Theory and Design of Integrated Optical Isolators
and Broadband Couplers Using Fresnel Zone Plates
by
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Abstract
This thesis is divided into two main sections: the first containing the analysis of the broadband vertical coupler, and the second involving the theory and design of the integrated optical isolators. In the first part we propose, theoretically investigate, and numerically demonstrate a compact (less than 10μm) broadband (more than 300nm) fiber-chip vertical coupler. The structure utilizes a Fresnel lens, or more advanced integrated optics, placed above a short, ridge and deep etched, vertical coupler in a Si waveguide. This optics is placed in order to match the radiating fields to the fiber mode. We use semivectorial simulations with a simple stochastic optimization to design a good integrated optics without cylindrical constrains. Three-dimensional Finite-Difference Time-Domain (FDTD) simulations reveal ~50% fiber coupling efficiency and a bandwidth of 200nm. In the second part we propose, theoretically investigate, and numerically demonstrate six designs of integrated optical isolators. We first derive analytically the value of the off-diagonal gyrotropic permittivity tensor element, $\varepsilon_g$. We then use this value to calculate a non-reciprocal phase shift in a Manganese, and a N/P doped silicon waveguide using analytic, perturbation, and a novel mode numeric approach. Finally, using the obtained magnitudes of the nonreciprocal phase shifts six integrated optical isolator designs are proposed.
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Chapter 1

Compact Broadband Vertical Coupler

1.1 Introduction

Many vertical fiber-chip coupler designs have been proposed and demonstrated in the last decade. It can be generally noted that these couplers are based on a grid coupling directly between the fiber and the on chip waveguide. Therefore, a constraint is applied on this grid. It must have at least the size of the fiber mode (~ 10μm) in order to support a similar field distribution and maximize the coupling efficiency. This dimension constraint, as we show here, fundamentally limits the coupler bandwidth to about 60nm FWHM, which is at the same order as the fabrication errors for the center wavelength.

The design proposed and numerically demonstrated in this thesis includes a vertical coupler, much smaller than the fiber mode dimensions, with integrated optics above it for coupling to a single mode optical fiber. Finite-Difference Time-Domain (FDTD) verifies this design to have a bandwidth of 300nm. In the telecom C-band, such a broadband coupler enables on chip Wavelength Division Multiplexing (WDM). Moreover, the size of this coupler takes an area smaller than the fiber tip placed above it.

This work is the first to separate between the coupling of the waveguide mode
(waveguide or vertical coupler) and the coupling of the fiber mode (fiber coupler), and hence the first to enable additional design flexibility of the vertical fiber-chip coupler. We show here the design and numerical evaluation of a method to couple a short vertical coupler with a fiber mode. The relation between the emitter length and the bandwidth will also be discussed in this thesis.

Our first concern, when designing a fiber coupler, is creating a mode, which is matched as closely possible to the fiber mode. To avoid the narrow band problem described above, and still be able to couple into a fiber mode, we propose here a split between the two problems. Step one, which is out of the scope of this thesis, is designing a short (low \( L_\# \)) vertical coupler and step two is coupling this vertical coupler to a fiber mode. In order to do so we simply apply some optics and enable propagation between the modes. Without loss of generality we can look at a waveguide to fiber coupling. By letting the short emitter field expand in the right amount of propagation, and then correct its phase using integrated Fresnel lens Figure 1-1a, we can practically match in a good agreement every short emitter field distribution to that of a fiber mode.

Further improvement to this design can be made using negative Fresnel lens right above the vertical coupler to cause the field shape to expand in a wider angle, achieving the fiber mode shape at a lower height. Figure 1-1b shows how this enables shorter distance to the original top lens plus fiber tip and eliminates the need of a thick oxide.

Figure 1-1: Two fiber couplers with (right) and without (left) the negative Fresnel lens for rapid expansion of the emitted field.
layer above the waveguide coupler.

To design the simplest version of the figure above, after choosing a waveguide vertical coupler we run a 3D FDTD simulation of the waveguide and emitter alone looking only on the amplitude distribution of the emitted light at different heights above the waveguide and emitter. Finding the height which gives the amplitude distribution best fit to the fiber mode field distribution gives us the right height to place the phase correcting lens. This lens role is to collimate the light in order to make it similar to the fiber mode, and it is being constructed using Fresnel lens. First curve in Fig 3 shows the power coupling efficiency of 86.3\% the fiber with lens and fiber tip height of 6.1\mu m above the waveguide and vertical coupler.

### 1.2 Vertical Coupler

For the purpose of analyzing a general vertical coupler, we consider a waveguide feeding a perturbation region with length L, and we monitor the wave emitting from that perturbation in a given direction with respect to the surface normal. It can be assumed, for simplicity, that the azimuth angle for the maximum coupling is set for
emission in the plane containing the waveguide direction and the chip surface normal. The emission is usually set to the first order refraction of the perturbation, so if the perturbation is designed for wavelength \( \lambda \) emitting in some angle \( \theta_{\text{max}} \), the phase of emission from lattice point 1 in the perturbation will be

\[
\phi(x_i) = 2\pi \left( ml + \frac{x_i \sin(\theta_{\text{max}})}{\lambda_0} \right),
\]

where \( m \) is the refraction order, \( x_i \) is the perturbation period position. For general wavelength the phase is multiplied by so that it can be written as

\[
\phi(x_i, \lambda) = \frac{2\pi}{\lambda} (ml\lambda_0 + x_i \sin(\theta_{\text{max}}))
\]

and the far field in the \( \theta_{\text{max}} \) direction is then proportional to the sum

\[
E_{FF} \propto \sum_l A(x_i) \exp \left( j(\phi(x_i, \lambda) - \frac{2\pi x_n \sin(\theta_{\text{max}})}{\lambda}) \right)
\]

\[
= \sum_{l=1}^{L_{\#}} A(x_i) \exp \left( j\frac{2\pi ml\lambda_0}{\lambda} \right).
\]

\( L_{\#} \) is the number of periods in the perturbation and \( A(x_i) \) is the amplitude of radiating field in the perturbations period \( l \). Assuming for simplicity that \( A(x_i) \) is constant (same for all \( x_i \)s), and \( m = 1 \), we compare the normalized intensity to \( \frac{1}{2} \), and numerically find the Full Width Half Maximum (FWHM) to equal.

Hence, for a common vertical coupler built with 20 to 25 periods with a central wavelength 1550nm, the FWHM is inherently limited to 60nm. A shorter coupler of four periods will enable FWHM > 300nm.

The vertical emitter size is \( 3\mu m \times 4.7\mu m \), with a grating period of 522nm.
1.3 Fresnel Zone Plate

Now we design the diverging lenses. As described above, these lenses allow for a shorter device length compared to their absence because they accelerate the natural divergence of the emitted mode. First, we approached this problem using analytic methods to obtain a baseline solution to work off of, then created a more accurate solution using numeric methods as described below.
1.3.1 Analytic Design

To design the diverging lens proposed here we first assume the emitter to be a point source emitter, for calculation simplicity. Next we define a target height of cladding above the waveguide and first lens. In addition, we assume a collimated beam at the output of the waveguide vertical coupler with a typical diameter $d$ corresponding to the vertical coupler length and width. The negative focal length of the bottom Fresnel lens will then be

$$f_1 \approx -\frac{h}{D/d - 1},$$

where $h$ is the second lens desired height above the first one and $D$ is the fiber mode diameter. The second lens focal length is then $f_2 = h + f_1$. If the vertical coupler length and width are not the same, we can add astigmatism to the first lens and calculate $f_{1\perp}$ and $f_{1\parallel}$ using the two dimensions $d_{\perp}$ and $d_{\parallel}$ separately. The lens will then have elliptical line curves when top viewed. We assume in this part a near-field approximation for the waveguide coupler to first lens and for the second lens to fiber tip. Although the assumptions given here are rough, $f_1$ and $f_2$ can be refined for better overlap to the fiber mode.

![Analytical diverging Fresnel Lens design](image)

Figure 1-5: Analytical diverging Fresnel Lens design

\[
\begin{align*}
    f^2 + r_n^2 &= \left(f + \frac{n\lambda}{2}\right)^2 \quad (9.8) \\
    r_n^2 &= n\lambda f + \frac{n^2\lambda^2}{4} \quad (9.9) \\
    r_n &\approx \sqrt{n\lambda f} \quad (9.10)
\end{align*}
\]
1.3.2 Numerical Design

Now that we have a baseline analytical design we can now optimize this design using numerical methods. This will be implemented by optimizing this analytical design by using stochastic optimization methods. This optimization step uses semi-vectorial simulation methods for speed enhancement. We then run the resultant lens mask through a low pass filter to avoid small points and sharp corners, and test it with a full vectorial FDTD simulation for verification.

In addition, by applying stochastic optimization when designing the lower lens, we can achieve better amplitude shape at the top lens corresponding to the fiber mode shape. In fact, it is not limited to any circular symmetry, and can take any deposited 2D shape of high indexed material in the lens plane.

Figure 1-6 shows the fabricated design variations, in order to compare simulation vs measurement for each step in the design processes.

<table>
<thead>
<tr>
<th>Design #</th>
<th>Lens_11</th>
<th>Lens_12</th>
<th>Lens_2</th>
<th>Simulation</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>-</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>I</td>
<td>F</td>
<td>-</td>
<td>F'</td>
<td>C</td>
</tr>
<tr>
<td>II</td>
<td>F</td>
<td>F</td>
<td>F'</td>
<td>C</td>
</tr>
<tr>
<td>III</td>
<td>-</td>
<td>-</td>
<td>F'</td>
<td>C</td>
</tr>
<tr>
<td>IV</td>
<td>GAO</td>
<td>-</td>
<td>F'</td>
<td>SV</td>
</tr>
<tr>
<td>V</td>
<td>GAO</td>
<td>GAO</td>
<td>F'</td>
<td>SV+FDTD</td>
</tr>
<tr>
<td>VI</td>
<td>GAO</td>
<td>GAO'</td>
<td>F'</td>
<td>SV</td>
</tr>
<tr>
<td>VII</td>
<td>Use VI</td>
<td>Use VI</td>
<td>PC</td>
<td>SV+FDTD</td>
</tr>
</tbody>
</table>

Figure 1-6: Fabricated Design Variations. F stands for analytic Fresnel Lens design. GAO stands for Genetic Algorithm Optimization. SV stands for semi-vectorial simulation, and FDTD stands for Finite Difference Time Domain simulation.

Fresnel Propagation Integral

The semi-vectorial Fresnel Propagation integral is used to speed up computational efficiency, show in Figure 1-7.

In the ideal case we would only use FDTD to calculate coupling efficiency, but
\[ U(x, y, z) = \frac{z}{i\lambda} \int \int U(\xi, \eta) \frac{e^{ikr_{01}}}{r_{01}^2} d\xi d\eta \]

Figure 1-7: Semi-vectorial Fresnel Propagation integral and Phase Transformation scheme.

the computational efficiency of today’s computers does not allow this to be a viable option. And since the propagation of the mode once it leave the vertical coupler is paraxial, this approximation is a reasonable substitution.

As shown in Figure 1-8, the simulation is carried out in a series of steps. Firstly, the amplitude and phase from a FDTD flux monitor placed directly above the vertical emitter is used as a starting field for the Fresnel Propagation Integral. Then using the integral we propagate it to the first diverging lens a distance \( L_1 \). Next we apply a phase shift to the propagated field, directly proportional to the dielectric arrangement of the first diverging lens. This process is carried out until we reach the optical fiber, wherein the field is saved. The outputted field is then used in a mode-overlap integral with the optical fiber mode to determine it’s coupling efficiency, which is later used as the optimization parameter in the genetic algorithm.

Figure 1-8: Mathematical representation of the Fresnel Propagation Integral.
Genetic Algorithm Optimization

Now that we have a computationally efficient algorithm for simulating the emission and coupling of the optical mode, we can now iterate this process to obtain local coupling maximums between the emitted mode and the fiber mode. Each diverging lens was created using a series of, $10\text{nm}$ spread randomly across each lens’ area trial pixels, will a minimum of $10\text{nm}$ feature size due to fabrication specifications. If the addition contributed to higher mode coupling then the change was accepted and subsequent pixels surrounding the chosen pixel is also tested. The genetic algorithm is shown in Figure 1-9.

![Diagrammatic representation of the genetic algorithm.](image)

**Figure 1-9:** Diagrammatic representation of the genetic algorithm.

After each generation the amplitude coupling efficiency either improves or stays the same. After running this overnight on a large multiprocessor machine, we can obtain a solution which corresponds to a local coupling maximum. After running this with different starting conditions, then simulating each local solution using FDTD we can obtain an accurate approximation to the global solution.

In Figure 1-10 we see such a solution which corresponds to changing both the first and second diverging lens in identically.

Figure 1-11 is also a solution where the first and second diverging lenses are optimized independently, allowing for an even greater coupling efficiency. This design
Figure 1-10: Local solution of diverging lens, where the first and second lens are identically optimized.

...turns out to yield the highest coupling efficiency and is used as the result in this thesis.

In Figure 1-12 we see a comparison between the mode amplitude of a optical fiber mode, compared to that of the emitted mode.

**Binary Phase Correction**

Now that we have a mode which the amplitudes are now of the same size, the last step- as described above- is to correct the phase of the emitted mode, a curved phase front, so that it’s optimized to match that of the fiber mode, a flat phase front.

For this a binary compensation algorithm is used. In the top most dielectric we can either decide for a 400nm Silicon Nitride dielectric to be present or not. And so we have developed a method for determining this. In the ideal case you would add
just the right amount of dielectric to phase shift the most “out of phase” parts of the mode so that they exactly match the parts of the mode “in phase”. But we cannot
do this, because the dielectric cannot be applied in an analogue fashion, but rather in a digital fashion.

Thus, as shown in Figure 1-13 we first plot the phase distribution of the emitted wave on a polar plot. We then find the global maximum, weighted by it’s standard deviation (fit with a Gaussian), and define an axis based on that maximum (shown as the solid red axis). Next choose the amount of desired phase shift we need for our application, $\alpha$, and create another axis (shown as the dashed blue line) that is rotated by an angle of $\alpha/2$ from the original (solid red) axis. Then any phase that is on the opposite side of the dashed blue line gets phase shifted, that is at every location that corresponds to a phase that is on the opposite side of the second axis goes through the dielectric material located at the third lens, where as the locations of the mode that is on the right side of the second axis pass through the parts of the third lens where the dielectric is absent.

In Figure 1-14 we see the result of this phase correction. The image on the left is the uncorrected phase that emerges form the emitter and diverging lenses. The image on the right corresponds to the corrected phase that passes through the third dielectric lens. Note that the color scheme for the z-axis on both images is the same, we see that the image on the right is but a small domain of the uncorrected phase.

Figure 1-12: Comparison of the mode amplitudes of a fiber mode and the mode emitted from Design #VI
Figure 1-13: Graphical representation of the Binary Phase Correction algorithm.

