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GAIN AND PHASE MARGIN FOR MULTILOOP LQG REGULATORS*

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Abstract -- Multiloop linear-quadratic state-feedback (LQSF) regulators are shown to be robust against a variety of large dynamical, time-varying, and nonlinear variations in open-loop dynamics. The results are interpreted in terms of the classical concepts of gain and phase margin, thus strengthening the link between classical and modern feedback theory.

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I. INTRODUCTION

Historically, feedback has been used in control system engineering as a means for satisfying design constraints requiring

1) stabilization of insufficiently stable systems,

2) reduction of system response to noise,

3) realization of a specific input/output relation (e.g., specified poles and zeroes), or

4) improvement of a system's robustness against variations in its open-loop dynamics.

Classical feedback synthesis techniques include procedures which ensure directly that each of these design constraints is satisfied [1] and [2]. Unfortunately, the direct methods of classical feedback theory become overwhelmingly complicated for all but the simplest feedback configurations. In particular, the classical theory cannot cope simply and effectively with multiloop feedback.

Linear-Quadratic-Gaussian (LQG) control theory has made relatively simple the solution of many multiloop control synthesis problems. The LQG technique [3] provides a straightforward means for synthesizing stable linear feedback systems which are insensitive to Gaussian white noise. Variations of the LQG technique have also been devised for the synthesis of feedback systems with specified poles [4, pp. 77-87], [5], [6]. Thus, the LQG technique is a valuable design aid for satisfying the first three of the aforementioned design constraints.

The results which follow show how the multivariable LQG design can satisfy constraints of the fourth type, i.e. constraints requiring a system to be robust against variations in open loop dynamics. The Linear-

Quadratic-State-Feedback regulator, which we refer to as the LQSF regulator, is considered. The robustness of LQSF regulator designs against variations in open-loop dynamics is measured in terms of multiloop generalizations of the classical notions of gain and phase margin. It is shown that LQSF multivariable designs have the property of an infinite gain margin and $\pm 60°$ phase margin for each control channel.

Such robustness results may appear incorrect at first glance, especially to control engineers familiar with classical servomechanism design. It should be noted that in classical servomechanism design the dimension of the compensators used (e.g. lead-lag networks) generally leads to conditionally stable systems, so that one may never have the infinite gain margin property. However, it should be stressed that when one uses full state-variable feedback one, in effect, introduces a multitude of zeroes in the compensator; it is this abundance of zeroes together with the Linear-Quadratic optimal design procedure that results in the surprising robustness properties of LQSF designs.

In order to provide a more detailed and realistic bridge between the classical and modern approaches, especially with respect to robustness issues, one has to examine the case in which not all state variables are available for feedback. In the modern control approach, one would then have to use a state reconstructor (Luenberger observer or constant gain Kalman filter). The results of this paper have obvious implications with respect to the robustness properties of Kalman filters, by duality. However, the overall robustness properties of the LQG design are not settled as yet; they will be addressed in a future publication. Also there are interesting and as yet unresolved issues of the robustness properties of output (or limited-state) variable feedback designs using quadratic performance criteria [31].

II. PREVIOUS WORK

The fundamental work on the robustness of feedback systems is due to Bode [1, pp. 451-88]. Employing the Nyquist stability criterion, Bode showed how the notions of gain and phase margin can be exploited to arrive at a simple and useful means for characterizing the classes of variations in open-loop dynamics which will not destabilize single-input feedback systems. The engineering implications of Bode's results are further developed by Horowitz [2]. Although the Nyquist criterion has been extended to multiloop feedback systems [7] and [8], there has as yet been only limited success in exploiting the multiloop version in the analysis of multiloop feedback system robustness [9] - [14].

Regarding the robustness properties specific to LQSF regulators, perhaps the most significant result is due to Anderson and Moore [4,pp.70-76]. Exploiting the fact that single-input LQSF regulators have a return-difference greater than unity at all frequencies [15], these authors show that single-input LQSF regulator designs have $\pm 60°$ phase margin, infinite gain margin, and 50% gain reduction tolerance. It has also been shown that the gain properties extend to memoryless nonlinear gains of the type shown in Figure 1 ([16] and [4, pp. 96-98]).* Related results by Barnett and Storey [18] and Wong [19] parameterize a class of linear, constant perturbations in feedback gain which will not destabilize a multiloop LQSF regulator. A generalization of the latter result to multiloop nonlinearities in optimal nonlinear state-feedback regulators with quadratic

*This result is attributed by Anderson [16] to Sage [17].

