

14.461 Lectures 14-18

Labor Market Search and Efficiency

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1 Diamond-Mortensen-Pissarides

1.1 Set Up

There is a continuum of measure 1 of homogeneous, risk-neutral workers and a continuum of larger measure of homogeneous risk-neutral firms. They have common discount factor β . At the beginning of each period, firms choose whether to pay a cost k and post a vacancy. There is free-entry which is going to pin down the measure of vacancies at each point in time. Once, firms have opened vacancies, there is random matching between unemployed workers at time t , u_t , and the vacancies opened at that time, v_t . A worker and firm matched at time t can produce y at each time $\tau \geq t$, until separation. The wage w_t that a worker matched at time t is going to receive each time $\tau \geq t$ until separation, is determined with generalized Nash bargaining at the time of the match. Separation is exogenous: at each time $t > \tau$ the employment relationship can end with probability s (notice that separation happens at the beginning of the period together with matching, so a match created at time t can be separated only from time $t + 1$ on). If a worker is unemployed he gets some utility b , which may be interpreted as leisure, home production, or, in some contexts, as unemployment benefit.

1.2 Matching

Matching is random and is represented by a matching function $m(u_t, v_t)$. We assume that the matching function is continuous, non-negative, increasing in both arguments and concave, with $m(0, v_t) = m(u_t, 0) = 0$ for all u_t, v_t . We also assume that it features

constant returns to scale. Define the market tightness $\theta_t \equiv v_t/u_t$. Given CRS, we can represent the probability a worker meets a firm and the probability a firm meets a worker as functions of θ_t only. In fact the probability a worker meets a firm is equal to

$$\frac{m(u_t, v_t)}{u_t} = m(1, \theta_t) \equiv \mu(\theta_t),$$

and the probability a firm meets a worker to

$$\frac{m(u_t, v_t)}{v_t} = m\left(\frac{1}{\theta_t}, 1\right) \equiv \frac{\mu(\theta_t)}{\theta_t}.$$

To guarantee that the function $\mu(\cdot)$ represents a probability we have to assume that $\mu(\theta) \leq \min\{\theta, 1\}$. Also, for tractability we assume that $\mu(\theta)$ is continuous and twice differentiable with $\mu'(\theta) > 0$ and $\mu''(\theta) < 0$ for all $\theta \in [0, \infty)$.

1.3 Bellman Equations

Let us define U_t the value of an unemployed worker at the beginning of time t , V_t the value of an employed worker at the beginning of time t , W_t the value of a firm with an open vacancy at the beginning of time t , J_t the value of a firm with a filled vacancy at the beginning of time t . Then,

$$\begin{aligned} U_t &= \mu(\theta_t)(w_t + \beta V_{t+1}) + (1 - \mu(\theta_t))(b + \beta U_{t+1}) \\ V_t &= s(b + \beta U_{t+1}) + (1 - s)(w_t + \beta V_{t+1}) \\ W_t &= -k + \frac{\mu(\theta_t)}{\theta_t}(y - w_t + \beta J_{t+1}) + \left(1 - \frac{\mu(\theta_t)}{\theta_t}\right)\beta \max\{W_{t+1}, 0\} \\ J_t &= s\beta \max\{0, W_{t+1}\} + (1 - s)(y - w_t + \beta J_{t+1}) \end{aligned}$$

Free-entry implies that

$$W_t = 0 \text{ for all } t.$$

Also, the law of motion for u_t is

$$u_{t+1} = u_t(1 - \mu(\theta_{t+1})) + (1 - u_t)s. \tag{1}$$

1.4 Steady State

In steady state, the Bellman equations become

$$U = \mu(\theta)(w + \beta V) + (1 - \mu(\theta))(b + \beta U) \quad (2)$$

$$V = s(b + \beta U) + (1 - s)(w + \beta V) \quad (3)$$

$$k = \frac{\mu(\theta)}{\theta}(y - w + \beta J) \quad (4)$$

$$J = (1 - s)(y - w + \beta J) \quad (5)$$

We assume that the wage is determined by generalized Nash bargaining solution with threat points equal to the outside option of the firm and the worker. If the firm does not meet a worker, she just gets 0, while if a worker does not meet a firm, he gets $b + \beta U$, where U is taken as given in the negotiation. If instead the firm and the worker bargain a wage ω , the worker gets $\omega + \beta \tilde{V}(\omega)$ and the firm gets $y - \omega + \beta \tilde{J}(\omega)$, where

$$\tilde{V}(\omega) \equiv \frac{s(b + \beta U) + (1 - s)\omega}{1 - (1 - s)\beta},$$

$$\tilde{J}(\omega) \equiv \frac{(1 - s)(y - \omega)}{1 - (1 - s)\beta}.$$

Hence, Nash Bargaining require

$$w = \max_{\omega} \left[\omega + \beta \tilde{V}(\omega) - b - \beta U \right]^{\eta} \left[y - \omega + \beta \tilde{J}(\omega) \right]^{1-\eta}. \quad (6)$$

We are interested in situations where

$$\omega + \beta \tilde{V}(\omega) \geq b + \beta U,$$

and

$$y - \omega + \beta \tilde{J}(\omega) \geq 0,$$

for some ω so that there is something to bargain over. A necessary and sufficient condition for that is

$$U \leq \frac{1}{\beta} \left[\frac{y}{1 - \beta} - b \right],$$

which requires

$$y \geq (1 - \beta)b$$

Definition 1 A steady state equilibrium is a vector (U, θ, w, u) such that vacancies are chosen to maximize firms' expected profits and wages are determined by Nash bargaining.

Combining (4) and (5) we obtain

$$k = \frac{\mu(\theta)}{\theta} \left[\frac{y - w}{1 - (1 - s)\beta} \right]. \quad (7)$$

As the wage increases, the expected profits for the firm decrease and less firms enter, driving θ down and hence the expected profits up.

The Nash bargaining problem (6) gives the foc

$$\eta \left[y - \omega + \beta \tilde{J}(\omega) \right] = (1 - \eta) \left[\omega + \beta \tilde{V}(\omega) - b - \beta U \right].$$

At the optimum $\omega = w$, $\tilde{V}(\omega) = V$, and $\tilde{J}(\omega) = J$. Also it must be that

$$\begin{aligned} w - b + \beta(V - U) &= \eta [y - b + \beta(V + J - U)] \\ y - \omega + \beta J &= (1 - \eta) [y - b + \beta(V + J - U)] \end{aligned}$$

and (combining them):

$$[w - b + \beta(V - U)] = \frac{\eta}{1 - \eta} (y - w + \beta J). \quad (8)$$

Combining the first equation with the expression for J and V we obtain

$$w = (1 - \beta)(b + \beta U) + \eta [y - (1 - \beta)(b + \beta U)] \quad (9)$$

where $(1 - \beta)(b + \beta U)$ represents the reservation wage.

