1 Lucas-Phelps islands

- Lucas 1972

- overlapping generations

- agents work at date t consume at date $t+1$

- preferences

$$E\left[C_{i,t+1} - \frac{1}{2}N_{i,t}^2\right]$$

agents work, accumulate money, spend, die

$$
\begin{aligned}
Y_{i,t} &= N_{i,t} \\
M_{i,t+1} &= P_{i,t} Y_{i,t} e^{\varepsilon_{t+1}} \\
P_{j,t+1} C_{i,t+1} &= M_{i,t+1}
\end{aligned}
$$

- $e^{\varepsilon_{t+1}} - 1$ is a proportional subsidy from the government (money injection)

$$M_{t+1} = M_t e^{\varepsilon_{t+1}}$$

- at date $t+1$ agent i consumes the output of agent j

- continuum of islands, $i \in [0, 1]$
- unit mass of agents on each
- old agents receive proportional transfer of money from govt'
- they travel to one island where they spend all their money
- prices $P_{i,t}$ determined in Walrasian equilibrium
- young agents decide their labor supply based on $P_{i,t}$

- old agents in island i are representative sample

- but different mass $\exp\left\{u_{i,t}\right\}$ in island i

- nominal demand demand in island i is

$$e^{u_{i,t}} \int_0^1 M_{i,t} di = e^{u_{i,t}} M_t$$

- $u_{i,t}$ normal with

$$\int_0^1 e^{u_{i,t}} di = 1$$

- normal idiosyncratic demand shock: $u_{i,t} \sim N\left(0, \sigma_u^2\right)$

- normal monetary shock: $\varepsilon_t \sim N\left(0, \sigma_\varepsilon^2\right)$

- total nominal demand is

$$D_{i,t} = e^{u_{i,t}+\varepsilon_t} M_{t-1}$$

in logs

$$d_{i,t} = m_{t-1} + u_{i,t} + \varepsilon_t$$

Market clearing

$$P_{i,t} N_{i,t} = D_{i,t}$$

Information structure

- all agents observe $\{M_{t-1}, M_{t-2}, ...\}$
- old agents observe $\varepsilon_t, P_{j,t}$
- young agents only observe $P_{i,t}$

observing $P_{i,t}$ and M_{t-1}—and knowing their own $N_{i,t}$—young agents can infer the sum

$$u_{i,t} + \varepsilon_t = p_{i,t} + n_{i,t} - m_{t-1}$$

Labor supply

agents solve

$$\max_{N_{i,t},C_{i,t+1}} \quad E\left[C_{i,t+1} - \frac{1}{2}N_{i,t}^2 \mid P_{i,t}, M_{t-1}, M_{t-2}, ...\right]$$

$$s.t. \quad P_{j,t+1}C_{i,t+1} = P_{i,t}N_{i,t}e^{\varepsilon_{t+1}}$$

substitute $C_{i,t+1}$ and obtain FOC:

$$E\left[\frac{P_{i,t}}{P_{j,t+1}}e^{\varepsilon_{t+1}} - N_{i,t} \mid P_{i,t}, M_{t-1}, M_{t-2}, ...\right] = 0$$

interpetation

$$\underbrace{N_{i,t}}_{\text{labor supply}} = \quad E[\underbrace{\frac{P_{i,t}}{P_{j,t+1}}}_{\text{exp.infl.}}e^{\varepsilon_{t+1}} \mid P_{i,t}, M_{t-1}, M_{t-2}, ...]$$

Equilibrium prices

Look for stationary equilibrium where

$$\frac{P_{i,t}}{M_{t-1}} = G\left(u_{i,t}, \varepsilon_t\right)$$

Decompose

$$\begin{aligned} N_{i,t} &= E[\frac{P_{i,t}}{P_{j,t+1}} e^{\varepsilon_{t+1}} \mid P_{i,t}, M_{t-1}, ...] = \\ &= E\left[\frac{P_{i,t}}{M_t} \mid P_{i,t}, M_{t-1}, ...\right] E\left[\frac{M_t}{P_{j,t+1}} e^{\varepsilon_{t+1}} \mid P_{i,t}, M_{t-1}, ...\right] \end{aligned}$$

In stationary equilibrium second piece is a constant

$$\xi = E\left[\frac{e^{\varepsilon_{t+1}}}{G\left(u_{j,t+1}, \varepsilon_{t+1}\right)}\right]$$

In the first piece, only information on M_t in the first piece is in $P_{i,t}$ and M_{t-1}.