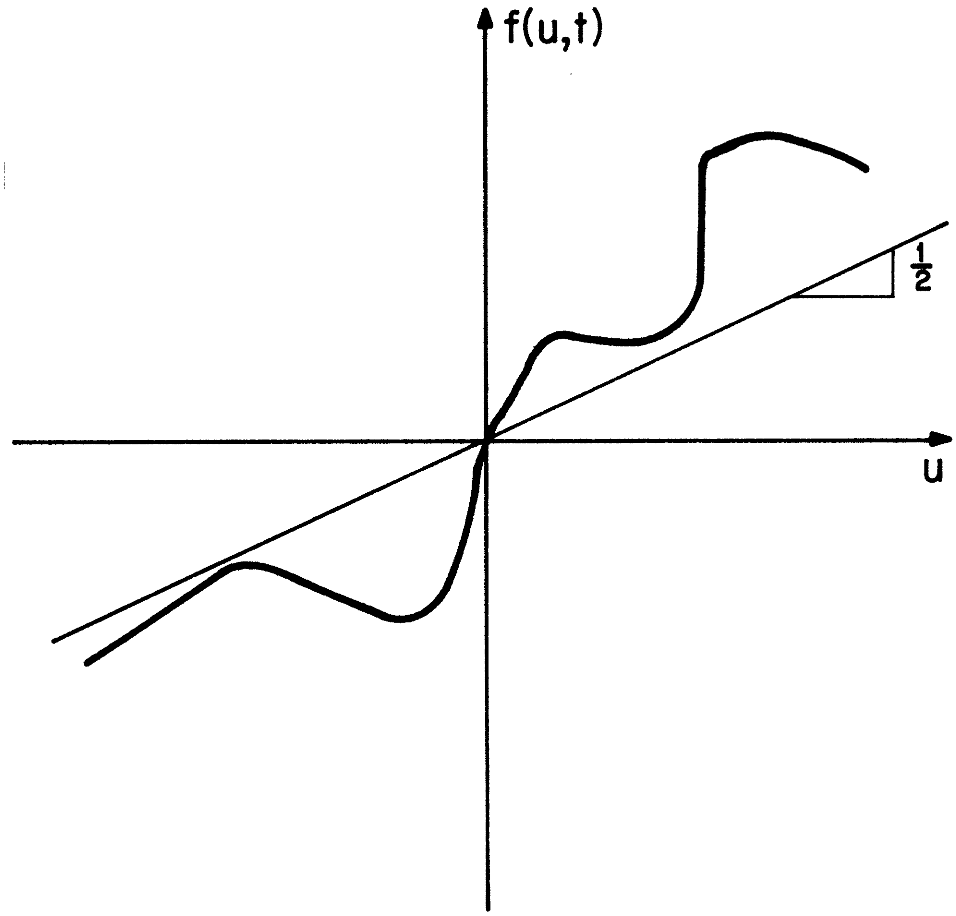

Fig. 1 Non-destabilizing Nonlinear Feedback Gain

performance index is incorrectly attributed to [16] by [20]. Insofar as the generalization stated in [16] applies to LQSF regulators, it is essentially equivalent to theorem 1 of this paper.

Various other results have been produced which are more or less indirectly related to the question considered here. Issues related to the inverse problem of optimal control, i.e. the characterization of the properties of optimal systems, are considered by [15], and [20] - [24]. The question of sensitivity in LQSF regulators is considered by [10], [15], and [25] - [28]. The stability conditions of Zames [29] and [30] involving loop gain, conicity, and positivity have many features in common with the results which are presented here.

III. DEFINITIONS AND NOTATION

The following conventions of notation and terminology are used:

(i) $\underline{A}^T$ $(\underline{x}^T)$ denotes the transpose of the matrix $\underline{A}$ (the vector $\underline{x}$).

(ii) $\underline{A}^*$ denotes the adjoint of the matrix $\underline{A}$ (i.e., the complex-conjugate of $\underline{A}^T$).

(iii) We say that the function $\underline{x}:[0, \infty) \to R^n$ is <u>square-integrable</u> if

$$\int_0^\infty \underline{x}^T(t)\ \underline{x}(t)\ dt < \infty.$$

(iv) The term <u>operator</u> is reserved for functions which map functions into functions. For example, a dynamical system may be viewed as an operator mapping input time-functions into output time-functions.

(v) We say that an operator $\underset{\sim}{N}$ with $\underset{\sim}{N}\ \underline{0} = \underline{0}$ is <u>norm-bounded</u> if there exists a constant $k < \infty$ such that

$$\int_0^\infty [(\underset{\sim}{N}\ \underline{u})\ (t)]^T\ [(\underset{\sim}{N}\ \underline{u})(t)]\ dt < k \int_0^\infty \underline{u}^T(t)\underline{u}(t)dt$$

for all square-integrable $\underline{u}$.

(vi) We say that an operator mapping input time-functions into output time-functions is <u>non-anticipative</u> if the value assumed by the output function at any time t_o depends only on the values of the input-function at times $t \leq t_o$.

(vii) If a function $\underline{x}:[0, \infty) \to R^n$ has the property that

$$\lim_{t \to \infty} \underline{x}(t) = \underline{0}$$

then we say that $\underline{x}$ is asymptotically stable. A system of ordinary differential equations is asymptotically stable if every solution is asymptotically stable.

