I. Dissipative Dynamics

I.A Brownian Motion of a Particle

Observations under a microscope indicate that a dust particle in a liquid drop undergoes a random jittery motion. This is because of the random impacts of the much smaller fluid particles. The theory of such (*Brownian*) motion was developed by Einstein in 1905 and starts with the equation of motion for the particle. The displacement $\vec{x}(t)$, of a particle of mass m is governed by,

$$m\,\ddot{\vec{x}} = -\frac{\dot{\vec{x}}}{\mu} - \frac{\partial \mathcal{V}}{\partial \vec{x}} + \vec{f}_{\text{random}}(t). \tag{I.1}$$

The three forces acting on the particle are:

 (i) A friction force due to the viscosity of the fluid. For a spherical particle of radius R, the mobility in the low Reynolds number limit is given by $\mu = (6\pi\bar{\eta}R)^{-1}$, where $\bar{\eta}$ is the specific viscosity.

 (ii) The force due to the external potential $\mathcal{V}(\vec{x})$, e.g. gravity.

(iii) A random force of zero mean due to the impacts of fluid particles.

The viscous term usually dominates the inertial one (i.e. the motion is overdamped), and we shall henceforth ignore the acceleration term. Eq.(I.1) now reduces to a *Langevin equation*,

$$\dot{\vec{x}} = \vec{v}(\vec{x}) + \vec{\eta}(t), \tag{I.2}$$

where $\vec{v}(\vec{x}) = -\mu \partial \mathcal{V}/\partial \vec{x}$ is the *deterministic* velocity. The *stochastic* velocity, $\vec{\eta}(t) = \mu\vec{f}_{\text{random}}(t)$, has zero mean,

$$\langle \vec{\eta}(t) \rangle = 0. \tag{I.3}$$

It is usually assumed that the probability distribution for the noise in velocity is Gaussian, i.e.

$$\mathcal{P}\left[\vec{\eta}(t)\right] \propto \exp\left[-\int d\tau\, \frac{\eta(\tau)^2}{4D} \right]. \tag{I.4}$$

Note that different components of the noise, and at different times, are independent, and the covariance is

$$\langle \eta_\alpha(t)\eta_\beta(t') \rangle = 2D\delta_{\alpha,\beta}\delta(t-t'). \tag{I.5}$$

The parameter D is related to *diffusion* of particles in the fluid. In the absence of any potential, $\mathcal{V}(\vec{x}) = 0$, the position of a particle at time t is given by

$$\vec{x}(t) = \vec{x}(0) + \int_0^t d\tau \vec{\eta}(\tau).$$

Clearly the separation $\vec{x}(t) - \vec{x}(0)$ which is the sum of random Gaussian variables is itself Gaussian distributed with mean zero, and a varaince

$$\left\langle (\vec{x}(t) - \vec{x}(0))^2 \right\rangle = \int_0^t d\tau_1 d\tau_2 \left\langle \vec{\eta}(\tau_1) \cdot \vec{\eta}(\tau_2) \right\rangle = 3 \times 2Dt.$$

For an ensemble of particles released at $\vec{x}(t) = 0$, i.e. with $\mathcal{P}\left(\vec{x}, t = 0\right) = \delta^3(\vec{x})$, the particles at time t are distributed according to

$$\mathcal{P}\left(\vec{x}, t\right) = \left(\frac{1}{\sqrt{4\pi Dt}}\right)^{3/2} \exp\left[-\frac{x^2}{4Dt}\right],$$

which is the solution to the diffusion equation

$$\frac{\partial \mathcal{P}}{\partial t} = D\nabla^2 \mathcal{P}.$$

A simple example is provided by a particle connected to a Hookian spring, with $\mathcal{V}(\vec{x}) = Kx^2/2$. The deterministic velocity is now $\vec{v}(\vec{x}) = -\mu K\vec{x}$, and the Langevin equation, $\dot{\vec{x}} = -\mu K\vec{x} + \vec{\eta}(t)$, can be rearranged as

$$\frac{d}{dt}\left[e^{\mu Kt}\vec{x}(t)\right] = e^{\mu Kt}\vec{\eta}(t). \tag{I.6}$$