Figure 1-14: Comparison of the uncorrected (does not pass through third lens) vs. the corrected phase (passed through third lens)

1.3.3 FDTD Simulation

Finally, after we’ve run the stochastic optimization as a semivectorial simulation for the large number of iterations involved in it, we verified it in a FDTD simulation to prove its efficiency. Figure 1-15 shows the coupling efficiency for design #VI.

1.4 Conclusions

In conclusion, we present a fiber-chip vertical coupler design that, by separating the waveguide coupling and the fiber coupling, enables broadband and small footprint. Using three-dimensional FDTD simulations, we show the coupling efficiency of the
Figure 1-15: Bandwidth normalized to the power up coupled from the emitter fiber coupler to reach 86.3% and an overall coupling bandwidth of 200nm, with a total efficiency of 53%. Furthermore, the proposed structure uses realistic dimensions and commonly available materials. While efficient fiber-chip vertical couplers have been demonstrated, to our knowledge this is the first proposed and numerically verified compact coupler with a bandwidth of 200nm and the first proposed structure of its kind.
Chapter 2

Integrated Optical Isolators

2.1 Introduction

Microphotonics is surpassing electronics in terms of bandwidth distance product and power consumption for telecom and datacom applications. As the bandwidth increases photonics will replace electronics.

Figure 2-1: Diagram shows an optical computing device presented at Intel Developer Forum
There is one problem however, photonics has one key feature missing: the isolator-the photonics equivalent of the diode in electronics. Isolators are important for many optical oscillation phenomena, for example in a laser, back reflection into the cavity corrupts the temporal amplitude. Back reflection also distorts spectral characteristics of integrated systems.

So called, bulk isolators are common devices used in labs across the world. Bulk isolators work by taking advantage of the non-reciprocal nature of the Faraday effect. One could harness the Faraday rotation to create integrated optical isolators, however high index contrast silicon waveguides are highly birefringent and thus in general cannot support these circularly polarized waves.

This work tries to address and solve the problem of creating an integrated isolator, by proposing various designs and theoretical methods. The main theme will be to use the so called nonreciprocal phase shift, between forward and backward traveling modes in a Manganese doped Silicon waveguide. This allows us to create an isolator which does not suffer from the problems that the previously designed integrated Faraday rotation isolators suffer from.

2.2 Theory

When a Silicon waveguide is doped with Manganese impurities, the waveguide becomes intrinsic magneto-optic material (rather than associated with the presence of ferromagnetic clusters or a foreign ferromagnetic phase). When a Magnetic field field is applied along the transverse (\(\hat{z}\)) direction of the waveguide, the doped Silicon waveguide becomes gyrotropic in nature, with it’s permittivity matrix matrix given by

\[
\tilde{\epsilon} = \begin{bmatrix}
\epsilon & i\epsilon_y & 0 \\
-i\epsilon_y & \epsilon & 0 \\
0 & 0 & \epsilon_z
\end{bmatrix}
\]  

as will be derived in the next section. These \(\epsilon_{x,y}\) and \(\epsilon_{y,z}\) elements couple the
$E_x$ and $E_y$ fields which leads to the Faraday rotation phenomenon in the transverse plane. When you, however, apply an external magnetic field in the transverse plane ($\hat{x}$ or $\hat{y}$) you get a permittivity tensor of one of the following forms:

\[
\hat{\varepsilon} = \begin{bmatrix}
\varepsilon & 0 & i\varepsilon_g \\
0 & \varepsilon & 0 \\
-i\varepsilon_g & 0 & \varepsilon_z \\
\end{bmatrix} \quad (2.2)
\]

\[
\hat{\varepsilon} = \begin{bmatrix}
\varepsilon & 0 & 0 \\
0 & \varepsilon & i\varepsilon_g \\
0 & -i\varepsilon_g & \varepsilon_z \\
\end{bmatrix} \quad (2.3)
\]

This is the nature of the so called nonreciprocal phase shift, which couples the $E_x$ and $E_z$ or the $E_y$ and $E_z$ fields as opposed to the $E_x$ and $E_y$ fields as in the case of the Faraday rotation. This phenomenon is summarized in Figure 2-2.

Figure 2-2: Matrix representations corresponding to externally applied magnetic field direction.

\[2.3\quad \varepsilon_g \text{ Calculation}\]

For the design of integrated optical isolators it is first important to get an accurate estimate of the off-diagonal elements of the material’s permittivity tensor. Without
the knowledge of this value, the rest of the calculations are meaningless. This number forms the basis of the rest of the designs.

In Manganese doped Silicon when an external magnetic field is applied the material becomes gyrotropic. When the DC magnetic field is applied along the transverse ($\hat{z}$) direction the permittivity tensor of the doped Silicon material takes the form

$$
\hat{c} = \begin{bmatrix}
\epsilon & i\epsilon_g & 0 \\
-i\epsilon_g & \epsilon & 0 \\
0 & 0 & \epsilon_z
\end{bmatrix}
$$

That is using the Lorentz Force Law and Newton's second law, we can setup a force balance equation:

$$
m \frac{dv}{dt} = q(E + v \times B).
$$

Assuming time-harmonic excitation by the external magnetic field, and assuming the wave frequency is much greater than the electron collision frequency, the collision effect can be neglected and the free electron plasma can be considered to exist in a lossless medium. We then see that

$$
-im\omega v = q(E + v \times B_0)
$$

Then,

$$
-i\omega v = \frac{q}{m}E + v \times \omega_c
$$

where $\omega_c = qB_0/m$. Then multiplying Equation 2.7 by $\omega_c$ we obtain that

$$
-i\omega v \cdot \omega_c = \frac{q}{m}E \cdot \omega_c
$$

Now we cross-multiply Equation 2.8 by $\omega_c$ to obtain
\[-i\omega \mathbf{v} \times \omega_c = \frac{q}{m} \mathbf{E} \times \omega_c + (\mathbf{v} \times \omega_c) \times \omega_c \quad (2.9)\]
\[= -\frac{q}{m} \omega_c \times \mathbf{E} + \omega_c (\mathbf{v} \cdot \omega_c) - \omega_c^2 \mathbf{v} \quad (2.10)\]
\[= -\frac{q}{m} \omega_c \times \mathbf{E} + \frac{iq}{\omega m} \omega_c (\omega_c \cdot \mathbf{E}) - \omega_c^2 \mathbf{v} \quad (2.11)\]

Now substituting Equation 2.11 into Equation 2.7 we see that

\[
\mathbf{J} = N q \mathbf{v} \quad (2.12)
\]
\[
= \frac{N q}{\omega^2 - \omega_c^2} \left[ -\frac{q}{m} \omega_c \times \mathbf{E} + \frac{iq}{\omega m} \omega_c (\omega_c \cdot \mathbf{E}) - \frac{iq}{m} \mathbf{E} \right] \quad (2.13)
\]
\[
= -\frac{i\omega \epsilon_0 \omega_p}{\omega^2 - \omega_c^2} \left[ -\frac{i}{\omega} \omega_c \times \mathbf{E} + \frac{\omega_c}{\omega^2} (\omega_c \cdot \mathbf{E}) - \mathbf{E} \right] \quad (2.14)
\]

Then from Ampere’s Law, we know that

\[
\nabla \times \mathbf{H} = -i\omega \epsilon_0 \mathbf{E} + \mathbf{J} \quad (2.15)
\]
\[
= -i\omega \left[ i\epsilon \mathbf{v} \times \mathbf{E} + \epsilon \mathbf{E} + \frac{\omega_p^2 \omega_c^2 \epsilon_0}{\omega^2 (\omega^2 - \omega_c^2)} \mathbf{v} \cdot \mathbf{E} \right] \quad (2.16)
\]
\[
= -i\omega \epsilon \mathbf{E} \quad (2.17)
\]

Thus we see that,

\[
\epsilon_g = \epsilon_0 \left[ \frac{-\omega_p \omega_c}{\omega (\omega^2 - \omega_c^2)} \right] \quad (2.18)
\]

where \(\omega_c = q B_0 / m\), and \(\omega_p = q \sqrt{N / m \epsilon_0}\). Thus since we know all of the constants on the L.H.S of Equation 2.18, we can obtain a value for \(\epsilon_g\) based on the strength of \(B_0\).

However, referring to the asymptote in Figure 2-3 a high \(\epsilon_g\) near those asymptotes is unusable for real devices.
2.4 Non-Reciprocal Phase Shift Calculation

Now that we have a calculated number for $\epsilon_g$, we are now in place to calculate what our nonreciprocal phase shift will be. We will tackle this using three different methods of increasing accuracy to learn more about the qualitative and quantitative aspects of the non-reciprocal phase shift for design purposes.

2.4.1 Effective Index Calculation

We solve the exact Eigenvalue Equation for the Slab Waveguide

By solving the dispersion relation, one can obtain the propagation constant.

Each region of the Ridge Waveguide is treated as a Slab Waveguide, obtaining a 3D waveguide.

Start with Maxwell's Eqs, with tensor permittivity

$$\nabla \times \mathbf{E} = -j\omega \mu_0 \mathbf{H}$$

$$\nabla \times \mathbf{H} = -j\omega \varepsilon_0 \mathbf{E}$$

(2.19)  
(2.20)
Solve for TM mode, by assuming

$$H_y = H_z = E_y = 0$$  \hspace{1cm} (2.21)

Derive vector wave eqn. by taking curl of Faraday's Law, and inserting Amperes Law

$$\nabla (\nabla \cdot \mathbf{E}^{II}) - \nabla^2 \mathbf{E}^{II} = \omega^2 \mu_0 \varepsilon_0 \mathbf{E}^{II}$$  \hspace{1cm} (2.22)

Get dispersion relation, by assuming $exp(jk^{II}x)$ dependence

$$(k^{II})^2 = \omega^2 \mu_0 \varepsilon_{eff} - \beta^2$$  \hspace{1cm} (2.23)

We then use trial wave functions of the form
Table 2.1: Nonreciprocal phase shift calculations using the analytic calculation.

<table>
<thead>
<tr>
<th>Waveguide Width (μm)</th>
<th>δβ (cm⁻¹)</th>
<th>Device Length (cm)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.045</td>
<td>0.716</td>
<td>4.4</td>
</tr>
<tr>
<td>0.1</td>
<td>0.290</td>
<td>10.8</td>
</tr>
<tr>
<td>0.12</td>
<td>0.202</td>
<td>15.6</td>
</tr>
</tbody>
</table>

\[
H_x^I = A e^{K_I y} e^{-j\beta z} \tag{2.24}
\]

\[
H_x^{III} = A e^{-K^{III} y} e^{-j\beta z} \tag{2.25}
\]

\[
H_x^{II} = [B \cos(k^{II} y) + C \sin(k^{II} y)] e^{-j\beta z} \tag{2.26}
\]

Solve for E fields from Amperes Law, evoke boundary conditions to get the determinantal equation by which we can calculate our non-reciprocal phase shift.

\[
0 = \tan(k^{II} \delta) \left[ \frac{1}{\epsilon_{eff}^2} \left( (k^{II})^2 - \frac{\beta^2}{\theta^2} \right) + j \frac{\beta}{\theta} \frac{1}{\epsilon_{eff}} \left( \frac{K^I}{\epsilon^I} - \frac{K^{III}}{\epsilon^{III}} \right) - \frac{K^I K^{III}}{\epsilon^I \epsilon^{III}} \right] \tag{2.27}
\]

\[
\frac{k^{II}}{\epsilon_{eff}} \left( \frac{K^I}{\epsilon^I} - \frac{K^{III}}{\epsilon^{III}} \right) \tag{2.28}
\]

where

\[
\theta = \frac{\epsilon_s}{j \epsilon_g} \tag{2.29}
\]

We see then that there are two solutions to Equation 2.27 for β traveling in opposite directions, which leads to a non-reciprocal phase shift.

### 2.4.2 Perturbation Method Calculation

First, we will make the standard assumption that the waveguide is homogeneous in the z-direction, and that the material is isotropic and lossless. The propagating modes, as before, can be described as

\[
E(x, y) \quad H(x, y) \quad e^{i(\beta z - \omega t)} \tag{2.30}
\]

where \( E \) and \( H \) are the components of the mode amplitude profile, \( \beta \) denotes the
Figure 2-5: Choice of waveguide width is tradeoff between \( \delta \beta \) and mode confinement.

Propagation constant, \( \omega = ck_0 \), is the angular frequency, and \( k_0 = \frac{2\pi}{\lambda} \) is the vacuum wave number with \( \lambda \) denoting the vacuum wavelength. Maxwell's Equations yield the following differential equations for the transverse Electric and Magnetic fields:

\[
\begin{bmatrix}
  k_0^2\epsilon + \partial_x \frac{1}{\epsilon} \partial_x \epsilon + \partial_y^2 \\
  \partial_y \frac{1}{\epsilon} \partial_x \epsilon - \partial_y \partial_x \\
  \partial_x \frac{1}{\epsilon} \partial_y \epsilon + \partial_y \frac{1}{\epsilon} \partial_y \epsilon + \partial_z^2
\end{bmatrix}
\begin{bmatrix}
  E_x \\
  E_y \\
  H_x \\
  H_y
\end{bmatrix}
= \beta^2
\begin{bmatrix}
  E_x \\
  E_y \\
  H_x \\
  H_y
\end{bmatrix}
\tag{2.31}
\]

The longitudinal \( z \) components are then determined by Maxwell's divergence equations:

\[
\nabla \cdot \epsilon \mathbf{E} = 0 
\tag{2.33}
\]
\[
\nabla \cdot \mu \mathbf{H} = 0 
\tag{2.34}
\]
where $\epsilon = n_0^2$ denotes the permittivity, $n_0$ is the isotropic refractive index, and $\mu$ is the permeability. At optical frequencies, $\mu \approx 1$. It turns out that Equations 2.31 and 2.32 are equivalent.

In fact, they completely describe the full vectorial modal fields $E$ and $H$. However, we usually see that one field component usually dominates and so if the non-dominant components are neglected, one can derive the simplified, so called, semi vectorial equations for the so called quasi-TE and quasi-TM modes:

\[
\text{quasi-TE: } E = \begin{bmatrix} 0 \\ E_y \\ E_z \end{bmatrix}, \quad H = \begin{bmatrix} H_x \\ H_y \\ H_z \end{bmatrix},
\]

(2.35)

corresponding to the wave-equation:

\[
\left( k_0^2 \epsilon + \partial_z^2 + \partial_y \frac{1}{\epsilon} \partial_y \epsilon \right) E_y = \beta_{TE}^2 E_y,
\]

(2.36)

\[
\text{quasi-TM: } E = \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}, \quad H = \begin{bmatrix} 0 \\ H_y \\ H_z \end{bmatrix}
\]

(2.37)

corresponding to the wave-equation:

\[
\left( k_0^2 \epsilon + \partial_z^2 + \partial_y \frac{1}{\epsilon} \partial_y \epsilon \right) H_y = \beta_{TM}^2 H_y
\]

(2.38)

For planar waveguides the partial derivatives with respect to $y$ vanish, yielding analytically solvable equations.

