Caustics and Evolutes

for Convex Planar Domains

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(1981)

Submitted to the
Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

Massachusetts Institute of Technology
June 1986

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ABSTRACT

The billiard ball map has been studied since the beginning of the century. Interest in it has recently increased, spurred by new mathematical techniques and the interest in the length spectrum associated to boundary value problems.

The goal of this study is to better understand convex planar domains in which the billiard ball map is integrable, that is, where there is a continuous family of caustics for the billiard ball map and this family includes the boundary. The principal result is the theorem of the third chapter which shows that the only smooth convex planar curves satisfying a certain group property are ellipses.

This group property and a curvature relating operator leading to both the property and to the proof of the theorem are defined in the second section and in the beginning of the third section.

Also important are the motivating example of chapter 4 which shows that the relation between caustics suggested by the theorem of the third chapter is special, and the example of chapter 5 in which the lengths of caustics are completely isolated.

Thesis Supervisor: Dr. Richard B. Melrose

Title: Professor of Mathematics

I would like to thank my parents and teachers who have enabled me to approach these studies, and my wife and friends who provided me with moral support to finish them. Special thanks go to Richard Melrose whose guidance was irreplaceable.

The first chapter of this work introduces the billiard ball map, invariant circles, and caustics, and explores some basic relations among them. The second chapter gives a characterization of caustics and builds an operator relating caustics which is based on this characterization. This chapter explains geometric relations between caustics in analytic terms and includes the main ingredients of the theorem of the third chapter.

The third chapter includes the principal calculation, giving an obstruction to satisfying the group property (also discussed in that chapter), and the fourth and fifth chapters provide relevant examples.

CONTENTS

Notation

Ω	a (planar, bounded, convex) domain.
β	the billiard ball map.
T^*X	the cotangent bundle of X.
$B^*_x X$	the unit ball in $T^*_x X$.
S^*X	the unit cotangent bundle of X.
ϕ_t	geodesic flow (in R^2).
$\partial\Omega$	the boundary of Ω (usually smooth).
ρ	projection $S^*_{\partial\Omega}\mathbf{R}^2 \longrightarrow B^*\partial\Omega (\Omega \subset R^2)$.

Chapter 1.

Introduction

The only proposition in this section is typical of the point of view that this work takes. Borrowed in part from algebraic geometry, this viewpoint agrees at times with the modern point of view ($B^*\partial\Omega$), and on other occasions with the century old point of view (everything is in the plane). The latter point of view is useful, and sometimes even elegant, as in the example of the fourth section.

1.1. The Billiard Ball Map

Given a convex planar domain, Ω, with smooth boundary, $\partial\Omega$, we define the **billiard ball map on** $\partial\Omega$,

$$\beta : B^*\partial\Omega \longrightarrow B^*\partial\Omega,$$

as follows:

Let S^*R^2 denote the unit cotangent bundle, and let $B^*\partial\Omega = \{\xi \in T^*\partial\Omega : |\xi| \leq 1\}$. We will view $B^*\partial\Omega$ as embedded in T^*R^2, and S^*R^2 as embedded in T^*R^2 (we want to be able to use geodesic

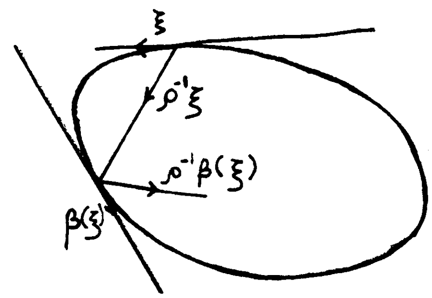


Figure 1.1: The billiard ball map

translation in R^2). Let ϕ_t denote geodesic flow in R^2, and $\pi : T^*R^2 \to R^2$ the projection.

There is an inward pointing normal, i(p)$\in S_pR^2$ (the unit tangent bundle's fiber at p) for p$\in \partial\Omega$. (If B$\subset R^2$ is any set containing Ω, and $\xi \in S_p^*R^2$ with ξ (i(p))>0, then $\pi(\phi_t\xi) \in \partial\Omega$ for some t>0 and if $\pi(\phi_t\xi) \in \partial$B then t>min{t> 0 | $\pi(\phi_t\xi) \in \partial\Omega$}.)

Given $\xi \in B_p^*\partial\Omega$, there is a unique inward pointing w $\in S_p^*R^2$ (w(i(p))$\geq$ 0) with ξ(v)=w(v) for v$\in T_p\partial\Omega \hookrightarrow T_pR^2$.

Define $\rho : S_{\partial\Omega}^*R^2 \to B^*\partial\Omega$ by $\rho_p(w) = \xi$ ifξ(v)=w(v), for all v$\in T_p\partial\Omega$. We have just shown that given an inward pointing orientation of $\partial\Omega$, ρ has an inverse, ρ^{-1}. If w is not tangent to $\partial\Omega$, that is w(i(p))>0, then there is a least t>0 with $\phi_tw \in S_{\partial\Omega}^*R^2$ (which we denote by t) and we set

$$\beta(\xi) = \rho\phi_t\rho^{-1}(\xi)$$

If w is tangent to $\partial\Omega$, that is w(i(p))=0 or $|\xi| = 1$, then we set $\beta(\xi) = \xi$.

1.2. Invariant Circles and Caustics

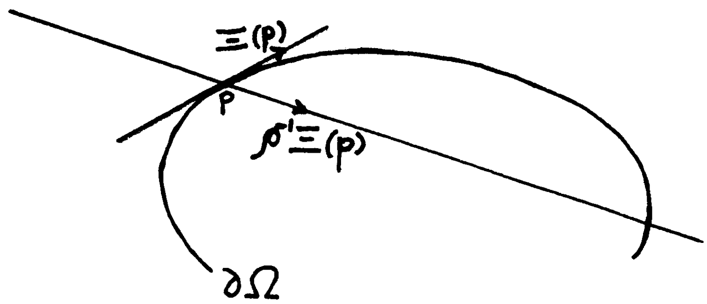
Definition 1.2.1: An **invariant circle** for the billiard ball map on $\partial\Omega, \beta$, is a smooth section, Ξ, of $B^*\partial\Omega \longrightarrow^{\pi} \partial\Omega$, such that if $\xi \in \Xi(\partial\Omega)$, then $\beta(\xi) \in \Xi(\partial\Omega)$.

Definition 1.2.2: A **caustic** for the billiard ball map on $\partial\Omega$ is a simple closed curve C$\subset \Omega$ such that if $\xi \in S^*$C and t(ξ) is the least t$\geq$ 0 with $\pi(\phi_t\xi) \in \partial\Omega$, then there is a $\xi' \in S^*$C (and a t(ξ'), the least t$\geq$ 0 with $\pi(\phi_t\xi') \in \partial\Omega$) with $\beta(\rho\phi_{t(\xi)}\xi) = \rho\phi_{t(\xi')}\xi', \rho$ as above.

$\rho^{-1}(\Xi(\partial\Omega)) \subset S^*_{\partial\Omega}R^2$ may be viewed as a (one parameter) family of lines (parameterized by any parameterization of $\partial\Omega$). To $\xi \in \rho^{-1}(\Xi(\partial\Omega))$ we associate the line $\{\pi\phi_t\xi \mid t \in R\}$, where ϕ_t is geodesic flow, and π is projection. (See the figure above.)

For convex Ω, we can see the correspondence of caustics to invariant circles directly. For $\xi \in \Xi(\partial\Omega)$, consider $\{\pi\phi_t(\rho^{-1}\xi) \mid t \in R\} \cap \partial\Omega = \{\pi(\xi), \pi(\beta(\xi))\}$. We claim that if $\pi\xi_2$ is between $\pi\xi_1$ and $\pi\beta(\xi_1)$ on $\partial\Omega$, then the line segments $\overline{\pi\xi_1\pi\beta(\xi_1)}$ and $\overline{\pi\xi_2\pi\beta(\xi_2)}$ must intersect ($\xi_1, \xi_2 \in \Xi(\partial\Omega)$). To see this, first note that q(p)=$\pi\beta\Xi$(p) is a continuous map (from $\partial\Omega$ to itself) and that q(p)=p only when $\beta\Xi(p) = \Xi(p)$, that is only when $|\Xi(p)|$=1 and the invariant circle, Ξ, corresponds to the caustic $\partial\Omega$. We show that q is one-to-one, and thus if $\pi\beta(\xi_2)$ is not outside the arc seg-

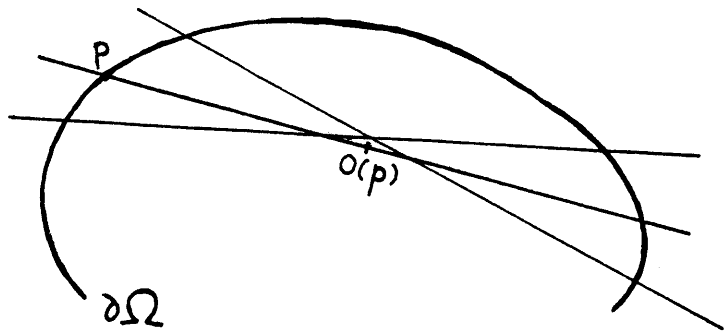
Figure 1.2: The correspondence of Ξ to a family of lines.



ment $\pi\xi_1\ \pi\beta(\xi_2)$, continuity of q is contradicted. To see that q is one-to-one assume q(p_1)=q(p_2). Then, since $\beta(\Xi(p_1)), \beta(\Xi(p_2)) \in \Xi(\partial\Omega)$, $\beta\Xi(p_1) = \beta\Xi(p_2)$, so$\Xi(p_1) = \Xi(p_2)$ and $p_1 = p_2$.

