14.461 Lectures 21-22
Money Search

Veronica Guerrieri

1 Kiyotaki and Wright

- time is discrete and runs forever
- there are three nondivisible goods 1, 2 and 3
- there is a continuum of infinitely lived agents of measure 1 who specialize in both consumption and production
- there are equal proportion of three types of agents, where type-i agent enjoys utility only from the consumption of good i and is able to produce only good $i^* \neq i$
- goods can be stored and each agent can store at most one good
- c_{ij} denotes the cost for agent of type i to store good of type j, with $c_{i3} > c_{i2} > c_{i1} > 0$ for all i

Type i expected utility is

$$E \sum \beta^t \left[I_i^U (t) U_i - I_{i^*}^D (t) D_i - I_{ij}^c (t) c_{ij} \right],$$

where U_i denotes the istantaneous utility from consuming good i and D_i the disutility from producing good i, and I_i^U, $I_{i^*}^D$, and I_{ij}^c are indicator function that are equal to one respectively if the agent consumes good i, if the agent produces i^* and if he stores good j. We assume that the gains from trade from consuming i and producing i^* are large enough that the agents want to be active in the economy, that is, we assume that for all i and k

$$u_i \equiv U_i - D_i > \frac{c_{ii^*} - c_{ik}}{1-\beta}.$$

Each period agents are matched randomly in pairs and must decide whether to trade or not. If an agent is lucky to get good i, he is going to consume and produce i^*. Hence, every agent i has one unit of one good other than i. The distribution of potential matches is characterized by the time path of $P(t) = \{p_{ij}(t)\}_{i,j \neq i}$ where $p_{ij}(t)$ is the proportion of type i agents holding good j in inventory at date t.

Each individual chooses a trading strategy to maximize his expected utility, taking as given the strategies of the other agents and $P(t)$. A trading strategy specify the rule determining under which circumstances an agent is willing to trade, in general as a function of the time and his whole history. However, we focus on steady-state equilibria where $P(t) = P$ for all t, and we restrict attention to strategies that depend only on the good j in storage and the good k of the agent met. Hence, the trading strategy of agent i is given by $\tau_i(j,k)$, which is equal to 1 if i wants to trade j for k. If type i with good j meets type h with good k, then there is trade iff $\tau_i(j,k)\,\tau_h(k,j) = 1$.

Definition 1 *A steady-state Nash equilibrium is a set of trading strategies $\{\tau_i\}$, one for each i, together with a steady-state distribution of inventories P, that satisfy 1) maximization, that is, each agent i chooses τ_i to maximize expected utility given the strategies of the others and P, and 2) rational expectations, that is, given $\{\tau_i\}$, P is the resulting steady-state distribution.*

Let $V_i(j)$ be the expected utility of agent i when he exits a trading opportunity with good j, given that he follows the maximizing strategy. First, if he exits with good i, then he consumes it and produce one unit of good i^*, that is,

$$V_i(i) = u_i + V_i(i^*).$$

The indirect utility of storing good $j \neq i$ is

$$V_i(j) = -c_{ij} + \max \beta E\left[V_i(j') \,|j\right].$$

Let $V_{ij} \equiv V_i(j)$. Then an optimal strategy requires

$$\tau_i(j,k) = 1 \text{ iff } V_{ij} > V_{ik}.$$

We assume that i will not trade if $V_{ij} = V_{ik}$. For $j = k$, trade is irrelevant and $\tau_i(j,k) = 0$. Clearly, agents of the same type do not trade in equilibrium.