Integrating the equation from 0 to t yields

$$e^{\mu Kt}\vec{x}(t) - \vec{x}(0) = \int_0^t d\tau e^{\mu K\tau}\vec{\eta}(\tau), \tag{I.7}$$

and

$$\vec{x}(t) = \vec{x}(0)e^{-\mu Kt} + \int_0^t d\tau e^{-\mu K(t-\tau)}\vec{\eta}(\tau). \tag{I.8}$$

Averaging over the noise indicates that the mean position,

$$\langle \vec{x}(t) \rangle = \vec{x}(0)e^{-\mu Kt}, \tag{I.9}$$

decays with a characteristic *relaxation time*, $\tau = 1/(\mu K)$. Fluctuations around the mean behave as

$$\left\langle \left(\vec{x}(t) - \langle\vec{x}(t)\rangle\right)^2 \right\rangle = \int_0^t d\tau_1 d\tau_2 e^{-\mu K(2t-\tau_1-\tau_2)} \overbrace{\langle\vec{\eta}(\tau_1)\cdot\vec{\eta}(\tau_2)\rangle}^{2D\delta(\tau_1-\tau_2)\times 3}$$

$$= 6D \int_0^t d\tau e^{-2\mu K(t-\tau)} \tag{I.10}$$

$$= \frac{3D}{\mu K}\left[1 - e^{-2\mu Kt}\right] \quad \overset{t\to\infty}{\longrightarrow} \quad \frac{3D}{\mu K}.$$

However, once the dust particle reaches equilibrium with the fluid at a temperature T, its probability distribution must satisfy the normalized Boltzmann weight

$$\mathcal{P}_{\text{eq.}}(\vec{x}) = \left(\frac{K}{2\pi k_B T}\right)^{3/2} \exp\left[-\frac{Kx^2}{2k_B T}\right], \tag{I.11}$$

yielding $\langle x^2 \rangle = 3k_B T/K$. Since the dynamics is expected to bring the particle to equilibrium with the fluid at temperature T, eq.(I.10) implies the condition

$$D = k_B T \mu. \tag{I.12}$$

This is the Einstein relation connecting the *fluctuations* of noise to the *dissipation* in the medium.

Clearly the Langevin equation at long times reproduces the correct mean and variance for a particle in equilibrium at a temperature T in the potential $\mathcal{V}(\vec{x}) = Kx^2/2$, provided that eq.(I.12) is satisfied. Can we show that the whole probability distribution evolves to the Boltzmann weight for any potential? Let $\mathcal{P}(\vec{x},t) \equiv \langle\vec{x}|\mathcal{P}(t)|0\rangle$ denote the probability density of finding the particle at $\vec{x}$ at time t, given that it was at 0 at $t = 0$. This probability can be constructed recursively by noting that a particle found at $\vec{x}$ at time $t+\epsilon$ must have arrived from some other point $\vec{x}'$ at t. Adding up all such probabilities yields

$$\mathcal{P}(\vec{x},t+\epsilon) = \int d^3\vec{x}'\, \mathcal{P}(\vec{x}',t)\, \langle\vec{x}|T_\epsilon|\vec{x}'\rangle, \tag{I.13}$$

where $\langle\vec{x}|T_\epsilon|\vec{x}'\rangle \equiv \langle\vec{x}|\mathcal{P}(\epsilon)|\vec{x}'\rangle$ is the transition probability. For $\epsilon \ll 1$,

$$\vec{x} = \vec{x}' + \vec{v}(\vec{x}')\epsilon + \vec{\eta}_\epsilon \quad, \tag{I.14}$$

where $\vec{\eta}_\epsilon = \int_t^{t+\epsilon} d\tau\vec{\eta}(\tau)$. Clearly, $\langle\vec{\eta}_\epsilon\rangle = 0$, and $\langle\eta_\epsilon^2\rangle = 2D\epsilon \times 3$, and following eq.(I.4),

$$p(\vec{\eta}_\epsilon) = \left(\frac{1}{4\pi D\epsilon}\right)^{3/2} \exp\left[-\frac{\eta_\epsilon^2}{4D\epsilon}\right]. \tag{I.15}$$

