Twisted Manolescu-Floer Spectra for Seiberg-Witten Monopoles

by

Tirasan Khandhawit

B.S., Duke University (2008)

Submitted to the Department of Mathematics
in partial fulfillment of the requirements for the degree of

Doctor of Philosophy

at the

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

June 2013

Author ....
Department of Mathematics
May 1, 2013

Certified by ....
Tomasz Mrowka
Singer Professor of Mathematics
Thesis Supervisor

Accepted by ....
Paul Seidel
Chairman, Department Committee on Graduate Students

Twisted Manolescu-Floer Spectra for Seiberg-Witten Monopoles

by

Tirasan Khandhawit

Submitted to the Department of Mathematics
on May 1, 2013, in partial fulfillment of the
requirements for the degree of
Doctor of Philosophy

Abstract

In this thesis, we extend Manolescus and Kronheimer-Manolescus construction of Floer homotopy type to general 3-manifolds. This Floer homotopy type is a candidate for an object whose suitable homology groups recover Floer homology. The main idea is to apply finite dimensional approximation technique and Conley index theory to Seiberg-Witten theory of 3-manifolds. Another part of the construction involves a concept of twisted parametrized spectra introduced by Douglas. We also provide explicit computation for the manifolds $S^1 \times S^2$ and T^3.

Thesis Supervisor: Tomasz Mrowka
Title: Singer Professor of Mathematics

Acknowledgments

Upon reaching this point in my journey of mathematics and life, I feel grateful to so many people. First, I am greatly indebted to my advisor Tom Mrowka for entrusting me this project. His encouragement and patience are invaluable to me throughout my graduate study.

I would like to thank Peter Kronheimer and Paul Seidel for serving on my thesis committees. I would also like to thank Peter Kronheimer and Mikio Furuta for helpful suggestions.

I feel very fortunate to become a part of a warm academic family. I enjoyed interaction with Ben Mares, Tim Nguyen, Steven Sivek, Nikhil Savale, Lucas Culler, Yasha Berchenko-Kogan, and Francesco Lin at MIT. Lenny Ng and Anda Degaratu, whom I met at Duke, also greatly inspire me.

I was surrounded by great friends at MIT and Boston area. I am glad to be able to meet Nan Li, Yan Zhang, Höskuldur Halldórsson, Jethro Van Ekeren, Josh Batson and other people at the department. I thank my wonderful housemates Sutheera Ratanasirintrawoot, Virot Chiraphadhanakul, Thiparat Chotibut, Wuttisak Trongsiriwat, Pichet Adstamongkonkul, as well as, Sira Sriswasdi. I apologize that I could not list all my Thai friends here.

I am also grateful to many people back in Thailand, especially all my math teachers. And finally, I cannot be here without my family and their limitless support.

Contents

Chapter 1

Introduction

1.1 Background

Since its invention by Floer [9] in 1988, Floer homology has been an important tool in the study of low-dimensional topology. The Seiberg-Witten version of Floer theory, also known as monopole Floer homology, is developed by Kronheimer and Mrowka in their book [18]. The theory associates abelian groups to 3-manifolds and homomorphisms to cobordisms between them.

There are now a number of Floer homology theories of 3-manifolds and 4-dimensional cobordisms ([14], [27]). Many important results come from these and equivalences between them. There is a natural question of building a space, or a more general object, whose homology is Floer homology. This *Floer homotopy type* would potentially give stronger invariants in low-dimensional topology and provide new insights and applications of Floer theories.

The quest for finding Floer homotopy type was started by Cohen, Jones, and Segal in [4]. Floer homotopy type for Seiberg-Witten theory was successfully constructed by Manolescu [22] for 3-manifolds with $b_1 = 0$ in 2003. Later, Kronheimer and Manolescu [17] extended the result to the case $b_1 = 1$ with nontorsion spin^c structure. Recently, Lipshitz and Sarkar [21] constructed a Khovanov homotopy type, a spectrum whose singular homology is Khovanov homology. This homotopy type can distinguish knots with the same Khovanov homology [32].

The idea of finite dimensional approximation is to approximate a map or a flow on an infinite-dimensional vector space by using sufficiently large finite-dimensional subspaces. The use of finite dimensional approximation in nonlinear analysis dates back to Švarc's work in 1964 [33]. Furuta [11] applied the technique to prove the 10/8-theorem. Later, Bauer and Furuta [2],[3] constructed a stable cohomotopy invariant from Seiberg-Witten theory on closed 4-manifolds. Similar ideas can be used to construct stable homotopy invariants from flows on a Hilbert space. Finite dimensional approximation for flows was studied by Gęba, Izydorek, and Pruszko [12] for a flow on a Hilbert space and by Manolescu [22] in the Seiberg-Witten case.

1.2 Overview

In Chapter 2, we provide a background in Conley index theory which will be used in this thesis. We also make a refinement and clarification of some results.

In Chapter 3, we study finite dimensional approximation on Hilbert spaces motivated by [12]. We hope that this gives a unifying approach for finite dimensional approximation in more general context.

In Chapter 4, we extend the construction of Seiberg-Witten-Floer stable homotopy type in [17], [22], [23], and [24]. We still apply finite dimensional approximation to the Seiberg-Witten flow on the Coulomb slice. The resulting object is a (pro)-spectrum $SWF(Y, \mathfrak{s})$, called the Manolescu-Floer spectrum. This can be viewed as a universal cover of the Floer homotopy type because there is still an action from harmonic gauge groups left.

In Chapter 5, we provide explicit calculation of the Manolescu-Floer spectrum for the manifolds $S^1 \times S^2$ and T^3.

In Chapter 6, we use the concept introduced in [8] to construct the twisted Manolescu-Floer spectrum $\widetilde{SWF}(Y, \mathfrak{s})$. We also discuss the process of taking homology of $\widetilde{SWF}(Y, \mathfrak{s})$ and try to compute relevant homology groups.

In the Appendix, we provide background and various computation in homology.

Chapter 2

Conley Theory

2.1 Basic Definitions

Let Ω be a locally compact, Hausdorff topological space. A *flow* on Ω is a continuous map $\gamma : \Omega \times \mathbb{R} \to \Omega$ such that $\gamma(x, 0) = x$ and $\gamma(x, s+t) = \gamma(\gamma(x, s), t)$ for all $x \in \Omega$ and s, t any real numbers. We will denote the image $\gamma(x, t)$ by $\gamma_t(x)$, or simply $x \cdot t$ when understood. Our context of interest will be a flow on a vector space generated by a gradient-like vector field.

The main objects in the study of Conley theory are isolating neighborhoods and isolated invariant sets. Let us introduce the following definitions.

Definition 1. Let M be a subset of Ω.

(i) Denote by $A^+(M) := \{x \in M | x \cdot \mathbb{R}^+ \subset M\}$, the invariant subset in positive time direction.

(ii) Denote by $A^-(M) = \{x \in M | x \cdot \mathbb{R}^- \subset M\}$, the invariant subset in positive time direction.

(iii) The *maximal invariant subset* of M is given by $\mathrm{Inv}(M) = \{x \in M | x \cdot \mathbb{R} \subset M\}$. Note that $\mathrm{Inv}(M) = A^+(M) \cap A^-(M)$.

(iv) For subsets $Z \subset Y \subset \Omega$, Z is *positively invariant* relative to Y if the condition $x \in Z$ and $x \cdot [0, t] \subset Y$ implies $x \cdot [0, t] \subset Z$.

(v) A compact subset X of Ω is called an *isolating neighborhood* if $\text{Inv}(X)$ is contained in $\text{Int}(X)$ the interior of X.

(vi) A compact subset S of Ω is called an *isolated invariant set* if there is an isolating neighborhood X so that $\text{Inv}(X) = S$.

Given an isolated invariant set or an isolating neighborhood, one will be able to extract some topological data, which can be viewed as a generalization of a Morse index (in the case the flow is generated by a gradient vector field). Now, we introduce an important concept of an index pair.

Definition 2. Let S be an isolated invariant set. A pair of compact subsets (N, L) is called an index pair for S if the following conditions hold

(i) $S \subset N \backslash L$ (this implies $S = \text{Inv}(N \backslash L)$),

(ii) L is positively invariant relative to N,

(iii) L is an exit set for N, i.e. if $x \in N$ but $x \cdot [0, \infty) \not\subset N$, then there exists $t > 0$ such that $x \cdot [0, t] \subset N$ and $x \cdot t \in L$.

For an isolating neighborhood X with $\text{Inv}(X) = S$, we will also called (N, L) an index pair for X if it is an index pair for S. This definition does not depend on X but sometimes it is convenient to emphasize the isolating neighborhood instead of the isolated invariant set. This leads to another definition and we hope to make some clarification here.

Definition 3. Let X be an isolating neighborhood with $\text{Inv}(X) = S$. A pair of compact subsets (N, L) is called an index pair for S relative to X if the following conditions hold

(i) $S \subset N \backslash L$,

(ii) N, L are positively invariant relative to X,

(iii) If $x \in N$ but $x \cdot [0, \infty) \not\subset X$, then there exists $t > 0$ such that $x \cdot [0, t] \subset N$ and $x \cdot t \in L$.

The last condition can be viewed as L is an exit set for N relative to X. The above definition is used by Conley in his original work [5], whereas the one in Definition 2 is used in [19] and [31].

Example 1. Consider a flow on $\mathbb{R}^2$ given by $(x, y) \cdot t = (2^{-t}x, 2^t y)$. The origin $(0, 0)$ is an isolated invariant set, as a hyperbolic fixed point. Indeed, any neighborhood of the origin is an isolating neighborhood. Let us pick the square $D = [-2, 2]^2$ for a fixed isolating neighborhood and consider its horizontal sides $L = [-2, 2] \times \{\pm 2\}$. One can see that (D, L) is an index pair for the origin.

Let also consider a smaller square $D_1 = [-1, 1]^2$ its horizontal sides $L_1 = [-1, 1] \times \{\pm 1\}$. The pair (D_1, L_1) is an index pair for the origin, however it is not an index pair relative to D.

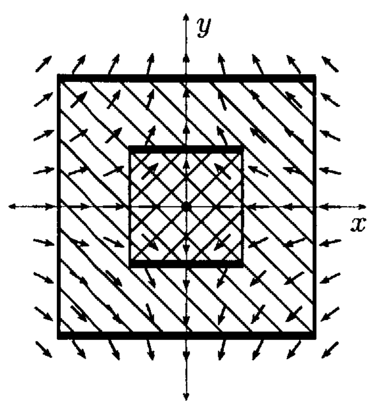


Figure 2-1: Examples of index pairs

We note that a condition for of a pair (N, L) in Definition 3 does not require the inclusion $L \subset N$. Nevertheless, we can consider the pair $(N \cup L, L)$ or the pair $(N, N \cap L)$, if one would demand the inclusion.

It's clear that an index pair relative to X is always an index pair for its isolated invariant set, but the converse does not hold in general as in the previous example. Still, one can modify an index pair to an index pair relative to an isolating neighborhood. We introduce another definition.

Definition 4. Let Y be a subset of X. We define the *minimal positively invariant* set of Y relative to X as the set $P(Y, X) = \{y \cdot t \mid y \in Y \text{and } y \cdot [0, t] \subset X\}$.

To obtain an index pair relative to X, the idea is to enlarge L by its maximal positively invariant set. We will deduce the statement by the following lemmas.

Lemma 1.

(i) If (N, L) is an index pair relative to X, then $L \cap A^+(X) = \emptyset$.

(ii) Let Y be a compact subset of X disjoint from $A^+(X)$, then the set $P(Y, X)$ is compact. Moreover, we have $P(Y, X) \cap A^+(X) = \emptyset$.

Proof.

(i) Suppose that $x \in L \cap A^+(X)$. Since $x \cdot \mathbb{R}^+ \subset X$, we have $x \cdot \mathbb{R}^+ \subset L$ because L is positively invariant. Since L is compact, the limit point $\lim_{t\to\infty} x \cdot t$ is also in L. This will contradict with $S \subset N \backslash L$ since the limit point lies in S.

(ii) (cf. [22]) We will show that $P(Y, X)$ is closed. Suppose that $x_n = y_n \cdot t_n$ converges to x for $y_n \in Y$ and $y_n \cdot [0, t_n] \subset X$. Since Y is compact, we pass to a subsequence that $y_n \to y$. We claim that the sequence $\{t_n\}$ is bounded, so that $t_n \to t$. By continuity, we see that $y \cdot [0, t] \subset X$ and $y_n \cdot t_n \to y \cdot t$. Hence $x \in P(Y, X)$ as desired.

If $\{t_n\}$ is not bounded, we can also choose a subsequence such that $y_{a_n} \cdot [0, n] \subset Y$. By continuity, we have $y \cdot [0, \infty) \subset X$ which contradicts to the fact that $Y \cap A^+(X) = \emptyset$.

□

Corollary 1. *Let X be an isolating neighborhood with* $\mathrm{Inv}(X) = S$ *and (N, L) be an index pair for S with $L, N \subset X$. Then, the pair $(N \cup P(L, X), P(L, X))$ is an index pair for S relative to X*

Proof. We have just shown that $P(L, X)$ is compact. The only remaining nontrivial part is to show that $N \cup P(L, X)$ is positively invariant relative to X. Suppose that $x \in N$ (it is obvious when $x \in P(L, X)$) and that $x \cdot [0, t] \subset X$, then we want to show that $x \cdot [0, t] \subset N \cup P(L, X)$. We can assume that $x \cdot \mathbb{R}^+ \not\subset N$ or we are done.

Since L is an exit set, there is t' such that $x \cdot [0, t'] \subset N$ and $x \cdot t' \in L$. The case $t' \geq t$ is trivial. If $t' < t$, we have that $x \cdot [t', t] \subset P(L, X)$ by its definition. Hence $x \cdot [0, t] \subset N \cup P(L, X)$. □

Two basic results in Conley theory are that, given a fixed isolated invariant set (or a fixed isolating neighborhood), an index pair always exists and that all the pointed spaces of the form $(N/L, [L])$, where (N, L) is an index pair, are homotopy equivalent. This leads to a definition

Definition 5. For an isolated invariant set (or an isolating neighborhood) S, we define its (homotopy) Conley index as a homotopy type of a pointed space $(N/L, [L])$ where (N, L) is an index pair of S.

To determine the Conley index of an isolated invariant set, one convenient way is to find a special isolating neighborhood such that every point on its boundary leaves the neighborhood immediately in one or another time direction. More precisely,

Definition 6. (cf. [6] , [30]) For a compact subset N, we define

$$n^+ := \{x \in \partial N \mid \exists\, \delta_0 > 0 \text{ such that } x \cdot (-\delta_0, 0) \cap N = \emptyset\},$$
$$n^- := \{x \in \partial N \mid \exists\, \delta_0 > 0 \text{ such that } x \cdot (0, \delta_0) \cap N = \emptyset\}.$$

Then, N is called an *isolating block* if $\partial N = n^+ \cup n^-$. The set n^- will also be called an exit set.

One immediately see that an isolating block N is an isolating neighborhood and the pair (N, n^-) is its index pair. To understand the homotopy type of $(N/n^-, [n^-])$, we introduce some useful lemmas.

Lemma 2. *[5] Let Y be a compact set and Z be a locally contractible compact subset of Y. Suppose that Z is contractible to a point y_0 of Y and that Z is a strong deformation retract of its neighborhood in Y. Then $(Y/Z, [Z])$ is homotopy equivalent to $(Y, y_0) \vee (S^1 \wedge (Z, z_0))$ for any point z_0 of Z.*

Note. The condition that Z is a strong deformation retract of its neighborhood in Y, namely (Y, Z) is an NDR-pair, is equivalent to that $Z \hookrightarrow Y$ is a cofibration.

The main idea is that Y/Z is homotopy equivalent to the mapping cone with the cone point as its based point. Then we use the contraction of Z to y_0 to collapse the mapping region to a point, so that the cone part becomes the (unreduced) suspension of Z joining to Y at y_0.

Consequently, if the pair (N, n^-) satisfies above conditions, one only needs to understand homotopy types of N and n^- individually. Furthermore, we have

Lemma 3. *[6] n^- is a strong deformation retract of $N - A^+(N)$.*

Let us introduce another useful concept of a product flow.

Definition 7. Let γ_1, γ_2 be a flow on Ω_1, Ω_2 respectively. A product flow $\gamma_1 \times \gamma_2$ is defined on $\Omega_1 \times \Omega_2$ by $(x_1, x_2) \cdot t \mapsto (\gamma_1(x_1, t), \gamma_2(x_2, t))$.

It is not hard to check that, for $i = 1, 2$, if X_i is an isolating neighborhood of S_i with respect to a flow γ_i on Ω_i, then $X_1 \times X_2$ is an isolating neighborhood of $S_1 \times S_2$ with respect to the product flow. Moreover, if (N_i, L_i) is an index pair for S_i, then $(N_1 \times N_2, N_1 \times L_2 \cup L_1 \times N_2)$ is an index pair of $S_1 \times S_2$. In other words, the Conley index of the product $S_1 \times S_2$ is given by a smash product $(N_1/L_1, [L_1]) \wedge (N_1/L_1, [L_1])$

Example 2. An important class of examples is a *linear flow* on a finite-dimensional vector space. Let L be a linear map on V and consider a flow γ_L given by a formula $\gamma_L(v, t) = e^{-tL}v$. For simplicity, we assume that L is self-adjoint and has no kernel. It is called a linear flow because $\gamma_L(v, t)$ is an integral curve of the following ODE

$$-\frac{\partial}{\partial t}\gamma_L(v, t) = L\,\gamma_L(v, t). \tag{2.1.1}$$

We can see that $\{0\}$ is an isolated invariant set of this flow. Let us decompose $V = V^+ \oplus V^-$ to the positive and the negative eigenspace of L. As in Example 1, one can check that $(B(V^+) \times B(V^-), B(V^+) \times S(V^-))$ is an index pair for $\{0\}$, where $B(W)$ and $S(W)$ denote a unit disk and a unit sphere in W respectively.

Consequently, the Conley index has a homotopy type of $B(V^-)/S(V^-)$ which can be identified with the sphere S^{V^-}, the compactification of V^- with a based point at infinity.

Note that one can replace (V^+, V^-) by any pair of maximal positive and maximal negative subspace of V with respect to L. The flow γ_L can be regarded as a product flow on $V^+ \times V^-$. One can decompose further to any direct sum of L-invariant subspaces.

Alternatively, we can show that $B(V)$ is an isolating block of the origin. On the unit sphere, a point is leaving $B(V)$ if $\langle Lv, v\rangle < 0$ and is entering $B(V)$ if $\langle Lv, v\rangle > 0$. When $\langle Lv, v\rangle = 0$, we consider the second derivative of the norm a trajectory $\gamma_L(t)$

$$\frac{d}{dt} \|\gamma_L(t)\|^2 = 4\langle L\gamma_L(t), L\gamma_L(t)\rangle = 4\,\|L\,\gamma_L(t)\|^2 .$$

This is always positive (except at the origin) because L has no kernel. It follows that a point on $S(V)$ with $\langle Lv, v\rangle = 0$ is also leaving $B(V)$ as a bounce-off point.

Note. In (2.1.1), we have a minus sign as we will be considering a downward gradient flow throughout this paper. A relevant functional on V is $f(v) = \frac{1}{2}\langle Lv, v\rangle$. The dimension of V^- agrees with the index of the origin as a critical point.

2.2 Conley Index as a Connected Simple System

We have mentioned that the homotopy type of $(N/L, [L])$ is an invariant for an isolated invariant set. It is sometimes useful to consider a collection of all index pairs and natural homotopy equivalences between them. One motivation is to reduce ambiguity of the choice of index pairs and maps when dealing with 'morphisms' between Conley indices.

For this purpose, we introduce a notion of a connected simple system. Roughly speaking, a connected simple system is a subcategory of a homotopy category of pointed topological spaces. The 'connected' part means that there is a map between every pair of objects and the 'simple' part means that all the maps between a pair of objects are in the same homotopy class of a homotopy equivalence.

Definition 8. A *connected simple system* $\mathcal{I} = (\mathcal{I}_o, \mathcal{I}_m)$ consists of a collection $\mathcal{I}_o$ of pointed spaces and a collection $\mathcal{I}_m$ of homotopy classes of pointed maps between them. In particular, for each X and Y in $\mathcal{I}_o$, a homotopy class $\mathcal{I}_m(X,Y) \in [X,Y]$ is specified with the following conditions

(i) $\mathcal{I}_m(X,X) = [id_X]$ for all $X \in \mathcal{I}_o$

(ii) If $\mathcal{I}_m(X,Y) = [f]$ and $\mathcal{I}_m(Y,Z) = [g]$, then $\mathcal{I}_m(X,Z) = [g \circ f]$ for each $X, Y, Z \in \mathcal{I}_o$.

Next, we describe a morphism between connected simple systems.

Definition 9. Let $\mathcal{I}, \mathcal{J}$ be connected simple systems. A morphism $\Phi : \mathcal{I} \to \mathcal{J}$ is a collection of homotopy classes of maps between their objects. For each $X \in \mathcal{I}_o$ and $X' \in \mathcal{J}_o$, a homotopy class $\Phi(X,X') \in [X,X']$ is specified with a property that

(i) If $\mathcal{I}_m(Y,X) = [f]$, $\Phi(X,X') = [\phi]$, and $\mathcal{J}_m(X',Y') = [g]$, then $\Phi(Y,Y') = [g \circ \phi \circ f]$ for any $X, Y \in \mathcal{I}_o$ and $X', Y' \in \mathcal{J}_o$.

One can see that choosing a homotopy class of maps between a certain $X \in \mathcal{I}_o$ and a certain $X' \in \mathcal{J}_o$ is sufficient to construct a morphism $\Phi : \mathcal{I} \to \mathcal{J}$.

With the above definitions, we can form a category **CSS** of connected simple systems.

Back to Conley theory, we describe natural maps between index pairs of an isolated invariant set. Given two index pairs (N_1, L_1) and (N_2, L_2), one can find $T \geq 0$ (sometimes called the common squeeze time) so that for all $t \geq T$ we have

(i) $x \cdot [-t,t] \subset N_1 \backslash L_1$ implies $x \in N_2 \backslash L_2$

(ii) $x \cdot [-t,t] \subset N_2 \backslash L_2$ implies $x \in N_1 \backslash L_1$

Then we have a continuous map $f : N_1/L_1 \times [T, \infty) \to N_2/L_2$ induced by the flow defined by

$$f([x],t) = \begin{cases} [x \cdot 3t] & \text{if } x \cdot [0,2t] \subset N_1 \backslash L_1 \text{ and } x \cdot [t,3t] \subset N_2 \backslash L_2, \\ [L_2] & \text{otherwise.} \end{cases}$$

We will refer to the maps in this form as the *flow map*. It is not hard to see that a composition of flow maps is also a flow map.

Definition 10. For an isolated invariant set (or an isolating neighborhood) S, we define its Conley index $\mathcal{I}(S)$ as a connected simple system whose objects consist of pointed spaces $(N/L, [L])$ arising from is index pairs (N, L) of S. The morphisms consist of homotopy classes of flow maps defined above. The above discussion guarantees that $\mathcal{I}(S)$ is a connected simple system.

We note that the definition for $\mathcal{I}(S)$ above is introduced by Salamon [31] and is different from Conley's original definition [5] since their choices of maps between index pairs are different. Nonetheless, they are shown to be equivalent by Kurland in [19]. As in [16], Conley theory can also be formulated as a functor from a category of isolated invariant sets to **CSS**.

Given a pointed space X, one can consider a connected simple system $[X]$ consisting of X as the only object with a class of the identity map. This gives a functor from $\mathbf{Top}_*$ to **CSS**.

Given two connected simple systems $\mathcal{I}$ and $\mathcal{J}$, one can form a smash product $\mathcal{I} \wedge \mathcal{J}$ whose objects and classes of maps are given by

$$(\mathcal{I} \wedge \mathcal{J})_o = \{X \wedge X' \mid X \in \mathcal{I} \text{ and } X' \in \mathcal{J}\}, \tag{2.2.1}$$

$$(\mathcal{I} \wedge \mathcal{J})_m(X \wedge X', Y \wedge Y') = \mathcal{I}_m(X,Y) \wedge \mathcal{J}_m(X',Y'). \tag{2.2.2}$$

This characterizes a Conley index for a product flow as

$$\mathcal{I}(S_1 \times S_2, \gamma_1 \times \gamma_2) = \mathcal{I}(S_1, \gamma_1) \wedge \mathcal{I}(S_2, \gamma_2). \tag{2.2.3}$$

Using above convention, we can also talk about a suspension of a connected simple system by saying $\Sigma\mathcal{I} = [S^1] \wedge \mathcal{I}$.

2.3 Attractor-Repeller Pairs

Under certain circumstance, Conley indices of isolated invariant sets can be related. We present two important constructions.

The first case is when an isolated invariant set has a decomposition as following

Definition 11. Let S be an isolated invariant set and define the ω-limit sets $\omega(U) := \bigcap_{t>0} \mathrm{cl}(U \cdot [t, \infty))$ and $\omega^*(U) := \bigcap_{t>0} \mathrm{cl}(U \cdot (-\infty, -t])$.

(i) A compact subset A of S is called an *attractor* (relative to S) if there is a neighborhood U of A in S such that the limit set $\omega(U) = A$.

(ii) A compact subset B of S is called a *repeller* (relative to S) if there is a neighborhood U of B in S such that the limit set $\omega^*(U) = B$.

(iii) For an attractor $A \subset S$, define $A^* := \{x \in S \,|\, \omega(x) \cap A = \emptyset\}$. This is a repeller and is called the *complementary repeller* of A. Moreover (A, A^*) is called an *attractor-repeller pair.*

Note that an attractor and a repeller are also isolated invariant sets. A basic result for an attractor-repeller pair (A, A^*) of S is that on can find compacts subset $N_3 \subset N_2 \subset N_1$ so that (N_2, N_3) is an index pair for A, (N_1, N_3) is an index pair for A, (N_1, N_2) is an index pair for A^*. The inclusions give a sequence of maps $N_2/N_3 \to N_1/N_3 \to N_1/N_2$ and one can also choose (N_1, N_2) to be an NDR-pair and define a map $N_1/N_2 \to \Sigma(N_2/N_3)$. This induces well-defined morphisms between Conley indices, i.e. they are independent of choices in the above constructions. In summary,

Proposition 1. *[31] The above construction induces a coexact sequence of connected simple systems*

$$\mathcal{I}(A) \to \mathcal{I}(S) \to \mathcal{I}(A^*) \to \Sigma\mathcal{I}(A) \to \cdots .$$

2.4 Equivariant Conley Index

The notion of equivariant Conley index was introduced by Floer in [10]. Let G be a compact Lie group and consider a G-equivariant flow on a G-space Ω. Most of the construction earlier can be done in the equivariant setting by replacing all the spaces and maps by equivariant ones.

For example, given a G-invariant isolating neighborhood or a G-invariant isolated invariant set, we define a G-index pair by a pair of G-spaces which is an index pair nonequivariantly (Definition 2 or 3). Indeed, given a nonequivariant index pair (N, L), we can consider a pair obtained by its orbit $(G \cdot N, G \cdot L)$. It is straightforward to check that this will be a G-index pair. A homotopy Conley index is the based G-space $(N/L, [L])$ and a collection of such spaces and flow maps forms an equivariant connected simple system.

One need to slightly modify Lemma 2 when entering equivariant setting because a G-space may not contain a G-based point (for example, when the G-action is free). Consequently, the reduced suspension for an unbased G-space may not defined, unlike the nonequivariant case where one can pick some point of the space for a based point. We restate the result

Lemma 4. *Let (Y, Z) be a G-NDR pair and suppose that Z is contractible (equivairantly) to a point y_0 of Y. Then a G-space $(Y/Z, [Z])$ is equivariantly homotopy equivalent to a G-space $(Y, y_0) \vee (\mathrm{S}Z, [Z \times \{0\}])$, where $\mathrm{S}Z := Z \times [0,1]/\{(z_1, 0) \sim (z_2, 0)$ and $(z_1, 1) \sim (z_2, 1)\}$ is the unreduced suspension.*

Note that $\mathrm{S}Z$ has at least two fixed points, so it is never free as a based G-space even if Z is free. For a basic example, we take an orthogonal G-representation V and consider the unit disk $D(V)$ and the unit sphere $S(V)$ as unbased G-spaces. The pair $(D(V), S(V))$ is a G-NDR pair and $D(V)/S(V)$ is homeomorphic to S^V, the one-point compactification of V with $\{\infty\}$ as a based point. Moreover, after adding a based point to $D(V)$ and $S(V)$, we have a cofiber sequence

$$S(V)_+ \to D(V)_+ \to S^V.$$

2.5 Duality

Given a flow γ, we can consider its reverse flow $-\gamma$, i.e. $-\gamma(t) = \gamma(-t)$. Notice that an isolating neighborhood and its isolated invariant set does not depend on the direction of the flow, but its Conley index can change. Under some circumstances, we will show that these indices are dual of each other in homotopy theoretic sense.

Consider a flow γ on a finite-dimensional smooth manifold M. Let X be an isolating neighborhood. Following Robbin and Salamon [29], we can find a special index pair (N, L) such that N is a submanifold (with corners) and its boundary decomposes to exit sets of γ and $-\gamma$. More precisely,

Proposition 2. *For an isolating neighborhood X, there is an index pair (N, L_+) such that N is a submanifold with boundary $\partial N = L_+ \cup L_-$ and (N, L_-) is an index pair of X with respect to $-\gamma$.*

Proof. Let S be the maximal invariant subset of X. First, recall that we can construct a smooth Lyapunov function $f : X \to \mathbb{R}$ such that $f(x) = 0$ on S and $f(t\cdot x)) < f(x)$ whenever $[0, t]\cdot x \subset X - S$ and $t > 0$ (cf. [29]).

For small $\delta > 0$, we choose sufficiently small $\epsilon > 0$ so that $f(-\delta \cdot x) - f(x) > 2\epsilon$ for all $x \in \partial X$ (We might need to extend f to a neighborhood of X that contains $[-\delta, \delta]\cdot X$). Then, we set

$$\begin{aligned} N &= \{x \in X | -\epsilon \leq f(x) \leq f(-\delta \cdot x) \leq \epsilon\} \\ &= X \cap f^{-1}[-\epsilon, \infty) \cap (f \circ (-\delta\cdot))^{-1}(-\infty, \epsilon]. \end{aligned}$$

From our choice of ϵ, we see that $N \subset \mathrm{Int}(X)$. By choosing a regular value, we can ensure that N is a submanifold with boundary $\partial N = L_+ \cup L_-$, where

$$L_+ = N \cap f^{-1}(-\epsilon)$$
$$L_- = N \cap \left(\delta \cdot f^{-1}(\epsilon)\right).$$

It is clear that $S \subset \mathrm{Int}(N - L_+)$ and that L_+ is positively invariant. Suppose that $x \in N$ but $t \cdot x \notin N$ for some $t > 0$. Since $N \subset \mathrm{Int}(X)$, we can assume that $[0,t] \cdot x \subset X$ so that $f(t \cdot x) < f(x)$. Similarly, we have $f((-\delta + t) \cdot x) < f(-\delta \cdot x) \leq \epsilon$. This implies that $f(t \cdot x) < -\epsilon$, so the path $[0,t] \cdot x$ passes through the set L_+. Hence (N, L_+) is an index pair for X with respect to γ. One can check that (N, L_-) is an index pair for the reverse flow by similar argument.

