))}{\Theta(d,q)} \left[-c(q) + W_t^S(m+d) - W_t^S(m)\right] + W_t^S(m)$$

2. Guess that the value function $W_t(m)$ is linear in m with coefficient ϕ_t. Define a stationary competitive search equilibrium where prices grow at the money growth rate, $\phi_t/\phi_{t+1} = \gamma$, and the distribution of money balances $F_t(\cdot)$ is degenerate at M_t at any t. In particular, you have to define three sets of optimality conditions (plus market clearing): the optimality condition for the agents in the centralized market, the optimality condition for agents who turn out to be buyers in the decentralized market, and the optimality condition for the agents who turn out to be sellers in the decentralized market. HINT: when you consider possible deviating terms of trade, agents know that the money distribution at the beginning of the day $F_t(\cdot)$ is degenerate at M_t. However, remember that you have to check that it is optimal to choose M_{t+1} in the centralized market at time t.

ANSWER. Call Y the set of posted terms of trades. You can write $W_t(m) = A + \phi_t m$.

For given M_t and ϕ_t, a competitive search equilibrium is a value W, two functions $V^S(m)$ and $V^B(m)$ on R_+, a measure λ with support Y, and a functions $\Theta(q,d)$ on R_+^2 satisfying:

(a) agents' optimality in the centralized market:

$$A = \max_{m' \geq 0, X \geq 0} \left\{U(X) - X - \phi_t m' + \beta\left[\alpha V_{t+1}^B(m') + (1-\alpha) V_{t+1}^S(m')\right]\right\},$$

(b) sellers'optimality condition: for any (q,d)

$$\frac{\mu(\Theta(q,d))}{\Theta(q,d)}\left[-c(q) + \phi_t d\right] + A + \phi_t M_t \leq V^S(M_t)$$

with equality if (p, q) is posted and

$$V^S(m) = \max_{p,q} \frac{\mu(\Theta(q,d))}{\Theta(q,d)} [-c(q) + \phi_t d] + A + \phi_t m;$$

(c) buyers' optimality: for any (q, d)

$$\mu(\Theta(q,d)) [u(q) - \phi_t d] + A + \phi_t M_t \le V^B(M_t)$$

with equality if $\Theta(p, q) < \infty$ and

$$V^B(m) = \max_{q,d \le m} \mu(\Theta(q,d)) [u(q) - \phi_t d] + A + \phi_t m;$$

(d) market clears

$$\begin{aligned} \int \Theta(d,q)\, d\lambda(q,d) &= \alpha \text{ and } \int d\lambda(q,d) = 1-\alpha \\ m' &= M_{t+1} \end{aligned}$$

3. Verify that the value function $W_t(m)$ is linear in m with coefficient ϕ_t. Also characterize the optimality condition in the centralized market for X and m' as a function of $\partial V_t^B(m')/\partial m'$, $\partial V_t^S(m')/\partial m'$, ϕ_t, and ϕ_{t+1}, using the fact that we are characterizing an equilibrium with degenerate distribution of money and $M_t > 0$ for all t.

ANSWER.

$$W_t(m) = \phi_t m + \max_{m' \ge 0, X \ge 0} \left\{ U(X) - X - \phi_t m' + \beta \left[\alpha V_{t+1}^B(m') + (1-\alpha) V_{t+1}^S(m') \right] \right\},$$

$$\begin{aligned} U'(X) &= 1 \\ \phi_t &\le \beta \left[\alpha \frac{\partial V_{t+1}^B(m')}{\partial m'} + (1-\alpha) \frac{\partial V_{t+1}^S(m')}{\partial m'} \right] \end{aligned}$$

with equality if $m' > 0$.

4. Using the definition derived in question 2, show that in equilibrium

$$\begin{aligned} V_t^B(m) &= \max_{\theta,d,q} \mu(\theta) [u(q) - \phi_t d] + A + \phi_t m \qquad \text{(P1)} \\ & s.t. \frac{\mu(\theta)}{\theta} [-c(q) + \phi_t d] + A + \phi_t M_t \ge V^S(M_t) \\ & d \le m \end{aligned}$$

ANSWER

Consider an equilibrium. Take any $(q, d) \in Y$, and let $\theta = \Theta(q, d)$. Then the first inequality constraint in P1 is satisfied, from seller's optimality condition. Now, suppose there exists θ', d', q' satisfying the constraint, and such that

$$\mu(\theta')[u(q') - \phi_t d'] + A + \phi_t m > V_t^B(m)$$

But from buyer's optimality

$$\mu(\Theta(q', d'))[u(q') - \phi_t d'] + A + \phi_t m \leq V^B(m)$$

which implies

$$\theta' > \Theta(q', d')$$

Then, from seller's optimality

$$\frac{\mu(\theta')}{\theta'}[-c(q') + \phi_t d'] + A + \phi_t m < \frac{\mu(\Theta(q', d'))}{\Theta(q', d')}[-c(q') + \phi_t d'] + A + \phi_t m \leq V^S(m)$$

which contradicts the initial assumption that θ', d', q' satisfy the constraint.

5. Find the foc for problem P1. Using them and the Envelope conditions for $V_t^B(m)$ and $V_t^S(m)$ substitute for $\partial V_t^B(m')/\partial m'$ and $\partial V_t^S(m')/\partial m'$ in the optimality condition for m' derived in question 3. Verify that an equilibrium with degenerate distribution exists.

ANSWER.

The EC's are

$$\frac{\partial V_t^B}{\partial m}(m) = \phi_t + \chi_{t+1}$$

where χ_{t+1} is the multiplier on the $d \leq m$ constraint.

$$\frac{\partial V_t^S}{\partial m} = \phi_t$$

So that

$$\phi_t = \beta\left[\alpha\left[\phi_{t+1} + \chi_{t+1}\right] + (1-\alpha)\phi_{t+1}\right]$$

Moreover from the FOC of P1

$$\begin{aligned} \frac{u'(q)}{c'(q)} &= \frac{\eta}{\theta} \\ \chi_t &= \mu(\theta)\phi_t\left[\frac{\eta}{\theta}-1\right] \\ u(q)-\phi_t d &= \frac{\eta}{\theta}\left[1-\frac{\mu(\theta)}{\theta\mu'(\theta)}\right][c(q)-\phi_t d] \end{aligned}$$

so that

$$\phi_t = \beta\phi_{t+1}\left[1+\alpha\mu(\theta)\left(\frac{u'(q)}{c'(q)}-1\right)\right]$$

6. Using the derivations in question 5, derive under which conditions the equilibrium achieve the first best. Discuss.

   ANSWER.

   From the equation above the first best is achieved iff $\gamma = \beta$ Friedman rule!

14.461 Advanced Macroeconomics I
Fall 2009