Equation (2) can be rewritten as

$$(1 - \beta)U = b + \mu(\theta)(w - b + \beta(V - U))$$

and combining this with (8) and (4) we obtain

$$(1 - \beta)U = b + \frac{\eta}{1 - \eta} \theta k, \quad (10)$$

so that the reservation wage becomes

$$(1 - \beta)(b + \beta U) = b + \frac{\eta}{1 - \eta} \beta \theta k.$$

As the market tightness increases, the expected utility for a worker increases and hence the reservation wage increases. Combining this with (9) we obtain

$$w = \eta (y + \beta \theta k) + (1 - \eta) b. \quad (11)$$

Finally we can find the equilibrium market tightness by equalizing demand and supply, that is, combining (11) and (7):

$$k = \frac{\mu(\theta)}{\theta} \left[\frac{y - \eta (y + \beta \theta k) - (1 - \eta) b}{1 - (1 - s) \beta} \right]. \quad (12)$$

The RHS is a non-increasing function of θ and hence there exists a unique θ that satisfy this equation! Then we can plug the equilibrium value for θ back into (10) and (11) to find the equilibrium values for U and w . To complete the equilibrium characterization we need to specify the steady state value of the unemployment rate. From the law of motion for the unemployment rate (1) we obtain that in steady state

$$u = \frac{s}{s + \mu(\theta)}.$$

Definition 2 *An equilibrium is a sequence $\{U_t, w_t, \theta_t, u_t\}_{t=0}^{\infty}$ such that at each point in time vacancies are chosen to maximize firms' expected profits, and wages in a match created at that time are determined by Nash bargaining.*

There is an equilibrium, where U_t , w_t , and θ_t are constant at the steady state level and the only interesting dynamics are the dynamics of u_t (and v_t). The only state variables of the equilibrium are u_t and v_t , and we can transform them into u_t and θ_t . Notice that, if U_t is constant the Nash bargaining solution implies that w_t is constant and then free entry implies that θ_t is constant as well. The only dynamics are the dynamics of the unemployment rate:

$$u_{t+1} = u_t (1 - \mu(\theta_t)) + (1 - u_t) s.$$

Proposition 1 *The equilibrium with U , w , and θ constant over time and equal to their steady state values is an equilibrium. If $1 - \eta = \theta \mu'(\theta) / \mu(\theta)$, it is the unique equilibrium.*

1.5 Social Planner

Consider a planner who decides how many vacancies to open and how much consumption to allocate to employed and unemployed workers at each point in time. Define c_t^u and c_t^e the consumption of unemployed and employed workers respectively. Then, at the beginning of time t , the value of an unemployed worker and of an employed worker respectively are

$$\begin{aligned} U_t &= \mu(\theta_t)(c_t^e + \beta V_{t+1}) + (1 - \mu(\theta_t))(c_t^u + \beta U_{t+1}) \\ V_t &= s(c_t^u + \beta U_{t+1}) + (1 - s)(c_t^e + \beta V_{t+1}) \end{aligned}$$

The intertemporal resource constraint can be written as

$$\begin{aligned} \sum_{t=0}^{\infty} \beta^t \{u_{t+1}y + (1 - u_{t+1})b - \theta_t u_t k\} &\geq \sum_{t=0}^{\infty} \beta^t \{u_{t+1}c_t^e + (1 - u_{t+1})c_t^u\} \\ &= u_0 U_0 + (1 - u_0) V_0 \end{aligned}$$

The Pareto frontier can be characterized by maximizing U_0 subject to $V_0 = \bar{V}$ and the resource constraint. Given that utility is transferable, the Pareto frontier is going to be linear. In particular, given that the resource constraint will hold with equality, the social planner will like to maximize the net present value of resources in the economy, that is, the LHS of the resource constraint.

Hence, the Planner Problem can be written in recursive terms as

$$P(u) = \max_{\theta, u'} (1 - u')y + u'b - k\theta u + \beta P(u')$$

subject to

$$u' = u(1 - \mu(\theta)) + (1 - u)s$$

Substituting the constraint into the objective we get

$$P(u) = \max_{\theta} [u\mu(\theta) + (1 - u)(1 - s)]y + [u(1 - \mu(\theta)) + (1 - u)s]b - k\theta u + \beta P(u')$$

The foc is

$$\mu'(\theta)(y - b - \beta P'(u')) = k$$

and the Envelope condition

$$P'(u) = [\mu(\theta) - (1 - s)]y + [1 - \mu(\theta) - s]b - k\theta + \beta P'(u')[1 - \mu(\theta) - s]$$

Guess:

$$P(u) = \alpha_0 + \alpha_1 u.$$

From the Envelope condition we get:

$$\alpha_1 = \frac{[1 - \mu(\theta) - s](b - y) - k\theta}{1 - \beta(1 - \mu(\theta) - s)}$$

and from the first order condition θ^* must satisfy

$$\frac{\mu'(\theta^*)(y - b + \beta k\theta^*)}{1 - \beta(1 - \mu(\theta^*) - s)} = k.$$

Plugging this back into the planner problem you can check that the guess is verified:

$$\begin{aligned} P(u) &= [u\mu(\theta) + (1 - u)(1 - s)]y + [u(1 - \mu(\theta)) + (1 - u)s]b - k\theta u \\ &\quad + \beta\alpha_0 + \beta\alpha_1[u(1 - \mu(\theta)) + (1 - u)s] \end{aligned}$$

Finally, we can rewrite the planner condition above as

$$k = \frac{\mu'(\theta)(y - b)}{1 - \beta(1 - s) + \beta\left[1 - \frac{\theta\mu'(\theta)}{\mu(\theta)}\right]\mu(\theta)}.$$

The equilibrium condition (12) can be rewritten as

$$k = \frac{\frac{\mu(\theta)}{\theta}(1 - \eta)(y - b)}{1 - \beta(1 - s) + \beta\eta\mu(\theta)}.$$

This implies that the equilibrium is constrained efficient if and only if

$$1 - \eta = \frac{\theta\mu'(\theta)}{\mu(\theta)},$$

which is called ‘‘Mortensen-Hosios condition’’.

Proposition 2 *The equilibrium does not generically achieve a constrained efficient allocation.*

There is efficiency only if a knife-edge condition is satisfied: the firm bargaining power is equal to the elasticity of the job finding function $\mu(\theta)$. If this elasticity is higher, more firms entry benefits workers more at the margin and hence a higher firms’ bargaining power internalizes this and generates more entry. If the firm bargaining power is too low, there would be too little entry, that is, market tightness would be too low, while if the firm bargaining power is too high, there would be too high market tightness.

1.6 Dynamic problem

Let us go back to the dynamic problem. To make it more tractable, notice that we can rewrite the original Bellman equations in terms of the net present value of wages and income as follows:

$$U_t = \mu(\theta_t)(\hat{w}_t + \beta V_{t+1}) + (1 - \mu(\theta_t))(b + \beta U_{t+1}) \quad (13)$$

$$V_t = s(b + \beta U_{t+1}) + (1 - s)\beta V_{t+1} \quad (14)$$

$$k = \frac{\mu(\theta_t)}{\theta_t}(\hat{y} - \hat{w}_t) \quad (15)$$

where

$$\hat{w}_t = \sum_{\tau=t}^{\infty} [(1-s)\beta]^{\tau-t} w_{\tau},$$

and

$$\hat{y} = \frac{y}{1 - (1-s)\beta}.$$

Given linear utility, what matters is the net present value of the wages in the employment relationship and the net present value of the production. Hence, we can assume that the bargaining for matches created at time t is over the net present value of wages \hat{w}_t .

From Nash Bargaining we have

$$\hat{w}_t = \max_{\omega} [\omega + \beta V_{t+1} - b - \beta U_{t+1}]^{\eta} [\hat{y} - \omega]^{1-\eta}$$

where V_{t+1} and J_{t+1} do not depend on ω !