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The transition rate is simply the probability of finding a noise of the right magnitude according to eq.(I.14), and

$$\langle \vec{x} \, | T(\epsilon)| \vec{x}' \rangle = p(\eta_\epsilon) = \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp \left[-\frac{(\vec{x} - \vec{x}' - \epsilon\vec{v}(\vec{x}'))^2}{4D\epsilon} \right]$$

$$= \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp \left[-\epsilon \frac{\left(\dot{\vec{x}} - \vec{v}(\vec{x}) \right)^2}{4D} \right]. \tag{I.16}$$

By subdividing the time interval t, into infinitesimal segments of size ϵ, repeated application of the above evolution operator yields

$$\mathcal{P}(\vec{x}, t) = \left\langle \vec{x} \left| T(\epsilon)^{t/\epsilon} \right| 0 \right\rangle$$

$$= \int_{(0,0)}^{(\vec{x},t)} \frac{\mathcal{D}\vec{x}(\tau)}{\mathcal{N}} \exp \left[-\int_0^t d\tau \frac{\left(\dot{\vec{x}} - \vec{v}(\vec{x}) \right)^2}{4D} \right]. \tag{I.17}$$

The integral is over all paths connecting the initial and final points; each path's weight is related to its deviation from the classical trajectory, $\dot{\vec{x}} = \vec{v}(\vec{x})$. The recursion relation (eq.(I.13)),

$$\mathcal{P}(\vec{x}, t) = \int d^3\vec{x}' \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} \exp \left[-\frac{(\vec{x} - \vec{x}' - \epsilon\vec{v}(\vec{x}'))^2}{4D\epsilon} \right] \mathcal{P}(\vec{x}', t - \epsilon), \tag{I.18}$$

can be simplified by the change of variables,

$$\vec{y} = \vec{x}' + \epsilon\vec{v}(\vec{x}') - \vec{x} \implies$$

$$d^3\vec{y} = d^3\vec{x}' \left(1 + \epsilon\nabla \cdot \vec{v}(\vec{x}') \right) = d^3\vec{x}' \left(1 + \epsilon\nabla \cdot \vec{v}(\vec{x}) + \mathcal{O}(\epsilon^2) \right). \tag{I.19}$$

Keeping only terms at order of ϵ, we obtain

$$\mathcal{P}(\vec{x}, t) = [1 - \epsilon\nabla \cdot \vec{v}(\vec{x})] \int d^3\vec{y} \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} e^{-\frac{y^2}{4D\epsilon}} \, \mathcal{P}(\vec{x} + \vec{y} - \epsilon\vec{v}(\vec{x}), t - \epsilon)$$

$$= [1 - \epsilon\nabla \cdot \vec{v}(\vec{x})] \int d^3\vec{y} \left(\frac{1}{4\pi D\epsilon} \right)^{3/2} e^{-\frac{y^2}{4D\epsilon}} \times$$

$$\left[\mathcal{P}(\vec{x}, t) + (\vec{y} - \epsilon\vec{v}(\vec{x})) \cdot \nabla\mathcal{P} + \frac{y_i y_j - 2\epsilon y_i v_j + \epsilon^2 v_i v_j}{2} \nabla_i\nabla_j\mathcal{P} - \epsilon\frac{\partial\mathcal{P}}{\partial t} + \mathcal{O}(\epsilon^2) \right]$$

$$= [1 - \epsilon\nabla \cdot \vec{v}(\vec{x})] \left[\mathcal{P} - \epsilon\vec{v} \cdot \nabla + \epsilon D\nabla^2\mathcal{P} - \epsilon\frac{\partial\mathcal{P}}{\partial t} + \mathcal{O}(\epsilon^2) \right]. \tag{I.20}$$

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Equating terms at order of ϵ leads to the *Fokker-Planck equation*,

$$\frac{\partial \mathcal{P}}{\partial t} + \nabla \cdot \vec{J} = 0, \qquad \text{with} \qquad \vec{J} = \vec{v}\mathcal{P} - D\nabla\mathcal{P} \quad . \tag{I.21}$$