Lemma 1 *Under our assumptions, each type i will accept good i, consume it and produce a new unit of good i^* whenever he has an opportunity, that is,* $\max_j V_{ij} = V_{ii}$ *for all i.*

Proof. Suppose that agent i prefers good k to i, that is, $V_{ik} = \max_j V_{ij}$ for $k \neq i$. Then, if i aquires k, he keeps it forever, so

$$V_{ik} = -\frac{c_{ik}}{1-\beta} \geq V_{ii} \geq u_i - \frac{c_{ii^*}}{1-\beta},$$

which implies

$$u_i \leq \frac{c_{ii^*} - c_{ik}}{1-\beta},$$

which contradicts our assumption! Then agent i always prefer good i. If he does not consume it

$$V_{ii} = -\frac{c_{ii}}{1-\beta} \geq u_i + V_{ii^*} \geq u_i - \frac{c_{ii^*}}{1-\beta},$$

which again lead to a contradiction, completing the proof.

Since agent i always wants to consume good i and produce i^*, we know that $\tau_i(j,i) = 1$. Therefore trade always happenms when a double coincidence happens, that is, type i with good j meets type j with good i. Moreover, trade will never happen if two agents i and j meet and have the same good k.

In order to find an equilibrium, one can guess the trading strategies (an ordering for the V) and then figure out the steady-state distribution P. Finally, check if the conjectured strategies maximixe expected utilities, given P and the others' strategies. Given that there is a finite number of possible strategies, we can completely characterize the set of equilibria.

1.1 Model A

Imagine type 1 produces good 2, type 2 produces good 3 and type 3 produces good 1. There are three possible cases: a "fundamental" equilibrium exists, a "speculative" equilibrium exists, no equilibria exist. In a "fundamental" equilibrium, agents always prefer a lower-storage-cost commodity to a higher-storage-cost one (unless the latter is their own consumption good!), so that agents need to look only at storage costs and utility values to decide trading strategies. In a speculative equilibrium, sometimes agents trade a lower- for a higher-storage-cost commodity simply because it has higher market value (the best way to trade for the good they want to consume).

In a fundamental equilibrium, agents always prefer their consumption good, and otherwise the lower-storage-cost one, that is, $V_{ii} = \max_j V_{ij}$ and $V_{12} > V_{13}$, $V_{21} > V_{23}$, and $V_{31} > V_{32}$.

Consider first agent of type I. When he exits a match with good 2 he pays c_{12} and next period he meets type I, II, and III with prob. $1/3$. If he meets type I, he cannot trade because so he keeps good 2 and leaves with V_{12}. If he meets type II, with probability p_{21} there is a double coincidence and he leaves with $u_1 + V_{12}$ and with probability p_{23} he has an option to leave with V_{12} or V_{13} (given that II wants always to trade good 3 for 2). If he meets III he cannot trade (given that III will never accept good 2) and he leaves with V_{12}. Then

$$V_{12} = -c_{12} + b\left[V_{12} + p_{21}\left(u_1 + V_{12}\right) + p_{23}\max\left(V_{12}, V_{13}\right) + V_{12}\right],$$

where $b = \beta/3$. Similarly

$$V_{13} = -c_{13} + b\left[V_{13} + V_{13} + p_{31}\left(u_1 + V_{12}\right) + p_{32}\max\left\{V_{12}, V_{13}\right\}\right]$$

Then

$$V_{12} - V_{13} = c_{13} - c_{12} + b\left[2\left(V_{12} - V_{13}\right) + \left(p_{21} - p_{31}\right)\left(u_1 - \max\left\{0, V_{13} - V_{12}\right\}\right)\right].$$

Either $V_{12} - V_{13} > 0$ and

$$(1 - 2b)\left(V_{12} - V_{13}\right) = c_{13} - c_{12} + \left(p_{21} - p_{31}\right) b u_1$$

or $V_{12} - V_{13} < 0$ and

$$\left(1 - 2b - b\left(p_{21} - p_{31}\right)\right)\left(V_{12} - V_{13}\right) = c_{13} - c_{12} + \left(p_{21} - p_{31}\right) b u_1$$

Given that

$$1 - 2b - b\left(p_{21} - p_{31}\right) > 1 - 2b - b > 1 - 3\beta/3 > 0,$$

$V_{12} - V_{13}$ has always the same sign of $c_{13} - c_{12} + \left(p_{21} - p_{31}\right) b u_1$ and hence $V_{12} > V_{13}$ iff $c_{13} - c_{12} > \left(p_{31} - p_{21}\right) b u_1$. This gives parameter values for P such that this is the best response of type I.