□

Example 3. One simple example is the case when $M = \mathbb{R}^2$ with a flow given by $t \cdot (x,y) = (2^t x, 2^{-t} y)$. The origin $(0,0)$ is an isolated invariant set, as a hyperbolic fixed point. We can take the square $X = [0,1]^2$ for its isolating neighborhood. We can use $f(x,y) = y^2 - x^2$ as a Lyapunov function in the proof above (see Figure 2-2).

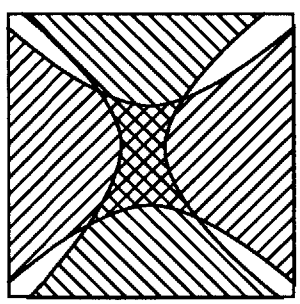


Figure 2-2: This demonstrates a special isolating neighborhood for the origin.

From now on, we will focus on a setting where we have an equivariant flow on a G-representation V. Note that, in the proof above, we can average f over the action of G so that N, L_+, and L_- are G-invariant manifolds.

Before proceeding, we recall some definitions from [25].

Definition 12. Let X and Y be based G-spaces and V be a representation of G. We say that X and Y are V- dual if there are G-maps

$$\epsilon : Y \wedge X \to S^V \text{ and } \eta : S^V \to X \wedge Y$$

such that the following diagrams are stably homotopy commutative.

$$\begin{array}{ccc} S^V \wedge X & \xrightarrow{\eta \wedge \mathrm{id}} & X \wedge Y \wedge X \\ & \searrow^{\tau} & \downarrow{\scriptstyle \mathrm{id} \wedge \epsilon} \\ & & X \wedge S^V \end{array}$$

and

$$\begin{array}{ccc} Y \wedge S^V & \xrightarrow{\mathrm{id} \wedge \eta} & Y \wedge X \wedge Y \\ {\scriptstyle \tau}\downarrow & & \downarrow{\scriptstyle \epsilon \wedge \mathrm{id}} \\ S^V \wedge Y & \xrightarrow{\sigma \wedge \mathrm{id}} & S^V \wedge Y, \end{array}$$

where τ is the transposition map and σ is the sign map $\sigma(v) = -v$.

We now state the duality result.

Proposition 3. *Let γ be a G-flow on a finite dimensional G-representation V and denote $-\gamma$ by its reverse flow. For a G-invariant isolating neighborhood X, the Conley indices $\mathcal{I}(X, \gamma)$ and $\mathcal{I}(X, -\gamma)$ are V-dual.*

Proof. We choose an index pair (N, L_+) as in the previous proposition. It is clear that (N, L_+) is a G-ENR pair. By results in [20] and [25], the unreduced mapping cones $C(N, L_+)$ and $C(V - L_+, V - N)$ are V-dual. Then, we notice that $C(N, L_+)$ has a homotopy type of N/L_+ while $C(V - L_+, V - N)$ has a homotopy type of N/L_-. $\square$

2.6 Maps to Conley indices

We will also need to construct maps from spaces to Conley indices. We begin with a lemma shown in [22].

Lemma 5. *Let X be an isolating neighborhood with* $\mathrm{Inv}(X) = S$. *If a pair (A, B) of compact subsets of X satisfies the following*

(i) If $x \in A^+(X) \cap A$, then $[0, \infty) \cdot x \cap \partial X = \emptyset$,

(ii) $B \cap A^+(X) = \emptyset$,

then there exists an index pair (N, L) of S such that $A \subset N$ and $B \subset L$.

In application, suppose we have a map $f : M \to X$ and a subspace K of M. If it turns out that the pair $(f(M), f(K))$ satisfies the hypothesis of the previous lemma, then we can find an index pair (N, L) and obtain a map induced from the inclusion $f : M/K \to N/L$.

It remains to show that this map is independent (up to homotopy) of the choice of index pairs. That is it is a well-defined map from M/K to a connected simple system.

Given two index pairs (N_1, L_1) and (N_2, L_2) with $A \subset N_1, N_2$ and $B \subset L_1, L_2$, we wish to show that the maps $F^t \circ \imath_1$ and $\imath_2$, in the diagram below, are homotopic

$$\begin{array}{ccc} A/B & \hookrightarrow & N_1/L_1 \\ & \searrow & \downarrow F_1^t \\ & & N_2/L_2, \end{array}$$

where $\imath_1, \imath_2$ are inclusions and F_1^t is a flow map.

The result from [19] implies that this is the case when we have the inclusion $(N_1, L_1) \subset (N_2, L_2)$. More precisely, there is a homotopy between the composition $N_1/L_1 \hookrightarrow N_2/L_2 \xrightarrow{F_2^t} N_1/L_1$ and the identity. Also, the inclusion $N_1/L_1 \hookrightarrow N_2/L_2$ is homotopic to a flow map $N_1/L_1 \xrightarrow{F_1^t} N_2/L_2$.

For a general case, we will construct a sequence of inclusions that relates (N_1, L_1) and (N_2, L_2) through index pairs which contain (A, B).

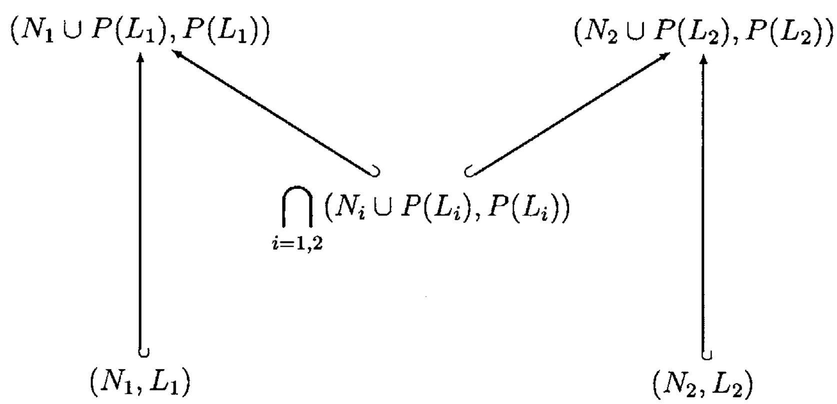
Since the subsets N_i and L_i are contained in X, we can consider a pair $(N_i \cup P(L_i, X), P(L_i, X))$ which is an index pair relative to X by from Lemma 1. Next, we will consider its intersection

Lemma 6. *If $(N_1, L_1), (N_2, L_2)$ are index pairs relative to X, then $(N_1 \cap N_2, L_1 \cap L_2)$ is also an index pair relative to X.*

Proof. The only nontrivial part here is to show the exit set property. Suppose that $x \in N_1 \cap N_2$ and $x \cdot \mathbb{R}^+ \not\subset X$. Then there exist t_1, t_2 such that $x \cdot [0, t_i] \subset N_i$ and $x \cdot t_i \in L_i$, where $i = 1, 2$. Let us assume that $t_1 \geq t_2$. Since N_2, L_2 are positively invariant relative to X, we see that $x \cdot [0, t_1] \subset N_2$ and $x \cdot t_1 \in L_2$ as well, thus we have the desired result. □

Remark. The intersection of two index pairs (with Definition 3) need not be an index pair in general.

Now we have a collection of inclusions of index pairs containing (A, B). This is shown in the diagram below (we abbreviate $P(L)$ for $P(L, X)$ in the diagram).



Chapter 3

Finite Dimensional Approxiamtion on Hilbert Spaces

3.1 Conley Theory on Hilbert Spaces

Conley index theory was developed for a flow on locally compact space, so one cannot apply it directly in infinite-dimensional setting. Gęba, Izydorek, and Pruszko [12] use finite-dimensional approximation and define a stable version of Conley index for a special class of flows.

Let H be a Hilbert space. A vector field on H is a continuous map from H to itself. We will be interested in a special class of vector fields.

Definition 13. We say that a vector field $F : H \to H$ is *permissible* if F admits a decomposition $F = L + K$ such that

(i) L is a self-adjoint Fredholm operator on H.

(ii) K is locally Lipschitz and compact (possibly nonlinear).

(iii) there exist positive constants c, d so that $\|K(x)\| \leq c\|x\| + \frac{d}{1+\|x\|}$ for all $x \in H$

We will also say that a pair (L, K), or simply $F = L + K$, is a *permissible* decomposition for F if it satisfies conditions above. In addition, L and K will be referred as a linear part and a compact part of F respectively.

The last condition implies that F is subquadratic i.e. there exist positive constants c, d such that $2\,|\langle F(x), x\rangle| \leq c\,|x|^2 + d$ for all $x \in H$.

Remark. The class of vector fields studied in [12] is slightly more general than our definition above. For example,

(i) For $F = L + K$, a linear part L need not be self-adjoint. We still require that any eigenspace of L is finite dimensional and its spectrum $spec(L)$ is isolated from the imaginary axis in the complex plane.

(ii) The last condition in Definition 13 can be omitted. However, one can always use a cut-off function to make F subquadratic when studying a fixed bounded subset of H.

Given two different permissible decomposition $F = L_1 + K_1 = L_2 + K_2$, we see that the difference $L_1 - L_2$ is compact. In fact, this gives a one-to-one correspondence between a set of permissible decompositions of F and a space of linear self-adjoint compact operators.

We will also extend this notion to a family of vector fields.

Definition 14. Let Λ be a metric space, regarded as a parameter space. We say that a family of vector fields $F : H \times \Lambda \to H$ is a permissible family of vector fields if there is a decomposition $F(x, \lambda) = L(x, \lambda) + K(x, \lambda)$ satisfy the following properties, where we sometimes write F_λ for the restriction $F(\cdot, \lambda)$ on a point in $\lambda \in \Lambda$,

(i) For each λ, the decomposition $F_\lambda = L_\lambda + K_\lambda$ is a permissible decomposition.

(ii) L_λ is a continuous family in the norm topology of bounded linear operators.

(iii) $K : H \times \Lambda \to H$ is compact.

(iv) There exist positive constants C, D so that $\|K(x, \lambda)\| \leq C\,\|x\| + \frac{D}{1+\|x\|}$ for all $(x, \lambda) \in H \times \Lambda$.

Such a decomposition is also called a permissible decomposition for the family.

We will now consider a family of flows on H generated by a family of vector fields F. More precisely, we look at a family of flows $\eta : H \times \mathbb{R} \times \Lambda \to H$ which is a solution of the following ODE

$$\frac{\partial}{\partial t}\eta(x,t,\lambda) = F(\eta(x,t,\lambda),\lambda) \tag{3.1.1}$$

$$\eta(x,0,\lambda) = x \tag{3.1.2}$$

Note. The subquadratic condition guarantees that η is defined for all time t (cf. [34]). Otherwise, we would only have a local flow.

One special case of a family of vector fields is a family obtained from a sequence of vector fields with an appropriate limit. We can identify and topologize $\mathbb{N}_\infty = \mathbb{N} \cup \{\infty\}$ as a subspace $\left\{\frac{1}{n} | n \in \mathbb{N}\right\} \cup \{0\}$ of the interval $[0,1]$.

It is straightforward to check the following condition for compactness of a map parametrized by $\mathbb{N}_\infty$.

Lemma 7. *A map $K : H \times \mathbb{N}_\infty \to H$ is compact if $K(\cdot,n) : H \to H$ is compact for each n and $K(\cdot,n)$ converges to $K(\cdot,\infty)$ pointwise uniformly on any fixed bounded subset of H.*

We now prove an important result for flows generated by permissible vector fields.

Proposition 4. *Let X be a closed and bounded subset of H and η be a family of flows generated by a family of permissible vector fields $F = L + K : H \times \Lambda \to H$. Then, the projection to second factor $pr_2 : \mathrm{Inv}(X \times \Lambda, \eta) \subset H \times \Lambda \to \Lambda$ is proper.*

Proof. Let $\{(x_n, \lambda_n)\}$ be a sequence in $\mathrm{Inv}(X \times \Lambda)$ with $\lambda_n \to \lambda$. Let $H = H_+ \oplus H_- \oplus H_0$ be the spectral decomposition corresponding to positive, negative, kernel part of L_λ respectively. Let $\pi_\pm, \pi_0$ be the orthogonal projection from H onto $H_\pm, H_0$.

We will show that the sequence $\{x_n\}$ has a Cauchy subsequence by decomposing x_n with respect to $H_\pm$ and H_0. Since the set $\mathrm{Inv}(X \times \Lambda)$ is a closed subset of a complete space, the Cauchy subsequence will be also convergent.

Since L_λ is a self-adjoint Fredholm operator, there is $\delta > 0$ such that the interval $(-\delta, \delta)$ contains no spectrum of L_λ except possibly 0. Then we have that

$$||e^{tL_\lambda}x|| \geq e^{t\delta}||x|| \text{ for all } x \in H_+.$$

Now let $\epsilon > 0$ be arbitrary. Since X is bounded, we assume that $X \subset B(R)$ a ball of radius R. We choose $T > 0$ so that $e^{T\delta} > \frac{3R}{\epsilon}$. Using an integral equation, we can write a formula for η as following

$$\eta(x, t, \lambda) = e^{tL_\lambda}x + e^{tL_\lambda}\int_0^t e^{-\tau L_\lambda}K(\eta(x, \tau, \lambda), \lambda)d\tau. \tag{3.1.3}$$

Denote $U(x, t, \lambda) = e^{tL_\lambda}\int_0^t e^{-\tau L_\lambda}K(\eta(x, \tau, \lambda), \lambda)d\tau$.

Claim. *The sequence $U(x_n, T, \lambda_n)$ has a Cauchy subsequence (cf. [28] Proposition A.18).*

Proof. Since F is subquadratic, we have

$$\frac{\partial}{\partial t}\|\eta(x, t, \lambda)\|^2 = 2\langle F(\eta(x, t, \lambda), \lambda), \eta(x, t, \lambda)\rangle \leq c\|\eta(x, t, \lambda)\|^2 + d$$

and consequently

$$\|\eta(x, t, \lambda)\|^2 \leq e^{ct}\|x\|^2 + \frac{d}{c}(e^{ct} - 1)$$

for positive time t. Then the set $\{(\eta(x_n, \tau, \lambda_n), \lambda_n) : n \in \mathbb{N} \text{ and } 0 \leq \tau \leq T\}$ is a bounded subset of $H \times \Lambda$, and so its image under K is precompact. Consequently the set $\{Te^{-\tau L_{\lambda_n}}K(\eta(x_n, \tau, \lambda_n), \lambda) : n \in \mathbb{N} \text{ and } 0 \leq \tau \leq T\}$ is also precompact.

Finally, we use the fact that, in a Hilbert space, a convex hull of a precompact set is precompact. We see that the sequence $U(x_n, T, \lambda_n)$ has a Cauchy subsequence as claimed.

□

By the claim, we can pass to a subsequence such that $\{U(x_n, T, \lambda_n)\}$ is Cauchy.

Since $\{(x_n, \lambda_n)\}$ is an element of an invariant set, $\eta(x_n, T, \lambda_n)$ lies in $X \subset B(R)$ as well. Thus,

$$\begin{aligned} ||e^{TL_\lambda}(x_m - x_n)|| &\leq ||e^{T(L_\lambda - L_{\lambda_m})}x_m|| + ||e^{T(L_\lambda - L_{\lambda_n})}x_n|| + ||\eta(x_m, \lambda_m, T)|| \\ &\quad + ||\eta(x_n, T, \lambda_n)|| + ||U(x_m, T, \lambda_m) - U(x_n, T, \lambda_n)|| \\ &\leq 3R \text{ for } m, n \text{ sufficiently large.} \end{aligned}$$

On the other hand,

$$||e^{TL_\lambda}(x_m - x_n)|| \geq ||e^{TL_\lambda}\pi_+(x_m - x_n)|| \geq \frac{3R}{\epsilon}||\pi_+(x_m) - \pi_+(x_n)||$$

Combining with the previous inequality, the sequence $\{\pi_+(x_n)\}$ has a Cauchy subsequence.

Similarly, the sequence $\{\pi_-(x_n)\}$ also has a Cauchy subsequence. On the other hand, the sequence $\{\pi_0(x_n)\}$ is a bounded sequence in a finite-dimensional Euclidean space, so it has a Cauchy subsequence as well.

Therefore the sequence $\{(x_n, \lambda_n)\}$ has a Cauchy subsequence and we finish the proof.

□

Definition 15. For a flow ϕ on a Hilbert space, a closed and bounded subset X of H is called *an isolating neighborhood* if $\text{Inv}(X, \phi) \subset \text{Int}(X)$.

From the previous proposition, we can deduce continuity of an isolating neighborhood for a family of flows. Denote the flow $\eta(\cdot, \cdot, \lambda) : H \times \mathbb{R} \to H$ by η_λ.

Corollary 2. *The set $\{\lambda \in \Lambda : X$ is an isolating neighborhood for the flow $\eta_\lambda\}$ is open.*

Proof. We will show that the compliment of this set is closed. Let $\{\lambda_n\}$ be a sequence in Λ such that there exists $\{x_n\}$ with $x_n \in \text{Inv}(X, \eta_{\lambda_n}) \cap \partial X$ and $\lambda_n \to \lambda$. From properness of the projection pr_2, there is a subsequence of $\{x_n\}$ with $x_n \to x$ for some $x \in H$. Since η is continuous and X is closed, we see that $x \in \text{Inv}(X, \eta_\lambda) \cap \partial X$. □

For a moment, we will consider a fixed isolating neighborhood X for a flow ϕ on H generated by a permissible vector field F with $F = L+K$ a permissible decomposition.

From Corollary 2, we know that X is also an isolating neighborhood for flows in a neighborhood of ϕ when it is a part of a family of flows. To strengthen this result, we will introduce some (pseudo)metrics for which X is an isolating neighborhood for flows generated by 'nearby' vector fields.

For compact maps K_1 and K_2, since X is bounded, we can define a pseudometric which depends on the set X

$$\rho_X(K_1, K_2) = \sup_{x \in X} \|(K_1 - K_2)x\| . \tag{3.1.4}$$

This gives a pseudometric for two permissible decomposition $F_1 = L_1 + K_1$ and $F_2 = L_2 + K_2$

$$\bar{\rho}_X(L_1 + K_1, L_2 + K_2) = \|L_1 - L_2\| + \rho_X(K_1, K_2) \tag{3.1.5}$$

As a consequence of Proposition 4, we can show that

Proposition 5. *There is $\epsilon > 0$ such that X is an isolating neighborhood for any flow η generated by $F_\eta = L_\eta + K_\eta$ with $\bar{\rho}_X(L_\eta + K_\eta, L + K) < \epsilon$.*

Proof. Suppose the statement is false. There is a sequence of permissible decomposition $F_n = L_n + K_n$ such that $\bar{\rho}_X(L_n + K_n, L + K) < \epsilon_n$ with $\epsilon_n \to 0$. There also exists a sequence $\{x_n\}$ such that $x_n \in \mathrm{Inv}(X, \eta_n)$ and x_n lies in the boundary ∂X , where η_n is generated by F_n.

We will show that the sequence $\{x_n\}$ is Cauchy and arrive at a contradiction. Consider

$$U_n(x,t) = e^{tL} \int_0^t e^{-\tau L_n} K_n(\eta_n(x,\tau)) d\tau, \tag{3.1.6}$$

$$U_{0,n}(x,t) = e^{tL} \int_0^t e^{-\tau L} K(\eta_n(x,\tau)) d\tau. \tag{3.1.7}$$

By compactness of K, we have that a sequence $\{U_{0,n}(x_n,t)\}$, for a fixed t, is

Cauchy. Since $\bar{\rho}_X(L_n + K_n, L + K)$ goes to 0, we see that $\|U_n(x_n, t) - U_{0,n}(x_n, t)\|$ also goes to 0, so that a sequence $\{U_n(x_n, t)\}$ is also Cauchy.

Using an integral equation for flows (3.1.3) and invariance of x_n, one can see that the quantity $\left\|e^{tL}(x_m - x_n)\right\|$ is uniformly bounded. Then, similar to the proof of Proposition 4, we can deduce that $\{x_n\}$ is Cauchy and complete the proof.

□

3.2 Compression of Vector Fields

In this section, let ϕ be a flow on H generated by a permissible vector field F and let X be an isolating neighborhood with respect to the flow ϕ.

For a finite-dimensional subspace $V \subset H$, we want to be able to study a flow on V approximating the flow ϕ on H. Let π_V be the orthogonal projection from H onto V. We also adopt a notation $\|V - W\| = \|\pi_V - \pi_W\|$ using operator norm.

Let $F = L + K$ be a permissible decomposition. We will be interested in a vector field of the form

$$F_V = \pi_V L \pi_V + (1 - \pi_V)L(1 - \pi_V) + \pi_V K, \tag{3.2.1}$$

and a flow ϕ_V generated by F_V. This vector field F_V can be considered as a compression of F on V

Consider a linear part $\pi_V L \pi_V + (1 - \pi_V)L(1 - \pi_V)$ of F_V. Viewing L as a block matrix on $H = V \oplus V^\perp$, the linear part of F_V is then the diagonal blocks. Whereas, the difference

$$L - (\pi_V L \pi_V - (1 - \pi_V)L(1 - \pi_V)) = \pi_V L(1 - \pi_V) + (1 - \pi_V)L\pi_V \tag{3.2.2}$$

gives the antidiagonal blocks.

Note that the difference $\pi_V L(1-\pi_V)+(1-\pi_V)L\pi_V$ is finite rank, so the linear part of F_V is also Fredholm. It is then straightforward to check that F_V is a permissible vector field.

Our goal is to approximate F by F_V so that X is an isolating neighborhood for ϕ_V as well. We can use a pseudometric defined in (3.1.5) and try to find F_V that satisfies hypothesis of Proposition 5.

For linear parts, we consider an operator norm of the difference as in (3.2.2)

$$\|\pi_V L(1-\pi_V) + (1-\pi_V)L\pi_V\| = \|\pi_V L(1-\pi_V) - (1-\pi_V)L\pi_V\| \tag{3.2.3}$$

$$= \|\pi_V L - L\pi_V\|, \tag{3.2.4}$$

where the last line holds since we can replace one of the blocks $(1-\pi_V)L\pi_V$ with $-(1-\pi_V)L\pi_V$ without changing the norm.

For compact parts, we need to consider a norm of $\|(1-\pi_{V_n})Kx\|$ for $x \in X$. We recall a following fact:

Lemma 8. *Suppose that $\{V_n\}$ is a sequence of subspaces of H such that $\pi_{V_n} \to 1$ pointwise and $K : H \to H$ is a compact map. Then $\pi_{V_n}K$ converges to K pointwise uniformly on any bounded set.*

Proof. We will prove this by contradiction. Let B be a bounded subset of H and suppose there exists a sequence $\{x_j\}$ in B such that $\left\|(1-\pi_{V_{n_j}})K(x_j)\right\| > \delta$ and n_j goes to infinity.

Since $\{x_j\}$ is bounded, $K(x_j)$ converges to some y in H. Then

$$\begin{aligned}
\left\|(1-\pi_{V_{n_j}})K(x_j)\right\| &\leq \left\|(1-\pi_{V_{n_j}})(K(x_j)-y)\right\| + \left\|(1-\pi_{V_{n_j}})y\right\| \\
&\leq \|K(x_j)-y\| + \left\|(1-\pi_{V_{n_j}})y\right\| \\
&< \delta \text{ for sufficiently large } j.
\end{aligned}$$

And we reach a contradiction.

□

This leads to a definition for subspaces that are good for finite-dimensional approximation.

Definition 16. Let $\{V_n\}$ be a sequence of finite-dimensional subspaces of H. We call that the sequence $\{V_n\}$ is an *asymptotically-invariant exhausting* sequence with respect to L if it satisfy

(i) $\pi_{V_n} L + L\pi_{V_n} \to 0$ in operator norm,

(ii) π_{V_n} converges to identity pointwise (i.e. in strong operator topology).

The term 'asymptotically-invariant' is referred to the second condition as a subspace V_n will be approaching an invariant subspace. The first condition is inspired by the following fact:

One can see that a sequence $\{V_n\}$ is asymptotically-invariant exhausting with respect to L implies that it is asymptotically-invariant exhausting with respect to L', when $L' - L$ is compact. Hence, if we fix a permissible vector field, the definition of an asymptotically-invariant exhausting sequence is independent of the choice of permissible decomposition of F.

We can conclude

Proposition 6. *Let $\{V_n\}$ be an asymptotically-invariant exhausting sequence of subspaces and let F_n be a family of vector fields parametrized by $\mathbb{N}_\infty$ defined by*

$$F_n = F_{V_n} = \pi_{V_n} L \pi_{V_n} + (1 - \pi_{V_n})L(1 - \pi_{V_n}) + \pi_{V_n} K,$$

and $F_\infty = F$. Then F is a permissible family of vector fields.

Consequently, we can show that X is an isolating neighborhood for the flow ϕ_{V_n} for sufficiently large n. In fact, we have a slightly stronger result.

Proposition 7. *Let $\{V_n\}$ be an asymptotically-invariant exhausting sequence. Then there are $\epsilon, N > 0$ such that X is an isolating neighborhood for the flow ϕ_W for any subspace W such that $\|W - V_n\| < \epsilon$ and $n \geq N$.*

Proof. Suppose the contrary, then we can find a sequence $\{\epsilon_m\}$ of positive real converging to 0, a sequence $\{n_m\}$ going to infinity, and a sequence of subspaces $\{W_m\}$ with $\|W_m - V_{n_m}\| < \epsilon_m$ such that X is not an isolating neighborhood for ϕ_{W_m}.

It is not hard to see that $\{W_m\}$ is also an asymptotically-invariant exhausting sequence. Then F_{W_m} is a permissible family of vector fields and we reach a contradiction from continuity of the Conley index.

$\square$

We now observe that V is an invariant subspace of the vector field F_V from the construction. Moreover, the vector field is equal to $\pi_V F$ on V. Consequently, the flow ϕ_V restricted on V does not depend on a decomposition of F.

If X is an isolating neighborhood for the flow ϕ_V, then so is $X \cap V$. We can then consider $X \cap V \subset V$ as an isolating neighborhood with respect to the flow ϕ_V on V. Since we are now in the finite-dimensional case and $X \cap V$ is compact, we obtain the Conley index

$$\mathcal{I}(X \cap V, \phi_V)$$

We will now establish relation between Conley indices of different spaces. Given an asymptotically-invariant exhausting sequence

Let V, W be finite-dimensional subspaces of H with an orthogonal decomposition $W = V \oplus U$. We consider a vector field $F_{V,W}$ given by

$$F_{V,W} = \pi_V L \pi_V + \pi_U L \pi_U + (1 - \pi_W) L (1 - \pi_W) + \pi_V K.$$

Since $\pi_W = \pi_V + \pi_U$, we have an identity $\pi_W L \pi_W - \pi_V L \pi_V - \pi_U L \pi_U = \pi_U L \pi_V + \pi_V L \pi_U$ and we obtain

$$\begin{aligned}
&\|L - \pi_V L \pi_V - \pi_U L \pi_U - (1 - \pi_W) L (1 - \pi_W)\| \\
&\leq \|(1 - \pi_W) L \pi_W + \pi_W L (1 - \pi_W)\| + \|\pi_U L \pi_V + \pi_V L \pi_U\| \\
&= \|(1 - \pi_W) L \pi_W - \pi_W L (1 - \pi_W)\| + \sqrt{2}\,\|\pi_U L \pi_V\| \\
&\leq \|L \pi_W - \pi_W L\| + \sqrt{2}\,\|(1 - \pi_V) L \pi_V\| \\
&= \|L \pi_W - \pi_W L\| + \|L \pi_V - \pi_V L\|
\end{aligned}$$

We have an identity, for $x \in U$,

$$\pi_U Lx = Lx + (\pi_U L - L\pi_U)x = Lx + (\pi_W L - L\pi_W)x - (\pi_V L - L\pi_V)x \qquad (3.2.5)$$

Denote U_0 by the kernel of the self-adjoint operator $\pi_U L \pi_U$ on U. We have a decomposition $U = U_0 \oplus U_1$ and also denote $e^{\pi_U L}$ by the flow generated by the linear vector field $\pi_U L$.

Proposition 8. *Let $\overline{F}$ be a family of vector fields over the interval $I = [0,1]$ given by $\overline{F}_s = (1-s)F_{V,W} + sF_W$ and Φ be the family of flows generated by $\overline{F}$. Suppose that $X \times I$ is an isolating neighborhood for Φ. Then*

$$\mathcal{I}(X \cap W, \phi_W) \cong \mathcal{I}(X \cap V, \phi_V) \wedge \mathcal{I}(B(R, U_1), e^{\pi_U L})$$

Proof. We see that the vector field $\overline{F}_s$ is W-invariant for each s, so we can consider the family of flow Φ on $W \times I$ and $(X \cap W) \times I$ becomes a compact isolating neighborhood. Denote $\phi_{V,W}$ by the flow generated by $F_{V,W}$. By continuity of the Conley index, we have

$$\mathcal{I}(X \cap W, \phi_W) \cong \mathcal{I}(X \cap W, \phi_{V,W}).$$

Next, we consider the flow $\phi_{V,W}$ on W. The vector field $F_{V,W}$ restricted to W becomes $\pi_V L \pi_V + \pi_U L \pi_U + \pi_V K$, which is linear on U-component as $W = V \oplus U$.

We observe that a vector in W with nonzero U_1-component will give an exponential flow on the U_1-component, and its trajectory cannot stay in the bounded set X for all time. Consequently, if $w \in \mathrm{Inv}(X \cap W, \phi_{V,W})$, then $x \in V \oplus U_0$, and $\mathrm{Inv}(X \cap W, \phi_{V,W}) = \mathrm{Inv}(X \cap (V \oplus U_0), \phi_{V,W})$.

Denote $B(R, U_1)$ by the ball of radius R in U_1 centered at the origin. We also see that the set $P = (X \cap (V \oplus U_0)) \times B(R, U_1)$ is an isolating neighborhood for $\phi_{V,W}$ and $\mathrm{Inv}(P, \phi_{V,W}) = \mathrm{Inv}(X \cap (V \oplus U_0), \phi_{V,W})$. Thus

$$\mathcal{I}(X \cap W, \phi_{V,W}) = \mathcal{I}(P, \phi_{V,W})$$

as they are isolating neighborhoods for the same isolated invariant set.

Consider a family of flows ψ_s on W generated by a family of vector fields

$$\widehat{F}_s(v+u) = \pi_V L \pi_V(v) + \pi_U L(u) + \pi_V K(v+su),$$

where $v \in V \oplus U_0$ and $u \in U_1$. Similarly, we can conclude that the invariant set of $\mathrm{Inv}(P, \psi_s)$ lies in $V \oplus U_0$ for each s. On $V \oplus U_0$, the vector field $\widehat{F}_s$ becomes $\pi_V F$. In particular, P is an isolating neighborhood for the flow ψ_s with the same invariant set for each s. By continuity of the Conley index, we have that

$$\mathcal{I}(P, \phi_{V,W}) = \mathcal{I}(P, \psi_1) \cong \mathcal{I}(P, \psi_0)$$

The flow ψ_0 on W can be decomposed as a product flow. One is the flow η generated by $\pi_V L \pi_V + \pi_V K$ on $V \oplus U_0$ and another is the flow $e^{\pi_U L}$ on U_1. Hence,

$$\mathcal{I}(P, \psi_0) \cong \mathcal{I}(X \cap (V \oplus U_0), \eta) \wedge \mathcal{I}(B(R, U_1), e^{\pi_U L})$$

Finally, one can regard the flow η as a family of flow on V parametrized by U_0. Since X is bounded, we see that $X \cap (V \times U_0) = X \cap (V \times B(R_0, U_0))$ for some R_0. We now have a product family of flows on V over the compact set $B(R_0, U_0)$ with $X \cap (V \times B(R_0, U_0)$ as an isolating neighborhood. Using local continuity of the Conley index, we have

$$\mathcal{I}(X \cap (V \oplus U_0), \eta) \cong \mathcal{I}(X \cap V, \phi_V)$$

□

Given an asymptotically-invariant exhausting sequence $\{V_n\}$, we will now construct a family of permissible vector fields parametrized by $\mathbb{N}_\infty \times [0,1]$. This will give isomorphisms between Coney indices of $\mathcal{I}(X \cap V_n, \phi_{V_n})$ up to suspensions for n sufficiently large. Consider a family of vector fields $F_{n,s}$ defined by

$$F_{n,s} = (1-s) F_{V_n, V_{n+1}} + s F_{V_{n+1}},$$

and $F_{\infty,s} = F$. We see that the linear part of $F_{n,s}$ converges to L.

