Effectiveness and Design of Sparse Process Flexibilities

by

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Abstract

The long chain has been an important concept in the design of flexible processes. This design concept, as well as other sparse flexibility structures, have been applied by the automotive and other industries as a way to increase flexibility in order to better match available capacities with variable demands. Numerous empirical studies have validated the effectiveness of these structures. However, there is little theory that explains the effectiveness of the long chain, except when the system size is large, i.e., by applying an asymptotic analysis.

Our attempt in this thesis is to develop a theory that explains the effectiveness of long chain and other sparse flexibility structures for finite size systems. We study the sales of sparse flexibility structures under both stochastic and worst-case demands. From our analysis, we not only provide rigorous explanation to the effectiveness of the long chain, but also refine guidelines in designing other sparse flexibility structures.

Under stochastic demand, we first develop two deterministic properties, supermodularity and decomposition of the long chain, that serve as important building blocks in our analysis. Applying the supermodularity property, we show that the marginal benefit, i.e., the increase in expected sales, increases as the long chain is constructed, and the largest benefit is always achieved when the chain is closed by adding the last arc to the system. Then, applying the decomposition property, we develop four important results for the long chain under IID demands: (i) an effective algorithm to compute the performance of long chain using only matrix multiplications; (ii) a proof on the optimality of the long chain among all 2-flexibility structures; (iii) a result that the gap between the fill rate of full flexibility and that of the long chain increases with system size, thus implying that the effectiveness of the long chain relative to full flexibility increases as the number of products decreases; (iv) a risk-pooling result implying that the fill rate of a long chain increases with the number of products, but this increase converges to zero exponentially fast.

Under worst-case demand, we propose the plant cover index, an index defined by a constrained bipartite vertex cover problem associated with a given flexibility structure. We show that the plant cover index allows for a comparison between the worst-case performances of two flexibility structures based only on their structures and is independent of the choice of the uncertainty set or the choice of the performance measure. More precisely, we show that if all of the plant cover indices of one structure are greater than or equal to the plant cover indices of the other structure, then the first structure is more robust than the second one, i.e. performs better in worst-case under any symmetric uncertainty set and a large class of performance measures. Applying this relation, we demonstrate the effectiveness of the long chain in worst-case performances, and derive a general heuristic that generates sparse flexibility structures which are tested to be effective under both stochastic and worst-case demands.

Finally, to understand the effect of process flexibility in reducing logistics cost, we study a model where the manufacturer is required to satisfy deterministic product demand at different distribution centers. Under this model, we prove that if the cost of satisfying product demands at distribution centers is independent of production plants or distribution centers, then there always exists a long chain that is optimal among 2-flexibility structures. Moreover, when all plants and distribution centers are located on a line, we provide a characterization for the optimal long chain that minimizes the total transportation cost. The characterization gives rise to a heuristic that finds effective sparse flexibility structures when plants and distribution centers are located on a 2-dimensional plane.

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[1]The map of USA is downloaded from http://www.miyufurniture.com/

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Chapter 1

Introduction

1.1 Objective

For many manufacturing firms, the ability to match demand and supply is key to their success. Failure to do so could lead to loss of revenue, reduced service levels, impact on reputation, and decline in the companys market share. Unfortunately, recent developments such as intense market competition, product proliferation and reduction in product life cycles have created an environment where customer demand is volatile and unpredictable. In such an environment, traditional operations strategies such as building inventory, investing in capacity buffers, or increasing committed response time to consumers do not offer manufacturers a competitive advantage. Therefore, many manufacturers have started to adopt an operations strategy known as process flexibility to better respond to market changes without significantly increasing cost, inventory, or response time (see [Simchi-Levi, 2010]).

Process flexibility is defined as the ability to "build different types of products in the same manufacturing plant or on the same production line at the same time" ([Jordan and Graves, 1995]). For example, in *full process flexibility structure*, each plant is capable of producing all products. In this case, when the demand for one product is higher than expected while the demand for a different product is lower than expected, a flexible manufacturing system can quickly make adjustments by shifting production capacities appropriately. By contrast, in a *dedicated structure* (sometimes called "no flexibility"), each plant is responsible for a single product and hence does not have the same ability to match supply with demand.

Because of its effectiveness in responding to uncertainties, process flexibility has gained significant attention various industries, from the automotive to the consumer packaged goods industry. Indeed, the plants for most of the automobile giants are much more flexible (in terms of process flexibility) today compared to twenty years ago ([Boudette, 2006]). Evidently, it is often too expensive to achieve a high degree of flexibility, for example full flexibility, and as a result, sparse or partial flexibility, is implemented instead.

Of course, there are many ways to implement sparse designs and the challenge is to identify an effective one. An important sparse flexibility structure analyzed in the literature and applied in practice by various companies is the *long chain* structure. The first to observe the power of the long chain were Jordan and Graves (see [Jordan and Graves, 1995]) who, through empirical analysis, showed that the long chain can provide almost as much benefit as full flexibility. In particular, [Jordan and Graves, 1995] found that for randomly generated demand, the expected amount of demand that can be satisfied by the long chain is very close to that of a full flexibility structure.

Unfortunately, with a few exceptions, there is very little theory to explain why long chain works so well. One objective of this thesis is to provide an answer to this question. We will develop results to rigorously explain the effectiveness of the long chain, and from those results, derive new insights and guidelines for designing sparse flexibility structures. First, we present a review of the existing literature for process flexibility structures.

1.2 Literature Review

The study of *process flexibility*, also known as "mix flexibility" or "short-term flexibility," began in the 1980s. Prior to the 1990s, research typically focused on the benefits, challenges and trade-offs between full flexibility and no flexibility (dedicated) systems (see the survey of [Sethi and Sethi, 1990]). Unfortunately, most companies are not interested in full flexibility because of its enormous implementation cost.

The seminal paper of [Jordan and Graves, 1995] is the first to consider the design and effectiveness of sparse process flexibility. Applying numerical analysis (simulation) to a stochastic demand model, Jordan and Graves demonstrate two important insights in process flexibility. First, they show that a sparse flexibility structure, known as the long chain, can provide almost as much benefit as full flexibility. Second, based on a planning model, they demonstrate that the concept of the long chain can be generalized to produce sparse flexibility structures that perform extremely well with realistic assumptions on demand uncertainties,.

Following the work of Jordan and Graves, researchers have attempted to explain analytically the observed effectiveness of the long chain and sparse flexibility structures. [Aksin and Karaesmen, 2007] shows that there is a decrease in marginal benefit associated with the increase in either the degree of flexibility or the capacities of the manufacturing plants. [Chou et al., 2010c] develops a method to compute the average demand satisfied by the long chain in asymptotic regime. Using this method, they show that for a certain class of demand distributions, the average sales associated with the long chain is very close to that of full flexibility when the system size goes to infinity. [Chou et al., 2010b] studies the effectiveness of sparse flexibility structures under the condition that secondary production is more expensive, and proves that long chain accrues at least 29.29 % of the benefit of full flexibility, when demand is normally distributed. Like [Jordan and Graves, 1995], all of these papers study flexibility structures under stochastic demand and focus on flexibility structures' average-case performances.

For worst analysis analysis, [Chou et al., 2011] uses graph expander to show that there exists sparse flexibility structures that can be arbitrarily close to the performance of full flexibility. In particular, [Chou et al., 2011] proves that when the system has n homogenous products, n homogenous plants and demand for each product is bounded by λ times the capacity of each plant, then an $(\alpha, \lambda, \Delta)$-expander performs within $(1 - \alpha\lambda)$-optimality of the full flexibility structure for any demand instance. [Chou et al., 2011] also generalizes the result to unbalanced systems, where the number of products is not equal to the number of plants.

Other than theoretical analysis, motivated from the chaining strategy in [Jordan and Graves, 1995], researchers have proposed various heuristics for generating effective sparse flexibility structures in the literature. [Mak and Shen, 2009] proposes a heuristic to find effective flexibility structures based on a relaxed stochastic programming problem. [Chou et al., 2010c] presents a constraint sampling method to find effective sparse flexibility structures while [Chou et al., 2011] presents an expansion heuristic that adds arc incident to the nodes with the lowest expansion ratio. In addition, [Deng and Shen, 2013] presents guidelines for creating flexibility structures under unbalanced networks, where the numbers of plants does not equal to the number of products.

Another line of research in process flexibility is to propose easy to compute measurements to rank the effectiveness of different flexibility structures. In the literature, these measurements are known as flexibility indices. The seminal paper of Jordan and Graves proposes an index that takes into account of both the structure of the flexibility network and the demand distribution. It is defined as the probability that the sales of a given structure is lower than that of full flexibility. Indices proposed by other researchers are only based on the flexibility structure. For example, [Iravani et al., 2005] introduces the structural flexibility matrix, where its (i, j)th entry is equal to the number of paths from the ith plant node to the jth product node in the flexibility structure. Using the structure flexibility matrix, the authors propose to use the mean and the largest eigenvalue of the matrix as flexibility indices. Finally, [Chou et al., 2008] proposes the expansion index, which is equal to the second smallest eigenvalue of the Laplacian of the given flexibility structure.

There has also been extensive research that has applied the concept of the long chain and limited flexibility in a variety of applications. For instance, flexibility structures was studied empirically by [Graves and Tomlin, 2003] in multistage supply chains; [Sheikhzadeh et al., 1998] and [Gurumurthi and Benjaafar, 2004] in queueing systems; [Hopp et al., 2004] in serial production lines, [Iravani et al., 2005] in queueing networks and [Wallace and Whitt, 2005] in call centers. In particular, [Iravani et al., 2005] suggests that the strength of a specific flexibility structure relative to other structures is mostly independent of the system specifications.

Moreover, the design and benefit of process flexibility has been studied under different objective and resource constraints. [Bassamboo et al., 2010] and [Bassamboo et al., 2012] study the optimal level of investment in (partially) flexible resources, when the cost of any flexible resource is linear. [Chou et al., 2013] studies the benefit of flexibility under a postponement model, and determines the optimal level of postponement with different flexibility structures. [Simchi-Levi et al., 2013] studies the increase in the resiliency of a supply chain when process flexibility and inventory is added into the system. Moreover, [Bish et al., 2005] studies the production swing of a make-to-order environment when flexibility is incorporated. For a more detailed review on the study of flexibilities, we refer the readers to the surveys of [Sethi and Sethi, 1990], [Buzacott and Mandelbaum, 2008] and [Chou et al., 2008].

1.3 Contributions

In this section, we outline the contributions of this thesis to the existing literature.

Deterministic Properties. This thesis establishes two important deterministic properties for the sales of the long chain process flexibility structures. First, the thesis identifies a supermodularity condition on a set of flexibility arcs in the long chain. Second, the thesis proves a decomposition property of the long chain, which decomposes the sales of the long chain into the differences between the sales of simpler flexibility structures. The decomposition allows us to characterize the sales of the long chain using a greedy algorithm, and therefore gives rise to a matrix multiplication method for computing the expected sales of the long chain when product demands are stochastic and independent. Both properties, supermodularity and decomposition are shown to hold under a fairly general setting, thus providing building blocks for studying more general models than the ones studied in the existing literature.

Average-Case Analysis. In our analysis of the expected sales of the long chain, we provide the first theoretical justifications to several empirically observed properties of the long chain under IID demand. First, we establish that if one starts with a dedicated structure and adds flexibilities to create a long chain, *the incremental benefits*, or the increase in expected sales, is always increasing. Second, we prove that the long chain is optimal among all 2-flexibility structures, structures that have degree two at each of the plant and product node. Third, we demonstrate that the difference between the fill rate of full flexibility structure and the fill rate of the long chain is increasing with the number of products. By combining our third result and a result from [Chou et al., 2010c], we can effectively demonstrate that a little bit of flexibility can go a long way. That is, under a large class of IID demand distributions, the expected sales of the long chain is almost as large as the expected sales of full flexibility under any finite size system.

Our analysis also identifies new insights in creating sparse flexibility structures. We prove that under IID demand, while the fill rate of a long chain increases with system size, the increase in fill rate, however, converges to zero exponentially fast. Thus, our result suggests that although the long chain is always the optimal 2-flexibility structure under IID demand, a structure that consists of several closed chains, where each chain connects a large number of plants and products, can perform just as well as the long chain.

Worst-Case Analysis. To analyze flexibility structures from a worst-case point of view, we study the worst-case performances of a flexibility structure over all demands that lie in a given uncertainty set. We introduce the "plant cover index", an index based only on the flexibility structure. The plant cover index allows for a comparison between the worst-case performances of two flexibility structures based only on their structures and independent of the choice of the uncertainty set or the worst-case performance measure. More specifically, we prove that if the plant cover indices for flexibility structure A is greater than or equal to the plant cover indices for flexibility structure B, the worst-case performance of A is always better than that of B, for all symmetric uncertainty sets, and a large class of worst-case performance measures.

The plant cover index is next applied as a tool in both theoretical analysis and empirical studies. In theoretical analysis, we prove that the long chain flexibility structure is most robust among all flexibility structures that have degree two at each of its product node, and all connected flexibility structures with $2n$ arcs. From numerical simulations, we find that the plant cover index is a useful index in determining the strength of flexibility structures in both average and worst-case. Moreover, we apply plant cover index to propose a data-independent heuristic that generates sparse flexibility structures effective from both average-case and worst-case point of views.

Distribution Systems. Finally, we propose a distribution systems model that extends the traditional plant/product model by introducing distribution centers. We prove that under deterministic demand, there exists a long chain structure that is optimal among all 2-flexibility structures if the supply chain costs are independent of either the plants or the distribution centers. When those conditions do not hold, we provide a counter example where there does not exist a long chain that is optimal among all 2-flexibility structures.

We also study the distribution systems model with only transportation cost, and the transportation cost is linear with the distance between plants and distribution centers. Our result identifies the long chain that minimizes transportation cost among 2-flexibility structures when all plants and distribution centers lie on a line. The analysis thus provides a simple guideline for designing flexibility structures in a manufacturing system that also takes transportation cost into consideration.

1.4 Basic Notations

Let $\mathbb{R}$ and $\mathbb{R}_+$ be the set of real and nonnegative real numbers respectively. Bold letters are reserved for vectors and matrices. For example, $\mathbf{x} \in \mathbb{R}^n$ is a vector with entries $x_1, x_2, ..., x_n$, and $\mathbf{A} \in \mathbb{R}^{mn}$ is a matrix with A_{ij} being the entry on its i-th row and j-th column.

We let m and n be the number of plants and products in the system. For arbitrary positive integers m and n, we use $A := \{a_1, a_2, ..., a_m\}$ to denote the set of plant nodes, and $B := \{b_1, b_2, ..., b_n\}$ to denote the set of product nodes. We assume that plant i has a fixed capacity of c_i for $1 \leq i \leq m$. A flexibility structure (aka flexibility design)

$\mathscr{A}$ is represented by a set of arcs connecting nodes in A to nodes in B. An arc $(a_i, b_j) \in \mathscr{A}$, represents that plant i is capable of producing project j under flexibility structure $\mathscr{A}$. When no ambiguity arise, we will sometimes use (i, j) interchangeably with (a_i, b_j) to simply the notation.

We note that $\mathscr{A}$ can be viewed as a bipartite graph whose partition has parts A and B. For any $u \in A \cup B$, define $N(u, \mathscr{A}) := \{v|(u, v) \text{ or } (v, u) \in \mathscr{A}\}$, i.e., $N(u, \mathscr{A})$ is the set of neighbors of u in the bipartite graph defined by $(A, B, \mathscr{A})$. Moreover, for set $S \subseteq A$ or $S \subseteq B$, we let $N(S, \mathscr{A}) := \cup_{u \in S} N(u, \mathscr{A})$. We define $I(\cdot)$ as an index function which maps nodes in $A \cup B$ to its indices, i.e. $I(a_i) = i$, $I(b_j) = j$ and $I((a_i, b_j)) = (i, j)$. Throughout the thesis, we will always assume that $|N(u, \mathscr{A})| \geq 1$ for all $u \in A \cup B$, that is, we assume no flexibility structure $\mathscr{A}$ has isolated plant or product nodes.

Because $\mathscr{A}$ can be viewed as a bipartite graph, we define an *undirected cycle* in $\mathscr{A}$ to be a set of arcs which forms a cycle when the arc orientations are ignored. A flexibility structure $\mathscr{A}$ is a long chain if its arcs form exactly one undirected cycle containing all plant and product nodes (see Figure 1-1 for an example). A *closed chain* is defined as an induced subgraph in $\mathscr{A}$ which forms an undirected cycle, while an *open chain* is an induced subgraph in $\mathscr{A}$ which forms an undirected line (one arc less than an undirected cycle). In Figure 1-1, an example of an open and a closed chain is presented.

Given an instance of the demand vector, $\mathbf{d}$, the total demand satisfied by a flexibility structure $\mathscr{A}$, denoted by $P(\mathbf{d}, \mathscr{A})$, is defined as the objective value of the

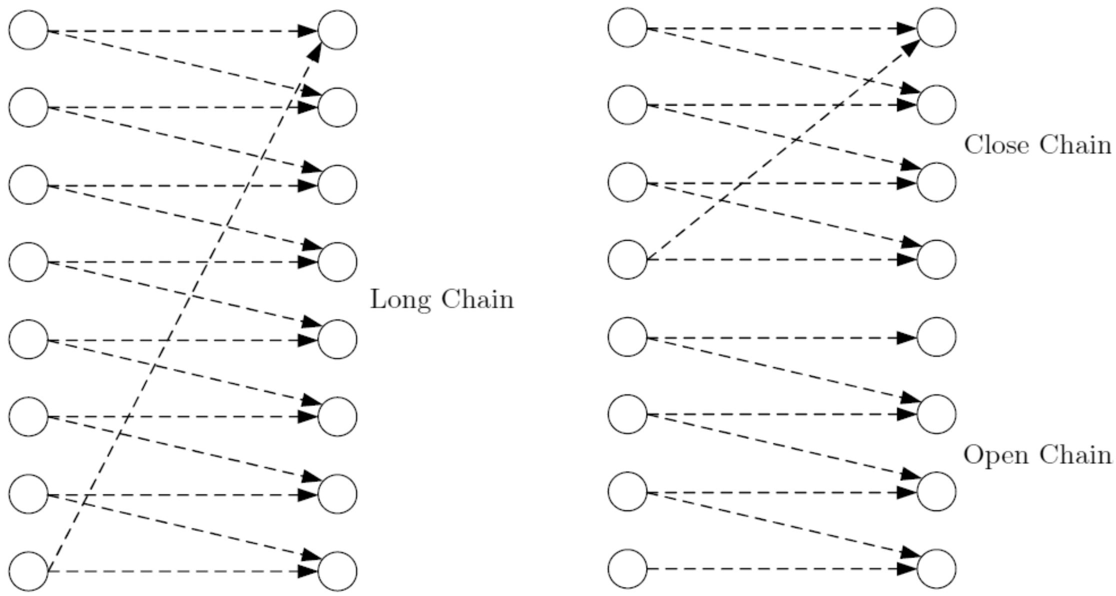


Figure 1-1: Configurations for Flexibility Designs

following linear program (LP):

$$
\begin{aligned}
P(\mathbf{d}, \mathscr{A}) := \max \quad & \sum_{(i,j)\in I(\mathscr{A})} f_{ij} \\
\text{s.t.} \quad & \sum_{a_i \in N(b_j, \mathscr{A})} f_{ij} \leq d_j, \ \forall j \in I(B) \\
& \sum_{b_j \in N(a_i, \mathscr{A})} f_{ij} \leq c_i, \ \forall i \in I(A) \\
& f_{ij} \geq 0, \forall (i,j) \in I(\mathscr{A}) \\
& \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}.
\end{aligned}
$$

We will refer to $P(\mathbf{d}, \mathscr{A})$ as *the sales of* $\mathscr{A}$ given $\mathbf{d}$.

When product demands are stochastic, $\mathbf{D}$ is used to denote the random vector of demands. $\mathbb{P}[.]$ and $\mathbb{E}[.]$ are used to denote the probability and expectation function of a random variable. In particular, $\mathbb{E}[P(\mathbf{D}, \mathscr{A})]$ is used to represent the expected sales of $\mathscr{A}$ given stochastic demand $\mathbf{D}$ (with plant capacity $\mathbf{c}$ fixed). The *fill rate* of a flexibility structure is defined as the ratio between the performance of a given flexibility structure and the total expected demand. When each product's expected demand is equal to one, the fill rate of a flexibility structure equals the per product expected sales.

We say that $\mathbf{D}$ is *exchangeable* if $[D_1, ..., D_n]$ equals to $[D_{\sigma(1)}, .., D_{\sigma(n)}]$ in distribution for any σ that is a permutation of $\{1, 2, ..., n\}$. We note that any independent and identically distributed (IID) demand is exchangeable but not all exchangeable demand are IID. For example, consider a random vector $\mathbf{D} = [D_1, ..., D_n]$ that is uniformly distributed on the linear polyhedron

$$\{(x_1, .., x_n) | \sum_{i=1}^{n} x_i = n, x_i \geq 0, \forall i = 1, .., n\}.$$

Clearly $\mathbf{D}$ is exchangeable, but the random variables in $\mathbf{D}$ are not independent, since they always sum up to n.

A system is defined to be balanced if $m = n$. In a balanced system with n plants and n products, we use $\mathscr{F}_n$ to denote the *full flexibility* structure, where $\mathscr{F}_n := \{(a_i, b_j) | \forall 1 \leq i, j \leq n\}$; $\mathscr{D}_n$ to denote the *dedicated* structure, where $\mathscr{D}_n := \{(a_i, b_i) | \forall 1 \leq i \leq n\}$; and $\mathscr{C}_n$ to denote the *long chain* structure, where $\mathscr{C}_n := \mathscr{D}_n \cup \{(a_1, b_2), (a_2, b_3), ..., (a_{n-1}, b_n), (a_n, b_1)\}$. We note that while our previous definition of the long chain is different from $\mathscr{C}_n$, any long chain that satisfies our previous definition can be represented by $\mathscr{C}_n$ under a specific relabeling of plants and products. Finally, we use $\mathscr{L}_n$ to denote the *open chain* structure, where $\mathscr{L}_n := \mathscr{C}_n \setminus \{(a_1, b_n)\}$.

In a balanced system, an arc $(a_i, b_j) \in \mathscr{A}$ is defined to be a *flexible arc* if $i \neq j$, and a *dedicated arc* if $i = j$. We say $\mathscr{A}$ is a 2-flexibility structure in a balanced system if any plant node and any product node is incident to exactly two arcs in $\mathscr{A}$. It can be seen that any 2-flexibility structure is the union of a number of closed chains.

1.5 Organization

The rest of the thesis is organized as follows.

In Chapter 2, we present two deterministic properties for the sales of the long chain. First, we prove that all flexible arcs in the long chain are supermodular with each other. Then, applying the supermodularity property, we derive a decomposition that proves the sales of $\mathscr{C}_n$ can be decomposed as a sum of n quantities, where each quantity is the difference of the sales associated with two open chains in $\mathscr{C}_n$. We show that the decomposition gives rise to an efficient method for computing the expected sales of the long chain under independent demand. Finally, we extend the supermodularity and the decomposition results to a more general model with different plant capacities, flexibility capacities and linear profits.

In Chapter 3, we apply the deterministic properties of the long chain developed in Chapter 2 to study the effectiveness of the long chain under stochastic demand. Our analysis gives rise to four important developments: (i) when a long chain is constructed from a dedicated structure, the incremental benefits of adding flexibilities are increasing and the largest benefit is always achieved when the chain is closed by adding the last arc to the system; (ii) the long chain is optimal among all 2-flexibility structures; (iii) the gap between the fill rate of full flexibility and that of the long chain increases with system size; and (iv) the fill rate of a long chain increases with the number of products, but this increase converges to zero exponentially fast.

In Chapter 4, we study the worst-case performances of the process flexibility structures under a demand uncertainty set. We first introduce the *plant cover index*, an index defined by a constrained bipartite vertex cover on the corresponding flexibility structures. Applying the plant cover index, we provide conditions under which one flexible structure is more robust than another under any symmetric uncertainty sets. Then, the condition is used to prove that the long chain compares favorably to other sparse flexibility structures with $2n$ flexibility arcs. Finally, we use simulation to show that the plant cover index can be used as a guideline to design sparse flexibility structures that works well under both worst-case and average-case performances.

In Chapter 5, we study sparse flexibility structures under an extended model with plants, products and distribution centers. We first prove that if the supply chain costs are independent of either plants or distribution centers, then there exist a long chain that is optimal among all 2-flexibility structures. Then, we restrict our model in the special case with transportation cost, and identify a long chain that minimizes the transportation cost among all 2-flexibility structures when plants and distribution centers are located on a line.

Finally, in Chapter 6, we conclude with a summary of the thesis and discuss several future directions.

The results from Chapter 2 and 3 have first appeared in [Simchi-Levi and Wei, 2012], while the results from Chapter 4 have first appeared in [Simchi-Levi and Wei, 2013].

Chapter 2

Supermodularity and Decomposition of Long Chain

This chapter establishes several building blocks to analyze and compute the effectiveness of the long chain. In Section 2.1 and 2.2, we study the sales of long chain facing an arbitrarily fixed demand instance $\mathbf{d}$. First, Section 2.1 establishes a result that shows that any pair of flexible arcs in a long chain are supermodular with each other; and in Section 2.2, using the supermodularity property as a key lemma, we derive a decomposition of the sales of the long chain for system size n, $\mathscr{C}_n$, as a sum of n quantities, where each quantity is the difference of the sales of two open chains. Then, in Section 2.3, applying the decomposition result, we develop a direct and efficient way to compute the expected sales of the long chain ($\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)]$) under IID demand. Finally, in Section 2.4, we establish several generalizations of the results in Sections 2.1-2.3.

Throughout this chapter, we assume the system is balanced ($m = n$) and consider a balanced system of size n. In Sections 2.1-2.3, we assume $c_i = 1$ for $1 \leq i \leq n$. This assumption is later relaxed in Section 2.4, where a more general model is considered.

2.1 Supermodularity

To establish the supermodularity property in the long chain, we start by formally defining the notion of supermodularity.

Definition 2.1. *A function $f(x, y)$ is said to be* supermodular *in x and y if for any real numbers x', x'', y', y'',*

$$f(\max\{x', x''\}, \max\{y', y''\}) + f(\min\{x', x''\}, \min\{y', y''\}) \geq f(x', y') + f(x'', y'').$$

Next, consider a flexibility structure $\mathscr{A}$, a demand instance $\mathbf{d}$ and two arcs, $\alpha, \beta \in \mathscr{A}$ with given non-negative capacities u_α and u_β. Define

$$\begin{aligned}
P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A}) := \max \quad & \sum_{(i,j)\in I(\mathscr{A})} f_{ij} \\
\text{s.t.} \quad & \sum_{a_i \in N(b_j, \mathscr{A})} f_{ij} \leq d_j, \, \forall j \in I(B) \\
& \sum_{b_j \in N(a_i, \mathscr{A})} f_{ij} \leq 1, \, \forall i \in I(A) \\
& f_\alpha \leq u_\alpha, \quad f_\beta \leq u_\beta, \\
& f_{ij} \geq 0, \forall (i,j) \in I(\mathscr{A}) \\
& \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}.
\end{aligned}$$

We prove that if $\mathscr{A} \subset \mathscr{C}_n$, then for any two flexible arcs α and β in $\mathscr{A}$, $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is supermodular in u_α and u_β. Note that if α and β are not in $\mathscr{A}$, then $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is equal to $P(\mathbf{d}, \mathscr{A})$ regardless of u_α and u_β and the supermodularity property holds. The interesting case arises when α and β are in $\mathscr{A}$.