(viii) If (S) denotes the system $\dot{\underline{x}}(t) = (\underset{\sim}{F}\ \underline{x})(t)$ where $\underset{\sim}{F}\ \underline{0} = \underline{0}$, we say that the pair $[\underline{H}, S]$ is detectable if, for each $\underline{x}: [0, \infty) \rightarrow R^n$ satisfying (S) with $\underline{x}$ not square-integrable, $\underline{H}\ \underline{x}$ is also not square-integrable. The significance of detectability is most apparent if we consider $\underline{x}(t)$ as a description of the internal dynamics of some physical system and $(\underline{H}\ \underline{x})(t)$ as the observed output. Viewed in this manner, detectability means essentially that unstable behavior in the system's internal dynamics always results in an output which is unstable. For example, if $\underline{H}$ is a non-singular square matrix, then $[\underline{H}, S]$ will be detectable.

(ix) We say that an operator mapping time-functions into time-functions is memoryless if the value assumed by its output function at any instant t_o depends only upon t_o and the instantaneous value of the input function at time t_o.

(x) $\underline{A} > \underline{0}$ ($\underline{A} \geq \underline{0}$) is used to indicate that the matrix $\underline{A}$ is positive definite (semi-definite).

(xi) We say that a rational transfer function P(s) is proper if P(s) has at least as many poles as zeroes.

IV. PROBLEM FORMULATION

The Linear-Quadratic-State-Feedback (LQSF) regulator problem can be formulated as follows

$$\min_{\underline{u}} \; J(\underline{x}, \underline{u})$$

subject to (4.1)

$$\dot{\underline{x}}(t) = \underline{A}\,\underline{x}(t) + \underline{B}\,\underline{u}(t) \; ; \; \underline{x}(0) = \underline{x}_o$$

$$\underline{x}(t) \in R^n \; , \; \underline{u}(t) \in R^m \; , \; \underline{A} \in R^{n\times n} \; , \; \underline{B} \in R^{n\times m}$$

where the performance index $J(\underline{x}, \underline{u})$ is given by

$$J(\underline{x}, \underline{u}) = \int_0^{\infty} [\underline{x}^T(t)\,\underline{Q}\,\underline{x}(t) + \underline{u}^T(t)\,\underline{R}\,\underline{u}(t)]dt$$

$$\underline{Q} = \underline{Q}^T \geq \underline{0} \; , \; \underline{R} = \underline{R}^T > \underline{0} \; . \tag{4.2}$$

The optimal control $\underline{u}^*(t)$ and the associated optimal state-trajectory $\underline{x}^*(t)$ are given by

$$\dot{\underline{x}}^*(t) = \underline{A}\,\underline{x}^*(t) + \underline{B}\,\underline{u}^*(t) \; ; \; \underline{x}^*(0) = \underline{x}_o$$

$$\underline{u}^*(t) = -\underline{H}\,\underline{x}^*(t) \equiv -\underline{R}^{-1}\underline{B}^T\underline{K}\,\underline{x}^*(t) \qquad (\Sigma^*)$$

where $\underline{K} = \underline{K}^T \geq \underline{0}$ satisfies the Riccati equation

$$\underline{0} = \underline{K}\,\underline{A} + \underline{A}^T\underline{K} - \underline{K}\,\underline{B}\,\underline{R}^{-1}\underline{B}^T\underline{K} + \underline{Q} \; . \tag{4.3}$$

The minimal value of the performance index is

$$J(\underline{x}^*, \underline{u}^*) = \underline{x}_o^T \underline{K}\, \underline{x}_o. \tag{4.4}$$

The class of systems considered here are perturbed versions of (Σ^*) satisfying

$$\frac{d}{dt}\tilde{\underline{x}}(t) = \underline{A}\,\tilde{\underline{x}}(t) + (\underline{B}\,\underset{\sim}{N}\,\tilde{\underline{u}})(t) \; ; \; \tilde{\underline{x}}(0) = \underline{x}_o$$
$$\tilde{\underline{u}}(t) = -\underline{H}\,\tilde{\underline{x}}(t) \qquad (\tilde{\Sigma})$$

where $\underline{A}$, $\underline{B}$, $\underline{x}_o$, and $\underline{H}$ are the same as in (Σ^*). We assume that $\underset{\sim}{N}$ is a norm-bounded, non-anticipative operator with $\underset{\sim}{N}\,\underline{0} = \underline{0}$ (see Figure 2).* It is further assumed that either $\underset{\sim}{N}$ is memoryless or that $\underset{\sim}{N}$ is linear-time-invariant with a rational transfer function matrix.

*The condition $\underset{\sim}{N}\,\underline{0} = \underline{0}$ is not restrictive since we can always consider the "DC" or steady-state effects separately as is common engineering practice.