Then the foc give us

$$\eta(\hat{y} - \hat{w}_t) = (1 - \eta)[\hat{w}_t - b + \beta(V_{t+1} - U_{t+1})].$$

Define

$$D_t \equiv U_t - V_t.$$

At the optimum

$$\hat{w}_t - b - \beta D_{t+1} = \eta[\hat{y} - b - \beta D_{t+1}], \quad (16)$$

$$\hat{y} - \hat{w}_t = (1 - \eta)[\hat{y} - b - \beta D_{t+1}]. \quad (17)$$

Combining equations (13) and (14) we obtain

$$D_t = \mu(\theta_t)(\hat{w}_t - b - \beta D_{t+1}) + (1 - s)(b + \beta D_{t+1}).$$

From (16)

$$\hat{w}_t - b - \beta D_{t+1} = \eta (\hat{y} - b - \beta D_{t+1})$$

Hence we can rewrite

$$D_t = \mu(\theta_t) \eta (\hat{y} - b - \beta D_{t+1}) + (1 - s) (b + \beta D_{t+1})$$

or

$$D_t = \mu(\theta_t) \eta (\hat{y} - b) + (1 - s) b + (1 - s - \mu(\theta_t) \eta) \beta D_{t+1}$$

Rewriting this one period forward

$$D_{t+1} = \mu(\theta_{t+1}) \eta (\hat{y} - b) + (1 - s) b + (1 - s - \mu(\theta_t) \eta) \beta D_{t+2}$$

From (15) and (17) we obtain

$$\begin{aligned} \beta D_{t+1} &= \hat{y} - b - \frac{k}{1 - \eta} \frac{\theta_t}{\mu(\theta_t)}, \\ \beta D_{t+2} &= \hat{y} - b - \frac{k}{1 - \eta} \frac{\theta_{t+1}}{\mu(\theta_{t+1})}. \end{aligned}$$

Plugging the last three equations together we obtain

$$\frac{k}{1 - \eta} \left[\frac{\theta_t}{\mu(\theta_t)} - \frac{\beta (1 - s - \mu(\theta_t) \eta) \theta_{t+1}}{\mu(\theta_{t+1})} \right] = (1 - \beta (1 - s)) \hat{y} - b$$

To characterize an equilibrium is enough to find a sequence $\{\theta_t\}_{t=0}^{\infty}$ that solves this difference equation. Notice that $\theta_t = \theta$ for all t is a solution and gives the steady state value of θ . This is consistent with Proposition 1. However, generically this may be not the only solution. I am going to show you later on that this it must be the only solution when $1 - \eta = \theta \mu'(\theta) / \mu(\theta)$.

2 Competitive Search

2.1 Set Up

As in the previous model, there is a continuum of measure 1 of risk-neutral homogeneous workers and a continuum of larger measure of risk-neutral homogeneous firms. The discount factor is β . Here, at the beginning of the period the firm decide to pay k to open a vacancy

and if she does so, she also posts a wage. Next, workers observe all the posted wages and decide where to apply. There is a potential submarket for any given $w \in R_+$. The matching function is the same as before. For any given wage w , there is an associated market tightness $\Theta(w)$ and $\mu(\Theta(w))$ denotes the probability a worker finds a job that posts w and $\mu(\Theta(w))/\Theta(w)$ denotes the probability a firm that posts w finds a worker applying for it. After matching, the employment relationship is implemented. A match created at time t has a probability s of being destroyed at each time $\tau > t$.

2.2 Equilibrium

We define a SS Competitive Search Equilibrium as follows.

Definition 3 *A SS CSE is an allocation $\{\bar{U}, W, \Theta\}$ with $U \in R_+$, $W \subset R_+$, and $\Theta : R_+ \mapsto [0, \infty]$ satisfying*

1. *firms' profit maximization and free-entry: for any $w \in R_+$*

$$\frac{\mu(\Theta(w))}{\Theta(w)}(y - w) - k \leq 0$$

with equality if $w \in W$;

2. *workers' optimal application: for any $w \in R_+$*

$$\mu(\Theta(w))(w + \beta\bar{V}) + (1 - \mu(\Theta(w)))(b + \beta\bar{U}) \leq \bar{U}$$

with equality if $\Theta(w) < \infty$, where

$$\bar{U} = \max \left\{ \sup_{w \in W} \mu(\Theta(w))(w + \beta\bar{V}) + (1 - \mu(\Theta(w)))(b + \beta\bar{U}), b + \beta\bar{U} \right\},$$

and $\bar{U} = b + \beta\bar{U}$ if $W = \emptyset$, and

$$\bar{V} = \frac{s(b + \beta\bar{U})}{1 - \beta(1 - s)}.$$

In equilibrium, firms profits are driven to zero by free-entry and workers apply to the contracts that offer them the highest possible expected utility. This definition of equilibrium embeds a notion of subgame perfection. The market tightness function impose restrictions

on beliefs about the market tightness of potential deviations. Imagine a deviation, as a measure ε of vacancies posting a wage w' . Then either $\Theta(w') = \infty$ and nobody is applying for that wage, or $\Theta(w') < \infty$ and it must be that workers obtain the same expected utility they would get in equilibrium. If they were getting something better, this would not be an equilibrium!

2.3 Characterization

We can now characterize the equilibrium

Proposition 3 *If $\{\bar{U}, W, \Theta\}$ is a CSE, then any $w \in W$ and $\theta = \Theta(w)$ solve*

$$\begin{aligned} \bar{U} &= \max_{w, \theta} \mu(\theta) (w + \beta \bar{V}) + (1 - \mu(\theta)) (b + \beta \bar{U}) & (P1) \\ \text{s.t. } & \frac{\mu(\theta)}{\theta} (y - w) = k, \end{aligned}$$

where

$$\bar{V} = \frac{s(b + \beta \bar{U})}{1 - \beta(1 - s)}.$$

Conversely, if some pair $\{w, \theta\}$ solves P1, then there exists a CSE $\{\bar{U}, W, \Theta\}$ such that $w \in W$ and $\theta = \Theta(w)$.

Proof. Step 1. Let $\{\bar{U}, W, \Theta\}$ be a CSE with $w \in W$ and $\theta = \Theta(w)$, then (w, θ) solve P1. First, profit maximization implies that the constraint is immediately satisfied. Second, suppose $\exists (w', \theta')$ that satisfy the constraint but achieve higher utility, that is,

$$\mu(\theta') (w' + \beta \bar{V}) + (1 - \mu(\theta')) (b + \beta \bar{U}) > \bar{U}.$$

Notice that for this condition to be satisfied it must be that $w' \geq b + \beta(\bar{U} - \bar{V})$, given that workers' optimal application requires $\bar{U} \geq b + \beta \bar{U}$. Workers' optimal application also implies

$$\mu(\Theta(w')) (w' + \beta \bar{V}) + (1 - \mu(\Theta(w'))) (b + \beta \bar{U}) \leq \bar{U}.$$

Given that $w' \geq b + \beta(\bar{U} - \bar{V})$, the LHS of the two conditions above is increasing in $\mu(\cdot)$ which implies that

$$\Theta(w') < \theta'.$$

Hence, from the assumption that (w', θ') satisfies the constraint, we obtain

$$\frac{\mu(\Theta(w'))}{\Theta(w')} (y - w') - k > \frac{\mu(\theta')}{\theta'} (y - w') - k = 0,$$

which contradicts profit maximization. This implies that (w, θ) solve problem P1!

