Weight structure on Kontsevich's noncommutative mixed motives

WEIGHT STRUCTURE ON NONCOMMUTATIVE MOTIVES

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Abstract. In this note we endow Kontsevich's category KMM_k of noncommutative mixed motives with a non-degenerate weight structure in the sense of Bondarko. As an application we obtain a convergent weight spectral sequence for every additive invariant (*e.g.* algebraic K-theory, cyclic homology, topological Hochschild homology, *etc.*), and a ring isomorphism between $K_0(\mathrm{KMM}_k)$ and the Grothendieck ring of the category of noncommutative Chow motives.

1. Weight structure

In his seminal talk [6], Kontsevich introduced the triangulated category KMM_k of noncommutative mixed motives (over a base commutative ring k) and conjectured the existence of a "different" t-structure on this category. In this note we formalize Kontsevich's beautiful insight and illustrate some of its important consequences. Recall from [8, 9] the construction of the additive category NChow_k of noncommutative Chow motives. Our formalization of the "different" t-structure is the following:

Theorem 1.1. *There exist two full subcategories* $\mathrm{KMM}_k^{w\geq 0}$ *and* $\mathrm{KMM}_k^{w\leq 0}$ *of* KMM_k *verifying the following seven conditions:*

(i) $\mathrm{KMM}_k^{w\geq 0}$ *and* $\mathrm{KMM}_k^{w\leq 0}$ *are additive and idempotent complete;*

(ii) $\mathrm{KMM}_k^{w\geq 0} \subset \mathrm{KMM}_k^{w\geq 0}[1]$ *and* $\mathrm{KMM}_k^{w\leq 0}[1] \subset \mathrm{KMM}_k^{w\leq 0}$*;*

(iii) *For every* $M \in \mathrm{KMM}_k^{w\geq 0}$ *and* $N \in \mathrm{KMM}_k^{w\leq 0}[1]$ *we have* $\mathrm{Hom}_{\mathrm{KMM}_k}(M, N) = 0$*;*

(iv) *For every* $M \in \mathrm{KMM}_k$ *there is a distinguished triangle*

$$N_2[-1] \longrightarrow M \longrightarrow N_1 \longrightarrow N_2$$

with $N_1 \in \mathrm{KMM}_k^{w\leq 0}$ *and* $N_2 \in \mathrm{KMM}_k^{w\geq 0}$*;*

(v) $\mathrm{KMM}_k = \cup_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\geq 0}[-l] = \cup_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\leq 0}[-l]$*;*

(vi) *There is a natural equivalence of categories* $\mathrm{NChow}_k \simeq \mathrm{KMM}_k^{w\geq 0} \cap \mathrm{KMM}_k^{w\leq 0}$*;*

(vii) $\cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\geq 0}[-l] = \cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\leq 0}[-l] = \{0\}$.

Items (i)-(iv) assert that the triangulated category KMM_k is endowed with a weight structure w (also known in the literature as a co-t-structure) in the sense of Bondarko [2, Def. 1.1.1]. Item (v) asserts that w is bounded, item (vi) that the heart of w can be identified with the category of noncommutative Chow motives, and item (vii) that w is non-degenerate. Theorem 1.1 should then be regarded as the noncommutative analogue of the Chow weight structure on Voevodsky's triangulated category of motives; consult [2, §6.5-6.6].

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2. Weight spectral sequences

Let dgcat_k be the category of (small) dg categories over a fixed base commutative ring k; consult Keller's ICM address [5].

Definition 2.1. Let $L(-) : \mathsf{dgcat}_k \to \mathcal{M}$ be a functor with values in a symmetric monoidal stable model category; see [4, §4 and §7]. We say that L is an *additive invariant* if it verifies the following three conditions:

(i) filtered colimits are mapped to filtered colimits;

(ii) *derived Morita equivalences* (*i.e.* dg functors which induce an equivalence on the associated derived categories; see [5, §4.6]) are mapped to weak equivalences;

(iii) *split exact sequences* (*i.e.* sequences of dg categories which become split exact after passage to the associated derived categories; see [7, §13]) are mapped to direct sums

$$0 \longrightarrow \mathcal{A} \rightleftarrows \mathcal{B} \rightleftarrows \mathcal{C} \longrightarrow 0 \quad \mapsto \quad L(\mathcal{A}) \oplus L(\mathcal{C}) \simeq L(\mathcal{B})$$

in the homotopy category $\mathsf{Ho}(\mathcal{M})$.