Given p$\in \partial\Omega$, take a sequence $\{p_k\} \subset \partial\Omega$ with $p_k \longrightarrow p$ as $k \longrightarrow \infty$, and $p_k \neq p$. Let $o_k = \overline{pq(p)} \bigcap \overline{p_k q(p_k)}$. $\{o_k\}$ converges to a unique point, o(p), since it is a sequence in the compact line segment $\overline{pq(p)}$ ($\partial\Omega$ is convex), and p'$\mapsto o(p', p) = \overline{pq(p)} \bigcap \overline{p'q(p')}$ is continuous in p' so there can be at most one accumulation point for $\{o_k\}$. For $\xi \in \Xi(\partial\Omega)$, we denote $o(\pi(\xi))$ by $c(\xi)$. C=$\{c(\xi) \mid \xi \in \Xi(\partial\Omega)\}$ is the desired caustic. C is a closed curve since c(ξ) is continuous and $\partial\Omega$ is closed. When $\partial\Omega$ is smooth and convex, (p,p')$\mapsto$ o(p,p') is smooth near $\partial\Omega$ (as shown in the next paragraph). Since $\partial\Omega$ is compact, C is smooth as long as (p,p')$\mapsto$ o(p,p') is injective. The condition for this map to be injective is an open condition, and, since $\partial\Omega$ is itself an invariant circle (Ξ_0(p)=(p,1)), for invari-

Figure 1.3: The construction of the caustic C.



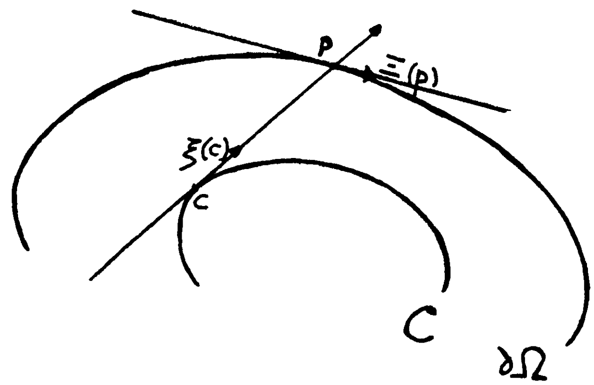
ant circles near Ξ_0 the corresponding caustic is smooth.

The role played by the curvature in this setting is clarified if we assume that the lines (corresponding to $\Xi(s), s \in \partial\Omega$) are given by $y = y(s) + m(s)x$ in the x-y plane near a fixed line $y = y(s_0) + m(s_0)x$ (we assume that $m(s) < \infty$ for s near s_0). Then the intersection of $y = y(s) + m(s)x$ and $y = y(s_0) + m(s_0)x$ is given by

$$x = \frac{y(s_0) - y(s)}{m(s) - m(s_0)}.$$

Both numerator and denominator approach zero as s approaches s_0, so this equation has a unique solution when $k(s_0) = \partial_s m(s_0) \neq 0$. (k is the curvature and for a closed curve if the curvature has constant sign then it must be positive.) So if the curvature is never zero, there is a well defined smooth convex caustic corresponding to Ξ. Finally, if Ξ is (C^2) near Ξ_0, then $0 < k(s)$ for any s. The construction of C (for convex Ω) shows that the tangent to C at

Figure 1.4: The integral curve corresponding to C.



c (when c=c(ξ) this tangent is $\{\pi\phi_t\rho^{-1}(\xi) \mid t \in R\}$) does not intersect C near c. Since C is closed, this implies that C is convex.

Conversely, given a smooth convex caustic, C, its tangents provide an invariant circle for the billiard ball map, β, on $\partial\Omega$. Orient C. At c$\in$C take $\tau(c) \in S_c$C (the unit tangent bundle). There is a unique $\xi \in S_c^*$C with $\xi(\tau(c))$=1. Regard $\xi \in T_c^*R^2$, under the inclusion $T_c^*C \hookrightarrow T_c^*R^2$. Let t(c) be the least t$\geq$ 0 with $\phi_t\xi \in T_{\partial\Omega}^*R^2$. For p$\in$ $\partial\Omega$ there is a (unique) c $\in$C with $\pi\phi_{t(c)}\xi(c)$=p, because C is convex. Let $\Xi(p) = \rho\phi_{t(c)}\xi(c)$. Since C is smooth, $c \mapsto \xi(c)$ is smooth, and since C is convex and smooth, p$\mapsto$ c is smooth. So Ξ is a smooth section of $B^*\partial\Omega \to^\pi \partial\Omega$. By definition, C is a caustic if and only if $\beta\rho\phi_{t(c)}\xi(c) = \rho\phi_{t(c')}\xi(c')$ for some c'$\in$ C, and hence, given p$\in$ $\partial\Omega$ as above, q=$\pi\rho\phi_{t(c')}\xi(c') \in \partial\Omega$ and $\beta(\Xi(p)) = \Xi(q)$. So $\beta(\Xi(\partial\Omega)) \subset \Xi(\partial\Omega)$ and Ξ is an invariant circle.

We summarize our findings in

Proposition 1: Given a convex planar domain Ω with a smooth boundary, $\partial\Omega$, there is a one-to-one correspondence between invariant circles for the billiard ball map on $\partial\Omega$ in a neighborhood of $\Xi_0(p) = (p,1)$ and convex smooth caustics for the billiard ball map on $\partial\Omega$ in a neighborhood of $\partial\Omega$.

pf: The correspondence given above is one-to-one because any curve in R^2 is completely determined by its tangent lines. (The construction of C from Ξ actually showed this.)

Note that if Ω were not convex, there would be a line segment $\overline{pq}$ contained in Ω (p,q$\in \partial\Omega$) so that for any point, a, on that segment there is another point, b, on $\partial\Omega$, with $\overline{ab}$ not contained in Ω. Hence, any caustic tangent to $\overline{pq}$ could not be convex. (The corresponding "invariant circle" would be discontinuous.)

Definition 1.2.3: The billiard ball map on $\partial\Omega$ is **integrable** if there is a neighborhood N of $S^*\partial\Omega$ in $B^*\partial\Omega$ which is included in the images of invariant circles for the billiard ball map on $\partial\Omega$. That is, for each $\xi \in$ N there is a p$\in \partial\Omega$ and an invariant circle Ξ with $\xi = \Xi$(p).

Definition 1.2.4: We say that the billiard ball map on $\partial\Omega$ is **integrable up to C**, or that it is **integrable between C and $\partial\Omega$**, if C is a caustic for the billiard ball map on $\partial\Omega$, and every point in Ω lying between C and $\partial\Omega$ belongs to some caustic.

Since Ω is compact, if the billiard ball map on $\partial\Omega$ is integrable, there is a caustic C whose corresponding integral curve has its image contained in the neighborhood N of the definition above, so that the billiard ball map on $\partial\Omega$ is integrable between C and $\partial\Omega$.

In what follows, we aim to understand convex planar curves for which the billiard ball map is integrable and an additional condition holds. The proposition of this section allows the characterization of caustics in the second section, and thus the characterization of curves for which the billiard ball map is integrable and the caustics have a special property.

Chapter 2.

Caustics and Evolutes

The treatment in this chapter is motivated by the first lemma within. The author first encountered the reflected distance in the dynamics of the billiard ball map in a paper by Lazutkin [L] where the length of a caustic and the rotation number of the map induced on that caustic by the billiard ball map are related through the reflected distance.

2.1. Caustics

Let Ω be a convex planar domain with a smooth boundary.

Let C be a caustic for the billiard ball map on $\partial\Omega$.

Orient C, that is, split S^*C to a disjoint union $S^*C = FS^*C \sqcup BS^*C$. Fix a point $p \in C$ and take $\xi_+(p) \in FS^*_pC$, $\xi_-(p) \in BS^*_pC$ (there is exactly one choice for each of these). Then we define points $q_+, q_- \in C$ which are the **forward** and **backward return points of p.** q_+ is such that there are $\xi_-(q_+) \in BS^*_{q+}C$ and

$t(\xi_-(q_+)) \geq 0$ and $t(\xi_+(p)) \geq 0$ with

$$\rho(\phi_{t(\xi_-(q+))}\xi_+(q_+)) =$$

$$-\rho(\phi_{t(\xi_+(p))}\xi_+(p)) \in B^*\partial\Omega,$$

and $q_- \in C$ is such that there are $\xi_+(q_-) \in FS^*_{q_-}C$ and $t(\xi_+(q_-))$, $t(\xi_-(p)) \geq 0$ with

$$\rho(\phi_{t(\xi_+(q_-))}\xi_+(q_-))$$

$$= -\rho(\phi_{t(\xi_-(p))}\xi_-(p)) \in B^*\partial\Omega.$$

Definition 2.1.1: For a,b $\in C$ denote by $|ab|$ the length of the arc segment of C between a and b. With $t(\xi_-(p))$, $t(\xi_+(p))$, $t(\xi_-(q_+))$, and $t(\xi_+(q_-))$ as above, we set

$$FQ(p, C, \partial\Omega) = t(\xi_+(p)) + t(\xi_-(q_+)) - |q_+p|$$

which we call **the forward reflected distance of C from $\partial\Omega$ at p.** (See the accompanying figure.)

$$BQ(p, C, \partial\Omega) = |pq_-| - t(\xi_+(q_-)) - t(\xi_-(p))$$

is **the backward reflected distance of C from $\partial\Omega$ at p.**

Note that we could define the "reflected distance" for any curve C inside Ω (C not necessarily a caustic) by replacing ρ and $-\rho$ above by π, and $B^*\partial\Omega$ by $\partial\Omega$, but this seems to be special to two dimensions.