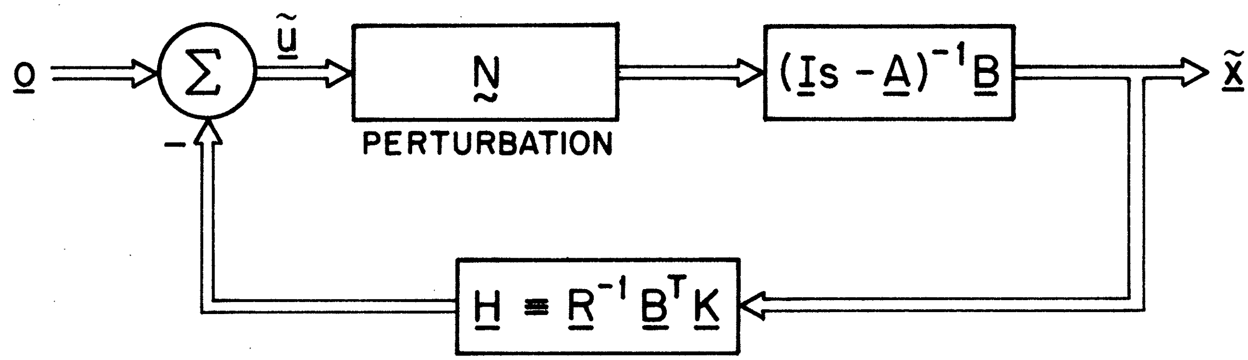

Fig. 2 Perturbed LQSF Regulator ($\tilde{\Sigma}$)

V. RESULTS

The two theorems which follow quantitatively characterize the tolerance of $(\tilde{\Sigma})$ to perturbations $\tilde{N}$. It is noted that the significance of these results is not restricted to systems with perturbations originating only at the point shown in Figure 2. Rather, it is only necessary that the system under consideration have open-loop input/output behavior which is the same as the open-loop behavior of $(\tilde{\Sigma})$. Both of the theorems which follow have interpretations in terms of generalizations of the classical notions of gain and phase margin. The proofs are given in the Appendix.

Theorem 1 -- (LQSF Multiloop Nonlinear Gain Tolerance)

Let the perturbation $\tilde{N}$ of $(\tilde{\Sigma})$ be a memoryless, time-varying nonlinearity,

$$(\tilde{N}\,\underline{u})(t) = \underline{f}(\underline{u}(t),\, t). \tag{5.1}$$

If there exists a constant $\beta \geq 0$ and a constant $k < \infty$ such that

$$k\,\underline{u}^T\underline{u} \geq \underline{u}^T\underline{f}(\underline{R}^{-1}\underline{u},\, t) \geq \frac{1+\beta}{2}\,\underline{u}^T\underline{R}^{-1}\underline{u} \tag{5.2}$$

for all $\underline{u} \in R^m$ and all $t \in [0, \infty)$, then

$$J(\underline{x}^*,\, \underline{u}^*) \geq \int_0^\infty [\tilde{\underline{x}}^T(t)\underline{Q}\,\tilde{x}(t) + \beta\,\tilde{\underline{u}}^T(t)\,\underline{R}\,\tilde{\underline{u}}(t)]dt \tag{5.3}$$

and if, additionally, $[\underline{Q}^{1/2}, \tilde{\Sigma}]$ is detectable then $(\tilde{\Sigma})$ is asymptotically stable. □

<u>Theorem 2</u> -- (LQSF) Multiloop Gain and Phase Margin)

Let the perturbation $\tilde{N}$ of $(\tilde{\Sigma})$ be a norm-bounded, linear, time-invariant operator $\tilde{L}$ with rational transfer function matrix $\underline{L}(s)$. If for some $\beta \geq 0$ and all ω

$$\underline{L}(j\omega)\underline{R}^{-1} + \underline{R}^{-1}\underline{L}^{*}(j\omega) - (1+\beta)\underline{R}^{-1} \geq \underline{0} \qquad (5.4)$$

and if $[\underline{Q}^{1/2}, \tilde{\Sigma}]$ is detectable, then $(\tilde{\Sigma})$ is asymptotically stable. □

VI. DISCUSSION

Theorems 1 and 2 characterize a wide class of variations in open-loop dynamics which can be tolerated by LQSF regulator designs. To appreciate the significance of these results and, in particular, their relation to classical gain and phase margin, it is instructive to consider the special case depicted in Figure 3 in which

$$\underline{Q} > \underline{0} , \tag{6.1}$$

$$\underline{R} = \text{diag}(r_1,\ldots,r_m) \equiv \begin{bmatrix} r_1 & 0 & \cdots & 0 \\ 0 & r_2 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & \cdots & r_m \end{bmatrix} \tag{6.2}$$

and the perturbation $\underset{\sim}{N}$ satisfies

$$\underset{\sim}{N}\,\underline{u} = \begin{bmatrix} \underset{\sim}{N}_1 u_1 \\ \cdot \\ \cdot \\ \cdot \\ \underset{\sim}{N}_m u_m \end{bmatrix} \tag{6.3}$$

so that the perturbations in the various feedback loops are non-interacting.