In general, the segment joining two Fredholm maps might not lie in the space of Fredholm maps. However, without loss of generality, we can consider n sufficiently large so that $L_{n,0}$ and $L_{n,1}$ lie in the convex neighborhood of L. With this, we can conclude that $F_{n,s}$ is a permissible family.

Hence, by continuity, there exists N so that X is an isolating neighborhood for the flow $\phi_{n,s}$ for all $s \in [0,1]$ and $n > N$.

Let us now consider a relation between Conley indices from Proposition 8. Suppose that $W = V \oplus U$ and let U^- be a negative definite subspace with respect to $\pi_U L \pi_U$. Then we have

$$\mathcal{I}(X \cap W, \phi_W) \cong \mathcal{I}(X \cap V, \phi_V) \wedge S^{U^-}. \tag{3.2.6}$$

We wish to associate a well-defined stable homotopy object to X. One candidate to consider is a coordinate-free spectrum E, which assigns a space $E(V)$ to each finite-dimensional subspace V of a fixed infinite-dimensional inner product space (namely, a universe) with a structure map

$$\Sigma^U E(V) \to E(W),$$

when $W = V \oplus U$.

But the relation in (3.2.6) has the term S^{U^-} rather than S^U. To resolve this, we could assign a space (or a connected simple system) $\Sigma^{V^+}\mathcal{I}(X \cap V, \phi_V)$ to V so that

$$\Sigma^U \left(\Sigma^{V^+}\mathcal{I}(X \cap V, \phi_V)\right) \cong \Sigma^{W^+}\left(\Sigma^{U^-}\mathcal{I}(X \cap V, \phi_V)\right) \cong \Sigma^{W^+}\mathcal{I}(X \cap W, \phi_W).$$

Hence, we have a spectrum $E(X)$ associated to X given by

$$E(X)(V) = \Sigma^{V^+}\mathcal{I}(X \cap V, \phi_V) \tag{3.2.7}$$

Note that the positive and negative space depends on a choice of a quadratic form on H, which usually arises from a self-adjoint operator. This choice only changes its associated spectrum by a suspension. Hence, we can view it as a choice of grading for a spectrum associated to X.

Alternatively, we can also consider the desuspension $\Sigma^{-V^-}\mathcal{I}(X \cap V, \phi_V)$. Similarly, we have $\Sigma^{-V^-}\mathcal{I}(X \cap V, \phi_V) \cong \Sigma^{-W^-}\mathcal{I}(X \cap W, \phi_W)$. This could be consider as a stable homotopy type of X.

Chapter 4

Manolescu-Floer Spectra for 3-manifolds

4.1 Preliminaries

Throughout the thesis, we will use [18] as a foundation for setting up Seiberg-Witten theory for 3-manifolds and 4-manifolds.

Let Y be a closed, oriented, Riemannian 3-manifold and equip Y with a fixed spin^c structure $\mathfrak{s}$. Let S be the associated spinor bundle and $\mathcal{A}(Y)$ the space of spin^c connections on S. We have Clifford multiplication $\rho : TY \to \mathrm{End}(S)$ which extends to differential forms. We also fix a reference spin^c connection B_0.

Denote the *configuration space* $\mathcal{C}(Y, \mathfrak{s}) = \mathcal{A}(Y) \oplus \Gamma(S)$ and consider the Chern-Simons-Dirac functional $\mathcal{L} : \mathcal{C}(Y, \mathfrak{s}) \to \mathbb{R}$ on this configuration space. For a pair of a spin^c connection B and a section Ψ of the spinor bundle, the functional is given by

$$\mathcal{L}(B, \Psi) = -\frac{1}{8}\int_Y (B^t - B_0^t) \wedge (F_{B^t} + F_{B_0^t}) + \frac{1}{2}\int_Y \langle D_B\Psi, \Psi\rangle \, d\mathrm{vol}, \qquad (4.1.1)$$

where B^t is the induced connection on $\Lambda^2 S$, F_{B^t} is the curvature 2-form, and D_B is the Dirac operator associated to B.

Recall that $\mathcal{A}(Y)$ is an affine space with the model space $\Omega^1(Y; i\mathbb{R})$ by an identification $B_0 + b \otimes 1_S$ for $b \in \Omega^1(Y; i\mathbb{R})$, so we have the tangent space

$$T_{(B,\Psi)}\mathcal{C}(Y, \mathfrak{s}) = C^\infty(Y; iT^*Y \oplus S)$$

with L^2-norm $\|b\|^2 + \|\psi\|^2$. Then, we can compute the gradient of $\mathcal{L}$

$$\operatorname{grad} \mathcal{L} = \left(\frac{1}{2} * F_{B^t} + \rho^{-1} (\Psi\Psi^*)_0 \ , \ D_B \Psi\right),$$

where the subscript 0 denotes the trace-free part of the Hermitian endomorphism $\Psi\Psi^*$. A critical point for this gradient vector field is also known as a solution of the 3-dimensional Seiberg-Witten equation

$$\begin{aligned} \frac{1}{2} * F_{B^t} + \rho^{-1} (\Psi\Psi^*)_0 = 0, \\ D_B \Psi = 0. \end{aligned} \tag{4.1.2}$$

One distinguished feature of the Seiberg-Witten theory is that one can interpret a trajectory of the downward gradient flow of the Chern-Simons-Dirac functional as a solution of the 4-dimensional Seiberg-Witten equation on the cylinder $I \times Y$.

In general, let X be an oriented, Riemannian 4-manifold, possibly with a boundary and equip X with a spin^c structure $\mathfrak{s}_X$. The spinor bundle S_X has a decomposition $S_X = S^+ \oplus S^-$ and the Clifford multiplication induces an isometry between self-dual 2-forms and skew-Hermitian endomorphisms of S^+,

$$\rho : \Lambda^+ \to \mathfrak{su}(S^+).$$

We have the Dirac operator $D_A^+ : \Gamma(S^+) \to \Gamma(S^-)$ for each A in the set of spin^c connection $\mathcal{A}_X$. We can now define the 4-dimensional Seiberg-Witten map on the space of 4-dimensional configuration $\mathcal{C}(X) = \mathcal{A}_X \oplus \Gamma(S^+)$, as following

$$\begin{aligned} \mathfrak{F} : \mathcal{C}(X) &\to i\Omega^2_+(X) \oplus \Gamma(S^-) \\ (A, \Phi) &\mapsto (\frac{1}{2} F^+_{A^t} - \rho^{-1} (\Phi\Phi^*)_0, D^+_A \Phi). \end{aligned}$$

The equation $\mathfrak{F}(A,\Phi)=0$ is known as the 4-dimensional Seiberg-Witten equation

$$\begin{aligned}\frac{1}{2}F^{+}_{A^t}-\rho^{-1}\left(\Psi\Psi^*\right)_0&=0,\\ D^{+}_{A}\Phi&=0.\end{aligned}\tag{4.1.3}$$

An important special case is when a 4-manifold is a cylinder $Z=\mathbb{R}\times Y$, where Y is a 3-manifold and Z is equipped with the induced spin^c structure from Y. A time-dependent pair $(B,\Psi)\in\mathcal{C}(Y,\mathfrak{s})$ gives rise to a configuration $(A,\Phi)\in\mathcal{C}(Z)$ and it satisfies

$$\begin{aligned}\frac{1}{2}\rho_Z(F^{+}_{A^t})-(\Phi\Phi^*)_0&=-\rho\left(\frac{d}{dt}B^t+*F_{B^t}+2\rho^{-1}\left(\Psi\Psi^*\right)_0\right),\\ D^{+}_{A}\Phi&=\frac{d}{dt}\Psi+D_B\Psi.\end{aligned}\tag{4.1.4}$$

This is another important feature of Seiberg-Witten theory; A trajectory of the downward gradient flow of the Chern-Simons-Dirac functional on a 3-manifold Y corresponds to a solution of the Seiberg-Witten equation on the cylinder $\mathbb{R}\times Y$.

To guarantee some generic conditions, we need to consider a perturbation of the Chern-Simons-Dirac functional. First, we can perturb $\mathcal{L}$ by a closed, real-valued 2-form ω by setting

$$\mathcal{L}_\omega(B,\Psi)=\mathcal{L}(B,\Psi)+\int_Y(B^t-B^t_0)\wedge i\omega$$

We will also need to perturb $\mathcal{L}_\omega$ by a class of functions called tame perturbations (cf. [18]). We will recall properties of tame perturbations later. Suppose that such a function $f:\mathcal{C}(Y,\mathfrak{s})\to\mathbb{R}$ has formal gradient $\mathfrak{q}$. Then a perturbed Chern-Simons-Dirac functional $\not{\mathcal{L}}$ is given by

$$\not{\mathcal{L}}=\mathcal{L}_\omega+f$$

with gradient

$$\operatorname{grad} \not{\mathcal{L}} = \operatorname{grad} \mathcal{L} - 2(*i\omega, 0) + \mathfrak{q}.$$

We now have the perturbed gradient flow equation

$$\begin{aligned} \frac{d}{dt} B^t &= - * F_{B^t} - 2\rho^{-1} (\Psi\Psi^*)_0 + 4 * i\omega - 2\mathfrak{q}^0(B, \Psi), \\ \frac{d}{dt} \Psi &= -D_B \Psi - \mathfrak{q}^1(B, \Psi), \end{aligned} \tag{4.1.5}$$

and the corresponding perturbed 4-dimensional Seiberg-Witten map on a cylinder $\mathfrak{F}_{\omega,\mathfrak{q}}$ given by

$$\mathfrak{F}_{\omega,\mathfrak{q}}(A, \Phi) = (\frac{1}{2} F^+_{A^t} - \rho^{-1} (\Phi\Phi^*)_0 - 2i\omega^+ + \hat{\mathfrak{q}}^0(A, \Phi), D^+_A \Phi + \hat{\mathfrak{q}}^1(A, \Phi)).$$

Here we will introduce another useful notion. For a configuration (A, Φ) on a 4-manifold X with boundary $\partial X = Y$, we define the (perturbed) topological energy $\mathcal{E}^{\text{top}}_{\omega,\mathfrak{q}}$ by

$$\mathcal{E}^{\text{top}}_{\omega,\mathfrak{q}}(A, \Phi) = \frac{1}{4} \int_X (F_{A^t} - 4i\omega) \wedge (F_{A^t} - 4i\omega) - \int_Y \langle \Phi|_Y, D_B \Phi|_Y \rangle + 2g(A, \Phi).$$

The topological energy depends only on topology of X and the restriction (B, Ψ) of (A, Φ) on the boundary as we have

$$\mathcal{E}^{\text{top}}_{\omega,\mathfrak{q}}(A, \Phi) = C - 2\not{\mathcal{L}}(B, \Psi),$$

where C is a topological constant. In the cylindrical case $Z = I \times Y$ and (A, Φ) arises from a trajectory $(B(t), \Psi(t))$ with $I = [t_1, t_2]$, the topological energy is simply twice the drop of the Chern-Simons-Dirac functional between two end points, i.e.

$$\mathcal{E}^{\text{top}}_{\omega,\mathfrak{q}}(A,\Phi) = 2(\pounds(B(t_1),\Psi(t_1)) - \pounds(B(t_2),\Psi(t_2))). \tag{4.1.6}$$

We now discuss the gauge group $\mathcal{G}(Y) = \text{Map}(Y, S^1)$ acting on the configuration space $\mathcal{C}(Y,\mathfrak{s})$. For $u : Y \to S^1$, the action is given by

$$u(B,\Psi) = (B - u^{-1}du \otimes 1_S, u\Psi).$$

We now look at the change of the perturbed functional under the action of $\mathcal{G}(Y)$. Note that the function g is gauge invariant by the construction.

$$\pounds(u(B,\Psi)) - \pounds(B,\Psi) = \left([u] \cup \left(4\pi[\omega] + 2\pi^2 c_1(\mathfrak{s})\right)\right)[Y], \tag{4.1.7}$$

where $[u] \in H^1(Y;\mathbb{Z})$ is the corresponding cohomology class of $u^{-1}du/(2\pi i)$ and $[\omega] \in H^2(Y;\mathbb{R})$ is the cohomology class of ω.

We see that the functional $\pounds$ is not necessarily invariant under the full gauge group depending on the form ω, although it is always invariant under the identity component $\mathcal{G}^e(Y)$ of $\mathcal{G}(Y)$. This leads to the following definition.

Definition 17. The *period class* of $\pounds$ is the cohomology class $c_{\pounds} = 4\pi[\omega] + 2\pi^2 c_1(\mathfrak{s})$ in $H^2(Y;\mathbb{R})$. We say that the perturbation $\pounds$ is *balanced* if the class $c_{\pounds}$ is zero, or equivalently $\pounds$ is fully gauge invariant. We also say that the perturbation $\pounds$ is *positively monotone* if $c_{\pounds} = tc_1(\mathfrak{s})$ for some positive real number t.

The main idea in constructing monopole Floer homology is to compute Morse homology of the quotient configuration space $\mathcal{B}(Y,\mathfrak{s}) = \mathcal{C}(Y,\mathfrak{s})/\mathcal{G}(Y)$ with respect to the downward gradient flow of the Chern-Simons-Dirac functional. The gradient vector field of the functional gives rise to the 3-dimensional (perturbed) Seiberg-Witten map:

$$\mathcal{F} : \mathcal{C}(Y, \mathfrak{s}) \to C^\infty(Y; iT^*Y \oplus S)$$
$$(B, \Psi) \mapsto (\frac{1}{2} * F_{B^t} + \rho^{-1} (\Psi\Psi^*)_0 + \mathfrak{q}^0(B, \Psi) - 2 * i\omega, D_B\Psi)$$

The Seiberg-Witten map is equivariant under the action of $\mathcal{G}(Y)$, where the induced action on a tangent vector is given by $u \cdot (b, \psi) = (b, u\psi)$.

4.1.1 Compactness and Boundedness Result

A fundamental result in Seiberg-Witten theory is the compactness of the solutions, modulo gauge transformations, of the Seiberg-Witten equation. Note that the gauge group $\mathcal{G}(X) = \text{Map}(X, S^1)$ on a 4-manifold and its action is defined in the same way as the 3-dimensional case.

Pick a reference connection A_0, we say that a configuration (A, Φ) satisfies the Coulomb-Neumann condition if

$$d^*(A^t - A_0^t) = 0, \text{ in } X,$$
$$\langle A^t - A_0^t, \vec{n} \rangle = 0, \text{ in } \partial X,$$

where $\vec{n}$ is the normal vector of the boundary. We also choose a basis $\{\gamma_j\}$ of the group $H_1(X; \mathbb{Z})/\text{torsion}$. Then we say that (A, Φ) satisfies the cycle condition if for all j

$$i \int PD(\gamma_j) \wedge (A^t - A_0^t) \in [0, 2\pi).$$

For any configuration (A, Φ), it turns out that one can always find a gauge transformation u such that $u \cdot (A, \Phi)$ satisfy both the Coulomb-Neumann condition and the cycle condition. Moreover, this gauge transformation is unique up to multiplying by constant functions.

We quote the main compactness result:

Proposition 9. *([18], Theorem 10.7.1) Let $Z = [t_1, t_2] \times Y$ and $\mathfrak{q}$ be a tame perturbation. Let $(A_n, \Phi_n) \in \mathcal{C}(Z)$ be a sequence of solutions $\mathfrak{F}_{\omega,\mathfrak{q}}(A_n, \Phi_n) = 0$ with a uniform bound on the topological energy*

$$\mathcal{E}^{top}_{\omega,\mathfrak{q}}(A_n, \Phi_n) \leq C.$$

Then there is a gauge transformation u_n such that a subsequence of $u_n \cdot (A_n, \Phi_n)$ converges in $L^2_{k+1}(Z')$, for any interior cylinder $Z' = [t_1', t_2'] \times Y$.

We now consider Seiberg-Witten trajectories on Y, that is a solution of Seiberg-Witten equations on $\mathbb{R} \times Y$. From now on, we will also assume that our perturbation $\mathfrak{q}$ is admissible. The main property we need is that

(i) A set of critical points $\operatorname{grad} \mathcal{L} = 0$ is finite modulo gauge transformation.

(ii) The moduli spaces of trajectories are regular.

With this hypothesis, we can deduce the following uniform boundedness result:

Proposition 10. *There is a uniform energy bound for Seiberg-Witten trajectories with finite energy.*

Proof. (Sketch) For a fixed interval I and a real number s, let A_s be a 4-dimensional configuration obtained from the trajectory on a translated interval $I + s$. Since the energy of the trajectory is finite, we see that

$$\mathcal{E}^{\text{top}}_{\omega,\mathfrak{q}}(A_s) \to 0 \quad \text{as } s \to \pm\infty.$$

This implies that A_s converges to a configuration with zero energy, which has to be a trivial trajectory at one of the critical points. Thus the energy of γ is bounded by a difference of $\mathcal{L}$ between two critical points. Using the fact that critical points are finite modulo gauge and the identity 4.1.7, we consider two cases.

If the perturbation is positively monotone, we pick representative $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ of critical points. Suppose that γ is a trajectory from $u \cdot \mathfrak{a}_i$ to $\mathfrak{a}_j$ (up to gauge) so that the dimension $\operatorname{gr}(u \cdot \mathfrak{a}_i, \mathfrak{a}_j)$ of the corresponding moduli space is nonnegative.

We also recall a formula

$$\mathrm{gr}(u \cdot \mathfrak{a}_i, \mathfrak{a}_j) = \mathrm{gr}(\mathfrak{a}_i, \mathfrak{a}_j) - 2\pi^2 \left([u] \cup c_1(\mathfrak{s})\right)[Y] \tag{4.1.8}$$

Thus we have that

$$\begin{aligned} \mathcal{L}(u \cdot \mathfrak{a}_i) - \mathcal{L}(\mathfrak{a}_j) &= t\left([u] \cup c_1(\mathfrak{s})\right)[Y] + \mathcal{L}(\mathfrak{a}_i) - \mathcal{L}(\mathfrak{a}_j) && (4.1.9) \\ &\leq \frac{t}{2\pi^2}\,\mathrm{gr}(\mathfrak{a}_i, \mathfrak{a}_j) + \mathcal{L}(\mathfrak{a}_i) - \mathcal{L}(\mathfrak{a}_j), && (4.1.10) \end{aligned}$$

as t is positive. The right hand side now depends only on $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$.

The balanced case is obvious, because $\mathcal{L}$ becomes gauge invariant.

□

The pointwise uniform bound follows immediately from Proposition 9.

Corollary 3. *There exists a constant C_k for each positive integer k such that, for any Seiberg-Witten trajectory γ with finite energy, we have a uniform bound $\|u_t \cdot \gamma(t)\|_{L^2_k} \leq C_k$ for all t, where $\gamma(t)$ is a restriction of γ on $\{t\} \times Y$ and $u_t \in \mathcal{G}(Y)$ is a gauge transformation in.*

4.2 The Coulomb Slice

Recall that the space $\mathcal{B}(Y, \mathfrak{s})$ can be identified with a Hilbert bundle over the Picard torus $H^1(Y; \mathbb{R})/H^1(Y; \mathbb{Z})$, which is not a vector space. In order to apply finite dimensional approximation technique, we will consider a slightly smaller gauge group so that the quotient space is an affine space and also good for setting up elliptic theory.

Consider the following subspace of $\mathcal{C}(Y, \mathfrak{s})$ defined by

$$\mathcal{K}_{B_0} = \left\{(B_0 + b \otimes 1, \Psi) \in \mathcal{C}(Y, \mathfrak{s}) | d^* b = 0\right\}.$$

This subspace is called a Coulomb slice. It is obtained as a quotient of $\mathcal{C}(Y, \mathfrak{s})$ by a gauge subgroup $\mathcal{G}^{\perp} = \{e^{\xi} \mid \xi \in C^{\infty}(Y; i\mathbb{R}) \text{ and } \int_Y \xi = 0\}$. We have a diffeomorphism

$$\begin{gathered} \mathcal{G}^{\perp} \times \mathcal{K}_{B_0} \to \mathcal{C}(Y, \mathfrak{s}) \\ \left(e^{\xi}, (B_0 + b \otimes 1, \Psi)\right) \mapsto \left(B_0 + (b - d\xi) \otimes 1, e^{\xi}\psi\right) \end{gathered}$$

The quotient group $\mathcal{G}(Y)/\mathcal{G}^{\perp}$ can be identified with $\mathcal{G}^h$, the group of harmonic maps from Y to S^1. With a fixed basepoint y_0 of Y, we can decompose

$$\mathcal{G}^h \cong H^1(Y; \mathbb{Z}) \times S^1,$$

where each integral cohomology class can be represented uniquely by a pointed harmonic map $u : Y \to S^1$ and the S^1 part represents constant maps. Consequently, the residual gauge group $H^1(Y; \mathbb{Z}) \times S^1$ acts on the Coulomb slice $\mathcal{K}_{B_0}$ and we have identification

$$\mathcal{B}(Y, \mathfrak{s}) \cong \mathcal{K}_{B_0}/(H^1(Y; \mathbb{Z}) \times S^1)$$

Next we describe how to find a representative in a Coulomb slice for each configuration $(B_0 + b \otimes 1, \Psi)$. This is the same as finding an element in $\mathcal{K}_{B_0}$ that lies in the same $\mathcal{G}^{\perp}$-orbit. This process will be called a (nonlinear) Coulomb projection.

Since an element of $\mathcal{G}^{\perp}$ is of the form e^{ξ} , we only have to solve for an imaginary-valued function ξ satisfying $d^*d\xi = d^*b$ and $\int_Y \xi = 0$. Such a function is unique and we will also denote it by $\xi(b)$, considered as a map

$$\xi : \Omega^1(Y; i\mathbb{R}) \to C^{\infty}(Y; i\mathbb{R}). \tag{4.2.1}$$

One can also regard the map ξ as a composition of d^* and the Green operator of d^*d, so that it becomes an operator of order -1.

We will shift our viewpoint from affine spaces their underlying vector spaces. Let $\mathcal{K} = \ker d^* \oplus \Gamma(S) \subset \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S)$, which we still call a Coulomb slice. Then we have an explicit formula for a Coulomb projection

$$\Pi : \Omega^1(Y; i\mathbb{R}) \oplus \Gamma(S) \to \mathcal{K} \tag{4.2.2}$$

$$(b, \Psi) \mapsto \big(b - d\xi(b), e^{\xi(b)}\Psi\big) \tag{4.2.3}$$

For a fixed B_0, we also denote

$$\Pi_{B_0}(B_0 + b \otimes 1, \Psi) = \Pi(b, \Psi).$$

Next, we will study a vector field on $\mathcal{K}$ induced from a vector field on $\mathcal{C}(Y, \mathfrak{s})$. There is an inclusion $\iota_{B_0} : \mathcal{K} \hookrightarrow \mathcal{C}(Y, \mathfrak{s})$ given by

$$\iota_{B_0}(b, \psi) = (B_0 + b \otimes 1, \psi),$$

so that $\Pi_{B_0} \circ \iota_{B_0} = \Pi$. For a time dependent (b, ψ), we look at the derivative

$$\begin{aligned} \frac{\partial}{\partial t}\Pi(b, \Psi) &= \frac{\partial}{\partial t}(b - d\xi(b), e^{\xi(b)}\Psi) \\ &= (\frac{\partial b}{\partial t} - d\xi(\frac{\partial b}{\partial t}), e^{\xi(b)}\xi(\frac{\partial b}{\partial t})\Psi + e^{\xi(b)}\frac{\partial \Psi}{\partial t}). \end{aligned}$$

This gives a formula for a push forward of a tangent vector (β, ψ) at $(B_0 + b \otimes 1, \Psi)$ by (note that $\xi(b) = 0$ because b is coclosed),

$$(\mathcal{D}_\Psi \Pi)(\beta, \psi) = (\beta - d\xi(\beta), e^{\xi(b)}\xi(\beta)\Psi + e^{\xi(b)}\psi). \tag{4.2.4}$$

$$= (\beta - d\xi(\beta), \xi(\beta)\Psi + \psi). \tag{4.2.5}$$

This map can be thought as a linearized Coulomb projection of a tangent space at $(B_0 + b \otimes 1, \Psi)$. We notice that the above formula does not depends only on Ψ.

Remark. When $B_1 = B_0 + b_0 \otimes 1$, we see that

$$\begin{aligned}
\frac{\partial}{\partial t}(\Pi_{B_0} \circ \iota_{B_1})(b,\psi) &= \frac{\partial}{\partial t}(b_0 + b - d\xi(b_0+b), e^{\xi(b_0+b)}\Psi) \\
&= (\frac{\partial b}{\partial t} - d\xi(\frac{\partial b}{\partial t}), e^{\xi(b_0)}(\xi(\frac{\partial b}{\partial t})\Psi + \frac{\partial \Psi}{\partial t})) \\
&= e^{\xi(b_0)}\frac{\partial}{\partial t}\Pi(b,\psi).
\end{aligned}$$

Given a vector filed $\mathcal{X} : \mathcal{C}(Y) \to T\mathcal{C}(Y)$, we can consider a composition $\Pi_{B_0}^*\mathcal{X} = (\mathcal{D}\Pi) \circ \mathcal{X} \circ \iota_{B_0}$ so we have the following diagram

$$\begin{array}{ccc}
\mathcal{C}(Y,\mathfrak{s}) & \xrightarrow{\mathcal{X}} & T\mathcal{C}(Y,\mathfrak{s}) \\
\uparrow{\scriptstyle \iota_{B_0}} & & \downarrow{\scriptstyle \mathcal{D}\Pi} \\
\mathcal{K} & \xrightarrow{\Pi_{B_0}^*\mathcal{X}} & \mathcal{K}.
\end{array}$$

The formula for $\Pi_{B_0}^*\mathcal{X}$ is given by

$$\Pi_{B_0}^*\mathcal{X}(b,\psi) = (\mathcal{X}_1(\iota_{B_0}(b,\psi)) - d\xi(\mathcal{X}_1(\iota_{B_0}(b,\psi))), \xi(\mathcal{X}_1(\iota_{B_0}(b,\psi)))\psi + \mathcal{X}_2(\iota_{B_0}(b,\psi))),$$

where $\mathcal{X} = (\mathcal{X}_1, \mathcal{X}_2)$.

To justify the construction, we consider a particular lift to $\mathcal{C}(Y,\mathfrak{s})$ for a time dependent $(b,\psi) \in \mathcal{K}$ given by

$$(B(t), \Psi(t)) = e^{\int_0^t \xi(\mathcal{X}_1(\iota_{B_0}(b,\psi))ds} \cdot (B_0 + b(t) \otimes 1, \psi(t)).$$

Then we have a derivative

$$\frac{\partial}{\partial t}(B,\Psi) = (\frac{\partial b}{\partial t} - d\xi(\mathcal{X}_1(\iota_{B_0}(b,\psi)), e^{\int_0^t \xi(\mathcal{X}_1(\iota_{B_0}(b,\psi)))ds}(\xi(\mathcal{X}_1(\iota_{B_0}(b,\psi)))\psi + \frac{\partial \psi}{\partial t})).$$

In particular, when $\frac{\partial}{\partial t}(b,\psi) = -\Pi_{B_0}^*\mathcal{X} + \mathcal{Y}$, we have

$$\frac{\partial}{\partial t}(B,\Psi) + \mathcal{X}(B,\Psi) = (\mathcal{Y}_1(b,\psi), e^{\int_0^t \xi(\mathcal{X}_1(b(s),\psi(s)))ds}\mathcal{Y}_2(b,\psi)).$$

The special case when $\mathcal{Y} = 0$ implies that a trajectory on $\mathcal{K}$ generated by a vector field $-\Pi^*_{B_0}\mathcal{X}$ is a Coulomb projection of a trajectory generated by $-\mathcal{X}$ on $\mathcal{C}(Y,\mathfrak{s})$. We will apply this principle to a Seiberg-Witten trajectory.

We would also like to recover $\Pi^*_{B_0}\mathcal{F}$ as a gradient vector field of $\mathcal{L}$ restricted to $\mathcal{K}$. To obtain this, we need another metric on $\mathcal{K}$ which is not the standard one. There is a decomposition of the tangent space of $\mathcal{C}(Y,\mathfrak{s})$ at (B,Ψ)

$$T_{(B,\Psi)}\mathcal{C}(Y,\mathfrak{s}) = \mathcal{J}_{(B,\Psi)} \oplus \mathcal{K}_{(B,\Psi)},$$

where $\mathcal{J}_{(B,\Psi)}$ is the image under linearization of the gauge group action. To define a new metric on $\mathcal{K}$, we use the L^2-norm of the projection onto $\mathcal{K}_{(B_0+b,\psi)}$ of a tangent vector as its norm under this new metric at (b,ψ). Since $\mathcal{L}$ is invariant under the identity component of the gauge group, it is not hard to check that the gradient of $\mathcal{L}$ on $\mathcal{K}$ with the above metric is $\Pi^*_{B_0}\mathcal{F}$.

There is an action of $\mathcal{G}(Y)$ on $\mathcal{K}$ induced by a gauge action on $T\mathcal{C}(Y)$, namely

$$u \cdot (b,\psi) = (b, u\psi). \tag{4.2.6}$$

When a vector field $\mathcal{X}$ is $\mathcal{G}(Y)$-equivariant, we can observe relationship between $\Pi^*_{B_0}\mathcal{X}$ and $\Pi^*_{u\cdot B_0}\mathcal{X}$, where u is a gauge transformation and $u \cdot B_0 = B_0 - u^{-1}du$ the gauge action on connections,

$$\begin{aligned}
\left((\mathcal{D}\Pi) \circ \mathcal{X} \circ \iota_{u\cdot B_0}\right)(b, u\psi) &= (\mathcal{D}\Pi)_{u\psi}\left(\mathcal{X}(u \cdot \iota_{B_0}(b,\psi))\right) \\
&= (\mathcal{D}\Pi)_{u\psi}\left(u \cdot \mathcal{X}(\iota_{B_0}(b,\psi))\right) \\
&= u \cdot (\mathcal{D}\Pi)_{\psi}\left(\mathcal{X}(\iota_{B_0}(b,\psi))\right).
\end{aligned}$$

In other words, we have

$$\Pi^*_{B_0}\mathcal{X}(b - u^{-1}du, u\psi) = u\,\Pi^*_{B_0}\mathcal{X}(b,\psi). \tag{4.2.7}$$

Recall that the 3-dimensional Seiberg-Witten map is given by

$$\mathcal{F}(B,\Psi) = (\frac{1}{2} * F_{B^t} + \rho^{-1}(\Psi\Psi^*)_0 - 2 * i\omega, D_B\Psi) + \mathfrak{q}(B,\Psi),$$

and is $\mathcal{G}(Y)$-equivariant. Since $d^*(*db) = 0$, we see that

$$\begin{aligned} \xi(\mathcal{F}_1(\iota_{B_0}(b,\psi))) &= \xi(\mathcal{F}_1(\iota_{B_0}(b,\psi)) - *db) && (4.2.8)\\ &= \xi(\frac{1}{2} * F_{B_0^t} + \rho^{-1}(\psi\psi^*)_0 - 2 * i\omega + \mathfrak{q}_1(B_0 + b\otimes 1, \psi)) && (4.2.9)\end{aligned}$$

We will want to decompose $\Pi^*_{B_0}\mathcal{F}$ to linear and nonlinear parts. One natural choice for a linear part is

$$L(b,\psi) = (*db, D_{B_0}\psi), \tag{4.2.10}$$

and we write

$$\Pi^*_{B_0}\mathcal{F} = L + Q. \tag{4.2.11}$$

From the above observation, the nonlinear part Q consists of constants, perturbations, and terms involving pointwise multiplication. Before proceeding, we will check that Q has nice compactness properties, allowing us to apply finite-dimensional approximation.