Step 2. For any (w, θ) solving P1, we can construct a CSE $\{\bar{U}, W, \Theta\}$ with $W = \{w\}$, with

$$\bar{U} = \mu(\theta) (w + \beta\bar{V}) + (1 - \mu(\theta)) (b + \beta\bar{U}),$$

and Θ such that

$$\mu(\Theta(w)) (w + \beta\bar{V}) + (1 - \mu(\Theta(w))) (b + \beta\bar{U}) = \bar{U}$$

for any $w \geq b + \beta(\bar{U} - \bar{V})$ and $\Theta(w) = \infty$ otherwise. Optimal workers' application is satisfied by construction. Let us now check that firms maximize profits. Suppose $\exists w'$ such that

$$\frac{\mu(\Theta(w'))}{\Theta(w')} (y - w') - k > 0.$$

Notice that, by the construction of Θ , it must be that $w' \geq b + \beta(\bar{U} - \bar{V})$. Define θ' so that

$$\frac{\mu(\theta')}{\theta'} (y - w') - k = 0.$$

Then by construction $\theta' \geq \Theta(w')$, which together with $w' > b + \beta(\bar{U} - \bar{V})$ gives

$$\mu(\theta') (w' + \beta\bar{V}) + (1 - \mu(\theta')) (b + \beta\bar{U}) > \bar{U}.$$

This is a contradiction because (w', θ') satisfy the constraint but achieve a higher value than \bar{U} , so that (w, θ) could not be the solution to P1! ■

Proposition 4 *There exists a unique SS CSE.*

Proof. Existence can simply be proved by showing that there exists a solution to P1 together with the second part of the previous proposition. The objective function in P1 is continuous in θ and w and the constraint set is compact given that the RHS of the constraint is also continuous in both arguments and non-empty ($\theta = 0$ and $w = y - k$).

To prove uniqueness, define the map T

$$T(U) = \max \mu(\theta) y - \theta k + \left(1 - \frac{\mu(\theta)(1 - \beta)}{1 - \beta(1 - s)} \right) (b + \beta U).$$

For given U , there exists a unique θ that solves this maximization problem (given that μ is strictly concave). Moreover

$$|T'(U)| = \left| \beta \left[1 - \mu(\theta) \left(\frac{1 - \beta}{1 - \beta(1 - s)} \right) \right] \right| \leq \beta.$$

Hence, T is a contraction and there exists a unique \bar{U} such that $T(\bar{U}) = \bar{U}$. ■

Next, let me define a CSE (natural extension of the definition of a SS CSE).

Definition 4 A CSE is a sequence of allocations $\{\bar{U}_t, \bar{V}_t, W_t, \Theta_t\}_{t=0}^{\infty}$ with U_t and V_t non-negative and bounded, $W_t \subset R_+$, and $\Theta_t : R_+ \mapsto [0, \infty]$ satisfying

1. firms' profit maximization and free-entry at any t : for any $w_t \in R_+$

$$\frac{\mu(\Theta_t(w_t))}{\Theta_t(w_t)} (y - w_t) - k \leq 0$$

with equality if $w_t \in W_t$;

2. workers' optimal application at any t : for any $w_t \in R_+$

$$\mu(\Theta_t(w_t)) (w_t + \beta \bar{V}_{t+1}) + (1 - \mu(\Theta_t(w_t))) (b + \beta \bar{U}_{t+1}) \leq \bar{U}_t$$

with equality if $\Theta(w) < \infty$, where

$$\bar{U}_t = \max \left\{ \sup_{w_t \in R_+} \mu(\Theta_t(w_t)) (w_t + \beta \bar{V}_{t+1}) + (1 - \mu(\Theta_t(w_t))) (b + \beta \bar{U}_{t+1}), b + \beta \bar{U}_{t+1} \right\},$$

and

$$\bar{V}_t = s (b + \beta \bar{U}_{t+1}) + (1 - s) \beta \bar{V}_{t+1}.$$

We next show that a CSE has no interesting dynamics, except for the unemployment rate.

Proposition 5 A CSE where $\bar{U}_t, \bar{V}_t, W_t$, and Θ_t are constant at the SS values is the unique CSE.

Sketch of the Proof. First, you can extend proposition 3 to show that a CSE can be characterized using the following problem

$$\Phi(D_{t+1}) = \max_{\theta_t} \mu(\theta_t) y - \theta_t k + (1 - s - \mu(\theta_t)) (b + \beta D_{t+1}),$$

where

$$D_t = \bar{U}_t - \bar{V}_t.$$

In particular, 1) given sequences $\{\bar{U}_t\}$ and $\{\bar{V}_t\}$, and hence $\{D_t\}$, the equilibrium sequence $\{\theta_t\}$ solves this maximization problem; and 2) given $\{\theta_t\}$, the equilibrium sequence $\{D_t\}$ solves $D_t = \Phi(D_{t+1})$ and $\{\bar{U}_t\}$ and $\{\bar{V}_t\}$ satisfy

$$\begin{aligned}\bar{U}_t &= \Phi(D_{t+1}) + \bar{V}_t, \\ \bar{V}_t &= s(b + \beta\bar{U}_{t+1}) + (1-s)\beta\bar{V}_{t+1}.\end{aligned}$$

Second, notice that from the definition of equilibrium, \bar{U}_t (and hence \bar{V}_t) must be bounded given that it is the continuation value for the unemployed workers and

$$0 \leq \bar{U}_t \leq \sum_{\tau=t}^{\infty} \beta^{\tau-t} y = \frac{y}{1-\beta}.$$

Hence, explosive paths cannot be an equilibrium. If \bar{U}_t and \bar{V}_t are constant at the steady state values, you can show that this is an equilibrium comparing the problem above and P1. Is this the unique equilibrium? If not, there exists an equilibrium with time-varying \bar{U}_t and \bar{V}_t . This would imply that D_t is time-varying as well. Notice that the maximization problem above has a unique maximum and $\Phi(D_{t+1})$ is a contraction given that

$$|\Phi'(D_{t+1})| = |(1-s-\mu(\theta_t))\beta| \leq \beta.$$

This implies that there exists a unique \bar{D} such that $\bar{D} = \Phi(\bar{D})$

$$D_t - \bar{D} = \Phi(D_{t+1}) - \Phi(\bar{D}),$$

and hence

$$\|D_t - \bar{D}\| < \beta \|D_{t+1} - \bar{D}\| < \dots < \beta^T \|D_{t+T} - \bar{D}\|.$$

If D_t is time-varying then $\|D_t - \bar{D}\| > \varepsilon$ for some t . This implies

$$\|D_{t+T} - \bar{D}\| > \beta^{-T} \varepsilon \rightarrow \infty.$$

and hence the sequence D_t is explosive. If D_t explodes, by construction, also \bar{U}_t and \bar{V}_t are explosive, which is a contradiction.

2.4 Constrained Efficiency

Propositions 3 and 5 show that we can use problem P1 to characterize any CSE, that is, the equilibrium θ must solve

$$\bar{U} = \max_{\theta} \mu(\theta) (y + \beta \bar{V}) - \theta k + (1 - \mu(\theta)) (b + \beta \bar{U}),$$

where

$$\bar{V} = \frac{s(b + \beta \bar{U})}{1 - \beta(1 - s)}.$$

Notice that this problem is equivalent to the following one:

$$D = \max_{\theta} \mu(\theta) y - \theta k + (1 - \mu(\theta) - s)b + \beta(1 - \mu(\theta) - s)D, \quad (18)$$

where

$$D \equiv \bar{U} - \bar{V}.$$

Recall that the planner problem is

$$P(u) = \max_{\theta, u'} u \mu(\theta) y + [u(1 - \mu(\theta)) + (1 - u)s]b - \theta u k + \beta P(u')$$

Notice that the objective function is slightly different from the one in section because now y is equal to the net present value of the production over the life of a match rather than per-period production. As before we can show that $P(u) = \alpha_0 + \alpha_1 u$, so that we can rewrite the planner problem as

$$P(u) = \alpha_0 + \left\{ \max_{\theta} \mu(\theta) y + [1 - \mu(\theta) - s]b - \theta k + \alpha_1 \beta [1 - \mu(\theta) - s] \right\} u.$$

To find the constrained efficient allocation it is then sufficient to solve the “mini-Bellman”:

$$\alpha_1 = \max_{\theta} \mu(\theta) y + (1 - \mu(\theta) - s)b - \theta k + \beta(1 - \mu(\theta) - s)\alpha_1.$$

This is exactly the same problem as 18 proving constrained efficiency!