Proposition 2.2. *Every additive invariant $L(-)$ gives rise to a triangulated functor $L(-) : \mathrm{KMM}_k \longrightarrow \mathsf{Ho}(\mathcal{M})$; which we still denote by $L(-)$.*

Consider the following compositions

$$L_n(-) : \mathrm{KMM}_k \xrightarrow{L(-)} \mathsf{Ho}(\mathcal{M}) \xrightarrow{\mathrm{Hom}(\mathbf{1}[n],-)} \mathsf{Ab} \qquad n \in \mathbb{Z}\,, \tag{2.3}$$

where $\mathbf{1}$ stands for the $\otimes$-unit of $\mathcal{M}$ and Ab for the category of abelian groups.

Example 2.4 (Algebraic K-theory). Recall from [5, §5.2] that the (connective) algebraic K-theory functor $K(-) : \mathsf{dgcat}_k \to \mathsf{Spt}$, with values in the category of spectra, satisfies the above conditions (i)-(iii) and hence is an additive invariant.

Example 2.5 (Hochschild and cyclic homology). Recall from [5, §5.3] that the Hochschild and cyclic homology functors $HH(-), HC(-) : \mathsf{dgcat}_k \to \mathcal{C}(k)$, with values in the category of complexes of k-modules, are additive invariants. In these examples the associated functors $HH_n(-)$ and $HC_n(-)$ take values in the abelian category k-Mod of k-modules.

Example 2.6 (Negative cyclic homology). Recall from [3, Example 7.10] that the mixed complex functor $C(-) : \mathsf{dgcat}_k \to \mathcal{C}(\Lambda)$, with values in the category of mixed complexes, satisfies the above conditions (i)-(iii) and hence is an additive invariant. Moreover, as explained in [3, Example 8.10], the associated functors $C_n(-)$ agree with the negative cyclic homology functors $HC_n^-(-)$.

Example 2.7 (Periodic cyclic homology). Periodic cyclic homology is *not* an additive invariant since its definition uses infinite products and these do not commute with filtered colimits. Nevertheless, it factors through KMM_k as follows: recall from [3, Example 7.11] that we have a 2-perioditization functor $P(-) : \mathcal{C}(\Lambda) \to k[u]$-Comod, with values in the category of comodules over the Hopf algebra $k[u]$. This functor preserves weak equivalences and hence by applying the above Proposition 2.2 to $C(-)$ we obtain the following composed triangulated functor

$$\mathrm{KMM}_k \overset{C(-)}{\longrightarrow} \mathsf{Ho}(\mathcal{C}(\Lambda)) \overset{P(-)}{\longrightarrow} \mathsf{Ho}(k[u]\text{-Comod})\,.$$

As explained in [3, Example 8.11], the associated functors $((P \circ C)(-))_n$ agree with the periodic cyclic homology functors $HP_n(-)$.

Example 2.8 (Topological Hochschild homology). Recall from [1] (see also [12, §8]) that the topological Hochschild homology functor $THH(-) : \mathsf{dgcat}_k \to \mathsf{Spt}$ is also an example of an additive invariant.

Recall from [11, Thm. 2.8] and [10, Prop. 2.5] the construction of the following natural transformations between additive invariants:

$$tr : K(-) \Rightarrow HH(-) \quad ch^{2i} : K(-) \Rightarrow HC(-)[-2i] \quad ch^{-} : K(-) \Rightarrow C(-)\,. \tag{2.9}$$

By first evaluating these natural transformations at a noncommutative mixed motive M, and then passing to the associated functors (2.3) we obtain, respectively, the Dennis trace maps, the higher Chern characters, and the negative Chern characters:

$$tr_n : K_n(M) \to HH_n(M) \quad ch_n^{2i} : K_n(M) \to HC_{n+2i}(M) \quad ch_n^{-} : K_n(M) \to HC_n^{-}(M)\,.$$

Theorem 2.10. *Under the preceding notations the following holds:*

(i) *To every noncommutative mixed motive M we can associate a cochain (weight) complex of noncommutative Chow motives*

$$t(M): \qquad \cdots \longrightarrow M^{(i-1)} \longrightarrow M^{(i)} \longrightarrow M^{(i+1)} \longrightarrow \cdots\,.$$

Moreover, the assignment $M \mapsto t(M)$ gives rise to a conservative functor from KMM_k *towards a certain weak category of complexes* $K_{\mathfrak{w}}(\mathrm{NChow}_k)$*; consult* [2, §3.1].

