BEHAVIOR OF SHELL STRUCTURES

BY

DAVID B. FARNSWORTH JR.

BACHELOR OF SCIENCE IN CIVIL ENGINEERING
MAY 1998
VANDERBILT UNIVERSITY

SUBMITTED TO THE DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF

MASTER OF ENGINEERING IN CIVIL AND ENVIRONMENTAL ENGINEERING
AT THE
MASSACHUSETTS INSTITUTE OF TECHNOLOGY

JUNE 1999

SIGNATURE OF AUTHOR______________________________
DAVID B. FARNSWORTH JR.
May 10, 1999

CERTIFIED BY______________________________
JEROME J. CONNOR
Professor, Department of Civil and Environmental Engineering
Thesis Supervisor

APPROVED BY______________________________
ANDREW J. WHITTLE
Chairman, Departmental Committee on Graduate Studies

BEHAVIOR OF SHELL STRUCTURES

BY

DAVID B. FARNSWORTH JR.

SUBMITTED TO THE DEPARTMENT OF CIVIL AND ENVIRONMENTAL ENGINEERING ON MAY 12, 1999
IN PARTIAL FULFILLMENT OF THE REQUIREMENTS FOR THE DEGREE OF MASTER OF ENGINEERING IN CIVIL AND ENVIRONMENTAL ENGINEERING

ABSTRACT

Shell structures have been used in construction for many years to span over large columnless spaces. New forms and variations on old ones have become popular in construction. The structural action of shells facilitates the efficient use of materials. The wide variety of shapes available results in aesthetically pleasing structures that enclose flexible space.

There are many approaches to the analysis and design of shell structures. The laws governing the behavior of general shell structures are given by the mathematically intensive general shell theory. The theory is valid for any shell under any loading condition, but requires the use of highly advanced mathematics to arrive at a solution. The membrane theory neglects the bending stresses in shells and simplifies the process of analysis considerably. But the membrane theory is only valid under special conditions.

This thesis aims to 1) provide the reader with an introduction to various methods of shell analysis, and 2) to examine the effects of certain parameters upon the occurrence of bending in shell structures. The use of the approximate method and computer based finite element analysis are explained in detail. The parametric analysis is intended to provide some insight as to the influence of certain geometrical properties on the stress systems of shell structures.

THESIS SUPERVISOR: JEROME J. CONNOR
TITLE: PROFESSOR OF CIVIL AND ENVIRONMENTAL ENGINEERING

ACKNOWLEDGMENTS

I would first like to thank Professor Connor for his guidance throughout the year. The time and effort that he has spent has contributed greatly to both my academic and personal growth. I could not have asked for a more enthusiastic advisor.

I would also like to thank my family for their love and support throughout the entire year. I could not have made it this far without them.

TABLE OF CONTENTS

TABLE OF FIGURES

CHAPTER 1

INTRODUCTION TO SHELL STRUCTURES

1.1 Introduction

The purpose of this thesis is to study the effects of boundary conditions on the stress distribution in shell structures and to provide a summary of analysis methods. In order to accomplish this objective, this paper will 1) discuss the general properties of shell structures, 2) introduce methods of analysis, 3) analyze a few simple problems to determine the relationship between certain variables and the stress distribution in shells.

Before the behavior of shell structures can be thoroughly investigated, a few key terms must be defined. The remainder of Chapter 1 introduces the definition of a shell, the geometric notation generally used to describe shells, and the force systems present in shells.

There are several approaches to the analysis of shells, all of which use the general theory of shells as a basis. Assuming small deflections and linear elastic behavior, the complete mathematical formulation governing the behavior of shells can be obtained by solving the equilibrium and force-displacement relationships. The derivation of the general equations is summarized and a discussion of the practical issues involving the use of the general theory is included in Chapter 2.

Membrane Theory is often used to analyze shell structures. Membrane Theory neglects certain terms and therefore greatly simplifies the complex mathematics that inhibit the use of the complete mathematical formulation based on the general shell theory. The assumptions made, however, render the membrane solution to be valid only under specific conditions. A description of the simplifications of the Membrane Theory and a discussion of the limitations are included in Chapter 3.

The approximate method provides a solution technique that can be applied to general shell problems without solving the complete formulation. In this approach, the solution to the complete mathematical formulation is split into two parts: the primary solution and the edge zone solution. The primary system solution is given by the Membrane Theory, but it will generally not be compatible with all boundary conditions. The edge zone solution is then determined to ensure compatibility. The details of this approach are fully explained in a Chapter 4.

Shell structures are ideal candidates for computer-based finite element analysis. A variety of finite element programs are available to the practicing engineer. Although the details may vary, the general approach is similar for all programs. This method first entails the creation of a computer model. The program then provides a solution and the results must be interpreted. A variety of topics ranging from modeling strategies to the convergence of the solution are introduced in Chapter 5.

The approximate method of analysis is illustrated by a case study in Chapter 6. The spherical shell under symmetrical loading is the subject of the case study. The geometry of the spherical shell allows for many simplifications that permit a clearer illustration of the stress distribution. A similar example is analyzed using the finite element program ADINA. The model conceptualization and generation are described, and the finite element based solutions are compared to those obtained with the approximate analytical formulation.

A parametric analysis provides the reader with some insight into factors that influence the bending stress distribution in the shell. The size of the edge zone defines the influence of boundary conditions upon the shell system. Several geometric properties are varied and the consequent effect on the size of the edge zone is discussed.

1.2 Definition of a Shell

The typical modern structure consists of an arrangement of beams and columns. The loads in such a structure are collected by the flooring or roof system and distributed into the beams. These beams then transmit the vertical load from the point of entry to the ends of the beam. At this point the load is transferred into the columns. The load is then transmitted along the column length to the foundation system, where it is distributed into the ground surface. In the analysis of the beams and columns, these elements are treated as lines with a certain cross section. The stresses acting on the cross section can then be calculated based upon a number of simplifying assumptions. These structural elements fall into the category of the linear, or one-dimensional, members. At the other end of the spectrum is the three dimensional elastic continua. Without any geometric limitations, no simplifying assumptions can be reasonably made (Flugge, 1973).

In between these two extremes lies the class of structures known as two-dimensional surface structures. Surface structures are, like linear members, capable of transmitting loads from one point to another. Whereas the load path in a linear member is along the line, the load path in a surface structure is along the surface. Included in this class of surface structures are plates and shells. Both are defined as having one dimension, namely the thickness, significantly smaller than the other two. The definition of a plate requires that its entire surface lie in a plane. On the other hand, a shell is defined as a curved surface structure (Pfluger, 1961).

1.3 Geometry of Shells

The shape and dimensions of a shell must be defined mathematically in order to establish the system of equations governing its behavior. In general, a shell structure is composed of solid material formed to a specific shape. This shape can be defined by the middle surface, which bisects the thickness of the shell at every point. The shell thickness is defined as the distance perpendicular to the middle surface between the outer surfaces of the shell. The direction that is perpendicular to the middle surface is referred to as the shell normal.

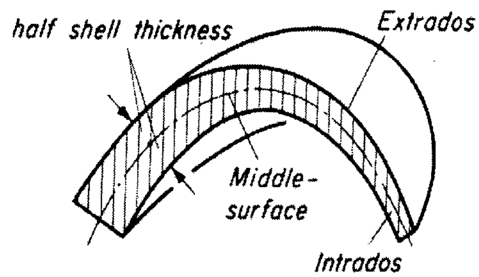


Figure 1-1 The shell middle surface

Shell structures can be of many shapes and forms. The middle surfaces can be defined analytically as a shell of revolution or a shell of translation. In addition, some shells are of forms that cannot be described analytically. Considering only one class of shell structures, however, can best provide an introduction to the behavior and analysis of shell structures. The shells of rotation are well suited for this purpose, as they minimize the complexity in geometry and notation and may therefore provide the reader with a clear insight to the behavior of shells.

A surface of revolution is created by the rotation of a curve about an axis lying in the same plane. The curve is referred to as the meridian. The axis is referred to as the shell axis. Parallels are the lines created by bisecting the middle surface of the shell with a plane that is perpendicular to the shell axis. The most common example of a shell of rotation is the dome.

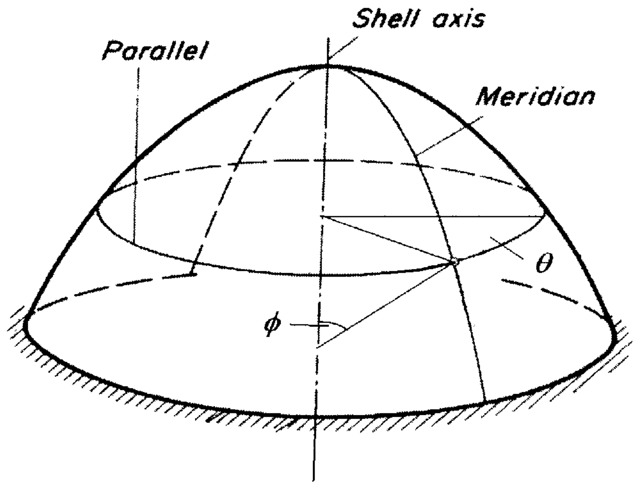


Figure 1-2 Surface of revolution

The location of any point on the dome can be given by the intersection of a meridian and a parallel as shown in the Figure1-2. The meridian is identified by the angle θ of its plane from some designated datum plane. The parallel is identified by the angle ϕ that the shell normal makes with the shell axis.

1.4 Force Systems in Shells of Rotation

Before the complete mathematical formulation is given, the basic force system present in shell structures is examined. This can be best accomplished by means of a comparison. Consider an arch and a dome as shown in Figure 1-3. A uniform load on the arch produces virtually no moment as long as the support does not constrain the normal displacement. If the loading is not uniformly distributed, however, then a bending moment is present. The dome carries the uniform loading much the same as the arch does. The forces in the dome that correspond to the forces in the arch are called the meridional forces.

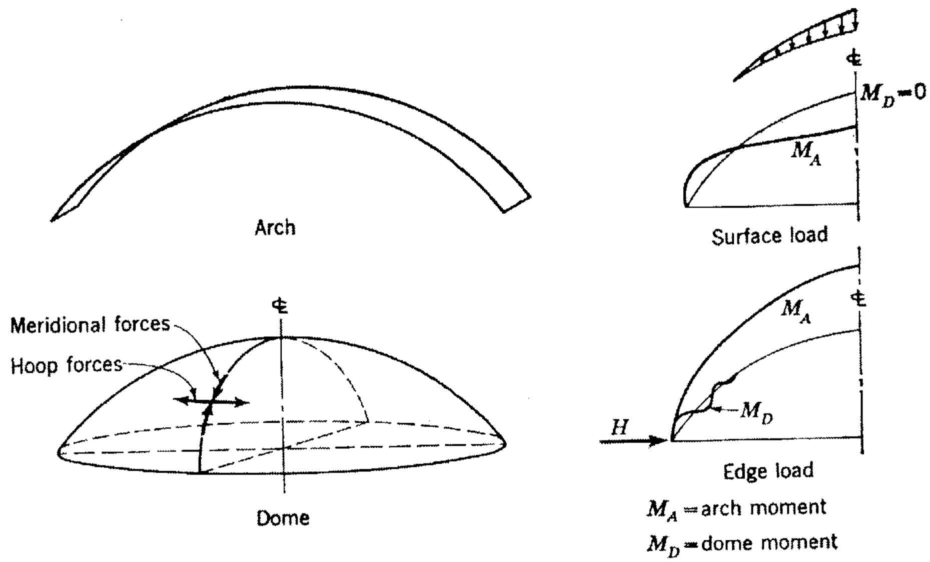


Figure 1-3 Dome vs. Arch

Also present in the dome are forces that act along the parallel direction. These forces are called hoop forces. As Figure 1-3 shows, these forces have no vertical component and therefore cannot directly resist the vertical loading.

The effect of the presence of these hoop forces is seen when observing a dome under partial load. Whereas the arch develops a bending moment, the dome carries the load without the occurrence of a bending moment. The hoop forces physically restrain each arch section of the dome from bending. It is from the existence of these hoop forces that shell structures derive their remarkable properties.

If a horizontal loading is instead applied at the base of the arch, an equal and opposite force at the other base must be present. The only means of transferring the load is up and over the apex and down to the opposite support. A bending moment therefore develops throughout the entire arch. The dome, on the other hand, subjected to a similar horizontal force, can handle this loading in a much different way. As the moment increases from the edge, the hoop forces again restrict the bending and quickly dampen out the moment.

Whereas the edge conditions effect the entire system in the arch, only a region is effected in the dome. This exhibits another phenomena of shell structures: the concept of an edge zone.

Many factors determine the size of the edge zone for a particular shell. This thesis aims to investigate the relationship between certain parameters and the occurrence of an edge zone. The parameters to be considered are the type of edge effect, the angle of opening, the shell thickness, and loading. Before the effects of these parameters on shell behavior can be investigated, however, the derivation of the general equations governing shell behavior and the various analysis methods to be used must be discussed.