So we have

$$N_{i,t} = \xi E\left[\frac{P_{i,t}}{M_t} \mid P_{i,t}, M_{t-1}\right]$$

From equilibrium condition we obtain

$$e^{u_{i,t}} M_t = P_{i,t} N_{i,t} = P_{i,t} \xi E \left[\frac{P_{i,t}}{M_t} \mid P_{i,t}, M_{t-1} \right]$$

in logs

$$m_t - p_{i,t} + u_{i,t} = (...) - E \left[m_t - p_{i,t} \mid p_{i,t}, m_{t-1} \right]$$

(constant terms in (...), depend on variances)

We obtain

$$p_{i,t} = \bar{p} + \frac{1}{2} \left(m_t + u_{i,t} \right) + \frac{1}{2} E \left[m_t | p_{i,t}, m_{t-1} \right]$$

We will see that this can be rewritten as

$$p_{i,t} - m_{t-1} = \bar{p} + \frac{1}{2} \left(\varepsilon_t + u_{i,t} \right) + \frac{1}{2} E \left[\varepsilon_t | \varepsilon_t + u_{i,t} \right]$$

confirming our conjecture that $P_{i,t}/M_t$ is only a function of ε_t and $u_{i,t}$

Agents observe m_{t-1} and nominal demand in their island

$$m_{t-1} + \varepsilon_t + u_{i,t}$$

Define

$$\overline{E}_t [m_t] = \int_0^1 E \left[m_t | m_{t-1}, \varepsilon_t + u_{i,t} \right] di$$

Then, averaging, we have

$$p_t = \bar{p} + \frac{1}{2} m_t + \frac{1}{2} \overline{E}_t [m_t]$$

Imperfect information

$$\overline{E}_t [m_t] \neq m_t$$

in particular

$$E \left[m_t | m_{t-1}, \varepsilon_t + u_{i,t} \right] = m_{t-1} + \beta (\varepsilon_t + u_{it})$$

where

$$\beta = \frac{\sigma_\varepsilon^2}{\sigma_\varepsilon^2 + \sigma_u^2}$$

so

$$\overline{E}_t [m_t] = m_{t-1} + \beta \epsilon_t \neq m_{t-1} + \varepsilon_t$$

We have

$$p_t = \bar{p} + m_{t-1} + \frac{1}{2}(1+\beta)\,\varepsilon_t$$

and output is

$$\begin{aligned} y_t &= m_t - p_t \\ &= \bar{y} + \frac{1}{2}(1-\beta)\,\varepsilon_t \end{aligned}$$

- larger $\sigma_\varepsilon^2/\sigma_u^2$ implies smaller real effects of monetary policy

- Phillips curve depends on the monetary regime

- OLS inflation on output gap

$$\pi_t = \frac{1+\beta}{1-\beta}\left(y_t - \bar{y}\right) - \frac{1}{2}\left(\beta - 1\right)\varepsilon_{t-1}$$

- as $\sigma_\varepsilon^2/\sigma_u^2 \to \infty$ we have $\beta \to 1$ and vertical Phillips curve

Wrapping up:

- imperfect information affects transmission of monetary shocks
- in particular, explains sluggish response of prices to m_t: short-run non-neutrality
- crucial formal step: agents can be wrong on average $m_t \neq \overline{E}_t [m_t]$
- policy regime affects inference and thus effects of shocks

2 Higher order expectations

- price setting firms with quadratic costs

$$\sum_{t=0}^{\infty} \beta^t E\left[\frac{P_{i,t}}{P_t} Y_{i,t} - \frac{1}{2}\left(Y_{i,t}\right)^2\right]$$

facing demand function

$$Y_{i,t} = \left(\frac{P_{i,t}}{P_t}\right)^{-\sigma} \frac{M_t}{P_t}$$

- have to set $P_{i,t}$ with imperfect knowledge of M_t

- optimality condition

$$E_{i,t}\left[(\sigma-1)\frac{1}{P_t}\left(\frac{P_{i,t}}{P_t}\right)^{-\sigma}\frac{M_t}{P_t}\right]=E_{i,t}\left[\sigma\frac{1}{P_{i,t}}\left(\frac{P_{i,t}}{P_t}\right)^{-2\sigma}\left(\frac{M_t}{P_t}\right)^2\right]$$

- if everything log-normal (or in approximation)

$$E_{i,t}\left[-\sigma p_{i,t}-(1-\sigma)\,p_t+m_t-p_t\right]=$$
$$=E_{i,t}\left[-(1+2\sigma)\,p_{i,t}+2\sigma p_t+2\,(m_t-p_t)\right]$$