Next, consider a typical agent II. We know that

$$V_{21} = -c_{21} + b\left[p_{12}\left(u_2 + V_{23}\right) + p_{13}\max\left(V_{21}, V_{23}\right) + V_{21} + p_{31}V_{21} + p_{32}\left(u_2 + V_{23}\right)\right]$$

and

$$V_{23} = -c_{23} + b\left[V_{23} + V_{23} + p_{31}\max\left(V_{21}, V_{23}\right) + p_{32}\left(u_2 + V_{23}\right)\right].$$

This implies that

$$\left(V_{21} - V_{23}\right)(1-b) = c_{23} - c_{21} + b\left[p_{12}\left(u_2 + V_{23}\right) + \left(p_{13} - p_{31}\right)\max\left(V_{21}, V_{23}\right) + p_{31}V_{21} - V_{23}\right]$$

so that if $V_{21} < V_{23}$

$$V_{21} - V_{23} = \frac{c_{23} - c_{21} + bp_{12}u_2}{1 - b\left(1 - p_{31}\right)},$$

which is a contradiction. Hence, $V_{21} > V_{23}$ for all parameter values and P. Similar argument for type III implies that $V_{31} > V_{32}$ for all parameter values and P. Hence, a fundamental equilibrium exists iff $c_{13} - c_{12} > \left(p_{31} - p_{21}\right)bu_1$, which means that the relative cost of storing good 3 rather than 2 is higher than the relative marketability of good 3 compared to good 2.

The steady state inventory distribution P can be summarized with three numbers, given that $p_{ii} = 0$ and $\sum_j p_{ij} = 1$. For the fundamental strategies $\left(p_{12}, p_{23}, p_{31}\right) = (1, .5, 1)$. In equilibrium, type I and III only trade for their consumption good, while agents of type II trade their production good 3 for good 1 whenever possible, acting as middle men, transferring good 1 from type III to type I. Hence, good 1 is the only medium of exchange (or commodity money).

If $c_{13} - c_{12} > \left(p_{31} - p_{21}\right)bu_1$, the best response for type I to fundamental play is to speculate and try to exchange good 2 for 3 which has higher marketability. The best response of type II and III to this strategy of type I is still fundamental play, that is, $V_{ii} = \max_j V_{ij}$ and $V_{12} < V_{13}$, $V_{21} > V_{23}$, and $V_{31} > V_{32}$ represent a speculative equilibrium iff $c_{13} - c_{12} > \left(p_{31} - p_{21}\right)bu_1$. The inventory distrubution implied is $\left(p_{12}, p_{23}, p_{31}\right) = (.5\sqrt{2}, \sqrt{2}-1, 1)$ and so the condition becomes $c_{13} - c_{12} < (\sqrt{2}-1)bu_1$. Now also type I agents play the role of middlemen in some trades, transferring good 3 from type II to III. Type II agents still use good 1 as a medium of exchange. Hence, in this equilibrium we have dual commodity monies, botth the most storable and the least storable goods are used in indirect trade. Type I use a medium of exchange an object that is dominated in rate of return!

We can show that there are no other set of stratedies that are consistent in equilibrium. Hence, in the intermediate region there is no pure strategy steady-state equilibrium in which all agents of the same type play the same strategy. There may be mixed strategy equilibria...

To sum up, if $c_{13} - c_{12} > .5bu_1$ there exists a fundamental equilibrium where only good 1 serves as commodity money, if $c_{13} - c_{12} > (\sqrt{2}-1)bu_1$ there exists a speculative equilibrium where both goods 1 and 3 serve as commodity monies, and if $.5bu_1 < c_{13} - c_{12} < (\sqrt{2}-1)bu_1$ there is no equilibirum.