CHAPTER 2

GENERAL SHELL THEORY

2.1 Introduction to General Shell Theory

The equations of equilibrium and the stress-strain law provide the basis for any method of analysis. For the analysis of shell structures, the general equations must be set up for the equilibrium of a differential element of the shell and also for the compatibility of the strains of adjacent elements. Before the system of equations governing shell behavior can be formulated, however, certain terms must be defined and the assumptions must be stated.

2.2 Definitions and Assumptions

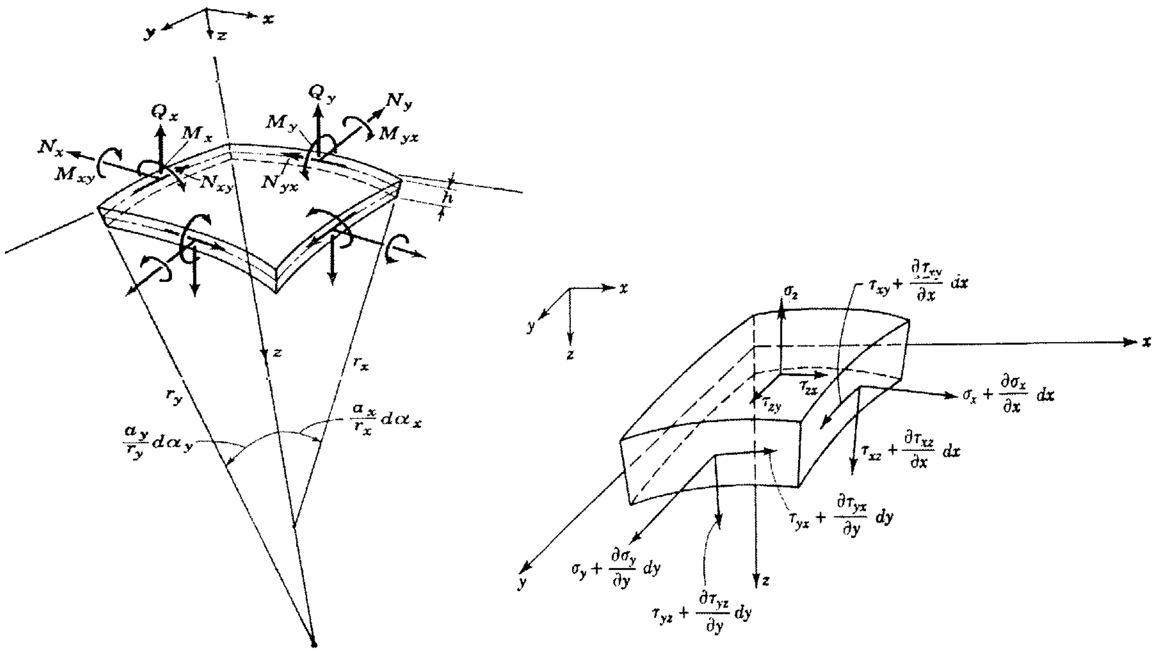


Figure 2-1 The differential shell element

The system of forces and moments that act upon the differential shell element can be described as stress resultants and stress couples acting per unit length of the shell middle surface. These forces are defined below and are shown as they act upon the element in Figure 2-1.

$$N_x = \int_h \sigma_x \left(1 - \frac{z}{r_y}\right) dz \qquad N_y = \int_h \sigma_y \left(1 - \frac{z}{r_x}\right) dz$$

$$N_{xy} = \int_h \tau_{xy} \left(1 - \frac{z}{r_y}\right) dz \qquad N_{yx} = \int_h \tau_{yx} \left(1 - \frac{z}{r_x}\right) dz$$

$$Q_x = \int_h \tau_{xz} \left(1 - \frac{z}{r_y}\right) dz \qquad Q_y = \int_h \tau_{yz} \left(1 - \frac{z}{r_x}\right) dz$$

$$M_x = \int_h \sigma_x z\left(1 - \frac{z}{r_y}\right)dz \qquad M_y = \int_h \sigma_y z\left(1 - \frac{z}{r_x}\right)dz$$

$$M_{xy} = -\int_h \tau_{xy} z\left(1 - \frac{z}{r_y}\right)dz \qquad M_{yx} = \int_h \tau_{yx} z\left(1 - \frac{z}{r_x}\right)dz \tag{2-1}$$

Because the shell thickness z is generally much smaller that the terms r_x and r_y the terms involving z/r_x and z/r_y can be neglected when either added to or subtracted from unity. And since $\tau_{xy} = \tau_{yx}$,

$$N_{xy} = N_{yx}$$

$$M_{xy} = - M_{yx} \tag{a}$$

It is assumed in the formulation that the deflections under loading are small so that the changes in shell geometry do not change the state of equilibrium. It is also assumed that the shell material behaves as a linear elastic material. This establishes a direct linear relationship between stress and strain in the shell. Two assumptions used in elementary beam theory are also applied here: 1) that plane sections remain plane after bending, and 2) deformations due to shear are neglected.

Now that the key terms have been defined and the assumptions have been stated, the general theory of shells can be formulated in the following five steps (Billington, 1965):

1) Determine the equilibrium equations for a shell element (five equations with eight unknowns).
2) Establish the strain - displacement relationships (six equations with three unknowns).
3) Establish the stress - strain relationships by assuming material properties (three equations with six unknowns) and then deriving force – strain equations (six equations with three unknowns).
4) Transform the force - strain relationships into force – displacement equations (six equations with six unknowns).

5) Obtain the complete formulation by combining the force – displacement equations with the equilibrium equations (11 equations with 11 unknowns).

2.3 Equilibrium

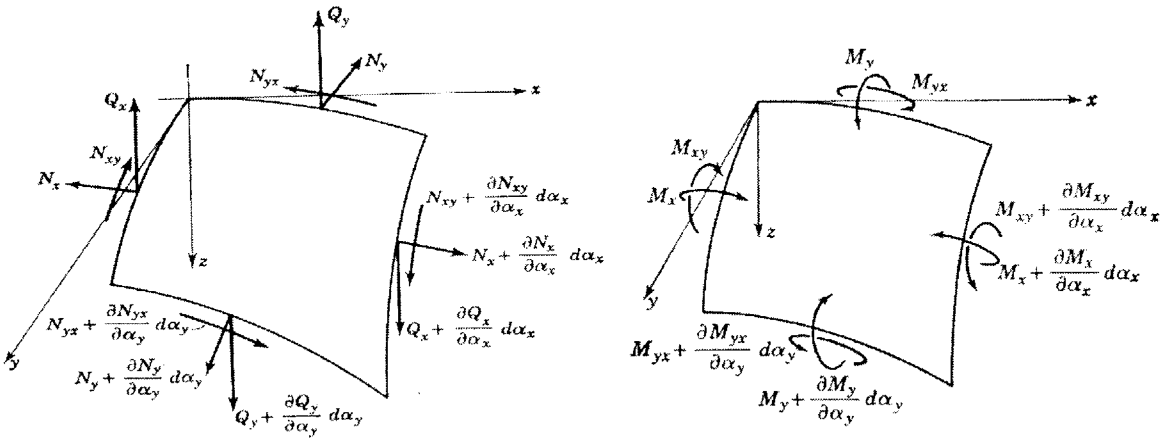


Figure 2-2 Stress resultants and couples

Figure 2.2 shows the differential shell element and the stress resultants. The equilibrium of the shell element is established with the six conditions:

$$\Sigma X = 0 \qquad \Sigma M_x = 0$$

$$\Sigma Y = 0 \qquad \Sigma M_y = 0$$

$$\Sigma Z = 0 \qquad \Sigma M_z = 0 \tag{2-2}$$

Because of (a) in (2-1), $\Sigma M_z = 0$ is dropped, and (2-2) reduces to five equations and 8 unknowns. The complete formulation of the equilibrium equations has been derived (Billington, 1965) and is given as:

$$\frac{\partial}{\partial \alpha_x}\left(N_x a_y\right) - N_y \frac{\partial a_y}{\partial \alpha_x} + N_{xy}\frac{\partial a_x}{\partial \alpha_y} + \frac{\partial}{\partial \alpha_y}\left(N_{yx} a_x\right) - Q_y \frac{a_x a_y}{r_{xy}} - Q_x \frac{a_x a_y}{r_x} + p_x a_x a_y = 0$$

$$\frac{\partial}{\partial \alpha_y}\left(N_y a_x\right) - N_x \frac{\partial a_x}{\partial \alpha_x} + N_{xy}\frac{\partial a_y}{\partial \alpha_x} + \frac{\partial}{\partial \alpha_x}\left(N_{xy} a_y\right) - Q_x \frac{a_x a_y}{r_{xy}} - Q_y \frac{a_x a_y}{r_y} + p_y a_x a_y = 0$$

$$\frac{\partial}{\partial \alpha_x}\left(Q_x a_y\right) + \frac{\partial}{\partial \alpha_y}\left(Q_y a_x\right) + N_x \frac{a_x a_y}{r_x} + N_{xy}\frac{a_x a_y}{r_{xy}} + N_{yx}\frac{a_x a_y}{r_{xy}} + N_y \frac{a_x a_y}{r_y} + p_z a_x a_y = 0$$

$$-\frac{\partial}{\partial\alpha_y}\left(M_y a_x\right)+M_x\frac{\partial a_x}{\partial\alpha_y}-M_{yx}\frac{\partial a_y}{\partial\alpha_x}+\frac{\partial}{\partial\alpha_x}\left(M_{xy}a_y\right)+Q_y a_x a_y=0$$

$$-\frac{\partial}{\partial\alpha_x}\left(M_x a_y\right)+M_y\frac{\partial a_y}{\partial\alpha_x}+M_{xy}\frac{\partial a_x}{\partial\alpha_y}-\frac{\partial}{\partial\alpha_y}\left(M_{xy}a_x\right)+Q_x a_x a_y=0 \qquad (2\text{-}3)$$

where a_x and a_y represent the radii of curvature for the shell in each respective direction.

2.4 Strain - Displacement Relationships

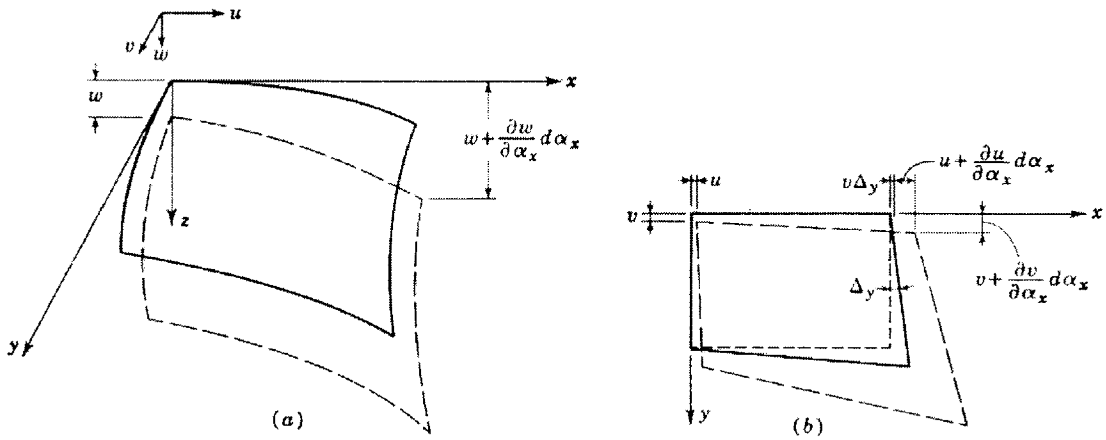


Figure 2-3 Shell element after deformation

Figure 2-3 shows the differential shell element after deformation. The displacements are given as the components *u,v,* and *w* that corrrespond to the directions *x,y,* and *z* respectively. The extensional strains at the middle surface are then defined as the strain in each direction, ε_{x0} and ε_{y0}, and angular shear strain γ_{xy0}, and are given as:

$$\varepsilon_{x0}=\frac{1}{a_x}\frac{\partial u}{\partial\alpha_x}+\frac{v}{a_x a_y}\frac{\partial a_x}{\partial\alpha_y}-\frac{w}{r_x}$$

$$\varepsilon_{y0}=\frac{1}{a_y}\frac{\partial v}{\partial\alpha_y}+\frac{u}{a_x a_y}\frac{\partial a_y}{\partial\alpha_x}-\frac{w}{r_y}$$

$$\gamma_{xy0} = \frac{1}{a_x}\frac{\partial v}{\partial \alpha_x} + \frac{1}{a_y}\frac{\partial u}{\partial \alpha_y} - \frac{u}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} - \frac{v}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x} - \frac{2w}{r_{xy}} \qquad (2\text{-}4)$$

The element also expereinces a strain due to bending. These strains are defined as the changes in curvature and are given as:

$$\chi_x = \frac{1}{\alpha_x}\frac{\partial \phi_x}{\partial \alpha_x} + \frac{\phi_y}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y}$$

$$\chi_y = \frac{1}{\alpha_y}\frac{\partial \phi_y}{\partial \alpha_y} + \frac{\phi_x}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x}$$

$$2\chi_{xy} = \frac{1}{\alpha_y}\frac{\partial \phi_x}{\partial \alpha_y} + \frac{1}{\alpha_x}\frac{\partial \phi_y}{\partial \alpha_x} - \frac{\phi_x}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} - \frac{\phi_y}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x}$$

where the rotatations are: (2-5)

$$\phi_x = \frac{u}{r_x} + \frac{\partial w}{a_x \partial \alpha_x} + \frac{v}{r_{xy}}$$

$$\phi_y = \frac{v}{r_y} + \frac{\partial w}{a_y \partial \alpha_y} + \frac{u}{r_{xy}}$$