In this case theorem 1 specializes to the following:

Corollary 3: If in the perturbed system $(\tilde{\Sigma})$ satisfies (6.1), (6.2), and (6.3) and each of the perturbations $\underset{\sim}{N}_i$ is memoryless with $(\underset{\sim}{N}_i u_i)(t) \equiv f_i(u_i(t), t)$ and for some $k > 0$, some $\beta \geq 0$ and all $t \in [0, \infty)$

$$f_i(0, t) = 0 \tag{6.4a}$$

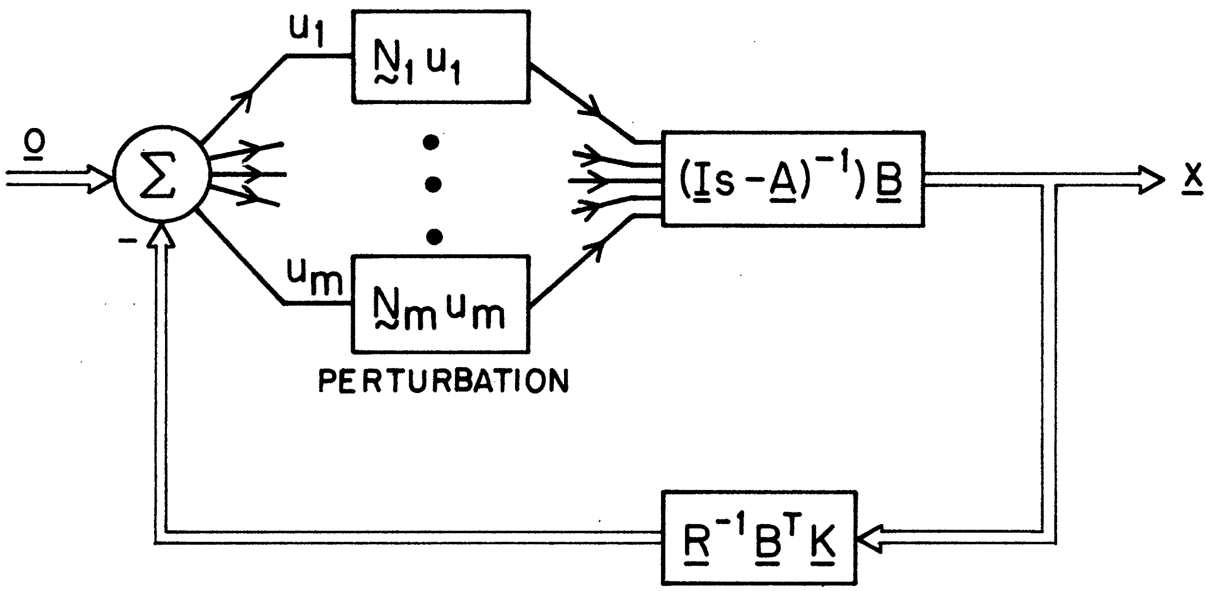

Fig. 3 LQSF Regulator with Non-interacting Perturbations in Each Control Loop

$$k \geq \frac{1}{u} f_i(u, t) \geq \frac{\beta + 1}{2} \text{ for all } u \neq 0 \tag{6.4b}$$

(see Figure 1), then $(\tilde{\sum})$ is asymptotically stable and (5.3) holds. □

Proof: This follows immediately from theorem 1. □

If we consider the case in which the N_i's of the system in Figure 3 are linear time-invariant operators, then theorem 2 becomes:

Corollary 4: If the perturbed system $(\tilde{\sum})$ satisfies (6.1), (6.2), and (6.3) and if each of the perturbations N_i is linear and time-invariant with proper rational transfer function $P_i(s)$, $Re[s_j] < 0$ for each pole s_j of $P_i(s)$, and $Re[P_i(j\omega) \geq 1/2$ for all ω, then $(\tilde{\sum})$ is asymptotically stable. □

Proof: The condition $Re[s_j] < 0$ assures that $\underset{\sim}{N}$ is norm-bounded. Taking $\underline{L}(s) = diag(P_i(s))$, the result follows immediately from theorem 2. □

From corollary 3, it is clear that the sufficient condition for stability

$$\frac{1}{u} f(u) > \frac{1}{2}, \tag{6.5}$$

proved in [4, pp. 96-98] and [16] for single-input LQSF regulators, generalizes to multiloop systems when $R = diag(r_1,...,r_m)$.

From corollary 4, the following two results follow directly:

Corollary 5: (LQSF $\pm 60°$ Multiloop Phase Margin): If $\underline{Q}$ and $\underline{R}$ satisfy (6.1) and (6.2), then a phase shift ϕ_i with $|\phi_i| \leq 60°$ in the respective feedback loops of each of the controls u_i will leave an LQSF regulator asymptotically stable. □

Proof: Take $P_i(j\omega) = e^{j\phi_i(\omega)}$. From corollary 4, we require $\cos \phi_i(\omega) \geq \frac{1}{2}$ or $|\phi_i(\omega)| \leq \cos^{-1}(1/2) = 60°$. □

Corollary 6: (Multiloop LQSF Infinite Gain Margin and 50% Gain Reduction Tolerance): If $\underline{Q}$ and $\underline{R}$ satisfy (6.1) and (6.2), then the insertion of linear constant gains $a_i \geq \frac{1}{2}$ into the feedback loops of the respective controls u_i will leave an LQSF regulator asymptotically stable. □ *

Proof: Follows trivially from corollary 4. □

Corollaries 5 and 6 are obvious multiloop generalizations of the previously established result [4, pp. 70-76] that single-input LQSF regulators have infinite gain margin, ±60° phase margin, and 50% gain reduction tolerance.