For this purpose, we show that $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is equivalent to a max-weight circulation problem, which allows us to apply a classical result from [Gale and Politof, 1981]. Define $G(\mathscr{A})$ to be the underlying graph for the max-weight circulation problem, which contains $\mathscr{A}$, an additional node s, an arc from s to each of the plant nodes, and an arc from each of the product nodes to s. The underlying graph of the max-weight circulation problem, $G(\mathscr{A})$, is illustrated in Figure 2-1 for $\mathscr{A} = \mathscr{C}_5$, of long chain for a balanced system of size five.

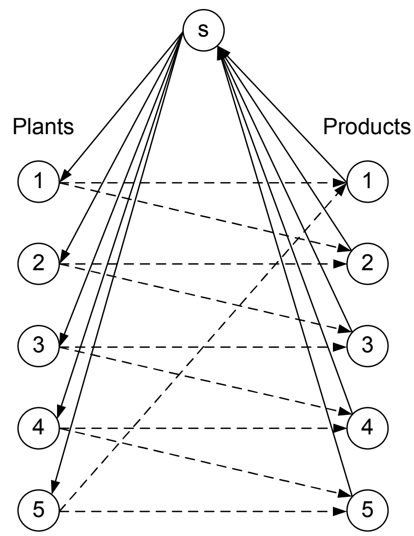


Figure 2-1: $G(\mathscr{C}_5)$ for the Max-weight Circulation associated to $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{C}_n)$

To complete the description of the max-weight circulation problem, we set the weight of each plant to product arc (that is, the arcs in $\mathscr{A}$) to 1 and the weight of every other arc to zero. The upper-bound (capacity) on the flow on an arc from s to plant i is set to be 1 for all $i = 1, 2, ..., n$; the upper-bound for the flow on an arc connecting product j to s is set to be d_j for all $j = 1, 2, ..., n$; the upper-bound for the flow on α (and β) to be u_α (and u_β), and the upper-bound for the flow on every other arc in $\mathscr{A}$ is set to be 1. Finally, we set the lower-bound for the flow on every arc in $G(\mathscr{A})$ to be 0.

In [Gale and Politof, 1981], Gale and Politof present the following definition.

Definition 2.2. *In a directed graph G, two arcs α, β are said to be **in series**, if for any cycle C containing both α and β, α and β have the same direction when we fix an orientation of C.*

Next, we show that any two flexible arcs from the set $\mathscr{C}_n$ are in series in graph $G(\mathscr{C}_n)$.

Lemma 2.1. *Let α and β be two flexible arcs in $\mathscr{C}_n$. Then α and β are in series in $G(\mathscr{C}_n)$, where $G(\mathscr{C}_n)$ is the underlying graph of the max-weight circulation problem for $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{C}_n)$.*

Proof. Let C be an arbitrary undirected cycle in $G(\mathscr{C}_n)$. If C does not contain node s, then C must be the undirected cycle which contains every plant to product arcs in $\mathscr{C}_n$. In that case, it is easy to verify that α and β have the same direction in C. Otherwise, suppose C contains s. In such a case, C can be decomposed into four pieces, X_1, X_2, α and β, where X_1, X_2 are the two paths between α and β. Without loss of generality, we assume X_1 contains s. Since α and β cannot be incident to the same node, both X_1 and X_2 are nonempty. As X_2 does not contain s, all arcs in X_2 are plant to product arcs (i.e. $X_2 \subset \mathscr{C}_n$). Because of the structure of $\mathscr{C}_n$, X_2 contains an odd number of arcs. Moreover, the path in $X_2 \cup \{\alpha\} \cup \{\beta\}$ has alternating directions for every two consecutive arcs and therefore, α and β have the same direction in C. This is illustrated by Figure 2-2. Since this is true for any arbitrary undirected cycle C, α and β are in series in $G(\mathscr{C}_n)$.

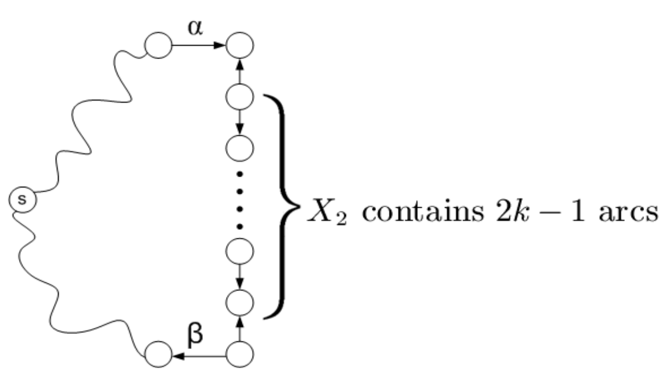


Figure 2-2: Illustration for the proof of Lemma 2.1

□

Lemma 2.1 allows us to apply the following important result of [Gale and Politof, 1981]. They show that if two arcs, α and β, in the underlying graph are in series, then the optimal flow of the max-weight circulation is supermodular with respect to the capacities of both arcs.

Theorem 2.1. *Let $\mathscr{A}$ be a flexibility structure for a balanced system of size n, and $\mathscr{A} \subset \mathscr{C}_n$. For any flexible arcs α, β in $\mathscr{A}$, $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is supermodular in u_α and u_β. Hence,*

$$P(\mathbf{d}, \mathscr{A}) + P(\mathbf{d}, \mathscr{A} \setminus \{\alpha, \beta\}) \geq P(\mathbf{d}, \mathscr{A} \setminus \{\alpha\}) + P(\mathbf{d}, \mathscr{A} \setminus \{\beta\}).$$

Proof. By construction, $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ can be computed by solving the max-weight circulation problem. Since $\mathscr{A} \subset \mathscr{C}_n$, the set arcs in $G(\mathscr{A})$ is a subset of the set of arcs in $G(\mathscr{C}_n)$. By Lemma 2.1, α and β are in series in $G(\mathscr{C}_n)$. Thus, α and β are in series in $G(\mathscr{A})$. Applying the main theorem in [Gale and Politof, 1981], we have that $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is supermodular in u_α and u_β. Hence

$$\begin{aligned} & P(1, 1, \mathbf{d}, \mathscr{A}) + P(0, 0, \mathbf{d}, \mathscr{A}) \geq P(0, 1, \mathbf{d}, \mathscr{A}) + P(1, 0, \mathbf{d}, \mathscr{A}) \\ \Longrightarrow \; & P(\mathbf{d}, \mathscr{A}) + P(\mathbf{d}, \mathscr{A} \setminus \{\alpha, \beta\}) \geq P(\mathbf{d}, \mathscr{A} \setminus \{\alpha\}) + P(\mathbf{d}, \mathscr{A} \setminus \{\beta\}). \end{aligned}$$

□

Theorem 2.1 thus suggests that any two flexible arcs in the long chain complement each other. That is, the existence of one flexible arc increases the marginal benefit that can be gained when the other flexible arc is added.

Furthermore, the supermodular result of [Gale and Politof, 1981] can be extended for two sets of arcs X and Y, where any pair of arcs in $X \cup Y$ are in series with each other. While that was not stated in the paper of [Gale and Politof, 1981], it was proven in [Granot and Veinott Jr., 1985] under more general settings. Here, we state Corollary 2.1 which is a special case of Theorem 17 from [Granot and Veinott Jr., 1985].

Corollary 2.1. *Let $\mathscr{A}$ be a flexibility structure for a balanced system of size n, and $\mathscr{A} \subset \mathscr{C}_n$. For any $X, Y \subseteq S$, where S is the set of all flexible arcs in $\mathscr{A}$, and demand instance $\mathbf{d}$,*

$$P(\mathbf{d}, \mathscr{A} \setminus (X \cap Y)) + P(\mathbf{d}, \mathscr{A} \setminus (X \cup Y)) \geq P(\mathbf{d}, \mathscr{A} \setminus X) + P(\mathbf{d}, \mathscr{A} \setminus Y).$$

2.2 Decomposition

In this section, we show that in a balanced system of size n, the sales of the long chain can be decomposed into the sum of n quantities, where each quantity is equal to the difference of sales of two open chains. Throughout the section, when some integer k appears in a statement, we are in fact referring to some $i \in \{1, ..n\}$ *congruent* to k *modulo* n. For example, if plant $n+3$ appears in a statement, then we are referring to plant 3; and if $f_{n+1,n+2}$, the flow from plant $n+1$ to product $n+2$ appears in a statement, then we are referring to $f_{1,2}$, the flow from plant 1 to product 2. Also, we define $\alpha_i = (a_i, b_{i+1})$ and $\beta_i = (a_i, b_i)$ for $i = 1, 2, ..., n$ (note that $\alpha_n = (a_n, b_1)$ as $n+1$ is congruent with 1 modulo n).

We first start the section with two lemmas. The first lemma states that if a flexible arc α is not required to achieve the optimal sale in $\mathscr{C}_n$, then α is not required to achieve the optimal sale in $\mathscr{C}_n \setminus X$, for any X that is a set of flexible arcs.

Lemma 2.2. *Suppose $P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha\}) = P(\mathbf{d}, \mathscr{C}_n)$, where α is a flexible arc in $\mathscr{C}_n$. Then, for any set $X \subseteq S$, where S is the set of all flexible arcs in $\mathscr{C}_n$, we have that*

$$P(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha\})) = P(\mathbf{d}, \mathscr{C}_n \setminus X)$$

Proof. If $\alpha \in X$, the result is trivial as $X \cup \{\alpha\} = X$. Otherwise, By Corollary 2.1,

$$\begin{aligned} &P(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha\})) + P(\mathbf{d}, \mathscr{C}_n) \geq P(\mathbf{d}, \mathscr{C}_n \setminus X) + P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha\}) \\ &\qquad \implies P(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha\})) \geq P(\mathbf{d}, \mathscr{C}_n \setminus X), \quad \text{since } P(\mathbf{d}, \mathscr{C}_n) = P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha\}). \end{aligned}$$

But by definition of $P(.)$, $P(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha\})) \leq P(\mathbf{d}, \mathscr{C}_n \setminus X)$, hence

$$P(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha\})) = P(\mathbf{d}, \mathscr{C}_n \setminus X).$$

□

The next lemma states that there always exists a flexible arc α where $P(\mathbf{d}, \mathscr{C}_n) = P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha\})$.

Lemma 2.3. *There exists some* $1 \leq i^* \leq n$ *such that* $P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\}) = P(\mathbf{d}, \mathscr{C}_n)$, *for any demand instance* $\mathbf{d}$.

Proof. Recall that

$$\begin{aligned} P(\mathbf{d}, \mathscr{C}_n) = \max \quad & \sum_{(i,j) \in I(\mathscr{C}_n)} f_{ij} \\ \text{s.t.} \quad & \sum_{a_i \in N(b_j, \mathscr{C}_n)} f_{ij} \leq d_j, \ \forall j \in I(B) \\ & \sum_{b_j \in N(a_i, \mathscr{C}_n)} f_{ij} \leq c_i, \ \forall i \in I(A) \\ & f_{ij} \geq 0, \forall (i,j) \in I(\mathscr{C}_n) \\ & \mathbf{f} \in \mathbb{R}^{|\mathscr{C}_n|}. \end{aligned}$$

The optimization problem associated with $P(\mathbf{d}, \mathscr{C}_n)$ clearly has an optimal solution because it is bounded. Let $\mathbf{f}^*$ be an optimal solution of $P(\mathbf{d}, \mathscr{C}_n)$. If $f^*_{ij} = 0$ for some $(i,j) \in \{\alpha_1, ..., \alpha_n\}$, then there is some i^* such that $f^*_{\alpha_{i^*}} = 0$ and this implies that $P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\}) = P(\mathbf{d}, \mathscr{C}_n)$.

Otherwise, $f^*_{ij} > 0$ for all $(i,j) \in I(\{\alpha_1, ..., \alpha_n\})$. Let $\mathbf{g}$ be the vector that

$$g_{ij} = \begin{cases} -1 & \text{if } (i,j) \in I(\{\alpha_1, ..., \alpha_n\}) \\ 1 & \text{if } (i,j) \in I(\{\beta_1, \beta_2, ..., \beta_n\}) \\ 0 & \text{otherwise.} \end{cases}$$

In network flow theory, $\mathbf{g}$ is known as an *augmenting cycle* for $\mathbf{f}^*$. Let $\delta^* = \min\{f_{ij}, (i,j) \in I(\{\alpha_1, ..., \alpha_n\})\}$. Note that $\mathbf{f}^* + \delta^* \mathbf{g}$ is a feasible and optimal solution of $P(\mathbf{d}, \mathscr{C}_n)$. Moreover, $f^*_{ij} + \delta^* g_{ij} = 0$ for some $(i,j) \in I(\{\alpha_1, ..., \alpha_n\})$. Thus, there is some i^* such that $f^*_{\alpha_{i^*}} + \delta^* g_{\alpha_{i^*}} = 0$ and this implies that $P(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\}) = P(\mathbf{d}, \mathscr{C}_n)$. □

Next, we show that the sales associated with $\mathscr{C}_n$ can be decomposed as a sum of n quantities, where each quantity is the difference of the sales associated with two open chains in $\mathscr{C}_n$.

Theorem 2.2. *For any fixed demand instance* $\mathbf{d}$ *on balanced system of size* n*, we have*

$$P(\mathbf{d},\mathscr{C}_n) = \sum_{i=1}^{n}(P(\mathbf{d},\mathscr{C}_n \setminus \{(a_i,b_{i+1})\}) - P(\mathbf{d},\mathscr{C}_n \setminus \{(a_{i-1},a_i),(a_i,b_i),(a_i,b_{i+1})\})).$$

Proof. For each $1 \leq k_1 \leq k_2 \leq n$, define $\mathscr{L}_{k_1\to k_2} = \{(a_i,b_i)|i = k_1, k_1+1, ..., k_2\} \cup \{(a_i,b_{i+1})|i = k_1, k_1+1, ..., k_2-1\}$, and for each $1 \leq k_2 < k_1 \leq n$, define $\mathscr{L}_{k_1\to k_2} = \{(a_i,b_i)|i = k_1, k_1+1..., n, 1, 2, ..., k_2\} \cup \{(a_i,b_{i+1})|i = k_1, ..., n, ..., k_2-1\}$. One can think of $\mathscr{L}_{k_1\to k_2}$ as the open chain connecting plant k_1 to product k_2 in the balanced system of size n. Also, since demand instance $\mathbf{d}$ is fixed, for the sake of succinctness, we use $P(\mathscr{A})$ to denote $P(\mathbf{d},\mathscr{A})$.

By definitions of α_i and β_i, we can rewrite $\mathscr{C}_n \setminus \{(a_i,b_{i+1})\}$ and $\mathscr{C}_n \setminus \{(i-1,i),(a_i,b_i),(a_i,b_{i+1})\}$ as $\mathscr{C}_n \setminus \{\alpha_i\}$ and $\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i,\beta_i\}$. For any $1 \leq i \leq n$, because $\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i\} = \{\beta_i\} \uplus \mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i,\beta_i\}$, where $\uplus$ represent the symbol for disjoint union,

$$P(\mathscr{C}_n \setminus \{\alpha_i\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i,\beta_i\}) = P(\mathscr{C}_n \setminus \{\alpha_i\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i\}) + \min\{1,d_i\}. \tag{2.1}$$

Lemma 2.3 shows that there is some i^* such that $P(\mathscr{C}_n) = P(\mathscr{C}_n \setminus \{\alpha_{i^*}\})$. Without loss of generality, we assume that $i^* = n$, as we can always relabel each plant (and product) i by $i - i^*$. Now, we have that for $i = 2, ..., n-1$,

$$\begin{aligned}
&P(\mathscr{C}_n \setminus \{\alpha_i\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i,\beta_i\}) \\
&= P(\mathscr{C}_n \setminus \{\alpha_i\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i\}) + \min\{1,d_i\} && \text{(by Equation (2.1))} \\
&= P(\mathscr{C}_n \setminus \{\alpha_i,\alpha_n\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i,\alpha_n\}) + \min\{1,d_i\} && \text{(by Lemma 2.2).}
\end{aligned}$$

Since $\mathscr{C}_n \setminus \{\alpha_i,\alpha_n\} = \mathscr{L}_{1\to i} \uplus \mathscr{L}_{(i+1)\to n}$, and $\mathscr{C}_n \setminus \{\alpha_{i-1},\alpha_i,\alpha_n\} = \mathscr{L}_{1\to(i-1)} \uplus$

$\mathscr{L}_{(i+1)\to n} \uplus \{\beta_i\}$, we have for $i = 2, ..., n-1$,

$$\begin{aligned}
&P(\mathscr{C}_n \setminus \{\alpha_i\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\}) \\
&= P(\mathscr{C}_n \setminus \{\alpha_i, \alpha_n\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \alpha_n\}) + \min\{1, d_i\} \\
&= P(\mathscr{L}_{1\to i}) + P(\mathscr{L}_{(i+1)\to n}) - \Big(P(\mathscr{L}_{1\to(i-1)}) + P(\mathscr{L}_{(i+1)\to n}) + \min\{1, d_i\}\Big) + \min\{1, d_i\} \\
&= P(\mathscr{L}_{1\to i}) - P(\mathscr{L}_{1\to(i-1)}). \qquad (2.2)
\end{aligned}$$

Also,

$$\begin{aligned}
&P(\mathscr{C}_n \setminus \{\alpha_1\}) - P(\mathscr{C}_n \setminus \{\alpha_n, \alpha_1, \beta_1\}) \\
&= P(\mathscr{C}_n \setminus \{\alpha_1\}) - P(\mathscr{C}_n \setminus \{\alpha_n, \alpha_1\}) + \min\{1, d_1\} \qquad \text{(by Equation (2.1))} \\
&= P(\mathscr{C}_n \setminus \{\alpha_1, \alpha_n\}) - P(\mathscr{C}_n \setminus \{\alpha_1, \alpha_n\}) + \min\{1, d_1\} \qquad \text{(by Lemma 2.2)} \\
&= \min\{1, d_1\}, \qquad (2.3)
\end{aligned}$$

and

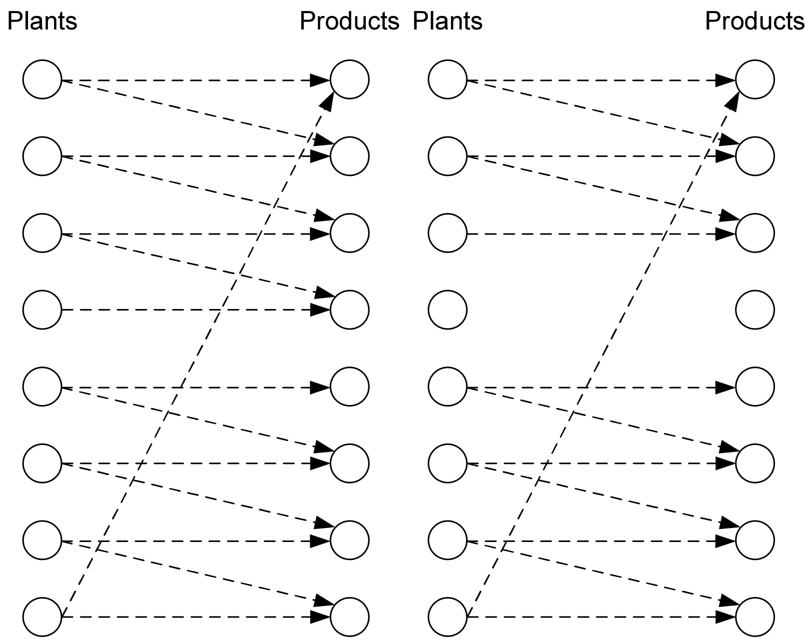
$$P(\mathscr{C}_n \setminus \{\alpha_n\}) - P(\mathscr{C}_n \setminus \{\alpha_{n-1}, \alpha_n, \beta_n\}) = P(\mathscr{L}_{1\to n}) - P(\mathscr{L}_{1\to(n-1)}). \qquad (2.4)$$

Now, applying Equations (2.2-2.4), we obtain that

$$\begin{aligned}
&\sum_{i=1}^{n}(P(\mathscr{C}_n \setminus \{\alpha_i\}) - P(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\})) \\
&= \min\{1, d_1\} + \sum_{i=2}^{n}(P(\mathscr{L}_{1\to i}) - P(\mathscr{L}_{1\to(i-1)})) \\
&= \min\{1, d_1\} + P(\mathscr{L}_{1\to n}) - P(\mathscr{L}_{1\to 1}) \\
&= P(\mathscr{L}_{1\to n}) \\
&= P(\mathscr{C}_n \setminus \{\alpha_n\}) \\
&= P(\mathscr{C}_n).
\end{aligned}$$

□

Recall that an open chain is an subgraph in $\mathscr{A}$ which has one arc less than a closed chain, thus, we have that $\mathscr{C}_n \setminus \{\alpha_i\}$ is an open chain connecting plant $i+1$ to product i, while $\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\}$ is an open chain connecting plant $i+1$ to product $i-1$ (see Figure 2-3). It turned out that the sales of open chains is much easier to analyze and compute. Indeed, in the next section, we will apply Theorem 2.2 to obtain an efficient method to compute $\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)]$ under IID demand.



(a) $\mathscr{C}_8 \setminus \{\alpha_4\}$ (b) $\mathscr{C}_8 \setminus \{\alpha_3, \alpha_4, \beta_4\}$

Figure 2-3: Flexibility Structures in the Decomposition

2.3 Computing Expected Sales

In this section, we present an algorithm to compute expected sales of the long chain, $\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)]$, under any IID demand $\mathbf{D}$. To derive the algorithm for computing $\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)]$, we first apply Theorem 2.2 to characterize the expected sales of the long chain using the difference between the expected sales of two open chains. Next, we introduce Algorithm 1, a greedy algorithm that computes the difference between the *sales* of two open chains under any deterministic demand instance. Finally, we apply Algorithm 1 to develop an efficient procedure to compute the difference between the expected sales of two open chains and therefore the expected sales of the long chain.

As defined in Section 1.4, $\mathbf{D}$ is *exchangeable* if $[D_1, ..., D_n]$ equals to $[D_{\sigma(1)}, .., D_{\sigma(n)}]$ in distribution for any σ that is a permutation of $\{1, 2, ..., n\}$. First, we state a characterization of $\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)]$ when $\mathbf{D}$ is *exchangeable*. Recall that we defined $\mathscr{L}_n = \mathscr{C}_n \setminus \{(a_n, b_1)\}$.

Theorem 2.3. *For any balanced system of size n with exchangeable demand $\mathbf{D}$, we have*

$$\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)] = n(\mathbb{E}[P(\mathbf{D}, \mathscr{L}_n)] - \mathbb{E}[P(\mathbf{D}, \mathscr{L}_{n-1})]).$$

Proof. Theorem 2.2 states that for any $\mathbf{d}$ which is an instance of $\mathbf{D}$,

$$P(\mathbf{d}, \mathscr{C}_n) = \sum_{i=1}^{n} (P(\mathbf{d}, \mathscr{C}_n \setminus \{(a_i, b_{i+1})\}) - P(\mathbf{d}, \mathscr{C}_n \setminus \{(i-1, i), (a_i, b_i), (a_i, b_{i+1})\})). \quad (2.5)$$

Since $\mathbf{D}$ is exchangeable, for any $1 \le i \le n$,

$$\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n \setminus \{(a_i, b_{i+1})\})] = \mathbb{E}[P(\mathbf{D}, \mathscr{L}_n)],$$
$$\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n \setminus \{(a_{i-1}, b_i), (a_i, b_i), (a_i, b_{i+1})\})] = \mathbb{E}[P(\mathbf{D}, \mathscr{L}_{n-1})].$$

Thus, integrating over all random instances of $\mathbf{D}$ on Equation (2.5), we have

$$\mathbb{E}[P(\mathbf{D}, \mathscr{C}_n)] = n(\mathbb{E}[P(\mathbf{D}, \mathscr{L}_n)] - \mathbb{E}[P(\mathbf{D}, \mathscr{L}_{n-1})]).$$

□

Recall that any IID demand is exchangeable, hence, by Theorem 2.3, we can also characterize the expected sales of a long chain by the expected sales of open chains under IID demand.

For the rest of this section, we will fix the demand vector, $\mathbf{D}$, to be a vector of IID entries, where each entry D_i has distribution D. For the sake of simplicity, we use $[\mathscr{A}]$ in place of $\mathbb{E}[P(\mathbf{D}, \mathscr{A})]$, as $\mathbf{D}$ is fixed.

Next, we start with a greedy algorithm that determines the optimal solution of $P(\mathbf{d}, \mathscr{L}_n)$.

Algorithm 1 Finding optimal solution $\mathbf{f}^*$ for $P(\mathbf{d}, \mathscr{L}_n)$

procedure SOLVE ($P(\mathbf{d}, \mathscr{L}_n)$)
  $f^*_{1,1} \leftarrow \min(1, d_1)$
  for $k = 2, ..., n$ **do**
    $f^*_{k-1,k} \leftarrow \min\{1 - f^*_{k-1,k-1}, d_k\}$
    $f^*_{k,k} \leftarrow \min\{1, d_k - f^*_{k-1,k}\}$
  end for
  return $\mathbf{f}^*$
end procedure

We note that a similar greedy style algorithm for computing the maximum sales of an open chain was mentioned in [Chou et al., 2010c]. We omit the proof for the correctness of Algorithm 1, since $P(\mathbf{d}, \mathscr{L}_k)$ is simply a max-flow problem on a path and it is well known that this problem can be solved using a greedy algorithm.

Given a random demand vector $\mathbf{D}$, let F_{ij} be the random flow on arc (i, j) returned by Algorithm 1, for $1 \leq i, j \leq n$. For each integer $1 \leq k \leq n-1$, define $W_k = 1 - F_{kk}$ and $W_0 = 0$. W_k can be thought of as the remaining capacity in plant k after the production of product k at plant k is determined.

To develop a method to compute the expected sales of the long chain, assume that the support of D lies in $\{\frac{i}{N} | i = 0, 1, 2, ...,\}$ for some integer $N \geq 1$. Under this assumption, we let $p_i = \mathbb{P}[D = \frac{i}{N}]$, for any $i = 0, 1, ..., 2N-1$, and $p_{2N} = \mathbb{P}[D \geq 2]$, where $\mathbb{P}[.]$ denotes the probability mass function.

Since the support of D lies in $\{\frac{i}{N} | i = 0, 1, 2, ...\}$, it is easy to see that the support of F_{kk} lies in $\{\frac{i}{N} | i = 0, 1, 2, ..., N\}$, as $0 \leq F_{kk} \leq \sum_i F_{ik} \leq 1$. But $W_k = 1 - F_{kk}$, so W_k also has a support set of $\{\frac{i}{N} | i = 0, 1, 2, ..., N\}$. As a result, the distribution of W_k can be described by a row vector $\mathbf{q}^k$ with $N + 1$ elements, where $q^k_i = \mathbb{P}[W_k = \frac{i}{N}]$, for $i = 0, 1, ..., N$. Then, we have

Lemma 2.4. $\mathbf{q}^{k+1} = \mathbf{q}^k \mathbf{A} = \mathbf{q}^0 \mathbf{A}^{k+1}$ *for* $0 \leq k \leq n-1$, *where*

$$\mathbf{A} = \begin{bmatrix} \sum_{i=N}^{2N} p_i & p_{N-1} & p_{N-2} & \cdots & p_1 & p_0 \\ \sum_{i=N+1}^{2N} p_i & p_N & p_{N-1} & \cdots & p_2 & p_0 + p_1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{2N-1} + p_{2N} & p_{2N-2} & p_{2N-3} & \cdots & p_{N+1} & \sum_{i=0}^{N-1} p_i \\ p_{2N} & p_{2N-1} & p_{2N-2} & \cdots & p_{N+1} & \sum_{i=0}^{N} p_i \end{bmatrix} \textit{ and } \mathbf{q}^0 = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \end{bmatrix}.$$

Proof. Since W_0 is 0 with probability 1, $\mathbf{q}^0 = [1\,0\,0\ldots 0]$. Because the demand is independent and W_k only depends on $D_1, \ldots D_k$, W_k is independent of D_{k+1}. Hence we have,

$$q_i^{k+1} = \mathbb{P}[W_{k+1} = i] = \sum_{j=0}^{N} \mathbb{P}[W_k = j]\mathbb{P}[D_{k+1} = N - i + j] = \sum_{j=0}^{N} q_j^k p_{N-i+j}, \forall 1 \leq i \leq N-1,$$
$$q_0^{k+1} = \mathbb{P}[W_{k+1} = 0] = \sum_{j=0}^{N} \mathbb{P}[W_k = j]\mathbb{P}[D_{k+1} \geq N + j] = \sum_{j=0}^{N} q_j^k \sum_{l=N+j}^{2N} p_{N+l},$$
$$q_N^{k+1} = \mathbb{P}[W_{k+1} = N] = \sum_{j=0}^{N} \mathbb{P}[W_k = j]\mathbb{P}[D_{k+1} \leq j] = \sum_{j=0}^{N} q_j^k \sum_{l=0}^{j} p_l.$$

This implies that $\mathbf{q}^{k+1} = \mathbf{q}^k \mathbf{A}$. □

A direct consequence of Lemma 2.4 is that the following matrix multiplications can be used to determine the expected sales of the long chain, when demands are IID and the support of a product demand is a subset of $\{\frac{i}{N} | i = 0, 1, 2, \ldots, \}$.