2.5 Stress - Strain Relationships

The strain relationships formulated above provide expressions for the strain at the middle surface of the shell. In order to derive the stress – strain relationships, expressions for the strain through the thickness of the shell must first be given. Since the element coordinate system is defined such that the z direction coincides with the shell normal, any point along the shell thickness is given by the z-coordinate. The strain at any point z in the shell is then:

$$\varepsilon_x = \varepsilon_{x0} - z\chi_x$$

$$\varepsilon_y = \varepsilon_{y0} - z\chi_y$$

$$\gamma_{xy} = \gamma_{xy0} - 2z\chi_{xy} \qquad (2\text{-}6)$$

It now becomes necessary to define certain material properties so that the relationships between strain and stress may be derived. As stated earlier, the formulation of general shell theory is based upon the assumption that the material is linearly elastic, isotropic, and homogeneous. The material properties that define a linear elastic isotropic material are Young's modulus *E* and Poisson's ratio ν. The stresses in the shell can then be related to the strains as:

$$\sigma_x = \frac{E}{1-\nu^2}\left(\varepsilon_x + \nu\varepsilon_y\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\left(\varepsilon_y + \nu\varepsilon_x\right)$$

$$\tau_{xy} = \frac{E}{2\left(1-\nu^2\right)}\gamma_{xy} \tag{2-7}$$

Substituting (2-6) into (2-7) and then the result into (2-1) obtains the stress resultants and stress couples:

$$N_x = K\left(\varepsilon_{x0} + \nu\varepsilon_{y0}\right)$$

$$N_x = K\left(\varepsilon_{y0} + \nu\varepsilon_{x0}\right)$$

$$N_{xy} = Gh\gamma_{xy0}$$

$$M_x = -D\left(\chi_x + \nu\chi_y\right)$$

$$M_y = -D\left(\chi_y + \nu\chi_x\right)$$

$$M_{xy} = -M_{yx} = D\left(1-\nu\right)\chi_{xy} \tag{2-8}$$

where

$$K = \frac{E}{1-\nu^2}$$

$$G = \frac{E}{2\left(1+\nu\right)}$$

$$D = \frac{Eh^3}{12\left(1-\nu^2\right)} \tag{2-9}$$

K is called the extensional rigidity, *G* the shear modulus, and *D* the bending rigidity tor the shell.

2.6 Force – Displacement Equations

The expressions in (2-4) and (2-5) for the extensional and bending strains as functions of the displacements can be substituted into (2-8) to get the force – displacement relationships:

$$N_x = K\left[\frac{1}{a_x}\frac{\partial u}{\partial \alpha_x} + \frac{v}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} - \frac{w}{r_x} + v\left(\frac{1}{a_y}\frac{\partial v}{\partial \alpha_y} + \frac{u}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x} - \frac{w}{r_y}\right)\right]$$

$$N_y = K\left[\frac{1}{a_y}\frac{\partial v}{\partial \alpha_y} + \frac{u}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x} - \frac{w}{r_y} + v\left(\frac{1}{a_x}\frac{\partial u}{\partial \alpha_x} + \frac{v}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} - \frac{w}{r_x}\right)\right]$$

$$N_{xy} = N_{xy} = Gh\left(\frac{1}{a_x}\frac{\partial v}{\partial \alpha_x} + \frac{1}{a_y}\frac{\partial u}{\partial \alpha_y} - \frac{u}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} - \frac{v}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x} - \frac{2w}{r_{xy}}\right)$$

(2-10)

$$M_x = -D\left[\frac{1}{\alpha_x}\frac{\partial \phi_x}{\partial \alpha_x} + \frac{\phi_y}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} + v\left(\frac{1}{\alpha_y}\frac{\partial \phi_y}{\partial \alpha_y} + \frac{\phi_x}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x}\right)\right]$$

$$M_x = -D\left[\frac{1}{\alpha_y}\frac{\partial \phi_y}{\partial \alpha_y} + \frac{\phi_x}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x} + v\left(\frac{1}{\alpha_x}\frac{\partial \phi_x}{\partial \alpha_x} + \frac{\phi_y}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y}\right)\right]$$

$$M_{xy} = -M_{yx} = \frac{D(1-v)}{2}\left(\frac{1}{\alpha_y}\frac{\partial \phi_x}{\partial \alpha_y} + \frac{1}{\alpha_x}\frac{\partial \phi_y}{\partial \alpha_x} - \frac{\phi_x}{a_x a_y}\frac{\partial a_x}{\partial \alpha_y} - \frac{\phi_y}{a_x a_y}\frac{\partial a_y}{\partial \alpha_x}\right)$$

2.7 Practical Issues

The six equations of (2-10) and the five equations of (2-3) provide a system of 11 equations with 11 unknowns: five stress resultants, three stress couples, and three displacements. The stresses and displacements of any shell can be obtained by reducing the system of equations in (2-3) and (2-10) to one equation with one unknown. This general theory is seldom used in practice however. The reduced equation is a linear eighth-order partial differential equation and is extraordinarily difficult to solve. As a result, most methods of analyses attempt to simplify the mathematics.

CHAPTER 3

THE MEMBRANE THEORY

3.1 Simplifications of the Membrane Theory

The equations of equilibrium for the shell element generally contain ten unknowns – the stress resultants and stress couples - and six equations. The system is therefore statically indeterminate. As a result, deformation must be considered in order to obtain a solution. The Membrane Theory avoids the complexities involved with solving the statically indeterminate system by reducing the number of unknowns from ten to four. This results in a statically determinate system that can be solved directly. The Membrane Theory accomplishes this by neglecting all normal shears, bending moments, and twisting moments in the shell. This simplification of the system is based on the tendency of the shell to resist loading by means of hoop and meridional forces as introduced in Chapter 1.4.

The equations of equilibrium given by (2-3) then reduce to:

$$\frac{\partial}{\partial \alpha_x}\left(N'_x a_y\right) - N'_y \frac{\partial a_y}{\partial \alpha_x} + N'_{xy}\frac{\partial a_x}{\partial \alpha_y} + \frac{\partial}{\partial \alpha_y}\left(N'_{yx} a_x\right) + p_x a_x a_y = 0$$

$$\frac{\partial}{\partial \alpha_y}\left(N'_y a_x\right) - N'_x \frac{\partial a_x}{\partial \alpha_x} + N'_{xy}\frac{\partial a_y}{\partial \alpha_x} + \frac{\partial}{\partial \alpha_x}\left(N'_{xy} a_y\right) + p_y a_x a_y = 0$$

$$\frac{N'_x}{r_x} + \frac{N'_{xy}}{r_{xy}} + \frac{N'_{yx}}{r_{xy}} + \frac{N'_y}{r_y} + p_z = 0 \qquad (3\text{-}1)$$

The prime marks indicate that the stress resultants act only in the plane of the shell and are only approximations of the actual stress resultants that would be obtained from the solution of (2-3) and (2-10). The solution can then be found for a specified geometry and loading by solving (3-1).

3.2 Limitations

The validity of the results obtained by the Membrane Theory depend upon a number of conditions. First, the boundary conditions must be compatible with the conditions of equilibrium. Second, the application of the loading must be compatible with the conditions of equilibrium. These conditions are well revealed through a simple static analysis of a few examples (Pfluger, 1961).

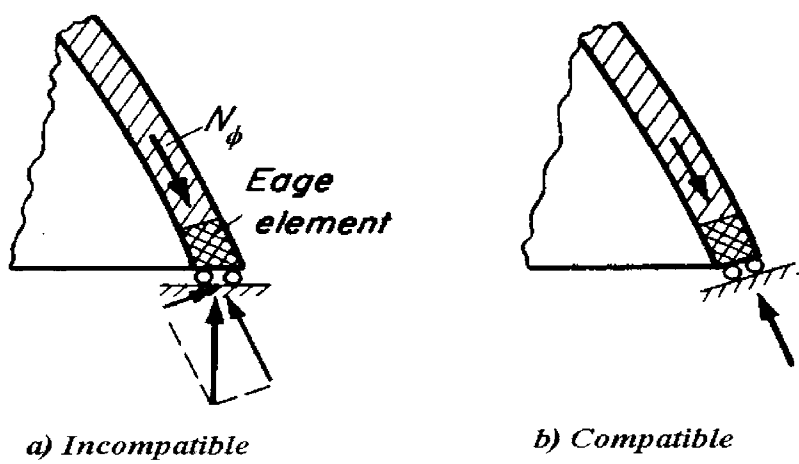
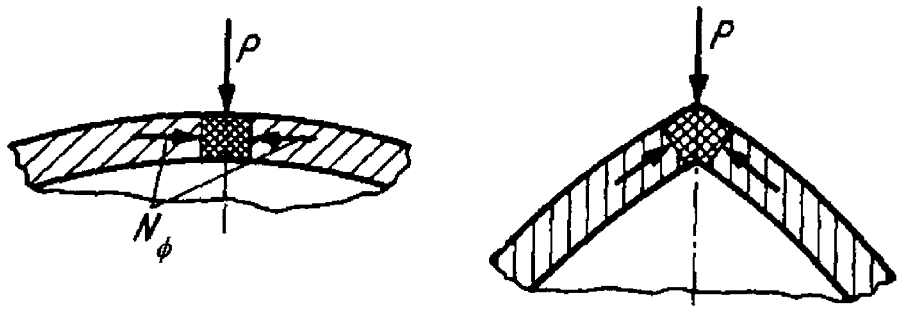


Figure 3-1 Compatibility of boundary conditions with the Membrane Theory

Figure 3-1 shows the shell edge with two different boundary conditions. The system depicted in (a) is free to translate horizontally. There is therefore a component of the reaction that acts normal to the shell surface at the edge. Yet the Membrane Theory neglects all forces that act out of plane, so the element is out of equilibrium. The boundary condition displayed in (b) supplies a reaction that acts only in the plane of the shell. Therefore, the support is compatible with both the Membrane Theory and the conditions of equilibrium.



a) Incompatible *b) Compatible*

Figure 3-2 Compatibility of point loads with membrane forces

Figure 3-2 shows a shell under a concentrated load at the apex. The system in (a) cannot resist the load without bending because the meridional forces of the infinitesimally small element at the apex have no vertical components. If, however, the size of the element taken were to be larger, a vertical reaction could be generated due to the curvature of the shell. This implies that the loading should be distributed smoothly over the shell surface for the Membrane Theory to be applicable. The membrane forces in the element at the apex in (b) have a a vertical component and may therefore be compatible with a point loading.

Now consider the case of a shallow shell subjected to a smoothly distributed load. As the radius of curvature of the shell approaches infinity, the shell becomes a plate. Under Membrane Theory, this shell possesses no internal force systems capable of contributing to the vertical equilibrium. Clearly, either normal shear or bending must occur in order for the plate to carry the loading. Thus the Membrane Theory is not applicable to plates.

As the previously discussed examples have shown, the Membrane Theory is not suitable for all shell analyses. While the assumptions of the Membrane Theory clearly simplify the mathematics involved in the solution of shell problems, they do not fully explain the behavior of shells. Under certain circumstances the Membrane Theory may provide the complete solution to a shell problem. Under different circumstances, however, serious incompatibilities may occur and a more rigorous analysis method must be employed.

CHAPTER 4

THE APPROXIMATE METHOD

4.1 Analysis Method

The complete mathematical formulation given in Chapter 2 theoretically provides an approach to the solution of general shell problems. In practice, however, the complex mathematics involved with this approach restricts its use. The Membrane Theory, on the other hand, greatly simplifies the solution process, but in doing so, it restricts its applicability to only a select few problems. Typically, the shell problems encountered by the practicing engineer do not fall under the category of shells that can be solved completely by the Membrane Theory. At the same time, the typical practicing engineer does not have the capability to solve the complex system given by the general theory. A method of analysis is needed that is both applicable to general shell problems and simplifies the mathematics involved in the solution. One such method is the so-called "approximate method" (Flugge, 1973).

The solution to the complete formulation would consist of two parts: the homogeneous solution and the particular solution. The particular solution includes all terms involving the loading and can for many shells be reasonably approximated by the Membrane Theory. The homogeneous solution then contains the relations that ensure the compatibility of the boundary conditions with the conditions of equilibrium. So, the approximate method of shell analysis generally consists of the following steps(Billington, 1965):

1) The particular solution is found by assuming that the load is resisted entirely by membrane forces.
2) The resulting forces and displacements at the boundary conditions will generally not be compatible with the solution obtained for the particular solution.
3) Therefore, additional forces and displacements must be introduced to the boundaries such that the total solution is compatible with the actual boundary conditions. These additional forces and displacements are called the edge effects.
4) The magnitude of the edge effects are calculated such to remove the errors of the particular solution at the boundaries.

The computation of the displacements due to the edge effects is based upon the fact that the resulting bending stresses decrease rapidly with increasing distance from the edge. As a result, the edge effects are only approximations and thus the method is called the "approximate method."