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*Corollary 6 is a special case of a result proved by Wong [19].

VII. CONCLUSIONS

Results have been generated which quantitatively characterize a wide class of variations in open-loop dynamics which will not destabilize LQSF regulators. A $\pm 60°$ phase margin property of LQSF regulators has been established for multiloop systems (corollary 5). The class of non-destabilizing linear feedback perturbations for multiloop LQSF regulators has been extended to include dynamical, transfer-function perturbations (theorem 2). A nonlinearity tolerance property for LQSF regulators has been proved (theorem 1). An upper bound on the performance index change in a perturbed LQSF system has been established (Eq. (5.3) in theorem 1 and corollary 3). The latter result can be interpreted as a measure of the stability of a perturbed LQSF regulator in comparison with the unperturbed regulator. The process of generating these results has brought pertinent previous results [4, pp. 70-76, 96-98], [16], [18] - [20] together under a unified theoretical framework.

The results presented show that modern multiloop LQSF regulators have excellent robustness properties as measured by the classical criteria of gain and phase margin, thus strengthening the link between modern and classical feedback theory. Additionally, these results show that multiloop LQSF regulator designs can tolerate a good deal of nonlinearity. The quantitative nature of the results suggests that they may be useful in the synthesis of robust controllers.

Although the results presented all specify that the tolerable perturbations be measured with respect to a perfect state-measurement LQSF system, it is apparent that statements may also be made about the general LQG regulator if the effect of the Kalman filter on the system's open-loop dynamics is viewed as a component of the perturbation $\tilde{N}$.

APPENDIX

Proofs of Theorems 1 and 2

We begin by introducing the following notation to facilitate the proofs:

(i) The inner-product space $L_2^n[0, \infty)$ is defined by

$$L_2^n[0, \infty) = \{\underline{x} | \underline{x}: [0, \infty) \to R^n, \int_0^\infty \underline{x}^T(t)\ \underline{x}(t)dt < \infty\} \qquad (A.1a)$$

$$<\underline{x}, \underline{y}> = \int_0^\infty \underline{x}^T(t)\ \underline{y}(t)dt \qquad (A.1b)$$

(ii) The extension $L_{2e}^n[0, \infty)$ of $L_2^n[0, \infty)$ is defined by

$$L_{2e}^n[0, \infty) = \{\underline{x} | \underline{x} = [0, \infty) \to R^n, \int_0^\tau \underline{x}^T(t)\underline{x}(t)dt < \infty \text{ for all } \tau\} \qquad (A.2a)$$

$$<x, y>_e = \begin{cases} <\underline{x}, \underline{y}> \text{ if the integral (A.1b) converges} \\ \infty \qquad \text{otherwise} \end{cases} \qquad (A.2b)$$

(iii) The linear truncation operator $\underset{\sim}{P}_\tau = L_{2e}^n[0, \infty) \to L_2^n[0, \infty)$

$$(\underset{\sim}{P}_\tau\ \underline{x})(t) = \begin{cases} \underline{x}(t) \text{ if } t \in [0, \tau] \\ 0 \qquad \text{otherwise} \end{cases} \qquad (A.3)$$

For brevity of notation we denote $\underset{\sim}{P}_\tau \underline{x}$ by $\underline{x}_\tau$.

The key result in the proofs of theorems 1 and 2 is the following:

Theorem A.1: If the perturbation $\underset{\sim}{N}$ of $(\tilde{\sum})$ is such that for some $\beta \geq 0$

$$< \underline{u}, (2\underset{\sim}{N} - (1 + \beta)\ \underline{I})\underline{R}^{-1}\underline{u} > \ \geq 0 \qquad (A.4)$$

for all $\underline{u} \in L_2^m[0, \infty)$, then (i)

$$\underline{x}_o^T \underline{K}\, \underline{x}_o \geq \langle \tilde{\underline{x}}, \underline{Q}\, \tilde{\underline{x}} \rangle + \beta \langle \tilde{\underline{u}}, \underline{R}\, \tilde{\underline{u}} \rangle \qquad \text{(A.5)}$$

where $\tilde{\underline{x}}$, $\tilde{\underline{u}}$ are the solution of $(\tilde{\Sigma})$, and (ii) if, additionally, $[\underline{Q}^{1/2}, \tilde{\Sigma}]$ is detectable, then $\tilde{\underline{x}}$ is asymptotically stable and square-integrable.