Proposition 6 *The CSE is constrained efficient.*

3 Asymmetric Information

In the previous lectures we focused on how the efficiency properties of search models can be affected by differences in wage determination, that is, in the way the surplus from a match is divided. However, often firms and workers have different information on the size of the surplus they can create and this asymmetric information may distort the allocation and, in particular, the level of unemployment in the economy.

3.1 Set Up

Time is discrete and the horizon infinite. There is a continuum of measure 1 of ex-ante homogeneous and risk-neutral workers and a continuum of larger measure of ex-ante homogeneous and risk-neutral firms. They have a common discount factor β . Workers can search freely, while firms have to pay a cost k to open a vacancy. The environment is very similar to the one above, except that now the present value of the surplus of a match is not y but $y - x$, where x represents the present value of the match-specific disutility for the worker, e.g. the cost of effort that the worker has to exert to make the match productive. When a match is formed, the value of x is drawn from the cdf $F(\cdot)$ with full support on $X \equiv [\underline{x}, \bar{x}]$ and is observed privately by the worker. Assume that $F(\cdot)$ is differentiable, with $f(\cdot)$ denoting the associated density function, and satisfies the monotone hazard rate condition

$$\frac{d(F(x)/f(x))}{dx} > 0.$$

At the beginning of each period t , firms can open a vacancy at cost k which entitles them to post an employment contract. Invoking the revelation principle, without loss of generality, we can restrict attention to the set Ω_t of incentive-compatible and individually rational direct revelation mechanisms at time t . A contract posted at time t is a map $C_t : X \mapsto [0, 1] \times R_+$ specifying the hiring probability $e_t(\tilde{x}) \in [0, 1]$ and the discounted present value of transfers $\omega_t(\tilde{x}) \geq 0$ from the firm to the worker for each matched worker at time t who reports type \tilde{x} . Let \mathbb{C}_t be the set of posted contracts. Each worker observes \mathbb{C}_t and chooses to apply for a contract $C_t \in \mathbb{C}_t$. As in the baseline competitive search model, each contract is associated with a market tightness $\Theta_t(C_t)$, so that a worker who applies to C_t has a probability $\mu(\Theta(C_t))$ of finding a firm and a firm posting C_t has probability $\mu(\Theta_t(C_t))/\Theta_t(C_t)$ to meeting a worker. The function $\mu(\cdot)$ satisfy the same assumptions

we have made before. After matching takes place, the draw x is realized and observed by the worker. Then, the worker can choose to make a report \tilde{x} and implement the contract. In this case, he is hired with probability $e(\tilde{x})$ and he gets a discounted present value of transfers equal to $\omega(\tilde{x})$. If the worker is hired the match is productive until separation. Otherwise, he can choose to walk away and join the pool of the unemployed who get b and search for a job next period.

3.2 Bellman Values

Let $v_t(x, \tilde{x})$ denote the expected utility of a worker of type x matched at time t who reports type \tilde{x} , that is

$$v_t(x, \tilde{x}) \equiv \omega_t(\tilde{x}) - e_t(\tilde{x})(x + b + \beta(U_{t+1} - V_{t+1})) + b + \beta U_{t+1}, \quad (19)$$

where V_{t+1} denotes the continuation utility of being employed (net of wages and disutility) and U_{t+1} is the continuation utility of being unemployed (both at the beginning of time $t + 1$).

An employment contract C_t is incentive-compatible iff

$$v_t(x, x) \geq v_t(x, \tilde{x}) \text{ for all } x, \tilde{x} \in X, \quad (\text{IC})$$

and individually rational iff

$$v_t(x, x) \geq b + \beta U_{t+1} \text{ for all } x \in X. \quad (\text{IR})$$

Following standard results in the mechanism design literature, IC and IR are equivalent to $e_t(\cdot)$ non-increasing together with the following two conditions:

$$v_t(x, x) = v_t(\bar{x}, \bar{x}) + \int_x^{\bar{x}} e_t(z) dz \text{ for all } x \in X \quad (\text{IC}')$$

and

$$v_t(\bar{x}, \bar{x}) \geq b + \beta U_{t+1}. \quad (\text{IR}')$$

Lemma 1 *Conditions IC and IR are equivalent to IC' and IR' together with $e_t(\cdot)$ non-increasing.*

Proof. Step 1. First we prove necessity. From IC we have that for any pair $\tilde{x} > x$

$$\begin{aligned} v_t(x, x) &\geq v_t(x, \tilde{x}), \\ v_t(\tilde{x}, \tilde{x}) &\geq v_t(\tilde{x}, x), \end{aligned}$$

and hence

$$v_t(x, x) - v_t(\tilde{x}, x) \geq v_t(x, x) - v_t(\tilde{x}, \tilde{x}) \geq v_t(x, \tilde{x}) - v_t(\tilde{x}, \tilde{x})$$

Using (19) we can rewrite this expression as

$$e_t(x) \leq \frac{v_t(x, x) - v_t(\tilde{x}, \tilde{x})}{(\tilde{x} - x)} \leq e_t(\tilde{x})$$

which implies that $e_t(\cdot)$ must be nonincreasing. Moreover, letting $\tilde{x} \rightarrow x$, we obtain

$$v_t(x, x) = v_t(\bar{x}, \bar{x}) + \int_x^{\bar{x}} e_t(z) dz.$$

Finally, this also implies that a necessary condition for IR is

$$v_t(\bar{x}, \bar{x}) \geq b + \beta U_{t+1}.$$

Step 2. Second, we prove sufficiency. Consider any x and \tilde{x} and assume without loss of generality that $x < \tilde{x}$. Using IC' and the fact that e is nonincreasing one can derive

$$v_t(x, x) - v_t(\tilde{x}, \tilde{x}) = \int_x^{\tilde{x}} e_t(z) dz \geq \int_x^{\tilde{x}} e_t(\tilde{x}) dz = e_t(\tilde{x})(\tilde{x} - x).$$

It follows that

$$\begin{aligned} v_t(x, x) &\geq v_t(\tilde{x}, \tilde{x}) + e_t(\tilde{x})(\tilde{x} - x) \\ &= \omega_t(\tilde{x}) - e_t(\tilde{x})[x + b + \beta(U_{t+1} - V_{t+1})] = v_t(x, \tilde{x}) \end{aligned}$$

Similarly we can obtain

$$v_t(\tilde{x}, \tilde{x}) \geq v_t(\tilde{x}, x).$$

This proves that IC must be satisfied. Finally if IR' is satisfied and e is nonincreasing, it must be that IR is satisfied, completing the proof. ■

3.3 Equilibrium

The definition of equilibrium in recursive terms is the natural generalization of the definition in the baseline model.