Theorem 2.4. $\frac{[\mathscr{C}_n]}{n} = [\mathscr{L}_n] - [\mathscr{L}_{n-1}] = \mathbf{q}^{n-1}\boldsymbol{\pi} = \mathbf{q}^0 A^{n-1}\boldsymbol{\pi}$, *where* $\boldsymbol{\pi}$ *is a vector of size* $N+1$ *and*

$$\pi_i = \sum_{j=1}^{N+i} j p_j + (N+i) \sum_{j=N+i+1}^{2N} p_j, \qquad \forall 1 \leq i \leq N$$

Proof. By Algorithm 1, $[\mathscr{L}_n]-[\mathscr{L}_{n-1}]$ can be written as the expectation of $F_{n-1n}+F_{nn}$, which is equal to $\mathbb{E}[\min\{1+W_{n-1}, D_n\}]$, thus

$$\mathbb{E}[\min\{1+W_{n-1}, D_n\}] = \sum_{i=0}^{N} \mathbb{P}[W_{n-1}=i]\mathbb{E}[\min\{D_n, \frac{N+i}{N}\}] = \sum_{i=0}^{N} q_i^{n-1}(\sum_{j=1}^{N+i} jp_j+(N+i)\sum_{j=N+i+1}^{2N} p_j).$$

Hence, we have that $[\mathscr{L}_n]-[\mathscr{L}_{n-1}] = \mathbf{q}^{n-1}\boldsymbol{\pi}$. Apply Theorem 2.3, and we are done. □

The matrix multiplication method developed here to compute the expected sales of the long chain is polynomial in N and n. Indeed, computing $\mathbf{q}^0\mathbf{A}^{n-1}\boldsymbol{\pi}$ requires $O(nN^2)$ operations if one sequentially evaluates $\mathbf{q}^0\mathbf{A}^i$ for $i = 1, ..., k$, or $O(N^{2.807}\log n)$ operations if one starts by determining $\mathbf{A}^{n-1}$ using the classical algorithm from [Strassen, 1969]. To the best of our knowledge, this is the first pseudo-polynomial time algorithm that computes the expected sales of a finite size long chain exactly when demand is discrete and IID. The other known algorithm to compute the expected sales of the long chain exactly is to solve the max-flow problem for all demand instances and sum them to determine the expected sales. This algorithm is exponential in n.

The matrix multiplication method can be applied to general IID demands as an approximation algorithm to compute the expected sales of long chains. In this case, one can approximate the expected sales of the long chain by discretizing the demand distribution on the set of $\{\frac{i}{N}|i = 0, 1, 2, ...,\}$ for some integer N. Clearly, as N increases, the error of the approximation decreases while the running time grows. Specifically, it is straightforward to show that the error of the approximation is bounded by $\frac{n}{2N}$. However, our computational experience suggests that the error is much smaller than this bound.

Moreover, the matrix multiplication method is fairly fast even for large N. For example, when $N = 1000$ and $n = 100$, $\mathbf{q}^0\mathbf{A}^{n-1}\boldsymbol{\pi}$ can be computed within 2 seconds using Matlab on a standard 2.1 GHz laptop. Hence, even for general IID demands, the matrix multiplication method can quickly approximate the expected sales of a large size long chain very accurately.

Figure 2-4 presents computational results obtained using the matrix multiplication method for three different IID demand distributions:

- **Normal**: Demand for a product is a discretized normal random variable with mean 1 and standard deviation of 0.33 on the support set of $\{\frac{i}{14}|i = 0, 1, ..., 28\}$; this distribution was originally applied in [Chou et al., 2010c] for their analysis of asymptotic behavior of long chains;

- **Uniform**: Demand for a product is uniformly distributed on the set $\{\frac{i}{10}|i = 0, 1, 2, ..., 9, 11, 12, ..., 20\}$;

- **Asymmetric**: Demand for a product is equal to $\frac{4}{5}$ with probability 0.4, 1 with probability 0.5 and 2 with probability 0.1.

For each distribution, Figure 2-4 depicts $\frac{[\mathscr{F}_n]}{n}$ (the per product expected sales of full flexibility), $\frac{[\mathscr{C}_n]}{n}$ (the per product expected sales of the long chain), and $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]}$ (the ratio between the expected sales of the long chain and the expected sales of full flexibility structure) for $n = 1, ..., 30$. Because demand is IID, $\frac{[\mathscr{C}_n]}{n}$ (and $\frac{[\mathscr{F}_n]}{n}$) is proportional to the fill rate of long chain (and full flexibility).

Figure 2-4 reveals several interesting observations. First, $\frac{[\mathscr{F}_n]}{n} - \frac{[\mathscr{C}_n]}{n}$, i.e., the gap between the fill rates of full flexibility and the long chain, is increasing, while the ratio, $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]}$, is decreasing. A similar observation on the ratio, using simulation results, is reported in [Chou et al., 2008]. In addition, Figure 2-4 suggests that the quantity $\frac{[\mathscr{C}_n]}{n}$, the fill rate of the long chain, is increasing but converges to a constant very quickly. These observations are discussed in detail Chapter 3.

The matrix multiplication method can be also used to compute the per product expected sales of the long chain for infinite size system. Observe that the matrix $\mathbf{A}$ is the transition matrix of a Markov chain with states $\frac{i}{N}$ for each $i = 0, 1, ..., N$. It can be shown that in the matrix $\mathbf{A}$, the communication class that contains state 0 is irreducible and aperiodic. Then, by the Perron-Frobenius theorem, see [Grimmett and Stirzaker, 1992], we have that $\lim_{n\to\infty} \mathbf{q}^0\mathbf{A}^{n-1} = \mathbf{q}^*$, where $\mathbf{q}^*\mathbf{A} = \mathbf{q}^*$ and $q_0^* > 0$. Thus, to compute $\lim_{n\to\infty} \frac{[\mathscr{C}_n]}{n}$, one can solve for $\mathbf{q}^*$ by finding the eigenvectors of $\mathbf{A}$, and then compute $\mathbf{q}^*\boldsymbol{\pi}$, which equals to $\lim_{n\to\infty} \frac{[\mathscr{C}_n]}{n}$. This provides another method

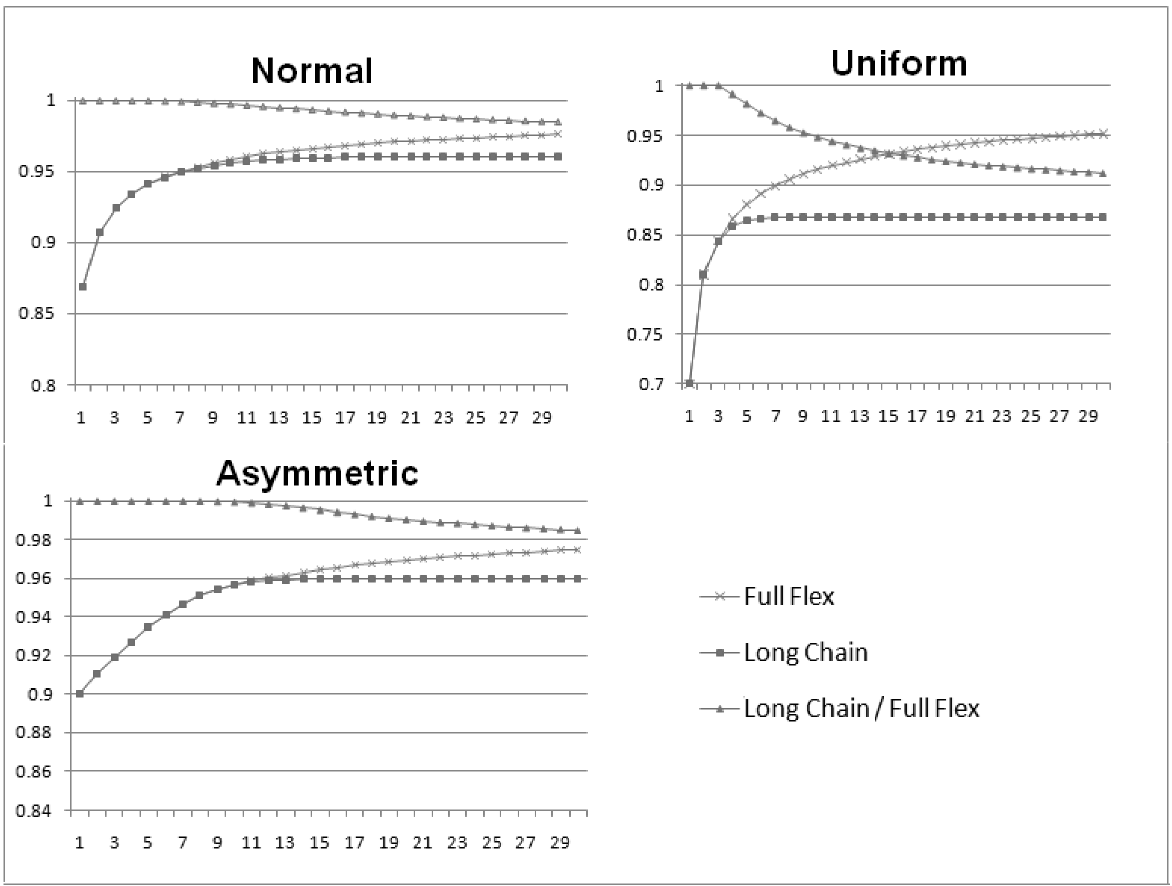


Figure 2-4: The Performance of Long Chains vs. The Performance of Full Flexibility

for computing $\lim_{n\to\infty} \frac{[\mathscr{C}_n]}{n}$ in addition to the result of Chou et al. [Chou et al., 2010c]. Interestingly, their procedure also involves discretizing demand and solving a system of linear equations.

2.4 Generalizations

In this section, we present a generalization of the results derived in Sections 2.1-2.3. In the generalization, we study a model where the firm maximizes its profit under a flexibility structure $\mathscr{A}$, instead maximizing its sales. We use $\tilde{P}(\mathbf{d}, \mathscr{A})$ to denote the profit under a flexibility structure $\mathscr{A}$ and demand instance $\mathbf{d}$. In particular,

$$\begin{aligned}
\tilde{P}(\mathbf{d}, \mathscr{A}) = \max \quad & \sum_{(i,j)\in I(\mathscr{A})} p_{ij} f_{ij} \\
\text{s.t.} \quad & \sum_{a_i \in N(b_j, \mathscr{A})} f_{ij} \le d_j, \, \forall j \in I(B) \\
& \sum_{b_j \in N(a_i, \mathscr{A})} f_{ij} \le c_i, \, \forall i \in I(A) \\
& 0 \le f_{ij} \le u_{ij}, \forall (i,j) \in I(\mathscr{A}) \\
& \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}.
\end{aligned}$$

$\tilde{P}(\mathbf{d}, \mathscr{A})$ can be interpreted as the maximum profit achieved by flexibility structure $\mathscr{A}$ given demand $\mathbf{d} \in \mathbb{R}^n_+$, with fixed plant capacity vector $\mathbf{c} \in \mathbb{R}^n_+$, linear profit vector $\mathbf{p} \in \mathbb{R}^{|\mathscr{A}|}_+$ and a flexibility capacity vector $\mathbf{u} \in \mathbb{R}^{|\mathscr{A}|}_+$ (that is, the production for each arc (i, j) in $\mathscr{A}$ is bounded by u_{ij}).

2.4.1 Generalized Supermodularity

For any flexibility structure $\mathscr{A}$, and any arcs $\alpha, \beta \in \mathscr{A}$, define

$$
\begin{aligned}
\tilde{P}_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A}) = \max \quad & \sum_{(i,j)\in I(\mathscr{A})} p_{ij} f_{ij} \\
\text{s.t.} \quad & \sum_{a_i \in N(b_j,\mathscr{A})} f_{ij} \le d_j, \ \forall j \in I(B) \\
& \sum_{b_j \in N(a_i,\mathscr{A})} f_{ij} \le c_i, \ \forall i \in I(A) \\
& f_\alpha \le u_\alpha, \quad f_\beta \le u_\beta, \\
& 0 \le f_{ij} \le u_{ij}, \forall (i,j) \in I(\mathscr{A}) \\
& \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}.
\end{aligned}
$$

Similarly to our analysis in Section 2.1, we can show that the optimization problem corresponding to $\tilde{P}_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is equivalent to a max-weight circulation problem with underlying graph $G(\mathscr{A})$. Indeed, it is easy to check that the equivalence hold, if we set the weight of each arc (a_i, b_j) in $\mathscr{A}$ to p_{ij}; the weight of every other arc to zero; the upper-bound (capacity) on the flow of each arc from s to plant i to c_i; the upper-bound on the flow of each arc connecting product j to s to d_j; the upper-bound on the flow of each arc (a_i, b_j) in $\mathscr{A}$ to u_{ij}; and the lower-bound on the flow of each arc in $G(\mathscr{A})$ to 0.

Because $\tilde{P}_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is equivalent to a max-weight circulation problem with underlying graph $G(\mathscr{A})$, and any pair of flexible arcs α and β in $\mathscr{C}_n$ are in series with each other, we have that α and β are supermodular with each other. Formally, we have the following statement.

Corollary 2.2. *Let $\mathscr{A}$ be a flexibility structure for a balanced system of size n, and $\mathscr{A} \subset \mathscr{C}_n$. For any flexible arcs α, β in $\mathscr{A}$, $\tilde{P}_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ is supermodular in u_α and u_β. Hence,*

$$\tilde{P}(\mathbf{d}, \mathscr{A}) + \tilde{P}(\mathbf{d}, \mathscr{A} \setminus \{\alpha, \beta\}) \ge \tilde{P}(\mathbf{d}, \mathscr{A} \setminus \{\alpha\}) + \tilde{P}(\mathbf{d}, \mathscr{A} \setminus \{\beta\}).$$

Corollary 2.2 states that under the more general profit model, the supermodularity of flexible arcs in the long chain still holds. Furthermore, we would like to note that the supermodularity of flexible arcs in $\mathscr{C}_n$ can be extended to even more general settings where the parameter on each arc in $\mathscr{C}_n$ lies in a lattice. For these extensions, we refer the readers to the work of Granot and Veinott [Granot and Veinott Jr., 1985].

2.4.2 Generalized Decomposition

In this subsection, we assume that for each $1 \leq i \leq n$, $p_{ii} \geq p_{ji}$ for all j, and $u_{ii} = c_i$. Intuitively, one can think of this assumption as having plant i to be the *primary* production plant for product i, where $p_{ii} \geq p_{ji}$ implies that it is cheaper to produce product i from plant i, and $u_{ii} = c_i$ implies that there is no real constraint on producing product i from plant i. Under this assumption, we present a corollary that decomposes $\tilde{P}(\mathbf{d}, \mathscr{C}_n)$, i.e. the profit of $\mathscr{C}_n$ under $\mathbf{d}$. For the sake of simplicity, in this subsection, we will use (i, j) in place of (a_i, b_j), for each $(a_i, b_j) \in \mathscr{C}_n$.

Corollary 2.3. *Suppose $p_{ii} \geq p_{ii+1}$ and $u_{ii} = c_i$ $\forall 1 \leq i \leq n$, we have*

$$\tilde{P}(\mathbf{d}, \mathscr{C}_n) = \sum_{i=1}^{n} \left(\tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_i\}) - \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i-1}, \beta_i, \alpha_i\}) \right).$$

Proof. The strategy of the proof is very similar to the proof of Theorem 2.2. First, we use an augmenting cycle argument similarly to the one applied in Lemma 2.3 in Section 2.2 to show that $\tilde{P}(\mathbf{d}, \mathscr{C}_n) = \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \alpha_{i^*})$, for some $1 \leq i^* \leq n$.

Let $\mathbf{f}^*$ be an optimal solution of $\tilde{P}(\mathbf{d}, \mathscr{C}_n)$. If $f^*_{\alpha_{i^*}} = 0$ for some i^*, then $\tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\}) = \tilde{P}(\mathbf{d}, \mathscr{C}_n)$. Otherwise, $f^*_{ij} > 0$ for all $(i, j) \in \{\alpha_1, ..., \alpha_n\}$, and let $\mathbf{g}$ be the vector that

$$g_{ij} = \begin{cases} -1 & \text{if } (i,j) \in \{\alpha_1, ..., \alpha_n\} \\ 1 & \text{if } (i,j) \in \{\beta_1, \beta_2, ..., \beta_n\} \\ 0 & \text{otherwise.} \end{cases}$$

Since $p_{\beta_k} \geq p_{\alpha_k}$, $\sum_{(i,j) \in \mathscr{C}_n} g_{ij} p_{ij} \geq 0$ and thus $\mathbf{g}$ is an *augmenting cycle* for $\mathbf{f}^*$. Let $\delta^* = \min\{f_{ij}, (i,j) \in \{\alpha_1, ..., \alpha_n\}\}$. Note that $\mathbf{f}^* + \delta^* \mathbf{g}$ is a feasible and optimal solution of

$\tilde{P}(\mathbf{d}, \mathscr{C}_n)$. Moreover, $f^*_{ij} + \delta^* g_{ij} = 0$ for some $(i, j) \in \{\alpha_1, ..., \alpha_n\}$. Thus, there is some i^* such that $f^*_{\alpha_{i^*}} + \delta^* g_{\alpha_{i^*}} = 0$ and this implies that $\tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\}) = \tilde{P}(\mathbf{d}, \mathscr{C}_n)$.

Applying Corollary 2.2, we have

$$\tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha_{i^*}\})) + \tilde{P}(\mathbf{d}, \mathscr{C}_n) \geq \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus X) + \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\})$$
$$\implies \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha_{i^*}\})) \geq \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus X), \quad \text{since } \tilde{P}(\mathbf{d}, \mathscr{C}_n) = \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus \{\alpha_{i^*}\}).$$

But by definition $\tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha_{i^*}\})) \leq \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus X)$, hence

$$\tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus (X \cup \{\alpha_{i^*}\})) = \tilde{P}(\mathbf{d}, \mathscr{C}_n \setminus X).$$

Now that we have established the generalized version of Lemma 2.2 and 2.3, the rest of the proof can then be completed following the same procedure as the proof of Theorem 2.2. Like Theorem 2.2, for each $1 \leq k_1 \leq k_2 \leq n$, define $\mathscr{L}_{k_1 \to k_2} = \{\beta_i | i = k_1, k_1 + 1, ..., k_2\} \cup \{\alpha_i | i = k_1, k_1 + 1, ..., k_2 - 1\}$, and for each $1 \leq k_2 < k_1 \leq n$, define $\mathscr{L}_{k_1 \to k_2} = \{\beta_i | i = k_1, k_1 + 1 ..., n, 1, 2, ..., k_2\} \cup \{\alpha_i | i = k_1, ..., n, ..., k_2 - 1\}$. Also, use $\tilde{P}(\mathscr{A})$ to denote $\tilde{P}(\mathbf{d}, \mathscr{A})$.

For any $1 \leq i \leq n$, because $\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i\} = \{\beta_i\} \uplus \mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\}$, where $\uplus$ represent the symbol for disjoint union,

$$\begin{aligned} &\tilde{P}(\mathscr{C}_n \setminus \{\alpha_i\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\}) \\ =&\tilde{P}(\mathscr{C}_n \setminus \{\alpha_i\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i\}) + p_{ii} \min\{c_i, d_i\}. \end{aligned} \tag{2.6}$$

We have already shown that there is some i^* such that $\tilde{P}(\mathscr{C}_n) = \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i^*}\})$.

Without loss of generality, we assume that $i^* = n$, and we have that for $i = 2, ..., n-1$,

$$
\begin{aligned}
&\tilde{P}(\mathscr{C}_n \setminus \{\alpha_i\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\}) \\
&= \tilde{P}(\mathscr{C}_n \setminus \{\alpha_i\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i\}) + p_{ii} \min\{c_i, d_i\} \\
&= \tilde{P}(\mathscr{C}_n \setminus \{\alpha_i, \alpha_n\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \alpha_n\}) + p_{ii} \min\{c_i, d_i\} \qquad \text{(by Lemma 2.2)} \\
&= \tilde{P}(\mathscr{L}_{1\to i}) + \tilde{P}(\mathscr{L}_{(i+1)\to n}) - \Big(\tilde{P}(\mathscr{L}_{1\to(i-1)}) + \tilde{P}(\mathscr{L}_{(i+1)\to n}) + p_{ii} \min\{c_i, d_i\}\Big) \\
&+ p_{ii} \min\{c_i, d_i\} \\
&= \tilde{P}(\mathscr{L}_{1\to i}) - \tilde{P}(\mathscr{L}_{1\to(i-1)}). \qquad (2.7)
\end{aligned}
$$

Also,

$$
\begin{aligned}
&\tilde{P}(\mathscr{C}_n \setminus \{\alpha_1\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_n, \alpha_1, \beta_1\}) \\
&= \tilde{P}(\mathscr{C}_n \setminus \{\alpha_1\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_n, \alpha_1\}) + p_{11} \min\{c_1, d_1\} \qquad \text{(by Equation (2.6))} \\
&= \tilde{P}(\mathscr{C}_n \setminus \{\alpha_1, \alpha_n\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_1, \alpha_n\}) + p_{11} \min\{c_1, d_1\} \qquad \text{(by Lemma 2.2)} \\
&= p_{11} \min\{c_1, d_1\}, \qquad (2.8)
\end{aligned}
$$

and

$$
\tilde{P}(\mathscr{C}_n \setminus \{\alpha_n\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{n-1}, \alpha_n, \beta_n\}) = \tilde{P}(\mathscr{L}_{1\to n}) - \tilde{P}(\mathscr{L}_{1\to(n-1)}). \qquad (2.9)
$$

Now, applying Equations (2.7-2.9), we obtain that

$$
\begin{aligned}
&\sum_{i=1}^{n} (\tilde{P}(\mathscr{C}_n \setminus \{\alpha_i\}) - \tilde{P}(\mathscr{C}_n \setminus \{\alpha_{i-1}, \alpha_i, \beta_i\})) \\
&= p_{11} \min\{c_1, d_1\} + \sum_{i=2}^{n} (\tilde{P}(\mathscr{L}_{1\to i}) - \tilde{P}(\mathscr{L}_{1\to(i-1)})) \\
&= p_{11} \min\{c_1, d_1\} + \tilde{P}(\mathscr{L}_{1\to n}) - \tilde{P}(\mathscr{L}_{1\to 1}) \\
&= \tilde{P}(\mathscr{L}_{1\to n}) \\
&= \tilde{P}(\mathscr{C}_n).
\end{aligned}
$$

□

2.4.3 Generalized Computation Method Under Independent Demand

Given any random demand distribution $\mathbf{D}$, we can integrate the equation presented in Corollary 2.3 over all instances of $\mathbf{D}$. This allows us to obtain a characterization for the expected profit of a long chain, which we present next.

Corollary 2.4. *Suppose $p_{ii} \geq p_{ii+1}$ and $u_{ii} = c_i$ $\forall 1 \leq i \leq n$, we have*

$$\mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n)] = \sum_{i=1}^{n} \Big(\mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n \setminus \{\alpha_i\})] - \mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n \setminus \{\alpha_{i-1}, \beta_i, \alpha_i\})] \Big).$$

Unfortunately, the greedy algorithm presented in Section 2.3 cannot be modified to compute the optimal solution of $\tilde{P}(\mathbf{d}, \mathscr{L}_n)$ in general. Thus we cannot formulate a similar algorithm to compute the expected profit for the general case. Nevertheless, the greedy algorithm does find the optimal solution of $\tilde{P}(\mathbf{d}, \mathscr{L}_n)$ successfully when its underlying optimization problem is unweighted, i.e. $p_{ij} = 1, \forall i, j$. That is, a greedy algorithm, similar Algorithm 1, can determine the optimal solution to the sales of an open chain, with plant capacity vector $\mathbf{c}$ and flexibility capacity vector $\mathbf{u}$.

The modified algorithm allows the same matrix multiplication method from Section 2.3 to be applied for computing the expected sales of long chain with independent but non-identical product demands, plant capacity $\mathbf{c}$ and flexibility capacity $\mathbf{u}$. In this case, similar to the multiplication procedures from Section 2.3, for each $1 \leq i \leq n$, $\mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n \setminus \{\alpha_i\})] - \mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n \setminus \{\alpha_{i-1}, \beta_i, \alpha_i\})]$ can be evaluated by computing $\mathbf{q}^0 \prod_{k=1}^{n-1} \mathbf{A}(\mathbf{k})$ for $n-1$ different matrices $\mathbf{A}(\mathbf{1}), \mathbf{A}(\mathbf{2}), ..., \mathbf{A}(\mathbf{n-1})$. Suppose we scale $\mathbf{c}$, $\mathbf{u}$ and $\mathbf{D}$ such that all entries of $\mathbf{c}$, $\mathbf{u}$ and the support of $\mathbf{D}$ are all integers, and let N denote the largest entry of $\mathbf{c}$. Then, we have that computing $\mathbf{q}^0 \prod_{k=1}^{n-1} \mathbf{A}(\mathbf{k})$ requires $O(nN^2)$ operations, while computing the sum of $\mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n \setminus \{\alpha_i\})] - \mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{C}_n \setminus \{\alpha_{i-1}, \beta_i, \alpha_i\})]$, for $1 \leq i \leq n$ requires $O(n^2N^2)$ operations.

Finally, we note that given an optimization problem $\tilde{P}(\mathbf{d}, \mathscr{A})$, the optimal flow on arc (a_i, b_j), f_{ij}^*, can be determined by greedy algorithm, for any (a_i, b_j) that is a *leaf arc*, where either a_i or b_j is incident to only one arc. Therefore, for any flexibility structure $\mathscr{A}$ such that any component in $\mathscr{A}$ contains only one cycle, we can sequentially allocate the production on the leaf arcs until all remaining arcs form several disjoint cycles, and then apply Algorithm 1 on each of those cycles. This implies that for any $\mathscr{A}$ where components in $\mathscr{A}$ contains only one cycle, we can compute $\mathbb{E}[\tilde{P}(\mathbf{D}, \mathscr{A})]$.

2.5 Conclusion

In this chapter, we develop two deterministic results that contribute to the existing literature on process flexibility. First, in Section 2.1, we derive the supermodularity of flexible arcs, which states that any two flexible arcs in a long chain are supermodular with each other. Then, in Section 2.2, using the supermodularity result as a key lemma, we derive a decomposition for the sales of the $\mathscr{C}_n$ as a sum of n quantities, where each quantity is equal to the difference between the sales of two open chains. Interestingly, while the decomposition is a deterministic result itself, in Section 2.3, we use it to establish to an efficient algorithm for computing the expected sales of the long chain, when demand is stochastic.