Definition 18. We say that Q is *quadratic-like* if , for trajectories $x_n(t)$ and $x(t)$ defined on compact interval, we have

(i) If $x_n^{(j)}(t)$ is uniformly bounded in L^2_{k-j+1} norm and $x_n^{(j)}(t)$ converges to $x^{(j)}(t)$ uniformly in L^2_{k-j} topology for $0 \leq j \leq s$, then $(\frac{\partial}{\partial t})^s Q(x_n(t))$ is uniformly bounded in L^2_{k-s} norm and also converges to $(\frac{\partial}{\partial t})^s Q(x(t))$ uniformly in L^2_{k-s-1} topology.

(ii) Q extends to a continuous map $L^2_k(Z)$, where Z is a cylinder $I \times Y$

Now we also recall some properties of a tame perturbation.

(i) For $k \geq 1$, $\mathfrak{q}$ defines a smooth vector field on $\mathcal{C}_k(Y)$. The first derivative $\mathcal{D}\mathfrak{q}$ extends to a smooth map

$$\mathcal{D}\mathfrak{q} : \mathcal{C}_k(Y) \to \mathrm{Hom}(\mathcal{T}, \mathcal{T}), \tag{4.2.12}$$

and its derivative is a multilinear map in $\mathrm{Mult}^s(\times_s \mathcal{T}, \mathcal{T})$ with a norm satisfying

$$\left\| D^s_{B,\Psi}\mathfrak{q} \right\| \leq C(1 + \|\Psi\|_{L^2})^s (1 + \|b\|_{L^2_{k-1}})^k (1 + \|\Psi\|_{L^2_{k,B}}). \tag{4.2.13}$$

(ii) For $k \geq 2$, the induced 4-dimensional perturbation $\widehat{\mathfrak{q}}$ extends to a smooth map

$$\widehat{\mathfrak{q}} : \mathcal{C}_k(Z) \to \mathcal{T}, \tag{4.2.14}$$

Lemma 9. *If Q is of the form $Q(b, \psi) = \mathfrak{q}(B_0 + b \otimes 1, \psi)$, then Q is quadratic-like.*

Proof. For $s = 0$, we observe that $Q(\gamma(b)) - Q(\gamma(a)) = \int_a^b \mathcal{D}_{\gamma(t)} Q(\gamma'(t))dt$. Taking a segment between two points, we see that Q is a bounded map on L^2_{k+1} and is also uniformly continuous on a bounded set.

Recall a formula for a derivative of Q

$$(\frac{\partial}{\partial t})^s Q(x(t)) = \sum_{m=1}^{s} \left(\sum_{\substack{\alpha_j \geq 1 \\ \alpha_1 + \ldots + \alpha_m = s}} c_\alpha \mathcal{D}^m_{x(t)} Q \left(x^{(\alpha_1)}(t), \ldots, x^{(\alpha_m)}(t) \right) \right). \tag{4.2.15}$$

Consider a term $\mathcal{D}^m_{x(t)}\mathfrak{q} \left(x^{(\alpha_1)}(t), \ldots, x^{(\alpha_m)}(t) \right)$ in the summand and suppose that $\alpha_1 = \max\{\alpha_j\}$. From the property of a tame perturbation, the derivative $\mathcal{D}^m_{x(t)}\mathfrak{q}$ is a multilinear map $\mathrm{Mult}^m(\times_m L^2_{k-\alpha_1}, L^2_{k-\alpha_1})$ whose norm is control by $L^2_{k-\alpha_1}$ of $x(t)$.

Since $1 \leq m \leq s \leq k-1$ and $\alpha_j \geq 1$ for each j, we see that $k - \alpha_1 \geq k - s + m - 1 \geq k - s \geq 1$. From convergence and boundedness of $x_n^{(j)}(t)$ in L^2_{k-j} topology, we can conclude that $(\frac{\partial}{\partial t})^s \mathfrak{q}(x_n(t))$ converges to $(\frac{\partial}{\partial t})^s \mathfrak{q}(x(t))$ uniformly in L^2_{k-s} topology (and indeed in L^2_{k-s-1} topology).

∎

It is clear that the quadratic-like property is closed under addition. We will show that it is also closed under multiplication.

Lemma 10. *If Q_1, Q_2 satisfy property is quadratic-like, then so does $Q_1 \sharp Q_2$.*

Proof. Recall a product rule for derivatives

$$(\frac{\partial}{\partial t})^s (Q_1(x(t)) \sharp Q_2(x(t))) = \sum_{\alpha_1+\alpha_2=s} c_\alpha (\frac{\partial}{\partial t})^{\alpha_1} Q_1(x(t)) \sharp (\frac{\partial}{\partial t})^{\alpha_2} Q_2(x(t)). \quad (4.2.16)$$

Consider the term $(\frac{\partial}{\partial t})^{\alpha_1} Q_1(x(t)) \sharp (\frac{\partial}{\partial t})^{\alpha_2} Q_2(x(t))$, from hypothesis, we have that $(\frac{\partial}{\partial t})^{\alpha_i} Q_i(x_n(t))$ converges to $(\frac{\partial}{\partial t})^{\alpha_i} Q_i(x(t))$ uniformly in $L^2_{k-\alpha_i-1}$ topology.

Without loss of generality, suppose that $\alpha_1 \leq \alpha_2$. We consider the following sequence of maps:

$$L^2_{k-\alpha_1} \times L^2_{k-\alpha_2} \to L^2_{k-\alpha_2} \hookrightarrow L^2_{k-s}. \quad (4.2.17)$$

$$L^2_{k-\alpha_1-1} \times L^2_{k-\alpha_2-1} \to L^2_{k-\alpha_2-1} \hookrightarrow L^2_{k-s-1}. \quad (4.2.18)$$

We observe that $k - \alpha_1 - 1 \geq k - s/2 - 1 \geq (k-1)/2 \geq 3/2$ and this inequality is strict when $k \geq 4$. Hence, we can apply Sobolev multiplication theorem so that the multiplication above is continuous. The last map is obtained from Sobolev embedding theorem since $\alpha_2 \leq s$.

For the 4-dimensional map, we can see that $\widehat{Q_1} \sharp \widehat{Q_2}$ is continuous from Sobolev multiplication.

□

Finally, we note that a differential operator of order 1 is quadratic-like. A composition of with a differential operator of order 0 also preserves the quadratic-like property.

4.3 Finite Dimensional Approximation

The main idea to construct the Seiberg-Witten-Floer homotopy type is to approximate the downward gradient flow on the Coulomb slice by compressing the flow to its finite dimensional subspaces.

The flow on $\mathcal{K}$ is generated by the vector field $-\mathcal{F}' = -L - Q$ defined previously. The linear part L is a self-adjoint operator and gives L^2 decomposition of $\mathcal{K}$ by its eigenspaces. To allow more classes of subspaces, we introduce a linear operator K, which is compact and self-adjoint.

The class of projections we consider is a a class of finite rank smoothing operator. In addition, we require the following properties for a sequence of projections $\{\pi_n\}$

(i) We normalize so that π_n has norm less than or equal to 1 in $\mathcal{B}(L^2_k)$.

(ii) π_n converges to 1 pointwise in L^2_k

(iii) $[L, \pi_n]$ is an operator on L^2_k and the norm converges to 0

These properties depend on the class of D modulo smoothing operators.

Let V_n be the image of π_n. We consider a flow on V_n generated by $-F_n$ where

$$F_n = \pi_n \mathcal{F}'. \tag{4.3.1}$$

We also call F_n a compression of $-\mathcal{F}'$ on V_n. Note that a flow generated by $-\mathcal{F}'$ is only partially defined, but its compression is well-defined on a finite dimensional subspace of smooth sections.

The following proposition and its proof is a slight adaptation of the result in [22].

Proposition 11. *Let $\mathcal{R}$ be a closed and bounded subset of $\mathcal{K}$. Suppose that $\mathcal{R}$ is an isolating neighborhood for $-\mathcal{F}'$ on $\mathcal{K}$. For sufficiently large n, $\mathcal{R} \cap V_n$ is an isolating neighborhood for a flow generated by $-F_n$.*

Proof. Suppose the contrary: there is a sequence such that $\mathcal{R} \cap V_n$ is not an isolating neighborhood. By definition, we have a trajectory $x_n : \mathbb{R} \to \mathcal{R} \cap V_n$ such that

$$-\frac{\partial}{\partial t} x_n(t) = \pi_n F(x_n(t))$$

and $x_n(0) \in \partial\mathcal{R}$ after reparametrization.

Consider the L^2_k-norm of the derivative

$$\left\| \frac{\partial}{\partial t} x_n(t) \right\|_{L^2_k} = \|\pi_n F(x_n(t))\|_{L^2_k} \tag{4.3.2}$$

Since $\mathcal{R}$ is bounded in L^2_{k+1}-norm, $\|\pi_n F(x_n(t))\|_{L^2_k}$ is uniformly bounded because F is quadratic-like. By Arzela-Ascoli theorem, a subsequence of x_n converges to a trajectory x in L^2_k-norm uniformly on compact intervals.

We observe that

$$\begin{aligned} F(x(t)) - \pi_n F(x_n(t)) = &\, D(x(t) - x_n(t)) + [D, \pi_n] x_n(t) \\ &+ \pi_n(Q(x(t)) - Q(x_n(t))) + (1 - \pi_n) Q(x(t)) \end{aligned}$$

This shows that $-\frac{\partial}{\partial t} x_n(t) = \pi_n F(x_n(t))$ converges to $F(x(t))$ L^2_{k-1}-norm uniformly on compact intervals. Thus, the derivatives $\frac{\partial}{\partial t} x_n(t)$ converges to $\frac{\partial}{\partial t} x(t)$ and we have

$$-\frac{\partial}{\partial t} x(t) = F(x(t))$$

Thus $x(t)$ is a Coulomb projection of a Seiberg-Witten trajectory with finite energy. By Corollary 3, we have that $x(t)$ is smooth and satisfies a priori bound.

Next, we will show that $(\frac{\partial}{\partial t})^s x_n$ converges to $(\frac{\partial}{\partial t})^s x$ in L^2_{k-s} uniformly on compact intervals. Note that we already have the case $s = 0, 1$ from above, so we will proceed with induction. Consider

$$-(\frac{\partial}{\partial t})^{s+1}(x - x_n) = (\frac{\partial}{\partial t})^s \left(F(x(t)) - \pi_n F(x_n(t))\right)$$

We can break apart the right hand side similar to above paragraph

$$
\begin{aligned}
(\frac{\partial}{\partial t})^s (F(x(t)) - \pi_n F(x_n(t))) = D((\frac{\partial}{\partial t})^s(x(t) - x_n(t))) + [D, \pi_n](\frac{\partial}{\partial t})^s x_n(t) \\
+ \pi_n((\frac{\partial}{\partial t})^s Q(x(t)) - (\frac{\partial}{\partial t})^s Q(x_n(t))) \\
+ (1 - \pi_n)(\frac{\partial}{\partial t})^s Q(x(t))
\end{aligned}
$$

With this, we can conclude that x_n converges to x in $L^2_k(Z)$, where $Z = I \times Y$. Finally, we apply elliptic estimate using an operator $-\frac{\partial}{\partial t} - D$ on the cylinder.

$$
\|x - x_n\|_{L^2_{k+1}(Z')} \leq \left\|(-\frac{\partial}{\partial t} - D)(x - x_n)\right\|_{L^2_k(Z)} + \|x - x_n\|_{L^2(Z)}
$$

We can break apart the first term on the right hand side as following

$$
(-\frac{\partial}{\partial t} - D)(x - x_n) = [D, \pi_n]x_n + \pi_n(Q(x) - Q(x_n)) + (1 - \pi_n)Q(x) \qquad (4.3.3)
$$

This also goes to 0 by the hypothesis and property $*$ of Q. Then, by bootstrapping, we have that $x_n(0)$ converges to $x(0)$ in L^2_{k+1} which is a contradiction because $x(0)$ cannot lie on the boundary of $\mathcal{R}$.

□

We point out two important quantities involving π_n in the above proof. The first one is the norm of the commutator

$$
[D, \pi_n] : L^2_s \to L^2_s \qquad (4.3.4)
$$

and the second quantity is

$$
\sup_{x \in \mathcal{R}} \|(1 - \pi_n)Q(x)\|_{L^2_s} \qquad (4.3.5)
$$

for $s = 0, \ldots, k+1$. These are the quantities we assume that they converge to zero in the hypothesis of the proposition. Analogous to Proposition 5 in the Hilbert space setup, we can generalize Proposition 11 as following

Proposition 12. *Let $\mathcal{R}$ be a closed and bounded subset of $\mathcal{K}$. Suppose that $\mathcal{R}$ is an isolating neighborhood for $-\mathcal{F}'$ on $\mathcal{K}$. There is $\epsilon > 0$ such that if $\mathcal{F}_1 = D_1 + Q_1$ satisfying*

$$\|D - D_1\|_{L^2_s} < \epsilon \quad \textit{and} \quad \sup_{x \in \mathcal{R}} \|(Q - Q_1)(x)\|_{L^2_s} < \epsilon$$

for $s = 0, \ldots, k+1$, then $\mathcal{R}$ is also an isolating neighborhood for $-\mathcal{F}_1$

Once we have finite dimensional approximation $\mathcal{I}(\mathcal{R} \cap V, \pi_V \mathcal{F})$, we can associate a spectrum to $\mathcal{R}$ in the same manner as (3.2.7)

Definition 19. Let $\mathcal{R}$ be a closed and bounded subset of $\mathcal{K}$. A spectrum $E(\mathcal{R}, \mathcal{F})$ associated to $\mathcal{R}$ with respect to a flow generated by $\mathcal{F}$ is given by

$$E(\mathcal{R}, \mathcal{F})(V) = \mathcal{I}(\mathcal{R} \cap V, \pi_V \mathcal{F}) \wedge S^{V^+},$$

where V^+ is a positive space with respect to a reference quadratic form.

4.4 Constructing Isolating Neighborhoods in the Coulomb Slice

Our next task is to find a suitable choice of isolating neighborhoods in the Coulomb slice for finite dimensional approximation. By Hodge decomposition, we have

$$\mathcal{K} = \Omega_h \oplus \Omega_\perp \oplus \Gamma(S),$$

where $\Omega_h \cong \mathbb{R}^{b_1}$ is the harmonic 1-forms and $\Omega_\perp \cong \mathrm{Im}(d^*)$ is its orthogonal complement. Then, we can view the Coulomb slice as a (trivial) bundle over $\mathbb{R}^{b_1}$.

The set of critical points and trajectories between them are bounded modulo the full gauge group. However there is a residual action of harmonic maps on the Coulomb slice, so that critical points and trajectories between them lie in a union of balls

$$Str(R) = \mathbb{Z}^{b_1} \cdot B(R).$$

which is no longer bounded.

To apply finite dimensional approximation, we will cut the set $Str(R)$ to obtain a bounded region by level sets of certain functions. Moreover, we will construct an appropriate filtration of bounded subsets of $Str(R)$. Each bounded region gives rise to a spectrum from finite dimensional approximation and the inclusion of subsets in the filtration will induce morphisms between these spectra. This construction generalizes the one in [17] from $b_1 = 1$ to a general 3-manifold.

To obtain a bounded subset, we will try to control the translational action of $\mathbb{Z}^{b_1}$ by functions which are quasiperiodic under translation by $H^1(Y;\mathbb{Z}) \cong \bigoplus_{j=1}^{b_1} \mathbb{Z}h_j$. For each h_i, there is a unique pointed harmonic map $u_j : Y \to S^1$ such that $ih_j = {u_j}^{-1}du_j$. For a vector $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \ldots, \alpha_{b_1}) \in \mathbb{R}^{b_1}$, we say that f has period $\boldsymbol{\alpha}$ if for each j

$$f(u_j \cdot x) = f(x) + \alpha_j.$$

With this notation, the Chern-Simons-Dirac functional $\mathcal{L}$ has period $\boldsymbol{\alpha}_{\mathcal{L}}$ where

$$\begin{aligned} \alpha_{\mathcal{L},j} &= ([u_j] \cup \boldsymbol{c}_{\mathcal{L}})\,[Y] \\ &= \left([u_j] \cup \left(4\pi[\omega] + 2\pi^2 c_1(\mathfrak{s})\right)\right)[Y]\,. \end{aligned}$$

Another important property for cut-off functions is transversality.

Definition 20. We say that a smooth function $f : \mathcal{K} \to \mathbb{R}$ is *positively (resp. negatively) transverse* to a vector field $\mathcal{X}$ if $\langle \operatorname{grad} f, \mathcal{X} \rangle \geq 0$ (resp. ≤ 0) on some level set $f^{-1}(a)$.

Remark. A transverse function for a vector field can be viewed as a dual notion of a pseudo-gradient vector field for a function.

This implies that for a function f positively transverse to $\mathcal{X}$, the value of f is decreasing along a nonconstant trajectory of a flow generated by $-\mathcal{X}$. We now consider a class of good functions for constructing the Seiberg-Witten flow.

Definition 21. A smooth function $f : \mathcal{K} \to \mathbb{R}$ is called a *positive (resp. negative) good* function if it satisfies

(i) f is positively (resp. negatively) transverse to grad $\mathcal{L}$.

(ii) grad $f =$ grad $\mathcal{L} + \delta$ for small δ

(iii) f has period $\boldsymbol{\alpha}$ for some $\boldsymbol{\alpha} \in \mathbb{R}^{b_1}$.

(iv) f is bounded on $B(R)$.

Denote $\mathcal{F}_+$ (resp. $\mathcal{F}_-$) by the set of positive (resp. negative) good functions.

We first show that the set of good functions is nonempty.

Proposition 13. *There exists a positive good function with period lying in a neighborhood of $\boldsymbol{\alpha}_{\mathcal{L}}$ in $\mathbb{R}^{b_1}$. Similarly, there exists a negative good function with period in a neighborhood of $-\boldsymbol{\alpha}_{\mathcal{L}}$.*

Proof. Let $p_j : \mathcal{K} \to \mathbb{R}$ be the projection to the j-th component of $H^1(Y;\mathbb{R}) \cong \bigoplus_{j=1}^{b_1} \mathbb{R}h_j$. Since the set of critical points of grad $\mathcal{L}$ is discrete, we can find a closed interval $[a_j, b_j] \subset [0,1]$ such that $p_j^{-1}[a_j, b_j]$ contains no critical point.

By properness property of the Seiberg-Witten equation, $\|\text{grad}\,\mathcal{L}(x)\| \geq \epsilon > 0$ for all $x \in p_j^{-1}[a_j, b_j]$. We can construct a smooth function $g : [a,b] \to \mathbb{R}$ satisfying $g(a) = 0$, $g(b) = \delta \neq 0$, and $|g'(x)| < \epsilon$ on (a,b). Then we extend the domain of g to $\mathbb{R}$ by setting $g(x+1) = g(x) + \delta$.

Now consider a function $f_j = g \circ p_j + \mathcal{L}$. We clearly see that f_j is bounded on $B(R)$ and

$$f_j(u_i \cdot x) = \begin{cases} f_j(x) + \alpha_{\mathcal{L},i} & \text{if } i \neq j \\ f_j(x) + \alpha_{\mathcal{L},j} + \delta & \text{if } i = j. \end{cases}$$

To show that f_j is positively transverse to $\mathcal{L}$, we have

$$\begin{aligned} \langle \text{grad}\, f_j, \text{grad}\,\mathcal{L} \rangle &= \langle \text{grad}(g \circ p_j), \text{grad}\,\mathcal{L} \rangle + \|\text{grad}\,\mathcal{L}\|^2 \\ &> -\epsilon \|\text{grad}\,\mathcal{L}\| + \|\text{grad}\,\mathcal{L}\|^2 > 0, \end{aligned}$$

when $\text{grad}(g \circ p_j)$ is nonzero. In the case that $\text{grad}(g \circ p_j)$ is zero, we simply have $\langle \text{grad}\, f_j, \text{grad}\, \mathcal{L} \rangle = \|\text{grad}\, \mathcal{L}\|^2 \geq 0$ and the equality holds only at the critical point of $\mathcal{L}$. Thus, f_j is a positive good function with period $\boldsymbol{\alpha}_{\mathcal{L}} + \delta \boldsymbol{e}_j$, where $\{\boldsymbol{e}_j\}$ is the standard basis of $\mathbb{R}^{b_1}$.

Finally, we observe that $t_1 f_1 + t_2 f_2$ is a positive good function with period $t_1\boldsymbol{\alpha}_1 + t_2\boldsymbol{\alpha}_2$ if f_1, f_2 are positive good functions with period $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ respectively and t_1, t_2 are positive numbers. Similarly, $-f$ is a negative good function with period $-\boldsymbol{\alpha}$ if f is a positive good function with period $\boldsymbol{\alpha}$.

□

Remark. In fact, there exists a positive good function with period lying in a small positive cone containing $\boldsymbol{\alpha}_{\mathcal{L}}$ in $\mathbb{R}^{b_1}$.

Consider a positive good function f with period $\boldsymbol{\alpha}$. We have

$$f\left((\sum_{j=1}^{b_1} c_j u_j) \cdot x \right) = f(x) + \boldsymbol{c} \cdot \boldsymbol{\alpha},$$

where $\boldsymbol{c} = (c_1, c_2, \ldots, c_{b_1})$ is regarded as a coefficient of translation. We see that the set of translations from $B(R)$ to $Str(R) \cap f^{-1}[m, n]$ is bounded by two hyperplanes whose normal vector is $\boldsymbol{\alpha}$.

Let $\{\boldsymbol{\alpha}_j\}$ be a basis of $\mathbb{R}^{b_1}$ such that each $\boldsymbol{\alpha}_j$ is sufficiently close to $\boldsymbol{\alpha}_{\mathcal{L}}$. Then, there exist positive good functions f_j with period $\boldsymbol{\alpha}_j$ by Proposition 13. From the above observation, we have

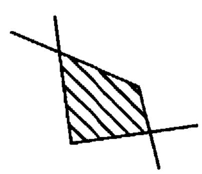


Figure 4-1: An example of a region bounded by hyperplane.

Lemma 11. *The set* $Str(R) \cap \left(\bigcap_{j=1}^{b_1} f_j^{-1}[a_j, b_j] \right)$ *is bounded.*

We set

$$\mathcal{R}_{m,n} = Str(R) \cap \left(\bigcap_{j=1}^{b_1} f_j^{-1}[m,n] \right).$$

For convenience, we consider an exhausting sequence $\{V_l\}$ of finite dimensional subspaces of $\mathcal{K}$ obtained from eigenspaces of L. Let ϕ_l be the compressed flow on V_l generated by the vector field $-L - \pi_l Q$. From Proposition 11, for sufficiently large l, the set $V_l \cap \mathcal{R}_{m,n}$ will be an isolating neighborhood for the flow ϕ_l . Denote $\mathcal{S}^l_{m,n}$ by the corresponding isolated invariant set and $\mathcal{I}(\mathcal{S}^l_{m,n})$ by its Conley index.

Note that $\mathcal{R}_{m,n} \subset \mathcal{R}_{m,n+1}$ and $\mathcal{R}_{m,n} \subset \mathcal{R}_{m-1,n+1}$ by definition. From our choice of the functions f_j, we observe that

$$\langle \pi \operatorname{grad} f_j, \pi \operatorname{grad} \pounds \rangle = \|\pi \operatorname{grad} \pounds\|^2 + \langle \pi\delta, \pi \operatorname{grad} \pounds \rangle,$$

on V_l. This tells us that the value of f_j is also decreasing along nonconstant trajectories of the compressed flows. In particular, we have that $\mathcal{S}^l_{m,n} \subset \mathcal{S}^l_{m,n+1}$ is an attractor subset. Thus we obtain a map $\mathcal{I}(\mathcal{S}^l_{m,n}) \to \mathcal{I}(\mathcal{S}^l_{m,n+1})$ induced by inclusion of index pairs, given by an attractor-repeller pair coexact sequence.

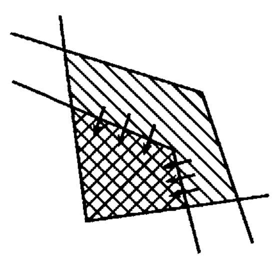


Figure 4-2: An example of a region bounded by hyperplane.

On the other hand, $\mathcal{S}^l_{m,n} \subset \mathcal{S}^l_{m-1,n}$ is a repeller subset, and we have a map in opposite direction. Together, we have a diagram

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{I}(\mathcal{S}^l_{m-1,n+1}) & \longrightarrow & \mathcal{I}(\mathcal{S}^l_{m,n+1}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & \mathcal{I}(\mathcal{S}^l_{m-1,n}) & \longrightarrow & \mathcal{I}(\mathcal{S}^l_{m,n}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

When we change from the subspace V_l to V_{l+1}, the Conley indices are related by $\mathcal{I}(\mathcal{S}^{l+1}_{m,n}) \cong \mathcal{I}(\mathcal{S}^l_{m,n}) \wedge S^W$. We then have a cube diagram

$$\begin{array}{ccccccc} \mathcal{I}(\mathcal{S}^{l+1}_{m-1,n+1}) & & \longrightarrow & & \mathcal{I}(\mathcal{S}^{l+1}_{m,n+1}) & & \\ \uparrow & \searrow & & & \uparrow & \searrow & \\ & & \mathcal{I}(\mathcal{S}^{l+1}_{m-1,n}) & \longrightarrow & & & \mathcal{I}(\mathcal{S}^{l+1}_{m,n}) \\ \uparrow & & \uparrow & & \uparrow & & \uparrow \\ \mathcal{I}(\mathcal{S}^{l}_{m-1,n+1}) & & \longrightarrow & & \mathcal{I}(\mathcal{S}^{l}_{m,n+1}) & & \uparrow \\ & \nwarrow & \uparrow & & & \nwarrow & \uparrow \\ & & \mathcal{I}(\mathcal{S}^{l}_{m-1,n}) & \longrightarrow & & & \mathcal{I}(\mathcal{S}^{l}_{m,n}) \end{array}$$

where the vertical maps are suspensions. This gives a diagram of spectra

$$\begin{array}{ccccccc} & & \vdots & & \vdots & & \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & E(\mathcal{R}_{m-1,n+1}) & \longrightarrow & E(\mathcal{R}_{m,n+1}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ \cdots & \longrightarrow & E(\mathcal{R}_{m-1,n}) & \longrightarrow & E(\mathcal{R}_{m,n}) & \longrightarrow & \cdots \\ & & \uparrow & & \uparrow & & \\ & & \vdots & & \vdots & & \end{array}$$

Definition 22. The Manolescu-Floer (pro-)spectrum $SWF_+^+(Y, \mathfrak{s})$ is given by a limit of the diagram of spectra $E(\mathcal{R}_{m,n})$ as $-m, n \to \infty$.

The notation SWF_+^+ means we use positive functions for both upper bound and lower bound for the level sets. Note that $SWF_+^+(Y, \mathfrak{s})$ is not a spectrum because there is an arrow that points opposite to the inclusion.

4.4.1 The Balanced Case

In the case of balanced perturbation, we have an alternative construction of the stable Conley index because the period $\boldsymbol{\alpha}_{\mathfrak{L}}$ is zero. As a consequence of Proposition 13, there exist both positive and negative good functions with period near the origin.

Pick an oriented basis $\{\boldsymbol{\alpha}_j\}$ of $\mathbb{R}^{b_1}$ such that the norm of $\boldsymbol{\alpha}_j$ is sufficiently small. Choose positive good functions f_j and negative good functions g_j both with period $\boldsymbol{\alpha}_j$. Now we consider the region

$$\widetilde{\mathcal{R}}_{m,n} = Str(R) \cap \left(\bigcap_{j=1}^{b_1} \left(f_j^{-1}(-\infty, n] \cap g_j^{-1}[m, \infty) \right) \right).$$

Similar to the balanced case, the region $\widetilde{\mathcal{R}}_{m,n}$ is bounded and we can apply finite dimensional approximation. Let $\{V_l\}$ be an exhausting sequence of $\mathcal{K}$ with the compressed flow ϕ_l. Again, $\widetilde{\mathcal{R}}^l_{m,n}$ will be an isolating neighborhood for sufficiently large l, and we denote $\widetilde{\mathcal{S}}^l_{m,n}$ by the corresponding isolated invariant set with $\mathcal{I}(\widetilde{\mathcal{S}}^l_{m,n})$ its Conley index.

The only difference in this case is that the set $\widetilde{\mathcal{S}}^l_{m,n}$ is an attractor subset of both $\widetilde{\mathcal{S}}^l_{m-1,n}$ and $\widetilde{\mathcal{S}}^l_{m,n+1}$. We then have a diagram

$$
\begin{array}{ccccccc}
 & & \uparrow & & \uparrow & & \\
\cdots & \longleftarrow & \mathcal{I}(\mathcal{S}^l_{m-1,n+1}) & \longleftarrow & \mathcal{I}(\mathcal{S}^l_{m,n+1}) & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & & \\
\cdots & \longleftarrow & \mathcal{I}(\mathcal{S}^l_{m-1,n}) & \longleftarrow & \mathcal{I}(\mathcal{S}^l_{m,n}) & \longleftarrow & \cdots \\
 & & \uparrow & & \uparrow & &
\end{array}
$$

We can define $SWF^+_-(Y,\mathfrak{s})$ similarly. But, notice that $SWF^+_-(Y,\mathfrak{s})$ is now a spectrum because all the direction of all arrows agree with the inclusion.

4.4.2 Duality

We denote $-Y$ by a manifold Y with reversed orientation.

When Y is equipped with a spin^c structure $\mathfrak{s}$, we can obtain a spin^c structure for $-Y$ with the same spinor bundle with Clifford multiplication $\rho_{-Y} = -\rho_Y$. Given a pair of perturbation $(\omega, \mathfrak{q})$ for Y, we also choose the pair $(\omega, -\mathfrak{q})$ for $-Y$

All the construction proceeds essentially in the same way. The main difference is that the signs of the Chern-Simons-Dirac functional its gradient change, i.e.

$$\mathcal{L}_{-Y} = -\mathcal{L}_Y \text{ and } \operatorname{grad}\mathcal{L}_{-Y} = -\operatorname{grad}\mathcal{L}_Y.$$

We now consider good funcions for $-Y$. Note that the Coulomb slice $\mathcal{K}$ of Y and $-Y$ are the same but the period $\boldsymbol{\alpha}_{\mathcal{L}_{-Y}}$ is equal to $-\boldsymbol{\alpha}_{\mathcal{L}_Y}$. Suppose that f is a positive good function for Y with period $\boldsymbol{\alpha}$, then we see that f is also a negative good function for $-Y$ with the same peiod.