Lemma 2: If C is a caustic for the billiard ball map on $\partial\Omega$ (Ω convex with smooth boundary), then the forward reflected distance,

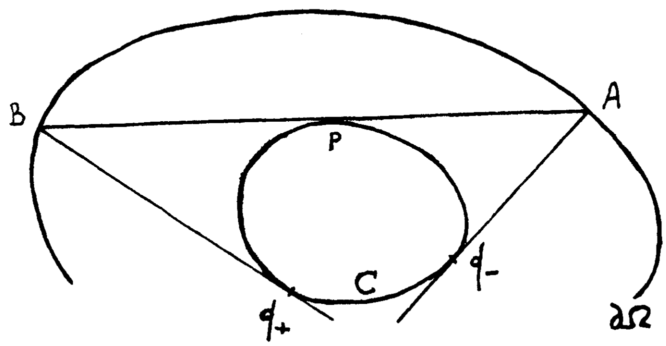


Figure 2.1: The reflected distance of C from $\partial\Omega$ at p.

FQ(p,C,$\partial\Omega$), is independent of the point p, as is the backward reflected distance.

proof: Consider FQ(p,C,$\partial\Omega$).

For each $\xi \in S^*C$ let $t(\xi)$ denote the first $t \geq 0$ with $\phi_t\xi \in S^*_{\partial\Omega}R^2$. Let

$$\rho : S^*_{\partial\Omega}R^2 \to B^*\partial\Omega$$

be as in the first section.

Fix p,p' $\in$C, let q,q' $\in$C be their forward return points, let $f \subset C$ be the arc segment between p and p', and let $b \subset C$ be the arc segment between q and q'. Consider the submanifolds $F, B \subset S^*R^2$ given by

$$F = \{\phi_t\xi \mid \xi \in FS^*_fC, 0 \leq t \leq t(\xi)\},$$

and

$$B = \{\phi_t\xi \mid \xi \in BS^*_bC, 0 \leq t \leq t(\xi)\}.$$

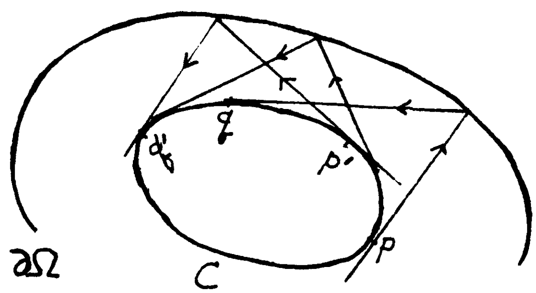


Figure 2.2: The forward and backward Lagrangians of C.

We define a map $\tau : F, B \to B^*\partial\Omega$. If $\xi \in F$ or $\xi \in B$, $\xi = \phi_{t.}\xi.$ for some $\xi. \in S^*C$, and $\tau(\xi) = \rho\phi_{t(\xi.-t.)}\xi$ (this is geodesic translation to $\partial\Omega$ followed by projection to $B^*\partial\Omega$).

It is known [GM] that the billiard ball map is symplectic with respect to the cannonical 2-form, ω, on $B^*\partial\Omega$. This symplectic structure was discovered by Birkhoff who viewed the billiard ball map as a twist map on an annulus. In this setting, the billiard ball is represented by its trajectory and the coordinates in the annulus are the angle which a trajectory makes with the tangent to the boundary and the point on the boundary at which it hits the boundary (given by arclength). In these coords, the 2-form is given by sin(angle)d(arclength)d(angle). We claim that F and B, the forward and backward flowouts above, are Lagrangian with respect to $\tau^*\omega$. To see this, note that $\tau^*\omega$ is invariant under ϕ_t, the geodesic flow, and that C is invariant under the billiard ball map.

So the only question is whether $\tau^*\omega$ is degenerate on $T^*_C R^2 \bigcap F$, and on $T^*_C R^2 \bigcap B$. But these intersections are in T^*C ($T^*C \subset T^*_C R^2$), and F and B are Lagrangian. Using Stoke's theorem we obtain

$$0 = \int_F \omega = \int_{\xi \in F(\partial F)} \imath(\xi)\omega,$$

and

$$0 = \int_B \omega = \int_{\xi \in B(\partial B)} \imath(\xi)\omega.$$

To conclude the proof for FQ, observe that

$$(*) \int_{\partial F} \imath(\xi)\omega + \int_{\partial B} \imath(\xi)\omega =$$

$$FQ(p, C, \partial\Omega) - FQ(p', C, \partial\Omega) + \int_{P \in A-B} |\rho(\xi_+(P))| - |\rho(\xi_-(P))|,$$

where A-B$\subset \partial\Omega$ is $\pi(B) \bigcap \partial\Omega = \pi(F) \bigcap \partial\Omega$, and for each P$\in$A-B, $\xi_+(P) \in F_P$ and $\xi_-(P) \in B_P$. When C is a caustic, $|\rho(\xi_+(P))| = |\rho(\xi_-(P))|$, so the integral on the left hand side of (*) is zero, and 0=FQ(p,C,$\partial\Omega$)-FQ(p',C,$\partial\Omega$).

The proof that BQ(p,C,$\partial\Omega$)=BQ(p',C,$\partial\Omega$) is exactly the same with the signs and the roles of p and q (and of p' and q') reversed. In fact, the argument above shows that $-BQ(q, C, \partial\Omega)$ equals $-BQ(q', C, \partial\Omega)$, and we may choose the q-s first, picking the p-s to be their backward return points.

2.2. Evolutes

Let Ω be a convex planar domain with a smooth boundary which is integrable up to C. In light of the lemma of the previous section, there is a relation between $\partial\Omega$ and each of the caustics for the billiard ball map on $\partial\Omega$.

Definition 2.2.1: For $Q \in R$, and $C \subset \Omega$, a caustic for the billiard ball map on $\partial\Omega$, we say that $\partial\Omega$ **is the Q-evolute of C** if $Q \geq 0$ and $FQ(p,C,\partial\Omega)=Q$ for $p \in C$, or if $Q < 0$ and $BQ(p,C,\partial\Omega)=Q$ for $p \in C$.

We identify smooth convex curves in R^2 by their curvature (see, for example, [MM]). That is, to each smooth convex curve we associate its curvature when the curve is parameterized by tangent angle, and to each $k \in C^\infty(R/2\pi Z; R)$, with

$$\int_0^{2\pi} \cos(t)\frac{dt}{k(t)} = \int_0^{2\pi} \sin(t)\frac{dt}{k(t)} = 0,$$

we associate the curve with coordinates

$$x(\theta) = \int_0^{\theta} \cos(t)\frac{dt}{k(t)},$$

$$y(\theta) = \int_0^{\theta} \sin(t)\frac{dt}{k(t)}.$$

We will call the space of all such curvatures $C^\infty_.(S; R^+)$.

In this setting we have

Definition 2.2.2:

$$\mathcal{L} : R \times C^\infty_.(S; R^+) \to C^\infty_.(S; R^+)$$

which takes (Q,k) to v, where v is the curvature of the Q-evolute of the curve C whose curvature is k, is called **the curvature relating operator.**

Proposition 3: In the setting of the previous two definitions

$$\mathcal{L}(-Q,k) = \mathcal{L}(Q,k).$$

Proof: If we change the orientation of C (whose curvature is k), we do not change the curvature of the evolute. But FQ(p,C,$\partial\Omega$)= -BQ(q,C,$\partial\Omega$) =FQ(p,$\bar{C}$, $\partial\Omega$) where q is the forward return point of p, and $\bar{C}$ is C with the reversed orientation.

2.3. Curvature Relations

We would like to compute at least part of the curvature relating operator, $\mathcal{L}$.

Let a be a simple closed (strictly) convex smooth planar curve given by its tangent angle ($0 \leq \theta \leq 2\pi$) and its curvature ($0 < k(\theta)$);

$$a(\theta) = (\int_0^\theta \cos(t)\frac{dt}{k(t)}, y_a^0 + \int_0^\theta \sin(t)\frac{dt}{k(t)}).$$

Let b be the Q-evolute of a ($0 < Q$), given by its tangent angle, ϕ, and curvature, v. Then

$$b(\phi) = a(\theta_1) + t_1(\cos\theta_1, \sin\theta_1) = a(\theta_2) + t_2(\cos\theta_2, \sin\theta_2),$$

and

$$Q = t_1 + t_2 - (s_2 - s_1),$$

where s is the arclength along a.