Definition 5 A CSE is a sequence $\{\bar{U}_t, \bar{V}_t, \mathbb{C}_t, \Theta_t\}_{t=0}^{\infty}$, with \bar{U}_t and \bar{V}_t bounded and non-negative, $\mathbb{C}_t \subset \Omega_t$, and $\Theta_t : \Omega_t \mapsto [0, \infty]$, such that

1. firms' profit maximization and free entry at all t : for any $C_t = \{e_t(\cdot), \omega_t(\cdot)\} \in \Omega_t$

$$\frac{\mu(\Theta_t(C_t))}{\Theta_t(C_t)} \int_{\underline{x}}^{\bar{x}} (e_t(x)y - \omega_t(x)) dF(x) - k \leq 0,$$

with equality if $C_t \in \mathbb{C}_t$;

2. workers' optimal application at all t : for any $C_t = \{e_t(\cdot), \omega_t(\cdot)\} \in \Omega_t$

$$\mu(\Theta_t(C_t)) \int [\omega_t(x) - e_t(x)(x + b + \beta(\bar{U}_{t+1} - \bar{V}_{t+1}))] dF(x) + (b + \beta\bar{U}_{t+1}) \leq \bar{U}_t$$

with equality if $\Theta(w) < \infty$, where

$$\bar{U}_t = \sup_{C'_t \in \mathbb{C}_t} \mu(\Theta_t(C'_t)) \int [\omega_t(x) - e_t(x)(x + b + \beta(\bar{U}_{t+1} - \bar{V}_{t+1}))] dF(x) + (b + \beta\bar{U}_{t+1})$$

or $\bar{U}_t = b + \beta\bar{U}_{t+1}$ if \mathbb{C}_t is empty, and

$$\bar{V}_t = s(b + \beta\bar{U}_{t+1}) + (1 - s)\beta\bar{V}_{t+1}.$$

In equilibrium, firms' profits are driven to 0 by free entry and workers obtain expected utility equal to \bar{U}_t at each time t . If a measure ε of firms deviate at time t and post a contract C'_t , they expect the associated market tightness (and hence their probability of finding a worker) to be such that workers would get the equilibrium expected utility \bar{U}_t . In equilibrium, it must be that any such deviation is unprofitable.

As in the baseline model, we can use a simple constrained maximization problem to characterize the equilibrium.

Proposition 7 If $\{\bar{U}_t, \bar{V}_t, \mathbb{C}_t, \Theta_t\}_{t=0}^{\infty}$ is a CSE, then any pair (C_t, θ) with $C_t \in \mathbb{C}_t$ and $\theta = \Theta(C_t)$ solves

$$\bar{U}_t = \max_{e_t(\cdot), \omega_t(\cdot), \theta_t} \mu(\theta_t) \int_{\underline{x}}^{\bar{x}} [\omega_t(x) - e_t(x)(x + b + \beta(\bar{U}_{t+1} - \bar{V}_{t+1}))] dF(x) + (b + \beta\bar{U}_{t+1}) \tag{P2}$$

subject to IC' , IR' , $e_t(x) \in [0, 1]$ and non-increasing for all $x \in X$, and

$$\frac{\mu(\theta_t)}{\theta_t} \int_{\underline{x}}^{\bar{x}} (e_t(x) y - \omega_t(x)) dF(x) - k = 0. \quad (20)$$

Conversely, if a sequence $\{C_t, \theta_t\}_{t=0}^{\infty}$ solves problem P2 at any t and $\bar{V}_t = s(b + \beta\bar{U}_{t+1}) + (1-s)\beta\bar{V}_{t+1}$, then there exists an equilibrium $\{U_t, V_t, C_t, \Theta_t\}_{t=0}^{\infty}$ such that $C_t \in \mathbb{C}_t$ and $\Theta_t(C_t) = \theta_t$ for any t .

The proof of this proposition is very similar to the one for Proposition 3 and therefore omitted. I suggest you to go over it as homework.

The next proposition shows that the equilibrium characterization can be further simplified, by noticing that without loss of generality we can restrict attention to wage contracts, that is, a flat wage that a worker can accept or reject. Notice that a wage contract is equivalent to a direct revelation mechanism characterized by a hiring cut-off rule, a flat transfer paid to employed workers and zero transfer paid to the workers who are not hired.

Proposition 8 Take any C_t and θ_t that solve problem P3 for given \bar{U}_{t+1} and \bar{V}_{t+1} . The contract $C_t = \{e_t(\cdot), \omega_t(\cdot)\}$ takes the form of a wage contract, that is,

$$e_t(x) = \begin{cases} 1 & \text{if } x \leq \hat{x}_t \\ 0 & \text{if } x > \hat{x}_t \end{cases} \quad \text{and } \omega_t(x) = \begin{cases} w_t & \text{if } x \leq \hat{x}_t \\ 0 & \text{if } x > \hat{x}_t \end{cases}$$

where $w_t = \hat{x}_t + b + \beta(\bar{U}_{t+1} - \bar{V}_{t+1})$ and the pair (\hat{x}_t, θ_t) solves

$$\Phi(\bar{U}_{t+1} - \bar{V}_{t+1}) = \max_{\hat{x}_t, \theta_t} \mu(\theta_t) \int_{\underline{x}}^{\hat{x}_t} [y - x - b - \beta(\bar{U}_{t+1} - \bar{V}_{t+1})] dF(x) - \theta_t k \quad (P3)$$

subject to

$$\frac{\mu(\theta_t)}{\theta_t} F(\hat{x}_t) [y - \hat{x}_t - b - \beta(\bar{U}_{t+1} - \bar{V}_{t+1})] = k.$$

Proof. Consider problem P2. First, integrating both sides of constraint IC' we obtain

$$\int_{\underline{x}}^{\bar{x}} v_t(x, x) dF(x) = v_t(\bar{x}, \bar{x}) + \int_{\underline{x}}^{\bar{x}} e_t(x) F(x) dx \quad (21)$$

given that using integration by parts

$$\begin{aligned} \int_{\underline{x}}^{\bar{x}} \left(\int_x^{\bar{x}} e_t(z) dz \right) f(x) dx &= \left| \left(\int_x^{\bar{x}} e_t(z) dz \right) F(x) \right|_{\underline{x}}^{\bar{x}} + \int_{\underline{x}}^{\bar{x}} e_t(x) F(x) dx \\ &= \int_{\underline{x}}^{\bar{x}} e_t(x) F(x) dx. \end{aligned}$$

Then combining (21) and condition IR', together with (19), we obtain

$$\int_{\underline{x}}^{\bar{x}} \left[\omega(x) - e(x) \left(x + b + \beta (\bar{U}_{t+1} - \bar{V}_{t+1}) + \frac{F(x)}{f(x)} \right) \right] dF(x) \geq 0.$$

Using condition (20) we can substitute for $\omega_t(x)$ and problem P2 can be rewritten as

$$\max_{e_t(x) \in [0,1], \theta_t} \mu(\theta_t) \int_{\underline{x}}^{\bar{x}} e_t(x) (y - x - b - \beta (\bar{U}_{t+1} - \bar{V}_{t+1})) dF(x) - \theta_t k + b + \beta \bar{U}_{t+1} \quad (\text{P2}')$$

subject to

$$\mu(\theta_t) \int_{\underline{x}}^{\bar{x}} e_t(x) \left[y - x - b - \beta (\bar{U}_{t+1} - \bar{V}_{t+1}) - \frac{F(x)}{f(x)} \right] dF(x) \geq \theta_t k \quad (22)$$

and $e_t(x)$ non-increasing. For given \bar{U}_{t+1} and \bar{V}_{t+1} , any non-increasing function $e_t(\cdot)$ and value θ_t that solve problem P2 also solve problem P2'. Moreover, for any function $e_t(\cdot)$ and value θ_t that solve problem P2', conditions IC' and (19) can be used to recover the function $\omega_t(\cdot)$ so that $e_t(\cdot)$, $\omega_t(\cdot)$, and θ_t solve P2.