Let $\{f_j\}$ be a positive good function basis for Y, then a collection $\{-f_j\}$ is a positive good basis for $-Y$. With the notation as before, we see that

$$\mathcal{R}(-Y)^l_{-n,-m} = \mathcal{R}(Y)^l_{m,n},$$

i.e. these two regions are the same. However, the compressed flows on $\mathcal{K}$ for $-Y$ are the reverse of those for Y. We observe that an isolating neighborhood and its isolated invariant set does not depend on the direction of the flow, but its Conley index can change. Denote $-\phi$ by the reverse of the flow ϕ, that is $-\phi(t) = \phi(-t)$. Then,

$$\mathcal{I}(\mathcal{S}(-Y)^l_{-n,-m}, \phi_{-Y}) = \mathcal{I}(\mathcal{S}(Y)^l_{m,n}, -\phi_Y).$$

From ???, this is a V_l dual of $\mathcal{I}(\mathcal{S}(Y)^l_{m,n}, \phi_Y)$. Then, we get a dual diagram

$$\begin{array}{ccccccc}
 & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & D^{V_l}\mathcal{I}(\mathcal{S}^l_{m-1,n+1}) & \longleftarrow & D^{V_l}\mathcal{I}(\mathcal{S}^l_{m,n+1}) & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
\cdots & \longleftarrow & D^{V_l}\mathcal{I}(\mathcal{S}^l_{m-1,n+1}) & \longleftarrow & D^{V_l}\mathcal{I}(\mathcal{S}^l_{m,n+1}), & \longleftarrow & \cdots \\
 & & \downarrow & & \downarrow & &
\end{array}$$

where we denote D^V by V-dual.

Chapter 5

Calculation

In this section, we present calculation of the Floer homotopy type in some special cases. The 3-manifolds in which we particularly interest are the product $S^1 \times S^2$ and the three-torus T^3.

5.1 Outline

We will first outline parts of strategies which apply to these cases. Since the 3-manifolds of interest have nonnegative scalar curvature, the corresponding solutions of the unperturbed Seiberg-Witten equations are reducible, and are gauge equivalent to flat connections. Recall that we have a decomposition of the Coulomb slice

$$\mathcal{K} = \Omega_h \oplus \Omega_\perp \oplus \Gamma(S).$$

With a based flat connection B_0 fixed, the zero locus of the induced vector field on the Coulomb slice is then the entire Ω_h-subspace.

To obtain a bounded isolating neighborhood and admissibility, we need to perturb the Chern-Simon-Dirac functional. We start by picking a Morse function f on the Picard torus (when $b_1 > 0$)

$$\mathbb{T} = H^1(Y; i\mathbb{R})/(2\pi i H^1(Y;\mathbb{Z})).$$

There is a projection map from the configuration space onto the harmonic part $\pi_h :$ $\mathcal{C}(Y, \mathfrak{s}) \to \Omega_h$ given by

$$(B_0 + b \otimes 1, \psi) \mapsto b_h,$$

where b_h is the harmonic part of b. Then, we have a composition $f \circ \pi_h : \mathcal{C}(Y, \mathfrak{s}) \to \mathbb{R}$ which will also be denoted by f. Consider a perturbed functional of the form

$$\not{\mathcal{L}} = \mathcal{L} + f. \tag{5.1.1}$$

Since the gradient of f only affect the Ω_h-component, reducible solutions of the perturbed equations correspond to critical points of f on $\mathbb{T}$. On a Coulomb slice, the induced flow on the Ω_h-subspace and the critical points are $\mathbb{Z}^{b_1}$-translation of those on the torus.

In the case of our interest, all critical points and trajectories between them will lie in the harmonic Ω_h-subspace. The appropriate isolating neighborhoods on the Coulomb slice to apply finite-dimensional approximation then arise as a tubular neighborhood of an isolating neighborhood on $\Omega_h \cong \mathbb{R}^{b_1}$ with respect to the downward gradient flow of f (See Figure 5-1).

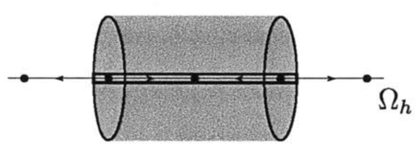


Figure 5-1: A tubular neighborhood as an isolating neighborhood.

In order to have an explicit description for Conley indices of these isolating neighborhoods, we use linear isomorphism on the vector space to reduce the flow to a linear one. For a general setting, let γ be a flow on a vector space V generated by a vector field F. Let A be a linear automorphism of V. We can consider a new flow γ_A on V given by $\gamma_A(x,t) = A^{-1}\gamma(Ax,t)$. It is straightforward to check that the two flows are equivalent (depicted by the diagram below) and that γ_A is generated by

$F_A = A^{-1} F A$.

$$\begin{array}{ccc} x & \xrightarrow{\gamma(\cdot,t)} & \gamma(x,t) \\ {\scriptstyle A^{-1}}\downarrow & & \downarrow{\scriptstyle A^{-1}} \\ A^{-1}x & \xrightarrow{\gamma_A(\cdot,t)} & V \end{array}$$

Next, we will use the fact that one can choose a radius of the tubular neighborhood for an isolating neighborhood to be arbitrarily small. After applying scaling automorphism in the normal direction, we normalize a radius of a tubular neighborhood to 1. On the other hand, the new vector field will be approaching a linear vector field as the radius gets smaller.

More explicitly, suppose that we have a model case of a vector space $H \oplus V$ with a vector field of the form $(\nabla f(h), L(h)v + Qv)$, where $L(h)$ is a self-adjoint linear operator depending on h and Q is quadratic on v-variable. With the formula above, the scaling process of vector fields will look like

Isolating neighborhood :	$\nu(\epsilon)$	$\xrightarrow{\text{Scaling}}$	$\nu(1)$
Vector field (on V) :	$L(h)v + \pi Qv$	$\xrightarrow{\text{Scaling}}$	$L(h)v + \epsilon\,\pi Qv$
Finite dimensional approximation	$\downarrow$		$\downarrow$
	$L(h)v + Qv$		$L(h)v$

This shows that the linearized vector field has the form $(\nabla f(h), L(h)v)$. A trajectory $(h(t), v(t))$ comes from a solution of a differential equation

$$-\frac{d}{dt}h(t) = \nabla f(h(t)) \tag{5.1.2a}$$

$$-\frac{d}{dt}v(t) = L(h(t))v(t). \tag{5.1.2b}$$

We see that the $h(t)$-part is just a gradient flow line flow line and the $v(t)$-part is a linear ODE with variable coefficient. This can be seen as a family version of a linear flow (Example 2).

Let also assume that we choose an isolating neighborhood $X$with respect to the gradient flow in the base space H. Its tubular neighborhood in $H \oplus V$ is simply $X \times B(1)$, where $B(1)$ is a unit disk in V whose boundary is a unit sphere $S(1)$. Under certain condition, this tubular neighborhood will also be an isolating block (Definition 6). Consequently, to compute its Conley index, we only need to determine an exit set which is on the boundary $(\partial X \times B(1)) \cup (X \times S(1))$.

The above equations allow us to analyze an exit set on $X \times S(1)$ by considering derivative of the norm of $v(t)$

$$\begin{aligned}\frac{d}{dt}\|v(t)\|^2 &= \langle \frac{d}{dt}v(t), v(t)\rangle \ + \langle v(t), \frac{d}{dt}v(t)\rangle \\ &= -2\langle L(h(t))v(t), v(t)\rangle.\end{aligned}$$

It follows that a point (h_0, v_0) on $X \times S(1)$ is entering (resp. leaving) the tubular neighborhood when the quantity $\langle L(h_0)v_0, v_0\rangle$ is positive (resp. negative). For the case that $\langle L(h_0)v_0, v_0\rangle$ is zero, we also need to check the second derivative to ensure that the point leaves the neighborhood immediately. One obtains

$$\begin{aligned}\frac{d^2}{dt^2}\|v(t)\|^2 &= -2\left(\langle \frac{d}{dt}(L(h(t)))v(t) + L(h(t))\dot{v}(t), v(t)\rangle + \langle L(h(t))v(t), \dot{v}(t)\rangle\right) \\ &= 2\langle\left(2(L(h(t)))^2 - \frac{d}{dt}(L(h(t)))\right)v(t), v(t)\rangle \\ &= 2\langle\left(2(L(h(t)))^2 + \nabla L(h(t)) \cdot \nabla f(h(t))\right)v(t), v(t)\rangle,\end{aligned}$$

where we have $\frac{d}{dt}(L(h(t))) = -\nabla L(h(t)) \cdot \nabla f(h(t))$ by the chain rule. We deduce that when the quantity $\langle (2(L(h_0))^2 + \nabla L(h_0) \cdot \nabla f(h_0))\, v_0, v_0\rangle$ is positive, the norm $\|v(t)\|$ has a local minimum at (h_0, v_0). If this condition holds for each point with $\langle L(h)v_0, v_0\rangle = 0$, then every point on $X \times S(1)$ will leave the neighborhood immediately in one or another time direction.

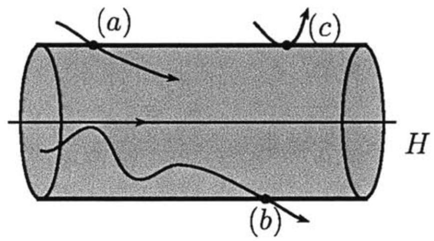


Figure 5-2: A tubular neighborhood as an isolating block.

Note that when h_0 is a critical point of ∇f, the flow on $\{h_0\} \times V$ is invariant and is a linear flow. The second derivative becomes $4\langle (L(h_0))^2 v_0, v_0\rangle$. For the tubular neighborhood to be an isolating block, we also require that the linear operator $L(h_0)$ at each critical point has no kernel.

In summary, we have deduced

Lemma 12. *In the above situation, suppose that X is an isolating block for the gradient flow on H. The tubular neighborhood $X \times B(1)$ is an isolating block for the flow generated by $(\nabla f(h), L(h)v)$ if the following conditions hold*

- *The operator $L(h_0)$ has no kernel when h_0 is a critical point for ∇f,*
- *The quantity $\langle (2(L(h_0))^2 + \nabla L(h_0) \cdot \nabla f(h_0))\, v_0, v_0\rangle$ is positive for each point on $X \times B(1)$ such that $\langle L(h)v_0, v_0\rangle = 0$.*

Under this hypothesis, its exit set can be described as a union of the set $\{(h, v) \in X \times B(1) \mid h$ lies in the exit set of $X\}$ and the set $\{(h, v) \in X \times S(1) \mid \langle L(h)v, v\rangle \leq 0\}$.

We remark that an exit set on $X \times S(1)$ can be viewed as an intersection of a unit sphere $S(1)$ and a nonpositive cone $\{v \mid \langle L(h)v, v\rangle \leq 0\}$ varying along the fiber. We will also refer to this by a *unit nonpositive cone*. An interesting example arises when an index of $L(h)$ changes so that the homotopy type of this cone is not constant (See Figure 5-3).

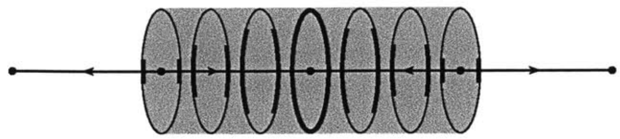


Figure 5-3: An exit set for a tubular neighborhood.

Denote by $\text{Cone}_{\leq}(V, L)$ a unit nonpositive cone in V with respect to L. We can simplify and decompose $\text{Cone}_{\leq}(V, L)$ as in the following lemma

Lemma 13. *Suppose that V can be decomposed as a direct sum $V_1 \oplus V_2$ of L-invariant subspaces. Then*

$$\text{Cone}_{\leq}(V, L) \cong \text{Cone}_{\leq}(V_1, L) \star \text{Cone}_{\leq}(V_2, L),$$

where $\star$ denotes the join operation.

Proof. Recall that there is a retraction from $\text{Cone}_{\leq}(V, L)$ to a unit sphere $S(V^-)$ of a maximal nonpositive subspace V^- of V. This retraction is given by $V \setminus V^+ \to S(V^-)$ where v is sent to $\pi_{V^-}v/\|\pi_{V^-}v\|$ a normalized orthogonal projection onto V^- and V^+ is its complementary maximal positive subspace.

Choose a maximal nonpositive subspace V_i^- of V_i then we see that $V_1^- \oplus V_2^-$ is a maximal nonpositive subspace of V and

$$\text{Cone}_{\leq}(V, L) \cong S(V_1^- \oplus V_2^-) \cong S(V_1^-) \star S(V_2^-) \cong \text{Cone}_{\leq}(V_1, L) \star \text{Cone}_{\leq}(V_2, L).$$

□

5.2 $S^1 \times S^2$

We will present a construction of SWF for the 3-manifold $S^1 \times S^2$. We will be mainly interested in the case when $c_1(\mathfrak{s})$ is torsion. When $c_1(\mathfrak{s})$ is nontorsion, there is no critical point and SWF is trivial.

Equip $S^1 \times S^2$ with a round metric, which has positive scalar curvature. Pick a flat connection B_0 which is trivial in S^1-variable and consider the perturbed functional $\not{\mathcal{L}} = \mathcal{L} + f$ as described earlier.

The vector field on the Coulomb slice with respect to the decomposition $\Omega_h \oplus \Omega_\perp \oplus \Gamma(S)$ is given by

$$(b_h, b_\perp, \psi) \mapsto (\nabla f + \pi_h \bar{\tau}(\psi), *db_\perp + \pi_\perp \bar{\tau}(\psi), \xi(\tau(\psi))\psi + D_{B_0+b}\psi), \tag{5.2.1}$$

where $\tau(\psi)$ denotes the quadratic term $\rho^{-1}(\psi\psi^*)_0$ and $\bar{\tau}(\psi)$ denotes the term $\tau(\psi) - d\xi(\tau(\psi))$. Then the scaled vector field of the direction normal to b_h is

$$(\nabla f + \epsilon^2 \pi_h \bar{\tau}(\psi), *db_\perp + \epsilon\, \pi_\perp \bar{\tau}(\psi), \epsilon^2 \xi(\tau(\psi))\psi + \epsilon\, b_\perp \psi + D_{B_0+b_h}\psi). \tag{5.2.2}$$

As $\epsilon \to 0$, we see that the limiting vector field is given by

$$(\nabla f, *db_\perp, D_{B_0+b_h}\psi), \tag{5.2.3}$$

which is linear except the term $D_{B_0+b_h}\psi$. This can be viewed as a linear flow with varying operators. We will focus on a flow generated by $(\nabla f, D_{B_0+b_h}\psi)$ on $\Omega_h \oplus \Gamma(S)$ since the flow on $\Omega_\perp$ is a linear flow independent of other components

In this case, one can explicitly describe finite dimensional model of Dirac operators using Fourier series and spherical harmonics. We first describe spinors of a spin bundle on S^2 which can be identified with $L \oplus L^{-1}$, where L is the canonical line bundle of $S^2 = \mathbb{C}P^1$. Following notations of [35], with the stereographic projection to $\mathbb{C}$ to give a chart on $S^2 - (0,0,1)$, an L^2-orthonormal basis of the space of sections of L is given by

$$Y^+_{l,m} := (-1)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!(l-m)!}{(l+\frac{1}{2})!(l-\frac{1}{2})!}} \sum_{r-s=m-\frac{1}{2}} \binom{l-\frac{1}{2}}{r}\binom{l+\frac{1}{2}}{s}\frac{z^r(-\bar{z})^s}{(1+z\bar{z})^l},$$

which is indexed by $l \in \mathbb{N} - \frac{1}{2}$ and $m \in \{-l, -l+1, \ldots, l\}$ with r, s nonnegative integers. Similarly, sections of L^{-1} has a basis indexed by the same set of (l, m) given by

$$Y^-_{l,m} := (-1)^{l-m}\sqrt{\frac{2l+1}{4\pi}}\sqrt{\frac{(l+m)!(l-m)!}{(l+\frac{1}{2})!(l-\frac{1}{2})!}}\sum_{r-s=m+\frac{1}{2}}\binom{l+\frac{1}{2}}{r}\binom{l-\frac{1}{2}}{s}\frac{z^r(-\bar{z})^s}{(1+z\bar{z})^l}.$$

The spin connection on $L \oplus L^{-1}$ is induced by the Levi-Civita connection. With respect to isotropic bases, the Clifford multiplication is given by

$$\alpha^+ = \frac{dz}{1+z\bar{z}} \mapsto \sigma_3 = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}, \quad \alpha^- = \frac{d\bar{z}}{1+z\bar{z}} \mapsto \sigma_2 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The spinor bundle of $S^1 \times S^2$ is a pull-back of the spin bundle on S^2. Let θ be a coordinate on S^1 factor. The Clifford multiplication extends by assigning

$$d\theta \mapsto \sigma_1 = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}.$$

Then, the Dirac operator on $S^1 \times S^2$ is given by (note that the sign is different from one in [35])

$$D_{B_0} = \sigma_1\frac{\partial}{\partial\theta} + \sigma_2\nabla_{(1+z\bar{z})\frac{\partial}{\partial\bar{z}}} + \sigma_3\nabla_{(1+z\bar{z})\frac{\partial}{\partial z}}.$$

Since a smooth function on S^1 has a Fourier series expansion, we can combine with the spherical harmonics above and introduce a basis for spinors

$$Y^1_{l,m,n} := \frac{1}{\sqrt{2}}e^{in\theta}\begin{bmatrix} Y^+_{l,m} \\ iY^+_{l,m} \end{bmatrix}, \quad Y^2_{l,m,n} := \frac{1}{\sqrt{2}}e^{in\theta}\begin{bmatrix} -Y^+_{l,m} \\ iY^+_{l,m} \end{bmatrix}.$$

The direct sum of these subspaces gives an L^2-decomposition for $\Gamma(S)$. It is not hard to check that, for an operator $D_{B_0+ia\,d\theta}$ and $a \in \mathbb{R}$,

$$D_{B_0+ia\,d\theta}Y^1_{l,m,n} = -(l+\frac{1}{2})Y^1_{l,m,n} + (n+a)Y^2_{l,m,n},$$
$$D_{B_0+ia\,d\theta}Y^1_{l,m,n} = (n+a)Y^1_{l,m,n} + (l+\frac{1}{2})Y^2_{l,m,n}.$$

Thus, on the subspace $V_{l,m,n} := span\left\{Y^1_{l,m,n}\,,\, Y^2_{l,m,n}\right\}$, the Dirac operator D_a acts as a matrix

$$\begin{bmatrix} -(l+\frac{1}{2}) & n+a \\ n+a & l+\frac{1}{2} \end{bmatrix}. \tag{5.2.4}$$

We see that D_a has no kernel for any a. Moreover, D_a has no eigenvalue in the interval $(-1/2, 1/2)$.

We now look at isolating neighborhoods for the vector field $(\nabla f, D_{B_0+b_h}\psi)$ on $\Omega_h \oplus \Gamma(S)$. The function f comes from a Morse function on S^1, which we will choose to be $f(\theta) = -\epsilon_0 \cos\theta$ for a sufficiently small $\epsilon_0 > 0$. On $\Omega_h \cong \mathbb{R}$, a set of critical points is $\{m\pi \,|m \in \mathbb{Z}\}$ and trajectories of the downward gradient flow travel from $m\pi$ to $(m \pm 1)\pi$ for each odd integer k. An isolating neighborhood we will consider is the interval $I_k = [-2k\pi - \delta_0, 2k\pi + \delta_0]$, which is an isolating block with no exit set for each integer m. Moreover, an isolated invariant set of I_k is an attractor relative to an an isolated invariant set of I_{k+1} with respect to inclusion $I_k \subset I_{k+1}$.

Note. To be consistent with the period of $\mathbb{T}$, we identify $a \in \mathbb{R}$ with $\frac{1}{2\pi} ia\, d\theta$ rather than $ia\, d\theta$. We denote $D_a := D_{B_0 + \frac{1}{2\pi} ia\, d\theta}$ so that the matrix of $D_{a+2\pi}$ on $Y^i_{l,m,n}$ and the matrix of D_a on $Y^i_{l,m,n+1}$ is the same.

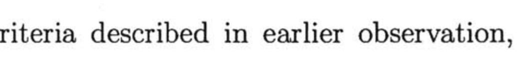


Figure 5-4: Isolating neighborhoods.

Claim. *Any tubular neighborhood gives an isolating block.*

Proof. We will use criteria described in earlier observation, in particular, we will check that $2D_a^2 + \nabla D_a \cdot \nabla f(a)$ is positive definite (which is stronger than the desired condition). Since D_a has no eigenvalue in the interval $(-1/2, 1/2)$, we have $\|D_a v\|^2 \geq \frac{1}{4}\|v\|^2$ for all a. On the other hand, the matrix of $\nabla D_a \cdot \nabla f(a)$ on $V_{l,m,n}$ is given by $\frac{\epsilon_0}{2\pi}\begin{bmatrix} 0 & -\sin a \\ -\sin a & 0 \end{bmatrix}$ whose eigenvalues are controlled by ϵ_0. Thus, for ϵ_0 small enough, $2D_a^2 + \nabla D_a \cdot \nabla f(a)$ is positive definite . ☐

We begin to describe Conley indices of these tubular neighborhoods. For a subset $X \subset \mathbb{R}^3$ and a subspace V of $\Gamma(S)$, we define

$$\nu(X,V) := \{(a,v) \in X \times V \mid \|v\| \le 1\} \tag{5.2.5}$$

$$n^-(X,V) := \{(a,v) \in X \times V \mid \|v\| = 1 \text{ and } \langle L(h)v, v\rangle \le 0\} \tag{5.2.6}$$

We consider $\nu(I_k, V)$ when V is a (finite) direct sum of those $Y^1_{l,m,n}$ and $Y^2_{l,m,n}$. The advantage of these subspaces is that they are invariant subspaces for all D_a simultaneously. The Conley index of $\nu(I_k, V)$ with respect to $(\nabla f(a), D_a v)$ restricted on $\mathbb{R} \times V$ will serve as a finite dimensional approximation of $\nu(I_k, \Gamma(S))$.

Since I_k has no exit point on its boundary, we have that $n^-(I_k, V)$ is an exit set for $\nu(I_k, V)$. From now on, we will also work in the S^1-equivariant context, where S^1 acts as a scalar multiplication by a unit complex number on $\Gamma(S)$. Both $\nu(I_k, V)$ and $l(I_k, V)$ are clearly S^1-invariant by definition.

We see that $n^-(I_k, V)$ is a submanifold of $\partial\nu(I_k, V)$, so $(\nu(I_k, V), l(I_k, V))$ is an S^1-NDR pair. Moreover, we easily see that $n^-(I_m, V)$ is contractible to a point in $\nu(I_k, V)$ and $\nu(I_k, V)$ deformation retracts to a point. By applying Lemma 4, we have that

$$\nu(I_k, V)/n^-(I_k, V) \cong \mathbf{S}n^-(I_k, V).$$

Recall that $\mathbf{S}$ denotes the unreduced suspension.

For $a \in I_k$, we first consider a fiber of $n^-(I_k, V)$ over a which is the unit nonpositive cone $\{v \in V \mid \|v\| = 1 \text{ and } \langle D_a v, v\rangle \le 0\}$. When D_a has no kernel, this is homotopy equivalent to the unit sphere of a maximal negative definite subspace V^- of V (with respect to D_a) just as in the case of a linear flow on a vector space.

Since D_a has no kernel for all a, the dimension of a maximal negative definite subspace V_a^- with respect to D_a is constant. Then we can choose a continuous family of V_a^- over $\mathbb{R}$, which is equivalent to choose a continuous function from $\mathbb{R}$ to an appropriate Grassmannian. This provides a retraction of the unit nonpositive cone to the unit sphere of V_a^- fiberwise. Hence we deduce that $n^-(I_k, V)$ is homotopy equivalent to a sphere bundle over I_k. Then we can deform $n^-(I_k, V)$ to a product

$I_k \times S(V_{a_0}^-)$ for some $a_0 \in I_k$.

Suppose that V has p summands of $Y^1_{l,m,n}$'s and $Y^2_{l,m,n}$'s each of which has one positive and one negative eigenvalue with respect to D_a. Then $\mathrm{S}n^-(I_k, V)$ is homotopy equivalent to the one-point compactification of $V_{a_0}^- \cong \mathbb{C}^p$. Hence, the (homotopy) Conley index is given by

$$\mathcal{I}(\nu(I_k, V)) = [S^{\mathbb{C}^p}].$$

Notice that, when $V = V_1 \oplus V_2$ of the same form

The inclusion $I_k \subset I_{k+1}$ as an attractor induces a map between Conley indices. We can see that $(\nu(I_k, V) \cup n^-(I_{k+1}, V), n^-(I_{k+1}, V))$ is an index pair for $\nu(I_k, V)$. As in Proposition 1, the map is induced by an inclusion of index pairs

$$\nu(I_k, V)/n^-(I_k, V)) \to \nu(I_{k+1}, V)/n^-(I_{k+1}, V)).$$

By naturality of the mapping cone construction, this is equivalent to a map

$$\mathrm{S}n^-(I_k, V) \to \mathrm{S}n^-(I_{k+1}, V),$$

induced by an inclusion $n^-(I_k, V) \subset n^-(I_{k+1}, V)$. Hence, the map

$$[S^{\mathbb{C}^p}] = \mathcal{I}(\nu(I_k, V)) \to \mathcal{I}(\nu(I_{k+1}, V)) = [S^{\mathbb{C}^p}]$$

is given by the identity map.

Roughly speaking, the Conley index $\mathcal{I}(\nu(I_k, V))$ is given by $S^{V_{a_0}^-}$. By choosing D_{a_0} as the reference quadratic form, the V-space assigned to the spectrum $E(\nu(I_k))$ is precisely S^V. By applying desuspension, we could say that the stable homotopy type of $\nu(I_k)$ is the 0-sphere.

Since the map induced by the inclusion $I_k \subset I_{k+1}$ is the identity map, we can conclude

Theorem 1. *The stable homotopy type of $SWF(S^1 \times S^2, \mathfrak{s})$ is S^0.*

5.3 T^3

5.3 The only interesting case is also the case when $c_1(\mathfrak{s})$ is torsion. We equip the torus with a flat metric. Since T^3 does not have positive scalar curvature, some ideas from the case $S^1 \times S^2$ will need to be modified.

Let B_0 be the trivial connection. Since the Dirac operator D_{B_0} has kernel, another kind of perturbation is required. As in [18], we consider the perturbed Chern-Simon-Dirac functional of the form

$$\not{\mathcal{L}} = \mathcal{L} - (\delta/2)\|\psi\|^2 + \epsilon f,$$

where δ and ϵ are sufficiently small positive number (the choice of ϵ depends on δ) and f is induced from a Morse function on T^3. Consequently, a linearized vector field has the form

$$(\epsilon \nabla f, *db_\perp, (D_{B_0+b_h} - \delta)\psi).$$

The spinor bundle is given by trivial bundle $T^3 \times \mathbb{C}^2$ and its section is a pair of complex-valued functions on T^3. For finite dimensional model, we can use Fourier series to write a function on T^3 as

$$\sum_{n_1,n_2,n_3 \in \mathbb{Z}} c_{n_1,n_2,n_3} e^{i(n_1\theta_1+n_2\theta_2+n_3\theta_3)}, \tag{5.3.1}$$

where $c_{\vec{n}}$ is a complex number.

The Clifford multiplication identifies the 1-form $d\theta_i$ with the Pauli matrix σ_i, that is

$$d\theta_1 \mapsto \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}, \quad d\theta_2 \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad d\theta_3 \mapsto \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}.$$

We can now describe Dirac operators explicitly in each Fourier mode. Denote V_{n_1,n_2,n_3} by a 2-dimensional subspace spanned by $(e^{i(n_1\theta_1+n_2\theta_2+n_3\theta_3)}, 0)$ and $(0, e^{i(n_1\theta_1+n_2\theta_2+n_3\theta_3)})$, then we have

$$\Gamma(S) = \bigoplus_{n_1,n_2,n_3 \in \mathbb{Z}} V_{n_1,n_2,n_3}$$

and it is easy to check that D_{B_0} acts on V_{n_1,n_2,n_3} as a matrix

$$\begin{bmatrix} -n_1 & -n_3 - n_2 i \\ -n_3 + n_2 i & n_1 \end{bmatrix}.$$

We can identify Ω_h with $\mathbb{R}^3$ so that $(b_1, b_2, b_3) \in \mathbb{R}^3$ corresponds to a connection $b_h = \frac{1}{2\pi} i(b_1 d\theta_1 + b_2 d\theta_2 + b_3 d\theta_3)$ so that the operator $D_{B_0+b_h} - \delta I$ (denoted by $D_{\vec{b}}$) acts on V_{n_1,n_2,n_3} (denoted by $V_{\vec{n}}$) as a matrix

$$\begin{bmatrix} -(n_1 + \frac{b_1}{2\pi}) - \delta & -(n_3 + \frac{b_3}{2\pi}) - (n_2 + \frac{b_2}{2\pi})i \\ -(n_3 + \frac{b_3}{2\pi}) + (n_2 + \frac{b_2}{2\pi})i & n_1 + \frac{b_1}{2\pi} - \delta \end{bmatrix}. \qquad (5.3.2)$$

The gauge translational action can be observed as the matrix of $D_{\vec{b}+2\pi\vec{u}}$ on $V_{\vec{n}}$ is the same as the matrix of $D_{\vec{b}}$ on $V_{\vec{n}+\vec{u}}$.

We now look at isolating neighborhoods for the vector field $(\epsilon\nabla f, (D_{B_0+b_h} - \delta)\psi)$ on $\Omega_h \oplus \Gamma(S)$. We can pick a Morse function to be $f(\theta_1, \theta_2, \theta_3) = -\cos\theta_1 - \cos\theta_2 - \cos\theta_3$. The flow on $\Omega_h \cong \mathbb{R}^3$ is basically the product of the one in the $S^1 \times S^2$ case. Similarly, a family of isolating neighborhoods we will consider consists of the cube $I_k = [-(2k + \frac{1}{2})\pi, (2k + \frac{1}{2})\pi]^3$, which is also an isolating block with no exit set. Moreover, an isolated invariant set of I_k is an attractor relative to an an isolated invariant set of I_{k+1} with respect to inclusion $I_k \subset I_{k+1}$.

The eigenvalues of the matrix (5.3.2) is $-\delta \pm \sqrt{(n_1 + \frac{b_1}{2\pi})^2 + (n_2 + \frac{b_2}{2\pi})^2 + (n_3 + \frac{b_3}{2\pi})^2}$. Since n_1, n_2, n_3 are integers, $D_{2\pi\vec{b}}$ has kernel on a small sphere $S^2_{2\pi\delta}$ centered at $2\pi(b_1, b_2, b_3)$ for each triple of integers (b_1, b_2, b_3). The point $2\pi(b_1, b_2, b_3)$ is an index 0 critical point and the kernel of $D_{2\pi\vec{b}}$ on $S^2_{2\pi\delta}$ centered at this point is a (complex) 1-dimensional subspace of $V_{-b_1,-b_2,-b_3}$.

Claim. *Any tubular neighborhood gives an isolating block.*

Proof. We will also be using criteria described in earlier observation. We will still show that $2D_{\vec{b}}^2 + \nabla D_{\vec{b}} \cdot \nabla f(\vec{b})$ is positive definite. However, the argument will be slightly more complicated since $D_{\vec{b}}$ can have kernel. By gauge translation, we can consider only $D_{\vec{b}}$ in the cube $[-\pi, \pi]^3$. On the subspace complementary to $V_{0,0,0}$, the

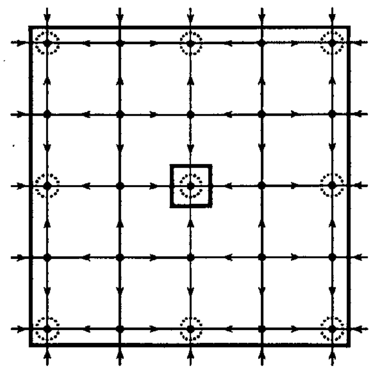


Figure 5-5: The flow on Ω_h

norm of an eigenvalue of $D_{\vec{b}}$ is bounded away from 0, so that the same argument in the case $S^1 \times S^2$applies.