Most of the results developed in the chapter can be used to study process flexibility under more general settings. For example, in Section 2.4, we show that the supermodularity and decomposition of a long chain can be extended to a more general setting where (i) plants have different capacities; (ii) sales is replaced by linear profit which depends on a plant-product combination; (iii) each plant-product combination has a given capacity limit.

Finally, it turns out that the supermodularity and decomposition results not only lead to an efficient algorithm for computing the expected sales of the long chain, but also allows us to analyze the long chain rigorously, and thus derive useful qualitative insights. This is discussed in the next chapter.

Chapter 3

Effectiveness of Long Chain under Stochastic Demand

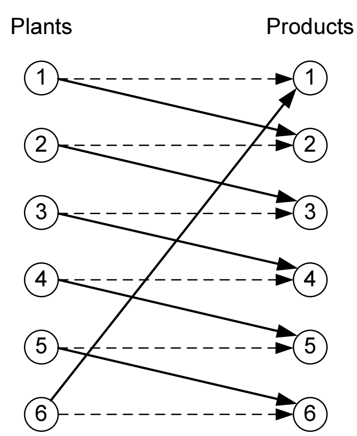
In this chapter, we focus on developing new theory to explains the effectiveness of the long chain. Utilizing the the two main results from Chapter 2, supermodularity and decomposition, we derive several theoretical properties to understand the strength and limitations of the long chain. In Section 3.1, we illustrate the importance of "closing the chain", by proving that as a long chain is constructed, the incremental benefits of adding flexibilities are increasing and the largest benefit is always achieved when the chain is closed by adding the last arc to the system. In Section 3.2, we prove that the gap between the fill rate of full flexibility and that of the long chain increases with system size, thus implying that the effectiveness of the long chain relative to full flexibility increases as the number of products decreases. Finally, in Section 3.3, we derive a risk-pooling result implying that the fill rate of a long chain increases with the number of products, but this increase converges to zero exponentially fast.

Throughout the chapter, we assume the system is balanced, that is, the number of plants is equal to the number of products and $c_i = 1$ for $1 \leq i \leq n$. In Section 3.1, we assume the stochastic demand vector, $\mathbf{D}$, to be exchangeable. In Sections 3.2-3.3, we further restrict $\mathbf{D}$ to be a vector of IID entries, where each entry D_i has distribution D. For the sake of simplicity, we will sometimes use $[\mathscr{A}]$ in place of $\mathbb{E}[P(\mathbf{D}, \mathscr{A})]$, when there is no ambiguity on the distribution of $\mathbf{D}$.

3.1 Incremental Benefits of Constructing Long Chain

This section is motivated by an observation that has been made in the literature ([Graves, 2008] and [Hopp et al., 2004]) regarding the expected sales of the long chain for a balanced system when products demands are IID. The observation states that if one starts with a dedicated structure and adds arcs to create the long chain, *the incremental benefits*, or the change in expected sales, associated with each added arc *is increasing.*

To illustrate this observation, consider an example with 6 plants and 6 products, where the demand for each product is equal to either 0.8, 1 or 1.2 with equal probabilities, and the capacity of each plant is 1. Then, we start with a dedicated flexibility structure (the dashed arcs in Figure 3-1(a)), and add arcs $(1,2)$, $(2,3)$, ..., $(5,6)$ and $(6,1)$ one at a time, until we complete the long chain. Each time we add such an arc, we determine the expected sales associated with the resulting structure at that time. Figure 3-1(b) displays the expected sales of the flexibility structures at different stages, as well as the incremental benefits when a new arc is added. As you can see, the *incremental benefits increase as we add more arcs.* The biggest impact, surprisingly, occurs when we add the last arc and close the long chain.



(a) Flexibility Structure

Structure	Performance	Incr. Benefit
Dedicated	5.6	
Added arc $(1,2)$	5.622	0.022
Added arc $(2,3)$	5.652	0.030
Added arc $(3,4)$	5.686	0.035
Added arc $(4,5)$	5.724	0.0379
Added arc $(5,6)$	5.765	0.0403
Added arc $(6,1)$	5.842	0.077
Full Flexibility	5.842	

(b) Incremental Benefits of Creating Long-Chain

Figure 3-1: The increase in Incremental Benefit

To formally prove this observation, we apply the supermodularity result we derived from Section 2.1. First, integrating over every demand instances over both side of the equation in Theorem 2.1, we get,

Corollary 3.1. *For any flexible arcs α, β in $\mathscr{A} \subset \mathscr{C}_n$, $\mathbb{E}[P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{D}, \mathscr{A})]$ is supermodular in u_α and u_β for any random demand $\mathbf{D}$.*

Next we apply Corollary 3.1 to formally prove the observation that the incremental benefits associated with adding arcs to the long chain is increasing. Consider the following sequence of flexibility structures: $\mathscr{L}_1^n, \mathscr{L}_2^n, \mathscr{L}_3^n, ..., \mathscr{L}_n^n, \mathscr{C}_n$, where we define $\mathscr{L}_1^n = \mathscr{D}_n$ and $\mathscr{L}_k^n = \mathscr{L}_k \cup \{(i,i) | i = k+1, ..., n\}$. In words, $\mathscr{L}_k^n$ is simply the open chain from plant 1 to product k plus the dedicated arcs connecting plants i to products i for all $k < i \leq n$. Finally, recall that $\mathscr{C}_n$ is the long chain of size n.

In the example of [Graves, 2008] and the table in Figure 3-1 of this chapter, one starts at $\mathscr{L}_1^n$ and add arcs sequentially to create $\mathscr{L}_2^n, .., \mathscr{L}_n^n, \mathscr{C}_n$. Now, we apply the supermodularity result to show that the incremental benefit, $[\mathscr{L}_k^n] - [\mathscr{L}_{k-1}^n]$, is nondecreasing with k.

Theorem 3.1. *For any balanced system of size n under exchangeable demand, we have*

$$[\mathscr{L}_2^n] - [\mathscr{L}_1^n] \leq [\mathscr{L}_3^n] - [\mathscr{L}_2^n] \leq ... \leq [\mathscr{L}_n^n] - [\mathscr{L}_{n-1}^n] \leq [\mathscr{C}_n] - [\mathscr{L}_n^n].$$

Proof. Proof of Theorem 3.1 Fix any $1 \leq k \leq n-1$. Let $\alpha = (1,2)$, $\beta = (k, k+1)$. By Corollary 3.1, we have

$$\begin{aligned} &\mathbb{E}[P_{\alpha,\beta}(1,1,\mathbf{D},\mathscr{L}_{k+1}^n)] + \mathbb{E}[P_{\alpha,\beta}(0,0,\mathbf{D},\mathscr{L}_{k+1}^n)] \\ \geq &\mathbb{E}[P_{\alpha,\beta}(1,0,\mathbf{D},\mathscr{L}_{k+1}^n)] + \mathbb{E}[P_{\alpha,\beta}(0,1,\mathbf{D},\mathscr{L}_{k+1}^n)] \end{aligned} \tag{3.1}$$

Setting $u_\alpha = 0$ is equivalent to deleting arc α in the optimization problem associated with $P_{\alpha,\beta}(u_\alpha, u_\beta, \mathbf{d}, \mathscr{A})$ while setting $u_\alpha = 1$ implies that this arc exists in the same model and its capacity is redundant. As a result, we have that,

$$\mathbb{E}[P_{\alpha,\beta}(1,1,\mathbf{D},\mathscr{L}_{k+1}^n)] = \mathbb{E}[P(\mathbf{D},\mathscr{L}_{k+1}^n)] = [\mathscr{L}_{k+1}^n], \tag{3.2}$$

$$\text{and} \quad \mathbb{E}[P_{\alpha,\beta}(1,0,\mathbf{D},\mathscr{L}_{k+1}^n)] = \mathbb{E}[P(\mathbf{D},\mathscr{L}_k^n)] = [\mathscr{L}_k^n]. \tag{3.3}$$

Let $\mathbf{D}_\sigma = [D_2, D_3, ..., D_n, D_1]$, then

$$\mathbb{E}[P_{\alpha,\beta}(0,0,\mathbf{D},\mathscr{L}^n_{k+1})] = \mathbb{E}[P(\mathbf{D}_\sigma,\mathscr{L}^n_{k-1})] = \mathbb{E}[P(\mathbf{D},\mathscr{L}^n_{k-1})] = [\mathscr{L}^n_{k-1}], \quad (3.4)$$

$$\text{and} \quad \mathbb{E}[P_{\alpha,\beta}(0,1,\mathbf{D},\mathscr{L}^n_{k+1})] = \mathbb{E}[P(\mathbf{D}_\sigma,\mathscr{L}^n_{k})] = \mathbb{E}[P(\mathbf{D},\mathscr{L}^n_{k})] = [\mathscr{L}^n_{k}], \quad (3.5)$$

where the second to last equality in (3.4) and (3.5) holds since the random vector $\mathbf{D}$ is exchangeable. Substituting Equations (3.2-3.5) into Inequality (3.1), we obtain that $[\mathscr{L}^n_{k+1}] - [\mathscr{L}^n_k] \geq [\mathscr{L}^n_k] - [\mathscr{L}^n_{k-1}]$, for $k = 2, .., n-1$.

Finally, to show $[\mathscr{L}^n_n] - [\mathscr{L}^n_{n-1}] \leq [\mathscr{C}_n] - [\mathscr{L}^n_n]$, let $\alpha = (1,2)$, $\beta = (n,1)$ and let $\mathbf{D}_\sigma = [D_2, D_3, ..., D_n, D_1]$. Then

$$\begin{aligned}
&\mathbb{E}[P_{\alpha,\beta}(1,1,\mathbf{D},\mathscr{C}_n)] = [\mathscr{C}_n],\\
&\mathbb{E}[P_{\alpha,\beta}(1,0,\mathbf{D},\mathscr{C}_n)] = [\mathscr{L}^n_n],\\
&\mathbb{E}[P_{\alpha,\beta}(0,0,\mathbf{D},\mathscr{C}_n)] = \mathbb{E}[P(\mathbf{D}_\sigma,\mathscr{L}^n_{n-1})] = [\mathscr{L}^n_{n-1}],\\
\text{and} \quad &\mathbb{E}[P_{\alpha,\beta}(0,1,\mathbf{D},\mathscr{C}_n)] = \mathbb{E}[P(\mathbf{D}_\sigma,\mathscr{L}^n_{n})] = [\mathscr{L}^n_{n}].
\end{aligned}$$

Since by Corollary 3.1,

$$\mathbb{E}[P_{\alpha,\beta}(1,1,\mathbf{D},\mathscr{C}_n)] + \mathbb{E}[P_{\alpha,\beta}(0,0,\mathbf{D},\mathscr{C}_n)] \geq \mathbb{E}[P_{\alpha,\beta}(1,0,\mathbf{D},\mathscr{C}_n)] + \mathbb{E}[P_{\alpha,\beta}(0,1,\mathbf{D},\mathscr{C}_n)],$$

we have that $[\mathscr{C}_n] - [\mathscr{L}^n_n] \geq [\mathscr{L}^n_n] - [\mathscr{L}^n_{n-1}]$. This completes the proof. □

Note that any IID demand $\mathbf{D}$ must also be exchangeable, but not all exchangeable demands are IID. Thus, the statement of Theorem 3.1 is in fact more general than the observation we mentioned at the beginning of he section.

Interestingly, observe that the proof of Theorem 3.1 requires the application of the supermodularity result (Theorem 2.1), which holds deterministically for any fixed demand instance. By contrast, Theorem 3.1 holds only stochastically under exchangeable demand but does not hold for any fixed demand instance.

3.1.1 Optimality

While the optimality of the long chain among all 2-flexibility structures has been long observed (see [Jordan and Graves, 1995]), the observation has not been theoretically justified. With Theorem 2.3 and 3.1, we can now prove the optimality of the long chain under exchangeable demand.

Corollary 3.2. *Consider a balanced system of size n under some exchangeable demand $\mathbf{D}$. Let $\mathbb{F}_2$ be the set of all 2-flexibility structures of the system. That is, $\mathbb{F}_2$ is the set of all flexibility structures where each plant node and each product node are incident to exactly two arcs. Then, we have*

$$[\mathscr{C}_n] = \arg\max_{\mathscr{A}\in\mathbb{F}_2}[\mathscr{A}].$$

In words, the long chain maximizes expected sales among all 2-flexibility structures in the system.

Proof. Consider a 2-flexibility structure $\mathscr{A} \in \mathbb{F}_2$. $\mathscr{A}$ must consists of several closed chains (i.e. induced subgraphs in $\mathscr{A}$ which form undirected cycles) denoted by $SC_1, SC_2, .., SC_k$. Let n_i be the number of products and plants in the closed chain SC_i. Since the system size is n, $\sum_{i=1}^{k} n_i = n$. Now, by Theorem 2.3, we have

$$\begin{aligned}
[\mathscr{A}] &= \sum_{i=1}^{k} n_i([\mathscr{L}_{n_i}] - [\mathscr{L}_{n_i-1}]) \\
&= \sum_{i=1}^{k} n_i([\mathscr{L}^n_{n_i}] - [\mathscr{L}^n_{n_i-1}] + \mathbb{E}[\min\{1, D_1\}]) \quad \text{(By definition of } \mathscr{L}^n_k) \\
&\le \sum_{i=1}^{k} n_i([\mathscr{L}^n_{n}] - [\mathscr{L}^n_{n-1}] + \mathbb{E}[\min\{1, D_1\}]) \quad \text{(by Theorem 3.1)} \\
&= \sum_{i=1}^{k} n_i([\mathscr{L}_{n}] - [\mathscr{L}_{n-1}]) \\
&= n([\mathscr{L}_{n}] - [\mathscr{L}_{n-1}]) = [\mathscr{C}_n].
\end{aligned}$$

□

3.2 Long Chain vs. Full Flexibility

In Figure 2-4, it was observed that the gap between the fill rates of full flexibility and that of the long chain, $\frac{[\mathscr{F}_n]}{n} - \frac{[\mathscr{C}_n]}{n}$, is increasing, while the ratio, $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]}$ is decreasing. In this section, we will formally prove the first part of the observation, and discuss some partial results related to the second part. We start by defining two random walks in Section 3.2.1. These random walks are applied to analyze the difference between the fill rates of the long chain and full flexibility in Section 3.2.2, as well as the ratio of the fill rate of long chain to that of full flexibility in Section 3.2.3.

3.2.1 Random Walks

We define two random walks, W_i and $\tilde{W}_i$, as follows:

Definition 3.1. *Let $W_0 = \tilde{W}_0 = 0$. For $i \geq 1$, define*

$$W_i = \begin{cases} 0 & \text{if } W_{i-1} + 1 - D_i < 0 \\ 1 & \text{if } W_{i-1} + 1 - D_i > 1 \\ W_{i-1} + 1 - D_i & \text{otherwise} \end{cases}$$

$$\tilde{W}_i = \begin{cases} 0 & \text{if } \tilde{W}_{i-1} + 1 - D_i < 0 \\ \tilde{W}_{i-1} + 1 - D_i & \text{otherwise} \end{cases}$$

W_i and $\tilde{W}_i$ are generalized random walks with random steps $1 - D_1, .., 1 - D_i$ and different sets of reflecting boundaries. W_i has reflecting boundaries of 0 and 1, while $\tilde{W}_i$ has a reflecting boundary only at 0. For any fixed vector $\mathbf{d}$ that is an instance of $\mathbf{D}$, define $W_i(\mathbf{d})$ (and $\tilde{W}_i(\mathbf{d})$) to be the instance of W_i (and $\tilde{W}_i$) corresponding to $\mathbf{d}$. Figure 3-2 illustrate an example of $W_i(\mathbf{d})$ and $\tilde{W}_i(\mathbf{d})$, with $i = 6$ and $\mathbf{d} = [0.6, 0.2, 1.2, 1.9, 1.4, 0.5]$.

The next lemma states several simple observations regarding $W_i(\mathbf{d})$ and $\tilde{W}_i(\mathbf{d})$ for any fixed vector $\mathbf{d}$.

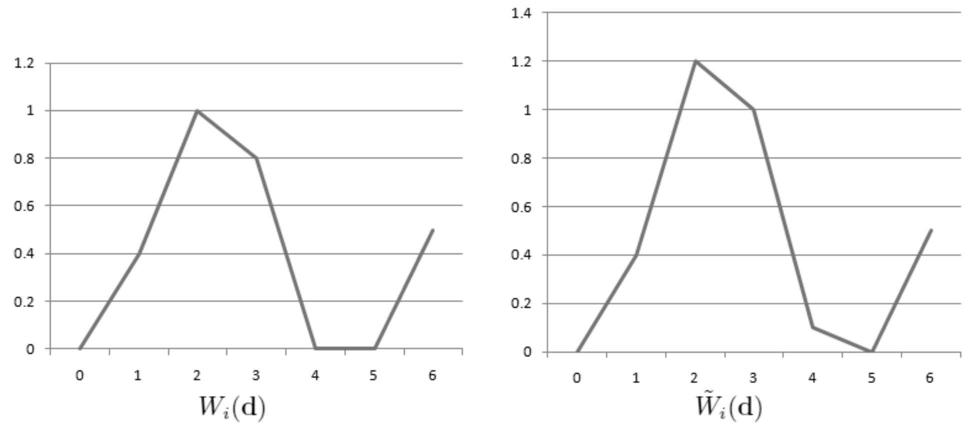


Figure 3-2: Illustration of $W_i(\mathbf{d})$ and $\tilde{W}_i(\mathbf{d})$

Lemma 3.1. *For any fixed vector* $\mathbf{d}$ *and* $i \geq j \geq 1$,

$$W_i(\mathbf{d}) \leq \tilde{W}_i(\mathbf{d}), \tag{3.6}$$

$$W_{i-j}(\mathbf{d}^j) \leq W_i(\mathbf{d}), \tag{3.7}$$

$$\text{and} \quad \tilde{W}_{i-j}(\mathbf{d}^j) \leq \tilde{W}_i(\mathbf{d}), \tag{3.8}$$

where $\mathbf{d}^j = [d_j, d_{j+1}, ...]$.

Proof. Since $\tilde{W}_i$ has no reflecting boundary at 1, $W_i(\mathbf{d}) \leq \tilde{W}_i(\mathbf{d})$. For Equation (3.7), observe that $W_{i-j}(\mathbf{d}^j)$ has the same step lengths as the last $i-j$ steps of $W_i(\mathbf{d})$. Since $W_j(\mathbf{d})$, the position of the random walk $W_i(\mathbf{d})$ after j steps, is greater or equal to 0, we have that $W_{i-j}(\mathbf{d}^j) \leq W_i(\mathbf{d})$. Similarly, we can also show that $\tilde{W}_{i-j}(\mathbf{d}^j) \leq \tilde{W}_i(\mathbf{d})$. □

The rest of this subsection establishes the connections between these random walks and sales (and expected sales) of the long chain and full flexibility. These connections would be used for the comparisons between the long chain and full flexibility in Section 3.2.2.

Similar to Section 2.2, in the rest of this subsection, when some integer k appears in a statement, we will in fact be referring to some $i \in \{1, ..n\}$ *congruent* to k *modulo* n. First, we show the relationship between $P(\mathbf{d}, \mathscr{C}_n)$ and random walks on $\mathbf{d}$.

Lemma 3.2. *Let* $\mathbf{d}^{i|n} = [d_i, d_{i+1}, ..., d_n, d_1, ..., d_{i-1}]$*, then*

$$P(\mathbf{d}, \mathscr{C}_n) = \sum_{i=1}^{n} \min\{1 + W_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\}$$

Proof. For each $1 \le i \le n$, it is not difficult to check that $\min\{1 + W_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\}$ is equal to the quantity $f^*_{n,n} + f^*_{n-1,n}$ returned by Algorithm 1 on $P(\mathbf{d}^{i|n}, \mathscr{L}_n)$. By Algorithm 1's greedy property, $f^*_{n,n} + f^*_{n-1,n} = P(\mathbf{d}^{i|n}, \mathscr{L}_n) - P(\mathbf{d}^{i|n}, \mathscr{L}_{n-1})$, which implies $\min\{1 + W_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\} = P(\mathbf{d}^{i|n}, \mathscr{L}_n) - P(\mathbf{d}^{i|n}, \mathscr{L}_{n-1})$. Finally, by definition, $P(\mathbf{d}^{i|n}, \mathscr{L}_n) = P(\mathbf{d}, \mathscr{C}_n \setminus \{(i-1, i)\})$ and $P(\mathbf{d}^{i|n}, \mathscr{L}_{n-1}) = P(\mathbf{d}, \mathscr{C}_n \setminus \{(i-2, i-1), (i-1, i-1), (i-1, i)\})$ and hence

$$\min\{1+W_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\} = P(\mathbf{d}, \mathscr{C}_n \setminus \{(i-1, i)\}) - P(\mathbf{d}, \mathscr{C}_n \setminus \{(i-2, i-1), (i-1, i-1), (i-1, i)\}). \tag{3.9}$$

Substitute Equation (3.9) to Theorem 2.2, and we have

$$P(\mathbf{d}, \mathscr{C}_n) = \sum_{i=1}^{n} \min\{1 + W_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\}.$$

□

We note that a similar observation to Lemma 3.2 was stated in [Chou et al., 2010c]. In particular, Chou et.al observed that $\min\{1 + W_{n-1}(\mathbf{d}), d_n\} = P(\mathbf{d}, \mathscr{L}_n) - P(\mathbf{d}, \mathscr{L}_{n-1})$. Our Lemma 3.2 goes a step further by applying Theorem 2.2.

Establishing the relationships between $P(\mathbf{d}, \mathscr{F}_n)$ and the random walks on $\mathbf{d}$ is more difficult. We do this by proving a lemma which shows that the sales associated with $\mathscr{F}_n$ is equal to the sales of $\mathscr{C}_n$ under a new demand $\tau(\mathbf{d})$, which is a linear transformation of $\mathbf{d}$. Specifically, we define $\tau(d_i) = \frac{d_i + (n-1)}{n}$, for $i = 1, 2, ..., n$.

Lemma 3.3. *For any demand instance* $\mathbf{d}$*,*

$$P(\tau(\mathbf{d}), \mathscr{C}_n) = P(\tau(\mathbf{d}), \mathscr{F}_n),$$

where $\tau(d_i) = \frac{d_i + (n-1)}{n}$*, for* $i = 1, 2, ..., n$.

Proof. By duality of linear programs,

$$\begin{aligned} P(\tau(\mathbf{d}), \mathscr{C}_n) = \min \quad & \sum_{1\le i\le n} p_i + \sum_{1\le j\le n} q_j \tau(d_j) \\ \text{s.t.} \quad & p_i + q_j \ge 1,\ \forall (i,j) \in \mathscr{C}_n \\ & p_i \ge 0, q_j \ge 0,\ \forall 1 \le i \le n, 1 \le j \le n \\ & \mathbf{p}, \mathbf{q} \in \mathbb{R}^n. \end{aligned} \tag{VC}$$

The linear program denoted by (VC) is an LP-relaxation of a min-weight bipartite vertex cover problem. Since the LP-relaxation of min-weight bipartite vertex cover is tight, it has an optimal solution $(\mathbf{p}^*, \mathbf{q}^*)$ where entries in $\mathbf{p}^*$ and $\mathbf{q}^*$ are either 0 or 1. Let $S = \{i | p_i^* = 0\}$ and $S' = \{j | q_j^* = 1\}$. Note that $N(S) \subseteq S'$ where $N(S)$ is the set of neighbors of S in $\mathscr{C}_n$. First, suppose $S \neq \emptyset$, and $S' \neq \{1, 2, ..., n\}$. Then, we must have $|S'| - 1 \ge |N(S)| - 1 \ge |S|$. Let $p_i^0 = 1$, $q_j^0 = 0$ for all $1 \le i, j \le n$. Clearly $(\mathbf{p}^0, \mathbf{q}^0)$ is a feasible solution of (VC). Also as $|S'| - 1 \ge |S|$,

$$\begin{aligned} n &\le (n - |S|) + (|S'| - 1) \\ &< (n - |S|) + \sum_{j \in S'} \frac{(n-1)}{n} && (\text{since } |S'| < n) \\ &\le \sum_{1\le i\le n} p_i^* + \sum_{j \in S'} \frac{d_j + (n-1)}{n} && (\text{since } d_j \ge 0) \\ &= \sum_{1\le i\le n} p_i^* + \sum_{1\le j\le n} \tau(d_j) q_j^*. \end{aligned}$$

But $\sum_{1\le i\le n} p_i^0 + \sum_{1\le j\le n} \tau(d_j) q_j^0 = n$, and this contradicts the optimality of $(\mathbf{p}^*, \mathbf{q}^*)$. Thus, one must have that either $S = \emptyset$ or $S' = \{1, 2, ..., n\}$. Therefore, $P(\tau(\mathbf{d}), \mathscr{C}_n) = \min\{n, \sum_{1\le j\le n} \tau(d_j)\} = P(\tau(\mathbf{d}), \mathscr{F}_n)$. □

Now, we can prove the lemma which establishes the relations between $P(\mathbf{d}, \mathscr{F}_n)$ and $\tilde{W}$.

Lemma 3.4. *Let* $\mathbf{d}^{i|n} = [d_i, d_{i+1}, ..., d_n, d_1, ..., d_{i-1}]$*, then*

$$P(\mathbf{d}, \mathscr{F}_n) = \sum_{i=1}^{n} \min\{1 + \tilde{W}_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\}$$

Proof. By Lemma 3.2 and 3.3, $P(\tau(\mathbf{d}), \mathscr{F}_n) = \sum_{i=1}^{n} \min\{1 + W_{n-1}(\tau(\mathbf{d}^{i|n})), \tau(d_{i-1})\}$. Note that for any $1 \leq j \leq n-1$, and $1 \leq i \leq n$,

$$W_j(\tau(\mathbf{d}^{i|n})) \leq \sum_{i \leq k \leq i+j} \max\{0, 1 - \frac{d_k + (n-1)}{n}\} \leq (n-1) \cdot \frac{1}{n} < 1$$

This implies that $W_j(\tau(\mathbf{d}^{i|n}))$ never touches the reflecting boundary at 1. Hence,

$$W_{n-1}(\tau(\mathbf{d}^{i|n})) = \tilde{W}_{n-1}(\tau(\mathbf{d}^{i|n})) \quad \forall 1 \leq i \leq n. \tag{3.10}$$

Since $1 - \tau(d_k) = \frac{1}{n}(1 - d_k)$, for any $1 \leq k \leq n$, we have

$$\tilde{W}_j(\tau(\mathbf{d}^{i|n})) = \frac{1}{n}\tilde{W}_j(\mathbf{d}^{i|n}) \quad \forall 1 \leq j \leq n-1. \tag{3.11}$$

Thus,

$$\begin{aligned}
P(\tau(\mathbf{d}), \mathscr{F}_n) &= \sum_{i=1}^{n} \min\{1 + W_{n-1}(\tau(\mathbf{d}^{i|n})), \tau(d_{i-1})\} \\
&= \sum_{i=1}^{n} \min\{1 + \tilde{W}_{n-1}(\tau(\mathbf{d}^{i|n})), \tau(d_{i-1})\} && \text{(By Equation (3.10))} \\
&= \sum_{i=1}^{n} \min\{1 + \frac{\tilde{W}_{n-1}(\mathbf{d}^{i|n})}{n}, \frac{d_{i-1} + n - 1}{n}\} && \text{(By Equation (3.11))} \\
&= \sum_{i=1}^{n} (\frac{n-1}{n} + \min\{\frac{1 + \tilde{W}_{n-1}(\mathbf{d}^{i|n})}{n}, \frac{d_{i-1}}{n}\}) \\
&= n - 1 + \frac{1}{n} \sum_{i=1}^{n} \min\{1 + \tilde{W}_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\}.
\end{aligned}$$

On the other hand,

$$\begin{aligned} P(\tau(\mathbf{d}), \mathscr{F}_n) &= \min\{n, \sum_{1 \le i \le n} \frac{d_i + n - 1}{n}\} \\ &= n - 1 + \frac{1}{n} \min\{n, \sum_{1 \le i \le n} d_i\} \\ &= n - 1 + \frac{1}{n} P(\mathbf{d}, \mathscr{F}_n) \end{aligned}$$

Therefore, we have that

$$n - 1 + \frac{1}{n} \sum_{i=1}^{n} \min\{1 + \tilde{W}_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\}) = n - 1 + \frac{1}{n} P(\mathbf{d}, \mathscr{F}_n)$$

which implies $P(\mathbf{d}, \mathscr{F}_n) = \sum_{i=1}^{n} \min\{1 + \tilde{W}_{n-1}(\mathbf{d}^{i|n}), d_{i-1}\})$. □

Integrating the equations in Lemma 3.2 and Lemma 3.4 over all instances in **D**, we obtain the lemma below, which relates the expectation of the two random walks with the expected sales of the long chain and that of full flexibility.