As described in the introduction, the magneto-optical properties of the material are described by the non-diagonal components of the permittivity tensor:
\[ \hat{\varepsilon} = n_0^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + K \begin{bmatrix} 0 & M_z & -M_y \\ -M_z & 0 & M_x \\ M_y & -M_x & 0 \end{bmatrix} \]

\[ = n_0^2 \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} + \Delta \hat{\varepsilon} \quad (2.39) \]

Where \( M_j, j = x, y, z \) denotes the vector components of the material’s magnetization. \( K \) is a complex material parameter \( K = K' + jK'' \), where \( K'' \) is related to the Faraday rotation by

\[ \Theta_{F,sat} = -k_0 \frac{K'' M_z}{2n_0}. \quad (2.41) \]

The real part \( K' \) determines the Faraday ellipticity, which is neglected in this thesis.

However, as stated earlier because of the inherent birefringence of silicon waveguides, we will be harnessing the nonreciprocal phase shift effect instead of the Faraday rotation effect. Thus, we see that the elements of the gyrotrropic part \( \hat{\varepsilon} \) of the permittivity tensor are small compared with \( n_0^2 \) and can thus be treated by perturbation theory. The component \( M_z \) which is parallel to the waveguide axis, gives rise to TE TM mode coupling and thus to mode conversion, which is similar to Faraday rotation in bulk media; this is discussed more in the Future Work section. The components \( M_x \) and \( M_y \), which are perpendicular to that axis, induce a change \( \delta \beta \) of the propagation constant \( \beta \), which depends on the propagation direction, forward or backward:

\[ \beta_{\text{forward}} = \beta + \delta \beta \quad (2.42) \]

\[ \beta_{\text{backward}} = \beta - \delta \beta \quad (2.43) \]
resulting in a difference $\Delta \beta = 2\delta \beta$ between the forward and backward propagation constants; $\beta$ denotes the unperturbed propagation constant. This nonreciprocal phase shift does not exist in the case of light propagation in bulk materials because the effect can be viewed as being immediately related to material discontinuities.

Similar to how this is treated in quantum-mechanics, perturbation theory for $\delta \beta$ yields

$$\delta \beta = \frac{\omega \varepsilon_0}{N} \int \int \mathbf{E}^* \Delta \varepsilon \mathbf{E} dxdy$$

(2.44)

normalized by the power flow in the $z$ direction:

$$N = \left( \int \int \mathbf{E} \times \mathbf{H}^* + \mathbf{E}^* \times \mathbf{H} \right) dxdy.$$  

(2.45)

We also see that the components $M_x$ and $M_y$ also cause TE-TM mode coupling that is usually negligible compared with the coupling produced by $M_z$. Now we can use the semi vectorial approximation for the quasi-TM and quasi-TE modes. Assume $\mathbf{M}$ is parallel to the $x$-axis, thus we see that

$$\mathbf{E}^* \Delta \varepsilon \mathbf{E} = \mathbf{E}^* \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & i\varepsilon_g \\ 0 & -i\varepsilon_g & 0 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

(2.46)

$$= \mathbf{E}^* \begin{bmatrix} 0 \\ i\varepsilon_g E_y \\ -i\varepsilon_g E_z \end{bmatrix}$$

(2.47)

$$= i\varepsilon_g E_z E_y^* - i\varepsilon_g E_x^* E_y$$

(2.48)

Now recall from our divergence relations Equation 2.33,
\[ \nabla \cdot \mathbf{E} = 0 \]  
\[ \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0 \]  

Since for the quasi-TE mode \( E_z = 0 \) this implies

\[ \frac{\partial E_y}{\partial y} = -\frac{\partial E_z}{\partial z} = \frac{\partial}{\partial z} (E \cdot e^{-i\beta z}) \]

\[ = i\beta_T E_z \]  

It then follows that

\[ E_z = \frac{-i}{\beta} \frac{\partial E_y}{\partial y} \]  

Now plugging Equation 2.53 into 2.50 yields

\[ \mathbf{E}^* \Delta \mathbf{E} = i\epsilon_g E_y^* \frac{i}{\beta} \partial_y E_y - i\epsilon_g \frac{i}{\beta} \partial_y E_y \]

\[ = \frac{2\epsilon_g}{\beta} E_y \partial_y E_y \]  

Now finally substituting Equation 2.55 into Equation 2.44 yields,

\[ \delta \beta = \frac{\omega\epsilon_0}{2N} \int \int \frac{2\epsilon_g}{\beta} E_y \partial_y E_y dxdy \]

simplifying we find that, the nonreciprocal phase shift for the TE mode takes the form of

\[ \delta \beta^{TE} = \frac{\omega\epsilon_0}{\beta_T E_N} \int \int \epsilon_g E_y \partial_y E_y dxdy. \]  

Where, we see that \( \epsilon_g \) present in the integral since it only multiplies that areas of the field that it interacts with. Using the same derivation as above we also see that if
\( M \) is parallel to the \( y \)-axis, the non-reciprocal phase shift for the TM mode is, where again we neglect the second order derivative with respect to \( x \),

\[
\delta \beta^{TM} = \frac{2 \beta^{TM}}{\omega \epsilon_0 N} \int \int \frac{\epsilon_g}{n_0^2} H_y \partial_x H_y dx dy. \tag{2.58}
\]

Thus, we see from our perturbative analysis that in order to achieve a large nonreciprocal phase shift according to Equations 2.57 and 2.58, it is essential to create a strong discontinuity of the Faraday rotation at the maximum of the mode intensities or to position such a discontinuity at the mode intensity maximum, respectively. Equations 2.57 and 2.58 show that the magnitude and sign of the nonreciprocal phase shifts \( \Delta \beta^{TE,TM} \) depend on the magnitude and sign of the Faraday rotation and on the geometry of the waveguide.

![Figure 2-6: Longitudinal electric field component changes sign, and so anti-symmetry in the mode cancels phase shift. Image from [7].](image)

Conclusively, we see the nonreciprocal manifests itself in the fact that in the TM wave the non-dominant longitudinal component of \( E \) is shifted by \( \pm 90^\circ \) from
Table 2.2: Nonreciprocal phase shift calculations using perturbation theory.

<table>
<thead>
<tr>
<th>Waveguide Width (μm)</th>
<th>δβ (cm⁻¹)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.045</td>
<td>0.854</td>
</tr>
<tr>
<td>0.1</td>
<td>0.332</td>
</tr>
<tr>
<td>0.12</td>
<td>0.250</td>
</tr>
</tbody>
</table>

the z component of E depending on the direction of propagation. We can see this as an elliptical polarization of the mode in the propagation plane (as compared to the transverse plane, as is the case in Faraday rotation). Thus, if the waveguide material is magneto-optic, applied transverse magnetic field can cause phase shifts of opposite signs for forward and backward propagating waves. Unfortunately, the longitudinal component of electric field in the fundamental TM wave changes sign near the middle of the waveguide Figure 2-6 and in a uniform symmetric waveguide, complete cancelation of the nonreciprocal effect ensues. This is why we must use the

\[
\begin{bmatrix}
\varepsilon_{xx} & 0 & i\varepsilon_g \\
0 & \varepsilon_{yy} & 0 \\
-i\varepsilon_g & 0 & \varepsilon_{zz}
\end{bmatrix}, \quad \text{or} \quad
\begin{bmatrix}
\varepsilon_{xx} & 0 & 0 \\
0 & \varepsilon_{yy} & i\varepsilon_g \\
0 & -i\varepsilon_g & \varepsilon_{zz}
\end{bmatrix}
\]

permittivity tensor components, which correspond to transverse magnetic fields, not longitudinal as in the case of the Faraday rotation.

### 2.4.3 Full anistropic Mode Solver Calculation

We start by writing Faraday’s and Ampere’s Laws in the frequency domain as

\[
\nabla \times \mathbf{E} = -j\omega \mu \mathbf{H} \quad (2.60)
\]

\[
\nabla \times \mathbf{H} = j\omega \mathbf{D} \quad (2.61)
\]

Where \(\omega\) is the angular frequency of the harmonic fields, and \(\mathbf{\mu}\) represents a diagonal permeability tensor with diagonal elements \(\mu_x, \mu_y, \text{ and } \mu_z\).

Thus we can write Equations 2.60 and 2.61 in terms of the following six component
equations, in terms of six electric and magnetic field components:

\[-j\omega\mu_x H_x = \frac{\partial E_y}{\partial y} + j\beta E_y\]  \hspace{1cm} (2.62)

\[-j\omega\mu_y H_y = -j\beta E_x - \frac{\partial E_z}{\partial x}\]  \hspace{1cm} (2.63)

\[-j\omega\mu_z H_z = \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\]  \hspace{1cm} (2.64)

\[j\omega D_z = \frac{\partial H_y}{\partial y} + j\beta H_y\]  \hspace{1cm} (2.65)

\[j\omega D_y = -j\beta H_x - \frac{\partial H_z}{\partial x}\]  \hspace{1cm} (2.66)

\[j\omega D_x = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\]  \hspace{1cm} (2.67)

\[j\omega D_z = \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}\]  \hspace{1cm} (2.68)

Figure 2-7: Yee Lattice

Therefore, we can map Equations 2.62 - 2.67 onto the Yee Lattice using Finite Difference discretization, yielding the corresponding Finite Difference equations:
-\( j \omega \mu_x^{(i,j+\frac{1}{2})} H_x^{(i,j+\frac{1}{2})} \) = \( \frac{E_x^{(i,j+1)} - E_x^{(i,j)}}{\Delta y} + j \beta E_y^{(i,j+\frac{1}{2})} \) \hspace{1cm} (2.69)

-\( j \omega \mu_y^{(i+\frac{1}{2},j)} H_y^{(i+\frac{1}{2},j)} \) = \( -j \beta E_x^{(i+\frac{1}{2},j)} - \frac{E_x^{(i+1,j)} - E_x^{(i,j)}}{\Delta x} \) \hspace{1cm} (2.70)

-\( j \omega \mu_z^{(i+\frac{1}{2},j+\frac{1}{2})} H_z \) = \( \frac{E_y^{(i+1,j+\frac{1}{2})} - E_y^{(i,j+\frac{1}{2})}}{\Delta x} - \frac{E_z^{(i+\frac{1}{2},j+1)} - E_z^{(i+\frac{1}{2},j)}}{\Delta y} \) \hspace{1cm} (2.71)

\( j \omega D_x^{(i+\frac{1}{2},j)} \) = \( \frac{H_x^{(i+\frac{1}{2},j+\frac{1}{2})} - H_x^{(i+\frac{1}{2},j-\frac{1}{2})}}{\Delta y} + j \beta H_y^{(i+\frac{1}{2},j)} \) \hspace{1cm} (2.72)

\( j \omega D_y^{(i,j+\frac{1}{2})} \) = \( -j \beta H_x^{(i,j+\frac{1}{2})} - \frac{H_x^{(i+\frac{1}{2},j+\frac{1}{2})} - H_x^{(i-\frac{1}{2},j+\frac{1}{2})}}{\Delta x} \) \hspace{1cm} (2.73)

\( j \omega D_z^{(i,j)} \) = \( \frac{H_y^{(i+\frac{1}{2},j)} - H_y^{(i,j-\frac{1}{2})}}{\Delta x} - \frac{H_z^{(i,j+\frac{1}{2})} - H_z^{(i,j-\frac{1}{2})}}{\Delta y} \) \hspace{1cm} (2.74)

This, can be rewritten in terms of finite difference operations \( \{ \hat{\partial}_x^E, \hat{\partial}_y^E, \hat{\partial}_x^H, \hat{\partial}_y^H \} \), defined as
\[
\begin{align*}
\delta^E_x E_{(i,j)} &= \frac{E_{(i+1,j)} - E_{(i,j)}}{\Delta x} \\
\delta^E_y E_{(i,j)} &= \frac{E_{(i,j+1)} - E_{(i,j)}}{\Delta y} \\
\delta^H_x H_{(i+\frac{1}{2},j+\frac{1}{2})} &= \frac{H_{(i+\frac{1}{2},j+\frac{1}{2})} - H_{(i-\frac{1}{2},j+\frac{1}{2})}}{\Delta x} \\
\delta^H_y H_{(i+\frac{1}{2},j+\frac{1}{2})} &= \frac{H_{(i+\frac{1}{2},j+\frac{1}{2})} - H_{(i+\frac{1}{2},j-\frac{1}{2})}}{\Delta y}
\end{align*}
\]

where \(\delta^E_{x,y}\) acts only on components of \(E\) and \(\delta^H_{x,y}\) acts only on components of \(H\).