(ii) *Every additive invariant $L(-)$ yields a convergent (weight) spectral sequence*

$$E_1^{pq}(M) = L_{-q}(M^{(p)}) \Rightarrow L_{-p-q}(M)\,. \tag{2.11}$$

Moreover, (2.11) *is functorial on M after the E_1-term.*

(iii) *The above natural transformations* (2.9) *respect the spectral sequence* (2.11).

Intuitively speaking, item (i) of Theorem 2.10 shows us that all the information concerning a noncommutative mixed motive can be encoded into a cochain complex. Items (ii) and (iii) endow the realm of noncommutative motives with a new powerful computational tool which is moreover well-behaved with respect to the classical Chern characters. We intend to develop this computational aspect in future work.

3. Grothendieck rings

As explained in [6, 8, 9], the categories KMM_k and NChow_k are endowed with a symmetric monoidal structure induced by the tensor product of dg categories. Hence, the Grothendieck group of KMM_k (considered as a triangulated category) and the Grothendick group of NChow_k (considered as an additive category) are endowed with a ring structure.

Theorem 3.1. *The equivalence of categories of item* (vi) *of Theorem 1.1 gives rise to a ring isomorphism*

$$K_0(\mathrm{NChow}_k) \xrightarrow{\sim} K_0(\mathrm{KMM}_k)\,.$$

Informally speaking, Theorem 3.1 shows us that "up to extension" the categories KMM_k and NChow_k have the same isomorphism classes.

4. Proofs

Proof of Theorem 1.1. Recall from [6] that a dg category $\mathcal{A}$ is called *smooth* if it is perfect as a bimodule over itself and *proper* if for each ordered pair of objects (x, y) in $\mathcal{A}$, the complex of k-modules $\mathcal{A}(x, y)$ is perfect. Recall also that Kontsevich's construction of KMM_k decomposes in three steps:

(1) First, consider the category KPM_k (enriched over spectra) whose objects are the smooth and proper dg categories, whose morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by the (connective) algebraic K-theory spectrum $K(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$, and whose composition is induced by the (derived) tensor product of bimodules.

(2) Then, take the formal triangulated envelope of KPM_k. Objects in this new category are formal finite extensions of formal shifts of objects in KPM_k. Let KTM_k be the associated homotopy category.

(3) Finally, pass to the pseudo-abelian envelope of KTM_k. The resulting category KMM_k is what Kontsevich named the category of noncommutative mixed motives.

Recall from [7, §15] the construction of the additive motivator of dg categories $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}$ and the associated base triangulated category $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$. The analogue of [3, Prop. 8.5] (with the above definition[1] of KMM_k and $\mathrm{Mot}^{\mathsf{loc}}_{\mathsf{dg}}(e)$ replaced by $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$) holds similarly. Hence, KMM_k can be identified with the smallest thick triangulated subcategory of $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$ spanned by the noncommutative motives of smooth and proper dg categories. Similarly, KTM_k can be identified with the smallest triangulated subcategory of $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$ spanned by the noncommutative motives of smooth and proper dg categories. In what follows, we will assume that these identifications have been made.

Now, recall from [8, 9] that the category NChow_k of noncommutative Chow motives is defined as the pseudo-abelian envelope of the category whose objects are the smooth and proper dg categories, whose morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by the Grothendieck group $K_0(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$, and whose composition is induced by the (derived) tensor product of bimodules.