Lemma 3.5. *Under IID demand, we have*

$$\begin{aligned} \frac{[\mathscr{C}_n]}{n} &= \mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}), D\}] \\ \frac{[\mathscr{F}_n]}{n} &= \mathbb{E}[\min\{1 + \tilde{W}_{n-1}(\mathbf{D}), D\}]. \end{aligned}$$

Proof. Since **D** is IID, we have that for any $1 \le i \le n$,

$$\begin{aligned} \mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}^{i|n}), D_{i-1}\}] &= \mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}), D\}] \\ \mathbb{E}[\min\{1 + \tilde{W}_{n-1}(\mathbf{D}^{i|n}), D_{i-1}\}] &= \mathbb{E}[\min\{1 + \tilde{W}_{n-1}(\mathbf{D}), D\}]. \end{aligned}$$

After integrating the equations in Lemma 3.2 and Lemma 3.4 over **D**, we get

$$\begin{aligned} [\mathscr{C}_n] &= n\mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}), D\}] \\ [\mathscr{F}_n] &= n\mathbb{E}[\min\{1 + \tilde{W}_{n-1}(\mathbf{D}), D\}]. \end{aligned}$$

□

3.2.2 Difference in Fill Rates

With Lemma 3.5 at hand, we now prove that the quantity $\frac{[\mathscr{F}_n]}{n} - \frac{[\mathscr{C}_n]}{n}$ is nondecreasing with n.

Theorem 3.2. *For any integer $n \geq 2$ and IID demand,*

$$\frac{[\mathscr{F}_n]}{n} - \frac{[\mathscr{C}_n]}{n} \leq \frac{[\mathscr{F}_{n+1}]}{n+1} - \frac{[\mathscr{C}_{n+1}]}{n+1} \leq \min\{1, \mathbb{E}[D]\} - \gamma,$$

where $\gamma = \lim_{k\to\infty} \frac{[\mathscr{C}_k]}{k}$.

Proof. First, we show that for any demand instance $\mathbf{d}$ of $\mathbf{D}$, we have,

$$\begin{aligned} &\min\{1 + \tilde{W}_{n-1}(\mathbf{d}^2), d_{n+1}\} - \min\{1 + W_{n-1}(\mathbf{d}^2), d_{n+1}\} \\ \leq &\min\{1 + \tilde{W}_n(\mathbf{d}), d_{n+1}\} - \min\{1 + W_n(\mathbf{d}), d_{n+1}\}, \end{aligned} \tag{3.12}$$

where $\mathbf{d}^2 = [d_2, d_3, ...]$. One can think of $W_{n-1}(\mathbf{d}^2)$ (and $\tilde{W}_{n-1}(\mathbf{d}^2)$) as the walk which started one time unit later than $W_n(\mathbf{d})$ (and $\tilde{W}_n(\mathbf{d})$). To prove Inequality (3.12), consider the following two cases.

Case 1: $W_i(\mathbf{d}^2) = 1$ for some $1 \leq i \leq n-1$. Then $W_i(\mathbf{d}^2) = W_{i+1}(\mathbf{d}) = 1$, and by the definition of W, we must have that $W_{n-1}(\mathbf{d}^2) = W_n(\mathbf{d})$. By Lemma 3.1, $\tilde{W}_{n-1}(\mathbf{d}^2) \leq \tilde{W}_n(\mathbf{d})$, and therefore we have

$$\begin{aligned} &\min\{1 + W_{n-1}(\mathbf{d}^2), d_{n+1}\} = \min\{1 + W_n(\mathbf{d}), d_{n+1}\} \\ &\min\{1 + \tilde{W}_{n-1}(\mathbf{d}^2), d_{n+1}\} \leq \min\{1 + \tilde{W}_n(\mathbf{d}), d_{n+1}\}, \end{aligned}$$

which implies that Inequality (3.12) holds.

Case 2: $W_i(\mathbf{d}^2) < 1$ for all $1 \leq i \leq n-1$. By definition of W and $\tilde{W}$, we have that $W_{n-1}(\mathbf{d}^2) = \tilde{W}_{n-1}(\mathbf{d}^2)$. By Lemma 3.1, $W_n(\mathbf{d}) \leq \tilde{W}_n(\mathbf{d})$, and therefore we have

$$\begin{aligned}
\min\{1 + W_{n-1}(\mathbf{d}^2), d_{n+1}\} &= \min\{1 + \tilde{W}_{n-1}(\mathbf{d}), d_{n+1}\} \\
\min\{1 + W_n(\mathbf{d}), d_{n+1}\} &\leq \min\{1 + \tilde{W}_n(\mathbf{d}), d_{n+1}\},
\end{aligned}$$

which again implies that Inequality (3.12) holds.

Since demand is IID, after integrating over $\mathbf{D}$ for both sides of Inequality (3.12), we get

$$\begin{aligned}
&\mathbb{E}[\min\{1 + \tilde{W}_{n-1}(\mathbf{D}), D_n\}] - \mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}), D_n\}] \\
&\leq \mathbb{E}[\min\{1 + \tilde{W}_n(\mathbf{D}), D_{n+1}\}] - \mathbb{E}[\min\{1 + W_n(\mathbf{D}), D_{n+1}\}].
\end{aligned}$$

Applying Lemma 3.5, we have that

$$\frac{[\mathscr{F}_n]}{n} - \frac{[\mathscr{C}_n]}{n} \leq \frac{[\mathscr{F}_{n+1}]}{n+1} - \frac{[\mathscr{C}_{n+1}]}{n+1}, \qquad \text{for all } n \geq 2. \tag{3.13}$$

Finally, Equation (3.13) implies that

$$\begin{aligned}
\frac{[\mathscr{F}_n]}{n} - \frac{[\mathscr{C}_n]}{n} &\leq \lim_{k\to\infty} (\frac{[\mathscr{F}_k]}{k} - \frac{[\mathscr{C}_k]}{k}) \\
&= \lim_{k\to\infty} (\mathbb{E}[\min\{\frac{\sum_{i=1}^{k} D_i}{k}, 1\}]) - \gamma \\
&= \min\{\mathbb{E}[D], 1\} - \gamma,
\end{aligned}$$

where the last equality holds because of the weak law of large numbers. □

Note that the fill rate of $\mathscr{C}_n$ (and $\mathscr{F}_n$) is equal to $\frac{[\mathscr{C}_n]}{n\mathbb{E}[D]}$ (and $\frac{[\mathscr{F}_n]}{n\mathbb{E}[D]}$). Thus, Theorem 3.2 implies that the smaller the system size, the smaller the gap between the fill rate of full flexibility and that of the long chain. This suggests that the long chain is more effective relative to full flexibility for smaller size systems. Moreover, Theorem 3.2 can be used to bound the gap between the fill rate of full flexibility and that of the long chain for systems of *any size*. For this purpose, we point out that [Chou et al., 2010c] shows that for many IID demand with $\mathbb{E}[D] = 1$, γ is close to one, implying that for any size system, the expected sales of the long chain is close to that of full flexibility.

For example, when D is normal with mean 1 and standard deviation of 0.33. Chou et al. [Chou et al., 2010c] showed that in this case $\gamma = 0.96$. Therefore, for this demand distribution, we have that the gap between the fill rate of full flexibility and that of the long chain for systems of any size is at most 4%.

3.2.3 Performance Ratio

The difference between the fill rate of the long chain and that of full flexibility is only one metric to evaluate the effectiveness of the long chain. A different metric, discussed in [Chou et al., 2008] and [Chou et al., 2010c], is to consider the ratio of the expected sales of long chain to that of full flexibility. Partial results related to the ratio are discussed next.

Applying the same argument as in the proof of Theorem 3.2, one can show that

$$\frac{\min\{1 + W_{n-1}(\mathbf{d}^2), d_{n+1}\}}{\min\{1 + \tilde{W}_{n-1}(\mathbf{d}^2), d_{n+1}\}} \geq \frac{\min\{1 + W_n(\mathbf{d}), d_{n+1}\}}{\min\{1 + \tilde{W}_n(\mathbf{d}), d_{n+1}\}}.$$

Unfortunately, one cannot integrate this inequality over $\mathbf{D}$ to obtain $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \frac{[\mathscr{C}_{n+1}]}{[\mathscr{F}_{n+1}]}$, as expectation does not preserve over multiplication and division. Indeed, it is not known whether $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \frac{[\mathscr{C}_{n+1}]}{[\mathscr{F}_{n+1}]}$ holds, although this inequality has been observed empirically in [Chou et al., 2008] and in this chapter, see Figure 2-4. Note that $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \frac{[\mathscr{C}_{n+1}]}{[\mathscr{F}_{n+1}]}$ for all $n \geq 2$ implies Theorem 3.2 and hence is a stronger statement.

Of course, if $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \frac{[\mathscr{C}_{n+1}]}{[\mathscr{F}_{n+1}]}$ holds, then it follows that

$$\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \lim_{k\to\infty} \frac{[\mathscr{C}_k]/k}{[\mathscr{F}_k]/k} = \lim_{k\to\infty} \frac{\gamma}{\min\{\mathbb{E}[D], 1\}}, \quad \forall n \geq 2. \tag{3.14}$$

where as before $\gamma = \lim_{k\to\infty} \frac{[\mathscr{C}_k]}{k}$, and $\lim_{k\to\infty} \frac{[\mathscr{F}_k]}{k} = \min\{\mathbb{E}[D], 1\}$ by the weak law of large numbers. This would provide a lower-bound on the ratio of the expected sales of the long chain to that of full flexibility for any system size.

Again, using the example from [Chou et al., 2010c], $\gamma = 0.96$ when D is normal with mean 1 and standard deviation of 0.33. Thus, if Inequality (3.14) holds, it would indicate that the expected sales of the long chain is at least 96% of that of full flexibility for any size system.

While we do not have a proof for $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \gamma$ when $\mathbb{E}[D] = 1$, we provide a lower-bound for $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]}$ that is almost equal to γ.

Corollary 3.3. *Suppose demand is IID and* $\mathbb{E}[D] = 1$, *then*

$$\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq 1 - \frac{(1-\gamma)n}{[\mathscr{F}_n]}$$

where $\gamma = \lim_{k\to\infty} \frac{[\mathscr{C}_k]}{k}$.

To explain the power of the lower-bound in Corollary 3.3, let $\delta_n = \frac{n}{[\mathscr{F}_n]} - 1$ which implies that $1 - \frac{(1-\gamma)n}{[\mathscr{F}_n]} = \gamma - \delta_n(1-\gamma)$. It can be shown that $\frac{n}{[\mathscr{F}_n]}$ is non-increasing with n (by applying for example Lemma 3.5), and hence, δ_n is non-increasing. Thus, if $\delta_k \approx 0$ for some small integer k, then Corollary 3.3 provides a lower-bound for $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]}$ that is close to γ for all $n \geq k$. Indeed, for many distributions with $\mathbb{E}[D] = 1$, $\delta_k \approx 0$ for small k. For example, suppose the distribution of D is normal with mean 1 and standard deviation 0.33, then $\frac{3}{[\mathscr{F}_3]} = 1.08$ which implies $\delta_3 = 0.08$ and [Chou et al., 2010c] shows that $\gamma = 0.96$. Applying Corollary 3.3, we have that

$$\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \gamma - \delta_3(1-\gamma) = 0.96 - 0.04 \times 0.08 = 0.9568, \quad \forall n \geq 3.$$

That is, when demand is normal with mean 1 and standard deviation 0.33, the long chain of any size greater than 2 achieves at least 95.68% of the expected sales of full flexibility. Next, we provide a proof for Corollary 3.3.

Proof of Corollary 3.3. By Theorem 3.2,

$$\frac{[\mathscr{F}_i]}{i} - \frac{[\mathscr{C}_i]}{i} \leq \frac{[\mathscr{F}_{i+1}]}{i+1} - \frac{[\mathscr{C}_{i+1}]}{i+1} \qquad \forall i \geq 1 \tag{3.15}$$

$$\implies \frac{[\mathscr{F}_i]}{i} - \frac{[\mathscr{F}_{i+1}]}{i+1} \leq \frac{[\mathscr{C}_i]}{i} - \frac{[\mathscr{C}_{i+1}]}{i+1} \qquad \forall i \geq 1 \tag{3.16}$$

Now, add Inequality (3.16) for all $i \geq n$, we have that

$$\frac{[\mathscr{F}_n]}{n} - \lim_{k\to\infty} \frac{[\mathscr{F}_k]}{k} \leq \frac{[\mathscr{C}_n]}{n} - \lim_{k\to\infty} \frac{[\mathscr{C}_k]}{k}. \tag{3.17}$$

But

$$\lim_{k\to\infty} \frac{[\mathscr{F}_k]}{k} = 1, \quad \lim_{k\to\infty} \frac{[\mathscr{C}_k]}{k} = \gamma,$$

and substituting those into Inequality (3.17), we have

$$\begin{aligned} \frac{[\mathscr{F}_n]}{n} - 1 &\leq \frac{[\mathscr{C}_n]}{n} - \gamma \\ \implies \frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} &\geq 1 - \frac{(1-\gamma)n}{[\mathscr{F}_n]}. \end{aligned}$$

□

3.3 Risk Pooling of Long Chain

In this section we focus on the per product expected sales (and fill rate, which is linearly proportion to the per product expected sales) of the long chain as a function of system size. We start by showing that $\frac{[\mathscr{C}_n]}{n}$ is nondecreasing with n under IID demand.

Theorem 3.3. *Under IID demand* $\mathbf{D}$*, we have* $\frac{[\mathscr{C}_n]}{n} \leq \frac{[\mathscr{C}_{n+1}]}{n+1}$*, for any integer* $n \geq 2$*.*

Proof. Since $\mathbf{D}$ is IID, the first n (and $n+1$) entries in $\mathbf{D}$ is exchangeable for balanced system of size n (and $n+1$). Thus, by Theorem 3.1, we have that $[\mathscr{L}_n^n] - [\mathscr{L}_{n-1}^n] \leq [\mathscr{L}_{n+1}^{n+1}] - [\mathscr{L}_n^{n+1}]$ which is equivalent to $[\mathscr{L}_n] - [\mathscr{L}_{n-1}] - \mathbb{E}[\min\{D,1\}] \leq [\mathscr{L}_{n+1}] - [\mathscr{L}_n] - \mathbb{E}[\min\{D,1\}]$ and hence implies $[\mathscr{L}_n] - [\mathscr{L}_{n-1}] \leq [\mathscr{L}_{n+1}] - [\mathscr{L}_n]$. Applying Theorem 2.3 completes the proof. □

The theorem thus states that $\frac{[\mathscr{C}_n]}{n}$, as well as the fill rate associated with a long chain, increases with the number of products, n. This phenomenon is analogous to the classical "risk-pooling" effect associated with demand aggregation, except that here we aggregate across capacities.

3.3.1 Exponential Convergence

Interestingly, Figure 2-4 not only suggests that $\frac{[\mathscr{C}_n]}{n}$ is nondecreasing with n, but also converges to a constant very quickly. This is shown in the next theorem, where we prove that the convergence rate is exponential for arbitrary IID, non-degenerate demands.

Theorem 3.4. *When demands are IID and non-degenerate, there exist constants $c < 0$ and $K > 0$ such that*

$$\frac{[\mathscr{C}_{n+1}]}{n+1} - \frac{[\mathscr{C}_n]}{n} \leq Ke^{cn},$$

for any $n \geq 2$.

Proof. From Lemma 3.5 we have, $\frac{[\mathscr{C}_n]}{n} = \mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}), D_n\}]$. Recall that $\mathbf{D}^2 = [D_2, D_3, ...]$. We have,

$$\begin{aligned}\frac{[\mathscr{C}_{n+1}]}{n+1} - \frac{[\mathscr{C}_n]}{n} &= \mathbb{E}[\min\{1 + W_n(\mathbf{D}), D_{n+1}\}] - \mathbb{E}[\min\{1 + W_{n-1}(\mathbf{D}^2), D_{n+1}\}] \\ &= \mathbb{E}[\min\{1 + W_n(\mathbf{D}), D_{n+1}\} - \min\{1 + W_{n-1}(\mathbf{D}^2), D_{n+1}\}] \\ &\leq \mathbb{P}[W_n(\mathbf{D}) \neq W_{n-1}(\mathbf{D}^2)],\end{aligned}$$

where the last inequality is true because $\min\{1+W_n(\mathbf{D}), D_{n+1}\} - \min\{1+W_{n-1}(\mathbf{D}^2), D_{n+1}\}$ never exceeds 1. Note that for any particular random instance $\mathbf{d}$, $W_n(\mathbf{d}) = W_{n-1}(\mathbf{d}^2)$ if $W_i(\mathbf{d}) = 0$ for some $1 \leq i \leq n$ or $W_i(\mathbf{d}^2) = 1$ for some $1 \leq i \leq n-1$. Thus,

$$\mathbb{P}[W_n(\mathbf{D}) \neq W_{n-1}(\mathbf{D}^2)] \leq \mathbb{P}[W_i(\mathbf{D}) > 0, W_i(\mathbf{D}^2) < 1, \forall 1 \leq i \leq n].$$

Therefore, we have

$$\frac{[\mathscr{C}_{n+1}]}{n+1} - \frac{[\mathscr{C}_n]}{n} \leq \mathbb{P}[W_i(\mathbf{D}) > 0, W_i(\mathbf{D}^2) < 1, \forall 1 \leq i \leq n].$$

Now, since $\mathbf{D}$ is non-degenerate and IID, there exists some t such that $p = \mathbb{P}[\sum_{j=1}^{t}(D_j - 1) \geq 1] > 0$. If some instance $\mathbf{d}$ satisfies the condition $W_i(\mathbf{d}) > 0, W_i(\mathbf{d}^2) < 1, \forall 1 \leq$

$i \leq n$, then we must have that $\sum_{j=2+(k-1)t}^{kt+1}(d_j - 1) < 1$ for any $1 \leq k \leq \lfloor \frac{n-1}{t} \rfloor$. Hence,

$$
\begin{aligned}
&\frac{[\mathscr{C}_{n+1}]}{n+1} - \frac{[\mathscr{C}_n]}{n} \\
&\leq \mathbb{P}[W_i(\mathbf{D}) > 0, W_i(\mathbf{D}^2) < 1, \forall 1 \leq i \leq n] \\
&\leq \mathbb{P}[\sum_{j=2+(k-1)t}^{kt+1}(D_j - 1) < 1, \forall 1 \leq k \leq \lfloor \frac{n-1}{t} \rfloor] \\
&= (1-p)^{\lfloor \frac{n-1}{t} \rfloor} \\
&\leq Ke^{cn},
\end{aligned}
$$

for some constants $K > 0$ and $c < 0$. □

Figure 2-4 and Theorem 3.4 show that $\frac{[\mathscr{C}_n]}{n} \approx \frac{[\mathscr{C}_{n+t}]}{n+t}$ for any t provided that n is large. Hence, it implies that in a system with a large number of plants and products, it is not necessary to have a long chain that connects all the plants and products. A collection of several chains, each of which with a large number of plants and products can be as effective.

Finally, we note that Theorem 3.4 can be applied to show that $\frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \geq \frac{[\mathscr{C}_{n+1}]}{[\mathscr{F}_{n+1}]}$ when n is large and D has mean 1 and finite variance. To see this, one needs to apply the central limit theorem to show that $\frac{[\mathscr{F}_{n+1})]}{n+1} - \frac{[\mathscr{F}_n]}{n} \geq K_1(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})$ for some constant $K_1 > 0$. Then, we have that

$$
\begin{aligned}
\frac{[\mathscr{C}_{n+1}]}{[\mathscr{F}_{n+1}]} &= \frac{[\mathscr{C}_{n+1}]}{n+1} / \frac{[\mathscr{F}_{n+1}]}{n+1} \\
&\leq (\frac{[\mathscr{C}_n]}{n} + Ke^{cn})/(\frac{[\mathscr{F}_n]}{n} + K_1(\frac{1}{\sqrt{n}} - \frac{1}{\sqrt{n+1}})) \\
\text{since } Ke^{cn} \ll K_1(\tfrac{1}{\sqrt{n}} - \tfrac{1}{\sqrt{n+1}}) \text{ for large } n, \quad &\leq (\frac{[\mathscr{C}_n]}{n})/(\frac{[\mathscr{F}_n]}{n}) \quad \text{for large } n, \\
&= \frac{[\mathscr{C}_n]}{[\mathscr{F}_n]} \quad \text{for large } n.
\end{aligned}
$$

3.4 Conclusion

This chapter provides several theoretical explanations on the effectiveness of the long chain, and through the theoretical justifications, refines some of the insights for designing sparse flexibility structures. Applying the supermodularity and the decomposition of the long chain, we derive four important results: First, we prove that under exchangeable demand, the incremental benefit, i.e. the increase in expected sales, is always increasing as the long chain is constructed, thus providing theoretical justification to the idea of "closing the chain", a concept that was empirically observed by authors such as [Hopp et al., 2004] and [Graves, 2008]. Second, we show that under exchangeable demand, the long chain is optimal among all 2-flexibility systems, a property of the long chain that has been widely assumed but, to the best of our knowledge, never rigorously proven before. Third, we prove that the difference between the fill rate of full flexibility design and the fill rate of the long chain is increasing with the number of products. This suggests that, relative to full flexibility design, the long chain is more effective for smaller size systems. Finally, we identify a risk pooling result where the fill rate of a long chain increases with system size. This increase in fill rate, however, is proved to converge to zero exponentially quickly. The last part of our analysis suggests that while the long chain is the optimal 2-flexibility design, a design with several closed chains, where each chain connects a substantial number of plants and products, performs just as well as the long chain.

It is appropriate to point out an important limitation of our model–we focus on a balanced system where the number of plants equals the number of products. Many real world systems do not satisfy this assumption, and typically, the number of products is much larger than the number of plants. One way to address this limitation is to apply the clustering method from [Jordan and Graves, 1995] to create an approximately balanced system and then follow the insights and guidelines developed here. Thus, the design principles emphasized in this chapter are also important for unbalanced systems as well. This includes the importance of "closing the chain", and that a design with several closed chains, where each chain connects a substantial number of plants and products, performs well for large size systems.

An alternative method for studying flexibility structures under an unbalanced system is to apply a worst-case analysis. That is, instead of assuming demand to be stochastic, we assume all demand instances lie in an uncertainty set, and study the sales of the worst-case demand within the uncertainty set. This method is discussed in the next chapter.

Chapter 4

Worst-case Analysis of Process Flexibility

In this chapter, we take a different approach by studying process flexibility structures from a worst-case, also referred to as *robust*, point of view. That is, we assume that any demand instance must belong to some uncertainty set, and study the worst-case performance of a given structure among all the demand instances in this uncertainty set. We will show that by analyzing flexibility structures from a worst-case point of view, we can gain additional insights on the effectiveness of sparse structures. This motivates a new method for generating flexibility structures that are effective from both worst-case and average-case performance measures.

To start the worst-case analysis, we first introduce a class of worst-case performance measures for flexibility structures which we call Γ. This class includes the minimum demand satisfied by the structure; the minimum ratio of the demand satisfied by the structure and that of full flexibility; and, the largest absolute gap between the demand satisfied by full flexibility and that of the specific structure under consideration. In what follows, we refer to these measures as the *robust measures* associated with a given structure. In Section 4.2, we introduce the *plant cover index*, an index defined by a corresponding constrained bipartite vertex cover problem. In the same section, we then apply the plant cover index to provide conditions under which one flexible structure is more robust than another among all uncertainty sets and for all measures in Γ. In Section 4.3, we show that an important flexibility structure called *the long chain* compares favorably in the robust measures to a class of sparse flexibility structures. In Section 4.4, we apply the theory developed in this chapter to design sparse flexibility structures which work well under both, worst-case and average-case performance measures. A numerical study underscores the power of our heuristic. In Section 4.5, we extend the results in Section 4.2.

4.1 Worst-case Performance Measures

We define a performance measure function $f(\cdot)$ to be a function that maps a demand vector $\mathbf{d} \in \mathbb{R}^n$ and a flexibility structure $\mathscr{A}$ to a real number. For example, the function $P(\cdot)$, defined as the sales of a flexibility structure given a demand instance $\mathbf{d}$ in Section 1.4, is a performance measure function. A robust measure (or *worst-case measure*), $R(\cdot)$, is a function which maps a given flexibility structure $\mathscr{A}$ and a set $S \subseteq \mathbb{R}^{+^n}$ to a real number. We refer to S as the *uncertainty set* and throughout this chapter, we assume S is always non-negative and compact. Each robust measure R has a corresponding performance measure function, f^R. Finally, for a given structure $\mathscr{A}$ and a uncertainty set S, define

$$R(\mathscr{A}, S) := \min_{\mathbf{d} \in S} f^R(\mathbf{d}, \mathscr{A}).$$

We assume that $f^R(\mathbf{d}, \mathscr{A})$ is always continuous in $\mathbf{d}$. This assumption guarantees that $R(\cdot)$ is well-defined. Next, we define three different robust measures, R^s, R^r and R^d with the following performance measure functions:

$$\begin{aligned} f^{R^s}(\mathbf{d}, \mathscr{A}) &:= P(\mathbf{d}, \mathscr{A}), \quad \forall \mathbf{d} \in \mathbb{R}^n \\ f^{R^r}(\mathbf{d}, \mathscr{A}) &:= \frac{P(\mathbf{d}, \mathscr{A})}{P(\mathbf{d}, \mathscr{F})}, \quad \forall \mathbf{d} \in \mathbb{R}^n \setminus \{\mathbf{0}\} \\ f^{R^d}(\mathbf{d}, \mathscr{A}) &:= P(\mathbf{d}, \mathscr{A}) - P(\mathbf{d}, \mathscr{F}), \quad \forall \mathbf{d} \in \mathbb{R}^n \end{aligned}$$

Given an uncertainty set S, observe that R^s is the smallest possible sales of structure

$\mathscr{A}$; R^r is the smallest possible ratio of the demand satisfied by $\mathscr{A}$ to that of demand satisfied by full flexibility; and finally, R^d is the most negative gap between demand satisfied by full flexibility and demand satisfied by $\mathscr{A}$. To ensure that $f^{R^r}(\mathbf{d}, \mathscr{A})$ is continuous in $\mathbf{d}$, we define $f^{R^r}(\mathbf{0}, \mathscr{A}) := 1$.

For any vector $\mathbf{d} \in \mathbb{R}^n$, define $\mathbf{d}^\sigma := [d_{\sigma(1)}, ..., d_{\sigma(n)}]^T$ where σ is a permutation of integers 1 to n. Moreover, define $Perm(\mathbf{d}) := \{\mathbf{d}^\sigma | \text{for all } \sigma \text{ that permutes 1 to } n\}$.