We will consider a set of pairs $(\vec{b}, v)$, where $\vec{b}$ lies on $S^2_{2\pi\delta}$ centered at the origin and v is a unit vector in $V_{0,0,0}$ which is also in the kernel of $D_{\vec{b}}$. One can check that the kernel of $\begin{bmatrix} -b_1 - \delta & -b_3 - b_2 i \\ -b_3 + b_2 i & b_1 - \delta \end{bmatrix}$ is given by $\frac{1}{\sqrt{2\delta(\delta - b_1)}} \begin{bmatrix} -b_1 + \delta \\ -b_3 + b_2 i \end{bmatrix}$ when $b_1 \neq \delta$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ otherwise. We also have that the matrix of $\nabla D_{\vec{b}} \cdot \nabla f(\vec{b})$ on $V_{0,0,0}$ is given by

$$\frac{\epsilon}{2\pi} \begin{bmatrix} -\sin b_1 & -\sin b_3 - \sin b_2 i \\ -\sin b_3 + \sin b_2 i & \sin b_1 \end{bmatrix}.$$

Roughly speaking, an element in the kernel of $D_{\vec{b}}$ corresponds to the larger eigenvalue of $D_{\vec{b}}$ so that it will correspond to the positive eigenvalue of $\nabla D_{\vec{b}} \cdot \nabla f(\vec{b})$. From above, we can calculate

$$\frac{2\pi}{\epsilon} \langle \nabla D_{\vec{b}} \cdot \nabla f(\vec{b}) v, v \rangle = \frac{1}{\delta}(b_1 \sin b_1 + b_2 \sin b_2 + b_3 \sin b_3) > \frac{\delta}{2},$$

where we use an approximation $x \sin x > x^2/2$ for small x. Consequently, we can choose a neighborhood of the pairs $(\vec{b}, v)$ such that the quantity $\frac{1}{\epsilon}\langle \nabla D_{\vec{b}} \cdot \nabla f(\vec{b}) v, v \rangle$ is bounded below by a positive constant (depended on δ). Hence $\langle \nabla D_{\vec{b}} \cdot \nabla f(\vec{b}) v, v \rangle$ is positive in this neighborhood regardless of ϵ.

On the other hand, the norm of an eigenvalue of $D_{\vec{b}}$ is bounded away from 0 outside this neighborhood. Similar to the case $S^1 \times S^2$, we can choose sufficiently small ϵ so that the term $\langle 2D_{\vec{b}}^2 v, v\rangle$ dominates the term $\langle \nabla D_{\vec{b}} \cdot \nabla f(\vec{b})v, v\rangle$. Therefore, we can ensure that $2D_{\vec{b}}^2 + \nabla D_{\vec{b}} \cdot \nabla f(\vec{b})$ is positive definite. □

Note. It is crucial that δ and ϵ are both positive. It is also important to perturb D_{B+b_h} by δI so that $D_{\vec{b}}$ has no kernel at a critical point of f.

We first try to understand an exit set $n^-(X, V_{0,0,0})$ (as defined in (5.2.6))of a tubular neighborhoods of some subset X in the subspace $V_{0,0,0} \cong \mathbb{C}^2$ of parallel sections. Recall that a matrix of $D_{2\pi\vec{b}}$ on $V_{0,0,0}$ is given by

$$\begin{bmatrix} -b_1 - \delta & -b_3 - b_2 i \\ -b_3 + b_2 i & b_1 - \delta \end{bmatrix}, \tag{5.3.3}$$

and its eigenvalues are $-\delta \pm \sqrt{b_1^2 + b_2^2 + b_3^2}$.

For a point $\vec{b}$ with $\|\vec{b}\| < 2\pi\delta$, the quadratic form associated to $D_{\vec{b}}$ is negative definite, so its unit nonpositive cone is the whole unit sphere $S^3 \subset \mathbb{C}^2$. When $\|\vec{b}\| = 2\pi\delta$, $D_{\vec{b}}$ has kernel but its unit nonpositive cone is still the sphere S^3.

On the other hand, for a point $\vec{b}$ with $\|\vec{b}\| > 2\pi\delta$, the quadratic form associated to $D_{\vec{b}}$ has signature $(1,1)$ and its unit nonpositive cone deformation retracts to a circle in S^3. This circle is the unit circle of a maximal negative definite subspace with respect to $D_{\vec{b}}$. We can choose this circle to be a rotation by unit complex numbers of a unit eigenvector corresponding to the negative eigenvalue.

With this we can deduce

Lemma 14.

(i) If X is a ball of radius R, then $n^-(X, V_{0,0,0})$ is homotopy equivalent to the unit 3-sphere $S^3 \subset \mathbb{C}^2$.

(ii) If X is a sphere of radius R greater than $2\pi\delta$, then $n^-(X, V_{0,0,0})$ is homotopy equivalent to the 3-sphere $S^3 \subset S^2 \times \mathbb{C}^2$ as the Hopf bundle.

Proof.

(i) This is trivial when $R \leq 2\pi\delta$ since $n^-(X, V_{0,0,0}) = X \times S^3$. If $R > 2\pi\delta$, we construct a deformation retraction to $\{0\} \times S^3$ by a map $H(\vec{b}, v, s) = (s\,\vec{b}, v)$. The only nontrivial part is to check is that a unit nonpositive cone at $\vec{b}$ contains in a unit nonpositive cone at $s\,\vec{b}$ when $s \in [0,1]$. By (5.3.3), we see that $D_{s\vec{b}} + \delta I = sD_{\vec{b}} + s\delta I$. Thus,

$$\langle D_{s\vec{b}}v, v\rangle \; = s\langle D_{\vec{b}}v, v\rangle \; + (1-s)(-\delta^2) \leq \max\{\langle D_{\vec{b}}v, v\rangle \;, -\delta^2\}.$$

(ii) For the matrix (5.3.3), one can find that an eigenvector corresponding to the eigenvalue $-\delta - R$ is given by $\begin{bmatrix} -b_1 - R \\ -b_3 + b_2 i \end{bmatrix}$ when $(b_1, b_2, b_3) \neq (-R, 0, 0)$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ otherwise. We see that this eigenvector is independent of δ and the norm of $\vec{b}$. This gives a continuous family of maximal negative definite subspaces over X, so that $n^-(X, V_{0,0,0})$ is a circle bundle over S^2. More explicitly, after normalizing $b_1^2 + b_2^2 + b_3^2 = 1$, a fiber over S^2 is an orbit of the vector $\frac{1}{\sqrt{2+2b_1}} \begin{bmatrix} -b_1 - 1 \\ -b_3 + b_2 i \end{bmatrix}$ (when $b_1 \neq -1$ and $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ otherwise) under multiplication by unit complex numbers. This is precisely the description of the Hopf bundle.

□

With the above lemma as a building block, we can now describe an exit set $n^-(X, V)$ and the Conley index of $\nu(X, V)$ in more general case. Another important ingredient is the gauge translational action, i.e. the matrix of $D_{\vec{b}+2\pi\vec{u}}$ on $V_{\vec{n}}$ is the same as the matrix of $D_{\vec{b}}$ on $V_{\vec{n}+\vec{u}}$.

For example, let us consider a square $[-\frac{1}{2}, 2\pi + \frac{1}{2}]^2 \times \{0\}$ and a subspace $V = \bigoplus_{i,j=0,1} V_{-i,-j,0}$. We observe that the situation in Lemma 14 occurs with the center shifted to $(2\pi i, 2\pi j, 0)$ on the subspace $V_{-i,-j,0}$ for each critical point. Near one of these points, the matrix of $D_{\vec{b}}$ on V has signature $(3, 5)$ since it is negative definite on $V_{-i,-j,0}$ and has a signature $(1, 1)$ on other summands. Outside this neighborhood, the matrix of $D_{\vec{b}}$ on V has signature $(4, 4)$. Consequently, the exit set will take the form of a bundle with fiber S^7 except 4 singular points whose fiber is S^9 (See Figure 5-6).

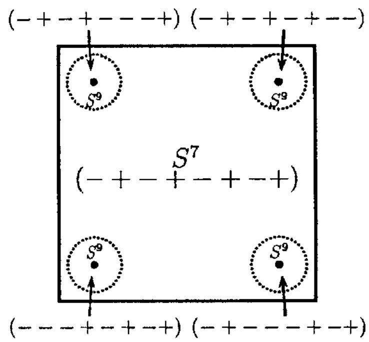


Figure 5-6: The signature of quadratic forms on Ω_h

Denote by Λ_k by the set $\{(n_1, n_2, n_3) \mid n_1, n_2, n_3 \in \{-k, -k+1, \ldots, k\}\}$ corresponding to all $(2k+1)^3$ critical points of f lying inside I_k. With the above argument, we can deduce that

Pick a point $\vec{b}_0$ in I_k away from the points in $\vec{\Lambda} := 2\pi(\Lambda_k \cap \Lambda)$. First, deform I_k to a wedge sum of 3-balls by choosing q copies of 2-spheres, each of which contains exactly one point in $\vec{\Lambda}$ and disjoint from each other except at the basepoint $\vec{b}_0$. This gives a homotopy equivalence between $n^-(I_k, V)$ and a bundle, denoted by E, with singularities over the wedge sum of 3-balls.

Note that the fiber of E at the basepoint $\vec{b}_0$ is the cone $\mathrm{Cone}_{\leq}(V, D_{\vec{b}_0})$ which is homotopy equivalent to the $(2p-1)$-sphere by choosing a negative eigenvector $v^-_{\vec{n},0}$ of $D_{\vec{b}_0}$ restricting on the subspace $V_{\vec{n}}$ for each $\vec{n} \in \Lambda$. Using a bump function near $\vec{b}_0$, we can deform a fiber of E in that neighborhood to S^{2p-1} as well.

Consider a 3-ball $B^3_{\vec{m}}$ in the wedge summand, corresponding to the ball containing $2\pi\vec{m} \in \vec{\Lambda}$. For a point $\vec{b}$ in $B^3_{\vec{m}}$, the operator $D_{\vec{b}}$ restricting on $V_{\vec{n}}$ has no kernel for $\vec{n} \neq \vec{m}$. Since $B^3_{\vec{m}}$ is contractible, we can deform the cone in the part of $\bigoplus_{\vec{n}\neq\vec{m}} V_{\vec{n}}$ to a constant family so that

$$\mathrm{Cone}_{\leq}(\bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}, D_{\vec{b}}) \cong \mathrm{Cone}_{\leq}(V_{\vec{m}}, D_{\vec{b}}) \star S(\bigoplus_{\vec{n}\neq\vec{m}} \mathbb{C}v^-_{\vec{n},0}),$$

from an identity in Lemma 13. By an argument in Lemma 14, we can construct a homotopy equivalence between the family of $\mathrm{Cone}_{\leq}(V_{\vec{m}}, D_{\vec{b}})$ over $B^3_{\vec{m}}$ and the sphere $S(V_{\vec{m}})$. Together, we see that $E|_{B^3_{\vec{m}}}$ is homotopy equivalent to the sphere $S(\mathbb{C}v^+_{\vec{m},0} \oplus \bigoplus_{\vec{n}\in\Lambda} \mathbb{C}v^-_{\vec{n},0})$, where $v^+_{\vec{m},0}$ is a vector linearly independent to $v^-_{\vec{m},0}$ in $V_{\vec{m}}$. We can move the wedge point $\vec{b}_0$ along with its fiber to $\{2\pi\vec{m}\} \times V$ so that the homotopy leaves this subspace fixed.

After applying this procedure to each 3-ball, we can conclude that $n^-(I_k, V)$ is homotopy equivalent to a "S^{2p-1}-sum" of q copies of S^{2p+1}. More explicitly, we have the sphere $S(\mathbb{C}v^+_{\vec{m},0} \oplus \bigoplus_{\vec{n}\in\Lambda} \mathbb{C}v^-_{\vec{n},0})$ for each $\vec{m} \in \Lambda_k \cap \Lambda$ and these spheres intersect each other exactly at $S(\bigoplus_{\vec{n}\in\Lambda} \mathbb{C}v^-_{\vec{n},0})$. Then, we see that the unreduced suspension of a S^{2p-1}-sum of S^{2p+1} is a $S^{\mathbb{C}^p}$-sum of $S^{\mathbb{C}^{p+1}}$.

Proposition 14. *Let $\Lambda \subset \mathbb{Z}^3$ be a subset of the cubic lattice. Suppose that Λ has p elements and $|\Lambda \cap \Lambda_k| = q$. Then, $n^-(I_k, \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}})$ can be described as an S^{2p-1}-bundle over I_k with q singular points whose fiber is S^{2p+1}. Moreover, if Λ' is a subset of the lattice disjoint from Λ and Λ_k, we have*

$$n^-(I_k, \bigoplus_{\vec{n}\in\Lambda\cup\Lambda'} V_{\vec{n}}) \cong n^-(I_k, \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}) \star S(\bigoplus_{\vec{n}\in\Lambda'} V^-_{\vec{n}}).$$

Consequently,

$$\mathbf{S}n^-(I_k, \bigoplus_{\vec{n}\in\Lambda\cup\Lambda'} V_{\vec{n}}) \cong \mathbf{S}n^-(I_k, \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}) \wedge (\bigwedge_{\vec{n}\in\Lambda'} S^{V^-_{\vec{n}}}).$$

Proof. We will only check the second part. From Lemma 13, we have

$$\mathrm{Cone}_{\leq}(\bigoplus_{\vec{n}\in\Lambda\cup\Lambda'} V_{\vec{n}}, D_{\vec{b}}) \cong \mathrm{Cone}_{\leq}(\bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}, D_{\vec{b}}) \star \mathrm{Cone}_{\leq}(\bigoplus_{\vec{n}\in\Lambda'} V_{\vec{n}}, D_{\vec{b}}).$$

Since Λ' is disjoint from Λ_k, we can choose a continuous family of maximal negative definite subspaces $V^-_{\vec{n},\vec{b}}$ depending on $\vec{b} \in I_k$ for each $\vec{n} \in \Lambda'$. Since I_k is contractible, we can deform a family of $S(V^-_{\vec{n},\vec{b}})$ to a constant family, so that $\mathrm{Cone}_{\leq}(\bigoplus_{\vec{n}\in\Lambda'} V_{\vec{n}}, D_{\vec{b}}) \cong$

$S(\bigoplus_{\vec{n}\in\Lambda'} V^-_{\vec{n},\vec{b}_0})$ for some $\vec{b}_0 \in I_k$. Hence,

$$n^-(I_k, \bigoplus_{\vec{n}\in\Lambda\cup\Lambda'} V_{\vec{n}}) \cong n^-(I_k, \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}) \star S(\bigoplus_{\vec{n}\in\Lambda'} V^-_{\vec{n}}).$$

□

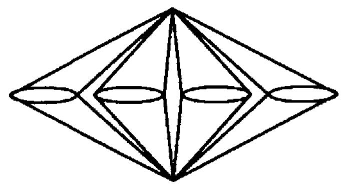


Figure 5-7: An S^0-sum as an unreduced suspension.

Since $\nu(I_k, V)$ is contractible, we also have

$$\mathcal{I}(\nu(I_k, V)) = [\mathbf{S}n^-(I_k, V)].$$

Similar to the $S^1 \times S^2$ case, a map between Conley indices $\nu(I_k, V) \to \nu(I_{k+1}, V)$ is induced by the inclusion $n^-(I_k, V) \subset n^-(I_{k+1}, V)$.

Let us try to compute homology of $\mathbf{S}n^-(I_k, V)$, where $V = \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}$ and denote $\vec{\Lambda} := 2\pi(\Lambda_k \cap \Lambda)$ for simplicity. First, we decompose I_k to a disjoint union of balls $\coprod_{\vec{b}\in\vec{\Lambda}} B(4\pi\delta, \vec{b})$ centered at critical points and a complement $I_k \setminus \coprod_{\vec{b}\in\vec{\Lambda}} B(3\pi\delta, \vec{b})$ so that its intersection is a union of 2-sphere centered at critical points. This gives a decomposition $n^-(I_k, V) = U_1 \cup U_2$ where

$$U_1 = n^-(I_k, V) \cap \coprod_{\vec{b}\in\vec{\Lambda}} B(4\pi\delta, \vec{b}) \times V \;\text{ and }\; U_2 = n^-(I_k, V) \cap (I_k \setminus \coprod_{\vec{b}\in\vec{\Lambda}} B(4\pi\delta, \vec{b})) \times V.$$

With p and q as in Proposition 14, we see that, up to homotopy equivalence, U_1 is a disjoint union of q copies of S^{2p+1} and U_2 is an S^{2p-1}-bundle over a wedge sum $\bigvee^q S^2$. The intersection $U_1 \cap U_2$ is a disjoint union of q copies of an S^{2p-1}-bundle over S^2. Note that the S^1-action on $n^-(I_k, V)$ and the above decomposition is free, so its S^1-homology is equivalent to nonequivariant homology of its quotient.

We apply the Mayer-Vietoris sequence. The quotient of U_1 is a disjoint union

$\coprod^q \mathbb{C}\mathrm{P}^p$. The quotient of U_2 is a $\mathbb{C}\mathrm{P}^{p-1}$-bundle over $\bigvee^q S^2$ whose homology is isomorphic to the product $H_*(\mathbb{C}\mathrm{P}^{p-1}) \otimes H_*(\bigvee^q S^2)$ given by Serre spectral sequence. Similarly, the quotient of $U_1 \cap U_2$ is a disjoint union of a $\mathbb{C}\mathrm{P}^{p-1}$-bundle over S^2, whose homology is also given by $H_*(\mathbb{C}\mathrm{P}^{p-1}) \otimes H_*(S^2)$.

$H_*^{S^1}(\_)$	0	2	...	$2p-2$	2p
$U_1 \cap U_2$	$\mathbb{Z}^q$	$\mathbb{Z}^{2q}$		$\mathbb{Z}^{2q}$	$\mathbb{Z}^q$
U_1	$\mathbb{Z}^q$	$\mathbb{Z}^q$		$\mathbb{Z}^q$	$\mathbb{Z}^q$
U_2	$\mathbb{Z}$	$\mathbb{Z}^{q+1}$		$\mathbb{Z}^{q+1}$	$\mathbb{Z}^q$

The inclusion map on homology is induced by $S^2 \hookrightarrow \bigvee^q S^2$ and $\mathbb{C}\mathrm{P}^{p-1} \hookrightarrow \mathbb{C}\mathrm{P}^p$. The Hopf map induces isomorphism in the top dimension $H_{2p}^{S^1}(U_1 \cap U_2) \to H_{2p}^{S^1}(U_1)$. Consequently, we have the homology of $n^-(I_k, V)$ is given by

$$H_m^{S^1}(n^-(I_k, V)) = \begin{cases} \mathbb{Z}, & m = 0, 2, \ldots, 2p-2 \\ \mathbb{Z}^q, & m = 2p \end{cases}.$$

To compute homology of $\mathrm{S}n^-(I_k, V)$, we recall a cofiber sequence

$$n^-(I_k, V)_+ \to \nu(I_k, V)_+ \to \mathrm{S}n^-(I_k, V).$$

Since $\nu(I_k, V)$ is contractible, we have

$$H_m^{S^1}(\mathrm{S}n^-(I_k, V)) = \begin{cases} \mathbb{Z}^{q-1}, & m = 2p+1 \\ \mathbb{Z}, & m = 2p+2, 2p+4, \ldots \end{cases}.$$

We can reproduce and generalize the computation using cyclic homology theory. This is further elaborated in Section A.1.

From Proposition 14, the Conley index $\mathcal{I}(\nu(I_k, V))$ is given by $\mathrm{S}n^-(I_k, \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}}) \wedge (\bigwedge_{\vec{n}\in\Lambda'} S^{V_{\vec{n}}^-})$. By choosing D_{a_0} as the reference quadratic form, we can describe the part $\mathrm{S}n^-(I_k, \bigoplus_{\vec{n}\in\Lambda} V_{\vec{n}})$ as the smash product of the S^0-sum of $S^{V_{\vec{n}}^+}$ over $\vec{n} \in \Lambda$ with $\bigwedge_{\vec{n}\in\Lambda} S^{V_{\vec{n}}^-}$. This gives a reformulation of $\mathcal{I}(\nu(I_k, V))$ as a suspension of the S^0-sum of $S^{V_{\vec{n}}^+}$ by S^{V^-}.

Similar to the $S^1 \times S^2$ case, we could say that the stable homotopy type of $\nu(I_k)$ is precisely the S^0-sum of $S^{V_{\vec{n}}^+}$ for $\vec{n}$ corresponding to each critical point in I_k.

Since the map induced by the inclusion $I_k \subset I_{k+1}$ is given by inclusion, we can conclude

Theorem 2. *The stable homotopy type of $SWF(T^3, \mathfrak{s})$ is the S^0-sum of $\mathbb{Z}^3$ copies of $S^{\mathbb{C}}$. Each copy corresponds to the positive eigenspace $V_{\vec{n}}^+$ for $\vec{n} \in \mathbb{Z}^3$.*

Chapter 6

Twisted Manolescu-Floer Spectra

6.1 Twisted Parametrized Spectra

The concept of twisted parametrized spectra was introduced by Douglas in [8]. Roughly speaking, a twisted parametrized spectrum is a bundle whose fiber is a spectrum twisted by automorphisms of the category of spectra.

When there is no twisting by automorphisms of the category of spectra, we recover a parametrized spectrum (cf. [26]). One can describe a parametrized spectrum over a space X as a sequence of parametrized spaces E_n over X related by fiberwise suspensions.

Consider a twisted parametrized spectrum over a circle whose monodromy around the circle is given by the suspension Σ. One may describe this locally as a parametrized spectrum over some open set of the circle, but cannot globally describe this as a sequence of ex-spaces over the circle globally because of the shift from monodromy.

The above example illustrates that one could formulate twisted parametrized spectra as sections of a "line bundle" of the category **Sp** of spectra.

Definition 23. Let X be a space. A *haunt* over X is a locally free rank-one module over the structure stack $\mathcal{O}_X$ of parametrized spectra over X. Given a haunt over X, a *twisted parametrized spectrum* is a global section of this haunt. With this setup, a twisted parametrized spectrum always comes with its underlying haunt.

One can think of the category **Sp** as a ring with units given by invertible spectra $Pic(S^0)$, also known as the Picard category of the sphere spectrum. We also know that $Pic(S^0)$ is weakly equivalent to $aut(\mathbf{Sp})$, the simplicial set of self equivalences of the category of spectra. Viewing $Pic(S^0)$ as a structure group, a haunt over X is classified by a homotopy class of maps $[X, BPic(S^0)]$.

We now return to Seiberg-Witten theory. Because the spectrum $SWF(Y,\mathfrak{s})$ is obtained by performing finite dimensional approximation on the Coulomb slice, the harmonic gauge group $\mathcal{G}^h$ has an induced action on $SWF(Y,\mathfrak{s})$. Using this action, we will form a twisted parametrized spectrum over the Picard torus $\mathbb{T}$.

Consider a bounded region $\mathcal{R}$ in the Coulomb slice. We will compare finite dimensional approximation of $u \cdot \mathcal{R}$, for $u \in \mathcal{G}^h$, with that of $\mathcal{R}$. Recall the equation (4.2.7) which relates induced gradient vector fields under gauge action

$$\Pi^*_{B_0}\mathcal{X}(b - u^{-1}du, u\psi) = u\,\Pi^*_{B_0}\mathcal{X}(b,\psi).$$

Let V be a finite-dimensional subspace of the Coulomb slice. The above relation implies equivalence between Conley indices

$$\mathcal{I}(u \cdot (\mathcal{R} \cap V), \Pi^*_{B_0}\mathcal{X}) = u \cdot \mathcal{I}(\mathcal{R} \cap V, \Pi^*_{B_0}\mathcal{X}). \tag{6.1.1}$$

This gives an action of u on Conley indices. If we assume that V contains a subspace of harmonic 1-forms, we see that $u \cdot V = u\,V$ where the right hand side means the partial action of u by multiplication in the spinor part. In particular, $u \cdot (\mathcal{R} \cap V) = (u \cdot \mathcal{R}) \cap (u\,V)$.

We now consider the spectrum the spectrum $E(\mathcal{R})$ by its V-space

$$E(\mathcal{R})(V) = \mathcal{I}(\mathcal{R} \cap V, \Pi^*_{B_0}\mathcal{X}) \wedge S^{V^+_{L_0}}.$$

The action of u also extends to the suspension part, given by $u \cdot S^{V^+_{L_0}} = S^{(uV)^+_{L_0}}$. This is motivated by the fact that the ambient subspace of $u \cdot (\mathcal{R} \cap V) = (u \cdot \mathcal{R})$ has become $u\,V$ instead of V. Note that the subspace $(uV)^+_{L_0}$ is not the same as $u(V)^+_{L_0}$ in general. From an identity $\langle (u^{-1}L_0u)\psi, \psi\rangle = \langle L_0u\psi, u\psi\rangle$, we can identify the positive space $(uV)^+_{L_0}$ with $V^+_{u\cdot L_0}$ where we define $u \cdot L_0 := u^{-1}L_0u$.

In summary, the action of u is given by the map

$$\mathcal{I}(\mathcal{R} \cap V, \Pi^*_{B_0}\mathcal{X}) \wedge S^{V^+_{L_0}} \mapsto \mathcal{I}((u \cdot \mathcal{R}) \cap (u\,V), \Pi^*_{B_0}\mathcal{X}) \wedge S^{V^+_{u\cdot L_0}}, \tag{6.1.2}$$

which we view as a combination of two actions: one on Conley indices and one on the (suspension) spheres. For the first part, we see that the Conley index is shifted by the action of u as in (6.1.1). This induces an automorphism on $SWF(Y, \mathfrak{s})$.

For the second part, we have a map between spheres

$$S^{V^+_{L_0}} \to S^{V^+_{u\cdot L_0}}$$

which comes from a projection $H^+_{L_{B_0}} \to H^+_{u\cdot L_0}$ between semi-infinite positive spaces. It is this part that creates a twist for a twisted parametrized spectrum. Since the difference of L_0 and $u \cdot L_0$ is compact, we know that this projection is Fredholm. We can then thought of the above map between spheres as an image under the J-homomorphism of this Fredholm map. Additionally, the space of Fredholm map has a homotopy type of $\mathbb{Z} \times BU$ so we have

$$\mathbb{Z} \times BU \xrightarrow{J} \mathbb{Z} \times BG \simeq Pic(S^0).$$

By the Bott periodicity theorem, the classifying space of $\mathbb{Z} \times BU$ is the infinite unitary group U. This says that a classifying map for this haunt over the Picard torus comes from the composite

$$\mathbb{T} \to U \to BPic(S^0).$$

We will specifically describe the construction using an open cover in this case when a torus is a base space. Consider an n-torus covered by 2^n open sets obtained from a product of the cover of a circle with two intervals $S^1 = U_0 \cup U_1$. We will denote a product of open sets by concatenating the subscript in a vector form, e.g. the cell $U_{n_1} \times U_{n_2} \times U_{n_3}$ will be denoted by $U_{(n_1,n_2,n_3)}$. We will also use a bold letter to denote a vector with $\mathbf{0} := (0,0,0)$ for the origin and use e_i for the vector with 1 at the i^{th} slot and 0 elsewhere.

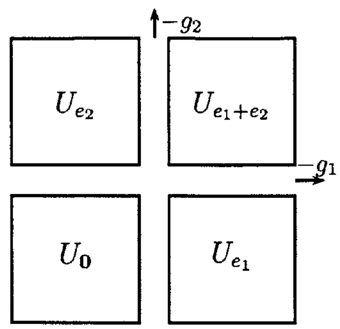


Figure 6-1: An open cover with transition functions for a torus.

For convenience, we retracts each intersection to an appropriate face of $U_{\mathbf{m}}$ viewed as an m-cube.

Definition 24. A twisted Manolescu-Floer spectrum $\widetilde{SWF}(Y,\mathfrak{s})$ is defined as a product parametrized spectrum $U_{\mathbf{m}} \times SWF(Y,\mathfrak{s})$ on each open set $U_{\mathbf{m}}$. The transition function is given as a product of transition functions on $T^{b_1} = (S^1)^{b_1}$ where a transition function g_i on the i^{th} copy of S^1 is given by the action 6.1.2 applied to u_i.

6.2 Homology of Twisted Parametrized Spectra

We now turn to discussion of homology theories for twisted parametrized spectra.

Let E be a parametrized spectrum over X and r be a spectrum. Recall that an r-homology for E can be defined as

$$r_*(E) = \pi_*(E/X \wedge r) = \pi_*((E \wedge_X r)/X),$$

where $\wedge_X$ is the fiberwise smash product and the process of quotient by the basepoint section $/X$ gives back an unparametrized object.

We first point out that the situation for twisted parametrized spectra is different from that of parametrized spectra. This is because a twisted parametrized spectrum does not arise from ex-spaces, and so one cannot collapse the base section to obtain globally defined homotopy type. One approach to define its homology is to first take a generalized homology functor to create a new haunt associated to the homology and then one can take the homotopy groups if the associated haunt is trivial.

Let R be a commutative ring spectrum. Regarding a category of spectra as a category of S^0-modules, we can use R-multiplication to pass this to a category of R-modules. Given a haunt, its associated R-haunt is its tensor stack with a stack of parametrized R-modules. This induces a map $Pic(S^0) \to Pic(R)$ as well as a map $BPic(S^0) \to BPic(R)$ between classifying spaces. See [8] for more details.

Let $\widetilde{E}$ be a twisted parametrized spectrum over X with $\mathcal{H}$ as the underlying haunt. We obtain a twisted parametrized R-module $\widetilde{E} \wedge_X R$ as a global section of the R-haunt $\mathcal{H}_R$. If the haunt $\mathcal{H}_R$ is trivial, then $\widetilde{E} \wedge_X R$ becomes a parametrized R-module whose homotopy groups now defined.

Definition 25. Suppose that the associated R-haunt of a haunt $\mathcal{H}$ over X is trivial with a trivialization τ. The R-homology group of a twisted parametrized spectrum $\widetilde{E}$ of the haunt $\mathcal{H}$ is given by

$$R_*^\tau(\widetilde{E}) = \pi_*(\tau(\widetilde{E} \wedge_X R)/X).$$

This group might depends on the trivialization τ.

One can check that the associated R-haunt is trivial by looking at a composite of the classifying map $X \to BPic(S^0) \to BPic(R)$, which could be null-homotopic even when the map $X \to BPic(S^0)$ is not.

When $c_1(\mathfrak{s})$ is torsion, we know that the classifying map $\mathbb{T} \to U$ factors through the inclusion $SU(2) \hookrightarrow U$ from Lemma 35.1.2 of [18]. Consequently, the classifying map for the R-haunt comes from the composite

$$\mathbb{T} \to SU(2) \hookrightarrow U \to BPic(S^0) \to BPic(R).$$

It turns out that, in many cases, a map $S^3 \simeq SU(2) \to BPic(R)$ is always null-homotopic. This amounts to checking that the homotopy group $\pi_3(BPic(R))$ is trivial. On the other hand, note that $\pi_3(BPic(S^0)) = \mathbb{Z}/2$.