Two assumptions can be relaxed under some parameter restrictions: agents need to produce to consume and cannot freely dispose of goods.

1.2 Model B

An alternative model assumes that $1^* = 3$, $2^* = 1$, and $3^* = 2$. It turns out that for all parameter values, there is a fundamental equilibrium where agents prefer the lower-storage-cost goods except for their consumption goods, that is, $V_{12} > V_{13}$, $V_{21} > V_{23}$, and $V_{31} > V_{32}$. Moreover under some parameter restriction, there is a speculative equilibrium where types II and III speculate while I play fundamental strategy, that is, $V_{12} > V_{13}$, $V_{21} < V_{23}$, and $V_{31} < V_{32}$.

1.3 Fiat Money

Now imagine that the economy is endowed with a fixed quantity M of a new object called good 0. It does not give utility or help in production, that is, is fiat money. Assume $c_{i0} = 0$ for all i and still agents can only store one unit of goods and hence, even if good 0 is divisible, they will always store only one good. The question is: there are equilibria where good 0 is used as medium of exchange? If ϕ units of good 0 are requested to buy one unit of each commodity, then the quantity of real balances in circulation is equal to $S = M/\phi$. Each agent holding fiat money will have exactly ϕ units stored, hence S is also the proportion of all agents holding good 0, that is $S = \sum_i p_{i0}/3$.

First, we show that there exist equilibria where fiat money does not circulate. Assume that $V_{ii^*} > D_i > 0$ so that no one wants to drop the economy. Then $V_{ij} > 0$ for all j such that $p_{ij} > 0$. If agent i believes that no one will accept fiat money in the future, then $V_{i0} = 0 < V_{ij}$ and nobody will hold money.

Next, let us suppose that everyone believes that others will accept fiat money and check that this is an equilibrium. First, agent i will prefer good 0 to any good other than his own consumption good. Clearly if i prefers good i to 0 (otherwise will keep 0 forever) and prefer good 0 to the other goods given that fundamentals and marketability go in the same direction (lowest storage cost and everybody accepts it). Then, we need to rank the other goods. This will depend on the distribution P that depends on teh real balances in circulation through $\pi(S)$.

Theorem 1 *Choose S and $\pi = \pi(S)$. Then if c_{13} and c_{32} are sufficiently large, there exists an equilibrium in which all agents play fundamental strategies, that is, $V_{ii} = \max_j V_{ij}$ for all i and $V_{10} > V_{12} > V_{13}$, $V_{20} > V_{21} > V_{23}$, and $V_{30} > V_{31} > V_{32}$.*

Now, everybody accept fiat money, type III accept it from I for good 1, type I accepts it from II for good 2, and type II accepts it both from I for good I and from III for good 3. So good 1 is a commodity money together with fiat money. However, fiat money is the only general medium of exchange.

1.4 Welfare

Steady state utility levels are given by

$$W_i = (1-\beta) \sum_j p_{ij} V_{ij}.$$

Suppose the agents were following the strategy of always trading regardless of the match, that is, $\tau_i(j,k) = 1$ for all i, j, k. They show that we can improve upon the equilibrium by introducing fiat money. Given that this equilibrium is a special case of the fiat money equilibrium with $S = 0$, it is enough to show that $\partial W_i / \partial S > 0$ at $S = 0$, which is the case for all i as long as u_i are not too large relative to c_{ij}. This is because, using fiat money reduces the inefficient storage of real commodities, but reducing the amount of real goods, the frequency of consumption also decreases. There is also a sense in which real money is neutral here. Welfare depends only on $S = M/\phi$ not on M, that is, the quantity equation holds exactly.