We get

$$t_1 = \frac{1}{\sin(\theta_2 - \theta_1)}\{\sin\theta_2 \int_{\theta 1}^{\theta 2} \cos(t)k^{-1}(t)dt$$

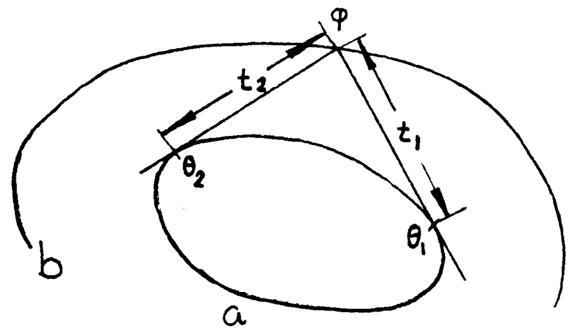


Figure 2.3: The curve b is the Q-evolute of a

$$-\cos\theta_2 \int_{\theta 1}^{\theta 2} \sin(t) k^{-1} dt\},$$

$$t_2 = \frac{1}{\sin(\theta_2 - \theta_1)} \{\cos\theta_1 \int_{\theta 1}^{\theta 2} \sin(t) k^{-1}(t) dt$$

$$-\sin\theta_1 \int_{\theta 1}^{\theta 2} \cos(t) k^{-1} dt\},$$

and

$$s_2 - s_1 = \int_{\theta 1}^{\theta 2} k^{-1}(t) dt.$$

Thus,

$$Q = \frac{1}{\sin(\theta_2 - \theta_1)} \int_{\theta 1}^{\theta 2} \{\sin(t - \theta_1) + \sin(\theta_2 - t) - \sin(\theta_2 - \theta_1)\} k^{-1}(t) dt.$$

Fix θ_1, and consider the left hand side of the equation for Q above as a function of θ_2. Set

$$\tilde{g}(\theta_2) = \int_{\theta 1}^{\theta 2} \{\sin(t - \theta_1) + \sin(\theta_2 - t) - sin(\theta_2 - \theta_1)\} k^{-1}(t) dt.$$

Let $\Delta = \theta_2 - \theta_1$, and $g(\Delta) = \tilde{g}(\theta_2)$.

It is clear that g is C^∞ in Δ (since $\partial_{\theta 2} = \partial_\Delta$ when θ_1 is fixed), and we find (using the calculation presented in the next section) that

$$g(\Delta) = \frac{1}{24} k^{-1}(\theta_1)\Delta^4 + O(\Delta^5).$$

It is also clear that when k is positive $\tilde{g}$ (and therefore g) is smooth in θ_1 – it involves only k^{-1} and its derivatives. We set $b(\theta_1) = (x(\theta_1), y(\theta_1))$, so that its curvature is

$$v(\theta_1) = (x'y'' - y'x'')/|(x', y')|^3.$$

Since, by assumption, a is a caustic for b, $\phi = \theta_1 + \frac{1}{2}\Delta$. Using this we can find an expression for $v(\phi)$ in terms of k and Δ (as we do in the following section to order 4). This expression is smooth in Δ and in θ_1, and hence, when Δ is sufficiently small, v is smooth in Δ and in ϕ.

We summarize in

Proposition 4: $\mathcal{L}$(Q,k) is a differential operator which is smooth in $Q^{2/3}$, for sufficiently small Q.

Proof: From the previous calculations $Q = \Delta^3(\theta_1)g(\Delta)$, with g smooth in Δ and in θ_1 and $g(0) = (12k)^{-1}(\theta_1)$. By the first proposition in the first chapter, if Q is sufficiently small, then $k > 0$, and because the region between C and $\partial\Omega$ is compact, $|g(\Delta) - (12k)^{-1}| < \min\{\frac{1}{2}(12k)^{-1}\}$, and $\Delta \sim Q^{1/3}$.

Setting $\Delta = cQ^{1/3} + f(Q, \Delta)$ shows that $f \sim Q^{2/3}$, and continuing in this fashion $\Delta = f(Q^{1/3})$ where f is smooth in $Q^{1/3}$ and in θ_1.

$\mathcal{L}$ is smooth in Δ and θ_1, again from the calculation, so $\mathcal{L}(Q,k)$ is smooth in $Q^{1/3}$ and in θ_1. Since $\phi = \theta_1 + \frac{1}{2}\Delta$, if Δ is sufficiently small, $\mathcal{L}$ is smooth in Δ and ϕ. But the proposition preceding the calculation shows that $\mathcal{L}(Q,k) = \mathcal{L}(-Q,k)$, and Q and $Q^{1/3}$ have the same sign. So $\mathcal{L}(Q,k)$ is in $C^\infty(S_\phi; R)$ and is smooth in $Q^{2/3}$.

Chapter 3.

Caustics in Integrable Domains

In this chapter we will take a closer look at the curvature relating operator in a region for which the billiard ball map is integrable. This will shed light on the structure of the curvature relating operator, and thus on the nature of such regions.

3.1. The Group Property

For the circle, parameterized by arclength, s, and with $(s,\theta) \in S^*_{S^1}R^2$, the billiard ball map is given by $\beta(s,\theta) = (s + 2\theta, \theta)$ (See figure). The caustics for the billiard ball map on the circle are concentric circles, so the billiard ball map on any given caustic is itself integrable and the caustics for the billiard ball map on the caustic are also caustics for the billiard ball map on the outer circle.

Definition 3.1.1: We say that the caustics for the billiard ball map on a circle satisfy the **group property**; the evolute of the evolute of a given caustic is also the evolute of that caustic.

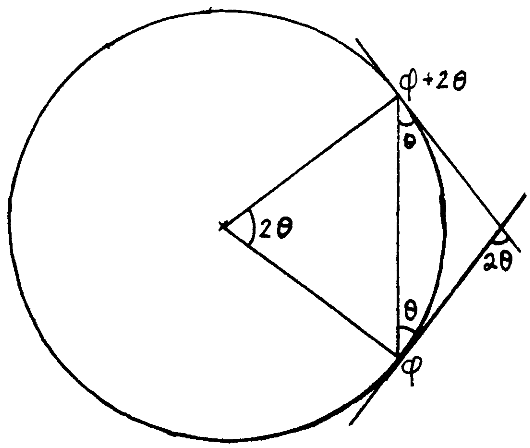


Figure 3.1: The billiard ball map on the circle.

The caustics for the billiard ball map on the ellipse also satisfy the group property for caustics not too far from the boundary [GM] and, indeed, if the ellipses are the only integrable planar curves, then all integrable planar curves do. However, the example of the next section does not satisfy the group property leading us to search for a characterisation of curves which satify the group property.

3.2. A Limiting Example

Here we construct an example for which the group property fails; there is a smooth curve with an evolute whose evolute is not an evolute of the original curve.

We wish to construct a smooth convex curve, C, which is the circle on one half and lies inside the circle on the other half (see

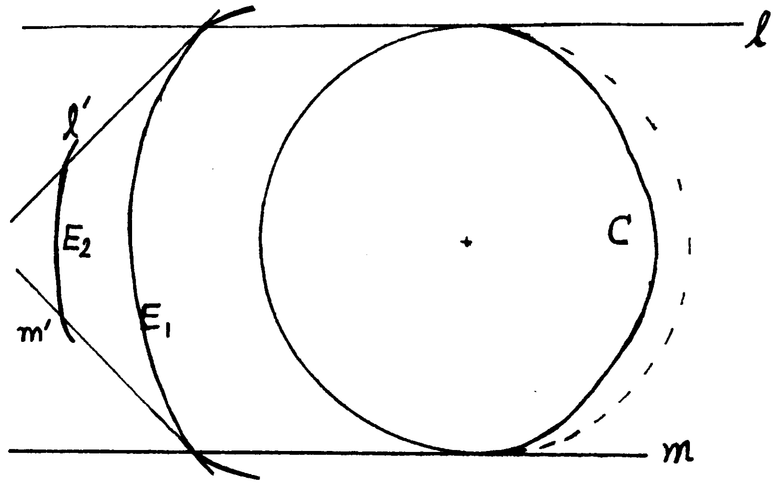


Figure 3.2: The curve, C, its evolutes and tangent lines.

figure).

We can do this by specifying the curvature, k, of this curve: let k be a C^∞ positive function on R with $k(\theta) = 1$ for $\pi \leq \theta \leq 2\pi$, and

$$\int_0^{2\pi} \cos t \frac{dt}{k(t)} = \int_0^{2\pi} \sin t \frac{dt}{k(t)} = 0,$$

and for $0 < \theta < \pi$,

$$\left(\int_0^{\theta} \cos t \frac{dt}{k(t)}\right)^2 + \left(\int_0^{\theta} \sin t \frac{dt}{k(t)}\right)^2 < 1.$$

This gives us the curvature of C, and hence C itself. In fact, we can construct such a C directly: Set

$$r(x) = \begin{cases} 1 - a \exp^{-\tan^2(x-\pi/2)} & , x \in [0, \pi) \\ 1 & , x \in [\pi, 2\pi) \end{cases}$$

with $0 < a < 1/(5M)$ where $M = \sup_{t \in R}(1+t^2)\exp^{-t^2} = 1$.