Proposition 4.1. *There is a natural fully-faithful functor*

$$\Phi : \mathrm{NChow}_k \longrightarrow \mathrm{KMM}_k \,. \tag{4.2}$$

Proof. Given dg categories $\mathcal{A}$ and $\mathcal{B}$, let $\mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$ be derived category of $\mathcal{A}$-$\mathcal{B}$-bimodules and $\mathrm{rep}(\mathcal{A}, \mathcal{B}) \subset \mathcal{D}(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$ the full triangulated subcategory spanned by those $\mathcal{A}$-$\mathcal{B}$-bimodules X such that for every object $x \in \mathcal{A}$, the associated $\mathcal{B}$-module $X(x, -)$ is perfect; consult [5, §4.2] for further details. Recall from [8, §5] the construction of the additive category Hmo_0: the objects are the dg categories, the morphisms from $\mathcal{A}$ to $\mathcal{B}$ are given by the Grothendieck group $K_0\mathrm{rep}(\mathcal{A}, \mathcal{B})$ of the triangulated category $\mathrm{rep}(\mathcal{A}, \mathcal{B})$, and the composition is induced by the (derived) tensor product of bimodules. There is a natural functor

$$\mathcal{U}_{\mathsf{A}} : \mathsf{dgcat}_k \longrightarrow \mathsf{Hmo}_0 \tag{4.3}$$

that is the identity on objects and which sends a dg functor $F : \mathcal{A} \to \mathcal{B}$ to the class of the corresponding $\mathcal{A}$-$\mathcal{B}$-bimodule. On the other hand, recall from [7, §15] the

[1] In [3, §8.2] we have considered non-connective algebraic K-theory since we were interested in the relation with Toën's secondary K-theory. However, Kontsevich's original definition is in terms of *connective* algebraic K-theory.

construction of the functor

$$\mathcal{U}_a : \mathsf{dgcat}_k \longrightarrow \mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)\,. \tag{4.4}$$

As proved in [8, Thms. 4.6 and 6.3], $\mathcal{U}_\mathsf{A}$ is the universal functor with values in an additive category which inverts derived Morita equivalences and sends split exact sequences[2] to direct sums (see condition (iii) of Definition 2.1). Since these conditions are satisfied by the functor $\mathcal{U}_a$ (see [7, Thm. 15.4]) and $\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$ is an additive category (since it is triangulated) we obtain an induced additive functor Ψ making the following diagram commute

$$\begin{array}{ccc} \mathsf{dgcat}_k & & \\ {\scriptstyle \mathcal{U}_\mathsf{A}}\downarrow & \searrow^{\mathcal{U}_a} & \\ \mathsf{Hmo}_0 & \xrightarrow[\Psi]{} & \mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)\,. \end{array}$$

Let us denote by $\mathsf{Hmo}_0^{\mathsf{sp}} \subset \mathsf{Hmo}_0$ the full subcategory of smooth and proper dg categories. When $\mathcal{A}$ (and $\mathcal{B}$) is smooth and proper we have a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Hmo}_0^{\mathsf{sp}}}(\mathcal{A},\mathcal{B}) := K_0\mathrm{rep}(\mathcal{A},\mathcal{B}) \simeq K_0(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})\,;$$

see [3, Lemma 4.9]. Hence, we observe that the category NChow_k of noncommutative Chow motives is the pseudo-abelian envelope of $\mathsf{Hmo}_0^{\mathsf{sp}}$. Since by construction the triangulated category $\mathrm{KMM}_k \subset \mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$ is idempotent complete, the composition $\mathsf{Hmo}_0^{\mathsf{sp}} \subset \mathsf{Hmo}_0 \stackrel{\Psi}{\longrightarrow} \mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)$ extends then to a well-defined additive functor

$$\Phi : \mathrm{NChow}_k \longrightarrow \mathrm{KMM}_k \subset \mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)\,.$$

Finally, the fact that Φ is fully-faithful follows from the following computation

$$\mathrm{Hom}_{\mathrm{Mot}^{\mathsf{add}}_{\mathsf{dg}}(e)}(\mathcal{U}_a(\mathcal{A}),\mathcal{U}_a(\mathcal{B})) \simeq K_0\mathrm{rep}(\mathcal{A},\mathcal{B}) \simeq K_0(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B})$$