Proof: For $\underline{K}$ the solution of (4.3) and $\tilde{\underline{x}}$ the solution of $(\tilde{\Sigma})$ with $\tilde{\underline{x}}(0) = \underline{x}_o$, we have that for every $\tau \in [0, \infty)$

$$\begin{aligned}
\underline{x}_o^T \underline{K}\, \underline{x}_o &= \tilde{\underline{x}}^T(\tau)\, \underline{K}\, \tilde{\underline{x}}(\tau) - \int_0^\tau \frac{d}{dt} (\tilde{\underline{x}}^T(t) \underline{K}\, \tilde{\underline{x}}(t))dt \\
&= \tilde{\underline{x}}^T(\tau)\, \underline{K}\, \tilde{\underline{x}}(\tau) - 2 \langle \underline{K}\, \tilde{\underline{x}}_\tau, (\underline{A} - \underline{B}\, \underset{\sim}{N}\, \underline{R}^{-1} \underline{B}^T \underline{K}) \tilde{\underline{x}}_\tau \rangle \\
&\geq -2 \langle \tilde{\underline{x}}_\tau, \underline{K}(\underline{A} - \underline{B}\, \underset{\sim}{N}\, \underline{R}^{-1} \underline{B}^T \underline{K})\, \tilde{\underline{x}}_\tau \rangle \\
&= \langle \tilde{\underline{x}}_\tau, (\underline{K}\, \underline{B}(2 \underset{\sim}{N} - \underline{I}) \underline{R}^{-1} \underline{B}^T \underline{K} + \underline{Q}) \tilde{\underline{x}}_\tau. \qquad \text{(A.6)}
\end{aligned}$$

Using (A.4) and the fact that $\tilde{u} = -\underline{R}^{-1} \underline{B}^T \underline{K}\, \tilde{\underline{x}}$, we have

$$\begin{aligned}
&\underline{x}_o \underline{K}\, \underline{x}_o - \langle \tilde{\underline{x}}_\tau, \underline{Q}\, \tilde{\underline{x}}_\tau \rangle - \beta \langle \tilde{\underline{u}}_\tau, \underline{R}\, \tilde{\underline{u}}_\tau \rangle \\
&\geq \langle \underline{x}_\tau, \underline{K}\, \underline{B}\, (2 \underset{\sim}{N} - (1 + \beta)\, \underline{I})\, \underline{R}^{-1} \underline{B}^T \underline{K}\, \tilde{\underline{x}}_\tau \rangle \\
&= \langle \underline{B}^T \underline{K}\, \tilde{\underline{x}}_\tau, (2 \underset{\sim}{N} - (1 + \beta)\, \underline{I})\, \underline{R}^{-1} \underline{K}\, \tilde{\underline{x}}_\tau \rangle \\
&\geq 0\,. \qquad \text{(A.7)}
\end{aligned}$$

Rearranging and taking the limit $\tau \to \infty$, (A.5) follows. Now, suppose for the purpose of argument that $\tilde{\underline{x}}$ is not square-integrable. Since $[\underline{Q}^{1/2}, \tilde{\Sigma}]$

is detectable, this means $\langle \underline{Q}^{1/2}\tilde{\underline{x}}_\tau , \underline{Q}^{1/2}\tilde{\underline{x}}_\tau \rangle$ increases without bound as τ increases, contradicting (A.5). Therefore, $\tilde{\underline{x}}$ is square-integrable. By hypothesis $\tilde{N}$ and hence $\underline{A} - \underline{B}\,\tilde{N}\,\underline{R}^{-1}\underline{B}^T\underline{K}$ are norm-bounded. Thus, $\dot{\tilde{\underline{x}}} = (\underline{A} - \underline{B}\,\tilde{N}\,\underline{R}^{-1}\underline{B}^T\underline{K})\tilde{\underline{x}}$ is also square-integrable. Since both $\tilde{\underline{x}}$ and $\dot{\tilde{\underline{x}}}$ are square-integrable, it follows (cf. [32, pp. 235-37]) that $\tilde{\underline{x}}$ is asymptotically stable. □

<u>Proof of Theorem 1</u>: Equation (5.2) ensures that (A.4) is satisfied. Since, for memoryless $\tilde{N}$, $\underline{x}$ is the state of $(\tilde{\Sigma})$ and since the initial time $t = 0$ is not distinguished, the asymptotic stability of $(\tilde{\Sigma})$ is assured if $\tilde{\underline{x}}$ is asymptotically stable for every initial state $\tilde{\underline{x}}(0) = \underline{x}_o$. Theorem 1 follows from (4.4) and theorem A.1. □