Since $Pic(R) \simeq Pic^0(R) \times BGL_1(R)$, where $Pic^0(R)$ is the equivalence classes of invertible R-modules and $GL_1(R)$ is the self-equivalences of R, we have

$$\begin{aligned}\pi_3(BPic(R)) &= \pi_2(Pic(R)) \\ &= \pi_2(BGL_1(R)) \qquad \text{(because } Pic^0(R) \text{ is discrete)} \\ &= \pi_1(GL_1(R)) = \pi_1(R).\end{aligned}$$

In the last line, the higher homotopy groups of $GL_1(R)$ agree with those of R because $GL_1(R)$ is the unit components of R. Thus, if $\pi_1(R) = R_1(S^0)$ is trivial, then so is the associated R-haunt.

As we have a parametrized spectrum, its homotopy groups can be computed using the generalized Serre spectral sequence.

Proposition 15. *(cf. [26]) Let E be a parametrized spectrum over a CW complex X. Under some technical hypothesis, there is a strongly convergent spectral sequence*

$$E^1_{p,q} \simeq \bigoplus_{p-cells} \pi_q(E_x) \longrightarrow \pi_{p+q}(E).$$

Furthermore, one can identify $E^1_{p,q}$ with $C^{cell}_p(X, \underline{\pi_q(E_x)})$ so that

$$E^2_{p,q} = H_p(X, \underline{\pi_q(E_x)}),$$

where $\underline{\pi_q(E_x)}$ denotes a coefficient system.

The spectral sequence arises from a filtration of the pull-back parametrized spectrum E^p over the p-skeleton of X. The E^1-page comes from a derived couple associated to a long exact sequence of the pair (E^p, E^{p-1}).

The case for nontorsion $c_1(\mathfrak{s})$ is more complicated in many aspects. First, the classifying map $\mathbb{T} \to U$ does not factor through $SU(2)$ anymore. However, the obstruction corresponds to a class $H^1(\mathbb{T})$ coming from spectral flow around a loop. One could hope to develop a theory for $\mathbb{Z}/l$-graded homology groups instead. Second, $SWF(Y, \mathfrak{s})$ is not a spectrum but a pro-spectrum, unless we require a perturbation to be balanced.

We can describe a trivialization explicitly for a haunt over a torus given by the open cover from previous section. This means we have a map from an open set $U_{\mathbf{m}}$ to the structure group compatible with transition functions. We only need to concern about the twist coming from the action $S^{V^+_{L_0}} \to S^{V^+_{u \cdot L_0}}$.

One can describe a process of constructing a global trivialization inductively on open sets similar to a cell by cell approach. Denote $\mathcal{U}_p$ by a set of $U_{\mathbf{m}}$ such that $\mathbf{m}$ contains exactly p 1's. Since we have a local trivialization, we can start by a map sending everything from $U_{\mathbf{0}}$ to the identity. Next, for each $U_{e_i} \in \mathcal{U}_1$, we have that U_{e_i} intersects with $U_{\mathbf{0}}$ at exactly two of the $(n-1)$-faces. The transition functions force that a trivialization on U_{e_i} would send one of this faces to the identity and another face to g_i. Thus, we have a trivialization if we can extend this boundary condition to the whole U_{e_i}. This is the same as finding a path from g_i to the identity. In other words, the first obstruction is whether g_i is homotopic to identity for each $i = 1, \ldots, n$.

For $U_{e_i+e_j} \in \mathcal{U}_2$, its intersection with $U_{\mathbf{0}}$ is given by four $(n-2)$-faces and the transition functions give a boundary condition on these faces. If we have a trivialization on $\mathcal{U}_1$ from the previous step, the trivialization on U_{e_i} and U_{e_j}, or viewed as a homotopy, will force more boundary condition on four $(n-1)$-faces of $U_{e_i+e_j}$. Thus we will have a global trivialization if we can continue to construct a map on $U_{e_i+e_j}$ with this boundary condition. This is equivalent to extending a loop to a disk.

Inductively, we can extend a trivialization on $\mathcal{U}_p$ to $\mathcal{U}_{p+1}$ if we can extend a map on ∂D^{p+1} to D^{p+1} for each open set in $\mathcal{U}_{p+1}$ which can be viewed as a $(p+1)$-cell. Indeed, if one has a trivialization, the map on ∂D^{p+1} can be viewed as the attaching map of the cell.

We consider the problem of extending a trivialization from ∂D^{p+1} to D^{p+1}. Recall that the twisting comes from a projection $H^+_{L_{B_0}} \to H^+_{u \cdot L_0}$ which can be regarded as an

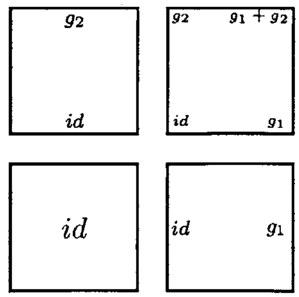


Figure 6-2: A trivialization for a torus.

element of the restricted general linear group. We can find an extension if the index bundle of the family over ∂D^{p+1} is trivial. This index bundle induces the Thom bundle and we can find an extension on a spectrum level if this Thom bundle is equivalent to a trivial bundle over ∂D^{p+1}. When passing to R-module, the condition for an extension becomes that this Thom bundle has an R-trivialization.

In summary, to be able to take R-homology, one only needs to find an R-orientation for the Thom bundle associated to each cell of the torus.

Proposition 16. *When $c_1(\mathfrak{s})$ is torsion, the twisted Manolescu-Floer spectrum $\widetilde{SWF}(Y,\mathfrak{s})$ is orientable for the S^1-equivariant Borel homology $H_*^{S^1}$, the c-homology $\widehat{H}_*^{S^1}$, and the Tate homology $\bar{H}_*^{S^1}$.*

6.3 The Manifolds $S^1 \times S^2$ and T^3 Revisited

As a continuation of Chapter 5, we will investigate the twisted spectra $\widetilde{SWF}(S^1 \times S^2)$ and $\widetilde{SWF}(T^3)$ and their homology groups in this section.

The homology theories of interest are the S^1-equivariant Borel homology $H_*^{S^1}$, coBorel homology (or c-homology) $\widehat{H}_*^{S^1}$, and Tate homology $\bar{H}_*^{S^1}$. See the Appendix for more background.

Consider the manifold $(S^1 \times S^2, \mathfrak{s})$ with the torsion spin^c structure. Since the action on Conley indices is trivial and the family of Dirac operator has no kernel, we can conclude that

Proposition 17. *The twisted spectrum $\widetilde{SWF}(S^1 \times S^2, \mathfrak{s})$ is equivalent to the trivial* *parametrized spectrum* $S^1 \times SWF(S^1 \times S^2, \mathfrak{s})$.

Corollary 4. *The homology groups of* $\widetilde{SWF}(S^1 \times S^2, \mathfrak{s})$ *is given by*

$$H_*^{S^1}(\widetilde{SWF}(S^1 \times S^2, \mathfrak{s})) = H_*(S^1) \otimes H_*^{S^1}(S^0)$$

and analogous statements hold for the $\widehat{H}_*^{S^1}$ *and* $\bar{H}_*^{S^1}$ *theories.*

For the torus T^3 with the torsion spin^c structure, the action on Conley indices is given by a shift in $\mathbb{Z}^3$ axis. We can choose a Dirac operator away from a critical of index 0 so that $\widetilde{SWF}(S^1 \times S^2, \mathfrak{s})$ is trivial on the 2-skeleton of T^3. This shows that the E^1-page of the Serre spectral sequence agrees with the one from twisted cellular homology for local coefficient.

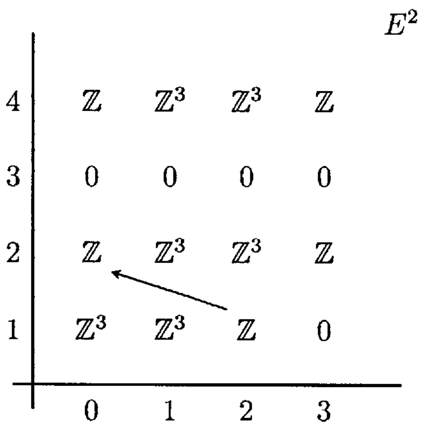
The detailed computation is given in Section A.2. We have that the E^2-page of $\widehat{H}_*^{S^1}(\widetilde{SWF}(T^3, \mathfrak{s}))$ is given by

	0	1	2	3
1	$\mathbb{Z}$	0	0	0
0	0	0	0	0
-1	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
-2	0	0	0	0
-3	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$

The E^2-page of $\bar{H}_*^{S^1}(\widetilde{SWF}(T^3, \mathfrak{s}))$ is given by

	0	1	2	3
2	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
1	0	0	0	0
0	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$
-1	0	0	0	0
-2	$\mathbb{Z}$	$\mathbb{Z}^3$	$\mathbb{Z}^3$	$\mathbb{Z}$

We see that the differential d^2 vanishes trivially for these two theories. This is not the case for the E^2-page of $H_*^{S^1}(\widetilde{SWF}(T^3, \mathfrak{s}))$



Conjecture 1. *The differential $d^2 : E^2_{2,1} \to E^2_{0,2}$ is an isomorphism. Consequently, we have an E^3-term*

Apart from the d^2 of $H_*^{S^1}(\widetilde{SWF}(T^3, \mathfrak{s}))$, the only higher differential left is d^3 on the E^3-page of each of the theories.

Conjecture 2. *The nontrivial differential d^3 is an isomorphism for each of the theories.*

Chapter 7

4-Manifolds with Boundaries

7.1 Preliminaries

Let X be a compact, connected, oriented, Riemannian 4-manifold with nonempty boundary $\partial X = Y$. We choose a metric so that a neighborhood of the boundary is isometric to the cylinder $I \times Y$ for some interval $I = (-C, 0]$. Let $\mathfrak{s}_X$ be a spin^c structure on X and $\mathfrak{s}$ be the induced spin^c structure on Y. Denote $S_X = S^+ \oplus S^-$ by the spinor bundle of X and S by the spinor bundle of Y.

We pick a perturbation $\mathfrak{q}$ on the boundary 3-manifold Y. This induces a perturbation on the cylinder, but not for a general 4-manifold. We will need a perturbation $\hat{\mathfrak{p}}$ on X supported in the collar neighborhood so that the restriction on $\{0\} \times Y$ is $\mathfrak{q}$. In addition, we assume that $\hat{\mathfrak{p}}$ is of the form

$$\hat{\mathfrak{p}} = \beta \mathfrak{q} + \beta_0 \mathfrak{p}_0 \tag{7.1.1}$$

in the collar neighborhood, where β is a cut-off function with value 1 near the boundary, β_0 is a bump function supported in $(-C, 0)$, and $\mathfrak{p}_0$ is a perturbation on Y.

As in the 3-dimensional case, it is important to impose some gauge fixing conditions to obtain a slice of the quotient configuration space. For a 1-form $a \in \Omega^1(X)$, we say that a satisfies the Coulomb condition if a is coclosed i.e. $d^*a = 0$.

However, the Coulomb condition is not sufficient to deduce Fredholm property for the boundary-value problems. One also need to impose a condition on the restriction of a on the boundary y. One of the standard boundary conditions is the Neumann boundary condition given by

$$\langle a, \vec{n} \rangle = 0 \text{ at } \partial X, \tag{7.1.2}$$

where $\vec{n}$ is the normal vector to the boundary.

In this paper, we will consider another boundary condition, which is can be viewed as the Coulomb condition on the boundary. We say that a 1-form a satisfies the boundary Coulomb condition if

$$d_3^*(a_{|\partial X}) = 0 \text{ at } \partial X, \tag{7.1.3}$$

where we write d_3^* here to emphasize that we restrict a on ∂X then take the 3-dimensional d^*.

One might view the condition $d_3^*(a_{|\partial X}) = 0$ as complementary to the Neumann condition. To see this, we use hodge theory to decompose the 1-form a in the collar neighborhood of ∂X as

$$a = \alpha_t + \beta_t + \gamma_t dt, \tag{7.1.4}$$

where α_t is an exact 1-form on Y, β_t is a coclosed 1-form on Y , γ_t is a 0-form on Y, and each of them is time dependent.

The Neumann condition simply means $\gamma_0 = 0$, whereas the boundary Coulomb condition means $\alpha_0 = 0$.

We now define the Coulomb slice for a 4-manifold X. Using above terminology, we denote a space of configuration whose 1-form part satisfies the Coulomb condition and the boundary Coulomb condition by

$$Coul^{CC}(X) = \left\{ (a, \phi) \in i\Omega^1(X) \oplus \Gamma(S^+) \mid d^*a = 0 \text{ and } d_3^*(a_{|\partial X}) = 0 \right\}, \tag{7.1.5}$$

and we also denote a space of configuration whose 1-form part satisfies the Coulomb-Neumann condition by

$$Coul^{CN}(X) = \left\{(a,\phi) \in i\Omega^1(X) \oplus \Gamma(S^+) \mid d^*a = 0 \text{ and } \langle a, \vec{n}\rangle = 0\right\}. \tag{7.1.6}$$

The advantage of using $Coul^{CC}(X)$ is that the restriction of its element is already in the 3-dimensional Coulomb slice $Coul(Y)$. On the other hand, for an element in $Coul^{CN}(X)$, its restriction is not necessarily in $Coul(Y)$ and one need to compose the restriction with the (nonlinear) Coulomb projection when applying finite dimensional approximation.

7.2 Atiyah-Patodi-Singer boundary-value problem

We will prove basic properties of the boundary-value problem coming from linearization of the Seiberg-Witten maps between Coulomb slices.

For a reference connection A_0, we have a linear map

$$D : Coul^{CC}(X) \to i\Omega^2_+(X) \oplus \Gamma(S^-) \oplus i\Omega^0(X) \tag{7.2.1}$$

$$(a,\phi) \mapsto (d^+a, D^+_{A_0}\phi, d^*a). \tag{7.2.2}$$

We study a map of the form

$$D \oplus (\Pi^- \circ r) : Coul^{CC}(X) \to i\Omega^2_+(X) \oplus \Gamma(S^-) \oplus i\Omega^0(X) \oplus Coul(Y), \tag{7.2.3}$$

where r denotes the restriction map and Π^- is an appropriate projection on $Coul(Y)$ for Atiyah-Patodi-Singer boundary condition. We will show that this map, extended to Sobolev completion, is Fredholm with a priori estimate.

Recall that, first, the restriction map extends to a continuous map between Sobolev spaces

$$r : L^2_s(X) \to L^2_{s-1/2}(Y).$$

We also recall that, on the 3-dimensional Coulomb slice $Coul(Y) = iKer(d^*) \oplus \Gamma(S)$, there is an operator

$$L_0 : iKer(d^*) \oplus \Gamma(S) \to iKer(d^*) \oplus \Gamma(S) \tag{7.2.4}$$

$$(b, \psi) \mapsto (*db, D_{B_0}\psi), \tag{7.2.5}$$

Denote H_0^- by its nonpositive eigenspace and Π_0^- by the projection onto H_0^-. The projection Π^- is chosen to be a projection commensurate to Π_0^-.

We also use the following facts. When Π is commensurate to Π_0, we have Index $Ind(D \oplus \Pi) = Ind(D \oplus \Pi_0) + Ind(\Pi\Pi_0)$

When an operator D of the form

$$D = D_1 \oplus D_2 : V \to W_1 \oplus W_2, \tag{7.2.6}$$

it is not hard to check that

$$\mathrm{Ker}(D) = \mathrm{Ker}(D_1|_{\mathrm{Ker}(D_2)}) = \mathrm{Ker}(D_2|_{\mathrm{Ker}(D_1)}), \text{ and} \tag{7.2.7}$$

$$\mathrm{Coker}(D) = \mathrm{Coker}(D_2) \oplus \mathrm{Coker}(D_1|_{\mathrm{Ker}(D_2)}) \tag{7.2.8}$$

$$= \mathrm{Coker}(D_1) \oplus \mathrm{Coker}(D_2|_{\mathrm{Ker}(D_1)}) \tag{7.2.9}$$

Proposition 18. *The map $D \oplus (\Pi^- \circ r)$ in (7.2.3) is Fredholm and we have an estimate*

$$\|x\| \le C\left(\|Dx\| + \|(\Pi \circ r)x\| + \|x\|_{L^2}\right) \tag{7.2.10}$$

Proof. The main idea is to apply the Atiyah-Patodi-Singer boundary-value problem (cf. [1]) and comparing two different semi-infinite subspaces as in [18]. One subspace arises from a spectral boundary condition while another comes from a semi-infinite subspace of the Coulomb slice of Y.

We consider an elliptic operator $\widetilde{D}$ coming from a linear part of the Seiberg-Witten map combined with the Coulomb gauge fixing

$$\widetilde{D} : i\Omega^1(X) \oplus \Gamma(S^+) \to i\Omega^2_+(X) \oplus \Gamma(S^-) \oplus i\Omega^0(X) \tag{7.2.11}$$

$$(a, \phi) \mapsto (d^+a, D^+_{A_0}\phi, d^*a). \tag{7.2.12}$$

One can write $\widetilde{D} = D_0 + K$, where K extends to an operator of order 0 and D_0 has the form

$$D_0 = \frac{d}{dt} + \widetilde{L}, \tag{7.2.13}$$

in the collar neighborhood (up to isomorphisms). The operator $\widetilde{L}$ is a first-order, self-adjoint elliptic operator given by

$$\widetilde{L} : i\Omega^1(Y) \oplus \Gamma(S) \oplus i\Omega^0(Y) \to i\Omega^1(Y) \oplus \Gamma(S) \oplus i\Omega^0(Y) \tag{7.2.14}$$

$$(b, \psi, c) \mapsto (*db - dc, D_{B_0}\psi, -d^*b), \tag{7.2.15}$$

which is a linear part of the 3-dimensional Seiberg-Witten map with the Coulomb gauge fixing. Using the Hodge decomposition, we can write $\widetilde{L}$ restricted to $i\Omega^1(Y) \oplus i\Omega^0(Y) = iIm(d) \oplus iKer(d^*) \oplus i\Omega^0(Y)$ as a block

$$\begin{bmatrix} 0 & 0 & -d \\ 0 & *d & 0 \\ -d^* & 0 & 0 \end{bmatrix}. \tag{7.2.16}$$

From the above decomposition, one can also view the domain of $\widetilde{L}$ as $Coul(Y) \oplus iIm(d) \oplus i\Omega^0(Y)$, so that $\widetilde{L} = L_0 \oplus L_1$ where L_1 has a block form

$$\begin{bmatrix} 0 & -d \\ -d^* & 0 \end{bmatrix}. \tag{7.2.17}$$

Now we apply the Atiyah-Patodi-Singer boundary-value problem. The operator

$$\widetilde{D} \oplus (\widetilde{\Pi}^- \circ r) : i\Omega^1(X) \oplus \Gamma(S^+) \to i\Omega^2_+(X) \oplus \Gamma(S^-) \oplus i\Omega^0(X) \oplus \widetilde{H}^- \qquad (7.2.18)$$

is Fredholm, where $\widetilde{H}^- \subset Coul(Y) \oplus iIm(d) \oplus i\Omega^0(Y)$ is the nonpositive eigenspace of $\widetilde{L}$ and $\widetilde{\Pi}^-$ is its spectral projection.

On the space $iIm(d) \oplus i\Omega^0(Y)$, let H_1^- is the nonpositive eigenspace of L_1 and Π_1^- be its spectral projection and denote Π_2 be a projection onto $iIm(d) \oplus \{const\}$. We see that $\widetilde{\Pi}^- = \Pi_0^- \oplus \Pi_1^-$ and it is commensurate to a projection $\Pi^- \oplus \Pi_1^-$.

Observe that dd^* is positive, self-adjoint on $iIm(d)$. Then, for each $b \in iIm(d)$, the pair $(b, d^*(dd^*)^{-1/2}b)$ lies in H_1^-. Moreover, $(0, c)$ lies in H_1^- when c is a constant function. Hence, H_1^- is complementary to $\{0\} \oplus \{const\}^{\perp}$.

Consequently, the kernel of $\Pi^- \oplus \Pi_2$ is complementary to the image of $\Pi^- \oplus \Pi_1^-$. By Proposition 17.2.6 of [18], the operator $\widetilde{D} \oplus ((\Pi^- \oplus \Pi_2) \circ r)$ is Fredholm.

Finally, we compare $\widetilde{D} \oplus ((\Pi^- \oplus \Pi_2) \circ r)$ with $\widetilde{D} \oplus ((\Pi^- \oplus \widetilde{\Pi}_2) \circ r)$ where $\widetilde{\Pi}_2$ is a projection onto $iIm(d) \oplus \{0\}$. By setting, $D_2 = d^* \oplus (\widetilde{\Pi}_2 \circ r)$, we see that $\mathrm{Ker}(D_2) = Coul^{CC}(X)$ and the cokernel of D_2 is the cokernel of d^* which has dimension $b_0(X)$. Thus, the map $D \oplus (\Pi^- \circ r)$ is Fredholm with index

$$\mathrm{Ind}(D \oplus (\Pi^- \circ r)) = \mathrm{Ind}(\widetilde{D} \oplus (\widetilde{\Pi}^- \circ r)) + \mathrm{Ind}(\Pi^- \Pi_0^-) + b_0(X) + b_0(Y) \qquad (7.2.19)$$

The estimate is also a consequence of the Atiyah-Patodi-Singer theorem combined with commensurate projections.

□

7.3 Finite Dimensional Approximation

We will apply finite dimensional approximation to the Seiberg-Witten map together with a boundary condition as in [22]. Consider the map

$$\mathfrak{F}_{\mathfrak{p}} \oplus (\Pi^- \circ r) : Coul^{CC}(X) \to i\Omega^2_+(X) \oplus \Gamma(S^-) \oplus Coul(Y), \qquad (7.3.1)$$

where $\mathfrak{F}_\mathfrak{p}$ is the Seiberg-Witten map with a decomposition $\mathfrak{F}_\mathfrak{p} = D + Q$. For convenience, denote $\mathcal{V}_X$ by $i\Omega^2_+(X) \oplus \Gamma(S^-)$.

We outline the construction which is based on [22]. First we pick a sufficiently large radius R. The image of the ball $B(R) \subset Coul^{CC}(X)$ under the restriction map is bounded. We can pick a bounded isolating neighborhood $\mathcal{R}$ containing this image. For each positive integer n, we consider a projection Π^-_n on $Coul(Y)$ commensurate to Π^-_0.

Since $D \oplus (\Pi^-_n \circ r)$ is Fredholm, we can pick a finite-dimensional subspace $V_n \oplus W_n$ of $\mathcal{V}_X \oplus Coul(Y)$ that contains the cokernel this map. We can also choose W_n so that $\mathcal{R} \cap W_n$ is an isolating neighborhood of the compressed flow on W_n. Let U_n be the preimage of $V_n \oplus W_n$ under $D \oplus (\Pi^-_n \circ r)$.

Consider a map $\mathfrak{F}_\mathfrak{p} \oplus (\Pi^-_n \circ r)$ on the ball

$$B(R, U_n) \to B(R', V_n) \times \mathcal{R}.$$

We will try to show that this gives rise to a quotient map

$$B(R, U_n)/S(R, U_n) \to B(\epsilon_n, V_n)/B(\epsilon_n, V_n)^C \wedge N/L,$$

where (N, L) is an index pair of $\mathcal{R} \cap W_n$ by applying Lemma 5. The map in the first factor sends everything outside the ball of radius ϵ_n to the basepoint. An important part is to check the hypothesis of Lemma 5 for existence of such (N, L). Consequently, this gives a map

$$S^{U_n} \to S^{V_n} \wedge \mathcal{I}(\mathcal{R} \cap W_n) \tag{7.3.2}$$

We show that this construction works when V_n, W_n are sufficiently large and ϵ_n is sufficiently small.

Lemma 15. *Let $\{x_n\}$ be a bounded sequence in L^2_{k+1} such that $(\hat{D} + \hat{\pi}_n \hat{Q})x_n \to 0$ in L^2_k. Suppose that there are half-trajectories $y_n : [0, \infty) \to W_n$ uniformly bounded in*

$L^2_{k+1/2}$ *and satisfy*

$$-\frac{\partial}{\partial t}y_n(t) = \pi_n F y_n(t), \tag{7.3.3}$$

together with $y_n(0) = (\pi_n \circ r)x_n$. *Then, after passing to a subsequence, the sequence* $\{x_n\}$ *converges to* x *in* L^2_{k+1} *and there exists a Seiberg-Witten half-trajectory* y *with* $y(0) = r(x)$.

Proof. First we consider the weak limit $x_n \rightharpoonup x$ in L^2_{k+1}. Then we have $x_n \to x$ in L^2_k by Rellich lemma. Since a linear map preserves weak limits and $\hat{Q}$ is continuous in L^2_k, we have $(\hat{D}+\hat{Q})x_n \rightharpoonup (\hat{D}+\hat{Q})x$ weakly in L^2_k .

On the other hand, we see that

$$\left\|(\hat{D}+\hat{Q})x_n\right\| \leq \left\|(\hat{D}+\hat{\pi}_n\hat{Q})x_n\right\| + \left\|(1-\hat{\pi}_n)\hat{Q}x_n\right\|. \tag{7.3.4}$$

The first term goes to 0 by hypothesis while the second term also goes to 0 because $(1-\hat{\pi}_n)$ converges to 0 uniformly on compact subset. Thus, $(\hat{D}+\hat{Q})x$ must equal to 0. Moreover

$$\|D(x_n - x)\| \leq \left\|(\hat{D}+\hat{Q})x_n\right\| + \left\|\hat{Q}x - \hat{Q}x_n\right\| \to 0. \tag{7.3.5}$$

Next, we move on to the restriction of x_n to 3-dimensional configuration. Similar to the proof of Proposition 11, there is a half-trajectory $y : [0,\infty) \to \mathcal{K}$ such that $y_n(t) \to y(t)$ in $L^2_{k+1/2}$ uniformly on compact subsets of the open half-line $(0,\infty)$ but only in $L^2_{k-1/2}$ on compact subsets of the closed half-line $[0,\infty)$. We also have

$$-\frac{\partial}{\partial t}y(t) = Fy(t). \tag{7.3.6}$$

Consider the exponential that $\frac{\partial}{\partial t}\left(e^{tD}\pi^-\right) = e^{tD}\pi^-(D+\frac{\partial}{\partial t})$

$$e^D\pi^-\gamma(1) - \pi^-\gamma(0) = \int_0^1 e^{tD}\pi^-\left(D\gamma(t) + \frac{\partial}{\partial t}\gamma(t)\right)dt \tag{7.3.7}$$

When $\gamma = y - y_n$, we see that

$$(D + \frac{\partial}{\partial t})(y_n - y) = (D\pi_n - \pi_n D)y_n + \pi_n(Qy - Qy_n) + (1 - \pi_n)Qy. \tag{7.3.8}$$

Since $L\pi_n - \pi_n L \to 0$ as a bounded operator on $L^2_{k+1/2}$ and y_n uniformly bounded, so that $(D\pi_n - \pi_n D)y_n(t) \to 0$ in $L^2_{k+1/2}$ uniformly on $[0, \infty)$.

For the other terms, we use the fact that $e^{tD}\pi^-$ and Q is a bounded map on $L^2_{k+1/2}$, so that

$$\left\|e^{tD}\pi^- \left(\pi_n(Qy(t) - Qy_n(t)) + (1 - \pi_n)Qy(t)\right)\right\|_{L^2_{k+1/2}} \leq R',$$

on $[0, 1]$ for some constant R'.

We fix $\delta > 0$. By continuity of Q, we have that $Qy_n(t) \to Qy(t)$ in $L^2_{k+1/2}$ uniformly on $[\delta, 1]$. We also have that y is smooth on $[\delta, 1]$ so that $\|y(t)\|_{L^2_{k+3/2}}$ is bounded on this interval. By compactness of Q, we get that $(1 - \pi_n)Qy(t) \to 0$ in $L^2_{k+1/2}$ uniformly on $[\delta, 1]$ as well.

Hence, for δ sufficiently small and n sufficiently large,

$$\int_0^1 \left\|e^{tD}\pi^- \left(\pi_n(Qy(t) - Qy_n(t)) + (1 - \pi_n)Qy(t)\right)\right\|_{L^2_{k+1/2}} dt \to 0 \tag{7.3.9}$$

We conclude that in $L^2_{k+1/2}$ topology

$$\begin{aligned} \left\|\pi^-(y(0) - y_n(0))\right\| &\leq \left\|e^D\pi^-(y(1) - y_n(1)) - \pi^-(y(0) - y_n(0))\right\| + \left\|e^D\pi^-(y(1) - y_n(1))\right\| \\ &\leq \int_0^1 \left\|e^{tD}\pi^-(D + \frac{\partial}{\partial t})(y_n(t) - y(t))\right\| dt + \left\|e^D\pi^-(y(1) - y_n(1))\right\|, \end{aligned}$$

and the last line goes to 0.

Since r is linear and by, we have that $r(x_n)$ converges weakly to $r(x)$ in $L^2_{k+1/2}$. In particular, $\pi^- y_n(0) = \pi^-\pi_n r(x_n)$ converges weakly to $\pi^- r(x)$ in $L^2_{k+1/2}$. Thus we must have $\pi^- r(x) = \pi^- y(0)$. The elliptic estimate implies that x_n converges to x in L^2_{k+1}, and we also have $r(x) = y(0)$.

□

We will need boundedness result for Seiberg-Witten half-trajectories result analogous to Corollary 3 for the case of 3-manifolds.

Proposition 19. *([18], Proposition 24.6.4) For $C > 0$, the space of (broken) X-trajectories with energy $< C$ is compact.*

We now state the main result.

Proposition 20. *For V_n, W_n sufficiently large and ϵ_n sufficiently small, the map $\mathfrak{F}_\mathfrak{p} \oplus (\Pi_n^- \circ r)$ satisfies the hypothesis of Lemma 5.*

Proof. (Sketch) We will prove by contradiction. Suppose there is a sequence of V_n, W_n, and ϵ_n not giving a pre-index pair. This gives a sequence $\{x_n\}$ of approximate solutions of the 4-dimensional Seiberg-Witten equation and a sequence $\{y_n\}$ of approximate Seiberg-Witten half-trajectories such that $r(x_n) = y_n(0)$. By Lemma 15, the sequence $\{x_n\}$ converges to a solution x and $\{y_n\}$ converges to a half-trajectory y with $r(x) = y(0)$. Together, we have an X-trajectory with finite energy. The contradiction arises from compactness and boundedness property from Proposition 19. □

It is not hard to see that such maps commute with maps between Conley indices of attractor-repeller pair. As a result, we have a map

$$\mathbf{S} \to SWF(Y).$$

Appendix A

Homology Computation

A.1 Equivariant Homology and Cyclic Homology

In this section, we will provide a background for equivariant homology theory. Although, we will mainly focus on the S^1-equivariant Borel homology and Tate homology, we will use a general framework set up by Greenlees and May in their book [13]. We will also introduce cyclic homology theories which is equivalent to S^1-equivariant homology theories but can be computed more explicitly.

Classically, the Borel homology of a G-space X is defined to be the homology of its homotopy quotient, i.e.

$$H_*^G(X) = H_*(X \times_G EG).$$

In general, given a G-spectrum k_G, we define the following spectra

$$f(k_G) = k_G \wedge EG_+, \quad c(k_G) = F(EG_+, k_G), \quad t(k_G) = F(EG_+, k_G) \wedge \widetilde{E}G,$$

where $F(EG_+, k_G)$ is the function spectrum and $\widetilde{E}G$ is the unreduced suspension of EG with one of the conepoints as the basepoint. These spectra give homology and cohomology theories for a G-spectrum (or a G-space) X by

$$f(k_G)_\alpha(X) = [S^\alpha, X \wedge f(k_G)]_G \quad \text{and} \quad f(k_G)^\alpha = [X, S^\alpha \wedge f(k_G)]_G.$$

The spectrum $f(k_G)$ is called the free G-spectrum associated to k_G and it follows that $f(k_G)$ represents a version of Borel homology. The spectrum $t(k_G)$ is called the Tate G-spectrum associated to k_G as it represents Tate homology and cohomology theories. The spectrum $c(k_G)$ is called the geometric completion of k_G.