4.2 Beam Analogy

The general procedure employed in the approximate method to analyze shell systems is called superposition. This method is frequently used to analyze indeterminate structures. The following example illustrates the basic concepts of this method.

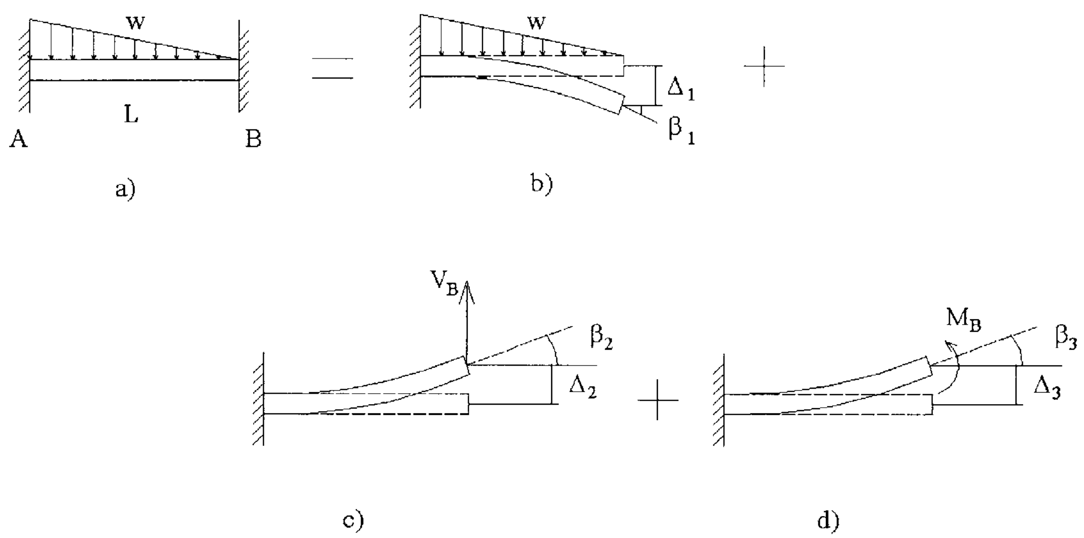


Figure 4-1 Beam analogy – approximate method

Consider a bending beam that is subjected to a distributed triangular load. Both ends are fixed as shown in Figure 4-1a). The reactions at each end consist of a force and a moment and the structure is therefore statically indeterminate. The problem can be solved in the following way:

1) Consider the system to be composed of the three determinate systems as shown in Figure 4-1.
2) Neglecting the fixity at B as shown in Figure 4-1b), first calculate the deflection and rotation at B due to the applied triangular loading. These values are found to be:

$$\Delta_1 = \frac{wL^3}{15EI} \qquad \beta_1 = -\frac{wL^2}{12EI} \tag{4-1}$$

3) Next consider the cantilever beam without the triangular loading as shown in Figures 4-1c) and d). Determine the displacements at B due to a normal load V_B and end moment M_B. The calculated response is given by:

$$\Delta_2 = -\frac{V_B L^3}{3EI} \qquad \beta_2 = \frac{V_B L^2}{2EI}$$

$$\Delta_3 = -\frac{M_B L^2}{2EI} \qquad \beta_3 = \frac{M_B L}{EI} \tag{4-2}$$

4) The boundary conditions of the system require that the displacement and the rotation at B both are zero. The values of the end actions at B can now be found by solving the compatibility equations of the system for the variables V_B and M_B. The compatibility equations are given by:

$$\Delta_1 + \Delta_2 + \Delta_3 = 0 \quad \rightarrow \quad \frac{wL^3}{15EI} - \frac{V_B L^3}{3EI} - \frac{M_B L^2}{2EI} = 0$$

$$\beta_1 + \beta_2 + \beta_3 = 0 \quad \rightarrow \quad -\frac{wL^2}{12EI} + \frac{V_B L^2}{2EI} + \frac{M_B L}{EI} = 0 \tag{4-3}$$

And solving (4-3) for V_B and M_B:

$$V_B = \frac{3w}{10} \qquad M_B = -\frac{wL}{15} \tag{4-4}$$

The negative sign on M_B indicates that the moment is applied in the opposite direction as previously shown in Figure 4-1d). It is important to note that superposition provides the exact solution to the beam example. The displacements due to the end actions are exactly those given by Bernoulli beam theory.

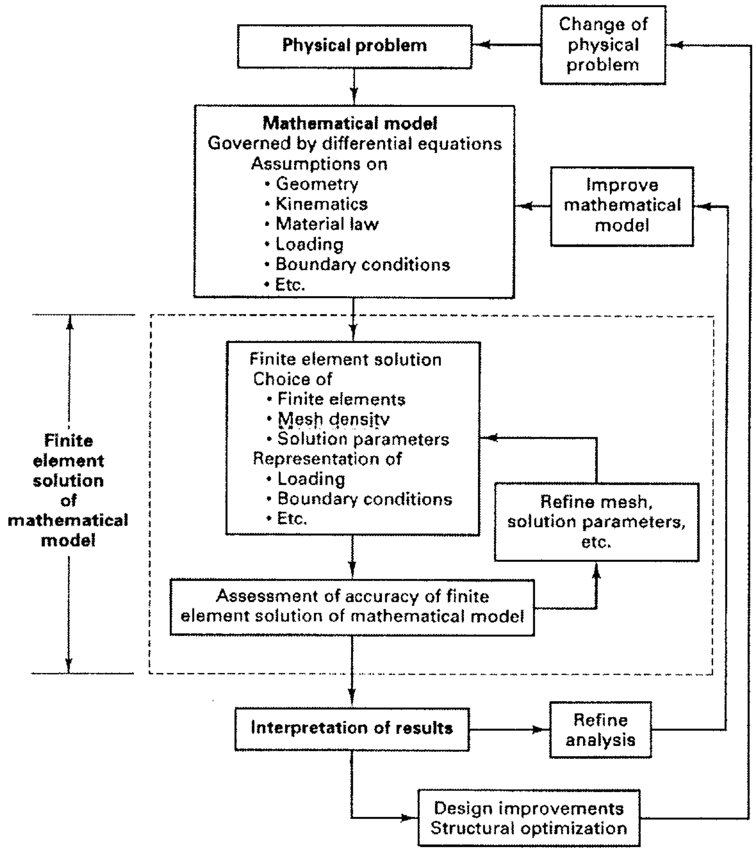
CHAPTER 5

COMPUTER BASED FINITE ELEMENT METHOD

5.1 Procedure

The analysis of complex structures is essentially carried out in the three following steps: 1) selection of a mathematical model, 2) solution of the model, and 3) the interpretation of the results. Computer based finite element programs are frequently used for the creation and solution of the model. It is often necessary to use a finite element program to analyze complex systems in a cost-effective manner. The recent advances in the computer industry have made the computer based finite element programs not only viable, but also easy to use. There are several programs available to the practicing engineer, but the general approach is roughly the same:

1) A model must be conceptualized that accurately represents the physical characteristics of the structure and the applied loading.
2) The model must then be created and solved in the finite element program.
3) The results must be interpreted.



The conceptualization of the representative model is required in both computer-based and conventional analyses. The engineer models a physical structure as an assemblage of elements with certain properties. Assumptions are made that simplify the mathematical model to a point where it can be solved. An example of this process in conventional analysis is the idealization of a beam. In reality, a beam is a three-dimensional object, yet in classical Bernoulli beam theory, certain assumptions are made that allow the beam to be modeled as a line with specified cross-sectional properties. The same procedure must also be followed before a complex structure may be analyzed using a finite element program. The accuracy of the solution is limited by the selection of the model. If a model is created that does not accurately represent the physical structure, the solution obtained does not accurately depict the behavior of the actual structure, but only of the model.

The creation of the model in the program generally begins with the definition of the geometry. Either a two- or three-dimensional coordinate system may be used, depending upon the conceptualization of the model. If, for example, a building is to be modeled as a three-dimensional wire-frame, then points are specified as nodes and lines are drawn to connect the nodes. For complex structures, however, the geometry is often imported from a CAD system.

Once the geometry of the model has been defined, it must be "meshed" with elements. The element types are chosen based upon a number of considerations and are then applied to the model. Element types range from three-dimensional solid elements to one-dimensional truss elements. Material properties must be specified and assigned to each element.

The loading and boundary conditions must also be applied to the model. Loading is normally specified as some combination of concentrated point loads, distributed line loads, or spatially distributed pressures. The boundary conditions are modeled by restricting the appropriate degrees of freedom at certain nodal points.

Once the model generation is complete, it can be solved. The computer programs employ a numerical procedure to arrive at a solution and an output file is generated. The numerical procedure is commonly known as the finite element method. The details of the numerical procedure have been presented (Bathe, 1996). Any further discussion of the theory behind the finite element method is beyond the scope of this thesis.

The solution output must then be interpreted. Most finite element programs display the results in some graphical form. If the solution does not make physical sense, it is then necessary to refine the model. Once an acceptable output has been obtained, the analysis is complete.

CHAPTER 6

CASE STUDY – SPHERICAL SHELL BY APPROXIMATE METHOD

6.1 Approximate Method – General Procedure

The approximate method of shell analysis can be best illustrated by a case study. This method provides the analyst with a solution that is both applicable to most shell problems and mathematically feasible. As stated earlier, the approximate method consists of the following steps:

1) The particular solution is first obtained by the Membrane Theory.
2) The errors of the membrane solution at the boundaries are calculated.
3) The effects due to unit edge loads at the boundaries are approximated.
4) The magnitudes of the edge effects are determined to ensure compatibility at the boundaries.

This case study focuses on a spherical shell with a fixed base due its simplicity in geometry. Before the approximate method can be applied to this problem, however, the applicable equations must be specialized for the case of the spherical shell.

6.2 Membrane Theory for Spherical Shells

The spherical shell is a surface of revolution with a constant radius of curvature a over the entire surface. It now becomes convenient to define the differential spherical shell element in polar coordinates as shown in Figure 6-1.

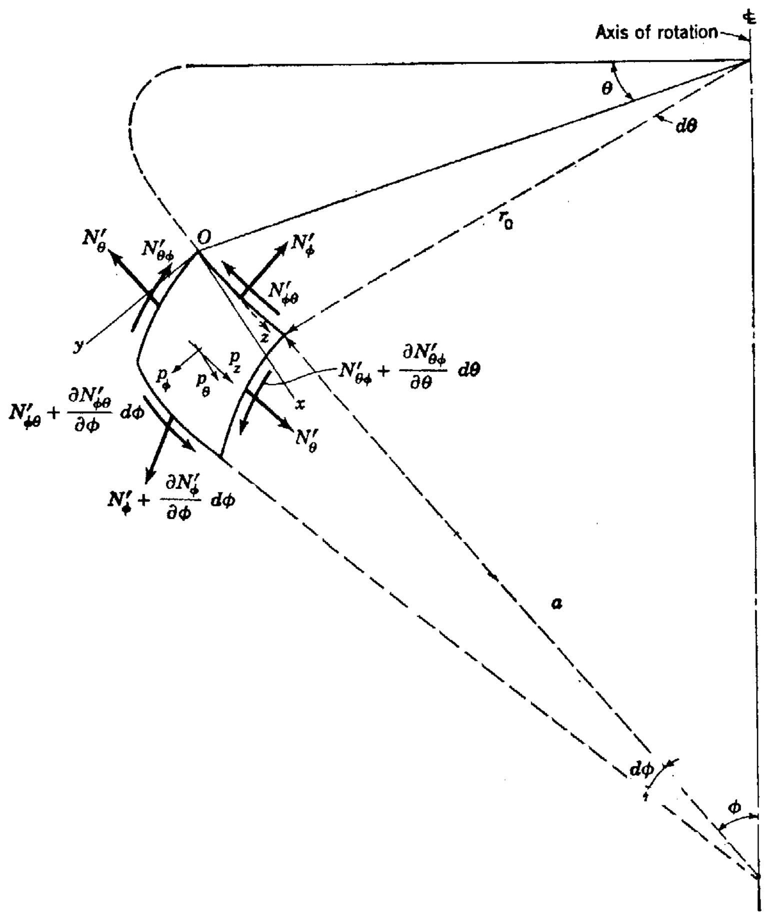


Figure 6-1 Spherical shell element and coordinate system

The terms in the generalized equations of the membrane theory can now be replaced for spherical shells as follows:

$$\alpha_x = 6 \qquad a_x = r_0 \qquad r_x = a \qquad r_0 = a\sin\phi \qquad N'_{xy} = N'_{\theta\phi}$$

$$\alpha_y = \phi \qquad a_y = a \qquad r_y = a \qquad N'_x = N'_\theta \qquad N'_y = N'_\phi$$

Therefore, the equations in (3-1) simplify to:

$$\frac{\partial N'_\theta}{\partial \theta}a - N'_{\theta\phi}\frac{\partial r_0}{\partial \phi} + \frac{\partial (N'_{\theta\phi} r_0)}{\partial \phi} + p_\theta r_0 a = 0$$

$$\frac{\partial (N'_\phi r_0)}{\partial \phi} - N'_\theta \frac{\partial r_0}{\partial \phi} + \frac{\partial N'_{\theta\phi}}{\partial \theta}a + p_\phi r_0 a = 0$$

$$\frac{N'_\theta + N'_\phi}{a} + p_z = 0 \qquad (6\text{-}1)$$

If the loading is also symmetrical about the shell axis, then no terms vary with θ and the Membrane Theory reduces to:

$$\frac{d(N'_\phi r_0)}{d\phi} - N'_\theta \frac{dr_0}{d\phi} + p_\phi r_0 a = 0$$

$$\frac{N'_\theta + N'_\phi}{a} + p_z = 0 \qquad (6\text{-}2)$$

The solution to (6-2) is then of the form:

$$N'_\phi = -\frac{R}{2\pi r_0 \sin\phi}$$

$$N'_\phi = \frac{R}{2\pi a \sin^2\phi} - p_z \frac{r_0}{\sin\phi} \qquad (6\text{-}3)$$

where R is the total vertical load above the cirle defined by the parallel circle ϕ and is given by:

$$R = \int_0^\phi (p_\phi \sin\phi + p_z \cos\phi)(2\pi r_0)a\,d\phi \qquad (a)$$

The Membrane Theory solution for the stresses of any spherical shell under axisymmetric loading can be found by first calculating R by (a) and then the stresses by (6-3).