The Fokker-Planck equation is simply the statement of conservation of probability. The probability current has a deterministic component $\vec{v}\mathcal{P}$, and a stochastic part $-D\nabla\mathcal{P}$. A *stationary distribution*, $\partial\mathcal{P}/\partial t = 0$, is obtained if the net current vanishes. It is now easy to check that the Boltzmann weight, $\mathcal{P}_{\text{eq.}}(\vec{x}) \propto \exp[-\mathcal{V}(\vec{x})/k_B T]$, with $\nabla\mathcal{P}_{\text{eq.}} = \vec{v}\mathcal{P}_{\text{eq.}}/(\mu k_B T)$, leads to a stationary state as long as the fluctuation–dissipation condition in eq.(I.12) is satisfied.

I.B Equilibrium Dynamics of a Field

The next step is to generalize the above formalism to a collection of degrees of freedom, most conveniently described by a continuous field. The procedure will be described in terms of the dynamics of a surface, although it is in fact quite general. Small fluctuations of the surface can be described by a height $h(\mathbf{x}, t)$. Specific examples are the distortions of a soap film or the fluctuations on the surface of water in a container. In both cases the minimum energy configuration is a flat surface (ignoring the small effects of gravity on the soap film). The energy cost of small fluctuations for a soap film comes from the increased area and *surface tension*, σ. Expanding the area in powers of the slope results in

$$\mathcal{H}_\sigma = \sigma \int d^D\mathbf{x} \left[\sqrt{1 + (\nabla h)^2} - 1 \right] \approx \frac{\sigma}{2} \int d^d\mathbf{x}\,(\nabla h)^2 \ . \tag{I.22}$$

For the surface of water there is an additional gravitational potential energy, obtained by adding the contributions from all columns of water as

$$\mathcal{H}_g = \int d^d\mathbf{x} \int_0^{h(\mathbf{x})} dh'\, \rho g h' = \frac{\rho g}{2} \int d^d\mathbf{x}\, h(\mathbf{x})^2 \ . \tag{I.23}$$

The total (potential) energy of small fluctuations is thus given by

$$\mathcal{H} = \int d^d\mathbf{x} \left[\frac{\sigma}{2}(\nabla h)^2 + \frac{\rho g}{2} h^2 \right], \tag{I.24}$$

with the second term absent for the soap film.

To construct a Langevin equation governing the dynamics of height fluctuations, first calculate the *force* on each surface element from the variations of the potential energy. The *functional derivative* of eq.(I.24) yields

$$F(\mathbf{x}) = -\frac{\delta \mathcal{H}[h]}{\delta h(\mathbf{x})} = -\rho g h + \sigma \nabla^2 h. \tag{I.25}$$

The straightforward analog of eq.(I.2) is

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = \mu F(\mathbf{x}) + \eta(\mathbf{x}, t), \tag{I.26}$$

with a random velocity, η, such that

$$\langle \eta(\mathbf{x}, t) \rangle = 0, \qquad \text{and} \qquad \langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle = 2D\delta(\mathbf{x} - \mathbf{x}')\delta(t - t'). \tag{I.27}$$

The Langevin equation,

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = -\mu\rho g h + \mu\sigma \nabla^2 h + \eta(\mathbf{x}, t), \tag{I.28}$$

is most easily solved by examining the Fourier components,

$$h(\mathbf{q}, t) = \int d^d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} h(\mathbf{x}, t), \tag{I.29}$$

which evolve according to

$$\frac{\partial h(\mathbf{q}, t)}{\partial t} = -\mu(\rho g + \sigma q^2)\, h(\mathbf{q}, t) + \eta(\mathbf{q}, t). \tag{I.30}$$

The Fourier transformed noise,

$$\eta(\mathbf{q}, t) = \int d^d\mathbf{x}\, e^{i\mathbf{q}\cdot\mathbf{x}} \eta(\mathbf{x}, t), \tag{I.31}$$

has zero mean, $\langle \eta(\mathbf{q}, t) \rangle = 0$, and correlations

$$
\begin{aligned}
\langle \eta(\mathbf{q}, t)\eta(\mathbf{q}', t') \rangle &= \int d^d\mathbf{x}d^d\mathbf{x}'\, e^{i\mathbf{q}\cdot\mathbf{x}+i\mathbf{q}'\cdot\mathbf{x}'} \overbrace{\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle}^{2D\delta^d(\mathbf{x}-\mathbf{x}')\ \delta(t-t')} \\
&= 2D\delta(t - t') \int d^d\mathbf{x}\, e^{i\mathbf{x}\cdot(\mathbf{q}+\mathbf{q}')} \\
&= 2D\delta(t - t')(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}').
\end{aligned}
\tag{I.32}
$$