2 Lagos and Wright

The main weakness of the previous model is that is not very useful to think about monetary policy because there is no theory of prices, but both M and P were taken as given. The strongest assumption in Kyotaki and Wright was the restriction on how much money agents can hold. In Lagos and Wrigth agents can decide to hold any amount of money $m \in R_+$. This could lead to the complication of tracking down $F(w)$, but the nice trick of the model that leads to a degenerate distribution of money is to use quasi-linear utility together with giving to the agents periodic access to a centralized market.

- time is discrete and runs forever
- there is a continuum $[0,1]$ of infinitely-lived agents with discount factor β
- each period is devided in two suberiods: day and night
- agents consume and supply labor in both subperiods
- preferences are $\mathcal{U}(x,h,X,H)$ where x and h are consumption and labor during the day, and X and H during the night
- assumption of quasi-linearity

$$\mathcal{U}(x,h,X,H) = u(x) - c(h) + U(X) - H,$$

  where u, c, and U are twice consinuously differentiable with $u' > 0$, $c' > 0$, $U' > 0$, $u'' < 0$, $c' \geq 0$, $U' \leq 0$, $u(0) = c(0) = 0$. Moreover, suppose there exists $q^* \in (0,\infty)$ and $X^* \in (0,\infty)$ such that $u'(q^*) = c'(q^*)$ and $U'(X^*) = 1$ with $U(X^*) > X^*$

- the goods traded during the day come in different varieties, while the good traded at night is a general good
- all goods traded in the day and in the night are perfectly divisible and not-storable
- there is another object, called money (total stock M), that is perfectly divisible and storable in any quantity $m \geq 0$.

Day. As in typical search model, during the day agents interact in a decentralized market with anonymous bilateral matching, where α is the meeting probability. Assume x comes in many varieties and each agent consumes only a subset. Moreover, each agent can transform labor one to one into one of the varieties he does not consume. Take two agents i and j that meet. There are four possible events:

1. with probability δ there is a double coincidence, that is, both consume what the other produces,

2. with probability σ there is a single coincidence, that is, only agent i consume what j produces,

3. with the same probability only agent j consume what i produces,

4. with probability $1-\delta-2\sigma$, neither wants to consume what the other produces

Night. Agents trade in a centralized market. With a Walrasian market specialization does not generate a double coincidence problem, hence we just assume that at night agents consume and produce a general good.

During the day the only feasible trades are barter in special goods and the exchange of special goods for money, and at night the only feasible trades involve general goods and money. Money is essential in the day.

2.1 Equilibrium

Let $F_t(\tilde{m})$ $(G_t(\tilde{m}))$ be the measure of agents starting the decentralized (centralized) market at t holding $m \leq \tilde{m}$. The initial distribution is given exogenously. For now assume that the total stock of money is fixed to M so that market clearing impose

$$\int m dF_t(m) = \int m dG_t(m) = M \text{ for all } t.$$

Let ϕ_t be the price of money in the centralized market ($1/\phi_t$ is the nominal price of general goods). At each t, given F_t, G_t, and ϕ_t, the only state variable for the agent problem is m. Call $V_t(m)$ $(W_t(m))$ the value function for an agent entering the decentralized (centralized) market at time t with m. In a bilateral meeting in the day market $h = x = q$ and hence we can denote by $q_t(m, \tilde{m})$ and $d_t(m, \tilde{m})$ the quantity and transfer in a bilateral meeting where the buyer has m and the seller $\tilde{m}$. Also denote by $B_t(m, \tilde{m})$ the payoff of an agent holding m who meets someone holding $\tilde{m}$ in a double coincidence meeting. Hence

$$\begin{aligned} V_t(m) &= \alpha\sigma \int \left(u\left[q_t(m,\tilde{m})\right] + W_t\left[m - d_t(m,\tilde{m})\right] \right) dF_t(\tilde{m}) \\ &+\alpha\sigma \int \left(-c\left[q_t(\tilde{m},m)\right] + W_t\left[m + d_t(\tilde{m},m)\right] \right) dF_t(\tilde{m}) \\ &+\alpha\delta \int B_t(m,\tilde{m})\, dF_t(\tilde{m}) \\ &+ (1 - \alpha 2\sigma - \alpha\delta) W_t(m). \end{aligned} \tag{1}$$