The displacements from the Membrane Theory for spherical shells under axisymmetric loading can be found by first finding the extensional strains and then calculating the displacements. For the purposes of this thesis, only the displacements at the edges are needed. These are given as the horizontal translation Δ_H and the rotation Δ_ϕ and are derived from the strain equations in (2-4) as:

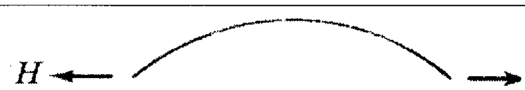
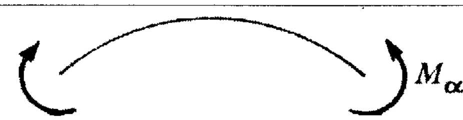
$$\Delta_H = \frac{a\sin\phi}{Eh}(N'_\theta - vN'_\phi)$$

$$\Delta_\phi = \frac{\cot\phi}{Eh}\left[\left(N'_\phi - N'_\theta\right)(1+v)\right] - \frac{1}{a}\frac{d}{d\phi}\left(\frac{\Delta_H}{\sin\phi}\right) \qquad (6\text{-}4)$$

6.3 Effect of Edge Forces on Spherical Shells

As previously stated, the approximate method consists of splitting the solution of the general theory solution into two parts: the membrane solution for the surface loading and the bending solution for edge effects. The general equations for the forces and displacements in spherical domes loaded by uniform edge forces have been approximated derived and are presented in Table 6-1 (Billington, 1965). The coordinate system and notation used in these equations is shown in Figure 6-2.

Table 6-1 Stresses and displacements resulting from edge effects

		
N_ϕ	$-\sqrt{2}\cot(\alpha-\psi)\sin\alpha e^{-\lambda\psi}\sin\left(\lambda\psi-\frac{\pi}{4}\right)H$	$-\frac{2\lambda}{a}\cot(\alpha-\psi)e^{-\lambda\psi}\sin(\lambda\psi)M_\alpha$
N_θ	$-2\lambda\sin\alpha e^{-\lambda\psi}\sin\left(\lambda\psi-\frac{\pi}{2}\right)H$	$-\frac{2\sqrt{2}\lambda^2}{a}e^{-\lambda\psi}\sin\left(\lambda\psi-\frac{\pi}{4}\right)M_\alpha$
M_ϕ	$-\frac{a}{\lambda}\alpha e^{-\lambda\psi}\sin(\lambda\psi)H$	$\sqrt{2}e^{-\lambda\psi}\sin\left(\lambda\psi+\frac{\pi}{4}\right)M_\alpha$
Δ_H	$\frac{2a\lambda\sin^2\alpha}{Eh}H$	$\frac{2\lambda\sin^2\alpha}{Eh}M_\alpha$
Δ_α	$\frac{2\lambda^2\sin\alpha}{Eh}H$	$\frac{4\lambda^3M_\alpha}{Eah}$

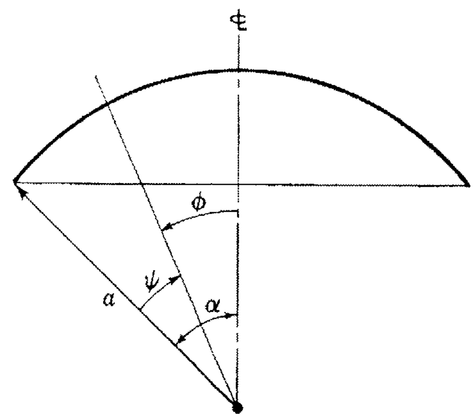


Figure 6-2 Notation used in equations of Table 6-1

The exponential decay term $e^{-\lambda\psi}$ in the equations of Table 7.1 can be physically explained as the damping in the system. The rate at which the edge effects are damped out in the system is dependent upon the damping parameter λ. The magnitude of λ is determined by the geometry and material properties of the of the shell as:

$$\lambda^4 = 3\left(1 - v^2\right)\left(\frac{a}{h}\right)^2 \tag{6-5}$$

6.4 Problem Definition

A spherical shell on a fixed base is subjected to a uniform gravity load over the entire dome surface. The dimensions of the shell, material properties, and loading are as follows:

a = 100 ft	- radius of curvature
h = 4 in.	- shell thickness
$\alpha = 30^{\circ}$	- angle of opening
$r_o = a\sin\alpha = 50$ ft	- radius of parallel at base
v = .167	- Poisson's ratio
q = 100 psf	- uniformly distributed load

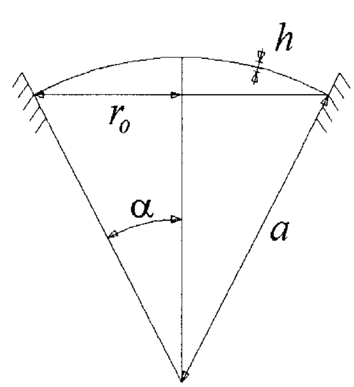


Figure 6-3 Case study

6.5 Membrane Solution

The membrane stress resultants shown in Table 6-2 are calculated using (6-3) where:

$$p_\phi = q \sin\phi$$

$$p_z = q\cos\phi$$

$$R = \int_0^\phi \left(p_\phi \sin\phi + p_z \cos\phi\right)\left(2\pi r_0\right) a\, d\phi$$

$$= 2\pi a^2 q \int_0^\phi \left(\sin\phi\right) d\phi$$

$$= 2\pi a^2 q\left(1-\cos\phi\right)$$

Therefore, the stress resultants at any angle ψ are given by:

$$N'_\phi = -aq\left(\frac{1}{1+\cos(\alpha-\psi)}\right)$$

$$N'_\phi = aq\left(\frac{1}{1+\cos(\alpha-\psi)} - \cos(\alpha-\psi)\right)$$

where: $\psi = \alpha - \phi$

Table 6-1 Membrane Theory solution for stress resultants and couples (Units: kips/ft and ft-kips/ft)

ψ (deg)	ψ (rad)	N'φ	N'θ	Mφ
0	0.000	-5.359	-3.301	0.00
1	0.017	-5.334	-3.412	0.00
2	0.035	-5.311	-3.519	0.00
5	0.087	-5.246	-3.817	0.00
10	0.175	-5.155	-4.241	0.00
20	0.349	-5.038	-4.810	0.00
30	0.524	-5.000	-5.000	0.00

6.6 Edge Zone Solution

The membrane solution is incompatible with the boundary conditions in the actual problem. The errors at the boundaries consist of the dome-edge translation and rotation. These values are obtained from (6-4) and are given as:

$$\Delta_{H0} = \frac{a^2 q}{Eh}\left(\frac{1+v}{1+\cos\alpha} - \cos\alpha\right)\sin\alpha = -3600\frac{q}{E}$$

$$\Delta_{\alpha 0} = -\frac{aq}{Eh}(2+v)\sin\alpha = 325\frac{q}{E}$$

Determining the edge forces and moments that are required to satisfy the equations of compatibility can eliminate the errors of the membrane solution. The general procedure is to first calculate the displacement and rotation of the edge due to unit edge forces, and then to determine the size of the correction forces by solving the equations of compatibility. The displacements and rotations due to unit edge forces are given by the equations in Table 6-1 as:

$$\Delta_H(H=1) = D_{11} = \frac{3400}{E}$$

$$\Delta_H(M_\alpha = 1) = D_{12} = D_{21} = \frac{1540}{E}$$

$$\Delta_\phi(M_\alpha = 1) = D_{22} = \frac{1390}{E}$$

The magnitude of the edge forces H and M_α, can now be found by solving the equations of compatibility. The fixed base boundary condition requires that the dome edge neither rotates nor translates. Thus, the equations of compatibility are:

$$\sum \Delta_H = 0 = HD_{11} + M_\alpha D_{12} + \Delta_{H0}$$

$$\sum \Delta_\alpha = 0 = HD_{21} + M_\alpha D_{22} + \Delta_{\alpha 0}$$

the solution of which leads to:

H = 235 lb/ft

M_α = -280 ft-lb/ft

The influence of these edge forces is restricted to an area of the shell called the edge zone. The equations in Table 6-1 can again be used to calculate the effect of the edge forces at any point in the shell. The results of these calculations are shown in Table 6-3.

Table 6-3 Effect of edge forces on stress resultants and couples (Units: ft-kips and ft-kips/ft)

	$N\phi$		$N\theta$		$M\phi$	
ψ (deg)	H	M	H	M	H	M
0	0.866	0.000	22.636	10.248	0.000	1.000
1	0.327	-0.212	14.074	3.715	0.573	0.881
2	-0.003	-0.274	7.229	-0.031	0.712	0.642
5	-0.195	-0.124	-1.236	-1.866	0.282	0.073
10	0.001	0.017	-0.301	0.007	-0.031	-0.027
20	-0.001	-0.001	0.000	-0.004	0.001	0.000
30	0.000	0.000	0.000	0.000	0.000	0.000

6.7 Complete Solution

The values for the stress resultants and stress couples can now be found throughout the entire shell by adding the edge zone solution to the membrane solution. These results are tabulated in Table 6.4.

Table 6-4 Complete Solution by Approximate Method (Units: ft-kips and ft-kips/ft)

ψ (deg)	$N'\phi$	$N'\theta$	$M\phi$
0	-5.156	-0.893	-0.282
1	-5.198	-1.166	-0.114
2	-5.234	-1.819	-0.014
5	-5.257	-3.581	0.045
10	-5.160	-4.314	0.000
20	-5.038	-4.809	0.000
30	-5.000	-5.000	0.000

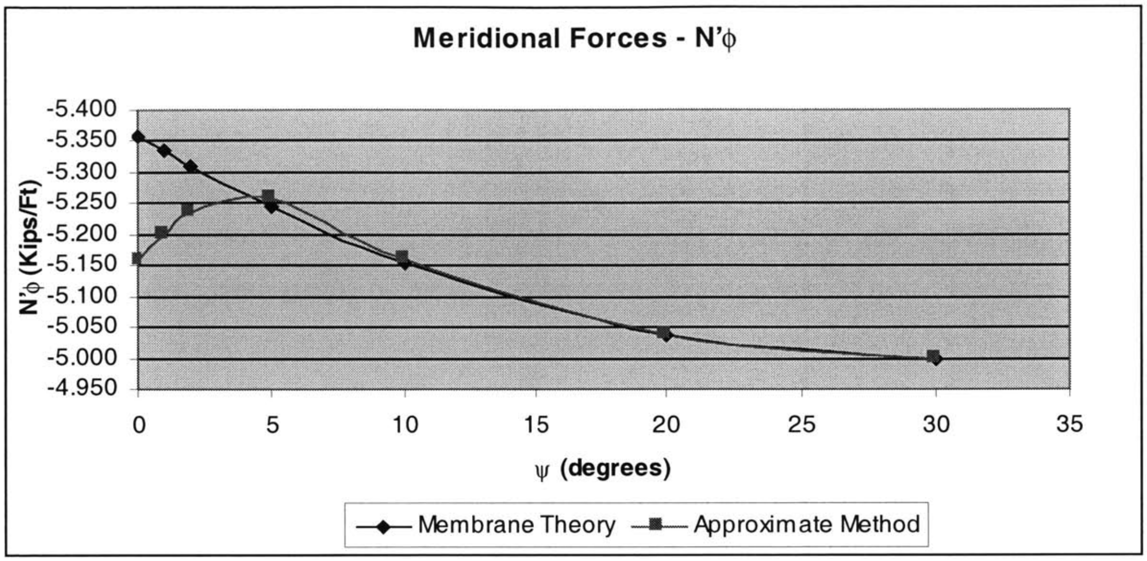


Figure 6-4 Meridional stress distribution by Membrane Theory and approximate method