Consider now the relaxed version of problem P2' where we do not impose the monotonicity condition on $e_t(\cdot)$. Pointwise maximization together with the monotone hazard rate assumption imply that there exists a threshold \hat{x}_t such that $e_t(x) = 1$ if $x \leq \hat{x}_t$ and 0 otherwise, where

$$\hat{x}_t = y - b - \beta (\bar{U}_{t+1} - \bar{V}_{t+1}) - \frac{\lambda}{1 + \lambda} \frac{F(\hat{x}_t)}{f(\hat{x}_t)}, \quad (23)$$

with λ being the Lagrangian multiplier attached (22). This implies that $e_t(\cdot)$ is non-increasing and that a solution to the relaxed version of problem P2' is also a solution to P2. Also, notice that once we impose the cut-off function, problem P2' reduces to P3. Finally, we have to recover $\omega_t(\cdot)$. Using expression (19) we obtain

$$v_t(x, x) = \begin{cases} \omega_t(x) - x + \beta V_{t+1} & \text{if } x \leq \hat{x}_t \\ \omega_t(x) + b + \beta U_{t+1} & \text{if } x > \hat{x}_t \end{cases}.$$

Using IC' we obtain

$$\omega_t(x) = \begin{cases} \omega_t(\bar{x}) + \hat{x} + b + \beta (U_{t+1} - V_{t+1}) & \text{if } x \leq \hat{x}_t \\ \omega_t(\bar{x}) & \text{if } x > \hat{x}_t \end{cases}.$$

We now need to pin down $\omega_t(\bar{x})$. If IR' is binding, it must be that $\omega_t(\bar{x}) = 0$. Moreover, IR' is binding iff (22) is binding. Imagine (22) is not binding and $\lambda = 0$. Then substituting for \hat{x}_t using (23) in the constraint of P3 we obtain

$$\frac{\mu(\theta_t)}{\theta_t} \int_{\underline{x}}^{\hat{x}_t} \left[\hat{x}_t - x - \frac{F(x)}{f(x)} \right] dF(x) \geq k.$$

From integration by parts $\int_{\underline{x}}^{\hat{x}_t} (\hat{x}_t - x) dF(x) = \int_{\underline{x}}^{\hat{x}_t} F(x) dx$ yielding a contradiction given that $k > 0$. This implies $\omega_t(\bar{x}) = 0$ for all t , completing the proof. ■

This proposition simplifies the equilibrium characterization. The two key frictions are asymmetric information and workers' limited commitment. The first implies that firms cannot price discriminate, and the second that the workers cannot commit to make payments to the firm if they are not hired. This is why in equilibrium firms post wage contracts, offering a flat wage such that the marginal worker \hat{x}_t is indifferent between working and being unemployed, that is, $w_t = \hat{x}_t + b + \beta(\bar{U}_{t+1} - \bar{V}_{t+1})$. Hence, at time t , all the inframarginal workers with $x < \hat{x}_t$ get some positive surplus $\hat{x}_t - x$. This implies that ex ante a worker expects to appropriate an average surplus equal to $\mu(\theta_t) \int_{\underline{x}}^{\hat{x}_t} F(x) dx$, which we name expected informational rents. The constraint of P3 can be rewritten as

$$\mu(\theta_t) \int_{\underline{x}}^{\hat{x}_t} [y - x - b - \beta(\bar{U}_{t+1} - \bar{V}_{t+1})] dF(x) \geq \theta_t k + \mu(\theta_t) \int_{\underline{x}}^{\hat{x}_t} \frac{F(x)}{f(x)} dF(x)$$

and tells us that the expected net surplus of a match needs to cover not only the vacancy creation cost but also the workers' expected informational rents. The workers' informational rents distort the equilibrium allocation in comparison to the first best, as emerges from equation (23). If we wanted to implement the first-best, the firms would not have enough surplus to cover the vacancy cost! You can easily show that.

Proposition 9 *There exists a CSE characterized by constant \bar{U}_t , \bar{V}_t , w_t , \hat{x}_t , and θ_t are constant and independent of u_0 . If*

$$\Phi'(D) + \beta(1 - s) \geq -\beta \text{ for all } D \geq 0,$$

then the equilibrium is unique.

The proof of this proposition is very similar to the proof sketched for Proposition 5. You can also look at the paper for a discussion on the possibility of cycles.

3.4 Constrained Efficiency

Let me consider a social planner who faces the same frictions of the market economy, that is, he does not observe the type of the workers and cannot force workers to work. Moreover, a worker can always decide to join the anonymous pool of unemployed, hide from the planner

and enjoy b , and search for a job next period. If a worker enters the unemployment pool, his history is indistinguishable from that of any other unemployed. Given these constraints together with the resource constraints, the planner decides how many vacancies to open and how to allocate consumption.

An allocation is defined as a sequence of $\{e_t(\cdot), c_t(\cdot), C_t^U, \theta_t\}$, where $e_t(\tilde{x})$ is the hiring decision for a worker matched at time t who reports \tilde{x} , and $c_t(\tilde{x})$ is the discounted present value of the consumption of the same worker, while C_t^U is the consumption for unemployed workers at time t , and θ_t the market tightness at time t .

Now the expected utility of a matched worker at time t of type x who reports \tilde{x} is

$$v_t(x, \tilde{x}) \equiv c_t(\tilde{x}) - e_t(\tilde{x})(x - \beta V_{t+1}) + (1 - e_t(\tilde{x}))(C_t^U + \beta U_{t+1}),$$

where

$$V_t = s(C_t^U + \beta U_{t+1}) + (1 - s)\beta V_{t+1}, \quad (24)$$

and

$$U_t = \mu(\theta_t) \int_{\underline{x}}^{\bar{x}} v_t(x, x) dF(x) + (1 - \mu(\theta_t))(C_t^U + \beta U_{t+1}). \quad (25)$$

Similarly to the equilibrium analysis, one can derive that an allocation is incentive-compatible iff $e_t(\cdot)$ is nonincreasing and

$$v_t(x, x) = v_t(\bar{x}, \bar{x}) + \int_x^{\bar{x}} e_t(z) dz.$$

Moreover, workers' limited commitment requires

$$v_t(\bar{x}, \bar{x}) \geq C_t^U + \beta U_{t+1}. \quad (26)$$

Also, it must be that

$$C_t^U \geq b, \quad (27)$$

otherwise unemployed workers would hide.

Finally, the planner can transfer resources intertemporally at the interest rate $\beta^{-1} - 1$ and the intertemporal resource constraint takes the form

$$P_0 \equiv \sum \beta^t \{u_t[\mu(\theta_t) \int_{\underline{x}}^{\bar{x}} [e_t(x)(y - b + C_t^U) - c_t(x)] dF(x) + b - C_t^U - \theta_t k] + (1 - u_t)s(b - C_t^U)\} \geq 0, \quad (28)$$

where

$$u_{t+1} = u_t \left[1 - \mu(\theta_t) \int_{\underline{x}}^{\bar{x}} e_t(x) dF(x) \right] + (1 - u_t) s. \quad (29)$$

Definition 6 An allocation $\{e_t(\cdot), c_t(\cdot), C_t^U, \theta_t\}$ is feasible if there exists a bounded sequence $\{U_t, V_t\}$ such that the following are satisfied for all t : (i) incentive-compatibility constraints summarized by (??) and $e_t(\cdot)$ non-increasing; (ii) participation constraints (26) and (27); (iii) resource constraint (28); and (iv) the law of motion for unemployment (29).