We see that \(\delta^E_{x,y}\) is a forwards difference operation, and \(\delta^H_{x,y}\) is a reverse difference operator. This reduces equations 2.69 - 2.74 to the following matrix equations

\[
\begin{bmatrix}
\mu_x H_x \\
\mu_y H_y \\
\mu_z H_z
\end{bmatrix}
= 
\begin{bmatrix}
0 & -j\beta I & \delta^E_y \\
-j\beta I & 0 & -\delta^E_x \\
-\delta^E_y & \delta^E_x & 0
\end{bmatrix}
\begin{bmatrix}
E_x \\
E_y \\
E_z
\end{bmatrix}
\]

\[
\begin{bmatrix}
D_x \\
D_y \\
D_z
\end{bmatrix}
= 
\begin{bmatrix}
0 & -j\beta I & \delta^H_y \\
-j\beta I & 0 & -\delta^H_x \\
-\delta^H_y & \delta^H_x & 0
\end{bmatrix}
\begin{bmatrix}
H_x \\
H_y \\
H_z
\end{bmatrix}
\]

Now, for a general anisotropic dielectric medium, the relation between \(E\) and \(D\) can be expressed as

\[
\begin{align*}
D_x &= \epsilon_{xx} E_x + \epsilon_{xy} E_y + \epsilon_{xz} E_z \\
D_y &= \epsilon_{yx} E_x + \epsilon_{yy} E_y + \epsilon_{yz} E_z \\
D_z &= \epsilon_{zx} E_x + \epsilon_{zy} E_y + \epsilon_{zz} E_z
\end{align*}
\]

which are used to define the nine \(\epsilon_{ij}\) elements of the permittivity tensor at a given point in space. Inserting these relations into equations 2.79 - 2.80 yields the following relation for a full tensor dialectic material,
\[-j\omega \mu_0 \mu_z H_x = j\beta E_y + \hat{\partial}_y^E E_x \]  \hspace{1cm} (2.84)
\[-j\omega \mu_0 \mu_y H_y = -j\beta E_x - \hat{\partial}_x^E E_x \]  \hspace{1cm} (2.85)
\[-j\omega \mu_0 \mu_x H_z = -\hat{\partial}_y^E E_x + \hat{\partial}_x^E E_y \]  \hspace{1cm} (2.86)
\[j\omega (\varepsilon_{xx} E_x + \varepsilon_{xy} E_y + \varepsilon_{xz} E_z) = j\beta H_y + \hat{\partial}_y^H H_x \]  \hspace{1cm} (2.87)
\[j\omega (\varepsilon_{yx} E_x + \varepsilon_{yy} E_y + \varepsilon_{yz} E_z) = -j\beta H_x - \hat{\partial}_x^H H_x \]  \hspace{1cm} (2.88)
\[j\omega (\varepsilon_{zx} E_x + \varepsilon_{zy} E_y + \varepsilon_{zz} E_z) = -\hat{\partial}_y^H H_x + \hat{\partial}_x^H H_y \]  \hspace{1cm} (2.89)

Note that in Equations 2.84 - 2.89 each \(\varepsilon_{ij} E_j\) or \(\mu_i H_j\) is a column vector with each element representing the value of \(\varepsilon^{ij} E^j\) or \(\mu^i H^j\) at a corresponding Yee-mesh field component at a particular point in the computational domain, namely

\[
\varepsilon_{ij} E_j = \begin{bmatrix}
\varepsilon^{ij} E_{1,1}^{j} \\
\varepsilon^{ij} E_{2,1}^{j} \\
\vdots \\
\varepsilon^{ij} E_{n,1}^{j} \\
\varepsilon^{ij} E_{1,2}^{j} \\
\varepsilon^{ij} E_{2,2}^{j} \\
\vdots \\
\varepsilon^{ij} E_{n,m}^{j}
\end{bmatrix} \hspace{1cm} (2.90)
\]

\[
\mu_i H_j = \begin{bmatrix}
\mu^i H_{1,1}^{j} \\
\mu^i H_{2,1}^{j} \\
\vdots \\
\mu^i H_{n,1}^{j} \\
\mu^i H_{1,2}^{j} \\
\mu^i H_{2,2}^{j} \\
\vdots \\
\mu^i H_{n,m}^{j}
\end{bmatrix} \hspace{1cm} (2.91)
\]
We can also observe the forward and backward difference matrices, expressed in equations 2.75 - 2.78 can be similarly represented in matrix form as

\[
\frac{\delta_x^E}{\Delta x} = \begin{bmatrix}
-1 & 1 \\
-1 & 1 \\
-1 & 1 \\
-1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
-1 & 1 \\
-1 & 1 \\
1 & 1
\end{bmatrix}
\]

(2.92)

\[
\frac{\delta_x^H}{\Delta x} = \begin{bmatrix}
1 & 1 \\
-1 & 1 \\
-1 & 1 \\
-1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
-1 & 1 \\
-1 & 1 \\
1 & 1
\end{bmatrix}
\]

(2.94)

\[
\frac{\delta_y^E}{\Delta y} = \begin{bmatrix}
-1 & 1 \\
1 & 1 \\
-1 & 1 \\
1 & 1 \\
-1 & 1 \\
\vdots & \vdots \\
1 & 1 \\
-1 & 1 \\
1 & 1
\end{bmatrix}
\]

(2.93)

Moreover, referencing figure 2-7 we observe that near the grid point \((i, j)\) the
values of $\varepsilon_{xj}$, $\varepsilon_{yj}$, and $\varepsilon_{zj}$ correspond to the material properties at the grid points $(i + \frac{1}{2}, j)$, $(i, j + \frac{1}{2})$, and $(i, j)$, respectively, which are the locations of $D_x$, $D_y$, and $D_z$, respectively.

Now, we are in position to create our Eigenvalue equation. By substituting the expressions for $E_x$ and $H_x$ from Equations 2.86 and 2.89, into Equations 2.84, 2.85, 2.87, and 2.88, we obtain an Eigenvalue equation for all four transverse field components:

$$
\beta \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix} = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} \\ M_{21} & M_{22} & M_{23} & M_{24} \\ M_{31} & M_{32} & M_{33} & M_{34} \\ M_{41} & M_{42} & M_{43} & M_{44} \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ H_x \\ H_y \end{bmatrix}
$$

(2.96)

where the sub matrices $M_{ij}(i, j = 1, 2, 3, 4)$ are defined as:
\[ M_{11} = -j \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \right) \partial_x^E \]  
\[ M_{12} = -j \left( \frac{\epsilon_{xy}}{\epsilon_{zz}} \right) \partial_x^E \]  
\[ M_{13} = - \left( \frac{1}{\omega \epsilon_{zz}} \right) \partial_x^E \partial_y^H \]  
\[ M_{14} = \left( \frac{1}{\omega \epsilon_{zz}} \right) \partial_x^E \partial_y^H + \omega \mu_y I \]  
\[ M_{21} = -j \left( \frac{\epsilon_{yy}}{\epsilon_{zz}} \right) \partial_y^E \]  
\[ M_{22} = -j \left( \frac{\epsilon_{xy}}{\epsilon_{zz}} \right) \partial_y^E \]  
\[ M_{23} = - \left( \frac{1}{\omega \epsilon_{zz}} \right) \partial_y^E \partial_y^H - \omega \mu_x I \]  
\[ M_{24} = \left( \frac{1}{\omega \epsilon_{zz}} \right) \partial_y^E \partial_x^H \]  
\[ M_{31} = -\omega \epsilon_{yz} I + \left( \frac{1}{\omega \mu_x} \right) \partial_x^H \partial_x^E + \omega \frac{\epsilon_{yz} \epsilon_{zz}}{\epsilon_{zz}} I \]  
\[ M_{32} = -\omega \epsilon_{yz} I - \left( \frac{1}{\omega \mu_x} \right) \partial_y^H \partial_x^E + \omega \frac{\epsilon_{yz} \epsilon_{zz}}{\epsilon_{zz}} I \]  
\[ M_{33} = -j \left( \frac{\epsilon_{yz}}{\epsilon_{zz}} \right) \partial_y^H \]  
\[ M_{34} = -j \left( \frac{\epsilon_{yz}}{\epsilon_{zz}} \right) \partial_x^H \]  
\[ M_{41} = \omega \epsilon_{xy} I + \left( \frac{1}{\omega \mu_x} \right) \partial_y^H \partial_x^E - \omega \frac{\epsilon_{xx} \epsilon_{zz}}{\epsilon_{zz}} I \]  
\[ M_{42} = \omega \epsilon_{xy} I - \left( \frac{1}{\omega \mu_x} \right) \partial_y^H \partial_x^E - \omega \frac{\epsilon_{xx} \epsilon_{zz}}{\epsilon_{zz}} I \]  
\[ M_{43} = j \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \right) \partial_x^H \]  
\[ M_{44} = -j \left( \frac{\epsilon_{xx}}{\epsilon_{zz}} \right) \partial_x^H \]  

Now since we are dealing with a full tensor material, we must adopt a more general PML which can handle anisotropic media, we will use that proposed by Teixeira and
Chew [33] as the absorbing boundary conditions. Thus, in the PML regions that permittivity and permeability tensors are given by,

\[
\hat{\epsilon}_{\text{PML}} = \begin{bmatrix}
\frac{\epsilon_{\text{xx}}}{s_x} & \frac{\epsilon_{\text{xy}}}{s_x} & \frac{\epsilon_{\text{xz}}}{s_x} \\
\frac{\epsilon_{\text{yx}}}{s_y} & \frac{\epsilon_{\text{yy}}}{s_y} & \frac{\epsilon_{\text{yz}}}{s_y} \\
\frac{\epsilon_{\text{zx}}}{s_z} & \frac{\epsilon_{\text{zy}}}{s_z} & \frac{\epsilon_{\text{zz}}}{s_z}
\end{bmatrix}
\] (2.114)

\[
\hat{\mu}_{\text{PML}} = \mu_0 \begin{bmatrix}
\frac{s_y s_z}{s_x} & 0 & 0 \\
0 & \frac{s_x s_z}{s_y} & 0 \\
0 & 0 & \frac{s_x s_y}{s_z}
\end{bmatrix}
\] (2.115)

where \(s_x, s_y\) and \(s_z\) are the so called complex PML parameters, where each is defined as

\[
s_j = 1 - i\alpha_j
\] (2.116)

for \(j = x, y, z\). The \(\alpha_j\)'s are chosen heuristically to control the field attenuation in the PML regions, for our application we choose:

\[
\alpha_j = \alpha_{j,\text{max}} \left(\frac{\rho}{d}\right)^2
\] (2.117)

for \(j = x, y \ (\alpha_z = 0)\), \(\rho\) is defined as the distance in the \(j^{\text{th}}\) direct from the start of the PML region, and \(\alpha_{j,\text{max}}\) is calculated using an assumed reflectivity value from the PML layer.

Now that we have designed a tool to calculate the non-reciprocal phase shift, we simply insert our calculated expression for \(\epsilon_g\) into the mode solver to determine the non-reciprocal phase shift, as shown in Figure 2-9. This will become increasingly more accurate by inserting an increasingly more accurate value for \(\epsilon_g\). The full Matlab of implementation of this full epsilon tensor mode solver is given in Appendix A.
Now that we have determined the strength of the non-reciprocal phase shift, which as we recall is dependent upon our calculation of $\epsilon_g$, we are now finally in a position to design an integrated optical isolator. In this section we will provide three device designs.

2.5 Integrated Isolator Designs

Now that we have determined the strength of the non-reciprocal phase shift, which as we recall is dependent upon our calculation of $\epsilon_g$, we are now finally in a position to design an integrated optical isolator. In this section we will provide three device designs.

2.5.1 Passive Non-Reciprocal Mach-Zehnder Phase Shifter Isolator

The first design implements a Mach-Zehnder interferometer.

The nonreciprocal mach-zehnder isolator relies on the nonreciprocal phase shift between forward and backward propagating modes. Using this effect, a waveguide MachZehnder interferometer can be designed such that in the forward direction the modes in the two arms are in phase yielding constructive interference, whereas in the backward direction the modes are out of phase by $180^\circ$ causing destructive interfer-
Figure 2-10: Diagram of the Mach-Zehnder isolator implementing nonreciprocal phase shift.

For this passive Mach-Zehnder isolator the magnetic field is produced by an external magnetic, in either the equatorial or polar configurations as described in Figure 2-2.

Figure 2-11: Two passive Mach-Zehnder isolator designs. Design 1 is in the Polar configuration, and design 2 is in the Equatorial configuration.

In Figure 2-11 the interferometer arms consist of nonreciprocal phase shift regions with opposite signs. They are in-plane magnetized perpendicular to mode propaga-
tion. As the magnetization is oppositely directed in the two arms, the nonreciprocal phase shift has an opposite sign so that the total length of the interferometer is only half of that of an interferometer with only one arm containing a nonreciprocal phase shift region.

In Figure 2-11 we see the use of $P$ and $N$ dopants, oppositely doped on either arm. In our application we also use Manganese dopants but doped only on one arm to create the non-reciprocal phase shift.

### 2.5.2 Non-reciprocal Ring Phase Shifter Isolator

The nonreciprocal ring resonator isolator works on a very similar principle as that of the Mach-Zehnder isolator. However, in the case of the ring we see a relative nonreciprocal phase shift depending on which direction the mode travels through the ring. We then, as in the case of the Mach-Zehnder isolator adjust the radius of the ring so that as light couples from the waveguide into the ring, we get constructive interferences, whereas in the opposite direction we get destructive interference.

As in the case of the Mach-Zehnder isolator we use an external magnetic to produce a magnetic field, and thus a gyrotropic media. In this case the magnet is located in the center of the ring, and the magnetic field is directed radially outward, so that at every point on the ring the magnetic field is in the equatorial configuration as described in Figure 2-2.

![Figure 2-12: Microring Isolator in the polar configuration.](image-url)
2.5.3 Active Non-reciprocal Mach-Zehnder Phase Shifter Isolator

This device is similar to the passive Mach-Zehnder isolator, except for the fact that the magnetic field does not come from an external off-chip magnet, but is instead generated on-chip through the use of metal coils, this is shown in Figure 2-13. This allows for a fully integrated solution, and since we can obtain a very dense and close magnetic field, higher performance.

Figure 2-13: Active Mach-Zehnder isolator in the polar configuration. A magnetic field into or out of the page, is generated by the metal coils.

2.6 Conclusions

In conclusion, we present six designs for integrated optical isolators, by separating the design process into $\epsilon_g$ calculation, nonreciprocal phase shift calculation (analytical, perturbative, and numeric), and finally isolator design. We see that the nonreciprocal phase shift calculations agree within 20% between the three methods, which is surprising and counts in a positive way toward the validity and consistency of each
2.7 Future Work

2.7.1 Experimental Determination of $\epsilon_g$

Since, as stated earlier, the accuracy of the isolator design is intimately tied to the accuracy of the calculated $\epsilon_g$, improving the accuracy of $\epsilon_g$ is paramount to the success of the device.

One such way to accomplish this is to get an estimate of the value of $\epsilon_g$ using experimental methods, which will give us another perspective on its value and see how it compares to the derived quantity.

In this experiment we will measure the faraday rotation in a bulk Silicon wafer doped with Manganese. The experimental setup diagram is shown in Figure 2-15 and a picture of the actual experimental setup is shown in Figure 2-14.
The reason for using Mn is that Mn-Doped Silicon creates an intrinsic magneto-optic material (rather than associated with the presence of ferromagnetic clusters or a foreign ferromagnetic phase).