Set $\alpha(x) = (r(x)\sin x, -r(x)\cos x) \in R^2$. Then α is C^∞ and it is convex – because $(1 - a\exp^{-\tan^2(x-\pi/2)})\sin x$ has a single local maximum at $x = \pi/2$.

Consider two evolutes of C, E_1 the Q_1 evolute of C (for some $Q_1 > 0$) and E_2 the Q_2 evolute of C ($Q_1 < Q_2$). Also consider the lines l and m tangent to C at $\theta = \pi$ and $\theta = 0$ respectively. (See previous figure.) Between the tangent lines, E_1 and E_2 are exactly the evolutes of a circle, and hence are circles themselves.

We claim that on the right of $E_1 \bigcap l$, E_1 is not a circle, that is, for points on E_1 near $E_1 \bigcap l$ but with smaller tangent angle, E_1 is not a circle.

Proof: This is a consequence of the discussion (in the first chapter) of caustics and integral curves. Because it is interesting in its own right, we state

Lemma 5: If C_1 and C_2 are both caustics for $\partial\Omega$, and they agree on a segment, σ, then they agree on the image of σ reflected from $\partial\Omega$. In the language of the section on caustics (the first section of the second chapter), if σ is a segment of C_1 with $\sigma \subset C_2$, and σ' is the set of forward or backward return points of σ, then $\sigma' \in C_1 \bigcap C_2$.

Proof: σ determines an integral curve segment, Σ, as in proposition 1, and $R_{\partial\Omega}(\Sigma)$ determines a curve segment, $\tilde{\sigma}$, where $R_{\partial\Omega}$ is reflection at $\partial\Omega$. Since C_1 and C_2 are caustics, $\tilde{\sigma} \subset C_1 \bigcap C_2$. But $\tilde{\sigma} = \sigma'$.

Thus, if E_1 were a circle in some interval on the right of $E_1 \bigcap l$, then C would agree with a circle in some interval to the right of $C \bigcap l$, which is a contradiction and proves the claim.

Now the evolute of E_1, call it E', is a circle between l' and m', where l' and m' are the tangents to E_1 at $E_1 \bigcap l$ and $E_1 \bigcap m$ (respectively). Using the previous claim, E' is not a circle (immediately) to the right of $E' \bigcap l'$.

If we choose E' to be that evolute of E_1 which agrees with E_2 between l' and m', then the above proves that E' and E_2 are not the same providing the wished for example.

3.3. The Curvature Relating Operator

For a curvature function, k, corresponding to a caustic which belongs to a collection of caustics satifying the group property, and any (sufficiently small) P and Q, there is an R with

$$\mathcal{L}(Q, \mathcal{L}(P,k))(\phi) = \mathcal{L}(R,k)(\phi), \forall \phi \in [0, 2\pi].$$

Since $\mathcal{L}(Q,k)$ is smooth in $Q^{2/3}$,

$$\mathcal{L} \sim \sum_{j=0}^{\infty} \mathcal{L}_0(k) Q^{2j/3},$$

which leads to

Definition 3.3.1: A simple smooth convex closed curve with curvature k such that for some fixed ϵ and any $0 < Q < \epsilon$ and any N, $\exists$ R and $c \in R$ with

$$\sum_{n=0}^{\infty} \mathcal{L}_n(\sum_{j=0}^{\infty} \mathcal{L}_j(k) Q^{2j/3}) Q^{2n/3} = \sum_{n=0}^{\infty} \mathcal{L}_n(k) R^{2n/3}$$

is said to satisfy **the formal group property.**

The immediate question is whether this is a property of $\mathcal{L}$, that is, it holds for all curvatures, or is a property which singles out curves satisfying the group property.

We answer this question by carrying out the computation of $\mathcal{L}$ to (as it turns out) almost two non-trivial terms.

We know that $\mathcal{L}(0,k) = k$ so $\mathcal{L}_0$ is the identity.

Fix $0 < P, Q$, set $v = \mathcal{L}(P,k)$ and $w = \mathcal{L}(Q,k)$, and assume that $w = \mathcal{L}(R,k)$ for some $0 < R$. w is C^∞ in $Q^{2/3}$ and in $R^{2/3}$, and $0 < w$ when $0 < k$ and P and Q are sufficiently small. In addition, v is smooth in $P^{2/3}$ and w is smooth in $P^{2/3}$ when P and Q are small (because then $0 < v$). Also, R=P when Q=0, and R=Q when P=0, so

$$R^{2/3} = P^{2/3} + Q^{2/3} + P^{2/3}Q^{2/3}G(k, P^{2/3}, Q^{2/3}),$$

with G a constant determined by the caustic with curvature k (G is a constant, since P, Q, and R are constants).

For $\mathcal{L}(Q, \mathcal{L}(P,k)) = \mathcal{L}(R,k)$ to hold to fourth order we must have (with $p = P^{2/3}$, $q = Q^{2/3}$, $r = R^{2/3}$),

$$\mathcal{L}_0 k + \mathcal{L}_1 k(p+q) + \mathcal{L}_1^2 kpq + \mathcal{L}_2 k(q^2+p^2)$$

$$= \mathcal{L}_0 k + \mathcal{L}_1 kr + \mathcal{L}_2 kr^2 + O_6$$

$$= \mathcal{L}_0 k + \mathcal{L}_1 k(p+q) + \mathcal{L}_1 pqG(k,0,0) + \mathcal{L}_2 k(p^2+pq+q^2) + O_6,$$

or

$$\mathcal{L}_1^2 k - \mathcal{L}_1 kG(k,0,0) - 2\mathcal{L}_2 k = 0.$$

We do not know G above, but in the fourth order terms G(k,0,0) affects the terms involving two or fewer derivatives and not those involving three or four derivatives. We also know that the above equation is satisfied by ellipses, so that if we find the terms in the equation which involve three and four derivatives, we can complete the equation by using the fact that it is satisfied by ellipses and including a constant multiple of $\mathcal{L}_1$.

For what follows we simplify the notation by writing θ for θ_1, and k for $k(\theta)$. When we wish to evaluate k at another point, say ϕ, we will write $k(\phi)$ only for the leading terms of an expression, with the understanding that the k-s following it are evaluated at the same point. Recalling our expression for Q,

$$Q = \frac{1}{\sin(\theta_2-\theta_1)}\left(\int_{\theta_1}^{\theta_2}\sin(\theta_2-t)k^{-1}(t)dt\right.$$
$$\left.+\int_{\theta_1}^{\theta_2}\sin(t-\theta_1)k^{-1}(t)dt\right)$$
$$-\int_{\theta_1}^{\theta_2}k^{-1}(t)dt.$$

We will consider θ_1 fixed, and solve for $\theta_2-\theta_1$ in terms of Q. We call the first second and third integrals above I1, I2, and I3 respectively. To simplify notation let ∂ denote differentiation with respect to θ_2, and set $\Delta = \theta_2 - \theta_1$.

$$\partial I1 = \cos\theta_2\int_{\theta_1}^{\theta_2}\cos t k^{-1}(t)dt + \sin\theta_2\int_{\theta_1}^{\theta_2}\sin t k^{-1}(t)dt,$$

and $\partial I1(\Delta=0)=0$.

$$\partial^2 I1 = k^{-1} - I1,$$

which allows us to easily find

$$I1 = \frac{1}{2!}k^{-1}\Delta^2 + \frac{1}{3!}(k^{-1})'\Delta^3$$

$$+\frac{1}{4!}[(k^{-1})'' - k^{-1}]\Delta^4 + \frac{1}{5!}[(k^{-1})^{(3)} - (k^{-1})']\Delta^5$$

$$+\frac{1}{6!}[(k^{-1})^{(4)} - (k^{-1})'' + k^{-1}]\Delta^6$$

$$+\frac{1}{7!}[(k^{-1})^{(5)} - (k^{-1})^{(3)} + (k^{-1})']\Delta^7 + O(\Delta^8),$$

where $f' = \partial f(\theta_1)$. Similarly,

$$I2 = \frac{1}{2}k^{-1}\Delta^2 + \frac{2}{3!}(k^{-1})'\Delta^3$$

$$+\frac{1}{4!}[3(k^{-1})'' - (k^{-1})]\Delta^4 + \frac{1}{5!}[4(k^{-1})^{(3)} - 4(k^{-1})']\Delta^5$$

$$+\frac{1}{6!}[5(k^{-1})^{(4)} - 10(k^{-1})'' + k^{-1}]\Delta^6$$

$$+\frac{1}{7!}[6(k^{-1})^{(5)} - 20(k^{-1})^{(3)} + 6(k^{-1})']\Delta^7 + O(\Delta^8),$$

and,

$$I3 = k^{-1}\Delta + \frac{1}{2}(k^{-1})'\Delta^2 + \frac{1}{3!}(k^{-1})''\Delta^3 + \frac{1}{4!}(k^{-1})^{(3)}\Delta^4$$

$$+\frac{1}{5!}(k^{-1})^{(4)}\Delta^5 + \frac{1}{6!}(k^{-1})^{(5)}\Delta^6 + O(\Delta^7).$$

Using

$$\frac{\Delta}{\sin\Delta} = 1 + \frac{1}{6}\Delta^2 + \frac{7}{360}\Delta^4 + O(\Delta^6),$$

$$Q = (I1 + I2)/(\sin\Delta) - I3$$

$$= \frac{1}{12}k^{-1}\Delta^3 - \frac{1}{24}k^{-2}k'\Delta^4$$

$$+(-\frac{1}{80}k^{-2}k'' + \frac{1}{40}k^{-3}(k')^2 + \frac{1}{120}k^{-1})\Delta^5$$

$$+\left(-\frac{1}{360}k^{-2}k^{(3)}+\frac{1}{60}k^{-3}k'k''-\frac{1}{60}k^{-4}(k')^3-\frac{1}{240}k^{-2}k'\right)\Delta^6+O(\Delta^7).$$

For convenience, we set q=$Q^{1/3}$ and denote θ_1 by θ. We are interested in $\mathcal{L}_1$ and in the first two terms of the group property, so we solve using the two highest derivative terms for each power of q greater than two, and we ommit products of more than two differentiated terms since they lead to lower order terms. In what follows, equivalence is equivalence modulo lower order and (three or more) mixed derivative terms.