<u>Proof of Theorem 2</u>: From (5.4) and Parseval's theorem it follows that, for every $\underline{u} \in L_2[0, \infty)$

$$\begin{aligned} &\langle \underline{u}, (2\,\tilde{N} - (1+\beta)\underline{I})\underline{R}^{-1}\underline{u} \rangle \\ &= \langle \underline{u}, (2\,\tilde{L} - (1+\beta)\underline{I})\underline{R}^{-1}\underline{u} \rangle \\ &= \int_{-\infty}^{\infty} \underline{U}^*(j\omega)(\underline{L}(j\omega)\underline{R}^{-1} + \underline{R}^{-1}\underline{L}^*(j\omega) - (1+\beta)\underline{R}^{-1})\underline{U}(j\omega)\,d\omega \\ &\geq 0 \end{aligned} \tag{A.8}$$

where $\underline{U}(j\omega)$ is the Fourier transform of $\underline{u}$. Thus (A.4) is satisfied. Since $[\underline{Q}^{1/2}, \tilde{\Sigma}]$ is detectable, theorem A.1 implies that $\tilde{\underline{x}}$ is asymptotically stable, regardless of the value of $\underline{x}_o$. It follows that the weighting pattern $\underline{W}(t)$ (i.e., the response of $(\tilde{\Sigma})$ to an impulse $\underline{I}_n\delta(t)$

where $\delta(t)$ is the Dirac delta function) is asymptotically stable.

From standard results on linear systems we have

$$\text{(i)} \quad \underline{\mathcal{W}}(s) = [\underline{I}s + \underline{A} - \underline{B}\ \underline{L}(s)\underline{R}^{-1}\underline{B}^T\underline{K}]^{-1} \tag{A.9}$$

where $\underline{\mathcal{W}}(s)$ is the Laplace transform of $\underline{W}(t)$,

$$\text{(ii)} \quad \underline{W}(t) = \sum_{s_i \in \mathcal{C}(W)} \underline{C}_i(t)e^{s_i t} \tag{A.10}$$

where $\underline{C}_i(t)$ are non-zero matrices of polynomials in t and $\mathcal{C}(W)$ is the set of characteristic frequencies of $\underline{W}(t)$, and

$$\text{(iii)} \quad P(\underline{W}) - Z(W) \subseteq \mathcal{C}(\underline{W}) \subseteq P(\underline{W}) \tag{A.11a}$$

where

$$Z(\underline{W}) \equiv \left\{ s_i \middle| \det[\underline{\mathcal{W}}(s_i)] = 0 \right\} \tag{A.11b}$$

$$P(\underline{W}) \equiv \left\{ s_i \middle| (\det[\underline{\mathcal{W}}(s_i)])^{-1} = 0 \right\} . \tag{A.11c}$$

(We call $Z(\underline{W})$ and $P(\underline{W})$ respectively the zeroes and the poles of $\underline{\mathcal{W}}(s)$.) Since $\underline{W}(t)$ is square-integrable,

$$\operatorname{Re}[s_i] < 0 \text{ for all } s_i \in \mathcal{C}(\underline{W}). \tag{A.12}$$

The dynamics of $(\tilde{\Sigma})$ are described (not necessarily minimally) by the differential equations

$$\begin{bmatrix} \underline{I}s - \underline{A} & -\underline{B} \\ \underline{L}_N(s)\underline{R}^{-1}\underline{B}^T\underline{K} & \underline{L}_D(s) \end{bmatrix} \begin{bmatrix} \underline{\tilde{x}} \\ \underline{\tilde{u}} \end{bmatrix} = \underline{0} \tag{A.13}$$

where $s = \frac{d}{dt}$, $\underline{L}_N(s)$ and $\underline{L}_D(s)$ are polynomial matrices satisfying $\underline{L}(s) = \underline{L}_D^{-1}\underline{L}_N(s)$, and the roots of $\det[\underline{L}_D(s)]$ are the poles of $\underline{L}(s)$. For $(\tilde{\Sigma})$ to be asymptotically stable, we require that the roots of the characteristic polynomial p(s) associated with (A.13) all have negative real parts. Using a well-known matrix identity, we have from (A.9) and (A.13)

$$p(s) \equiv \det\begin{bmatrix} \underline{I}s - \underline{A} & -\underline{B} \\ \underline{L}_N(s)\underline{R}^{-1}\underline{B}^T\underline{K} & \underline{L}_D(s) \end{bmatrix}$$

$$= \det[\underline{L}_D(s)] \cdot \det[\underline{I}s - \underline{A} + \underline{B}\,\underline{L}(s)\underline{R}^{-1}\underline{B}^T\underline{K}]$$

$$= \frac{\det[\underline{L}_D(s)]}{\det[\underline{W}(s)]} \tag{A.14}$$

and therefore

$$\det[\underline{W}(s)] = \frac{\det[\underline{L}_D(s)]}{p(s)} . \tag{A.15}$$

From (A.11) and (A.15) it follows that, except for those roots of p(s) which cancel with the roots of the polynomial $\det[\underline{L}_D(s)]$, all roots of the characteristic polynomial p(s) are contained in $C(\underline{W})$. Since $\underset{\sim}{L}$ is norm-bounded, it follows that all the roots of $\det[\underline{L}_D(s)]$ have negative real parts. Thus any cancellations in (A.15) can involve only roots with negative real parts. From (A.12) we conclude that all the roots of the characteristic polynomial p(s) have negative real parts and, hence, $(\tilde{\Sigma})$ is asymptotically stable. □

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