The linearly polarized light incident on the sample is assumed to be propagating in the z direction and polarized along the x, y plane corresponding to the phase of the fiber polarization modulator, $\Omega$. The electric field of the light beam can be expressed in Jones matrix form as

$$E_0 = \begin{bmatrix} \cos(\Omega t) \\ \sin(\Omega t) \end{bmatrix} A_0 e^{-i\omega t + ikz}$$  \hspace{1cm} (2.118)$$

where $A_0$ is the amplitude of the electric field of the beam. Due to Faraday rotation the polarization of the light will be rotated by a small angle $\theta$, which can be given by

$$E = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix} A_0 e^{-i\omega t + ikz}$$ \hspace{1cm} (2.119)$$

After the light travels through the analyzer, which is situated at an angle $\phi$ relative to the polarizer, we can represent the resultant field as

$$E = \begin{bmatrix} \cos(\phi - \theta) \cos(\phi) \\ \cos(\phi - \theta) \sin(\theta) \end{bmatrix} A_0 e^{-i\omega t + ikz}$$ \hspace{1cm} (2.120)$$

the intensity measured at the detector is now given by

$$I = \cos^2(\phi - \theta) A_0^2$$ \hspace{1cm} (2.121)$$

We realize that maximum difference between the lowest and highest intensity ($\Delta I$)
will happen at some optimal angle between the analyzer and the polarizer. Thus, by taking the first derivative of $I$ with respect to $\theta$, we see that

$$\frac{\partial I}{\partial \theta} = \sin(2\phi - 2\theta)A_0^2.$$ \hspace{1cm} (2.122)

Because $\theta$ is small ($\theta \ll 1^\circ$), it can easily be seen that a maximum $\Delta I$ is obtained when $\phi = 45^\circ$, which means the intensity measured by the detector as expressed by Equation 2.121 is reduced to (subject to $\phi = 45^\circ$)

$$I = \frac{1}{2}(1 + 2\theta)A_0^2.$$ \hspace{1cm} (2.123)

Taking into consideration that the ac polarization modulation is sinusoidal, and because $\theta$ is proportional to $B$, the Faraday rotation angle can be written in the form $\theta = \theta_0 \sin(\Omega t)$. Hence we see that Equation 2.123 can be simplified to

$$I = \frac{1}{2}(1 + 2\theta_0 \sin(\Omega t))A_0^2 = I_0 + \Delta I \sin(\Omega t).$$ \hspace{1cm} (2.124)

By measuring the relative change of the light intensity $\frac{\Delta I}{I_0}$, the Faraday rotation angle can be determined by

$$\theta_0 = \frac{12\theta_0 A_0^2}{2 A_0^2} = \frac{1}{2} \frac{\Delta I}{I_0}.$$ \hspace{1cm} (2.125)

We can then easily obtain the Verdet coefficient from the follow relation

$$\theta_0 = VB_0 L,$$ \hspace{1cm} (2.126)

where $\theta_0, L, B$ are all experimentally determined. Using these estimates I predict a realistic measurement of $\frac{\Delta I}{I_0}$ can be sensitive on the order on $\pm 10^{-5}$, which corresponds to a difference of the order $10^{-10}$ in the refractive index of left and right circularly polarized light in the medium using this technique. This implies a rotation of the plane of polarization of the order of $10^{-5}$ radians is measurable by this technique due to the elimination of broadband noise by the lock-in amplifier.

We then use this derived value of the Verdet constant to calculate an experiment-
tally obtained value for $\epsilon_g$
Bibliography


Appendix A

Full Permittivity Tensor Mode Solver Matlab Code

derivative_mesh.m: This function aligns the field and tensor components correctly on the Yee lattice

```matlab
1 % create NSEWP
2 function [indooN, indooS, indooE, indooW, indooP, indooNW, indooNE, indooSW, indooSE...]
3     indeoN, indeoS, indeoE, indeoW, indeoP, indeoNW, indeoNE, indeoSW, indeoSE...
4     indeeN, indeeS, indeeE, indeeW, indeeP, indeeNW, indeeNE, indeeSW, indeeSE] = derivative_mesh(ind2, nx, ny)
5 % This function aligns the field and tensor components correctly on the Yee
6 % lattice
7 %
8 % USAGE:
```
% [indooN, indooS, indooE, indooW, indooP, indooNW, indooNE, indooSW, indooSE...] 
% indoeN, indoeS, indoeE, indoeW, indoeP, indoeNW, indoeNE, indoeSW, indoeSE... 
% indeeN, indeeS, indeeE, indeeW, indeeP, indeeNW, indeeNE, indeeSW, indeeSE] = derivative_mesh(ind2, nx, ny)

% INPUT:
% % nx, ny - size of index mesh
% % ind2 - index mesh (doubled in size and interpolated

% OUTPUT:
% %
% % indices of correct gridding for derivative computations on Yee lattice

% AUTHOR: Brad G Cordova (bcordova@mit.edu)
% Version 1.0 (created July 2012)

indoo = ind2(1:2:end,1:2:end);
indoe = ind2(1:2:end,2:2:end);
indeo = ind2(2:2:end,1:2:end);
indee = ind2(2:2:end, 2:2:end);

%% padding
indoo = [indoo(:,1), indoo, indoo(:, ny)];
indoo = [indoo(1,:), indoo; indoo(nx,:)];
indeo = [indeo(:,1), indeo, indeo(:, ny)];
indeo = [indeo(1,:), indeo; indeo(nx,:)];
indeo = [indeo(:,1), indeo, indeo(:, ny)];
indeo = [indeo(1,:), indeo; indeo(nx,:)];
indee = [indee(:,1), indee, indee(:, ny)];
indee = [indee(1,:), indee; indee(nx,:)];
size(indoo);

%%
indooN = zeros(1, nx*ny);
indooS = zeros(1, nx*ny);
indooE = zeros(1, nx*ny);
indooW = zeros(1, nx*ny);
indooP = zeros(1, nx*ny);
indooNW = zeros(1, nx*ny);
indooNE = zeros(1, nx*ny);
indooSW = zeros(1, nx*ny);
indooSE = zeros(1, nx*ny);

indeeN = zeros(1, nx*ny);
indeeS = zeros(1, nx*ny);
indeeE = zeros(1, nx*ny);
indeeW = zeros(1, nx*ny);
indeeP = zeros(1, nx*ny);
indeeNW = zeros(1, nx*ny);
indeeNE = zeros(1, nx*ny);
```matlab
indoeSW = zeros(1, nx*ny);
indoeSE = zeros(1, nx*ny);

indeoN = zeros(1, nx*ny);
indeoS = zeros(1, nx*ny);
indeoE = zeros(1, nx*ny);
indeoW = zeros(1, nx*ny);
indeoP = zeros(1, nx*ny);
indeoNW = zeros(1, nx*ny);
indeoNE = zeros(1, nx*ny);
indeoSW = zeros(1, nx*ny);
indeoSE = zeros(1, nx*ny);

% NSEWP
indooN(:, :) = indoo(2:nx+1, 3:ny+2);
indooS(:, :) = indoo(2:nx+1, 1:ny);
indooE(:, :) = indoo(3:nx+2, 2:ny+1);
indooW(:, :) = indoo(1:nx, 2:ny+1);
```
\[ \text{indeoN}(:) = \text{indeo}(2:nx+1,3:ny+2); \]
\[ \text{indeoN}(:) = \text{indeo}(2:nx+1,1:ny); \]
\[ \text{indeoE}(:) = \text{indeo}(3:nx+2,2:ny+1); \]
\[ \text{indeoW}(:) = \text{indeo}(1:nx,2:ny+1); \]
\[ \text{indeoP}(:) = \text{indeo}(2:nx+1,2:ny+1); \]
\[ \text{indeoNW}(:) = \text{indeo}(1:nx,3:ny+2); \]
\[ \text{indeoNE}(:) = \text{indeo}(3:nx+2,3:ny+2); \]
\[ \text{indeoSW}(:) = \text{indeo}(1:nx,1:ny); \]
\[ \text{indeoSE}(:) = \text{indeo}(3:nx+2,1:ny); \]
\[ \text{indeeN}(:) = \text{indee}(2:nx+1,3:ny+2); \]
\[ \text{indeeS}(:) = \text{indee}(2:nx+1,1:ny); \]
\[ \text{indeeE}(:) = \text{indee}(3:nx+2,2:ny+1); \]
\[ \text{indeeW}(:) = \text{indee}(1:nx,2:ny+1); \]
\[ \text{indeeP}(:) = \text{indee}(2:nx+1,2:ny+1); \]
\[ \text{indeeNW}(:) = \text{indee}(1:nx,3:ny+2); \]
\[ \text{indeeNE}(:) = \text{indee}(3:nx+2,3:ny+2); \]
\[ \text{indeeSW}(:) = \text{indee}(1:nx,1:ny); \]
\[ \text{indeeSE}(:) = \text{indee}(3:nx+2,1:ny); \]
indeeP( :) = indee(2:nx+1, 2:ny+1);
indeeNW( :) = indee(1:nx, 3:ny+2);
indeeNE( :) = indee(3:nx+2, 3:ny+2);
indeeSW( :) = indee(1:nx, 1:ny);
indeeSE( :) = indee(3:nx+2, 1:ny);
**index_mesh.m**: This function aligns the field and tensor components correctly on the Yee lattice

```matlab
function [dx dy nx ny ind] = index_mesh(dxy, width, height, epsilon, d_PML, PML_row, PML_column)
% This function creates an index mesh for the finite difference
% mode solver. The function will accommodate a generalized input matrix.
%
% USAGE:
%
% [dx dy nx ny ind] = index_mesh(dxy, width, height, epsilon)
% [dx dy nx ny ind] = index_mesh(dxy, width, height, epsilon, d_PML, PML_row, PML_column)
%
% INPUT
%
% dxy - vector [dx dy] horizontal and vertical grid spacing
% epsilon - structure of permittivities
% width - matrix defining horizontal widths of permittivity structure
% height - matrix defining vertical heights of permittivity structure
%
% OUTPUT
%
% dx, dy - horizontal and vertical grid spacing
% nx, ny - size of index mesh
% ind - index mesh
```
dx = dxy(1);
dy = dxy(2);

numx = round(width/dx);
numy = round(height/dy);
nx = sum(numx);
ny = sum(numy);

AA = zeros(nx*ny,1);
kk = 0;

for ii = 1:size(numx,2)
    for ss = 1:size(numx,2)
        for jj = 1:size(numy,2)
            AA(kk+1: kk + numy(jj),1) = epsilon(jj,ii)*ones(numy(jj),1);
            kk = kk + numy(jj);
        end
    end
end

ind = AA;
if nargin==7 %Add PML to index mesh
nx_PML = round(d_PML/dx);
ny_PML = nx_PML;
ind = reshape(ind,ny,nx)';

triu_5_row = triu(ones(nx_PML,ny_PML));
triu_5_col = triu(ones(nx_PML,ny_PML),1)';

triu_6_col = flipud(triu(ones(nx_PML,ny_PML),1));
triu_6_row = flipud(triu(ones(nx_PML,ny_PML))');

triu_7_row = flipud(triu(ones(nx_PML,ny_PML)));  
triu_7_col = flipud(triu(ones(nx_PML,ny_PML),1)');

triu_8_col = triu(ones(nx_PML,ny_PML),1); 
triu_8_row = triu(ones(nx_PML,ny_PML))';

%PML boundary labels 

PML_1 = flipud(PML_row(:,ny_PML+1:ny_PML+ny),1);
PML_2 = PML_row(:, ny_PML+1:ny_PML+ny, 2);
PML_3 = flipr(PML_column(nx_PML+1:nx_PML+nx, :, 3));
PML_4 = PML_column(nx_PML+1:nx_PML+nx, :, 4);

PML_5 = flipud(PML_row(:, 1:ny_PML, 5)) .* triu_5_row + flipr(PML_column(1:nx_PML, :, 5)) .* triu_5_col;
PML_6 = PML_row(:, 1:ny_PML, 6) .* triu_6_row + flipr(PML_column(1:nx_PML, :, 6)) .* triu_6_col;
PML_7 = flipud(PML_row(:, 1:ny_PML, 7)) .* triu_7_row + PML_column(1:nx_PML, :, 7) .* triu_7_col;
PML_8 = PML_row(:, 1:ny_PML, 8) .* triu_8_row + PML_column(1:nx_PML, :, 8) .* triu_8_col;

PML_top = PML_3;
PML_bot = PML_4;
PML_left = [PML_5 PML_1 PML_7];
PML_right = [PML_6 PML_2 PML_8];

ind = [PML_top ind PML_bot];
ind = [PML_left; ind; PML_right];
% ind = [flipr(PML_column_left) ind PML_column_right];

% ind = [flipud(PML_row_top); ind; (PML_row_bottom)];

[nx ny] = size(ind);

% figure; imagesc(abs(ind))

ind = reshape(ind', nx*ny, 1);

end
**eigen.modes.m**: This function aligns the field and tensor components correctly on the Yee lattice

```matlab
function [Ex Ey Ez Hx Hy Hz, neff, nx, ny] = eigen.modes(dxy, width, height, lambda, guess_index, n_modes, epsilon, mu, PML)

% This function computes all field components of a dielectric waveguide constructed from an arbitrary
% permitivity tensor, using the finite difference method.

% USAGE:

% [Ex Ey Ez Hx Hy Hz, neff, nx, ny] = eigen.modes(dxy, width, height, lambda, ...
%       guess_index, n_modes, epsilon, mu)

% [Ex Ey Ez Hx Hy Hz, neff, nx, ny] = eigen.modes(dxy, width, height, lambda, ...
%       guess_index, n_modes, epsilon, mu, PML)

% INPUT:

% lambda – optical wavelength
% guess_index – scalar shift to apply when calculating the eigenvalues.
% This routine will return the eigenpairs which have an effective index closest to this guess
```
% n_modes - the number of modes to calculate
% dxy - vector [dx dy] horizontal and vertical grid spacing
% epsilon - structure of permittivities
% mu - structure of permeabilities

% OUTPUT:

% % Hx, Hy, Hz - calculated magnetic field components.
% % Ex, Ey, Ez - calculated electric field components.
% % nx, ny - size of index mesh
% % neff - vector of modal effective indices
%

% % AUTHOR: Brad G Cordova (bcordova@mit.edu)
% % Version 1.0 (created July 2012)

dx = dxy(1); dy = dxy(2);
c_0 = 299792458;
omega = 2*pi*c_0/lambda;

if nargin==9 %if PML

PML_row_epsilon = PML.row_epsilon;
PML_column_epsilon = PML.column_epsilon;
PML_row_mu = PML.row_mu;
PML_column_mu = PML.column_mu;
dPML = PML.width;

for ii = 1:9
[dx dy nx ny epsilon_mesh(:,ii)] = index_mesh(dxy, width, height, epsilon(:,ii),dPML,PML_row_epsilon, PML_column_epsilon);
epsilon_mesh2 = reshape(epsilon_mesh(:,ii),ny, nx)';
[X,Y] = meshgrid(1:ny, (1:nx)');
[MM,NN] = meshgrid(1:0.5:ny,(1:0.5:nx)');
epsilon_mesh2 = interp2(X, Y, epsilon_mesh2, MM, NN); % linear interpolation of the ind_mesh
[mx, my] = size(epsilon_mesh2);
epsilon_mesh2 = [epsilon_mesh2, epsilon_mesh2(:,my)]; % add column
epsilon_mesh2 = [epsilon_mesh2; epsilon_mesh2(mx,:)]; % add row
epsilon_mesh_double(:,:,ii) = epsilon_mesh2;
[mx, my] = size(epsilon_mesh_double(:,:,ii));
end

figure(100);
subplot(121)
imagesc(real(epsilon_mesh_double(:,1,:))); title('Real Index Profile');
subplot(122)
imagesc(imag(epsilon_mesh_double(:,1,:))); title('Imaginary Index Profile');

for ii = 1:9
[dx dy nx ny mu_mesh(:,ii)] = index_mesh(dxy, width, height, 
mu(:,:,ii),dPML,PML_row_mu,PML_column.mu);
u_mesh2 = reshape(mu_mesh(:,ii),ny, nx)';
[X,Y] = meshgrid(1:ny, (1:nx)');
[MM,NN] = meshgrid(1:0.5:ny,(1:0.5:nx)');
mu_mesh2 = interp2(X, Y, mu_mesh2, MM, NN); % linear interpolation of the ind mesh
[mx, my] = size(mu_mesh2);
mu_mesh2 = [mu_mesh2, mu_mesh2(:,my)]; % add column
mu_mesh2 = [mu_mesh2; mu_mesh2(mx,:)]; % add row
mu_mesh_double(:, :, ii) = mu_mesh2;
[mx, my] = size(mu_mesh_double(:, :, ii));
end