We say that a performance measure function $f(\cdot)$ is *permutation consistent* with $P(\cdot)$ if the following holds: fix any $\mathbf{d} \in \mathbb{R}^n$, then for any $\mathbf{d}^1$,$\mathbf{d}^2 \in Perm(\mathbf{d})$, and for any structures $\mathscr{A}_1$, $\mathscr{A}_2$,

$$f(\mathbf{d}^1, \mathscr{A}_1) > f(\mathbf{d}^2, \mathscr{A}_2) \iff P(\mathbf{d}^1, \mathscr{A}_1) > P(\mathbf{d}^2, \mathscr{A}_2).$$

We claim that $f^{R^s}(\cdot), f^{R^r}(\cdot)$ and $f^{R^d}(\cdot)$ are all permutation consistent with $P(\cdot)$. By definition, this is true for f^{R^s}. It is also true for f^{R^r} and f^{R^d} by observing that $P(\mathbf{d}^1, \mathscr{F}) = P(\mathbf{d}^2, \mathscr{F})$, for any $\mathbf{d}^1$,$\mathbf{d}^2 \in Perm(\mathbf{d})$.

Define Γ to be the set of all robust measures with performance measure functions that are permutation consistent with $P(\cdot)$. Note that Γ contains a large number of interesting robust measures. In particular, the three robust measures we introduced, R^s, R^r and R^d are all in Γ.

Let $\min^i(\mathbf{d})$ to be the i-th smallest element in the set $\{d_1, d_2, ..., d_n\}$. For any uncertainty set S, we say that S is *symmetric* if for any $\mathbf{d} \in S$, $\mathbf{d}^\sigma \in S$ for any permutation σ. Throughout the chapter, we will assume that all the uncertainty sets are symmetric unless stated otherwise. This assumption implies that the products are homogenous and the worst-case performance will not change if the products are relabeled. In particular, if demands of products is an exchangeable random vector, i.e., the random demand vector $\mathbf{D}$ has the same distribution as $\mathbf{D}^\sigma$, then the support of $\mathbf{D}$ is a symmetric set.

For flexibility structures $\mathscr{A}_1$ and $\mathscr{A}_2$, we say that $\mathscr{A}_1$ is *more robust* than $\mathscr{A}_2$ if **for any** $R \in \Gamma$ and **any symmetric set** S, we have $R(\mathscr{A}_1, S) \geq R(\mathscr{A}_2, S)$. Moreover, we say $\mathscr{A}_1$ is *strictly more robust* than $\mathscr{A}_2$ if $\mathscr{A}_1$ is more robust than $\mathscr{A}_2$, and there exists some symmetric set S and $R \in \Gamma$ that $R(\mathscr{A}_1, S) > R(\mathscr{A}_2, S)$.

4.2 Plant Cover Index

First, we start the section by defining *the plant cover index at k* for flexibility structure $\mathscr{A}$. This index is defined by the minimum plant capacities required to create a vertex cover on $\mathscr{A}$, given that vertex cover contains exactly k product vertices. This index, denoted by $\delta^k(\mathscr{A})$, is the objective value of the following integer program.

$$\begin{aligned}\delta^k(\mathscr{A}) := \min \sum_{i=1}^{m} c_i p_i \\ \text{s.t.} \quad \sum_{j=1}^{n} q_j = k, \\ p_i + q_j \geq 1, \forall (i,j) \in I(\mathscr{A}) \\ \mathbf{p} \in \{0,1\}^m, \mathbf{q} \in \{0,1\}^n.\end{aligned}$$

Note that it is straightforward to check that $\delta^0(\mathscr{A}) = \sum_{i=1}^{m} c_i$ and $\delta^n(\mathscr{A}) = 0$. Observe also that the following holds.

Remark 4.1.

$$\delta^k(\mathscr{A}) = \min_{S \subseteq B, |S| = n-k} \sum_{a_i \in N(S, \mathscr{A})} c_i.$$

4.2.1 Plant cover indices and Robust Measure

We start with a lemma that shows that $R^s(\mathscr{A}, S)$ can be upper-bounded by the sum of $\delta^k(\mathscr{A})$ and $\min^i(\mathbf{d})$, $i = 1, ..., k$, for any $\mathbf{d} \in S$. Recall that $R^s(\mathscr{A}, S) = \min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A})$.

Lemma 4.1. *For any fixed $\mathbf{d} \in S$ and any integer $0 \leq k \leq n$,*

$$R^s(\mathscr{A}, S) \leq \delta^k(\mathscr{A}) + \sum_{i=1}^{k} \min{}^i(\mathbf{d}).$$

Proof. By definition

$$\begin{aligned} P(\mathbf{d}, \mathscr{A}) = \max \quad & \sum_{(i,j)\in I(\mathscr{A})} f_{ij} \\ \text{s.t.} \quad & \sum_{i \in I(N(j,\mathscr{A}))} f_{ij} \leq d_j, \forall j \in I(B) \\ & \sum_{j \in I(N(i,\mathscr{A}))} f_{ij} \leq c_i, \forall i \in I(A) \\ & f_{ij} \geq 0, \forall (i,j) \in I(\mathscr{A}) \\ & \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}, \end{aligned}$$

and by the classical max-flow min-cut theorem, we have

$$\begin{aligned} P(\mathbf{d}, \mathscr{A}) = \min & \sum_{i=1}^{m} c_i p_i + \sum_{j=1}^{n} q_j d_j \\ \text{s.t.} \quad & p_i + q_j \geq 1, \forall (i,j) \in I(\mathscr{A}) \\ & \mathbf{p} \in \{0,1\}^m, \mathbf{q} \in \{0,1\}^n. \end{aligned}$$

By definition of $\delta^k(\mathscr{A})$, we can find vectors $\mathbf{p}' \in \{0,1\}^m$, $\mathbf{q}' \in \{0,1\}^n$ such that $\sum_{i=1}^m c_i p'_i = \delta^k(\mathscr{A})$, $\sum_{j=1}^n q'_j = k$ and $\mathbf{p}'$, $\mathbf{q}'$ are feasible for the optimization problem associated with $P(\mathbf{d}, \mathscr{A})$. Define σ to be a permutation of 1 through n such that $q'_j = 1$ if and only if $d_{\sigma(j)} \in \{\min^i(\mathbf{d}) | 1 \leq i \leq k\}$. Then, we have that

$$\sum_{i=1}^{m} c_i p'_i + \sum_{j=1}^{n} q'_j d_{\sigma(j)} = \delta^k(\mathscr{A}) + \sum_{i=1}^{k} \min{}^i(\mathbf{d})$$

Therefore, $P(\mathbf{d}^\sigma, \mathscr{A}) \leq \delta^k(\mathscr{A}) + \sum_{i=1}^k \min^i(\mathbf{d})$. Since $\mathbf{d}^\sigma \in S$ as S is symmetric, $R^s(\mathscr{A}, S) \leq P(\mathbf{d}^\sigma, \mathscr{A})$ and we are done. □

Next, we show that the inequality in Lemma 4.1 can be achieved as an equality for some integer k and some vector $\mathbf{d}$.

Proposition 4.1. *Let* $\boldsymbol{\tau} = \arg\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A})$. *Then*

$$R^s(\mathscr{A}, S) = \delta^k(\mathscr{A}) + \sum_{i=1}^{k} \min{}^i(\boldsymbol{\tau})$$

for some nonnegative integer $k \leq n$.

Proof. By definition of $R^s(\mathscr{A}, S)$, $P(\boldsymbol{\tau}, \mathscr{A}) = R^s(\mathscr{A}, S)$. Since

$$\begin{aligned} P(\boldsymbol{\tau}, \mathscr{A}) = \max \quad & \sum_{(i,j) \in I(\mathscr{A})} f_{ij} \\ \text{s.t.} \quad & \sum_{i \in I(N(j,\mathscr{A}))} f_{ij} \leq \tau_j, \forall j \in I(B) \\ & \sum_{j \in I(N(i,\mathscr{A}))} f_{ij} \leq c_i, \forall i \in I(A) \\ & f_{ij} \geq 0, \forall (i,j) \in I(\mathscr{A}) \\ & \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}, \end{aligned}$$

we can apply the max-flow min-cut theorem, to obtain

$$\begin{aligned} P(\boldsymbol{\tau}, \mathscr{A}) = \min & \sum_{i=1}^{m} c_i p_i + \sum_{j=1}^{n} q_j \tau_j \\ \text{s.t.} \quad & p_i + q_j \geq 1, \forall (i,j) \in I(\mathscr{A}) \\ & \mathbf{p} \in \{0,1\}^m, \mathbf{q} \in \{0,1\}^n. \end{aligned}$$

Let $\mathbf{p}^*$, $\mathbf{q}^*$ be the optimal solution to the optimization problem above, and define $k := \sum_{j=1}^{n} q_j^*$. Then, we must have $\sum_{j=1}^{n} q_j^* \tau_j \geq \sum_{j=1}^{k} \min^j(\boldsymbol{\tau})$ and $\sum_{i=1}^{m} c_i p_i^* \geq \delta^k(\mathscr{A})$. Hence, we have that

$$R^s(\mathscr{A}, S) = P(\boldsymbol{\tau}, \mathscr{A}) = \sum_{i=1}^{m} c_i p_i^* + \sum_{j=1}^{n} q_j^* \tau_j \geq \delta^k(\mathscr{A}) + \sum_{i=1}^{k} \min{}^i(\boldsymbol{\tau})$$

But by Lemma 4.1, $R^s(\mathscr{A}, S) \leq \delta^k(\mathscr{A}) + \sum_{i=1}^{k} \min^i(\boldsymbol{\tau})$ and hence we have $R^s(\mathscr{A}, S) = \delta^k(\mathscr{A}) + \sum_{i=1}^{k} \min^i(\boldsymbol{\tau})$. □

Proposition 4.1 demonstrates that there is a strong connection between $\delta^k(\mathscr{A})$ and $R^s(\mathscr{A}, S)$ for any S. Using this connection, we next show that for any two flexibility structures $\mathscr{A}_1$, $\mathscr{A}_2$, if the plant cover indices of $\mathscr{A}_1$ are always greater than or equal to the plant cover indices of $\mathscr{A}_2$, then for any robust measure $R \in \Gamma$ and any symmetric uncertainty set S, $\mathscr{A}_1$'s worst-case performance is greater than or equal to the worst-case performance of $\mathscr{A}_2$.

Theorem 4.1. *Suppose $\mathscr{A}_1$ and $\mathscr{A}_2$ are two flexibility structures with $\delta^k(\mathscr{A}_1) \geq \delta^k(\mathscr{A}_2)$ for all $1 \leq k \leq n$. Then $\mathscr{A}_1$ is more robust than $\mathscr{A}_2$. That is, for any symmetric set S and any $R \in \Gamma$,*

$$R(\mathscr{A}_1, S) \geq R(\mathscr{A}_2, S).$$

Proof. Let $\boldsymbol{\tau} = \arg\min_{\mathbf{d} \in S} f^R(\mathbf{d}, \mathscr{A}_1)$, and let $S' := Perm(\boldsymbol{\tau})$. Then, since f^R is permutation consistent with $P(\cdot)$, $P(\boldsymbol{\tau}, \mathscr{A}_1) = \min_{\mathbf{d} \in S'} P(\mathbf{d}, \mathscr{A}_1)$. Applying Proposition 4.1, we have $P(\boldsymbol{\tau}, \mathscr{A}_1) = \delta^k(\mathscr{A}_1) + \sum_{j=1}^{k} \min^j(\boldsymbol{\tau})$ for some k.

By the condition in the theorem, $\delta^k(\mathscr{A}_1) + \sum_{j=1}^{k} \min^j(\boldsymbol{\tau}) \geq \delta^k(\mathscr{A}_2) + \sum_{j=1}^{k} \min^j(\boldsymbol{\tau})$. By Lemma 4.1, $\delta^k(\mathscr{A}_2) + \sum_{j=1}^{k} \min^j(\boldsymbol{\tau}) \geq \min_{\mathbf{d} \in S'} P(\mathbf{d}, \mathscr{A}_2)$. Hence, $P(\boldsymbol{\tau}, \mathscr{A}_1) \geq \min_{\mathbf{d} \in S'} P(\mathbf{d}, \mathscr{A}_2)$. Since f^R is permutation consistent with $P(\cdot)$,

$$R(\mathscr{A}_1, S) = f^R(\boldsymbol{\tau}, \mathscr{A}_1) \geq \min_{\mathbf{d} \in S'} f^R(\mathbf{d}, \mathscr{A}_2) \geq \min_{\mathbf{d} \in S} f^R(\mathbf{d}, \mathscr{A}_2) = R(\mathscr{A}_2, S).$$

□

Theorem 4.1 shows that if $\delta^k(\mathscr{A}_1) \geq \delta^k(\mathscr{A}_2)$ for all $1 \leq k \leq n$, then $\mathscr{A}_1$ is more robust than $\mathscr{A}_2$. The next result implies that if $\delta^k(\mathscr{A}_1) > \delta^k(\mathscr{A}_2)$ for some k, then $\mathscr{A}_1$ is strictly more robust than $\mathscr{A}_2$.

Proposition 4.2. *Suppose there exists $1 \leq k \leq n$, with $\delta^k(\mathscr{A}_1) < \delta^k(\mathscr{A}_2)$. Then, for any $R \in \Gamma$, there exists a symmetric set S such that*

$$R(\mathscr{A}_1, S) < R(\mathscr{A}_2, S).$$

Proof. Let $\boldsymbol{\tau}$ be the vector with first k entries equal to 0 and each of the next $n - k$

entries equal to $\sum_{i=1}^{m} c_i$. Let $S = Perm(\boldsymbol{\tau})$. By Lemma 4.1 and Proposition 4.1, we have that

$$\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}_1) = \delta^k(\mathscr{A}_1)$$
$$\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}_2) = \delta^k(\mathscr{A}_2).$$

$\delta^k(\mathscr{A}_1) < \delta^k(\mathscr{A}_2)$ implies that $\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}_1) < \min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}_2)$. Because f^R is permutation consistent with $P(\cdot)$, we have

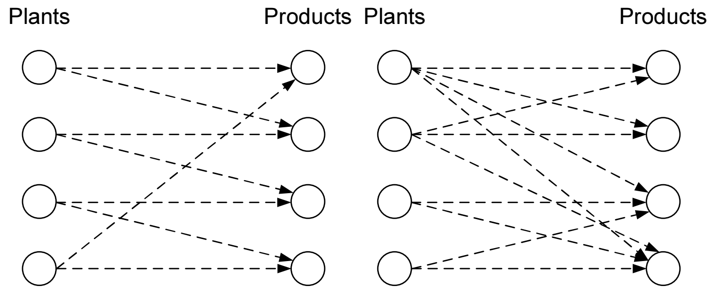
$$R(\mathscr{A}_1, S) < R(\mathscr{A}_2, S).$$

□

Observe that Proposition 4.2 also implies that the converse of Theorem 4.1 is true. That is, if the statement $\delta^k(\mathscr{A}_1) \geq \delta^k(\mathscr{A}_2)$ is not true for some $1 \leq k \leq n$, then for any $R \in \Gamma$, there exists some symmetric set S such that $R(\mathscr{A}_1, S) < R(\mathscr{A}_2, S)$. Thus, we have our next corollary, which states that $\delta^k(\mathscr{A}_1) \geq \delta^k(\mathscr{A}_2)$ for $1 \leq k \leq n$ is equivalent with $\mathscr{A}_1$ being more robust than $\mathscr{A}_2$.

Corollary 4.1. *$\mathscr{A}_1$ is more robust than $\mathscr{A}_2$ if and only if $\delta^k(\mathscr{A}_1) \geq \delta^k(\mathscr{A}_2)$ for $1 \leq k \leq n$.*

An interesting question is whether better worst-case performance implies better average-case performance. Specifically, when $\mathscr{A}_1$ is strictly more robust than $\mathscr{A}_2$, we know that the worst-case performance of $\mathscr{A}_1$ is always better (and sometimes strictly better) than the worst-case performance of $\mathscr{A}_2$. The question is whether this implies that the expected sales of $\mathscr{A}_1$ is greater than or equal to the expected sales of $\mathscr{A}_2$ for any IID demand distribution. We answer the question in the negative through a counterexample. Consider, $n = m = 4$, $c_i = 1$ for $i = 1, 2, 3, 4$ and flexibility structures $\mathscr{A}_1$ and $\mathscr{A}_2$ in Figure 4-1. It is easy to check that $\delta^k(\mathscr{A}_1) = \delta^k(\mathscr{A}_2)$ for $k = 0, 1, 3, 4$ and $\delta^2(\mathscr{A}_1) > \delta^2(\mathscr{A}_2)$. However, we find that the expected sales of $\mathscr{A}_1$ (equal to 3) which is less than the expected sales of $\mathscr{A}_2$ (equal to 3.125) when demand for each products is IID and equal to 0 or 2 with equal probability.



(a) $\mathscr{A}_1$ (b) $\mathscr{A}_2$

Figure 4-1: Designs $\mathscr{A}_1$ and $\mathscr{A}_2$

4.2.2 Hardness Result

In the previous subsection, we have established a connection between $\delta^k(\mathscr{A})$ and $R(\mathscr{A}, S)$, for any flexibility structure $\mathscr{A}$ and any symmetric uncertainty set S. Here, we will establish that computing $R(\mathscr{A}, S)$ is an NP-hard problem for any $R \in \Gamma$. To establish the hardness result, we begin with a lemma, which is an immediate consequence of the work of Kuo and Fuchs [Kuo and Fuchs, 1987].

Lemma 4.2. *Given non-negative integers k, t and some flexibility structure $\mathscr{A}$, determining whether $\delta^k(\mathscr{A}) \leq t$ is NP-hard.*

Proof. Consider the case $c_i = 1$ for all $1 \leq i \leq m$. In this case, note that $\delta^k(\mathscr{A}) \leq t$ if and only if there is a vertex cover $V_A \cup V_B$, where $V_A \subseteq A$, $|V_A| \leq t$ and $V_B \subseteq B$, $|V_B| \leq k$. By [Kuo and Fuchs, 1987], it is NP-hard to determine if there exists such a vertex cover. Thus, we have that determining whether $\delta^k(\mathscr{A}) \leq t$ is NP-hard. □

Having established Lemma 4.2, we now prove that computing $R(\mathscr{A}, S)$ for any $R \in \Gamma$ is NP-hard.

Corollary 4.2. *For any robust measure $R \in \Gamma$, determining whether $R(\mathscr{A}, S) \leq t$ is NP-hard.*

Proof. We prove this result by showing that for $c_i = 1$ for all $1 \leq i \leq m$, the problem of determining if $\delta^k(\mathscr{A}) \leq t$ for some integer t can be reduced to the problem of determining if $R(\mathscr{A}, S) \leq t'$ for some $t' \in \mathbb{R}$ and $S \subseteq \mathbb{R}^n$.

We can assume $t < m$, since $\delta^k(\mathscr{A}) \leq m$. Let $\mathbf{e}$ be the vector such that $e_j = 0$ for $1 \leq j \leq k$ and $e_j = m$ for $k+1 \leq j \leq n$. Let $S := Perm(\mathbf{e})$, where $Perm(\mathbf{e}) = \{\mathbf{e}^\sigma | \text{for all permutation } \sigma\}$. Also, let

$$\mathscr{A}' := \{(a_i, b_j) | 1 \leq i \leq m, 1 \leq j \leq k\} \cup \{(a_i, b_j) | 1 \leq i \leq t, k+1 \leq j \leq n\}.$$

It is easy to check that $\mathbf{e} \in \arg\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}')$ and $P(\mathbf{e}, \mathscr{A}') = t$. Since f^R is permutation consistent, $\min_{\mathbf{d} \in S} f^R(\mathbf{d}, \mathscr{A}') = f^R(\mathbf{e}, \mathscr{A}')$.

Now, let $t' := f^R(\mathbf{e}, \mathscr{A}')$. Since f^R is permutation consistent with $P(\cdot)$, we have that for any $\mathbf{d} \in S$, $f^R(\mathbf{d}, \mathscr{A}) \leq t'$ if and only if $P(\mathbf{d}, \mathscr{A}) \leq t$. Thus, determining whether $\min_{\mathbf{d} \in S} f^R(\mathbf{d}, \mathscr{A}) = R(\mathscr{A}, S) \leq t'$ is equivalent to determining whether $\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}) \leq t$.

By construction of S, Lemma 4.1, and Proposition 4.1, we have that $\min_{\mathbf{d} \in S} P(\mathbf{d}, \mathscr{A}) = \delta^k(\mathscr{A})$. Therefore, we have that determining whether $R(\mathscr{A}, S) \leq t'$ is at least as hard as determining whether $\delta^k(\mathscr{A}) \leq t$, which is NP-hard.

□

We would like to point out that while Lemma 4.2 shows that computing $\delta^k(\mathscr{A})$ is NP-hard, off-the-shelf solvers can compute these quantities fairly quickly. The reason for this is probably because the optimization problem that defines $\delta^k(\mathscr{A})$ reduces to a bipartite vertex cover problem (which has a tight integrality gap) when we relax the constraint $\sum_{j=1}^n q_j = k$ with a Lagrangian multiplier. Indeed, in our computational tests, the binary program solver in cplex has consistently solved $\delta^k(\mathscr{A})$ for systems with around 20 plant nodes and 20 product nodes within 1 second. Finally, researchers in computer science have developed algorithms to compute $\delta^i(\mathscr{A})$ that work quite efficiently in practice (see [Fernau and Niedermeier, 2001], [Bai and Fernau, 2008]).

4.3 Sparse Flexibility Structures and Long Chain

In this section, we apply the results of the previous section to analyze the worst-case effectiveness of sparse flexibility structures. In particular, we are interested in the long chain flexibility structure, which has been studied extensively in the literature from the average-case point-of-view. As is typical in the analysis of the long chain, see for example [Jordan and Graves, 1995], we consider in this section balanced systems (i.e. $m = n$) and assume $c_i = 1$ for all $1 \leq i \leq n$. Recall that $\mathscr{C}_n$ is used to denote the long chain in a system of size n.

Consider the class of all flexibility structures in which the degree of each product node is exactly two. The theorem below shows that the long chain is most robust among this class of structures.

Theorem 4.2. *The long chain flexibility structure, $\mathscr{C}_n$, is more robust than $\mathscr{A}$ if for any $u \in B$, $|N(u, \mathscr{A})| = 2$.*

Proof. It's easy to check that $\delta^k(\mathscr{C}_n) = n - k + 1$ for $1 \leq k \leq n - 1$, and $\delta^n(\mathscr{C}_n) = 0 = \delta^n(\mathscr{A})$. To prove Theorem 4.2, it is sufficient to show that for all $1 \leq k < n$, we can find some $S \subset B$, $|S| = k$, such that $|N(B \setminus S, \mathscr{A})| \leq n - k + 1$, as $\delta^k(\mathscr{A}) \leq |N(B \setminus S, \mathscr{A})|$.

Suppose the graph formed by $\mathscr{A}$ consists of c connected bipartite components. For $1 \leq i \leq c$, let $A_i \subset A$, $B_i \subset B$ be the set of vertices of the i-th component. Without loss of generality, we also assume that $|A_i| - |B_i|$ is non-decreasing with i. Because $\sum_{i=1}^{c}(|A_i| - |B_i|) = 0$, this assumption implies that $\sum_{i=1}^{t} |A_i| \leq \sum_{i=1}^{t} |B_i|$ for any $t \leq c$.

We now show that for any i, and any $1 \leq l \leq |B_i|$, there exists some $T \subseteq B_i$, $|T| = l$ such that $|N(T, \mathscr{A})| \leq l + 1$. This is done by induction on l. For $l = 1$, take any $u \in B_i$ and let $T := \{u\}$ and $|N(T, \mathscr{A})| = 2$. Suppose the statement is true for some $l < |B_i|$, then we can find set $T^l \subset B_i$, $|T^l| = l$ and $|N(T^l, \mathscr{A})| \leq l + 1$. Since the vertices in $A_i \cup B_i$ form a connected component, and $T^l \subsetneq B_i$, there exists some $u \in N(T^l, \mathscr{A})$ such that (u, v) is an arc for some $v \notin T^l$. Since $|N(v, \mathscr{A})| = 2$ and $u \in N(T^l, \mathscr{A})$, we must have that $|N(S^l \cup \{v\}, \mathscr{A})| \leq l + 2$. Thus, by induction, we have that for any $1 \leq l \leq |B_i|$, there exists some $T \subseteq B_i$, $|T| = l$ such that $|N(T, \mathscr{A})| \leq l + 1$.

For any $1 \leq k < n$, find the largest t such that $\sum_{i=1}^{t} |B_i| < n - k$. By the choice of t, we have $t < c$ and $n - k - \sum_{i=1}^{t} |B_i| \leq |B_{t+1}|$. Thus, we can find some set T where $|T| = n - k - \sum_{i=1}^{t} |B_i|$, $T \subseteq B_{t+1}$ and $|N(T, \mathscr{A})| \leq n - k - \sum_{i=1}^{t} |B_i| + 1$. Finally, let $S := (B_{t+1} \cup B_{t+2} \cup ... \cup B_c) \setminus T$, and we have that

$$|N(B \setminus S, \mathscr{A})| = |N(T, \mathscr{A})| + \sum_{i=1}^{t} |A_i| \leq n - k - \sum_{i=1}^{t} |B_i| + 1 + \sum_{i=1}^{t} |B_i| \leq n - k + 1.$$

Since $S \subset B$ and $|S| = n - \sum_{i=1}^{t} |B_i| - (n - k - \sum_{i=1}^{t} |B_i|) = k$, we are done.

□

Interestingly, Theorem 3.2 suggests that long chain is at least as good as any 2-flexibility structures from average-case point-of-view for any exchangeable demand distribution. Theorem 4.2 complements this result by noting that any 2-flexibility structure satisfies the condition $|N(u, \mathscr{A})| = 2$, $\forall u \in B$. Hence, Theorem 4.2 implies that the long chain is also the most robust 2-flexibility structure.

A natural generalization of Theorem 4.2 is to consider the class of flexibility structures with $2n$ arcs, rather than the class of all flexibility structures in which the degree of each product node is exactly two. To our surprise, this generalization does not hold. Indeed, in Figure 4-2, we provide flexibility structure $\mathscr{A}$ with $n = 9$ nodes and 18 arcs, where $\delta^4(\mathscr{A}) > \delta^4(\mathscr{C}_n)$. Note that by Corollary 4.1, we immediately have that the long chain is not more robust than $\mathscr{A}$.

The example shown by Figure 4-2 may motivate a claim that there exists a structure with $2n$ arcs which is strictly more robust than the long chain. This is also not true, and it is a simple consequence of Theorem 4.2. By Theorem 4.2, if a structure $\mathscr{A}$ with $2n$ arcs has that $\delta^k(\mathscr{A}) > \delta^k(\mathscr{C}_n)$ for some k, then there is some node $u \in B$ where $|N(u, \mathscr{A})| = 1$. But this implies that $\delta^{n-1}(\mathscr{A}) = 1 < 2 = \delta^{n-1}(\mathscr{C}_n)$. Hence, there is no structure with $2n$ arcs that is strictly more robust than $\mathscr{C}_n$. That is, $\mathscr{C}_n$ is in some sense a "Pareto optimal" structure among all flexibility structures with $2n$

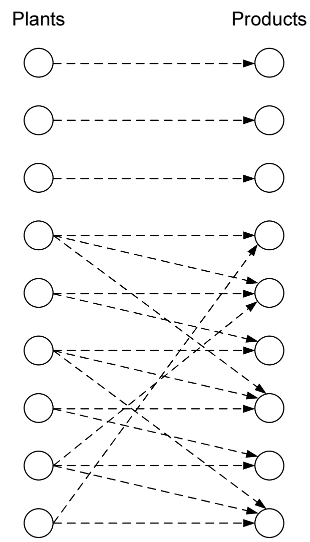


Figure 4-2: A flexibility structure $\mathscr{A}$ with 9 plants/products and 18 arcs

arcs when considering worst-case performance analysis.