We know that Δ is a smooth function of q, so we can solve for it iteratively (setting $\Delta=(12)^{1/3}k^{1/3}q+b(k)q^2$, and solving for b(k) by plugging into the above equation, etc.).

$$\Delta\equiv(12)^{1/3}k^{1/3}q+\frac{1}{6}(12)^{2/3}k^{-1/3}k'q^2+\left(\frac{3}{5}k''-\frac{1}{5}k^{-1}(k')^2\right)q^3$$

$$+(12)^{1/3}\left(\frac{2}{15}k^{1/3}k^{(3)}-\frac{1}{10}k^{-2/3}k'k''\right)q^4+O(q^5).$$

Recall that $b(\phi)=a(\theta)+t_1(\cos\theta,\sin\theta)$, which when we solve for t_1, will give us an expression for b, parameterized (unfortunately) by θ. This will enable us to compute the curvature of b.

Denote t_1 by t, and recall that

$$t=\frac{1}{\sin(\Delta)}\int_\theta^{\theta+\Delta}\sin(\theta+\Delta-s)k^{-1}ds.$$

so that (using our expansions for I_1 and for $1/\sin(\Delta)$)

$$t=\frac{1}{2}k^{-1}\Delta-\frac{1}{6}k^{-2}k'\Delta^2+\frac{1}{24}(-k^{-2}k''+2k^{-3}(k')^2+k^{-1})\Delta^3$$

$$+\frac{1}{120}\left(-k^{-2}k^{(3)}+6k^{-3}k'k''-6k^{-4}(k')^3-\frac{7}{3}k^{-2}k'\right)\Delta^4+O(\Delta^5).$$

And,

$$t \equiv \frac{1}{2}(12)^{1/3}k^{-2/3}q - \frac{1}{12}(12)^{2/3}k^{-4/3}k'q^2$$
$$+(-\frac{1}{5}k^{-1}k'' + \frac{7}{3}k^{-2}(k')^2)q^3$$
$$+(12)^{1/3}(-\frac{1}{30}k^{-2/3}k^{(3)} + \frac{1}{10}k^{-5/3}k'k'')q^4 + O_5.$$

(Here O_n denotes terms of order n and greater.)

Let X and Y denote the coordinates of b, and x and y denote the coordinates of the curve a, and let v denote the curvature function for b. Then

$$v = \frac{X'Y'' - X''Y'}{|(X', Y')|^3},$$

and using $b = a + t(\cos\theta, \sin\theta)$

$$v(\theta) = k + kk't - \frac{1}{2}k^3t^2 - k^3(t')^2 - 3k^2k'tt' - k^3tt''$$
$$-\frac{3}{2}k^3k't^3 + 3k^4tt't'' + 6k^3k't(t')^2 + 8k^4(t')^3$$
$$+\frac{3}{8}k^5t^4 + \frac{15}{2}k^4k't^3t' + 3k^5t^2(t')^2 - 10k^4k't(t')^3$$
$$-9k^5(t')^4 + \frac{3}{2}k^5t^3t'' - 6k^5t(t')^2t'' + O_5$$
$$\equiv k + kk't - \frac{1}{2}k^3t^2 - k^3(t')^2 - 3k^2k'tt' - k^3tt'' + 3k^4tt't''$$
$$\equiv k(\theta) + \frac{1}{2}(12)^{1/3}k^{1/3}k'q$$
$$+(12)^{2/3}(\frac{1}{6}k^{2/3}k'' + \frac{1}{36}k^{-1/3}(k')^2 - \frac{1}{8}k^{5/3})q^2$$
$$+(\frac{1}{2}kk^{(3)} + \frac{3}{10}k'k'')q^3 + (12)^{1/3}(\frac{7}{60}k^{1/3}k'k^{(3)} + \frac{1}{10}k^{4/2}k^{(4)})q^4 + O_5.$$

The problem is, of course, that this gives $v(\phi)$ in terms of a function (of q, and depending on k) at θ. To deal with this we recall that

$$\phi = (\theta_2 + \theta_1)/2 = \theta + \frac{1}{2}\Delta.$$

So,

$$\phi \equiv \theta + \frac{1}{2}(12)^{1/3}k^{1/3}k'q + \frac{1}{12}(12)^{2/3}k^{-1/3}k'q^2 + (\frac{3}{10}k'' - \frac{1}{10}k^{-1}(k')^2)q^3$$
$$+(12)^{1/3}(\frac{1}{15}k^{1/3}k^{(3)} - \frac{1}{20}k^{-2/3}k'k'')q^4 + O_5,$$

which yields (solving iteratively, and expanding the expressions involving $k(\theta)$ with each iteration)

$$\theta \equiv \phi - \frac{1}{2}(12)^{1/3}k^{1/3}(\phi)q + (-\frac{1}{20}\ddot{k} + \frac{1}{10}k^{-1}(\dot{k})^2)q^3 + O_5.$$

Here $\dot{k}$ denotes differentiation with respect to ϕ, which will be denoted in square brackets for higher derivatives (e.g. $k^{[3]}$). Using this (and in the process of obtaining it) we also find

$$k(\theta) \equiv k(\phi) - \frac{1}{2}(12)^{1/3}k^{1/3}\dot{k}q + \frac{1}{8}(12)^{2/3}k^{2/3}\ddot{k}q^2$$
$$+(-\frac{1}{4}kk^{[3]} - \frac{1}{20}\dot{k}\ddot{k})q^3 + \frac{1}{32}(12)^{1/3}k^{4/3}k^{[4]}q^4 + O_5,$$

$$k'(\theta) \equiv \dot{k}(\phi) - \frac{1}{2}(12)^{1/3}k^{1/3}\ddot{k}q + \frac{1}{8}k^{2/3}k^{[3]}q^2 - \frac{1}{4}kk^{[4]}q^3 + O_4,$$

(where we use $k'(\theta) = \partial_\phi(k(\theta)) \times (1/\partial_\phi\theta)$,)

$$k''(\theta) \equiv \ddot{k}(\phi) - \frac{1}{2}(12)^{1/3}k^{1/3}k^{[3]}q + \frac{1}{8}(12)^{2/3}k^{2/3}k^{[4]}q^2 + O_3,$$

$$k^{(3)}(\theta) \equiv k^{[3]}(\phi) - \frac{1}{2}(12)^{1/3}k^{[4]}q + O_2,$$

$$k^{1/3}(\theta) \equiv k^{1/3}(\phi) - \frac{1}{6}(12)^{1/3}k^{-1/3}\dot{k}q$$
$$+(12)^{2/3}(\frac{1}{24}\ddot{k} - \frac{1}{36}k^{-1}(\dot{k})^2)q^2$$
$$+(-\frac{1}{12}k^{1/3}k^{[3]} + \frac{3}{20}k^{-2/3}\dot{k}\ddot{k})q^3 + O_4,$$

$$k^{-1/3}(\theta) \equiv k^{-1/3}(\phi) + \frac{1}{6}(12)^{1/3}k^{-1}\dot{k}q$$
$$+(12)^{2/3}(-\frac{1}{24}k^{-2/3}\ddot{k} + \frac{1}{18}k^{-5/3}(\dot{k})^2)q^2 + O_3,$$

and finally, $k^{-1}(\theta) = k^{-1}(\phi) +$ derivative terms $\times q$.

We now substitute in the previous equation for v to get

$$v(\phi) \equiv k(\phi) + (12)^{2/3}(\frac{1}{24}k^{2/3}\ddot{k} - \frac{1}{18}k^{-1/3}(\dot{k})^2 - \frac{1}{8}k^{5/3})q^2$$
$$+(12)^{1/3}(\frac{1}{160}k^{4/3}k^{[4]} - \frac{3}{40}k^{1/3}\dot{k}k^{[3]})q^4 + O(q^5).$$

To obtain the equation $\mathcal{L}_1^2 - \mathcal{L}_2 = 0$ we find

$$\dot{v} \equiv \dot{k} + (12)^{2/3}(\frac{1}{24}k^{2/3}k^{[3]} - \frac{1}{12}k^{-1/3}\dot{k}\ddot{k})q^2 + O_4,$$

$$\ddot{v} \equiv \ddot{k} + (12)^{2/3}(\frac{1}{24}k^{2/3}k^{[4]} - \frac{1}{18}k^{-1/3}\dot{k}k^{[3]})q^2 + O_3,$$

$$v^{2/3} \equiv k^{2/3} + (12)^{2/3}(\frac{1}{36}k^{1/3}\ddot{k} - \frac{1}{27}k^{-2/3}(\dot{k})^2)q^2 + O_4,$$

$$(\dot{v})^2 \equiv (\dot{k})^2 + (12)^{2/3}\frac{1}{12}k^{2/3}\dot{k}k^{[3]}q^2 + O_4,$$

and $v^{-1/3} \equiv k^{-1/3}$+derivative terms $\times q^2$. (Here all terms are evaluated at ϕ.)