% figure; imagesc(mu_mesh_double(:, :, 5));

else
for ii = 1:9
[dx dy nx ny epsilon_mesh(:, ii)] = index_mesh(dxy, width, height, epsilon(:, :, ii));
epsilon_mesh2 = reshape(epsilon_mesh(:, :, ii), ny, nx)';
[X,Y] = meshgrid(1:ny, (1:nx)');
[MM,NN] = meshgrid(1:0.5:ny,(1:0.5:nx)');
epsilon_mesh2 = interp2(X, Y, epsilon_mesh2, MM, NN); % linear interpolation of the ind mesh
[mx, my] = size(epsilon_mesh2);
epsilon_mesh2 = [epsilon_mesh2, epsilon_mesh2(:,my)]; % add column
epsilon_mesh2 = [epsilon_mesh2; epsilon_mesh2(mx,:)]; % add row
epsilon_mesh_double(:, :, ii) = epsilon_mesh2;
[mx, my] = size(epsilon_mesh_double(:, :, ii));
end

figure(100); imagesc(real(epsilon_mesh_double(:, :, 1)'));

for ii = 1:9
    [dx dy nx ny mu_mesh(:, ii)] = index_mesh(dxy, width, height, mu(:, :, ii));
    mu_mesh2 = reshape(mu_mesh(:, ii), ny, nx)';
    [X,Y] = meshgrid(1:ny, (1:nx)');
    [MM,NN] = meshgrid(1:0.5:ny,(1:0.5:nx)');
    mu_mesh2 = interp2(X, Y, mu_mesh2, MM, NN); % linear interpolation of the ind_mesh
    [mx, my] = size(mu_mesh2);
    mu_mesh2 = [mu_mesh2, mu_mesh2(:,my)]; % add column
    mu_mesh2 = [mu_mesh2; mu_mesh2(mx,:)]; % add row
    mu_mesh_double(:, :, ii) = mu_mesh2;
    [mx, my] = size(mu_mesh_double(:, :, ii));
end

% figure; imagesc(mu_mesh_double(:, :, 5));
end

fprintf(1,'Creating Index Mesh... \n');

[exx_oo_N, exx_oo_S, exx_oo_E, exx_oo_W, exx_oo_P, exx_oo_NW, exx_oo_NE, exx_oo_SW, exx_oo_SE ...
    exx_oe_N, exx_oe_S, exx_oe_E, exx_oe_W, exx_oe_P, exx_oe_NW, exx_oe_NE, exx_oe_SW, exx_oe_SE ...
exx_eo_N, exx_eo_S, exx_eo_E, exx_eo_W, exx_eo_P, exx_eo_NW, exx_eo_NE, exx_eo_SW, exx_eo_SE ...

exx_ee_N, exx_ee_S, exx_ee_E, exx_ee_W, exx_ee_P, exx_ee_NW, exx_ee_NE, exx_ee_SW, exx_ee_SE] = derivative_mesh(
    epsilon_mesh_double(:,:1), nx, ny);

[ exy_oo_N, exy_oo_S, exy_oo_E, exy_oo_W, exy_oo_P, exy_oo_NW, exy_oo_NE, exy_oo_SW, exy_oo_SE ...

  exy_oe_N, exy_oe_S, exy_oe_E, exy_oe_W, exy_oe_P, exy_oe_NW, exy_oe_NE, exy_oe_SW, exy_oe_SE ...

  exy_eo_N, exy_eo_S, exy_eo_E, exy_eo_W, exy_eo_P, exy_eo_NW, exy_eo_NE, exy_eo_SW, exy_eo_SE ...

  exy_ee_N, exy_ee_S, exy_ee_E, exy_ee_W, exy_ee_P, exy_ee_NW, exy_ee_NE, exy_ee_SW, exy_ee_SE] = derivative_mesh(
    epsilon_mesh_double(:,:4), nx, ny);

[ exz_oo_N, exz_oo_S, exz_oo_E, exz_oo_W, exz_oo_P, exz_oo_NW, exz_oo_NE, exz_oo_SW, exz_oo_SE ...

  exz_oe_N, exz_oe_S, exz_oe_E, exz_oe_W, exz_oe_P, exz_oe_NW, exz_oe_NE, exz_oe_SW, exz_oe_SE ...

  exz_eo_N, exz_eo_S, exz_eo_E, exz_eo_W, exz_eo_P, exz_eo_NW, exz_eo_NE, exz_eo_SW, exz_eo_SE ...

  exz_ee_N, exz_ee_S, exz_ee_E, exz_ee_W, exz_ee_P, exz_ee_NW, exz_ee_NE, exz_ee_SW, exz_ee_SE] = derivative_mesh(
    epsilon_mesh_double(:,:7), nx, ny);

[ eyx_oo_N, eyx_oo_S, eyx_oo_E, eyx_oo_W, eyx_oo_P, eyx_oo_NW, eyx_oo_NE, eyx_oo_SW, eyx_oo_SE ...

  eyx_oe_N, eyx_oe_S, eyx_oe_E, eyx_oe_W, eyx_oe_P, eyx_oe_NW, eyx_oe_NE, eyx_oe_SW, eyx_oe_SE ...
eyx_eo_N, eyx_eo_S, eyx_eo_E, eyx_eo_W, eyx_eo_P, eyx_eo_NW,
eyx_eo_NE, eyx_eo_SW, eyx_eo_SE...
eyx_ee_N, eyx_ee_S, eyx_ee_E, eyx_ee_W, eyx_ee_P, eyx_ee_NW,
eyx_ee_NE, eyx_ee_SW, eyx_ee_SE] = derivative_mesh(
epsilon_mesh_double (::2), nx, ny);

[eyy_oo_N, eyy_oo_S, eyy_oo_E, eyy_oo_W, eyy_oo_P, eyy_oo_NW,
eyy_oo_NE, eyy_oo_SW, eyy_oo_SE...
eyy_oe_N, eyy_oe_S, eyy_oe_E, eyy_oe_W, eyy_oe_P, eyy_oe_NW,
eyy_oe_NE, eyy_oe_SW, eyy_oe_SE...
eyy_eo_N, eyy_eo_S, eyy_eo_E, eyy_eo_W, eyy_eo_P, eyy_eo_NW,
eyy_eo_NE, eyy_eo_SW, eyy_eo_SE...
eyy_ee_N, eyy_ee_S, eyy_ee_E, eyy_ee_W, eyy_ee_P, eyy_ee_NW,
eyy_ee_NE, eyy_ee_SW, eyy_ee_SE] = derivative_mesh(
epsilon_mesh_double (::5), nx, ny);

[eyz_oo_N, eyz_oo_S, eyz_oo_E, eyz_oo_W, eyz_oo_P, eyz_oo_NW,
eyz_oo_NE, eyz_oo_SW, eyz_oo_SE...
eyz_oe_N, eyz_oe_S, eyz_oe_E, eyz_oe_W, eyz_oe_P, eyz_oe_NW,
eyz_oe_NE, eyz_oe_SW, eyz_oe_SE...
eyz_eo_N, eyz_eo_S, eyz_eo_E, eyz_eo_W, eyz_eo_P, eyz_eo_NW,
eyz_eo_NE, eyz_eo_SW, eyz_eo_SE...
eyz_ee_N, eyz_ee_S, eyz_ee_E, eyz_ee_W, eyz_ee_P, eyz_ee_NW,
eyz_ee_NE, eyz_ee_SW, eyz_ee_SE] = derivative_mesh(
epsilon_mesh_double (::8), nx, ny);

[ezx_oo_N, ezx_oo_S, ezx_oo_E, ezx_oo_W, ezx_oo_P, ezx_oo_NW,
ezx_oo_NE, ezx_oo_SW, ezx_oo_SE...
ezx_oe_N, ezx_oe_S, ezx_oe_E, ezx_oe_W, ezx_oe_P, ezx_oe_NW,
ezx_oe_NE, ezx_oe_SW, ezx_oe_SE...
ezx_eo_N, ezx_eo_S, ezx_eo_E, ezx_eo_W, ezx_eo_P, ezx_eo_NW, ezx_eo_NE, ezx_eo_SW, ezx_eo_SE...

ezx_eo_N, ezx_eo_S, ezx_eo_E, ezx_eo_W, ezx_eo_P, ezx_eo_NW, ezx_eo_NE, ezx_eo_SW, ezx_eo_SE] = derivative_mesh(
    epsilon_mesh_double(:, :, 3), nx, ny);

ezy_oe_N, ezy_oe_S, ezy_oe_E, ezy_oe_W, ezy_oe_P, ezy_oe_NW, ezy_oe_NE, ezy_oe_SW, ezy_oe_SE...

ezy_oe_N, ezy_oe_S, ezy_oe_E, ezy_oe_W, ezy_oe_P, ezy_oe_NW, ezy_oe_NE, ezy_oe_SW, ezy_oe_SE...

ezy_eo_N, ezy_eo_S, ezy_eo_E, ezy_eo_W, ezy_eo_P, ezy_eo_NW, ezy_eo_NE, ezy_eo_SW, ezy_eo_SE]
    = derivative_mesh(
    epsilon_mesh_double(:, :, 6), nx, ny);

ezz_oe_N, ezz_oe_S, ezz_oe_E, ezz_oe_W, ezz_oe_P, ezz_oe_NW, ezz_oe_NE, ezz_oe_SW, ezz_oe_SE...

ezz_oe_N, ezz_oe_S, ezz_oe_E, ezz_oe_W, ezz_oe_P, ezz_oe_NW, ezz_oe_NE, ezz_oe_SW, ezz_oe_SE...

ezz_eo_N, ezz_eo_S, ezz_eo_E, ezz_eo_W, ezz_eo_P, ezz_eo_NW, ezz_eo_NE, ezz_eo_SW, ezz_eo_SE]
    = derivative_mesh(
    epsilon_mesh_double(:, :, 9), nx, ny);

[mux_oe_N, mux_oe_S, mux_oe_E, mux_oe_W, mux_oe_P, mux_oe_NW, mux_oe_NE, mux_oe_SW, mux_oe_SE...

[mux_oe_N, mux_oe_S, mux_oe_E, mux_oe_W, mux_oe_P, mux_oe_NW, mux_oe_NE, mux_oe_SW, mux_oe_SE...
mu_x_eo_N, mu_x_eo_S, mu_x_eo_E, mu_x_eo_W, mu_x_eo_P,
mu_x_eo_NW, mu_x_eo_NE, mu_x_eo_SW, mu_x_eo_SE...
mu_x_eo_N, mu_x_eo_S, mu_x_eo_E, mu_x_eo_W, mu_x_eo_P,
mu_x_eo_NW, mu_x_eo_NE, mu_x_eo_SW, mu_x_eo_SE]
= derivative_mesh (mu_mesh_double (:,:1),nx,ny);

mu_y_oo_N, mu_y_oo_S, mu_y_oo_E, mu_y_oo_W, mu_y_oo_P,
mu_y_oo_NE, mu_y_oo_SW, mu_y_oo_SE...
mu_y_oe_N, mu_y_oe_S, mu_y_oe_E, mu_y_oe_W, mu_y_oe_P,
mu_y_oe_NW, mu_y_oe_NE, mu_y_oe_SW, mu_y_oe_SE...
mu_y_oe_N, mu_y_oe_S, mu_y_oe_E, mu_y_oe_W, mu_y_oe_P,
mu_y_oe_NW, mu_y_oe_NE, mu_y_oe_SW, mu_y_oe_SE...
mu_y_oe_N, mu_y_oe_S, mu_y_oe_E, mu_y_oe_W, mu_y_oe_P,
mu_y_oe_NW, mu_y_oe_NE, mu_y_oe_SW, mu_y_oe_SE]
= derivative_mesh (mu_mesh_double (:,:5),nx,ny);

mu_z_oo_N, mu_z_oo_S, mu_z_oo_E, mu_z_oo_W, mu_z_oo_P,
mu_z_oo_NE, mu_z_oo_SW, mu_z_oo_SE...
mu_z_oe_N, mu_z_oe_S, mu_z_oe_E, mu_z_oe_W, mu_z_oe_P,
mu_z_oe_NW, mu_z_oe_NE, mu_z_oe_SW, mu_z_oe_SE...
mu_z_oe_N, mu_z_oe_S, mu_z_oe_E, mu_z_oe_W, mu_z_oe_P,
mu_z_oe_NW, mu_z_oe_NE, mu_z_oe_SW, mu_z_oe_SE...
mu_z_oe_N, mu_z_oe_S, mu_z_oe_E, mu_z_oe_W, mu_z_oe_P,
mu_z_oe_NW, mu_z_oe_NE, mu_z_oe_SW, mu_z_oe_SE]
= derivative_mesh (mu_mesh_double (:,:9),nx,ny);

A11_P = 1j./dx .* ezx_oo_P ./ ezz_oo_P;
\[ A_{11}.E = -1j./dx .* ezx.oo.E ./ ezz.oo.E; \]
\[ A_{12}.P = 1j./dx .* ezy.oo.P ./ ezz.oo.P; \]
\[ A_{12}.E = -1j./dx .* ezy.oo.E ./ ezz.oo.E; \]
\[ A_{13}.P = 1./((\omega.*ezz.oo.P) ./ (dx.*dy)); \]
\[ A_{13}.S = -1./((\omega.*ezz.oo.P) ./ (dx.*dy)); \]
\[ A_{13}.SE = 1./((\omega.*ezz.oo.E) ./ (dx.*dy)); \]
\[ A_{13}.E = -1./((\omega.*ezz.oo.E) ./ (dx.*dy)); \]
\[ A_{14}.P = -(1 ./ (\omega.*ezz.oo.E) + 1 ./ (\omega.*ezz.oo.P)) ./
          (dx.*dx) + \omega .* \mu_y.oe.P; \]
\[ A_{14}.E = 1./((\omega.*ezz.oo.E) ./ (dx.*dx)); \]
\[ A_{14}.W = 1./((\omega.*ezz.oo.P) ./ (dx.*dx)); \]
\[ A_{21}.P = 1j./dy .* ezx.oo.P ./ ezz.oo.P; \]
\[ A_{21}.N = -1j./dy .* ezx.oo.N ./ ezz.oo.N; \]
\[ A_{22}.P = 1j./dy .* ezy.oo.P ./ ezz.oo.P; \]
\[ A_{22}.N = -1j./dy .* ezy.oo.N ./ ezz.oo.N; \]
\[ A_{23}.P = (1 ./ (\omega .* ezz.oo.P) + 1 ./ (\omega .* ezz.oo.N)) ./
          (dy.*dy) - \omega .* \mu_x.oe.P; \]
\[ A_{23}.N = -1./((\omega.*ezz.oo.N) ./ (dy.*dy)); \]
\[ A_{23}.S = -1./((\omega.*ezz.oo.P) ./ (dy.*dy)); \]
\[ A_{24}.P = -1./((\omega.*ezz.oo.P) ./ (dy.*dx)); \]
\[ A_{24}.N = 1./((\omega.*ezz.oo.N) ./ (dy.*dx)); \]
\[ A_{24}.W = 1./((\omega.*ezz.oo.P) ./ (dy.*dx)); \]
\[ A_{24}.NW = -1./((\omega.*ezz.oo.N) ./ (dy.*dx)); \]
A31_P = -omega .* eyx_oe_P - 1./((omega.*mu_zee_E)./(dx.*dy)) + eyz_oe_P .* omega .* ezx_oo_P ./ ezz_oo_P;
A31_N = 1./((omega.*mu_zee_E)./(dx.*dy));
A31_W = 1./((omega.*mu_zee_P)./(dx.*dy));
A31_NW = -1./((omega.*mu_zee_P)./(dx.*dy));