Each Fourier mode in eq.(I.30) now behaves as an independent particle connected to a spring as in eq.(I.6). Introducing a decay rate

$$\gamma(\mathbf{q}) \equiv \frac{1}{\tau(\mathbf{q})} = \mu(\rho g + \sigma q^2), \tag{I.33}$$

the evolution of each mode is similar to eq.(I.8), and follows

$$h(\mathbf{q}, t) = h(\mathbf{q}, 0)e^{-\gamma(\mathbf{q})t} + \int_0^t d\tau \ e^{-\gamma(\mathbf{q})(t-\tau)}\eta(\mathbf{q}, \tau). \tag{I.34}$$

Fluctuations in each mode decay with a different *relaxation time* $\tau(\mathbf{q})$; $\langle h(\mathbf{q}, t)\rangle = h(\mathbf{q}, 0)\exp[-t/\tau(\mathbf{q})]$. The competition between surface tension and gravity introduces a *capillary length*, $\lambda_c \approx \sqrt{\sigma/\rho g}$. For most liquids λ_c is of the order of a few millimeters. (It is λ_c that sets the characteristic size of rain drops or ripples on the surface of a pond.) On length scales larger than λ_c (or $q \ll 1/\lambda_c$), the relaxation time saturates to $\tau_{\max} = 1/(\mu\rho g)$. On the other hand, for the soap film where gravity is not important, the characteristic time scale grows with wavelength as $\tau(q) \approx (\mu\sigma q^2)^{-1}$. The divergence of the time scale is usually described by a *dynamic exponent* z, as $\tau \propto \lambda^z$. The value of $z = 2$ for the soap film is characteristic of diffusion processes.

The connected height-height correlation functions are obtained from

$$\langle h(\mathbf{q}, t)h(\mathbf{q}', t)\rangle_c = \int_0^t d\tau_1 d\tau_2 e^{-\gamma(\mathbf{q})(t-\tau_1)-\gamma(\mathbf{q}')(t-\tau_2)} \overbrace{\langle \eta(\mathbf{q}, \tau_1)\eta(\mathbf{q}', \tau_2)\rangle}^{2D\delta(\tau_1-\tau_2)(2\pi)^d\delta^d(\mathbf{q}+\mathbf{q}')}$$

$$= (2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')\, 2D\int_0^t d\tau e^{-2\gamma(\mathbf{q})(t-\tau)} \tag{I.35}$$

$$= (2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')\frac{D}{\gamma(\mathbf{q})}\left(1 - e^{-2\gamma(\mathbf{q})t}\right)$$

$$\overset{t\to\infty}{\longrightarrow} \ (2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')\frac{D}{\mu(\rho g + \sigma q^2)}.$$

However, direct diagonalization of the Hamiltonian in eq.(I.24) gives

$$\mathcal{H} = \int \frac{d^d\mathbf{q}}{(2\pi)^d}\frac{(\rho g + \sigma q^2)}{2}|h(\mathbf{q})|^2, \tag{I.36}$$

leading to correlation functions

$$\langle h(\mathbf{q})h(\mathbf{q}')\rangle = (2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}')\frac{k_B T}{\rho g + \sigma q^2}. \tag{I.37}$$

Comparing equations (I.35) and (I.37) indicates that the long-time dynamics reproduce the correct equilibrium behavior if the fluctuation–dissipation condition, $D = k_B T \mu$, is satisfied. In fact it is possible to obtain the correct equilibrium weight with $\mathbf{q}$ dependent mobility and noise, as long as the generalized fluctuation–dissipation condition,

$$D(\mathbf{q}) = k_B T \mu(\mathbf{q}), \tag{I.38}$$

holds. Physically, correlations in noise at different locations are generated if the impact of particles from the surrounding fluid exerts a force over many surface elements.