Thus,

$$\mathcal{L}_1^2 \equiv (12)^{1/3}(\frac{1}{48}k^{4/3}k^{[4]} - \frac{1}{12}k^{1/3}\dot{k}k^{[3]}),$$

and

$$0 = \mathcal{L}_1^2 - 2\mathcal{L}_2 \equiv (12)^{1/3}(\frac{1}{120}k^{4/3}k^{[4]} + \frac{1}{15}k^{1/3}\dot{k}k^{[3]}).$$

This resulting equation must be satisfied by the ellipse, whose curvature, given in terms of its tangent angle and the major and minor axes, is

$$k(\theta) = (ab)^{-2}(a^2\sin^2\theta + b^2\cos^2\theta)^{3/2},$$

and so,

Theorem 6: The equation satisfied by the curvature of a convex smooth curve whose evolutes satisfy the group property is

$$k^{(4)} = -8k^{-1}\dot{k}k^{(3)} + k^{-1}(\ddot{k})^2 + \frac{20}{3}k^{-2}(\dot{k})^2\ddot{k} + 4\ddot{k} - \frac{44}{9}k^{-3}(\dot{k})^4 - 36k^{-1}(\dot{k})^2$$
$$+a(k)(\frac{1}{24}k^{-2/3}\ddot{k} - \frac{1}{18}k^{-5/3}(\dot{k})^2 - \frac{1}{8}k^{1/3}),$$

where a(k) is a constant depending on the caustic with curvature k, and the equation is chosen so that it is 0 for ellipses.

We know that this equation is satisfied by the curvature of any ellipse and its rotations, and we think that these are the only convex closed curves whose curvatures satisfy this equation.

We can show this if a(k)=0 as follows. We first observe that the curvature of any simple closed curve (parameterized by tangent angle) is periodic (with period 2π), and that if we set $y = -\theta$ and consider the curvature as a function of y, the equation remains the same (a(k) is a constant).

Next observe that the equation is homogeneous (if $k(\theta)$ satisfies the equation, so does $ck(\theta)$), so, given that k is never zero, we may assume that $k(0) = 1$. Since the curve is closed, its curvature must have a minimum, so by rotating the curve we may assume that $\dot{k}(0) = 0$, and that $\ddot{k}(0) \geq 0$. (We may now rescale so that $k(0) = 1$ still holds.) Finally, because the equation is invariant under the change $y = -\theta$ and $\dot{k}(0) = 0$, $k^{(3)}(0) = -k^{(3)}(0)$, and $k^{(3)}(0) = 0$.

Looking at the curvature of the ellipse as given above, we see that $k(0) = a/b^2$ can be scaled to become 1, the curvature has a minimum at $\theta = 0$, $\ddot{k}(0) = \frac{3}{a} - 3$ can be set to any non-negative value, and $k^{(3)}(0) = 0$. Since the group property equation has unique positive solutions (it is well posed), the curvature of any closed curve satisfying the group property equation agrees, after rotation and rescaling, with an ellipse, and thus represents an ellipse.

Alternatively, as noted in the previous section, to be the curvature of a simple closed curve, the curvature (given as a function of tangent angle) must be periodic (of period 2π) and must satisfy the closure conditions

$$\int_0^{2\pi} \cos tk^{-1}(t)dt = \int_0^{2\pi} \sin tk^{-1}(t)dt = 0.$$

We could make use of this by letting $r(\theta)$ be the radius of curvature $(= k^{-1}(\theta))$. Then r is periodic of period 2π and is continuous (since k is assumed to be strictly positive and smooth), so

$$r(\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos(n\theta) + b_n sin(n\theta),$$

and the closure conditions simply require that $a_1 = b_1 = 0$. We want to show that the closure conditions are not satisfied by a fourth solution of the group property equation, which we could do by plugging the sum representation of r into the group property equation for r – the equation obtained by substituting r^{-1} for k in the group property equation.

It is my hope that this result will be usefull in understanding the spectral properties of ellipses and that integrable curves can be shown to satisfy the group property so that the conjecture that the only integrable curves in two space are ellipses can be proved.

Chapter 4.

Reflection From a Boundary with Negative Curvature

In this chapter we consider an example of geodesics reflected away from the boundary of a convex region. Let X be the torus $X = R^2/Z^2$ endowed with the metric from R^2, and let $\Omega \subset X$ be an open convex region with a smooth boundary, $\partial\Omega$. We define the billiard ball map on outward pointing elements of $B^*\partial\Omega$ (or $S^*_{\partial\Omega}R^2$) as in the first chapter, for those elements which return to $\partial\Omega$.

Remark: In this example any geodesic leaving $\partial\Omega$ returns to $\partial\Omega$, that is, if $\xi \in T^*_{\partial\Omega}(X\backslash\Omega)$ then $\exists t > 0 (t < \infty)$ with $\phi_t\xi \in T^*_{\partial\Omega}X$, where ϕ_t is geodesic flow induced from R^2. **Proof:** If ξ is outward pointing (as assumed), then its orbit under geodesic flow is either rational – in which case the orbit is closed (or returns to $\partial\Omega$ before closing), or it is irrational – in which case the orbit is dense and, again, must hit $\partial\Omega$ in finite time.

For convenience we will consider the torus with Ω deleted to

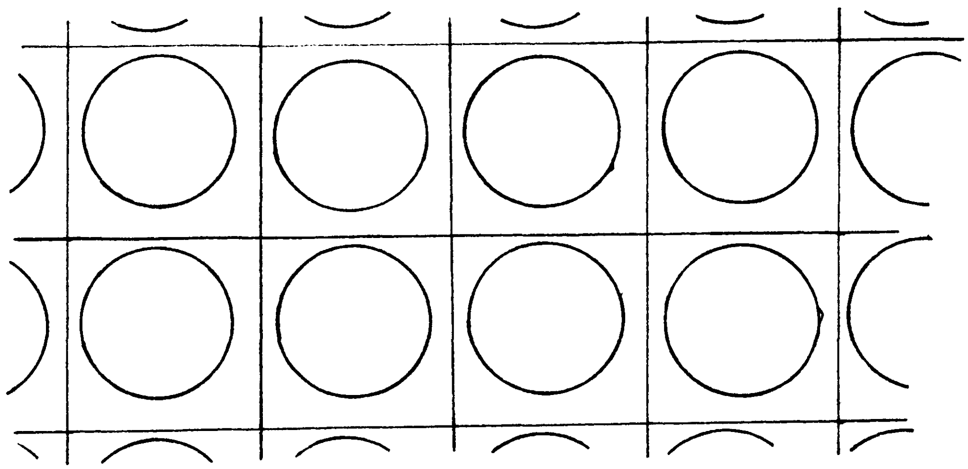


Figure 4.1: The flat torus with a convex region deleted.

be R^2 with Z^2 copies of Ω deleted. (See figure.) It is clear that we may assume that $\Omega \bigcap([0,1]\times[0,1]) \subset (0,1)\times(0,1)$. This viewpoint is our standing assumption for the remainder of this example. In this setting, we code each copy of Ω by the coordinates (n,m) if it is contained in (n,n+1)×(m,m+1) and we call it **the (n,m) copy of Ω.**

We restrict our attention to $S^*_{\partial\Omega}(X \setminus \Omega)$.

Definition 4..2: For $\xi \in S^*_p R^2$ let $\tau(\xi) \in S_p R^2$ be such that $\xi(\tau(\xi)) = 1$. For $\xi_1, \xi_2 \in S^*_p R^2$ we call $\arccos \xi_2(\tau(\xi_1)) = \xi_1(\tau(\xi_2))$ **the angle between ξ_1 and ξ_2**, which we denote by $\xi_1 \circ \xi_2$.

Let $p_1, p_2 \in \partial\Omega, \xi_j \in S^*_{\partial\Omega}(X \setminus \Omega), \xi_1 \neq \xi_2$, then either there are $t_1, t_2 \in R$ such that $\pi(\phi_{t_1}(\xi_1)) = \pi(\phi_{t_2}(\xi_2))$, or there are no such t_1 and t_2. In fact, since Ω is convex, $t_1 < 0$ if and only if $t_2 < 0$.