A32_P = -omega .* eyy_oe_P + (1./((omega.*mu_zee_P)+1./((omega.*mu_zee_W)))./(dx.*dx)) + eyz_oe_P .* omega .* ezy_oo_P ./ ezz_oo_P;
A32_E = -1./((omega.*mu_zee_P)./(dx.*dx));
A32_W = -1./((omega.*mu_zee_W)./(dx.*dx));

A33_P = -lj./dy .* eyz_oe_P ./ ezz_oo_P;
A33_S = lj./dy .* eyz_oe_S ./ ezz_oo_S;

A34_P = lj./dx .* ezy_oo_P ./ ezz_oo_P;
A34_W = -lj./dx .* ezy_oo_W ./ ezz_oo_W;

A41_P = omega .* exx_oe_P - (1./((omega.*mu_zee_P)+1./((omega.*mu_zee_S)))./(dy.*dy)) - exz_ee_P .* omega .* ezx_oo_P ./ ezz_oo_P;
A41_N = 1./((omega.*mu_zee_P)./(dy.*dy));
A41_S = 1./((omega.*mu_zee_S)./(dy.*dy));

A42_P = omega .* exy_oe_P + 1./((omega.*mu_zee_P)./(dx.*dy)) - exz_ee_P .* omega .* ezy_oo_P ./ ezz_oo_P;
A42_S = -1./((omega.*mu_zee_S)./(dx.*dy));
A42_E = -1./((omega.*mu_zee_P)./(dx.*dy));
A42_SE = 1./((omega.*mu_zee_S)./(dx.*dy));

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A43_P = 1j./dy .* exz_eo_P ./ ezz_oo_P;
A43_S = -1j./dy .* exz_eo_S ./ ezz_oo_S;

A44_P = -1j./dx .* exz_eo_P ./ ezz_oo_P;
A44_W = 1j./dx .* exz_eo_W ./ ezz_oo_W;

ii = zeros(nx,ny);
ii (: ) = 1:nx*ny;
iall = zeros(1,nx*ny);
is = zeros(1,nx*(ny-1));
in = zeros(1,nx*(ny-1));
ie = zeros(1,(nx-1)*ny);
iw = zeros(1,(nx-1)*ny);
inw= zeros(1,(nx-1)*(ny-1));
ine= zeros(1,(nx-1)*(ny-1));
isw= zeros(1,(nx-1)*(ny-1));
ise= zeros(1,(nx-1)*(ny-1));

iall (: ) = ii;
is (: ) = ii(1:nx, 1:ny-1);
in (: ) = ii(1:nx, 2:ny);
iw (: ) = ii(1:nx-1,1:ny);
ie (: ) = ii(2:nx, 1:ny);
inw (: ) = ii(1:nx-1,2:ny);
ine (: ) = ii(2:nx,2:ny);
isw (: ) = ii(1:nx-1,1:ny-1);
ise (: ) = ii(2:nx, 1:ny-1);
A11 = sparse ([iall, iw], ...
    [iall, ie], ...
    [A11_P(iall), A11_E(iw)]);

A12 = sparse ([iall, iw], ...
    [iall, ie], ...
    [A12_P(iall), A12_E(iw)]);

A13 = sparse ([iall, iw, in, inw], ...
    [iall, ie, is, ise], ...

A14 = sparse ([iall, iw, ie], ...
    [iall, ie, iw], ...
    [A14_P(iall), A14_E(iw), A14.W(ie)]);

A21 = sparse ([iall, is], ...
    [iall, in], ...
    [A21_P(iall), A21.N(is)]);

A22 = sparse ([iall, is], ...
    [iall, in], ...
    [A22_P(iall), A22.N(is)]);

A23 = sparse ([iall, is, in], ...
    [iall, in, is], ...

A24 = sparse ([iall, ie, is, ise], ...

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A31 = sparse ([iall, ie, is, ise], ...)
    [iall, iw, in, inw], ...
    [A31.P(iall), A31.W(ie), A31.N(is), A31.NW(ise)]);

A32 = sparse ([iall, iw, ie], ...)
    [iall, ie, iw, ], ...
    [A32.P(iall), A32.E(iw), A32.W(ie)]);

A33 = sparse ([iall, in], ...)
    [iall, is], ...
    [A33.P(iall), A33.S(in)]);

A34 = sparse ([iall, ie], ...)
    [iall, iw], ...
    [A34.P(iall), A34.W(ie)]);

A41 = sparse ([iall, is, in], ...)
    [iall, in, is], ...
    [A41.P(iall), A41.N(is), A41.S(in)]);

A42 = sparse ([iall, iw, in, inw], ...)
    [iall, ie, is, ise], ...
    [A42.P(iall), A42.E(iw), A42.S(in), A42.SE(inw)]);

A43 = sparse ([iall, in], ...)
    [iall, is], ...
    [A43.P(iall), A43.S(in)]);
A44 = sparse ([iall, ie], ...)  
   [iall, iw], ...  
   [A44_P(iall), A44_W(ie)]);

A = sparse([A11 A12 A13 A14; A21 A22 A23 A24; A31 A32 A33 A34  
               ; A41 A42 A43 A44]);  %Eigenvalue Matrix Equation

fprintf(1, 'Solving Eigenvalue Equation for Transverse Fields  
 ... \n')
guess = (2*pi*guess_index/lambda);  
options.tol = 1e-8;  
options.disp = 0;  
tic
[v, d] = eigs(A, speye(size(A)), n_modes, guess, options);  
toc
neff = lambda*diag(d)/(2*pi);  
Beta = diag(d);

for ii=1:n_modes
    Ex_tmp(:, :ii) = reshape((v(1:nx*ny, ii)), nx, ny);
    Ey_tmp(:, :ii) = reshape((v(nx*ny+1:2*nx*ny, ii)), nx, ny);
    Hx_tmp(:, :ii) = reshape((v(2*nx*ny+1:3*nx*ny, ii)), nx, ny);
    
    %

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\[ \text{Hy\_tmp}(::, ii) = \text{reshape}((v(3*nx*ny+1:4*nx*ny, ii)), nx, ny); \]

end

fprintf(1, 'Calculating Longitudinal Fields ... \n\n')

\[ [\text{Ez\_tmp}, \text{Hz\_tmp}] = \text{longitudinal\_fields}(\omega, dx, dy, \mu_z, ee\_P, \text{ezx\_oo\_P}, \text{ezy\_oo\_P}, \text{ezz\_oo\_P}, \text{Ex\_tmp}, \text{Ey\_tmp}, \text{Hx\_tmp}, \text{Hy\_tmp}); \]

for ii=1:n\_modes
    fprintf(1, 'mode \%d: n\_eff = \%e, beta = \%2.12e \n', ii, neff(ii), Beta(ii))
end

%\%

\%\%

for ii=1:n\_modes
    \text{Ex}(::, ii) = \text{Ex\_tmp}(::, ii);
    \text{Ey}(::, ii) = \text{Ey\_tmp}(::, ii);
    \text{Ez}(::, ii) = \text{Ez\_tmp}(::, ii);

    \text{Hx}(::, ii) = \text{Hx\_tmp}(::, ii);
    \text{Hy}(::, ii) = \text{Hy\_tmp}(::, ii);
    \text{Hz}(::, ii) = \text{Hz\_tmp}(::, ii);
end
longitudinal_fields.m: This function aligns the field and tensor components correctly on the Yee lattice.

```matlab
function [Ez,Hz] = longitudinal_fields(omega,dx,dy,mu_z_ee_P,
ezx_oo_P,ezy_oo_P,ezz_oo_P,Ex,Ey,Hx,Hy)

% This function computes the longitudinal field components of a dielectric waveguide constructed from an arbitrary
% permitivity tensor, using the finite difference method.
%
% USAGE:
% %
% % % [Ez,Hz] = longitudinal_fields(omega,dx,dy,mu_z_ee_P,
ezx_oo_P,ezy_oo_P,ezz_oo_P,Ex,Ey,Hx,Hy)
%
% % INPUT:
% %
% % Ex, Ey - calculated transverse electric field components
% % Hx, Hy - calculated transverse magnetic field components
% % omega - optical angular frequency
% % dx - horizontal grid spacing (vector or scalar)
% % dy - vertical grid spacing (vector or scalar)
% % mu_z_ee_P - z-component of mu tensor
% % exz_oo_P - zx component of epsilon tensor
% % ezy_oo_P - zy component of epsilon tensor
% % ezz_oo_P - zz component of epsilon tensor
%
% % OUTPUT:
% %
```

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% Hz - calculated longitudinal magnetic field. This output will
% have the same dimensions as Hx and Hy.
% Ez - calculated longitudinal magnetic field. This output will
% have the same dimensions as Ex and Ey.

%%
%% AUTHOR: Brad G Cordova (bcordova@mit.edu)
%% Version 1.0 (created July 2012)


[nx,ny,nz] = size(Ex);

mu_z_ee_P = reshape(mu_z_ee_P,nx,ny);
ezx_oo_P = reshape(ezx_oo_P,nx,ny);
ezy_oo_P = reshape(ezy_oo_P,nx,ny);
ezz_oo_P = reshape(ezz_oo_P,nx,ny);

for ii=1:nz

Ex_j = pad_j(Ex(:, :, ii));
Hx_j = pad_j(Hx(:, :, ii));
Ey_i = pad_i(Ey(:, :, ii));
Hy_i = pad_i(Hy(:, :, ii));

Hz(:, :, ii) = -1./(1j*omega*mu_z_ee_P) .* ( 1/dx*(Ey_i(3:nx +2,:) - Ey_i(2:nx+1,:)) - 1/dy*(Ex_j(:,3:ny+2) - Ex_j(:,2: ny+1)) );

end
Ez(:, :, ii) = 1./(1j*omega*ezz_oo_P) .* (1/dx*(Hy_i(2:nx+1,:)
- Hy_i(1:nx,:)) - 1/dy*(Hx_j(:, 2:ny+1) - Hx_j(:, 1:ny)) )

... - 1./ezz_oo_P .* (ezx_oo_P .* Ex(:, :, ii) +
  ezy_oo_P .* Ey(:, :, ii)));

end
pml_tensor.m: This function aligns the field and tensor components correctly on the Yee lattice

function [PML_row_epsilon PML_column_epsilon PML_row_mu
   PML_column_mu] = pml_tensor(dxy, width, height,
   pml_parameters, epsilon_tensor, mu_tensor)

2 % This function creates the PML boundary tensor using analytic function continuation into the complex plane.
3 %
4 % USAGE:
5 %
6 % [PML_row_epsilon PML_column_epsilon ...
7 %    PML_row_mu PML_column_mu] = pml_tensor(dxy, width, height, ...
8 %       pml_parameters, epsilon_tensor, mu_tensor)
9 %
10 % INPUT:
11 %
12 % epsilon_tensor - permittivity tensor
13 % mu_tensor - permeability tensor
14 % dxy - vector [dx dy] horizontal and vertical grid spacing
15 % width - matrix defining horizontal widths of permittivity structure
16 % height - matrix defining vertical heights of permittivity structure
17 %
18 % OUTPUT:
19 %
20 % PML_row_epsilon PML_column_epsilon - row and column of PML permittivity tensor
% PML_row_mu PML_column_mu - row and column of PML permeability tensor

% AUTHOR: Brad G Cordova (bcordova@mit.edu)
% Version 1.0 (created July 2012)

fprintf(1,'Creating PML Tensor... 
');

dx = dxy(1);
dy = dxy(2);

numx = round(width/dx);
umy = round(height/dy);

nx = sum(numx);
ny = sum(numy);

width_PML = pml_parameters.width;
height_PML = pml_parameters.height;

alpha_1_max_PML = pml_parameters.alpha_1_max;
alpha_2_max_PML = pml_parameters.alpha_2_max;
alpha_3_max_PML = pml_parameters.alpha_3_max;
alpha_4_max_PML = pml_parameters.alpha_4_max;

d_PML = width_PML;
nx_PML = round(width_PML/dx);
ny_PML = round(height_PML/dy);
for ii = 1:9
    for jj=1:nx_PML
        for qq=1:8
            rho = jj*dx;
            alpha_x = alpha_x_max_PML.*(rho./dPML).^2;
            alpha_y = alpha_y_max_PML.*(rho./dPML).^2;
            alpha_1 = alpha_1_max_PML.*(rho./dPML).^2;
            alpha_2 = alpha_2_max_PML.*(rho./dPML).^2;
            alpha_3 = alpha_3_max_PML.*(rho./dPML).^2;
            alpha_4 = alpha_4_max_PML.*(rho./dPML).^2;
            s1 = 1 - 1j*alpha_1;
            s2 = 1 - 1j*alpha_2;
            s3 = 1 - 1j*alpha_3;
            s4 = 1 - 1j*alpha_4;
            s_PML = [s1 s2 s3 s4; s1 s2 s3 s4];
            sx = s_PML(qq,1);
            sy = s_PML(qq,2);
            sz = 1;
            epsilon_PML_tensor = [(sy*sz/sx)*epsilon_tensor(1,1) sz*
                                epsilon_tensor(1,2) sy*epsilon_tensor(1,3);
                                sz*epsilon_tensor(2,1) (sz*sz/sx/)
                                sy*epsilon_tensor(2,2) sx*
                                epsilon_tensor(2,3);
                                sy*epsilon_tensor(3,1) sx*
                                epsilon_tensor(3,2) (sx*sy/
                                sz)*epsilon_tensor(3,3)];
PML_row_epsilon(jj,:,qq) = epsilon_PML_tensor(ii)*ones(1,ny+2*ny_PML);
PML_column_epsilon(:,jj,qq) = epsilon_PML_tensor(ii)*ones(nx+2*nx_PML,1);
end
end

for jj=1:nx_PML
    for qq=1:8
        rho = jj*dx;
        % alpha_x = alpha_x_max_PML.*(rho./d_PML).^2;
        % alpha_y = alpha_y_max_PML.*(rho./d_PML).^2;
        alpha_1 = alpha_1_max_PML*(rho./d_PML).^2;
        alpha_2 = alpha_2_max_PML*(rho./d_PML).^2;
        alpha_3 = alpha_3_max_PML*(rho./d_PML).^2;
        alpha_4 = alpha_4_max_PML*(rho./d_PML).^2;
        s1 = 1 - 1j*alpha_1;
        s2 = 1 - 1j*alpha_2;
        s3 = 1 - 1j*alpha_3;
        s4 = 1 - 1j*alpha_4;
        s_PML = [s1 1; s2 1; 1 s3; 1 s4; s1 s3; s2 s3; s1 s4; s2 s4];
        sx = s_PML(qq,1);
        sy = s_PML(qq,2);
        sz = 1;
\begin{verbatim}
mu_PML_tensor = [ (sy*sz/sx)*mu_tensor(1,1)  0
                  0
                  0
                  (sz*sx/sy
                   )*mu_tensor(2,2)  0;
                  0
                  0
                  (sx*sy/
                   sz)*mu_tensor(3,3)];