Starting with a flat interface, $h(\mathbf{x}, t = 0) = h(\mathbf{q}, t = 0) = 0$, the profile at time t is

$$h(\mathbf{x}, t) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} e^{-i\mathbf{q}\cdot\mathbf{x}} \int_0^t d\tau e^{-\mu(\rho g + \sigma q^2)(t-\tau)} \eta(\mathbf{q}, t). \tag{I.39}$$

The average height of the surface, $\bar{H} = \int d^d\mathbf{x} \langle h(\mathbf{x}, t) \rangle / L^d$ is zero, while its overall width is defined by

$$w^2(t, L) \equiv \frac{1}{L^d} \int d^d\mathbf{x} \langle h(\mathbf{x}, t)^2 \rangle = \frac{1}{L^d} \int \frac{d^d\mathbf{q}}{(2\pi)^d} |h(\mathbf{q}, t)|^2, \tag{I.40}$$

where L is the linear size of the surface. Using eq.(I.35), we find that the width grows as

$$w^2(t, L) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{D}{\gamma(\mathbf{q})} \left(1 - e^{-2\gamma(\mathbf{q})t}\right). \tag{I.41}$$

There are a range of time scales in the problem, related to characteristic length scales through eq.(I.33). The shortest time scale, $t_{\min} \propto a^2/(\mu\sigma)$, is set by an atomic size a. The longest time scale is set by either the capillary length (λ_c) or the system size (L). For simplicity we shall focus on the soap film where the effects of gravity are negligible and $t_{\max} \propto L^2/(\mu\sigma)$. We can now identify three different ranges of behavior in eq.(I.41):

(a) For $t \ll t_{\min}$, none of the modes has relaxed since $\gamma(\mathbf{q})t \ll 1$ for all $\mathbf{q}$. Each mode grows diffusively, and

$$w^2(t, L) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{D}{\gamma(\mathbf{q})} 2\gamma(\mathbf{q})t = \frac{2Dt}{a^d}. \tag{I.42}$$

(b) For $t \gg t_{\max}$, all modes have relaxed to their equilibrium values since $\gamma(\mathbf{q})t \gg 1$ for all $\mathbf{q}$. The height fluctuations now saturate to a maximum value given by

$$w^2(t, L) = \int \frac{d^d\mathbf{q}}{(2\pi)^d} \frac{D}{\mu\sigma q^2}. \tag{I.43}$$

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The saturated value depends on the dimensionality of the surface, and in a general dimension d behaves as

$$w^2(t,L) \propto \frac{D}{\mu\sigma} \begin{cases} a^{2-d} & \text{for } d > 2, \quad (\chi = 0) \\ \ln(L/a) & \text{for } d = 2, \quad (\chi = 0^+) \\ L^{2-d} & \text{for } d < 2, \quad (\chi = \frac{2-d}{2}) \end{cases}, \tag{I.44}$$

where we have defined a *roughness exponent* χ that governs the divergence of the width with system size via $\lim_{t\to\infty} w(t,L) \propto L^\chi$. (The symbol 0^+ is used to indicate a logarithmic divergence.) The exponent of $\chi = 1/2$ in $d = 1$ indicates that the one dimensional interface fluctuates like a random walk.

(c) For $t_{\min} \ll t \ll t_{\max}$ only a fraction of the shorter length scale modes are saturated. The integrand in eq.(I.41) (for $g = 0$) is made dimensionless by setting $y = \mu\sigma q^2 t$, and

$$w^2(t,L) \propto \frac{D}{\mu\sigma} \int dq \, q^{d-3} \left(1 - e^{-2\mu\sigma q^2 t}\right)$$

$$\propto \frac{D}{\mu\sigma} \left(\frac{1}{\mu\sigma t}\right)^{\frac{d-2}{2}} \int_{t/t_{\max}}^{t/t_{\min}} dy \, y^{\frac{d-4}{2}} \left(1 - e^{-2y}\right). \tag{I.45}$$