Definition 7 For given u_0 , an allocation is constrained efficient if it maximizes U_0 subject to $V_0 \geq \bar{V}$ and feasibility.

For any given u_0 , the dual problem of the planner requires to minimize P_0 subject to $U_0 \geq \bar{U}$, $V_0 \geq \bar{V}$, and feasibility. The recursive version of this problem is

$$P(U_t, V_t, u_t) = \max_{\substack{e_t(\cdot), c_t(\cdot), C_t^U, \theta_t \\ U_{t+1}, V_{t+1}, u_{t+1}}} u_t \left[\mu(\theta_t) \int_{\underline{x}}^{\bar{x}} [e_t(x)(y - b + C_t^U) - c_t(x)] dF(x) + b - C_t^U - \theta_t k \right] \\ + (1 - u_t) s (b - C_t^U) + \beta P(U_{t+1}, V_{t+1}, u_{t+1})$$

subject to (??) and $e_t(\cdot)$ non-increasing, (26) and (27), (29), and the promise keeping constraints (24) and (25).

Proposition 10 The constrained efficient allocation $\{e_t^*(\cdot), c_t^*(\cdot), C_t^{U*}, \theta_t^*\}$ is characterized by

$$e_t^*(x) = \begin{cases} 1 & \text{if } x \leq \hat{x}_t^* \\ 0 & \text{if } x > \hat{x}_t^* \end{cases} \quad \text{and} \quad c_t^*(x) = \begin{cases} c_t^* & \text{if } x \leq \hat{x}_t^* \\ 0 & \text{if } x > \hat{x}_t^* \end{cases},$$

where $c_t^* = \hat{x}_t^* + C_t^{U*} + \beta(U_{t+1}^* - V_{t+1}^*)$ and \hat{x}_t^* , θ_t^* , and C_t^{U*} solve

$$P(U_t, V_t, u_t) = \max_{\substack{\hat{x}_t, \theta_t, C_t^U \\ U_{t+1}, V_{t+1}, u_{t+1}}} u_t [\mu(\theta_t) \int_{\underline{x}}^{\hat{x}} \left(y - x - b - \beta(U_{t+1} - V_{t+1}) - \frac{F(x)}{f(x)} \right) dF(x) \\ + b - C_t^U - \theta_t k] + (1 - u_t) s (b - C_t^U) + \beta P(U_{t+1}, V_{t+1}, u_{t+1})$$

subject to

$$\begin{aligned} [(1 - u_t) \nu_t] \quad V_t &\leq s (C_t^U + \beta U_{t+1}) + (1 - s) \beta V_{t+1}, \\ [u_t \eta_t] \quad U_t &\leq \mu(\theta_t) \int_{\underline{x}}^{\hat{x}} \frac{F(x)}{f(x)} dF(x) + C_t^U + \beta U_{t+1}, \\ [\pi_t] \quad u_{t+1} &= u_t \left[1 - \mu(\theta_t) \int_{\underline{x}}^{\hat{x}} dF(x) \right] + (1 - u_t) s, \\ [\chi_t] \quad C_t^U &\geq b. \end{aligned}$$

This proposition allows to simplify the planner problem and to make it more directly comparable with the problem that characterizes the CSE. The proof of this proposition follows a similar logic than the proof of Proposition 8 and hence I leave it to you as homework. Notice that the main difference is that the planner problem depends on the state variable u_t and hence may have different dynamics from the equilibrium. In fact, we can show that the CSE is generically constrained inefficient.

Proposition 11 *If $u_0 \neq u^{SS}$, then the competitive search equilibrium is constrained inefficient.*

Proof. Proceeding by contradiction, suppose the CSE is constrained efficient for a given initial value $u_0 \neq u^{SS}$. One can show that the foc of the two problems yield a contradiction!

Using the foc of problem P3, for given \bar{U} and \bar{V} , an equilibrium can be characterized by a \hat{x} , θ and λ such that

$$y - \hat{x}_t - b - \beta (\bar{U}_{t+1} - \bar{V}_{t+1}) - \frac{\lambda_t}{1 + \lambda} \frac{F(x)}{f(x)} = 0$$

$$\mu'(\theta_t) \int_{\underline{x}}^{\hat{x}_t} \left[y - x - b - \beta (\bar{U}_{t+1} - \bar{V}_{t+1}) - \frac{\lambda_t}{1 + \lambda_t} \frac{F(x)}{f(x)} \right] dF(x) = k$$

$$\mu(\theta_t) \int_{\underline{x}}^{\hat{x}_t} \left[y - x_t - b - \beta (\bar{U}_{t+1} - \bar{V}_{t+1}) - \frac{F(x)}{f(x)} \right] dF(x) = \theta_t k.$$

The foc of the planner problem are

$$y - \hat{x}_t - b + C_t^U - (1 - \eta_t) \frac{F(\hat{x}_t)}{f(\hat{x}_t)} - \pi_t = 0$$

$$\mu'(\theta_t) \int_{\underline{x}}^{\hat{x}} \left(y - x - b + \beta (U_{t+1} - V_{t+1}) - (1 - \eta_t) \frac{F(x)}{f(x)} - \pi_t \right) dF(x) = \theta_t k$$

$$\chi_t = u_t (1 - \eta_t) + (1 - u_t) (1 - \nu_t) s$$

$$P_U - u_t \mu(\theta_t) F(\hat{x}_t) + (1 - u_t) s \nu_t + u_t \eta_t = 0$$

$$P_V + u_t \mu(\theta_t) F(\hat{x}) + (1 - u_t) (1 - s) \nu_t = 0$$

$$P_u = \pi_t$$

Suppose the CSE is constrained efficient. Then all these equations must be satisfied. First combining the first two foc with the equilibrium conditions we get

$$\begin{aligned} \left(1 - \eta_t - \frac{\lambda_t}{1 + \lambda}\right) \frac{F(\hat{x}_t)}{f(\hat{x}_t)} + \pi_t &= 0 \\ \int_{\underline{x}}^{\hat{x}} \left(1 - \eta_t - \frac{\lambda_t}{1 + \lambda}\right) \frac{F(x)}{f(x)} dF(x) + \pi_t \int_{\underline{x}}^{\hat{x}} dF(x) &= 0 \end{aligned}$$

Given that

$$\frac{F(\hat{x}_t)}{f(\hat{x}_t)} \left(\frac{\int_{\underline{x}}^{\hat{x}} \frac{F(x)}{f(x)} dF(x)}{\int_{\underline{x}}^{\hat{x}} dF(x)} \right)^{-1} > 1$$

then it must be that $\eta_t = 1 - \lambda/(1 + \lambda)$ and $\pi_t = 0$.

From the Envelope conditions

$$\begin{aligned} P_U &= -u_t \eta_t \\ P_V &= -(1 - u_t) \nu_t \end{aligned}$$

Then

$$\begin{aligned} u_{t+1} (1 - \eta_{t+1}) - (1 - u_t) s (1 - \nu_t) - u_t (1 - \eta_t) &= 0 \\ -(1 - u_{t+1}) \nu_{t+1} + u_t \mu(\theta_t) F(\hat{x}) + (1 - u_t) (1 - s) \nu_t &= 0 \end{aligned}$$

Combining the two we get

$$\nu_t (1 - u_t) = (1 - u_t) - \frac{\lambda}{s} (u_{t+1} - u_t)$$

and this leads to

$$(u_{t+1} - u_t) [\mu(\theta) F(\hat{x}) + s] = 1$$

which gives a contradiction if $u_t \neq u^{SS}$. ■

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