- we get

$$p_{i,t} = \frac{\sigma}{1+\sigma} E_{i,t}\left[p_t\right] + \frac{1}{1+\sigma} E_{i,t}\left[m_t\right]$$

optimal price weighted average of expected price of other price setters

$$\xi = \frac{1}{1+\sigma}$$

$$p_{i,t} = (1-\xi) E_{i,t}\left[p_t\right] + \xi E_{i,t}\left[m_t\right]$$

- higher σ higher weight on other price setters prices: *strategic complementarity*

- now specify the information structure

- simple static case

$$m_t = u_t$$

- private signal

$$z_{i,t} = u_t + v_{i,t}$$

- Undetermined coefficients

$$p_t = \phi m_t$$

- substituting gives

$$\begin{aligned} p_{i,t} &= \left(\left(1-\xi\right)\phi+\xi\right)E_{i,t}\left[m_t\right] \\ &= \left(\left(1-\xi\right)\phi+\xi\right)\beta z_{i,t} \end{aligned}$$

- aggregating gives

$$p_t = \left(\left(1-\xi\right)\phi+\xi\right)\beta m_t$$

- fixed point

$$\phi = \left(\left(1-\xi\right)\phi+\xi\right)\beta$$

- solution

$$\phi = \frac{\xi\beta}{1-\left(1-\xi\right)\beta}$$

- the response of prices to a monetary shock now depends on how informative is the signal and on the degree of strategic complementarity
  - less informative signal→smaller price response (bigger quantity response)
  - more strategic complementarity→smaller price response (bigger quantity response)
- Alternative interpretation: use notation

$$\begin{aligned}\bar{E}_t[X_t] &= \int E_{i,t}[X_t]\,di \\ \bar{E}_t^{(j)}[X_t] &= \int E_{i,t}^{(j-1)}[X_t]\,di \text{ (higher order exp)}\end{aligned}$$

$$p_t = (1-\xi)\,\bar{E}_t[p_t] + \xi\bar{E}[m_t]$$

$$
\begin{aligned}
p_t &= \sum_{j=0}^{\infty} \xi \left(1-\xi\right)^j E_t^{(j)} \left[m_t\right] \\
&= \xi\beta \sum \left(1-\xi\right)^j \beta^j m_t
\end{aligned}
$$

- (obviously gives the same ϕ)

2.1 Dynamics

- process for money

$$m_t = m_{t-1} + u_t$$

- price setters observe

$$z_{i,t} = m_t + v_{i,t}$$

- need to form expectations about m_t and p_t

- Conjecture: state variables

$$\begin{bmatrix} m_t \\ p_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi_m & \phi_p \end{bmatrix} \begin{bmatrix} m_{t-1} \\ p_{t-1} \end{bmatrix} + \begin{bmatrix} u_t \\ \phi_u u_t \end{bmatrix}$$

- Kalman filter

$$E_{i,t}\begin{bmatrix} m_t \\ p_t \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi_m & \phi_p \end{bmatrix} E_{i,t-1}\begin{bmatrix} m_{t-1} \\ p_{t-1} \end{bmatrix} + K\left(z_{i,t} - E_{i,t-1}m_t\right)$$

- Integrating across agents we get

$$\begin{bmatrix} m_{t|t} \\ p_{t|t} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \phi_m & \phi_p \end{bmatrix}\begin{bmatrix} m_{t-1|t-1} \\ p_{t-1|t-1} \end{bmatrix} + K\left[m_t - m_{t|t-1}\right]$$

- Now use the optimality condition

$$p_t = (1-\xi)\,p_{t|t} + \xi m_{t|t}$$

(as in usual method of undetermined coefficients) to get

$$
\begin{aligned}
p_t &= \left[\begin{array}{cc} \xi & 1-\xi \end{array}\right] \left[\begin{array}{c} m_{t|t} \\ p_{t|t} \end{array}\right] = \\
&= \xi m_{t-1|t-1} + (1-\xi)\left(\phi_m m_{t-1|t-1} + \phi_p p_{t-1|t-1}\right) + \\
&\quad + \left[\begin{array}{cc} \xi & 1-\xi \end{array}\right] K \left[m_{t-1} + u_t - m_{t-1|t-1}\right]
\end{aligned}
$$

- here is were we use Woodford's trick: use

$$p_{t-1|t-1} = \frac{p_{t-1} - \xi m_{t-1|t-1}}{1-\xi}$$

- now we have p_t in terms of the state variables p_{t-1} and m_{t-1}:

$$p_t = \phi_m m_{t-1} + \phi_p p_{t-1} + \phi_u u_t = \phi_p p_{t-1} + \left(\begin{array}{cc} \xi & 1-\xi \end{array}\right) K \left[m_{t-1} + u_t\right] \tag{1}$$

if the following condition is satisfied

$$\xi + (1-\xi)\left(\phi_m - \phi_p \xi / (1-\xi)\right) - \left(\begin{array}{cc} \xi & 1-\xi \end{array} \right) K = 0 \qquad (2)$$

this condition makes the term with $m_{t-1|t-1}$ disappear.