One of of the properties of these three spectra associated to k_G is that they form a cofibration sequence

$$f(k_G) \to c(k_G) \to t(k_G)$$

and consequently give rise to a long exact sequence of homology groups

$$\ldots \to f(k_G)_n(X) \to c(k_G)_n(X) \to t(k_G)_n(X) \to f(k_G)_{n-1}(X) \to \ldots.$$

From now on, we will be mainly concerned with the case $G = S^1$ and $k_G = H\mathbb{Z}$ the Eilenberg-MacLane spectrum regarded as an S^1-spectrum with trivial action. The associated homology and cohomology theories in this case can be computed using cyclic homology and cohomology theory introduced by Jones in [15].

Definition 26. Let $P := \mathbb{Z}[u, u^{-1}]$ where $\deg(u) = 2$ and C be a chain complex with a degree one operator J such that $dJ + Jd = 0$ and $J^2 = 0$. Define a differential on $P \otimes C$ by the formula

$$d(up \otimes x) = p \otimes J(x) + up \otimes d(x),$$

where $p \in \mathbb{Z}[u, u^{-1}]$. This also gives a differential on a quotient complex $P^+ \otimes C$ and a subcomplex $P^- \otimes C$ where P^- is the negative degree part of P and $P^+ := P/P^- = \mathbb{Z}[u]$.

Let X be a pointed CW-complex with a cellular action by S^1. The circle action induces a degree one operator J on the reduced cellular chain complex $C_*(X)$ given by $J(x) = f_*(c \otimes x)$ where c is the single 1-cell of S^1 and f is the action $S^1 \times X \to X$. One can check that $dJ + Jd = 0$ and $J^2 = 0$, so we have a complex $P \otimes C_*(X)$ as in the above definition. It turns out that the homology of this complexes agree with appropriate S^1-equivariant homology theories.

Proposition 21. *([13], Theorem 14.2) There are isomorphisms*

$$H_n^{S^1}(X) := H_n(P^+ \otimes C_*(X)) \cong f(H\mathbb{Z})_{n+1}(X),$$
$$\widehat{H}_n^{S^1}(X) := H_{n-1}(P^- \otimes C_*(X)) \cong c(H\mathbb{Z})_{n+1}(X),$$
$$\bar{H}_n^{S^1}(X) := H_n(P \otimes C_*(X)) \cong t(H\mathbb{Z})_{n+2}(X).$$

The homology $H_*^{S^1}, \widehat{H}_*^{S^1}, \bar{H}_*^{S^1}$ *will be called Borel homology, c-homology, and Tate homology respectively. The c-homology is sometimes referred as coBorel homology.*

We first remark that we slightly change the notation and the grading from [13] to align with Floer homology groups. Our $\bar{H}_n^{S^1}$ is their $\widehat{H}_{n+1}^{S^1}$ and our $\widehat{H}_n^{S^1}$ is their $\check{H}_n^{S^1}$, so the long exact sequence has a form

$$\ldots \to \bar{H}_n^{S^1}(X) \to H_n(X) \to \widehat{H}_n^{S^1}(X) \to \bar{H}_{n-1}^{S^1}(X) \to \ldots.$$

Second, we can define these homology groups with coefficient in an arbitrary abelian group. The above isomorphisms still hold, but we need to replace $H\mathbb{Z}$ with a spectrum corresponding to an appropriate Mackey functor.

For example, the homology groups for S^0 are given by

	...	-4	-3	-2	-1	0	1	2	3	4	...
$\bar{H}_*^{S^1}(S^0)$	...	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
$H_*^{S^1}(S^0)$	...	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...
$\widehat{H}_*^{S^1}(S^0)$	...	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	0	0	0	0	...

Now, we consider the one-point compactification $S^{\mathbb{C}}$ of the complex plane with a circle action given by complex multiplication. We can give $S^{\mathbb{C}}$ a CW-structure with a cellular action by S^1. First, give $S^{\mathbb{C}}$ a cell decomposition with 2 cells in each dimension just as the 2-sphere. Denote $x^\pm, y^\pm, z^\pm$ by its 0-cells, 1-cells, and 2-cells respectively. The standard differential of the cellular chain complex is given by

$$dy^+ = x^+ - x^-, \quad dz^+ = y^+ + y^-,$$
$$dy^- = x^- - x^+, \quad dz^- = -y^+ - y^-.$$

The circle action by complex multiplication acts nontrivially on the 1-cells. Viewing x^- as $\{\infty\}$ or the north pole and x^+ as $\{0\}$ or the south pole, we see that y^+ travels from $\{\infty\}$ to $\{0\}$ along the negative imaginary axis and y^- travels from $\{0\}$ to $\{\infty\}$ along the positive imaginary axis. The cells z^+ and z^- are right and left hemispheres given an outward normal orientation. As in Figure A-1, the S^1-action is the counterclockwise rotation around the z-axis. We see that the image of y^+ under rotation gives a sphere with outward normal, while the image of y^- gives a sphere with inward normal. Thus, the action of J is given by

$$J(y^+) = z^+ + z^-, \quad J(y^-) = -z^+ - z^-.$$

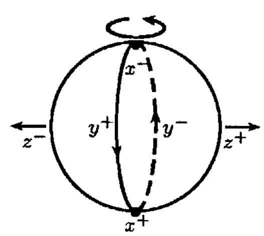


Figure A-1: A CW-structure on $S^{\mathbb{C}}$ with a cellular action by S^1.

Let X be the S^0-sum of N copies of the spheres $S^{\mathbb{C}}$ as described in Section 5.3. The above CW-structure of $S^{\mathbb{C}}$ gives a cell decomposition of X by given by two 0-cells $\{x^+, x^-\}$, $2N$ 1-cells $\{y_j^+, y_j^-\}$, and $2N$ 2-cells $\{z_j^+, z_j^-\}$ for $j = 1, \ldots, N$ with differential

$$dy_j^+ = x^+ - x^-, \quad dz_j^+ = y_j^+ + y_j^-,$$
$$dy_j^- = x^- - x^+, \quad dz_j^- = -y_j^+ - y_j^-$$

and the J-action $J(y_j^\pm) = \pm(z_j^+ + z_j^-)$. The first few terms of the chain complex $P^+ \otimes C_*(X)$ is given by

$$\mathbb{Z} \xleftarrow{\epsilon} C_0(X) \leftarrow C_1(X) \oplus u\mathbb{Z} \leftarrow C_2(X) \oplus uC_0(X) \leftarrow uC_1(X) \oplus u^2\mathbb{Z} \leftarrow \ldots,$$

where $\mathbb{Z}$ is the augmented group for a reduced cellular chain complex. The u-term does not affect the differential except when pairing with a chain from $C_1(X)$, e.g.

$$d(uy_i^+) = z_i^+ + z_i^- + u(x^+ - x^-).$$

With this, one can find the homology groups $H_*(P^+ \otimes C_*(X))$

	1	2	3	4	5	6	7	...
$H_*(P^+ \otimes C_*(X))$	$\mathbb{Z}^{N-1}$	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...

The group $H_1(P^+ \otimes C_*(X))$ consists of classes $[\sum c_j y_j^+]$ such that $\sum c_j = 0$ and, for each positive even integer k, the group $H_k(P^+ \otimes C_*(X))$ is generated by a class $[u^{k/2}(x^+ - x^-)]$.

For the complex $P^- \otimes C_*(X)$, the first few terms of the chain complex is given by

$$\ldots \leftarrow \frac{1}{u^3}C_2(X) \oplus \frac{1}{u^2}C_0(X) \leftarrow \frac{1}{u^2}C_1(X) \oplus \frac{1}{u}\mathbb{Z} \leftarrow \frac{1}{u^2}C_2(X) \oplus \frac{1}{u}C_0(X) \leftarrow \frac{1}{u}C_1(X) \leftarrow \frac{1}{u}C_2(X).$$

Similarly, we can find that its homology groups are given by

	0	-1	-2	-3	-4	-5	-6	...
$H_*(P^- \otimes C_*(X))$	$\mathbb{Z}^N$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	...

The group $H_0(P^- \otimes C_*(X))$ consists of classes $[\sum c_j u^{-1}(z_j^+ + z_j^-)]$ and, for each negative even integer k, the group $H_k(P^- \otimes C_*(X))$ is generated by a class $[u^{k/2}(x^+ - x^-)]$. Finally, we have the chain complex of $P \otimes C_*(X)$ is given by

$$\ldots \leftarrow u^n C_1(X) \oplus u^{n+1}\mathbb{Z} \leftarrow u^n C_2(X) \oplus u^{n+1} C_0(X) \leftarrow \ldots,$$

with its homology groups

	...	-3	-2	-1	0	1	2	3	...
$H_*(P \otimes C_*(X))$	...	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	...

We observe that $\bar{H}_*^{S^1}(X) = \bar{H}_*^{S^1}(S^0)$. This is because the Tate homology only detects the singular part under the circle action and the action on X is free except at two fixed points.

A.2 Twisted Cellular Homology

In this section, we introduce a twisted cellular homology theory to compute homology with local coefficient. Some treatments can be found in [7].

We will specialize to the homology of the three-torus with a local system of equivariant homology of another space. In our case, we can start at a chain level by setting up a twisted chain complex. Let X be a space and suppose that X has a CW structure with a cellular $\mathbb{Z}^3$-action. On the cellular chain level, this means that there is a $\mathbb{Z}^3$-action on each chain group $C_j(X)$ and the action intertwines with the differential. With this we will define a chain complex which is a product of $C_*(T^3)$ and $C_*(X)$ but twisted by the $\mathbb{Z}^3$-action. This chain complex can be viewed as coming from a twisted CW-structure of the homotopy quotient $X \times_{\mathbb{Z}^3} E\mathbb{Z}^3$ of X.

Since the total space $E\mathbb{Z}^3$ is $\mathbb{R}^3$, we can consider a standard lattice CW-structure on $\mathbb{R}^3$ with a free cellular $\mathbb{Z}^3$-action. We first consider a cell structure of the real line whose 0-cells and 1-cells are given by e^0_n and e^1_n for each $n \in \mathbb{Z}$. The differential is given by

$$d(e^1_n) = e^0_{n+1} - e^0_n.$$

This gives a product cell structure on $\mathbb{R}^3$ and we will denote a product of cells by concatenating a superscript and putting a subscript in a vector form, e.g. the cell $e^0_{n_1} \times e^1_{n_2} \times e^0_{n_3}$ will be denoted by $e^{010}_{(n_1,n_2,n_3)}$. We will also use a bold letter to denote a vector and, in particular, use bold letters $\mathbf{i}, \mathbf{j}, \mathbf{k}$ for the standard basis

$$\mathbf{i} := (1,0,0), \quad \mathbf{j} := (0,1,0), \quad \mathbf{k} := (0,0,1)$$

and $\mathbf{0} := (0,0,0)$ for the origin. With this notation, the boundary of the cube is given by a formula

$$de^{111}_{\mathbf{0}} = e^{011}_{\mathbf{i}} - e^{011}_{\mathbf{0}} - e^{101}_{\mathbf{j}} + e^{101}_{\mathbf{0}} + e^{110}_{\mathbf{k}} - e^{110}_{\mathbf{0}}.$$

A differential for a twisted complex is induced by the differential on $C_*(\mathbb{R}^3) \otimes C_*(X)$

quotient by the diagonal action. For example, we have

$$\begin{aligned} d(e_{\mathbf{0}}^{110} \otimes x) &= (e_{\mathbf{i}}^{010} - e_{\mathbf{0}}^{010} - e_{\mathbf{j}}^{100} + e_{\mathbf{0}}^{100}) \otimes x - e_{\mathbf{0}}^{110} \otimes dx \\ &= e_{\mathbf{0}}^{010} \otimes \mathbf{i}x - e_{\mathbf{0}}^{010} \otimes x - e_{\mathbf{0}}^{100} \otimes \mathbf{j}x + e_{\mathbf{0}}^{100} \otimes x - e_{\mathbf{0}}^{110} \otimes dx. \end{aligned} \tag{A.2.1}$$

This allows us to compute a homology with local coefficient system.

Suppose further that X has a circle action which induces a degree one operator J on $C_*(X)$ as in the previous section. If the circle action and the operator J is compatible with the $\mathbb{Z}^3$-action, we can consider the product of complexes $C_*(\mathbb{R}^3)$ and $P \otimes C_*(X)$ twisted by a $\mathbb{Z}^3$-action (as well as P^+ and P^-).

Let X be a S^0-sum of $\mathbb{Z}^3$ copies of $S^{\mathbb{C}}$ with a CW-structure described in the previous section. We can index the 1-cells and 2-cells $\{y_{\mathbf{n}}^+, y_{\mathbf{n}}^-, z_{\mathbf{n}}^+, z_{\mathbf{n}}^-\}$ by an element of $\mathbb{Z}^3$, so that the action of $\mathbf{m} \in \mathbb{Z}^3$ is translation of a subscript $\mathbf{n} \mapsto \mathbf{m} + \mathbf{n}$ and the action is trivial on 0-cells.

We will use a spectral sequence of a filtered complex to compute the homology of this twisted cellular homology. Similar to the case of a double complex, one of the canonical filtration is given by

$$F_p C_{p+q} = \bigoplus_{\substack{r+s=p+q \\ r \leq p}} C_r(\mathbb{R}^3) \otimes_{\mathbb{Z}^3} C_s(P \otimes C_*(X)).$$

By shifting a subscript of each term from $C_r(\mathbb{R}^3)$ to $\mathbf{0}$ as in (A.2.1), we see that the E^1-page is obtained by taking homology in vertical direction. We can identify the E^1-term as

$$E^1_{p,q} = C_p(T^3) \otimes H_q(P \otimes C_*(X)).$$

Remark. This spectral sequence only agrees with the Serre spectral sequence of $\widetilde{SWF}(T^3, \mathfrak{s})$ only on the E^1-page and the groups on E^2-page. The higher differentials are not necessarily the same. This is because the differential here only accounts for the $\mathbb{Z}^3$-action on X but not the action on the category of spectra. However, one could still hope to modify the differential to make the chain complex equivalent to the Morse-Floer complex for Floer homology.

We will start with the local coefficient system in $H_*(P^- \otimes C_*(X))$. From the result in Section A.1, the E^1-page is given by

	0	1	2	3
0	$C_0(T^3)\otimes(\bigoplus^{\mathbb{Z}^3}\mathbb{Z})$	$C_1(T^3)\otimes(\bigoplus^{\mathbb{Z}^3}\mathbb{Z})$	$C_2(T^3)\otimes(\bigoplus^{\mathbb{Z}^3}\mathbb{Z})$	$C_3(T^3)\otimes(\bigoplus^{\mathbb{Z}^3}\mathbb{Z})$
-1	0	0	0	0
-2	$C_0(T^3)\otimes\mathbb{Z}$	$C_1(T^3)\otimes\mathbb{Z}$	$C_2(T^3)\otimes\mathbb{Z}$	$C_3(T^3)\otimes\mathbb{Z}$
-3	0	0	0	0
-4	$C_0(T^3)\otimes\mathbb{Z}$	$C_1(T^3)\otimes\mathbb{Z}$	$C_2(T^3)\otimes\mathbb{Z}$	$C_3(T^3)\otimes\mathbb{Z}$

We will proceed by computing the differential d^1. For a negative even integer q, a generator of $H_q(P^+ \otimes C_*(X))$ is represented by a cycle $u^{q/2}(x^+ - x^-)$. Since this is invariant under the $\mathbb{Z}^3$-action, the horizontal differential $d^1 : E^1_{p,2q} \to E^1_{p-1,2q}$ is just the differential of $C_*(T^3)$ which is zero. Thus, for such q, we have

$$E^2_{p,q} = H_p(T^3).$$

On the other hand, recall that the group $H_0(P^- \otimes C_*(X))$ has a basis given by $\{[u^{-1}(z^+_{\mathbf{n}} + z^-_{\mathbf{n}})]\}_{\mathbf{n}\in\mathbb{Z}^3}$, each of which is not invariant under the $\mathbb{Z}^3$-action. As a result, the differential d^1 for $E^1_{p,0}$ is not as trivial as in other groups. To simplify the computation, we will slightly change the viewpoint by shifting the subscript of the term $(z^+_{\mathbf{n}} + z^-_{\mathbf{n}})$ to $\mathbf{0}$ instead. In other word, we rewrite an element of $E^1_{p,0}$ as

$$e_{\mathbf{0}} \otimes (\sum_{\mathbf{n}\in\mathbb{Z}^3} c_{\mathbf{n}}[u^{-1}(z^+_{\mathbf{n}} + z^-_{\mathbf{n}})]) = (\sum_{\mathbf{n}\in\mathbb{Z}^3} c_{\mathbf{n}} e_{\mathbf{n}}) \otimes [u^{-1}(z^+_{\mathbf{0}} + z^-_{\mathbf{0}})].$$

We can then identify $E^1_{p,0}$ with the (lattice) cellular chain group $C_p(\mathbb{R}^3)$ and the differential d^1 can be identified with the standard differential on $C_*(\mathbb{R}^3)$. Therefore, we have

$$E^2_{p,0} = \begin{cases} \mathbb{Z}, & p = 0 \\ 0, & p \neq 0. \end{cases}$$

Because there is a shift in the grading, the homology of T^3 with local coefficient in the c-homology of X, denoted by $H_p(T^3, \underline{\widehat{H}}^{S^1}_q(X))$, is the group $E^2_{p,q-1}$ of above spectral sequence.

Next, we look at the spectral sequence of local system in $H_*(P^+ \otimes C_*(X))$ whose E^1-page is given by

4	$C_0(T^3) \otimes \mathbb{Z}$	$C_1(T^3) \otimes \mathbb{Z}$	$C_2(T^3) \otimes \mathbb{Z}$	$C_3(T^3) \otimes \mathbb{Z}$
3	0	0	0	0
2	$C_0(T^3) \otimes \mathbb{Z}$	$C_1(T^3) \otimes \mathbb{Z}$	$C_2(T^3) \otimes \mathbb{Z}$	$C_3(T^3) \otimes \mathbb{Z}$
1	$C_0(T^3) \otimes (\bigoplus^{\mathbb{Z}^3-1} \mathbb{Z})$	$C_1(T^3) \otimes (\bigoplus^{\mathbb{Z}^3-1} \mathbb{Z})$	$C_2(T^3) \otimes (\bigoplus^{\mathbb{Z}^3-1} \mathbb{Z})$	$C_3(T^3) \otimes (\bigoplus^{\mathbb{Z}^3-1} \mathbb{Z})$
	0	1	2	3

Analogously, the differential d^1 for $E^1_{p,q}$ is zero when $q \neq 1$. We recall that the group $H_1(P^+ \otimes C_*(X))$ consists of classes $[\sum_{\mathbf{n} \in \mathbb{Z}^3} c_{\mathbf{n}} y^+_{\mathbf{n}}]$ with $\sum_{\mathbf{n} \in \mathbb{Z}^3} c_{\mathbf{n}} = 0$ with finitely many nonzero terms. We also shift $y^+_{\mathbf{n}}$ to $y^+_{\mathbf{0}}$ similar to the P^- case, but we will identify $E^1_{p,1}$ with a subgroup of $C_p(\mathbb{R}^3)$ instead.

We define a *balanced* chain group $C^{bal}_p(\mathbb{R}^3)$ as a kernel of the generalized augmented map $\epsilon : C_p(\mathbb{R}^3) \to C_p(T^3)$ which adds up all the coefficient of the term in the same $\mathbb{Z}^3$-orbit. It is straightforward to see that a boundary of a balanced chain is also balanced, and $C^{bal}_*(\mathbb{R}^3)$ becomes a subcomplex of $C_*(\mathbb{R}^3)$. In fact, a boundary of any chain is balanced because it holds for each of a generator. We now identify a complex $(E^1_{*,1}, d^1)$ with the balanced chain complex $(C^{bal}_*(\mathbb{R}^3), d)$.

Observe that a p-cycle of $C_*(\mathbb{R}^3)$ is always balanced for $p > 0$. Because we know that $H_p(\mathbb{R}^3) = 0$, any p-cycle is a boundary and is therefore balanced from the previous paragraph. Consequently, if a chain is a boundary of an unbalanced chain, it cannot be a boundary of a balanced chain.

We claim that $H_0(C_*^{bal}(\mathbb{R}^3))$ has a basis $\{[e_{\mathbf{i}}^{000} - e_{\mathbf{0}}^{000}], [e_{\mathbf{j}}^{000} - e_{\mathbf{0}}^{000}], [e_{\mathbf{k}}^{000} - e_{\mathbf{0}}^{000}]\}$, where we can view each element as $de_{\mathbf{0}}^{100}, de_{\mathbf{0}}^{010}$, and $de_{\mathbf{0}}^{001}$ respectively. A group of balanced 0-cycles is generated by $\{e_{\mathbf{n}}^{000} - e_{\mathbf{0}}^{000}\}_{\mathbf{n}\neq\mathbf{0}}$ and we can choose a path from $\mathbf{n}$ to $\mathbf{0}$ as a 1-chain. By adding an appropriate linear combination of $e_{\mathbf{0}}^{100}, e_{\mathbf{0}}^{010}$, and $e_{\mathbf{0}}^{001}$, we can make this 1-chain balanced. This shows that $e_{\mathbf{n}}^{000} - e_{\mathbf{0}}^{000}$ is homologous to a linear combination of $de_{\mathbf{0}}^{100}, de_{\mathbf{0}}^{010}$, and $de_{\mathbf{0}}^{001}$. On the other hand, a linear combination of $e_{\mathbf{0}}^{100}, e_{\mathbf{0}}^{010}$, and $e_{\mathbf{0}}^{001}$ is not balanced unless it is zero, so $\{[de_{\mathbf{0}}^{100}], [de_{\mathbf{0}}^{010}], [de_{\mathbf{0}}^{001}]\}$ is linearly independent.

Similarly, one can check that $H_p(C_*^{bal}(\mathbb{R}^3))$ is generated by a boundary of generators of $C_{p+1}(T^3)$. In summary, we have

$$E_{p,1}^2 = \begin{cases} \mathbb{Z}^3, & p = 0,1 \\ \mathbb{Z}, & p = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Lastly, the spectral sequence of local system in $H_*(P \otimes C_*(X))$ is simpler than the others because a generator of the homology group $H_*(P \otimes C_*(X))$ is of the form $[u^k(x^+ - x^-)]$ which is invariant under $\mathbb{Z}^3$-action. The E^2-page is given by

$$E_{p,q}^2 = H_p(T^3) \otimes H_q(P \otimes C_*(X)).$$

A.3 $RO(G)$-graded Equivariant Homology

Besides an integer grading, an equivariant homology theory also comes with $RO(G)$-grading. This can be seen from the definition $k_\alpha(X) = [S^\alpha, X \wedge k]_G$ where we can let α be any virtual representation of G.

Let X be a G-space, k_G be a G-spectrum, and $f(k_G) := k_G \wedge EG_+$. In our case, $G = S^1$ and k_G is the Eilenberg-Maclane spectrum so that $f(k_G)$ represents the Borel homology. The action of S^1 on X is either trivial or free (induced by multiplication by unit complex numbers). We will try to compute $f(k_G)^G_{\mathbb{C}+n}(X) = [S^{\mathbb{C}+n}, X \wedge f(k_G)]_G$ by relating it to integer-graded groups and nonequivariant homology groups.

The sphere $S^{\mathbb{C}+n}$ can be viewed as the unreduced suspension of a unit sphere $S(\mathbb{C}\oplus\mathbb{R}^n) \cong S(\mathbb{C})\star S(\mathbb{R}^n)$. Then the orbit space $S^{\mathbb{C}+n}/G$ is the unreduced suspension of $\{pt\} \star S(\mathbb{R}^n) \cong D(\mathbb{R}^n)$ while the fixed point space is the unreduced suspension of $S(\mathbb{R}^n)$. We deduce that, for spaces,

$$[S^{\mathbb{C}+n}, X]_G = [(D^{n+1}, S^n), (X, X^G)]. \tag{A.3.1}$$

We claim that this extends to stable category, i.e.

$$[S^{\mathbb{C}+n}, X \wedge f(k_G)]_G = [(D^{n+1}, S^n), (X \wedge f(k_G), (X \wedge f(k_G))^G)] \tag{A.3.2}$$

$$= \pi^{st}_{n+1}(X \wedge f(k_G), (X \wedge f(k_G))^G). \tag{A.3.3}$$

Next, we use the long exact sequence of relative homotopy groups

$$\to \ \pi^{st}_{n+1}(B) \to \pi^{st}_{n+1}(A) \to \pi^{st}_{n+1}(A, B) \to \pi^{st}_n(B) \to \pi^{st}_n(A) \to .$$

Since the action on S^n is free, we have $\pi^{st}_n((X \wedge f(k_G))^G) = f(k_G)^G_n(X)$ and $\pi^{st}_n(X \wedge f(k_G)) = \pi^{st}_n(X \wedge k_G) = k_n(X)$ because EG is contractible. Therefore, we have a long exact sequence

$$\to f(k_G)^G_{n+1}(X) \to k_{n+1}(X) \to f(k_G)^G_{\mathbb{C}+n}(X) \to f(k_G)^G_n(X) \to k_n(X) \to .$$

After plugging in Borel homology, we have

$$\to H^{S^1}_n(X) \to H_{n+1}(X) \to f(k_G)^G_{\mathbb{C}+n}(X) \to H^{S^1}_{n-1}(X) \to H_n(X) \to .$$

Similar argument applies to the c-homology and Tate homology.

Bibliography

[1] M. F. Atiyah, V. K. Patodi, and I. M. Singer. Spectral asymmetry and Riemannian geometry. I. *Math. Proc. Cambridge Philos. Soc.*, 77:43–69, 1975.

[2] S. Bauer and M. Furuta. A stable cohomotopy refinement of Seiberg-Witten invariants. I. *Invent. Math.*, 155(1):1–19, 2004.

[3] Stefan Bauer. A stable cohomotopy refinement of Seiberg-Witten invariants. II. *Invent. Math.*, 155(1):21–40, 2004.

[4] R. L. Cohen, J. D. S. Jones, and G. B. Segal. Floer's infinite-dimensional Morse theory and homotopy theory. In *The Floer memorial volume*, volume 133 of *Progr. Math.*, pages 297–325. Birkhäuser, Basel, 1995.

[5] C. Conley. *Isolated invariant sets and the Morse index*, volume 38 of *CBMS Regional Conference Series in Mathematics.* American Mathematical Society, Providence, R.I., 1978.

[6] C. Conley and R. Easton. Isolated invariant sets and isolating blocks. *Trans. Amer. Math. Soc.*, 158:35–61, 1971.

[7] J. F. Davis and P. Kirk. *Lecture notes in algebraic topology*, volume 35. American Mathematical Society Providence, 2001.

[8] C. L. Douglas. Twisted parametrized stable homotopy theory, 2005. arXiv:math/0508070.

[9] A. Floer. An instanton-invariant for 3-manifolds. *Comm. Math. Phys.*, 118(2):215–240, 1988.

[10] Andreas Floer. A refinement of the Conley index and an application to the stability of hyperbolic invariant sets. *Ergodic Theory Dynam. Systems*, 7(1):93–103, 1987.

[11] M. Furuta. Monopole equation and the $\frac{11}{8}$-conjecture. *Math. Res. Lett.*, 8(3):279–291, 2001.

[12] K. Gęba, M. Izydorek, and A. Pruszko. The Conley index in Hilbert spaces and its applications. *Studia Math.*, 134(3):217–233, 1999.

[13] J. P. C. Greenlees and J. P. May. Generalized Tate cohomology. *Mem. Amer. Math. Soc.*, 113(543):viii+178, 1995.

[14] M. Hutchings. Embedded contact homology and its applications, 2010. arXiv:1003.3209.

[15] J. D. S. Jones. Cyclic homology and equivariant homology. *Inventiones Mathematicae*, 87(2):403–423, 1987.

[16] T. Kaczyński and M. Mrozek. Connected simple systems and the Conley functor. *Topol. Methods Nonlinear Anal.*, 10(1):183–193, 1997.

[17] P. Kronheimer and C. Manolescu. Periodic floer pro-spectra from the seiberg-witten equations, 2002. arXiv:math/0203243.

[18] P. Kronheimer and T. Mrowka. *Monopoles and three-manifolds*, volume 10 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2007.

[19] H. L. Kurland. On the two definitions of the Conley index. *Proc. Amer. Math. Soc.*, 106(4):1117–1130, 1989.

[20] L. G. Lewis, Jr., J. P. May, M. Steinberger, and J. E. McClure. *Equivariant stable homotopy theory*, volume 1213 of *Lecture Notes in Mathematics*. Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure.

[21] R. Lipshitz and S. Sarkar. A Khovanov homotopy type, 2011. arXiv:1112.3932.

[22] C. Manolescu. Seiberg-Witten-Floer stable homotopy type of three-manifolds with $b_1 = 0$. *Geom. Topol.*, 7:889–932 (electronic), 2003.

[23] C. Manolescu. *A spectrum valued TQFT from the Seiberg-Witten equations*. ProQuest LLC, Ann Arbor, MI, 2004. Thesis (Ph.D.)–Harvard University.

[24] C. Manolescu. A gluing theorem for the relative Bauer-Furuta invariants. *J. Differential Geom.*, 76(1):117–153, 2007.

[25] J. P. May. *Equivariant homotopy and cohomology theory*, volume 91 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1996. With contributions by M. Cole, G. Comezaña, S. Costenoble, A. D. Elmendorf, J. P. C. Greenlees, L. G. Lewis, Jr., R. J. Piacenza, G. Triantafillou, and S. Waner.

[26] J. P. May and J. Sigurdsson. *Parametrized homotopy theory*, volume 132 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2006.

[27] P. Ozsváth and Z. Szabó. Holomorphic disks and topological invariants for closed three-manifolds. *Annals of mathematics*, pages 1027–1158, 2004.

[28] P. H. Rabinowitz. *Minimax methods in critical point theory with applications to differential equations*, volume 65 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 1986.

[29] Joel W. Robbin and Dietmar Salamon. Dynamical systems, shape theory and the Conley index. *Ergodic Theory Dynam. Systems*, 8*(Charles Conley Memorial Issue):375–393, 1988.

[30] Krzysztof P. Rybakowski. On the homotopy index for infinite-dimensional semiflows. *Trans. Amer. Math. Soc.*, 269(2):351–382, 1982.

[31] Dietmar Salamon. Connected simple systems and the Conley index of isolated invariant sets. *Trans. Amer. Math. Soc.*, 291(1):1–41, 1985.

[32] C. Seed. Computations of the Lipshitz-Sarkar Steenrod square on Khovanov homology, 2012. arXiv:1210.1882.

[33] A. S. Švarc. On the homotopic topology of Banach spaces. *Dokl. Akad. Nauk SSSR*, 154:61–63, 1964.

[34] C. Troestler and M. Willem. Nontrivial solution of a semilinear Schrödinger equation. *Comm. Partial Differential Equations*, 21(9-10):1431–1449, 1996.

[35] J. Várilly. Dirac operators and spectral geometry, 2006. `http://toknotes.mimuw.edu.pl/sem3/files/Varilly_dosg.pdf`.