for every smooth and proper dg category $\mathcal{A}$; see [7, Prop. 16.1]. □

Let us now verify the conditions of Bondarko's [2, Thm. 4.3.2 II] with $\underline{C}$ the triangulated category $\mathrm{KTM}_k \subset \mathrm{KMM}_k$ and H the essential image of the composition $\mathsf{Hmo}_0^{\mathsf{sp}} \subset \mathrm{NChow}_k \to \mathrm{KMM}_k$. By construction, H generates KTM_k in the sense of [2, page 11]. Moreover, given any two smooth and proper dg categories $\mathcal{A}$ and $\mathcal{B}$, we have the following computation:

$$\mathrm{Hom}_{\mathrm{KMM}_k}(\Phi(\mathcal{A}),\Phi(\mathcal{B})[-n]) \simeq \begin{cases} K_n(\mathcal{A}^{\mathsf{op}} \otimes^{\mathbb{L}} \mathcal{B}) & n \geq 0 \\ 0 & n < 0\,. \end{cases} \tag{4.5}$$

This follows from [7, Prop. 16.1] combined with the specific construction of Φ. Hence, $H \subset \mathrm{KTM}_k$ is negative in the sense of [2, Def. 4.3.1(1)]. The conditions of [2, Thm. 4.3.2 II] are then satisfied and so we conclude that there exists a unique bounded weight structure w on KTM_k whose heart is the pseudo-abelian envelope of H. Note that the heart is then equivalent to NChow_k under the above fully-faithful functor (4.2). Since the weight structure w is bounded, [2, Prop. 5.2.2] implies that w can be extended from KTM_k to KMM_k. The heart remains exactly the same since the category NChow_k is by construction idempotent complete. By [2, Defs. 1.1.1 and 1.2.1] we then conclude that conditions (i)-(vi) of Theorem 1.1 are

[2]This condition can equivalently be formulated in terms of a general semi-orthogonal decomposition in the sense of Bondal-Orlov; see [8, Thm. 6.3(4)].

verified, where $\mathrm{KMM}_k^{w\geq 0}$ (resp. $\mathrm{KMM}_k^{\leq 0}$) is the smallest idempotent complete and extension-stable subcategory of KMM_k (see [2, Def. 1.3.1]) containing the objects $\Phi(\mathrm{NChow}_k)[n], n \leq 0$ (resp. $\Phi(\mathrm{NChow}_k)[n], n \geq 0$). It remains then to verify condition (vii). We start by showing the equality $\cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\geq 0}[-l] = \{0\}$.

Proposition 4.6. *For every noncommutative mixed motive M there exists an integer $j \in \mathbb{Z}$ (which depends on M) such that for every $N \in \Phi(\mathrm{NChow}_k)$ we have*

$$\mathrm{Hom}_{\mathrm{KMM}_k}(N, M[i]) = 0 \quad \text{when} \quad i > j\,. \tag{4.7}$$

Proof. Let $\mathcal{C}$ be a full subcategory of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ containing the zero object. Let us denote by $\mathcal{C}[\mathbb{Z}]$ the category $\cup_{n\in\mathbb{Z}}\mathcal{C}[n]$, by $\mathcal{C}^\natural$ the idempotent completion of $\mathcal{C}$ inside $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$, and by $\mathrm{Ext}(\mathcal{C})$ the subcategory of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ formed by the objects $\mathcal{O}$ for which there exists a distinguished triangle

$$M_1 \longrightarrow \mathcal{O} \longrightarrow M_2 \longrightarrow M_1[1] \tag{4.8}$$

with M_1 and M_2 in $\mathcal{C}$. Note that $\mathcal{C} \subseteq \mathrm{Ext}(\mathcal{C})$. Consider the following

Vanishing Condition: there exists an integer $j \in \mathbb{Z}$ such that for every object $N \in \Phi(\mathrm{NChow}_k)$ we have

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)}(N, \mathcal{O}[i]) = 0 \quad \text{when} \quad i > j\,.$$

We now show that if by hypothesis the above vanishing condition holds for every object $\mathcal{O}$ of $\mathcal{C}$, then it holds also for every object of the following categories:

(1) The category $\mathcal{C}[\mathbb{Z}]$: this is clear since every object in $\mathcal{C}[\mathbb{Z}]$ is of the form $\mathcal{O}[n]$, with n and integer and $\mathcal{O} \in \mathcal{C}$;
(2) The category $\mathrm{Ext}(\mathcal{C})$: by construction every object $\mathcal{O}$ of $\mathrm{Ext}(\mathcal{C})$ fits in the above distinguished triangle (4.8). Let j_1 and j_2 be the integers of the vanishing condition which are associated to M_1 and M_2, respectively. Then, by choosing $j := \max\{j_1, j_2\}$ we observe that the object $\mathcal{O}$ also verifies the above vanishing condition;
(3) The category $\mathcal{C}^\natural$: this is clear since every object in $\mathcal{C}^\natural$ is a direct summand of an object in $\mathcal{C}$; recall that $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ admits arbitrary sums and so every idempotent splits.