- Condition (2) pins down ϕ_p (which, fortunately, is not pinned down by matching coefficients in (1)!)

- So we have

$$\phi_m = \phi_u = \phi = \left(\begin{array}{cc} \xi & 1-\xi \end{array} \right) K$$

and from (2), after some algebra,

$$\phi_p = 1 - \phi$$

- Now we have a map: $\phi \rightarrow$Kalman gains $K \rightarrow \phi'$

  ...and we need to find a fixed point of it

- Implications for price and output dynamics

$$\begin{aligned} m_t &= m_{t-1} + u_t \\ p_t &= (1-\phi)\, p_{t-1} + \phi\, (m_{t-1} + u_t) \end{aligned}$$

  and

$$\begin{aligned} y_t &= m_t - p_t = (1-\phi)\,(m_{t-1} + u_t) - (1-\phi)\, p_{t-1} \\ &= (1-\phi)\,(y_{t-1} + u_t) \end{aligned}$$

- The parameter ϕ determines both the impact effect of the shock u_t on prices and the persistence

- Computational experiments (see matlab codes): higher σ_v^2 and lower ξ increase ϕ

- (See the paper for a closed form expression for ϕ as a function of σ_u^2/σ_v^2 in equations 3.6 and 3.7 with $\phi = \hat{k}$)

2.2 Tools: Kalman filter

A simple intro to the Kalman filter. State space representation

$$X_t = AX_{t-1} + U_t.$$

Information set $\{Y_t, Y_{t-1}, ...\}$ where

$$Y_t = FX_t + V_t.$$

We want to derive the steady state dynamics of

$$X_{t|t} \equiv E_t [X_t] .$$

Assumption: U_t and W_t are mutually independent, each of them is an i.i.d. Gaussian vector with mean zero and variance-covariance matrices Σ_U and Σ_W. (can be extended to U_t and W_t correlated)

Updating rule

$$
\begin{aligned}
X_{t|t} &= X_{t|t-1} + K\left(Y_t - Y_{t|t-1}\right) \\
&= AX_{t-1|t-1} + K\left(Y_t - Y_{t|t-1}\right) \\
&= (I - KF)\,AX_{t-1|t-1} + KY_t
\end{aligned}
$$

(alternative common approach focuses on one-step-ahead forecast $X_{t+1|t}$ here we focus on $X_{t|t}$)

Define

$$
P = Var_{t-1}\left[X_t\right]
$$

then K satisfies

$$
K = PF'\left(FPF' + \mathbf{\Sigma}_V\right)^{-1}
$$

(from orthogonality condition)

We need to find expressions for the matrix P. Bayesian updating for the variances gives

$$\hat{P} \equiv Var_t\left[X_t\right] = P - PF'\left(FPF' + \mathbf{\Sigma}_V\right)^{-1} FP,$$

and the dynamics of X_t imply that

$$\begin{aligned} Var_t\left[X_{t+1}\right] &= A\hat{P}A' + \mathbf{\Sigma}_U = \\ &= A\left[P - PF'\left(FPF' + \mathbf{\Sigma}_V\right)^{-1} FP\right] A' + \mathbf{\Sigma}_U \end{aligned}$$

so imposing steady state for learning we have

$$P = A\left[P - PF'\left(FPF' + \mathbf{\Sigma}_W\right)^{-1} FP\right] A' + \mathbf{\Sigma}_U$$

(Riccati equation for P)

Example: State law of motion:

$$x_t = x_{t-1} + \varepsilon_t$$

Observation equation:

$$y_t = x_t + \eta_t$$

Now

$$k = \frac{P}{P + \sigma_\eta^2} = \frac{1/\sigma_\eta^2}{1/P + 1/\sigma_\eta^2}$$

and P satisfies

$$P = \frac{P}{P + \sigma_\eta^2}\sigma_\eta^2 + \sigma_\varepsilon^2$$

simple quadratic equation

$$P = \frac{1}{2}\sigma_\varepsilon^2 + \frac{1}{2}\sqrt{4\sigma_\varepsilon^2\sigma_\eta^2 + \left(\sigma_\varepsilon^2\right)^2}$$

Dynamics

$$x_{t|t} = (1 - k)\, x_{t-1|t-1} + k y_t$$

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