PML_row_mu(jj,:,qq) = mu_PML_tensor(ii)*ones(1,ny+2*ny_PML);
PML_column_mu(:,jj,qq) = mu_PML_tensor(ii)*ones(nx+2*nx_PML
                   ,1); \%make bigger to account for expanded PML
end
end
end
\end{verbatim}
mode_contour.m: This function aligns the field and tensor components correctly on the Yee lattice

function mode_contour(x, y, mode, dB, xyrange)
% Produces a contour plot (in dB) of one field component of the
% mode of an optical waveguide.
% USAGE:
% contourmode(x, y, mode);
% contourmode(x, y, mode, dBrange);
% contourmode(x, y, mode, dBrange, xyrange);
% INPUT:
% x, y - vectors describing horizontal and vertical grid points
% mode - the mode or field component to be plotted
% dBrange - contour levels to plot (in dB), with 0 dB corresponding
% to the level |mode| = 1. default = (0: -3: -45)
% xyrange - axis range to use (optional)
% EXAMPLE: Make a contour plot of the magnetic field component Hx,
% with contours from 0 dB down to -50 dB, relative to the maximum
% value, in 5 dB increments.
% contourmode(x, y, Hx/max(abs(Hx(:))), (0: -5: -50));
NOTES:

(1) This function uses the current color map to determine the colors of each contour, with 0 dB corresponding to the maximum color and -dbmax corresponding to the minimum color.

You can use the 'colormap' command to change the current color map.

(2) The aspect ratio of the plot box is automatically adjusted so that the horizontal and vertical scales are equal.

(3) The mode is not normalized or scaled in any way.

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x = real(x);
y = real(y);

if (nargin < 5)
    xyrange = [min(x),max(x),min(y),max(y)];
end

if (size(mode) == [length(x)-1,length(y)-1])
    x = (x(1:end-1) + x(2:end))/2;
    y = (y(1:end-1) + y(2:end))/2;
end
if (nargin < 4) || isempty(dB)
    dB = (0:-3:-45);
end

% Compute and plot contours

c = contourc(x,y,20*log10(abs(transpose(mode))),dB);
cmap = colormap;
ii = 1;
cla;
while (ii < length(c)),
    level = c(1,ii);
n = c(2,ii);
jj = 1+round((length(cmap)-1)*(level - min(dB))/(max(dB)-min(dB)));
    color = cmap(jj,:);
    line(c(1,ii+1:ii+n),c(2,ii+1:ii+n), 'Color',color);
    ii = ii+n+1;
end
axis(xyrange);
set(gca, 'PlotBoxAspectRatio', [xyrange(2)-xyrange(1) xyrange(4)-xyrange(3) 1],...
    'Box', 'on');
mode_image.m: This function aligns the field and tensor components correctly on the Yee lattice

1 function [xf, yf, modebmp] = mode_image(x, y, mode, dx, dy)
2 % Produces a properly scaled color plot of a two-dimensional mode. This routine is especially useful when x and y are
3 % non-uniformly spaced vectors. In this case, the mode is
4 % interpolated over a uniformly-spaced grid before producing
5 % an image plot. The output can be directly saved to a file
6 % using the imwrite() function.
7 %
8 % USAGE:
9 %
10 % [xf, yf, modebmp] = imagemode(x, y, mode);
11 % [xf, yf, modebmp] = imagemode(x, y, mode, dx, dy);
12 %
13 % INPUT:
14 %
15 % x, y - vectors describing horizontal and vertical grid points
16 % mode - the mode or field component to be plotted
17 % dx, dy (optional) - fine grid spacing at which to oversample
18 % (interpolate) the mode. If left unspecified, this routine
19 % will use the smallest value of diff(x) and diff(y).
20 %
21 % OUTPUT:
22 %
23 % xf, yf - points at which the mode was interpolated
25 % modebmp - 8-bit unsigned integer array representing the mode
26 % image
27 %
28 % AUTHOR: Brad G Cordova (bcordova@mit.edu)
29 % Version 1.0 (created July 2012)
30
31 x = real(x);
32 y = real(y);
33
34 if (size(mode) == [length(x)-1,length(y)-1])
35     x = (x(1:end-1) + x(2:end))/2;
36     y = (y(1:end-1) + y(2:end))/2;
37 end
38
39 if (nargin == 3)
40     [dx,ix] = min(diff(x));
41     [dy,iy] = min(diff(y));
42     xf = (min(x):dx:max(x))';
43     yf = (min(y):dy:max(y));
44 % line up with finest portion of grid
45     delta = dx*(interp1(xf,(1:length(xf)),x(ix+1)) - ...
46             round(interp1(xf,(1:length(xf)),x(ix+1))));
47     xf = xf + delta;
48     delta = dy*(interp1(yf,(1:length(yf)),y(iy+1)) - ...
49             round(interp1(yf,(1:length(yf)),y(iy+1))));
50     yf = yf + delta;
51 % eliminate points outside of range
52     kv = find((min(x) < xf) & (xf < max(x)));
53     xf = xf(kv);
kv = find((min(y) < yf) & (yf < max(y)));  

yf = yf(kv);

else
    xf = (min(x):dx:max(x))';
    yf = (min(y):dy:max(y));
end

cmax = size(colormap,1)-1;

modebmp = uint8(transpose(interp2(y,x, ...  
    abs(cmax*mode),yf,xf)));
image(xf,yf,flipud(modebmp));
set(gca,'YDir','normal');
v = [min(xf),max(xf),min(yf),max(yf)];
axis(v);
set(gca,'PlotBoxAspectRatio',[v(2)-v(1) v(4)-v(3) 1]);
**pad.i.m:** This function aligns the field and tensor components correctly on the Yee lattice

```matlab
function B = pad_i(A)
%This function pads a matrix vertically
[nx, ny] = size(A);
B = [A(1,:);A;A(nx,:)];
end
```
pad.j.m: This function aligns the field and tensor components correctly on the Yee lattice

    function B = pad.j(A)
    %This function pads a matrix horizontally
    [nx,ny] = size(A);
    B = [A(:,1), A, A(:,ny)];
    end
simple.wg.m: This function aligns the field and tensor components correctly on the Yee lattice

% Simple waveguide example for arbitrary permittivity tensor
% mode solver

% AUTHOR: Brad G Cordova (bcordova@mit.edu)
% Version 1.0 (created July 2012)

% All units are arbitrary but self consistent (and in this case all in SI)

clear all;
close all;
clc;
tic

%path where mode solver tools are located
addpath('./tools')

dx = 0.01e-6;
dy = dx;
dxy = [dx dy];

n_modes = 5;
guess_index = 2.42;

% basic parameters

lambda = 1550e-9; % free space wavelength
mu_0 = 4*pi*1e-7; % permeability of free space
\[ \epsilon_0 = 8.85418782 \times 10^{-12}; \quad \text{permittivity of free space} \]

\[ c_0 = 299792458; \quad \text{speed of light in vacuum} \]

\[ \omega = 2\pi c_0 / \lambda; \quad \text{angular frequency of free space light} \]

\[ k_0 = 2\pi / \lambda; \quad \text{free space wave vector} \]

\[ n_{Si} = 3.48; \quad \text{index of refraction of Si} \]

\[ n_{SiO} = 1.445; \quad \text{index of refraction of SiO}_2 \]

\[ \text{doping-concentration} = 1e26; \quad \text{linspace}(1e24, 1e30, 10) \]

\[ \alpha = 8.5e-18 \times \text{doping-concentration} / (1e6); \quad \text{electrons} \]

\[ \% \alpha = 6.0e-18 \times \text{doping-concentration} / (1e6); \quad \text{holes} \]

\[ \alpha = \alpha \times 100; \]

\[ \alpha_{dB} = \alpha / 100 \times 10 \times \log_{10}(\exp(1)); \quad \text{dB cm}^{-1} \]

\[ \epsilon_{Si} = n_{Si}^2 \times \epsilon_0; \]

\[ \epsilon_{SiO} = n_{SiO}^2 \times \epsilon_0; \]

\[ k_{\alpha} = \alpha \times \lambda / (4\pi); \]

\[ n_{Si} = (3.48 + 1j \times k_{\alpha}); \]

\[ \epsilon_{Si-complex} = n_{Si}^2 \times \epsilon_0; \]

\[ \% \text{Faraday Rotation} \]

\[ B_0 = 1; \quad \% \text{T} \]

\[ N = 5e24; \]

\[ q = 1.60217646e-19; \]

\[ m = 9.10938188e-31; \]

\[ \epsilon_0 = 8.85418782e-12; \]
\texttt{FR = B0*(u*lambda^2+v/lambda^2)*doping\_concentration/(2.01e23);}

\texttt{epsilon\_g = 3.48*FR*lambda/pi*epsilon\_0;}

\texttt{\% fprintf(1,'epsilon\_g = \%e \n\n',epsilon\_g/epsilon\_0);}
epsilon_Si_complex];

epsilon_tensor_core2 = [epsilon_Si  0
                     1j*epsilon_g;
                     0  epsilon_Si
                     0;
             -1j*epsilon_g  0
                     epsilon_Si];

epsilon_tensor_SiO = [epsilon_SiO  0  0;
                     0  epsilon_SiO  0;
                     0  0  epsilon_SiO];

mu0_tensor = [mu.0  0  0;
              0  mu.0  0;
              0  0  mu.0];

%define width and height of each segment
width = [1e-6, 0.48e-6, 1e-6];
height = [1e-6...
          0.22e-6...
          1e-6];

%epsilon structure
%here you create the structure corresponding to width and height grid variables
for ii = 1:9
epsilon(:, :, ii) = [epsilon_tensor_SiO(ii), epsilon_tensor_SiO(ii), epsilon_tensor_SiO(ii);
epsilon_tensor_SiO(ii),
epsilon_tensor_core(ii),
epsilon_tensor_SiO(ii);
epsilon_tensor_SiO(ii), epsilon_tensor_SiO(ii),
epsilon_tensor_SiO(ii), epsilon_tensor_SiO(ii)];

% in this definition it goes xx, yx, zx, xy,
yy, zy, zx, xy, zz
end

% mu structure
% make sure mu and epsilon have the same number of matrix
% elements corresponding to width and height grid variables
for ii = 1:9
mu(:, :, ii) = [mu0_tensor(ii), mu0_tensor(ii), mu0_tensor(ii);
mu0_tensor(ii), mu0_tensor(ii), mu0_tensor(ii);
mu0_tensor(ii), mu0_tensor(ii), mu0_tensor(ii)];

% in this definition it goes xx, yx, zx, xy, yy, zy,
, zx, xy, zz
end

%%
% Calculation of eigenmodes, and the output of fields
eigen_modes(dxy, width, height, lambda, guess_index, n_modes,
epsilon, mu);

% beta_1 = 7.124457737634e+06;
% beta_2 = 7.118010554696e+06;
% nrps = abs(beta_1-beta_2)/(1e6)*180/pi %degrees per micron

%%
%Plotting routine
for ii=1:n_modes
figure

colormap(hot)

x = (1:nx)*dx;
xc = (1:nx)*dx - dx/2;
y = (1:ny)*dy;
yc = (1:ny)*dy - dy/2;

Ex = Ex_tmp(:, :, ii);
Ey = Ey_tmp(:, :, ii);
Ez = Ez_tmp(:, :, ii);

Hx = Hx_tmp(:, :, ii);
Hy = Hy_tmp(:, :, ii);
Hz = Hz_tmp(:, :, ii);
if max(max(abs(Hx))) > max(max(abs(Hy)))
    hn = max(max(abs(Hx)));  
else
    hn = max(max(abs(Hy)));  
end

if max(max(abs(Ex))) > max(max(abs(Ey)))
    en = max(max(abs(Ex)));  
else
    en = max(max(abs(Ey)));  
end

subplot(231)
mode_image(x,y,Ex/en);
title('Ex'); colorbar; axis xy; axis image;xlabel('x (m)');
ylabel('y (m)');

subplot(232)
mode_image(x,y,Ey/en);
title('E_y'); colorbar; axis xy; axis image;xlabel('x (m)');
ylabel('y (m)');

subplot(233)
mode_image(x,y,Ez/en);
title('E_z'); colorbar; axis xy; axis image;xlabel('x (m)');
ylabel('y (m)');

subplot(234)
mode_image(x,y,Hx/hn);
title('H.x'); colorbar; axis xy; axis image; xlabel('x (m)'); ylabel('y (m)');

subplot(235)
mode_image(x,y,Hy/hn);
title('H.y'); colorbar; axis xy; axis image; xlabel('x (m)'); ylabel('y (m)');

subplot(236)
mode_image(x,y,Hz/hn);
title('H.z'); colorbar; axis xy; axis image; xlabel('x (m)'); ylabel('y (m)');
end

for ii=1
figure
load('colormaps.mat')
colormap(whtred)
set(gcf, 'color', 'w');

x = (1:nx)*dx;
xc = (1:nx)*dx - dx/2;
y = (1:ny)*dy;
cy = (1:ny)*dy - dy/2;

Ex = Ex_tmp(:, :, ii);
Ey = Ey_tmp(:, :, ii);
Ez = Ez_tmp(:, :, ii);
Hx = Hx_tmp(:, :, ii);
Hy = Hy_tmp(:, :, ii);
Hz = Hz_tmp(:, :, ii);

if max(max(abs(Hx))) > max(max(abs(Hy)))
    hn = max(max(abs(Hx)));  
else
    hn = max(max(abs(Hy)));  
end

if max(max(abs(Ex))) > max(max(abs(Ey)))
    en = max(max(abs(Ex)));  
else
    en = max(max(abs(Ey)));  
end

mode_image(x, y, Ex/en);
title('E_x'); colorbar; axis xy; axis image;xlabel('x (m)');
ylabel('y (m)');  
end