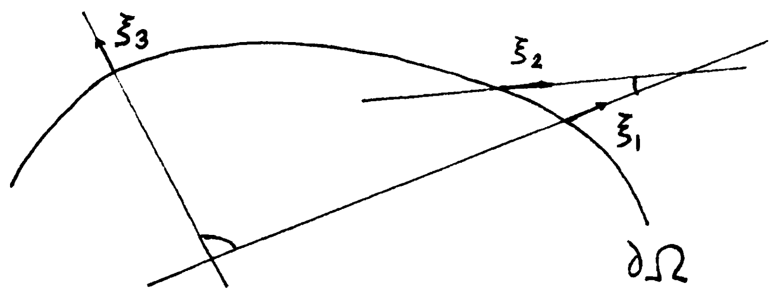


Figure 4.2: The angle between ξ_1 and ξ_2 is negative, while $\xi_1 \circ \xi_2 > 0$.

Definition 4..3: If t_1 and t_2 as above exist, we call

$$-\text{sign}(t_2) \arccos \phi_{t_2}\xi_2(\tau(\phi_{t_1}\xi_1)) = -\text{sign}(t_1) \arccos \phi_{t_1}\xi_1(\tau(\phi_{t_2}\xi_2))$$

the angle between ξ_1 and ξ_2. If no such t_1 and t_2 exist, the angle between ξ_1 and ξ_2 is zero (they are parallel). See figure. We denote this angle by $\xi_1 \circ \xi_2$ (as well).

In the setting of $S^*_{\partial\Omega}(X \setminus \Omega)$, which is our setting for the remainder of this example, we will view the billiard ball map, β, as

$$\rho^{-1} \circ \beta \circ \rho : S^*_{\partial\Omega}(X \setminus \Omega) \to S^*_{\partial\Omega}(X \setminus \Omega).$$

We will still call it the billiard ball map and denote it by β.

The key tool for this example is

Lemma 7: Let $\xi_1, \xi_2 \in S^*_{\partial\Omega}(X \setminus \Omega)$ be such that $\pi\beta(\xi_1)$ and $\pi\beta(\xi_2)$ are contained in the same copy of $\partial\Omega$ ($\pi : T^*R^2 \to R^2$ the projection.)

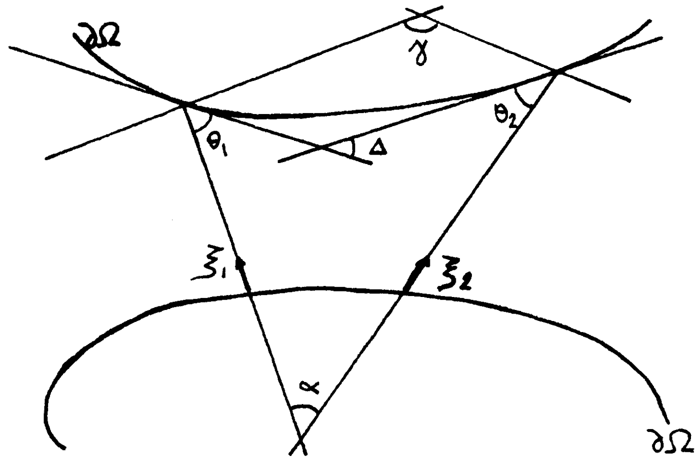


Figure 4.3: The first case: $t \leq 0$.

Then the angle between $\beta(\xi_1)$ and $\beta(\xi_2)$ is at least as large as the angle between ξ_1 and ξ_2. Moreover, if $\beta(\xi_1) \circ \beta(\xi_2) = \xi_1 \circ \xi_2$, then $\beta(\xi_1) = \xi_1$ and $\beta(\xi_2) = \xi_2$.

Proof: Assume first that $\xi_1 \circ \xi_2 \geq 0$, that is, that $\pi\phi_{t_1}\xi_1 = \pi\phi_{t_2}\xi_2$ for $t_1, t_2 \leq 0$ (see figure). Then (with angles as in the figure) $\gamma + \theta_1 + \theta_2 + (\pi - \Delta) = 2\pi$ and $\alpha + (\theta_1 + \Delta) + (\pi - \Delta) + (\theta_2 + \Delta) = 2\pi$, so $\gamma = \alpha + 2\Delta$.

By convexity of $\partial\Omega$, $\Delta > 0$, so $\gamma > \alpha$, unless $\Delta = \pi$, that is, ξ_1 and ξ_2 are tangent to $\partial\Omega$ ($\Delta \leq \pi$ since ξ_1 and ξ_2 originate in the same copy of $\partial\Omega$.)

In the second case, assume that $\xi_1 \circ \xi_2 < 0$ and that if $t(\xi)$ is the least t>0 such that $\pi\phi_{t(\xi)}\xi \in \partial\Omega$, then $t_1 < t(\xi_1), t_2 < t(\xi_2)$ (see figure). If the tangents at $\pi\beta(\xi_1)$ and $\pi\beta(\xi_2)$ were simply the line $\overline{\pi\beta(\xi_1)\pi\beta(\xi_2)}$, the angle between $\beta(\xi_1)$ and $\beta(\xi_2), \tilde{\gamma}$, would be smaller than γ (the actual angle), using the convexity of $\partial\Omega$. But

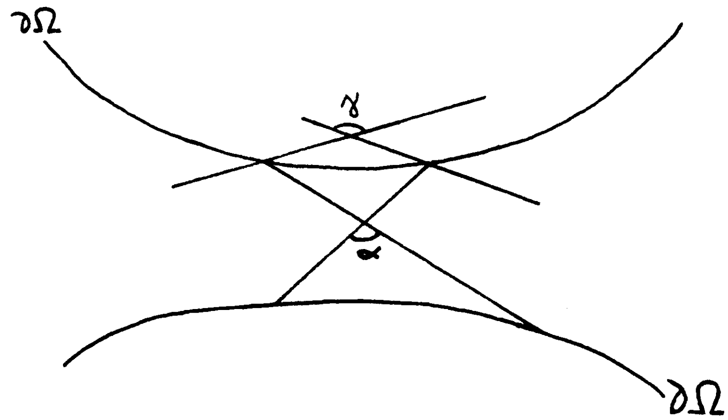


Figure 4.4: The second case: $t(\xi) > t > 0$.

$\tilde{\gamma} = \alpha$ and hence $\beta(\xi_1) \circ \beta(\xi_2) = \gamma > 0$.

Finally, assume that $\xi_1 \circ \xi_2 < 0$ and $t_1 \geq t(\xi_1)$ and $t_2 \geq t(\xi_2)$. We may assume $\beta(\xi_1) \circ \beta(\xi_2) < 0$ for otherwise we are done. Reversing the direction of the flow, and comparing the situation to that of the first and second cases, we see that $\alpha > \gamma$. Thus $\xi_1 \circ \xi_2 = -\alpha < -\gamma = \beta(\xi_1) \circ \beta(\xi_2)$.

The lemma enables us to prove

Theorem 8: Lengths of closed geodesics in $(R^2/Z^2) \setminus \Omega$ are isolated, that is, for any given length $l \in R$, there are at most finitely many geodesics of length l, and there is a $\delta > 0$ such that there are no geodesics of length l' with $l' \in (l - \delta, l) \bigcup (l, l + \delta)$.

Proof: Using the preceding lemma, we show that closed geodesics which are not tangent to $\partial\Omega$ are completely determined by the sequence of copies of $\partial\Omega$ which they visit: If the closed geodesics

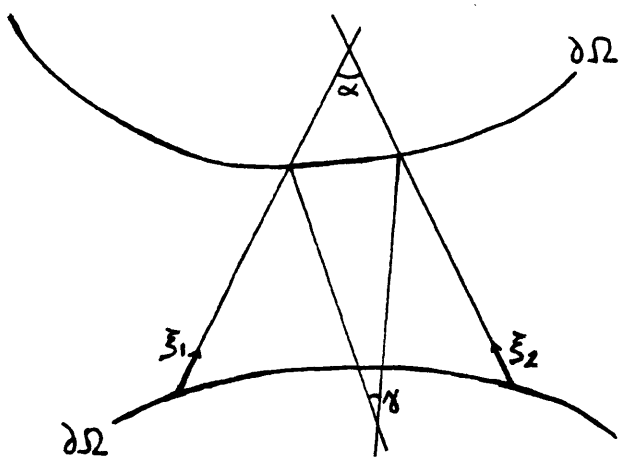


Figure 4.5: The third case: $t > t(\xi) > 0$.

starting out at ξ_1 and at ξ_2 visit the same copies of $\partial\Omega$ and have periods m and n, then $\beta^{mn}(\xi_1) \circ \beta^{mn}(\xi_2) > \xi_1 \circ \xi_2$, but $\beta^{mn}(\xi_j) = \xi_j$, so either $\xi_1 = \xi_2$ or they are parallel and m=n=1.

Fix $l > 0$. At most $(l+1)^2$ copies of Ω intersect with a circle of radius $l/2$ from any given point. Hence there are at most $(l+1)^2!$ arrangements of copies of Ω. There may be at most 2 geodesics that are tangent to $\partial\Omega$ for each pair of copies of $\partial\Omega$ – they are represented by a line tangent to the two copies in the R^2 representation of this situation. Thus there are at most

$$\binom{(l+1)^2}{2} + (l+1)^2!$$

closed geodesics of length bounded by l.

Note that two geodesics which are reflected infinitely often from the same copies of $\partial\Omega$ might be different – they may, for example, oscillate between two copies of $\partial\Omega$ until they become parallel.

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