The final integral is convergent for $d < 2$, and dominated by its upper limit for $d \geq 2$. The initial growth of the width is usually described by an exponent β defined through $\lim_{t\to 0} w(t,L) \propto t^\beta$, and

$$w^2(t,L) \propto \begin{cases} \frac{D}{\mu\sigma} a^{2-d} & \text{for } d > 2, \quad (\beta = 0) \\ \frac{D}{\mu\sigma} \ln(t/t_{\min}) & \text{for } d = 2, \quad (\beta = 0^+) \\ \frac{D}{(\mu\sigma)^{d/2}} t^{(2-d)/2} & \text{for } d < 2, \quad (\beta = (2-d)/4) \end{cases}. \tag{I.46}$$

The dependencies on space and time in the height–height correlation function can be summarized by the *dynamic scaling* form

$$\left\langle [h(\mathbf{x},t) - h(\mathbf{x}',t')]^2 \right\rangle = |\mathbf{x} - \mathbf{x}'|^{2\chi} g\left(\frac{|t-t'|}{|\mathbf{x}-\mathbf{x}'|^z}\right). \tag{I.47}$$

Since equilibrium equal time correlations only depend on $|\mathbf{x} - \mathbf{x}'|$, $\lim_{y\to 0} g(y)$ should be a constant. On the other hand correlations at the same point can only depend on time, requiring that $\lim_{y\to\infty} g(y) \propto y^{2\chi/z}$, and leading to the exponent identity $\beta = \chi/z$.

The single particle Fokker-Planck equation (I.21) can be generalized to describe the evolution of the whole probability functional, $\mathcal{P}([h(\mathbf{x})],t)$, as

$$\frac{\partial \mathcal{P}([h(\mathbf{x})],t)}{\partial t} = -\int d^d\mathbf{x} \frac{\delta}{\delta h(\mathbf{x})} \left[\frac{\partial h(\mathbf{x},t)}{\partial t} \mathcal{P} - D\frac{\delta \mathcal{P}}{\delta h(\mathbf{x})}\right]. \tag{I.48}$$

For the equilibrium Boltzmann weight

$$\mathcal{P}_{\text{eq.}}[h(\mathbf{x})] \propto \exp\left[-\frac{\mathcal{H}[h(\mathbf{x})]}{k_B T}\right] \propto \exp\left[-\frac{\sigma}{2k_B T}\int d^d\mathbf{x}(\nabla h)^2\right], \tag{I.49}$$

the functional derivative results in

$$\frac{\delta\mathcal{P}_{\text{eq.}}}{\delta h(\mathbf{x})} = -\nabla\cdot\frac{\delta\mathcal{P}_{\text{eq.}}}{\delta(\nabla h)} = \frac{\sigma}{k_B T}\left(\nabla^2 h\right)\mathcal{P}_{\text{eq.}}. \tag{I.50}$$

The total probability current,

$$J[h(\mathbf{x})] = \left[\mu\sigma\nabla^2 h - \frac{D\sigma}{k_B T}\nabla^2 h\right]\mathcal{P}_{\text{eq.}}, \tag{I.51}$$

vanishes if the fluctuation–dissipation condition, $D = \mu k_B T$, is satisfied. Once again, the Einstein equation ensures that the equilibrium weight indeed describes a steady state.

I.C Dynamics of a Conserved Height

The prescription for dynamics that leads to the Langevin equations (I.25)–(I.27), does not conserve the net height, $\int d^d\mathbf{x}\,h(\mathbf{x}, t)$. Although this quantity is on average zero, it undergoes stochastic fluctuations in time. In dealing with a volume of liquid, if particle exchange with the surrounding gas via evaporation and condensation is negligible, the total height of the liquid must be conserved, i.e.

$$\frac{d}{dt}\int d^d\mathbf{x}\,h(\mathbf{x}, t) = \int d^d\mathbf{x}\,\frac{\partial h(\mathbf{x}, t)}{\partial t} = 0. \tag{I.52}$$

How can we construct a dynamical equation that satisfies eq.(I.52)? The integral clearly vanishes if the integrand is a total divergence, i.e.