Moreover, the value of entering a centralized market is

$$W_t(m) = \max_{X,H,m'} \left\{ U(X) - H + \beta V_{t+1}(m') \right\} \tag{2}$$

subject to

$$\begin{aligned} X &= H + \phi_t m - \phi_t m' \\ X &\geq 0 \\ 0 &\leq H \leq \bar{H} \\ m' &\geq 0 \end{aligned}$$

where $\bar{H}$ is an upper bound on hours. They assume an interior solution for X and H, characterize the equilibrium, and then check that $0 < H < \bar{H}$ is satisfied.

Let us now determine the terms of trade in the decentralized market. In double-coincidence meetings we use symmetric Nash bargaining solution with threat point given by the continuation utilities $W_t(m)$ and $W_t(\tilde{m})$. Regardless of money holdings, this implies that each give each other q^* and no money changes hand, so that

$$B_t(m,\tilde{m}) = u(q^*) - c(q^*) + W_t(m).$$

In single coincidence meetings, we use generalized Nash solution with buyer's bargaining power equal to $\theta > 0$. Then

$$(q,d) = \max \left[u(q) + W_t(m-d) - W_t(m)\right]^{\theta} \left[-c(q) + W_t(\tilde{m}+d) - W_t(\tilde{m})\right]^{1-\theta}, \tag{3}$$

subject to $d \leq m$ and $q \geq 0$.

Definition 2 *An equilibrium is a list of* $\{V_t, W_t, X_t, H_t, m'_t, q_t, d_t, \phi_t, F_t, G_t\}$ *where for all* t, $V_t(m)$ *and* $W_t(m)$ *are the value functions,* $X_t(m)$, $H_t(m)$, *and* $m'_t(m)$ *are the decision rules in the centralized market,* $q_t(m,\tilde{m})$ *and* $d_t(m,\tilde{m})$ *are the terms of trade in the decentralized market,* ϕ_t *is the price in the centralized market, and* $F_t(m)$ *and* $G_t(m)$ *are the distribution of money holdings before and after the decentralized market. For all* t*:*

1. *given prices and distributions, the value functions and decision rules satisfy (1) and (2);*
2. *given the value functions, the terms of trade in the decentralized market solve (3);*
3. $\phi_t > 0$ *(focus on monetary equilibria);*
4. *centralized money market clears;*
5. $\{F_t, G_t\}$ *consistent with initial conditions and evolution of money holdings implied by trades in the centralized and decentralized markets.*

Recall that we guess (and then verify) that $0 \le H \le \bar{H}$. First, let us look at the centralized market problem:

$$W_t(m) = \phi_t m + \max_{X \ge 0, m' \ge 0} \left\{ U(X) - X - \phi_t m' + \beta V_{t+1}(m') \right\}.$$

This implies that $X(m) = X^*$ for all m and that $m'(m)$ does not depend on m either. Hence, W_t is linear in m with slope ϕ_t. Hence the bargaining problem in the decentralized market can be rewritten as

$$(q_t, d_t) = \max_{q_t \ge 0, d_t \le m} \left[u(q) - \phi_t d \right]^{\theta} \left[-c(q) + \phi_t d \right]^{1-\theta}.$$

The solution is

$$q_t(m,\tilde{m}) = \begin{cases} \hat{q}_t(m) & \text{if } m < m_t^* \\ q^* & \text{if } m \ge m_t^* \end{cases} \quad \text{and } d_t(m,\tilde{m}) = \begin{cases} m & \text{if } m < m_t^* \\ m_t^* & \text{if } m \ge m_t^* \end{cases},$$

where $\hat{q}_t(m)$ solves

$$\phi_t m = z(q_t) \equiv \frac{\theta c(q_t) u'(q_t) + (1-\theta) u(q_t) c'(q_t)}{\theta u'(q_t) + (1-\theta) c'(q_t)},$$

and $m_t^* = z(q^*)/\phi_t$. Hence, we can notice that the terms of trade only depend on the money of the buyer, that is, can be denoted, with some abuse of notation, by $q_t(m)$ and $d_t(m)$. Moreover, we can show that $q_t(m)$ is strictly increasing for all $m < m_t^*$ and continuous at m_t^* and is constant at $q_t(m) = q^*$ for all $m \geq m_t^*$.