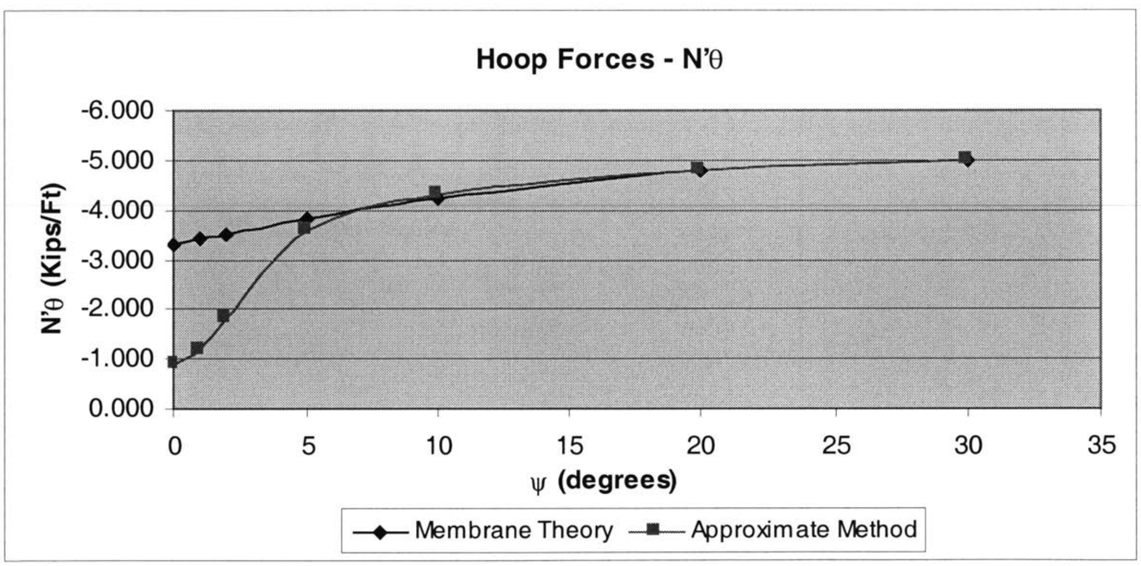


Figure 6-5 Hoop stress distribution for Membrane Theory and approximate method

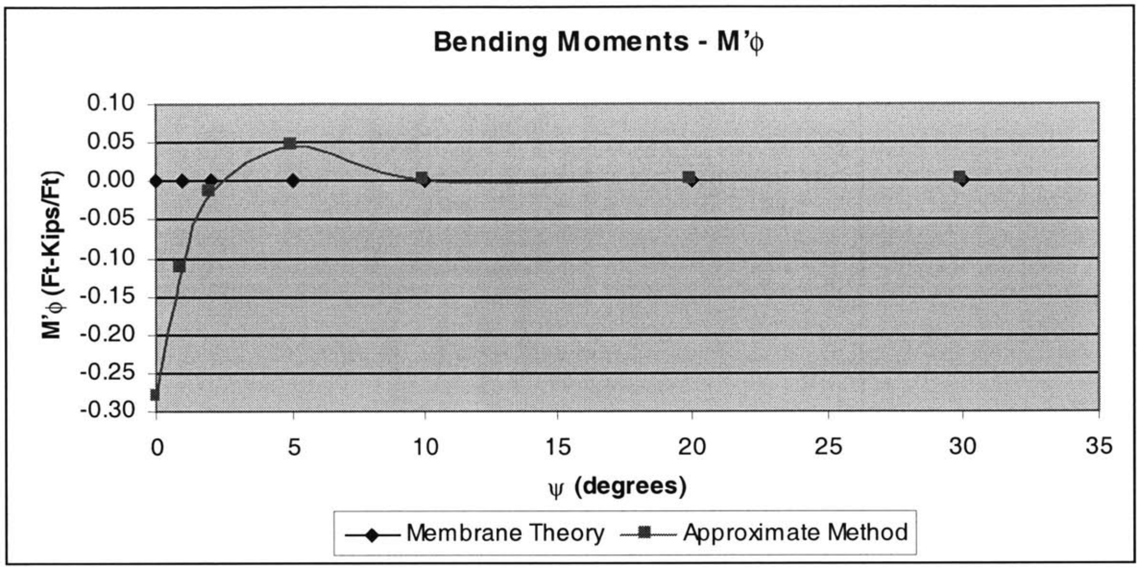


Figure 6-6 Bending moment distribution by Membrane Theory and approximate method

Figure 8-2, Figure 8-3, and Figure 8-4 display the stress resultants in the shell as obtained by both the Membrane Theory, and the approximate method. Notice how the two solutions are only different for the area of the shell near the edge. The edge effects are damped out rapidly. The area effected by the edge forces is known as the edge zone. The influence of several parameters upon the size of the edge zone is discussed in Chapter 8.

CHAPTER 7

CASE STUDY: COMPUTER BASED FINITE ELEMENT ANALYSIS

7.1 Computer Based Finite Element Analysis

The objective of this chapter is to illustrate use of the finite element program ADINA for the analysis of shell structures. An ADINA analysis typically begins with the definition of the actual problem to be solved. A model must be created that accurately represents the problem. Next, the model must be solved. Finally, the solution to the model must be interpreted. If the results are not satisfactory, the model must be refined. In ADINA, the convergence of the solution generally depends upon the type of elements used, the mesh density, and the number of nodes per element.

7.2 Problem Definition and Model Conceptualization

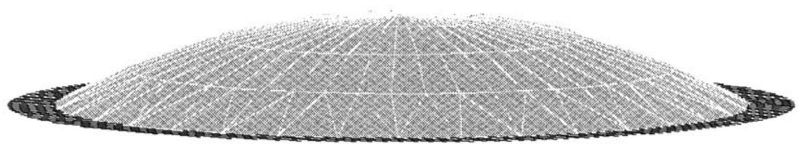


Figure 7-1 Three-dimensional physical problem

The problem to be analyzed is a spherical concrete dome with a rigidly supported edge. A uniform gravity load of 90 pounds per square foot is distributed over the entire shell surface. The radius of curvature of the shell mid-surface is 94.5 feet. The radius of the circle that defines the edge is 44.25 feet. The rise of the dome is roughly 11 feet. The shell thickness is only 4 inches.

A complex three-dimensional problem is difficult to construct and costly to solve. The symmetry of both the structure and the loading allows for a great simplification. The three-dimensional problem can be accurately represented by a two-dimensional axisymmetric analysis.

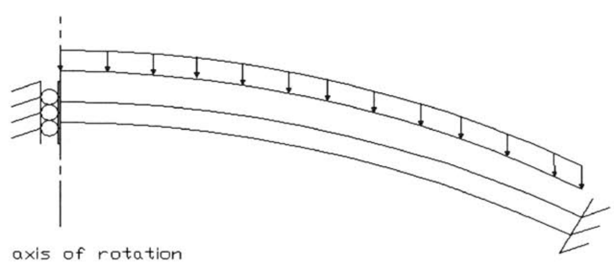


Figure 7-2 Two-dimensional axisymmetric problem

The simplified axisymmetric problem is depicted in Figure 7-2. Boundary conditions shown represent the rigid support at the edge and the conditions of symmetry at the apex.

In other words, the apex is free to translate in the z-direction, but it is fixed in both the y-translation and x-rotation.

7.3 Model Generation and Solution

The selection of elements is very important in attaining an accurate solution to the simplified problem. The best element for this problem is the ADINA axisymmetric shell element. This element is a two-dimensional isobeam element that accounts for the axisymmetric hoop strain and stress that occurs in the model. There are 2-, 3-, and 4-node isobeam elements. Isobeam elements are more effective than axisymmetric 2-D solid elements for this problem, because the 2-D solid elements often lock in shear when used to model a thin shell (ADINA R&D Inc, 1998).

Geometry and boundary conditions are defined in ADINA as specified above. The master degrees of freedom in the system are restricted to z-translation, y-translation, and x-rotation. Concrete can be modeled in the analysis as a linearly elastic isotropic material with an elastic modulus of 1000 ksi and a Poisson's ratio of .1667. The shell thickness is specified as 4 inches. The loading is applied as the z-component of a 90 psf pressure along the line of elements.

The elements are defined in ADINA as axisymmetric isobeam elements and are set to calculate the stress and strain response of the structure. The mesh to be used is specified by dividing the line into elements. The line can then be meshed with 2-, 3-, or 4-node elements. The initial mesh to be analyzed consists of ten 2-node elements. More refined meshes are also generated by increasing the number of nodes per element and by increasing the mesh density. Datafiles are created and a solution is obtained for each of the meshes.

7.4 Interpretation of Results

The solution for each mesh can then be examined in ADINA-PLOT. The deformed shape of the shell can be compared to the original shell geometry. Stress distribution in the shell, however, is the objective of this project. Stresses at various points can be displayed by ADINA in a list. An analytical solution to this problem has been obtained to use as a basis for comparison. The analytic solution has been obtained by an approximate method as explained in Chapter 6.

The complete approximate solution is then composed of the following three stress resultants: 1) hoop stress – N_θ, 2) meridional stress – N_ϕ, and 3) bending moment – M_ϕ.

In ADINA, the stresses corresponding to the hoop and meridional stresses are stress-TT and stress-RR, respectively. These stresses are calculated along the elements at various integration points. The quantity and location of integration points in each isobeam element vary depending upon the number of nodes.

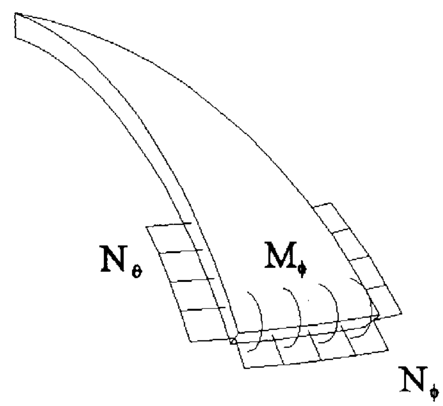


Figure 7-3 Stress resultants and couples

Bending moments are not explicitly shown in the output of ADINA for the isobeam elements, but can be estimated based on the distribution of stress-RR through the thickness of the shell. This meridional stress distribution is given by the stress at the integration points of the isobeam element. The 2-, 3-, and 4-node elements all use Gauss 2-point integration along the s-direction. Integration along the r-direction is taken at 1 point for the 2-node element, 2 points for the 3-node element, and 3 points for the 4-node element. Figure 7-4 displays the location of the integration points for the isobeam elements.

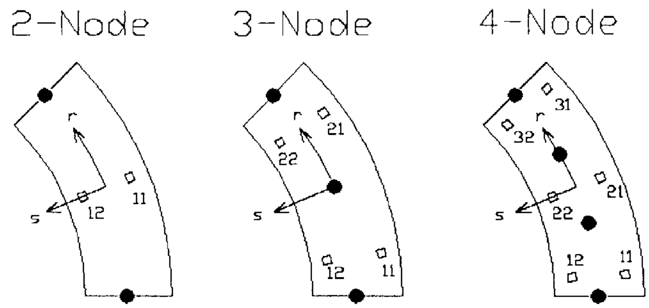


Figure 7-4 Gauss Integration points for the isobeam elements

Since the stresses are given at each of the integration points, the finite element solution can be compared to the solution obtained by the approximate method only after converting the stresses RR and TT into stress resultants and bending moments as given by the approximate method solution. The stress resultants and bending moments for each solution are shown in the graphs on the following pages. The effect of the mesh density and number of nodes per element on convergence of the finite element solution is also discussed for each resultant.

7.5 Bending Moment

The moment distribution in the finite element solution is very close to the approximate method solution regardless of the number of nodes per element and mesh density. The moment is a maximum of roughly -0.2 ft-kips/ft at the boundary edge, when $\psi = 0^{\circ}$. It is completely damped out by $\psi = 10^{\circ}$.