Let us now apply the above general arguments to the category $\mathcal{C} = \Phi(\mathrm{NChow}_k)$. By computation (4.5) the above vanishing condition holds for every object (with $j = 0$). Recall that KMM_k is the smallest thick triangulated subcategory of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ spanned by the objects $N \in \Phi(\mathrm{NChow}_k)$. Hence, every object $M \in \mathrm{KMM}_k$ belongs to the category obtained from $\Phi(\mathrm{NChow}_k)$ by applying the above constructions (1)-(3) a *finite* number of times (the number of times depends on M). As a consequence, we conclude that M satisfies the above vanishing condition and so the proof is finished. □

Let $M \in \cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\geq 0}[-l]$. Note that equality (4.7) can be re-written as

$$\mathrm{Hom}_{\mathrm{KMM}_k}(N[-i], M) = 0 \quad \text{when} \quad i > j\,. \tag{4.9}$$

Since $\mathrm{KMM}_k^{w\geq 0}$ is the smallest idempotent complete and extension stable subcategory of KMM_k containing the objects $\Phi(\mathrm{NChow}_k)[n], n \leq 0$, we conclude from (4.9) that $\mathrm{Hom}_{\mathrm{KMM}_k}(\mathcal{O}, M) = 0$ for every object $\mathcal{O}$ belonging to $\mathrm{KMM}_k^{w\geq 0}[-l]$ with $l > j$. Since by hypothesis $M \in \mathrm{KMM}_k^{w\geq 0}[-l]$ we then conclude by the Yoneda lemma that $M = 0$ in $\mathrm{KMM}_k^{w\geq 0}[-l]$ (with $l > j$) and hence in $\cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\geq 0}[-l]$. Let us now prove the equality $\cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\leq 0}[-l] = \{0\}$.

Proposition 4.10. *For every non-trivial noncommutative mixed motive M there exists an integer $j \in \mathbb{Z}$, an object $N \in \Phi(\mathrm{NChow}_k)$, and a non-trivial morphism $f : N[j] \to M$.*

Proof. We prove the following equivalent statement: if $\mathrm{Hom}_{\mathrm{KMM}_k}(N[n], M) = 0$ for every integer $n \in \mathbb{Z}$ and object $N \in \Phi(\mathrm{NChow}_k)$, then $M = 0$. Recall that KMM_k is the smallest thick triangulated subcategory of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ spanned by the objects $N \in \Phi(\mathrm{NChow}_k)$. The class of objects $\mathcal{O}$ in $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$ satisfying the equalities

$$\mathrm{Hom}_{\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)}(\mathcal{O}[n], M) = 0 \qquad n \in \mathbb{Z}$$

is clearly stable under extensions and direct factors. Since by hypothesis it contains the objects $N \in \Phi(\mathrm{NChow}_k)$ it contains also all the objects of the category KMM_k. Hence, by taking $\mathcal{O} = M$ and $n = 0$, the identity morphism of M allows us to conclude that $M = 0$. □

Let $M \in \cap_{l\in\mathbb{Z}} \mathrm{KMM}_k^{w\leq 0}[-l]$. If by hypothesis M is non-trivial, then the morphism f of Proposition 4.10 gives rise to to a non-trivial morphism

$$0 \neq f[-j] : N \longrightarrow M[-j]\,. \tag{4.11}$$

Since by construction N belongs to $\mathrm{KMM}_k^{w\geq 0}$, condition (iii) of Theorem 1.1 combined with the non-trivial morphism (4.11) implies that $M[-j] \notin \mathrm{KMM}_k^{w\leq 0}[1]$. Hence, $M \notin \mathrm{KMM}_k^{w\leq 0}[1+j]$ and so we obtain a contradiction with our hypothesis. This allows us to conclude that $M = 0$ and so the proof of Theorem 1.1 is finished.