While the long chain does not always achieve the best worst-case performance among all structures with $2n$ arcs, the next result shows that the long chain is the optimal structure among all *connected* structures with $2n$ arcs.

Corollary 4.3. *The long chain flexibility structure, $\mathscr{C}_n$, is more robust than $\mathscr{A}$, if $|\mathscr{A}| = 2n$, and the bipartite graph with vertex sets A, B and arc set $\mathscr{A}$ is connected.*

Proof. For $n = 1$, it is simple to check that Corollary 4.3 holds. Suppose $\mathscr{A}^*$ is a counterexample to Corollary 4.3 in the *smallest* system (the smallest n^* where there is a counterexample). Clearly, $n^* > 1$. Since $\mathscr{A}^*$ is a counterexample, there exists some $1 \leq k^* < n^*$ such that $\delta^{k^*}(\mathscr{A}^*) > n^* - k^* + 1$. By Theorem 4.2, we know there must exists some $u \in B$, with $|N(u, \mathscr{A}^*)| = 1$. Let $v = N(u, \mathscr{A}^*)$, and let G be the bipartite graph with vertex sets A, B, and arc set $\mathscr{A}^*$. Because G is connected, we must have $|N(v, \mathscr{A}^*)| \geq 2$.

Let $\mathscr{A}' := \{(v', u') | (v', u') \in \mathscr{A}^*, u' \neq u, v' \neq v\}$. Consider the bipartite graph G' with vertex sets $A \setminus v$, $B \setminus u$, and arc set $\mathscr{A}'$. Suppose G' has z components, then we must have $|N(v, \mathscr{A}^*)| \geq z + 1$. In this case, we can add $z - 1$ arcs to G' so that G' is a connected bipartite graph. Let $\mathscr{A}''$ be the arc set that contains $\mathscr{A}'$ and the $z - 1$ added arcs. Note that $|\mathscr{A}''| \leq 2(n^* - 1)$.

By construction, the bipartite graph with vertex sets $A\backslash u$, $B\backslash v$ and arc set $\mathscr{A}''$ is connected. Because $1 \leq k^* < n^*$, the minimality assumption on $\mathscr{A}^*$ and Observation 4.1, there exists some $S \subseteq B\backslash v$, with $|S| = n^* - k^* - 1$ and $|N(S, \mathscr{A}'')| \leq n^* - k^*$. But this implies that $S \cup \{v\} \subseteq B$, $|S \cup \{u\}| = n^* - k^*$ and $|N(S \cup \{u\}, \mathscr{A}'')| \leq n^* - k^* + 1$. By Observation 4.1, $|N(S \cup \{u\}, \mathscr{A}'')| \leq n^* - k^* + 1$ implies that $\delta^{k^*}(\mathscr{A}^*) \leq n^* - k^* + 1$. This contradicts our assumption that $\delta^{k^*}(\mathscr{A}^*) > n^* - k^* + 1$ and therefore, we have that Corollary 4.3 must be true. □

While the long chain does not always achieve the best worst-case performance among all (non-connected) structures with $2n$ arcs, computational experiments suggest that it is very effective. The next proposition provides one way to explain this observation by showing that the plant cover indices of the long chain are greater than or equal to that of any other structure with $2n$ arcs for the last $\sqrt{n}$ of plant cover indices.

Proposition 4.3. *In a balanced system with $n > 1$ plants and products, for any integer $0 \leq k \leq \sqrt{n}$ we have,*

$$\delta^{n-k}(\mathscr{C}_n) \geq \delta^{n-k}(\mathscr{A}),$$

for any $\mathscr{A}$ such that $|\mathscr{A}| = 2n$.

Proof. It's easy to check that $\delta^{n-k}(\mathscr{C}_n) = k + 1$ for any $k < n$. We will prove that for any $|\mathscr{A}| \leq 2n$, and for any integer $1 \leq k \leq \sqrt{n}$, we can always find some $S \subseteq B$, with $|S| = k$ and $|N(S, \mathscr{A})| \leq k + 1$. Note that by Observation 4.1, this immediately implies that $\delta^{n-k}(\mathscr{A}) \leq |N(S, \mathscr{A})| \leq k + 1 \leq \delta^{n-k}(\mathscr{C}_n)$.

Suppose there exists a counterexample $\mathscr{A}^*$ in a balanced system of size n. That is, there exists some k, $1 \leq k \leq \sqrt{n}$, for which we cannot find $S \subseteq B$ with $|S| = k$ and $|N(S, \mathscr{A}^*)| \leq k + 1$. Without loss of generality, assume $\mathscr{A}^*$ is such a structure in the *smallest* balanced system. Let k^*, $1 \leq k^* \leq \sqrt{n}$ be the integer for which we cannot find any $S \subseteq B$ with $|S| = k^*$ and $|N(S, \mathscr{A}^*)| \leq k^* + 1$. Also, let $B_1 := \{u | u \in B, |N(u, \mathscr{A}^*)| = 1\}$, $B_2 := \{u | u \in B, |N(u, \mathscr{A}^*)| = 2\}$ and $B_3 :=$

$\{u|u \in B, |N(u, \mathscr{A}^*)| \geq 3\}$.

Suppose we have some u, $u \in B_1$ with $(v, u) \in \mathscr{A}^*$ and $|N(v, \mathscr{A}^*)| \geq 2$. Let $\mathscr{A}' = \{(v', u')|(v', u') \in \mathscr{A}^*, u' \neq u, v' \neq v\}$. Then $\mathscr{A}'$ is a structure in a balanced system of size $n-1$, and $|\mathscr{A}'| \leq 2n-2$. By our assumption on the minimality of n, we can find some $S \subseteq B \setminus u$, $|S| = k^* - 1$ and $|N(S, \mathscr{A}')| \leq k^*$. But this implies that $|N(S \cup \{u\}, \mathscr{A}^*)| \leq k^* + 1$, and we have a contradiction. Thus, for any $u \in B_1$ with $(v, u) \in \mathscr{A}^*$ we have $N(v, \mathscr{A}^*) = 1$. That is, any plant v that is a neighbor of some $u \in B_1$in $\mathscr{A}^*$ has a degree one.

Suppose there exists $B_C \subset B_2$ such that all arcs incident to B_C form a single cycle. Then clearly, $|N(B_C, \mathscr{A}^*)| = |B_C|$. If $|B_C| \geq k^*$, then it is easy to check that we can find $S \subseteq B_C$ with $|S| = k^*$ and $|N(S, \mathscr{A}^*)| \leq k^* + 1$, which leads to a contradiction. If $|B_C| < k^*$, then let $\mathscr{A}' := \{(v', u')|(v', u') \in \mathscr{A}^*, u' \notin B_C, v' \notin N(B_C, \mathscr{A}^*)\}$. In this case, $|\mathscr{A}'| = |\mathscr{A}^*| - 2|B_C| \leq 2(n - |B_C|)$, and $\mathscr{A}'$ is a flexibility structure defined on a system with $n - |B_C|$ plants and $n - |B_C|$ products. By the minimality of n, we can find some $S \subseteq B \setminus |B_C|$ such $|S| = k^* - |B_C|$ and $N(S, \mathscr{A}') \leq k^* - |B_C| + 1$. This implies that $N(S \cup B_C, \mathscr{A}') \leq k^* + 1$, which is again a contradiction. Hence, there is no $B_C \subset B_2$ such that all arcs incident to nodes B_C form a cycle.

Let G_2 be the bipartite graph with node sets $A_2 = N(B_2, \mathscr{A}^*)$, B_2 and arc set $\mathscr{A}_2 = \{(v', u')|(v', u') \in \mathscr{A}^*, u' \in B_2\}$. Because there does not exist any $B_C \subset B_2$ such that all arcs incident to nodes B_C forms one cycle, G_2 contains no cycles. This implies that G_2 contains a number of components, T_1, T_2, ..., T_z, with each component T_i, we have that $|T_i \cap B_2|$, i.e., the number of product nodes in T_i, is exactly one less than $|T_i \cap A_2|$, i.e., the number of plant nodes in T_i. Note that any v that is a neighbor of $u \in B_1$ is not in T_i for all $1 \leq i \leq z$, this implies that $z = \sum_{i=1}^{z} |A \cap T_i| - |B \cup T_i| \leq (n - |B_1|) - |B_2| \leq |B_3|$. Because $|\mathscr{A}^*| = 2n$, the average degree of nodes in B is 2. This implies that $|B_3| \leq |B_1|$, and therefore $z \leq |B_1|$. Because $|B_1| \leq k^* \leq \sqrt{n}$, it is easy to check that $|B_1|(k^* - |B_1| + 2) \leq n$. Therefore, we have

$$\begin{aligned}
& |B_1|(k^* - |B_1| + 2) \leq n, \\
\Longrightarrow \quad & |B_1|(k^* - |B_1|) \leq n - 2|B_1|, \\
\Longrightarrow \quad & k^* - |B_1| \leq \frac{n - 2|B_1|}{|B_1|}, \\
\Longrightarrow \quad & k^* - |B_1| \leq \frac{n - |B_1| - |B_3|}{|B_1|}, \text{ since } |B_3| \leq |B_1|, \\
\Longrightarrow \quad & k^* - |B_1| \leq \frac{|B_2|}{z}.
\end{aligned}$$

This implies that $\sum_{i=1}^{z} |T_i \cap A_2|/z$ is at least $k^* - |B_1|$ and hence there exists $1 \leq i^* \leq z$ such that T_{i^*} has $k^* - |B_1|$ plant nodes. Therefore, we can find a set $S \subset T_{i^*} \cap B$ such that $|N(S, \mathscr{A}^*)| \leq k^* - |B_1| + 1$, which implies that $|N(S \cap B_1)| \leq k^* + 1$. This leads to a contradiction. Hence, we must have that for any $0 \leq k \leq \sqrt{n}$, we can find some $S \subseteq B$, $|S| = k$ with $\delta^{n-k}(\mathscr{A}^*) \leq |N(S, \mathscr{A}^*)| \leq k + 1$.

□

Finally, we show that when the performance function is the ratio of sales of a specific structure to that of full flexibility, the long chain has a better worst-case performance compared to any structure with $2n$ arcs, independent of whether the structure forms a connected bipartite graph.

Proposition 4.4. *Consider a symmetric set S satisfying the additional requirement that any $\mathbf{d} \in S$, $\mathbf{d}' \in S$ for all $\mathbf{d}' \leq \mathbf{d}$. For such S, we have*

$$R^r(\mathscr{C}_n, S) \geq R^r(\mathscr{A}, S),$$

for any $\mathscr{A}$ with $|\mathscr{A}| = 2n$.

Proof. By Theorem 4.2, we know $R^r(\mathscr{C}_n, S) \geq R^r(\mathscr{A}, S)$, if for any $u \in B$, $|N(u, \mathscr{A})| = 2$. Thus, we only need to consider the case where there exists some $u \in B$ such that $|N(u, \mathscr{A})| = 1$. In that case, since S is symmetric, without loss of generality, let u be product node b_1. Also, define $d_{\max} := \max\{d_1 | \mathbf{d} \in S\}$.

If $d_{\max} \leq 1$, then for any $\mathbf{d} \in S$, $\mathbf{d} \leq [1,1,...,1]^T$ and thus

$$\frac{P(\mathbf{d}, \mathscr{C}_n)}{P(\mathbf{d}, \mathscr{F})} \geq \frac{P(\mathbf{d}, \mathscr{D})}{P(\mathbf{d}, \mathscr{F})} = \frac{\sum_{i=1}^n \min\{d_i, 1\}}{\min\{\sum_{i=1}^n d_i, n\}} = \frac{\sum_{i=1}^n d_i}{\sum_{i=1}^n d_i} = 1 \geq \frac{P(\mathbf{d}, \mathscr{A})}{P(\mathbf{d}, \mathscr{F})},$$

which implies $R^r(\mathscr{A}, S) \leq R^r(\mathscr{C}_n, S)$.

Otherwise, $d_{\max} > 1$, let $\mathbf{d}^* := [d_{\max}, 0, 0, ..., 0]^T$. By the assumption on S, $\mathbf{d}^* \in S$. Hence,

$$R^r(\mathscr{A}, S) \leq \frac{P(\mathbf{d}^*, \mathscr{A})}{P(\mathbf{d}^*, \mathscr{F})} \leq \frac{\min\{1, d_{\max}\}}{\min\{n, d_{\max}\}} = \frac{1}{\min\{n, d_{\max}\}}.$$

However, for any $\mathbf{d} \in S$, let $X := \{i | d_i < 1\}$, $Y := \{i | d_i > 1\}$, then

$$\frac{P(\mathbf{d}, \mathscr{C}_n)}{P(\mathbf{d}, \mathscr{F})} \geq \frac{P(\mathbf{d}, \mathscr{D})}{P(\mathbf{d}, \mathscr{F})} = \frac{\sum_{i=1}^n \min\{d_i, 1\}}{\min\{\sum_{i=1}^n d_i, n\}} \geq \frac{\sum_{i \in X} d_i + |Y|}{\sum_{i=1}^n d_i} = \frac{\sum_{i \in X} d_i + |Y|}{\sum_{i \in X} d_i + \sum_{i \in Y} d_i}.$$

But $\frac{\sum_{i \in X} d_i}{\sum_{i \in X} d_i} = 1 > \frac{1}{d_{\max}}$, and $\frac{|Y|}{\sum_{i \in Y} d_i} \geq \frac{|Y|}{|Y| d_{\max}} \geq \frac{1}{d_{\max}}$, hence,

$$\frac{P(\mathbf{d}, \mathscr{C}_n)}{P(\mathbf{d}, \mathscr{F})} \geq \frac{\sum_{i \in X} d_i + |Y|}{\sum_{i \in X} d_i + \sum_{i \in Y} d_i} \geq \frac{1}{d_{\max}}.$$

Thus, we also have that $R^r(\mathscr{C}_n, S) \geq R^r(\mathscr{A}, S)$. □

4.4 Numerical Experiments

In this section, we present computational experiments to show that the plant cover indices not only reveal the strength of a structure from the worst-case point-of-view, but also from the average-case point-of-view. Motivated by this finding, we present a heuristic that applies the plant cover indices to generate flexibility structures that are effective in both worst-case and average-case performance measures.

4.4.1 Plant Cover Indices and Expected Sales

Our objective in this section is to test the following hypothesis: Given a pair of flexibility structures $\mathscr{A}_1$ and $\mathscr{A}_2$, if $\mathscr{A}_1$ is strictly more robust than $\mathscr{A}_2$, then the expected sales of $\mathscr{A}_1$, $\mathbb{E}[P(\mathbf{D}, \mathscr{A}_1)]$, is greater than the expected sales of $\mathscr{A}_2$, $E[P(\mathbf{D}, \mathscr{A}_2)]$, for some IID demand distribution $\mathbf{D}$.

To test this hypothesis, we randomly generated 50 flexibility structures as follows. We start with a structure that is analogous to the dedicated structure, where each product node b_j is neighbor to only one plant node, and each plant node a_i is connected to exactly c_i product nodes. Then, we select the next K arcs uniformly at random and add them to the system. In our first test, we have $m = n = 10$, $K = 18$ and $c_i = 1$ for $i = 1, 2, ..., 10$. Hence, we test a balanced system with homogenous plants. In our second test, we have $m = 7$, $n = 14$, $K = 10$ and $c_1 = c_2 = 3$, $c_3 = c_4 = c_5 = 2$ and $c_6 = c_7 = 1$. In this case, we test an unbalanced system with varying capacities.

The test is designed as follows. For the 50 randomly generate flexibility structures we identify every pair of structures, $\mathscr{A}_1$ and $\mathscr{A}_2$ where $\delta^k(\mathscr{A}_1) \geq \delta^k(\mathscr{A}_2)$ for all $1 \leq k \leq n$, and for some k^*, $1 \leq k^* \leq n$, $\delta^{k^*}(\mathscr{A}_1) > \delta^{k^*}(\mathscr{A}_2)$. For each pair, we compute the difference between sales of $\mathscr{A}_1$ and $\mathscr{A}_2$ with $500n$ demand instances, where the demand of each product is generated IID at random with random distribution $N(1, 0.5)$ truncated at 0. If our test suggests (with 95% confidence) that the expected sales of $\mathscr{A}_1$ is higher than the expected sales of $\mathscr{A}_2$ given that the demand for each product is IID, we then say that the plant cover indices are *consistent* with the expected sales in the stochastic setting; if our test suggests (again, with 95% confidence) that expected sales of $\mathscr{A}_1$ is lower than the expected sales of $\mathscr{A}_2$, we then say that the plant cover indices are *inconsistent* with the expected sales in the stochastic setting; otherwise, we say the test is *inconclusive*. The results of these tests are presented in Table 4.1 under the row "Coordinates".

In addition, we perform a similar test for every pair of structures, $\mathscr{A}_1$ and $\mathscr{A}_2$, where we compare every pair of structures $\mathscr{A}_1$ and $\mathscr{A}_2$ where $\sum_{k=1}^{n} \delta^k(\mathscr{A}_1) > \sum_{k=1}^{n} \delta^k(\mathscr{A}_2)$ instead of comparing each δ^k. The results of these tests are presented in Table 4.1 under the row "Sum".

As you can see, the numerical results suggest that the plant cover index is not only powerful from worst-case performance point-of-view (Theorem 4.1), but also from average-case point-of-view.

System	Testing Metric	Consistent	Inconsistent	Inconclusive
m=10, n=10, K=18	Coordinates	685	19	13
	Sum	998	48	25
m=7, n=14, K=10	Coordinates	220	0	0
	Sum	1067	46	1

Table 4.1: Consistency between average sales and plant cover indices

4.4.2 Generating Effective Flexibility Structure

Section 4.4.1 demonstrates that plant cover indices can be a strong indicator of the effectiveness of a flexility structure. In this section, we propose a heuristic that applies the plant cover indices to generate an effective flexibility structure.

Consider an initial structure and a budget of K additional arcs. The plant cover heuristic adds K arcs sequentially through K steps. At each step, for the current flexibility structure $\mathscr{A}$, the heuristic computes $\delta^k(\mathscr{A})$ for all $1 \leq k \leq n$. Then, the heuristic finds the arc that is estimated to have the biggest impact on $\sum_{k=1}^n \delta^k(\mathscr{A})$ and adds this arc to $\mathscr{A}$. The heuristic is formally described as Algorithm 2.

Algorithm 2 Finding Effective Flexibility Design using Plant Cover Indices

Given: flexibility structure $\mathscr{A}$ in a m plants n products system and integer K.
for $t = 1, 2, \ldots, K$ **do**
    Find $\delta^1(\mathscr{A})$, $\delta^2(\mathscr{A})$, ..., $\delta^n(\mathscr{A})$, and their corresponding optimal solutions $(\mathbf{p}^1, \mathbf{q}^1)$, $(\mathbf{p}^2, \mathbf{q}^2)$, ..., $(\mathbf{p}^n, \mathbf{q}^n)$.
    Let $\Psi(x, y) = 1$ if $x = y = 0$ and $\Psi(x, y) = 0$ otherwise. For each $1 \leq i \leq m$, $1 \leq j \leq n$, compute $W(i, j) = \sum_{k=1}^n \Psi(p_i^k, q_j^k)$.
    Find an arc set S which for any arc $(a_{i^*}, b_{j^*}) \in S$,
$p(i^*, j^*) = \max\{W(i, j) | 1 \leq i \leq m, 1 \leq j \leq n\}$.
    Randomly select an arc from S, add this arc to $\mathscr{A}$.
end for
Return $\mathscr{A}$.

There are other variants of Algorithm 2 which one may consider. In particular, one can compute $W(i, j) = \sum_{k=1}^k f(\Psi(p_i^k, q_j^k), k)$ for some function $f(.,.)$. In Algorithm 2, we used function $f(\Psi(p_i^k, q_j^k), k) = \Psi(p_i^k, q_j^k)$. Another natural candidate for f is

$f(\Psi(p_i^k, q_j^k), k) = \Psi(p_i^k, q_j^k)c_i$. Indeed, finding the $f(\Psi(p_i^k, q_j^k), k)$ that optimize the performance of our heuristic is an interesting open question.

Table 4.2 presents some numerical results with the plant cover heuristic by comparing its performance to that of other structures. These structures include: (i) The structure with the highest expected sales among 50 randomly generated structures; (ii) Incomplete 3-chain, which is our attempt to construct a 3-chain structure described in [Chou et al., 2011] using K available arcs (see Figure 4-3); (iii) The structure generated by the expander heuristic in [Chou et al., 2011]; (iv) Full flexibility structure.

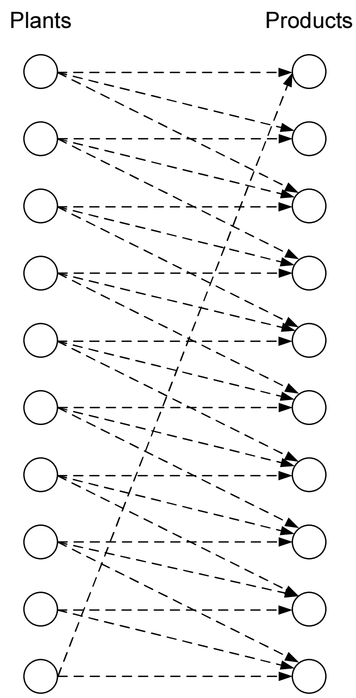
Like our tests in Section 4.4.1, we consider two set of tests, where the first set has $m = n = 10$, $K = 18$, $c_i = 1$ for $1 \leq i \leq 10$ and the second set has $m = 7, n = 14$, $K = 10$, $c_1 = c_2 = 3$, $c_3 = c_4 = c_5 = 2$ and $c_6 = c_7 = 1$. In Table 4.2, we present the average sales of each structure and the worst ratio of the sales of the structure under consideration to that of full flexibility among all $500n$ demand instances.

System	Design	Average Sales	Worst Ratio to Full Flex.
m=10, n=10, K=18	Plant Cover	9.3940	89.77%
	Random	9.3499	82.82%
	3-Chain	9.3548	82.58%
	Expander	8.9629	74.08%
	Full Flexibility	9.4088	100%
m=7, n=14, K=10	Plant Cover	13.2436	88.21%
	Random	13.089	79.82%
	3-Chain	13.1675	81.85%
	Expander	13.067	75.97%
	Full Flexibility	13.2864	100%

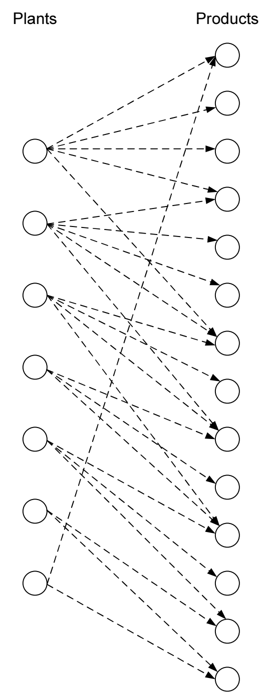
Table 4.2: Comparison between the Plant Cover Heuristics and others heuristics

Finally, we analyze the performance of the plant cover heuristic with different number of arcs added to the system and compare its performance to that of the expander heuristic of [Chou et al., 2011]. Figure 4-4 plots the ratio between the expected sales of the structure generated by the plant cover heuristic to that of full flexibility for $K = 1 : 20$ in the $m = 7, n = 14$, $c_1 = c_2 = 3$, $c_3 = c_4 = c_5 = 2$ and $c_6 = c_7 = 1$ system. One can see that the ratio between the expected sales of the plant cover heuristic structure to that of full flexibility exceeds 99% when more than

9 arcs are added to the system. Figure 4-4 also plots the ratio between the expected sales of the structure generated by the expander heuristic to that of full flexibility for $K = 1 : 20$. As you can see, in this test setting, the performance of a structure generated by the plant cover heuristic using K arcs is comparable to the performance of a structure generated by the expander heuristic using $K + 2$ arcs.



(a) $m = n = 10$



(b) $m = 7, n = 14$

Figure 4-3: Incomplete 3-Chains

As Table 4.2 and Figure 4-4 show, when demand is IID, the plant cover heuristic can find effective flexibility structures that perform well in average-case and worst-case. In general, we expect this heuristic to work well when the products' expected demands do not vary significantly. However, when expected demand varies, the plant cover heuristic does not necessarily perform well since it ignores demand information. By contrast, the expander heuristics is shown to work well when different products have varying expected demand.

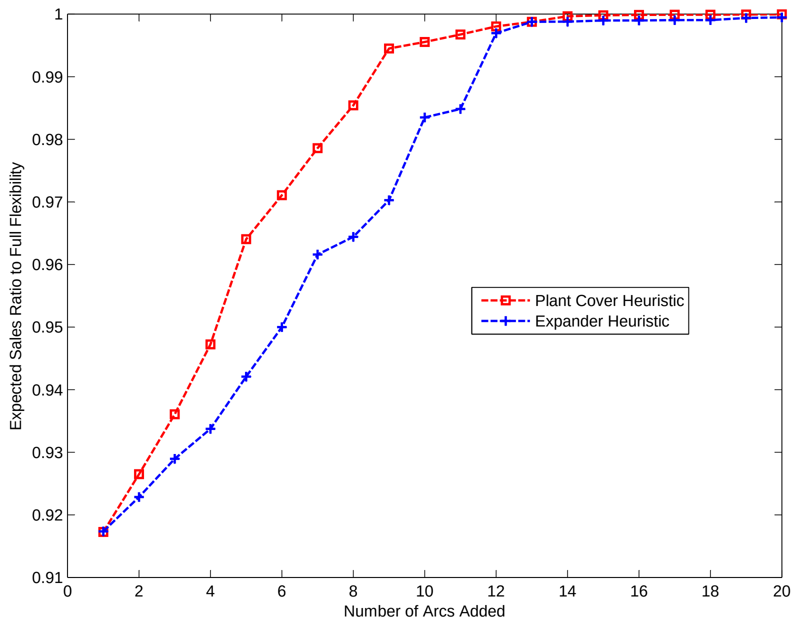


Figure 4-4: Designs Generated by Plant Cover and Expander Heuristics

4.5 Extensions

In this section, we presents two extensions of Theorem 4.1.

4.5.1 Additional Production Constraints

First, we consider a model where the sales of $\mathscr{A}$ not only depends on demands and plants capacities, but also on another class of linear constraints. In this case, $P(\mathbf{d}, \mathscr{A})$ is defined as the objective value of the following LP.

$$
\begin{aligned}
P(\mathbf{d}, \mathscr{A}) = \max \quad & \sum_{(i,j)\in I(\mathscr{A})} f_{ij} \\
\text{s.t.} \quad & \sum_{i\in I(N(j,\mathscr{A}))} f_{ij} \le d_j, \ \forall j \in I(B) \\
& \sum_{j\in I(N(i,\mathscr{A}))} f_{ij} \le c_i, \ \forall i \in I(A) \\
& \sum_{(i,j)\in I(\mathscr{A})} \Phi_{hij} f_{ij} \le \phi_h, \forall h = 1, 2, ..., H \\
& 0 \le f_{ij}, \forall (i,j) \in I(\mathscr{A}) \\
& \mathbf{f} \in \mathbb{R}^{|\mathscr{A}|}.
\end{aligned}
$$

For example, in some applications, an added flexibility arc (a_i, b_j) can only be utilized for p $(p < 1)$ fraction of the capacity at plant i. In that case, we would have the additional constraint $f_{ij} \le pc_i$.