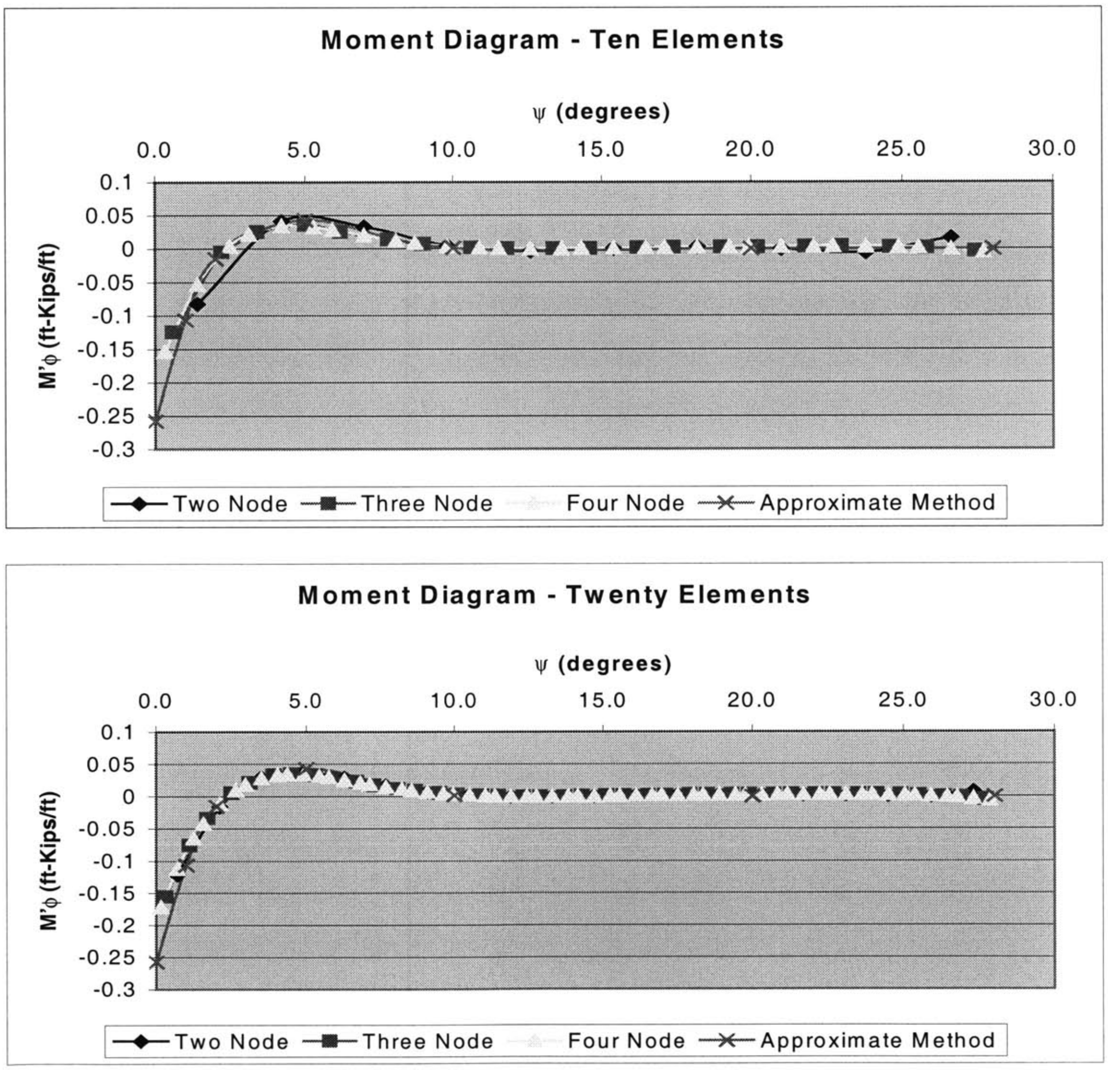
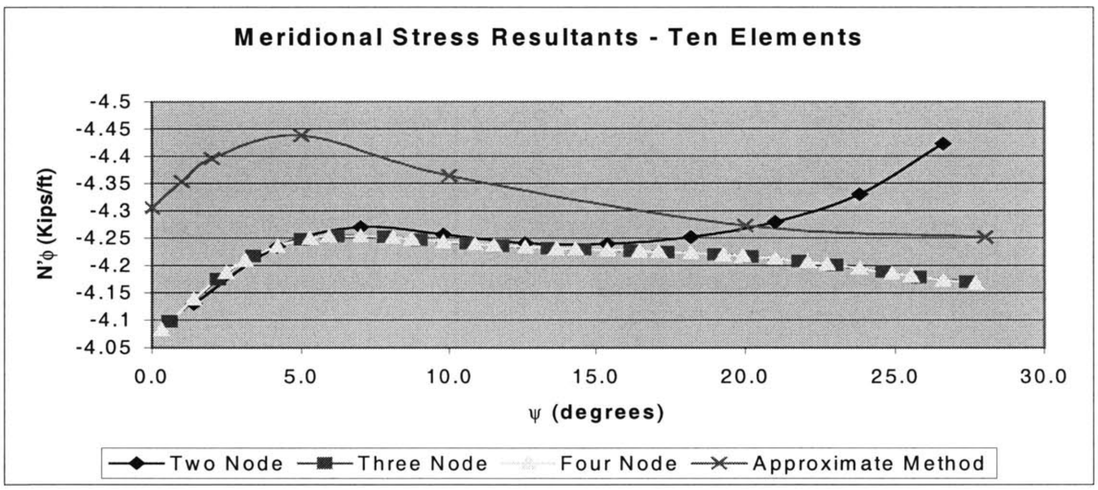


Figure 7-5 Moment diagrams for ten and twenty element meshes

7.6 Meridional Stress

The meridional stress resultants of both the ten and twenty element meshes generally follow the same trend as the approximate method solution. Values for the FEM solutions are slightly lower though. The behavior of the 2-node elements varies near the apex of the dome. This is due to the singularity of the model at the apex and the linear distribution of stresses in the 2-node elements. The singularity occurs because the area upon which the stress acts approaches zero at the axis of symmetry. The 3- and 4-node elements can more effectively deal with the singularity at the apex due to their higher order stress distributions along the r-direction.



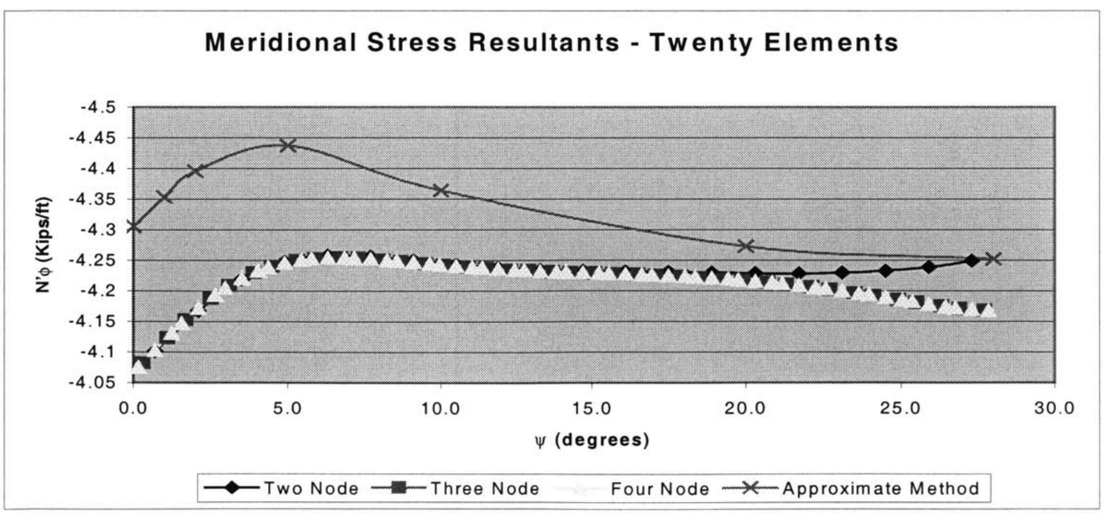
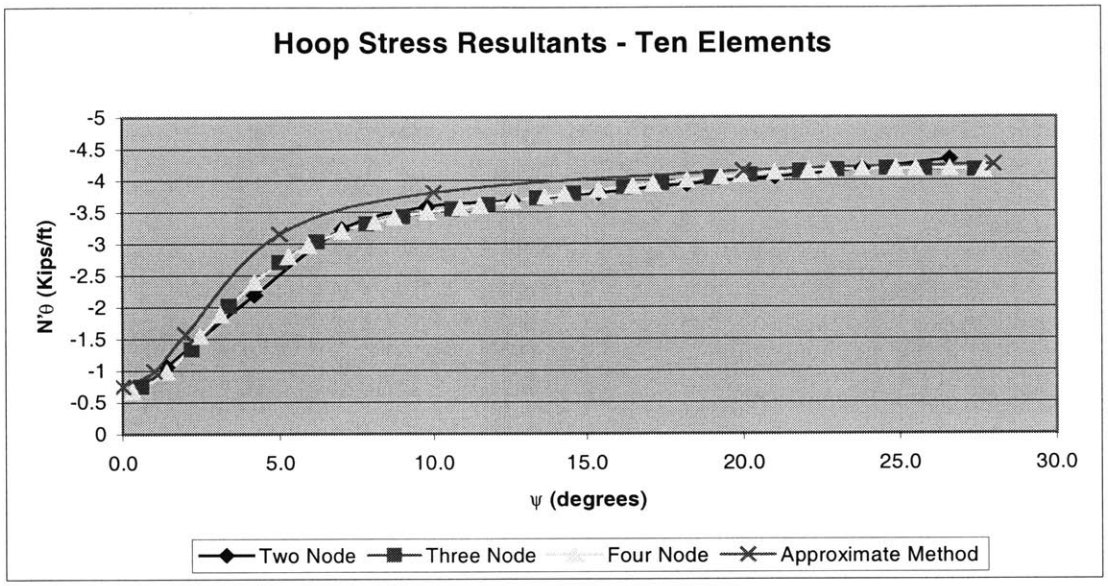


Figure 7-6 Meridional stress distribution for ten and twenty element meshes

7.7 Hoop Stress

The hoop stress solution is roughly equivalent for the different meshes used and the approximate method. The general trend in the hoop stress distribution is to increase radically from the edge and then level off at a peak value of around 4 kips/ft near the apex of the dome.



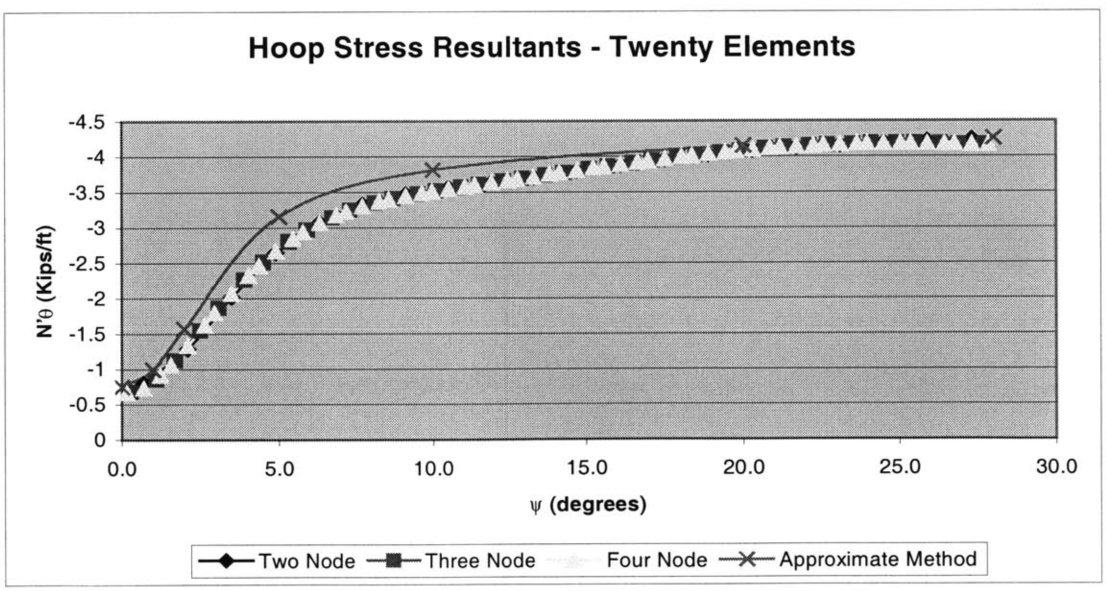


Figure 7-7 Hoop stress distribution for ten and twenty element meshes

CHAPTER 8

PARAMETERS AFFECTING THE SIZE OF THE EDGE ZONE

8.1 The Edge Zone Solution

The edge zone can be defined as the area in a shell in which bending is present. The size of the edge zone is dependent upon numerous parameters. The bending moment in a spherical shell is a function of the shell geometry, Poisson's ratio, and the edge forces at the boundary conditions. The edge forces, however, are dependent upon the loading and geometry of the shell. As the previous examples have shown, the bending moment is damped out as the distance from the edge increases. It is the purpose of this chapter to illustrate the effect of certain geometric parameters on the size of the edge zone.

The edge zone solution is generally a result of boundary conditions. Under the Membrane Theory, the entire shell is assumed to be free from bending. If the shell boundary conditions are compatible with the membrane solution, then there is no bending in the shell at any point, and the edge zone is non-existent. If on the other hand, the boundary conditions are not compatible with the membrane solution, then there is an additional edge zone solution that includes bending. The equation for the bending in a shell is given from Table 6-1 as:

$$M_{\phi} = \left(\frac{a}{\lambda} \sin \alpha e^{-\lambda\psi} \sin \lambda\psi \right) H + \left(\sqrt{2} e^{-\lambda\psi} \sin\left(\lambda\psi + \frac{\pi}{4} \right) \right) M_{\alpha} \tag{8-1}$$

where:

$$\lambda^4 = 3\left(1 - v^2\right)\left(\frac{a}{h}\right)^2$$

8.2 Effect of Curvature on the Edge Zone

Considering the problem analyzed in Chapter 8, the effect of changing the shell geometry upon the edge zone is examined below. The equation for the bending moment is given in Table (8-1). The damping coefficient λ determines the distance from the edge at which the edge effects are effectively diminished. The values of H and M_{α} determine the initial magnitude of the oscillating moment. By changing λ, the designer can define the region of the shell that will experience bending.

The effect of the curvature on the distribution of bending in the shell can be examined by changing the angle of opening. The span of the dome is held constant at r_o = 50 ft. The radius of curvature a of the dome can then be varied by changing the opening angle α. When α is small, the dome is very shallow. When α is 90°, the shell is an exact hemisphere.