Going back to the decentralized value function we have

$$V_t(m) = v_t(m) + \phi_t m + \max_{m' \geq 0} \{-\phi_t m' + \beta V_{t+1}(m')\},$$

where

$$\begin{aligned} v_t(m) &= \alpha\sigma [u(q_t(m)) - \phi_t d_t(m)] + \alpha\sigma \int (\phi_t d_t(\tilde{m}) - c(q_t(\tilde{m}))) \, dF_t(\tilde{m}) \\ &\quad + \alpha\delta [u(q^*) - c(q^*)] + \phi_t m + U(X^*) - X^* \end{aligned}$$

We can focus on equilibria with degenerate distribution of money $F(\cdot)$ (LW show some conditions under which in any equilibrium the money distribution has to be degenerate). The foc with respect to m' is given by

$$\phi_t \geq \beta V'_{t+1}(m'),$$

with complementarity slackness $m' \geq 0$. If the money distribution is degenerate, then market clearing requires $m' = M$ and hence the foc has to hold with equality. Moreover, the Envelope condition is

$$V'_t(m) = \alpha\sigma [u'(q_t(m)) q'_t(m) - \phi_t d'_t(m)] + \phi_t.$$

Hence, combining the two and using the bargaining solution we get

$$\phi_t = \beta \left[\alpha\sigma u'(q_{t+1}(M)) q'_{t+1}(M) + (1 - \alpha\sigma) \phi_{t+1}\right].$$

One can verify that as long as $\phi_t/\phi_{t+1} > \beta$, in the decentralized market when a single coincidence problem arise, the liquidity constraint is going to be binding and $d_t(m) = m$ for all t.

Now assume that M can change over time with new money injected as lump sum transfers in the centralized market. Then, the equilibrium condition can be generalized to

$$\frac{z(q_t)}{M_t} = \beta \frac{z(q_{t+1})}{M_{t+1}} \left[\alpha\sigma \frac{u'(q_{t+1})}{z'(q_{t+1})} + 1 - \alpha\sigma\right].$$

A monetary equilibrium is now characterized by any path for $\{q_t\}$ that satisfy the above equation. If $M_{t+1} = (1+\tau) M_t$ with τ constant, we can consider a steady state where q and $\phi M = z(q)$ are constant, that is, where $\phi_t/\phi_{t+1} = 1+\tau$. A necessary condition for the equilibrium to exist is $\phi_t/\phi_{t+1} \geq \beta$ and hence $\tau \geq \beta - 1$ (the lower bound being the Friedman rule).

The steady-state condition is now

$$\frac{u'(q)}{z'(q)} = 1 + \frac{1+\tau-\beta}{\beta\alpha\sigma}.$$

According to the Fisher equation, the nominal interest rate i must be such that

$$1+i = (1+r)(1+\pi)$$

where $\pi = \tau$ is the equilibrium inflation rate and

$$r = \frac{1-\beta}{\beta}$$

then

$$\frac{u'(q)}{z'(q)} = 1 + \frac{i}{\alpha\sigma}.$$

If $\theta = 1$ then $z(\theta) = c(\theta)$ and the efficient outcome q^* is obtained if and only if $i = 0$. If $\theta < 1$, then $q < q^*$ (hold-up problem) at the Friedman rule, but still the Friedman rule is constrained efficient!

14.461 Advanced Macroeconomics I
Fall 2009