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = -\nabla\cdot\vec{j} + \eta(\mathbf{x}, t). \tag{I.53}$$

The noise itself must be a total divergence, $\eta = -\nabla\cdot\vec{\sigma}$, and hence in Fourier space,

$$\langle\eta(\mathbf{q}, t)\rangle = 0, \quad\text{and}\quad \langle\eta(\mathbf{q}, t)\eta(\mathbf{q}', t')\rangle = 2Dq^2\delta(t - t')(2\pi)^d\delta^d(\mathbf{q} + \mathbf{q}'). \tag{I.54}$$

We can now take advantage of the generalized Einstein relation in eq.(I.38) to ensure the correct equilibrium distribution by setting,

$$\vec{j} = \mu\nabla\left(-\frac{\delta\mathcal{H}}{\delta h(\mathbf{x})}\right). \tag{I.55}$$

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A similar question arises in dealing with the dynamics of a binary mixture undergoing phase separation. The order parameter $\phi(\mathbf{x})$, the difference between the densities of the two species, may now be conserved. The standard procedure and terminology for such situations is provided by Hohenberg and Halperin. Given a Hamiltonian $\mathcal{H}[\phi]$, the Langevin dynamics of the field $\phi(\mathbf{x}, t)$ is constructed from

$$\frac{\partial \phi(\mathbf{x}, t)}{\partial t} = -\hat{\mu}\left(\frac{\delta \mathcal{H}}{\delta \phi(\mathbf{x})}\right) + \eta(\mathbf{x}, t), \tag{I.56}$$

with

$$\langle \eta(\mathbf{x}, t) \rangle = 0, \quad \text{and} \quad \langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle = 2\hat{D}\left(\delta^d(\mathbf{x} - \mathbf{x}')\delta(t - t')\right). \tag{I.57}$$

In **model A** dynamics the field ϕ is *not conserved*, and $\hat{\mu} = \mu$ and $\hat{D} = D$ are constants. In **model B** dynamics the field ϕ is *conserved*, and $\hat{\mu} = -\mu\nabla^2$ and $\hat{D} = -D\nabla^2$.

Let us now go back to the example of a conserved volume of fluid whose surface fluctuations are subject to the Hamiltonian (I.24). The equation of motion constructed from model B dynamics is[†]

$$\frac{\partial h(\mathbf{x}, t)}{\partial t} = \mu\rho g\nabla^2 h - \mu\sigma\nabla^4 h + \eta(\mathbf{x}, t). \tag{I.58}$$

The evolution of each Fourier mode is given by

$$\frac{\partial h(\mathbf{q}, t)}{\partial t} = -\mu q^2(\rho g + \sigma q^2)h(\mathbf{q}, t) + \eta(\mathbf{q}, t) \equiv -\frac{h(\mathbf{q}, t)}{\tau(\mathbf{q})} + \eta(\mathbf{q}, t). \tag{I.59}$$

Because of the constraints imposed by the conservation law, the relaxation of the surface is more difficult, and slower. The relaxation times diverge even in the presence of gravity, and depending on wavelength we can define dynamic exponents z, via

$$\tau(\mathbf{q}) = \frac{1}{\mu q^2(\rho g + \sigma q^2)} \approx \begin{cases} q^{-2} & \text{for } q \ll \lambda_c^{-1} & (z = 2) \\ q^{-4} & \text{for } q \gg \lambda_c^{-1} & (z = 4) \end{cases}. \tag{I.60}$$

The equilibrium behavior is unchanged, and

$$\lim_{t\to\infty}\left\langle |h(\mathbf{q}, t)|^2 \right\rangle = \frac{Dq^2}{\mu q^2(\rho g + \sigma q^2)} = \frac{D}{\mu(\rho g + \sigma q^2)}, \tag{I.61}$$

as before. Thus the same static behavior can be achieved by different dynamics. The static exponents (e.g. χ) are determined by the equilibrium (stationary) state and are unchanged, while the dynamic exponents may be different. As a result, dynamical critical phenomena involve many more universality classes than the corresponding static ones.

[†] While model A dynamics provides a reasonably accurate description of the relaxation of a soap film, model B dynamics is not particularly useful for describing surface waves. As conservation of momentum in the fluid is an important constraint not included here, the following results are merely intended as an illustration.