Proof of Proposition 2.2. The category dgcat_k carries a (cofibrantly generated) Quillen model structure whose weak equivalences are precisely the derived Morita equivalences; see [8, Thm. 5.3]. Hence, it gives rise to a well-defined Grothendieck derivator $\mathsf{HO}(\mathsf{dgcat}_k)$; consult [3, Appendix A] for the notion of Grothendieck derivator. Since by hypothesis $\mathcal{M}$ is stable and $L(-)$ satisfies conditions (i)-(iii) of Definition 2.1, we obtain then a well-defined additive invariant of dg categories $\mathsf{HO}(\mathsf{dgcat}_k) \to \mathsf{HO}(\mathcal{M})$ in the sense of [7, Notation 15.5]. By the universal property of [7, Thm. 15.4] this additive invariant factors through $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}$ giving rise to a homotopy colimit preserving morphism of derivators $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}} \to \mathsf{HO}(\mathcal{M})$ and hence to a triangulated functor $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e) \to \mathsf{Ho}(\mathcal{M})$ on the underlying base categories. As explained in the proof of Theorem 1.1, the category KMM_k can be identified with a full triangulated subcategory of $\mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e)$. The composition obtained

$$L(-) : \mathrm{KMM}_k \subset \mathrm{Mot}_{\mathsf{dg}}^{\mathsf{add}}(e) \longrightarrow \mathsf{Ho}(\mathcal{M})$$

is then the triangulated functor mentioned in Proposition 2.2.

Proof of Theorem 2.10. As explained in the proof of Theorem 1.1, the category KMM_k is endowed with a non-degenerate bounded weight structure w whose heart is equivalent to the category NChow_k of noncommutative Chow motives. In particular we have the following equalities

$$\mathrm{KMM}_k^{+} = \mathrm{KMM}_k = \mathrm{KMM}_k^{-}\,;$$

see [2, Def. 1.3.5]. Hence, item (i) follows from the combination of [2, Thm. 3.2.2 II] with [2, Thm. 3.3.1]. By Proposition 2.2 every additive invariant $L(-)$ gives rise to a triangulated functor $L(-) : \mathrm{KMM}_k \to \mathsf{Ho}(\mathcal{M})$ and hence to a composed functor

$$\mathrm{KMM}_k \xrightarrow{L(-)} \mathsf{Ho}(\mathcal{M}) \xrightarrow{\mathrm{Hom}(\mathbf{1},-)} \mathsf{Ab}\,. \tag{4.12}$$

Note that (4.12) is *homological*, *i.e.* it sends distinguished triangles to long exact sequences, and that we have the following identifications:

$$\mathrm{Hom}(\mathbf{1}, L(M[-i])) \simeq \mathrm{Hom}(\mathbf{1}, L(M)[-i]) \simeq \mathrm{Hom}(\mathbf{1}[i], L(M)) = L_i(M)\,. \tag{4.13}$$

Since the weight structure w is bounded, we have $\mathrm{KMM}_k = \mathrm{KMM}_k^b$; see [2, Def. 1.3.5]. Hence, item (ii) follows from [2, Thm. 2.3.2 II and IV] (with $H =$ (4.12)) and from the above identifications (4.13). Finally, item (iii) follows from [2, Thm. 2.3.2 III] since all the Chern characters (2.9) are natural transformations of additive invariants.

Remark 4.14. As the above proof clearly shows, Theorem 2.10(ii) applies also to periodic cyclic homology; see Example 2.7.

Proof of Theorem 3.1. As explained in [3, Thm. 7.5] the functor (4.4) is symmetric monoidal. Since (4.3) is also symmetric monoidal we conclude from the construction of (4.2) that this latter functor is also symmetric monoidal. Recall from the proof of Theorem 1.1 that the category KMM_k is endowed with a bounded weight structure w. Since the functor (4.3) is symmetric monoidal the heart $\Phi(\mathrm{NChow}_k)$ of w is then a full additive symmetric monoidal subcategory of KMM_k. Hence, the result follows from the combination of [2, Thm. 5.3.1] with [2, Remark 5.3.2].

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