For each value of α, the horizontal force H and the moment M_{α} at the edge must first be calculated. The bending moment can then be found at any parallel defined by the angle ψ by calculating (8-1). The point at which the bending moment is effectively damped out marks the border of the edge zone. The angle ψ ranges from zero at the edge to α at the apex of the dome. Since α changes in each case, it is convenient to obtain the bending moment at different values of ψ/α, which are expressed as percentages.

The bending moment distributions in shells of various opening angles are plotted in Figure 8-1 and Figure 8-2. The effect of the angle of the shell on the size of the edge zone is clear: as the angle decreases and thus the shell becomes shallower, a larger percentage of the dome experiences bending moments. If the angle of the shell were to be infinitesimally small, then the shell would be a flat circular plate, and bending would be experienced throughout the entire surface.

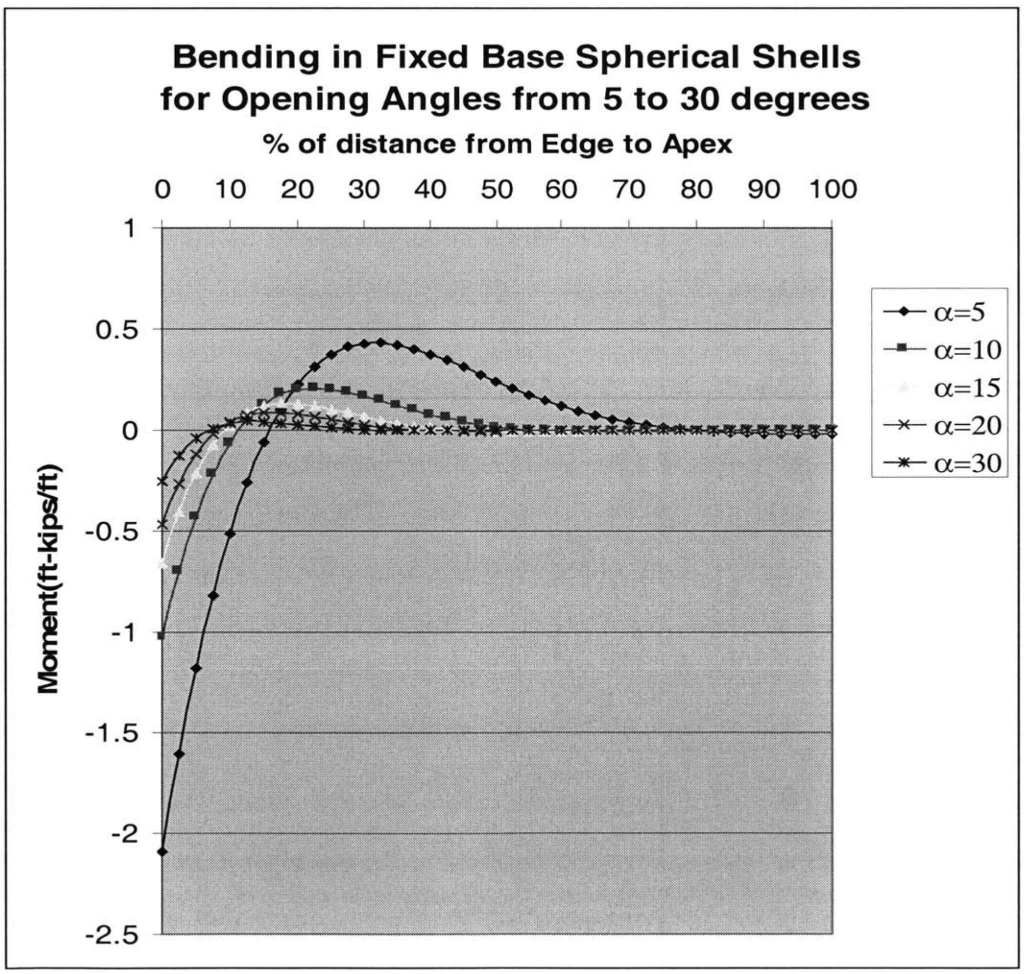


Figure 8-1 Influence of curvature upon edge zone size for shallow shells

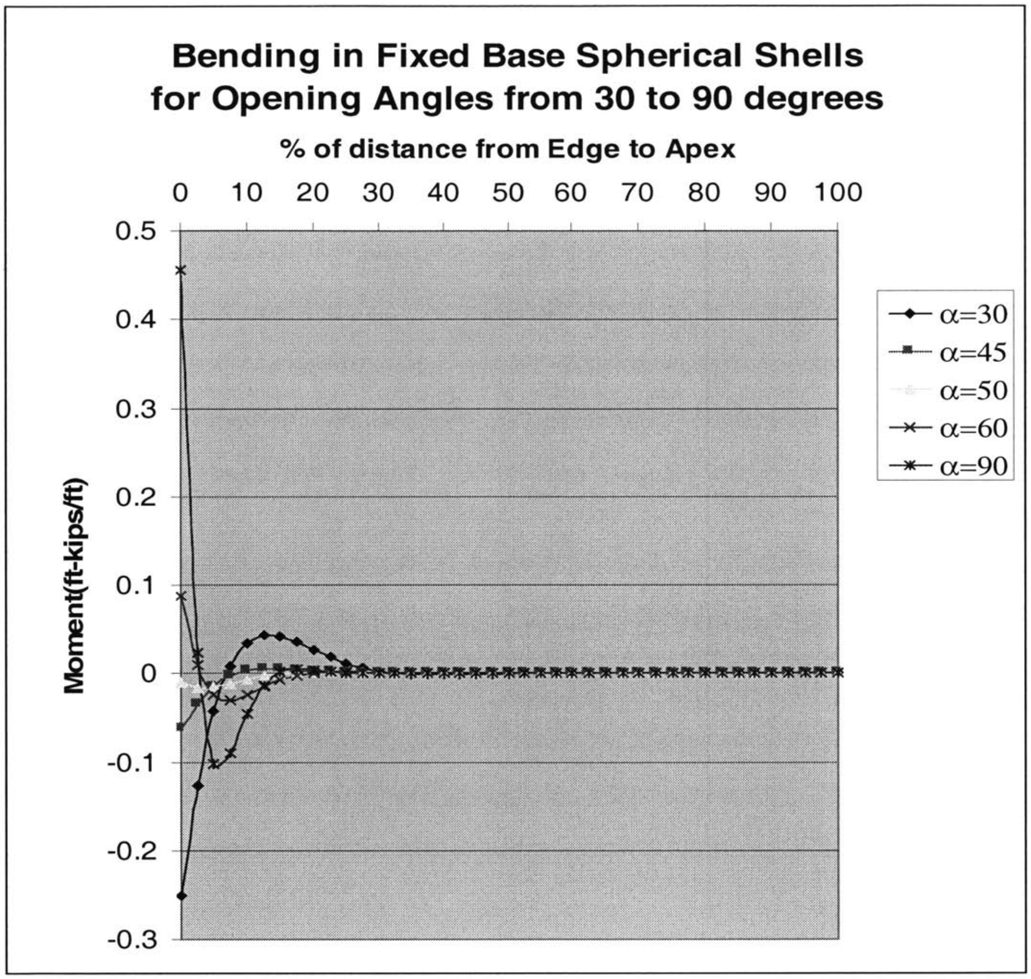


Figure 8-2 Influence of curvature upon edge zone size for deeper shells

A change in the sign of the moment occurs somewhere between $\alpha = 45^o$ and $\alpha = 60^o$. This is in the general area of the shell in which the hoop stress approaches zero. Above this point, the hoop stress is compressive, and below this point, the hoop stress is tensile. At this point the errors due to the membrane theory are relatively small, and only small edge forces are required to ensure compatibility. As a result, the edge zone solution has a small effect on the overall system. Yet the small effect is nonetheless distributed over a larger area of the shell than that of the larger edge moment in the 60^o shell. This illustrates that the size of the edge zone is independent of the magnitude of the edge forces. Only the magnitude of the oscillating moment distribution is dependent upon the magnitude of the edge forces.

8.3 Effect of Shell Thickness on the Edge Zone

Now consider the effect of the shell thickness on the edge zone. The span is again held constant at r_o = 50 ft. The shell angle is also held constant at 30°. The shell thickness h can now be varied from 2 inches to 2 feet. The results are plotted in Figure 9-3.

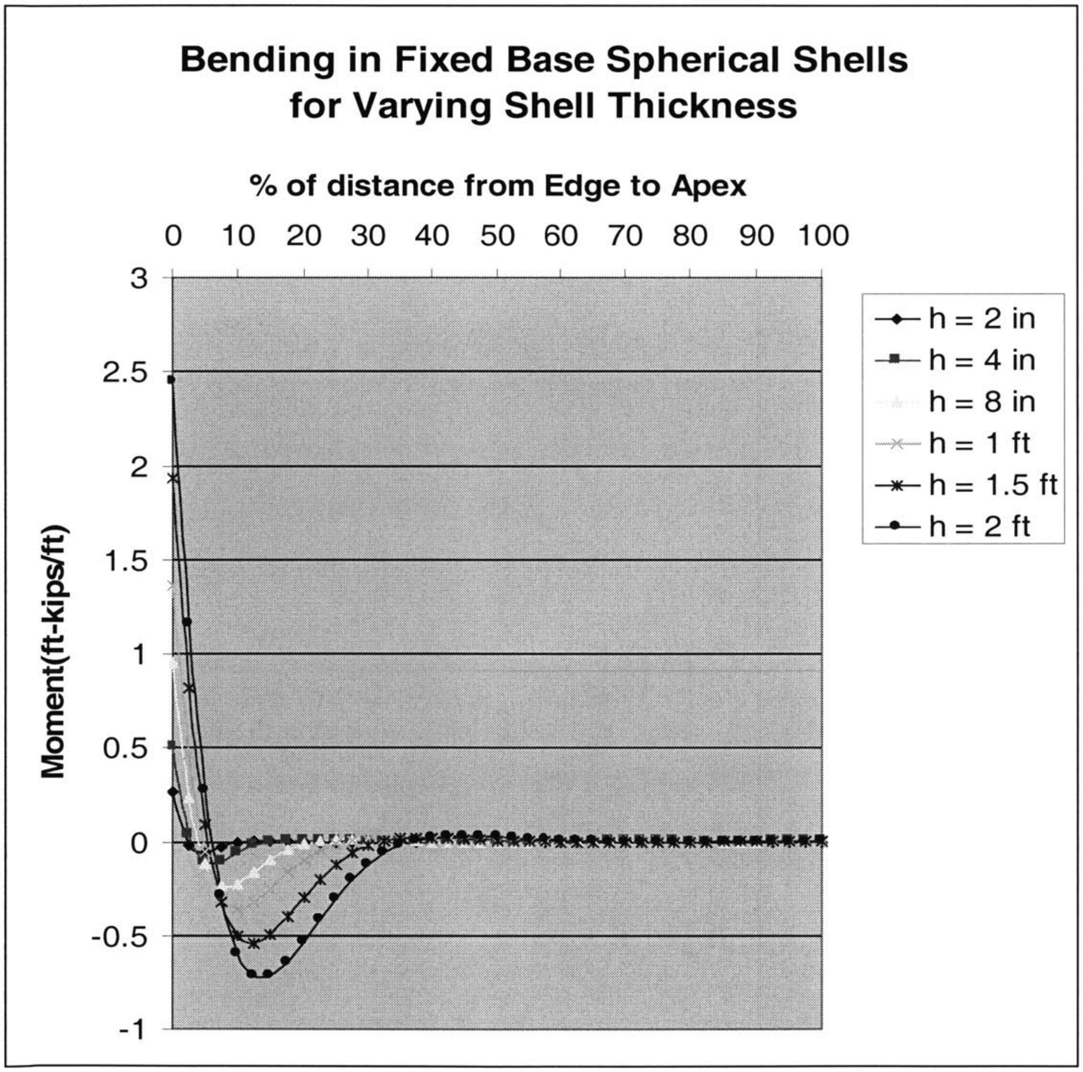


Figure 8-3 Influence of shell thickness upon edge zone size

The plots in Figure 9-3 show that the edge zone size increases with increasing shell thickness. The bending rigidity of the shell is directly related to the shell thickness. As the thickness increases, the bending rigidity of the shell increases. If the shell is very thin, its bending stiffness is small, and the loading will be resisted mostly by membrane forces. As the thickness of the shell is increased, the bending rigidity increases and more of the shell will begin to resist the loading through bending action as well as membrane action.

REFERENCES

ADINA R&D Inc., *ADINA Theory and Modeling Guide Vol. 1,* ADINA R&D, Inc., Watertown, MA, 1998.

Bathe, K.J., *Finite Element Procedures,* Prentice Hall, New Jersey, 1996.

Beles, A., and Soare, M., *Elliptic and Hyberbolic Paraboloidal Shells Used in Constructions,* Editura Academiei Romane, Bucharest, 1964.

Billington, D., *Thin Shell Concrete Structures,* McGraw-Hill, Inc., New York, 1965.

Flugge, W. *Stresses in Shells,* 2nd Edition, Springer-Verlag, New York, 1973.

Pfluger, A., *Elementary Statics of Shells,* 2nd Edition, F.W. Dodge Corporation, New York, 1961.