Under this setting, we define the plant cover index, $\delta^k(\mathscr{A})$, for $0 \le k \le n$, as follows.

$$
\begin{aligned}
\delta^k(\mathscr{A}) := \min & \sum_{i=1}^{m} c_i p_i + \sum_{h=1}^{H} \phi_h z_h \\
\text{s.t.} \quad & \sum_{j=1}^{n} q_j = k, \\
& p_i + q_j + \sum_{h=1}^{H} \Phi_{hij} z_h \ge 1, \forall (i,j) \in I(\mathscr{A}) \\
& z_h \ge 0, \forall\, 1 \le h \le H, \\
& \mathbf{p} \in [0,1]^m, \mathbf{q} \in \{0,1\}^n, \mathbf{z} \in \mathbb{R}^h.
\end{aligned}
$$

Next, note that the dual of the LP of $P(\mathbf{d}, \mathscr{A})$ can be written as follows:

$$\begin{aligned} P(\mathbf{d}, \mathscr{A}) = \max \quad & \sum_{i=1}^{m} c_i p_i + \sum_{j=1}^{n} d_j q_j + \sum_{h=1}^{H} \phi_h z_h \\ \text{s.t.} \quad & p_i + q_j + \sum_{h=1}^{H} \Phi_{hij} z_h \geq 1, \forall (i,j) \in I(\mathscr{A}) \\ & z_h \geq 0, \forall\, 1 \leq h \leq H, \\ & \mathbf{p} \in [0,1]^m, \mathbf{q} \in [0,1]^n, \mathbf{z} \in \mathbb{R}^h. \end{aligned}$$

Consider the case where the dual problem has no integrality gap with respect to $\mathbf{q}$, that is, the objective value is unchanged when we relax the integrality constraint on $\mathbf{q}$ and use $\mathbf{q} \in [0,1]^n$ instead. In this case, we can apply the same proof techniques as in Section 4.2, and develop the same result as Theorem 4.1 for this more general settings.

The dual problem has a tight integrality gap with respect to $\mathbf{q}$, when the system of inequalities,

$$\sum_{i \in I(N(j,\mathscr{A}))} f_{ij} \leq d_j, \; \forall j \in I(B) \tag{4.1}$$

$$\sum_{j \in I(N(i,\mathscr{A}))} f_{ij} \leq c_i, \; \forall i \in I(A) \tag{4.2}$$

$$\sum_{i,j \in I(\mathscr{A})} \Phi_{hij} f_{ij} \leq \phi_h, \forall h = 1, 2, ..., H, \tag{4.3}$$

is *totally dual integral* (see Section 8.6 of [Bertsimas and Weismantel, 2008] for a more detailed discussion of this topic). In particular, if all inequalities in (4.3) are of the form $f_{ij} \leq r_{ij}$, then we have that the system of inequalities in (4.1-4.3) is totally dual integral. Observe that the constraint $f_{ij} \leq r_{ij}$ is equivalent in our model to requiring that for some h, $1 \leq h \leq H$, $\Phi_{hij} = 1$ and $\Phi_{hi'j'} = 0$ for all other $(i', j') \in I(\mathscr{A})$.

Using the plant cover indices defined under this more general setting, we can define a similar plant cover heuristic like the one proposed in Section 4.4.2. When the dual of the LP defining $P(\mathbf{d}, \mathscr{A})$ has a tight integrality gap with respect to $\mathbf{q}$, we expect the effectiveness of the plant cover heuristics in this more general setting to be comparable with the effectiveness of the plant cover heuristics in Section 4.4.2.

Finally, we note that when the dual of the LP defining $P(\mathbf{d}, \mathscr{A})$ does have a duality gap with respect to $\mathbf{q}$, then the theoretical results from Section 4.2 no longer hold. In this case, we expect the plant cover heuristic to work reasonably well when the integrality gap is reasonably small.

4.5.2 General Uncertainty Sets

All the results we established so far assumed that the uncertainty set S is symmetric. Can we say anything if S is not symmetric? Clearly, if we do not impose any constraint on S, there isn't much we can do. Next, consider the following definition.

Definition 4.1. *We say S is* min-consistent *with $\boldsymbol{\mu} \in \mathbb{R}^n$, if for any $X_1, X_2 \subset \{1, 2, ..., n\}$, we have*

$$\sum_{i \in X_1} \mu_i \leq \sum_{i \in X_2} \mu_i \iff \min\{\sum_{i \in X_1} d_i | \mathbf{d} \in S\} \leq \min\{\sum_{i \in X_2} d_i | \mathbf{d} \in S\}.$$

Note that if S is symmetric, then S is min-consistent with $\mathbf{e}$ where $e_1 = e_2 = ... = e_n = 1$.

In this case, we define the vertex cover index, $\delta^k(\mathscr{A}, \boldsymbol{\mu})$, as

$$\begin{aligned} \delta^k(\mathscr{A}, \boldsymbol{\mu}) := \min & \sum_{i=1}^{m} c_i p_i \\ \text{s.t.} \quad & \sum_{j=1}^{n} q_j \mu_j \leq k, \\ & p_i + q_j \geq 1, \forall (i,j) \in I(\mathscr{A}) \\ & \mathbf{p} \in \{0,1\}^m, \mathbf{q} \in \{0,1\}^n. \end{aligned}$$

Next, we state a worst-case analysis result that is analogous to Theorem 4.1, in the setting where the uncertainty set is min-consistent with $\boldsymbol{\mu}$. The proof of Corollary 4.4 is left in Appendix A.

Corollary 4.4. *Let $\boldsymbol{\mu}$ be an arbitrary non-negative vector in $\mathbb{R}^n$. Suppose $\delta^k(\mathscr{A}_1, \boldsymbol{\mu}) \geq \delta^k(\mathscr{A}_2, \boldsymbol{\mu})$ for all $k \in \{\sum_{j \in X} \mu_j | X \subseteq \{1, 2, ..., n\}\}$. Then for any uncertainty set S that is* min-consistent *with $\boldsymbol{\mu}$, we have $R^s(\mathscr{A}_1, S) \geq R^s(\mathscr{A}_2, S)$.*

We would like to point out that the extension results in this subsection are weaker than Theorem 4.1. By allowing the demand set to be non-symmetric, it is no longer true that if $\delta^k(\mathscr{A}_1, \boldsymbol{\mu}) \geq \delta^k(\mathscr{A}_2, \boldsymbol{\mu})$, then $R(\mathscr{A}_1, S) \geq R(\mathscr{A}_2, S)$ for any robust measure $R \in \Gamma$.

While we would like to use Corollary 4.4 as a guideline to develop practical heuristics for generating process flexibility designs, several key difficulties arise. First, it is unclear how to come up with the vector $\boldsymbol{\mu}$. Second, it may take a long time to compute all $k \in \{\sum_{j \in X} \mu_j | X \subseteq \{1, 2, ..., n\}\}$, as the set $\{\sum_{j \in X} \mu_j | X \subseteq \{1, 2, ..., n\}\}$ can contain up to 2^n values. Thus, at this moment, while Corollary 4.4 is an interesting theoretical property it does not provide useful guidelines for generating effective process flexibility designs when product demands are not homogenous.

4.6 Discussion and Conclusion

This chapter studies the worst-case performance of process flexibility when demand can take values in an uncertainty set. We prove that the worst-case performance, i.e., the robustness, of a flexibility structure relative to that of other structures is largely independent of the choice of uncertainty sets, or the performance measures. To establish this result, we introduce the plant cover index, an index that only depends on the flexibility structure. We prove that if all of the plant cover indices of one flexibility structure, $\mathscr{A}_1$, are greater than or equal to the plant cover indices of another structure, $\mathscr{A}_2$, then the worst-case performance of $\mathscr{A}_1$ is greater than or equal to the worst-case performance of $\mathscr{A}_2$, for all symmetric uncertainty sets and a large class of performance functions.

Using the condition established with plant cover index, we prove that the long chain flexibility structure is more robust than any structure that has degree two on all of its product nodes, or forms a connected bipartite graph with $2n$ arcs. Because long chain flexibility structure has an attractive average-case performance, see [Simchi-Levi and Wei, 2012], we investigate whether in general a structure with high plant cover indices performs well not only in worst-case but also in average-case. We answer this question in the affirmative using a numerical study, and propose a plant cover heuristics that generates flexibility structures. In our computational results, the plant cover heuristic is shown to be effective from both average-case and worst-case performances measures. Finally, we present two simple extensions to our chapter, illustrating that our analysis can be applied to more general settings.

Finally, we would like to point out the connection between the plant cover index, and the graph expanders studied in [Chou et al., 2011]. In [Chou et al., 2011], $\mathscr{A}$ is $(\alpha, \lambda, \Delta)$-expander if

$$\min_{1 \leq k \leq \alpha n} \frac{\delta^{n-k}(\mathscr{A})}{k} \geq \lambda$$

and $|N(u, \mathscr{A})| \leq \Delta$ for any $u \in A \cup B$. In some sense, the plant cover index can be seen as a more precise indicator of the robustness of $\mathscr{A}$, compared with the expander parameters α and λ. Thus, this chapter complements the study of [Chou et al., 2011] by showing that in the case which the expander parameters α and λ are not enough to identify the effective sparse flexibility structures, plant cover index can be used as a suitable alternative. Moreover, the plant cover heuristics proposed in this chapter complements nicely the expander heuristic [Chou et al., 2011], as the plant cover heuristics works nicely in the setting when the product demands are IID or homogenous, where the expander heuristics works well when there exist large differences between the expected demands for different products demand.

Chapter 5

Process Flexibility under Distribution Systems

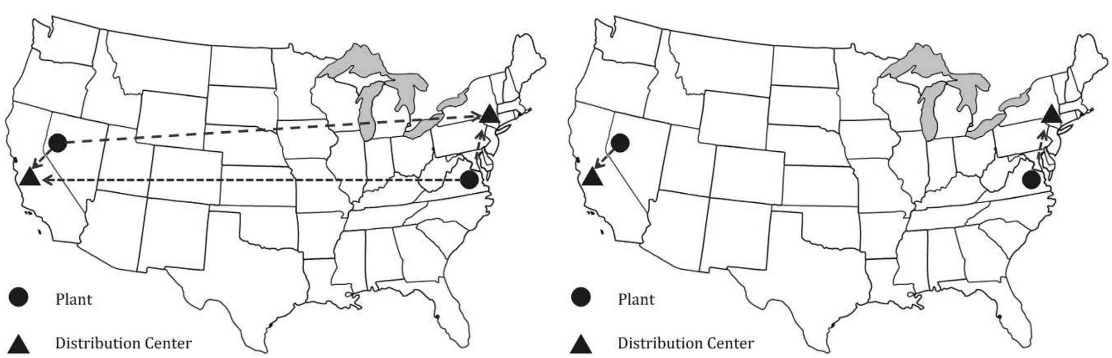
The previous chapters studied the effectiveness of process flexibility and long chain in matching available capacity with uncertain demands. As observed in [Simchi-Levi, 2010], process flexibility has other benefits, such as reducing production and transportation costs. Motivated by this observation, in this chapter, we study the design and effectiveness of process flexibility in reducing the firm's total logistics/supply chain costs.

This chapter is organized as follows. In Section 5.1, we begin with a motivating example to illustrate how process flexibility can reduce supply chain/transportation costs, and formally define our model. In Section 5.2, we study the model under general linear supply chain costs, and prove that if the supply chain costs are independent of either plants or distribution centers, then there exist a long chain that is optimal among all 2-flexibility structures. In Section 5.3, we study our model in the special case where the transportation cost is directly proportional to the geometric distance between plants and distribution centers. We then prove that if all of the plants and distributions centers lie on a line, then there exists a long chain that is optimal among all 2-flexibility structures. Moreover, our proof provides guidelines on how to find a effective long chains (in reducing transportation cost) under the general case where plants and distributions centers lie on a plane.

5.1 Motivating Example and Model

We first start with a simple example to illustrate the reduction in supply chain cost under deterministic demand. In the example, a firm has two plants, and two distributions centers, as shown on Figure 5-1. The firm also produces two types of products, and at each distribution center, the demand of each type of product is one. We suppose that one pair of plant and distribution center is on the east coast of United States, and another pair is on the west coast (see Figure 5-1).

If each plant is dedicated to producing just one type of product, the firm would then have to ship one unit of product from each plant to each distribution center. As illustrated on the left hand side of Figure 5-1, the firm have to ship one product from its east coast plant to its west coast distribution center, and from its west coast plant to its east coast distribution center. However, if each plant has the flexibility of producing both products, then the firm can supply its distribution centers on the east and west coasts with their nearby plants, and thus achieve a significant reduction in transportation cost.



(a) System without Flexibility (b) System with Flexibility

Figure 5-1: Reducing Transportation Cost with Flexibility [1]

The transportation example shows that having flexibility can greatly reduce transportation and thus supply chain cost under deterministic demand. Next, we will formally introduce the distribution system model for this chapter. Like the previous chapters, we will consider a balanced system of size n, that is, a system with n plants and n products. In this chapter, in addition to plants and products, distribution centers are also included. We use n' to denote the number of distribution centers, and $B' = \{b'_1, b'_2, ..., b'_{n'}\}$ to denote the set of distribution centers. Our model assumes that products are first produced at plants, and then transported to distribution centers to satisfy demand. The demand is deterministic and d_{jk} is used to denote the demand of product j at distribution center k. Also, we use p_{ijk} to denote the total cost for producing one unit of product j from plant i, and then shipping to distribution center k. The total supply chain cost under flexibility design $\mathscr{A}$ is defined as

$$
\begin{aligned}
P^D(\mathscr{A}) = \min \; & \sum_{i,j,k} p_{ijk} f_{ijk} \\
\text{s.t.} \quad & \sum_{i} f_{ijk} = d_{jk}, \forall 1 \le j \le n, 1 \le k \le n', \\
& \sum_{j,k} f_{ijk} \le c_i, \forall 1 \le i \le n, \\
& f_{ijk} \ge 0, \forall 1 \le i \le n, 1 \le j \le n, 1 \le k \le n', \\
& f_{ijk} = 0, \forall (a_i, b_j) \notin \mathscr{A}, \\
& \mathbf{f} \in \mathbb{R}^{n^2 n'}
\end{aligned} \tag{P}
$$

We will refer to $P^D(\mathscr{A})$ as the total supply chain cost with flexibility structure $\mathscr{A}$ under our distribution system model. In the linear program defining $P^D(\mathscr{A})$, f_{ijk} can be thought of the amount of product j produced from plant i and and shipped to distribution center k.

Because we do not impose symmetry in vector $\mathbf{p}$, we will consider any long chain of size n, that is, any structure $\mathscr{A}$ where all of its arcs form one undirected cycle containing all plants and all products. Note that for any $\mathscr{A}$ that is a long chain, then there exists permutations σ, σ' that $\{(a_{\sigma(i)}, b_{\sigma'(j)}) | (a_i, b_j) \in \mathscr{A}\} = \mathscr{C}_n$. Finally, recall that a flexibility structure is a 2-flexibility structure if each plant node and each product node are incident to exactly two arcs, and the set of all 2-flexibility structures is denoted by $\mathbb{F}_2$.

5.2 Long Chain under General Costs

In this section, we focus on the design of sparse flexibility structures under general supply chain cost vector $\mathbf{p}$.

First, we prove that if the supply chain cost is independent with the production plants, i.e., $p_{ijk} = p_{ijk'}$ for all $1 \leq i, j \leq n$, and $1 \leq k, k' \leq n'$, then there always exists a long chain that is optimal among 2-flexibility structures. We note that a preliminary version of this proof was discovered by [Chou et al., 2010a].

Proposition 5.1. *If $p_{ijk} = p_{ijk'}$, $\forall k, k' = 1, .., n'$, and $\forall i, j$. Then there always exists some 2-flexibility structure $\mathscr{A}^o$ where $\mathscr{A}^o$ is a long chain, and $P^D(\mathscr{A}^o) = \min_{\mathscr{A} \in \mathbb{F}_2} P^D(\mathscr{A})$.*

Proof. Let $p_{ij} = p_{ij1}$, for any i, j. It is easy to check that $P^D(\mathscr{A})$ is equal to

$$
\begin{aligned}
P^D(\mathscr{A}) = \min \quad & \sum_{i,j,k} p_{ij} f_{ij} \\
\text{s.t.} \quad & \sum_i f_{ij} = \sum_k d_{jk}, \forall 1 \leq j \leq n, \\
& \sum_j f_{ij} \leq c_i, \forall 1 \leq i \leq n, \\
& f_{ij} \geq 0, \forall 1 \leq i, j \leq n, \\
& f_{ij} = 0, \forall (a_i, b_j) \notin \mathscr{A}, \\
& \mathbf{f} \in \mathbb{R}^{n^2}.
\end{aligned}
$$

Suppose $\mathscr{A}^*$ is an optimal structure among all 2-flexibility structures. Wlog, we may assume $\mathscr{D}_n \subset \mathscr{A}^*$, since we can always relabel the plants to satisfy this condition. Let $\mathscr{A}^1, \mathscr{A}^2, ..., \mathscr{A}^t$ be disjoint components of $\mathscr{A}^*$ such that $\mathscr{A}^1 \cup \mathscr{A}^2 \cup ... \cup \mathscr{A}^t = \mathscr{A}^*$. Because $\mathscr{A}^*$ is a 2-flexibility structure, each component $\mathscr{A}^i$ must form a single undirected cycle. In particular, if $t = 1$, then $\mathscr{A}^* = \mathscr{A}^1$ must be a long chain.

If $t \geq 2$, wlog, assume $\{(a_i, b_i) | 1 \leq i \leq t_1\} = \mathscr{A}^1 \cap \mathscr{D}_n$, $\{(a_i, b_i) | t_1 + 1 \leq i \leq t_2\} = \mathscr{A}^2 \cap \mathscr{D}_n$. Let $S^1 = \mathscr{A}^1 \setminus \{(a_i, b_i) | 1 \leq i \leq t_1\}$, $S^2 = \mathscr{A}^2 \setminus \{(a_i, b_i) | t_1 + 1 \leq i \leq t_2\}$. Next, we prove that there always exist $\alpha^1 \in \mathscr{A}^1$, $\alpha^2 \in \mathscr{A}^2$ such that $P^D(\mathscr{A}^1) =$

$P^D(\mathscr{A}^1 \setminus \{\alpha^1\})$ and $P^D(\mathscr{A}^2) = P^D(\mathscr{A}^2 \setminus \{\alpha^2\})$.

Let $\mathbf{f}^1$ be an optimal solution of $P^D(\mathscr{A}^1)$. If $f^*_{\alpha^1} = 0$ for some $\alpha^1 \in \mathscr{A}^1$, then $P^D(\mathscr{A}^1) = P^D(\mathscr{A}^1 \setminus \{\alpha^1\})$. Otherwise, $f^1_{ij} > 0$ for all $(a_i, b_j) \in \mathscr{A}^1$, and let $\mathbf{g}$ be the vector that

$$g_{ij} = \begin{cases} -1 & \text{if } (a_i, b_j) \in S^1 \\ 1 & \text{if } (a_i, b_j) \in \mathscr{A}^1 \setminus S^1 \\ 0 & \text{otherwise.} \end{cases}$$

If $\sum_{i,j} g_{ij} p_{ij} \leq 0$, let $\delta^1 := \min\{f^1_{ij}, (a_i, b_j) \in S^1\}$. Then, $\mathbf{f}^1 + \delta^1 \mathbf{g}$ is feasible for $P^D(\mathscr{A}^1)$, and because $\sum_{i,j} g_{ij} p_{ij} \leq 0$, $\mathbf{f}^1 + \delta^1 \mathbf{g}$ must also be optimal solution of $P^D(\mathscr{A}^1)$. Because $f^1_{ij} + \delta^1 g_{ij} = 0$ for some $(a_i, b_j) \in S^1$, therefore, there always exists some $\alpha^1 \in S^1$ such that $P^D(\mathscr{A}^1) = P^D(\mathscr{A}^1 \setminus \{\alpha^1\})$. Likewise, if $-\sum_{i,j} g_{ij} p_{ij} \leq 0$, we can show that there always exists some $\alpha^1 \in \mathscr{A}^1 \setminus S^1$ such that $P^D(\mathscr{A}^1) = P^D(\mathscr{A}^1 \setminus \{\alpha^1\})$. Therefore, we proved that there always exists some $\alpha^1 \in \mathscr{A}^1$ such that $P^D(\mathscr{A}^1) = P^D(\mathscr{A}^1 \setminus \{\alpha^1\})$.

Similarly, we can also prove that there exists some $\alpha^2 \in \mathscr{A}^2$ such that $P^D(\mathscr{A}^2) = P^D(\mathscr{A}^2 \setminus \{\alpha^2\})$. Suppose $\alpha^1 = (a_{i_1}, b_{j_1}) \in S^1$, $\alpha^2 = (a_{i_2}, b_{j_2}) \in S^2$, with $P^D(\mathscr{A}^1) = P^D(\mathscr{A}^1 \setminus \{\alpha^1\})$ and $P^D(\mathscr{A}^2) = P^D(\mathscr{A}^2 \setminus \{\alpha^2\})$. Let $\alpha^3 = (a_{i_1}, b_{j_2})$ and $\alpha^4 = (a_{i_2}, b_{j_1})$, and let $\mathscr{A}^{1'} := (\mathscr{A}^1 \setminus \{\alpha^1\}) \cup (\mathscr{A}^2 \setminus \{\alpha^2\}) \cup \{\alpha^3, \alpha^4\}$. Then observe that $\mathscr{A}^{1'}$ forms exactly one undirected cycle (see Figure 5-2), and $P^D(\mathscr{A}^{1'}) \geq P^D((\mathscr{A}^1 \setminus \{\alpha^1\}) \cup (\mathscr{A}^2 \setminus \{\alpha^2\})) = P^D(\mathscr{A}^1 \setminus \{\alpha^1\}) + P^D(\mathscr{A}^2 \setminus \{\alpha^2\}) = P^D(\mathscr{A}^1) + P^D(\mathscr{A}^2)$. Therefore, we have $\mathscr{A}^{**} = \mathscr{A}^{1'} \cup \mathscr{A}^3 \cup ... \cup \mathscr{A}^t$ is also an optimal structure among all 2-flexibility structures. Because $\mathscr{A}^{**}$ has one less disjoint components than $\mathscr{A}^*$, by induction, we can show that there exist some $\mathscr{A}^o$ where $\mathscr{A}^o$ is connected and is an optimal structure among all 2-flexibility structures. Finally, if $\mathscr{A}^o$ is a connected 2-flexibility structure, then it must be a long chain, and therefore, the proof is complete. □

Next, we prove that if the supply chain cost is independent with the production plants, i.e., $p_{ijk} = p_{i'jk}$ for any $1 \leq i, i', j \leq n$ and $1 \leq k \leq n'$, then there also exists a long chain that is optimal among all 2-flexible structures.

Proposition 5.2. *Suppose* $p_{ijk} = p_{i'jk}$, $\forall i, i' = 1, .., n$, *and* $\forall 1 \leq j \leq n, 1 \leq k \leq n'$.

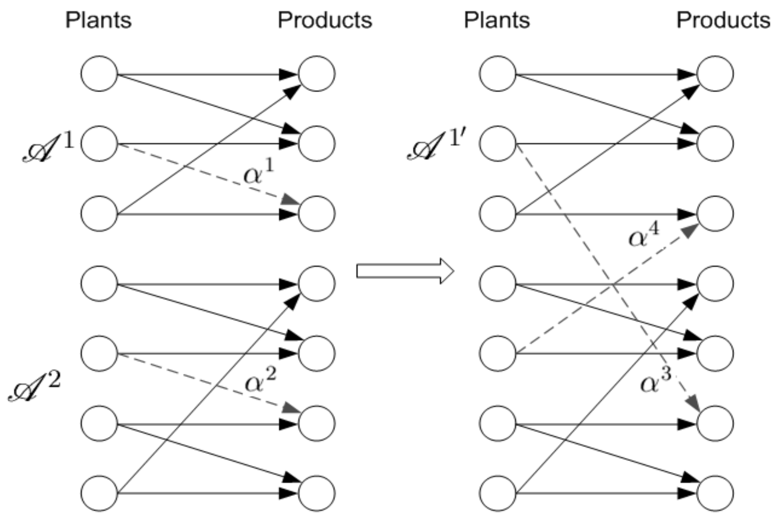


Figure 5-2: Illustration of $\mathscr{A}^1$, $\mathscr{A}^2$ and $\mathscr{A}^{1'}$

Then there always exists some 2-flexibility structure $\mathscr{A}^o$ where $\mathscr{A}^o$ is a long chain, and $P^D(\mathscr{A}^o) = \min_{\mathscr{A} \in \mathbb{F}_2} P^D(\mathscr{A})$.

Proof. Because $p_{ijk} = p_{i'jk}$, let $p_{jk} := p_{1jk}$. We claim that $P^D(\mathscr{A})$ is equal to the objective of the following linear program:

$$
\begin{aligned}
P^D(\mathscr{A}) = \min \quad & \sum_{j,k} p_{jk} y_{jk} && (P') \\
\text{s.t.} & \sum_k y_{jk} = \sum_i x_{ij}, \forall 1 \le j \le n, \\
& y_{jk} = d_{jk}, \forall 1 \le j \le n, 1 \le k \le n', \\
& \sum_j x_{ij} \le c_i, \forall 1 \le i \le n, \\
& x_{ij} \ge 0, \ \forall 1 \le i, j \le n, \\
& x_{ij} = 0, \forall (i,j) \notin \mathscr{A}, \\
& y_{jk} \ge 0, \ \forall 1 \le j \le n, 1 \le k \le n', \\
& \mathbf{x} \in \mathbb{R}^{n^2}, \ \mathbf{y} \in \mathbb{R}^{nn'}.
\end{aligned}
$$

To prove our claim, we need to show that the above linear program, (P'), has the same optimal value as (P), original linear program defining $P^D(\mathscr{A})$. For this, we note that for any $\mathbf{f}$ that is feasible to (P), if we define $x_{ij} = \sum_k f_{ijk}$ for all $1 \le i, j \le n$ and $y_{jk} = \sum_i f_{ijk}$, for all $1 \le j \le n, 1 \le k \le n'$, then $(\mathbf{x},\mathbf{y})$ is a feasible solution for (P'), with objective value

$$\sum_{j,k} p_{jk} y_{jk} = \sum_{i,j,k} p_{ij} f_{ijk}.$$

Also for any $(\mathbf{x},\mathbf{y})$ that is feasible for $\mathbf{f}$, we can always find a nonnegative $\mathbf{f}$ with $f_{ijk} = x_{ij} \frac{y_{jk}}{\sum_t y_{jt}}$, with objective value

$$\sum_{i,j,k} p_{ij} f_{ijk} = \sum_{j,k} p_{jk} (\sum_i x_{ij}) \frac{y_{jk}}{\sum_t y_{jt}} = \sum_{j,k} p_{jk} y_{jk}.$$

Thus, we have that (P') and (P) have the same optimal values.

Let $\mathscr{A}^*$ be a 2-flexibility structure where $P^D(\mathscr{A}^*) = \min_{\mathscr{A} \in \mathbb{F}_2} P^D(\mathscr{A})$. Then, let $(\mathbf{x}^*, \mathbf{y}^*)$ be a pair of optimal solution for the linear program (P') defining $P^D(\mathscr{A}^*)$. Now, define $EQ(\mathscr{A})$ to be the following set of equations:

$$EQ(\mathscr{A}) = \left(\begin{array}{l} \sum_i x_{ij} = \sum_k y^*_{jk}, \forall j, \\ \sum_j x_{ij} \le c_i, \forall i, \\ x_{ij} \ge 0, \ \forall i, j, \\ x_{ij} = 0, \ \forall (i,j) \notin \mathscr{A}, \\ \mathbf{x} \in \mathbb{R}^{n^2}. \end{array} \right)$$

Note that equations in $EQ(\mathscr{A})$ is simply a flow feasibility problem. Thus, by a similar augmenting flow argument used in the proof of Proposition 5.1, we can prove that if $\mathscr{A}^*$ has multiple disjoint components, then there exists $\mathscr{A}^o$ where $\mathscr{A}^o$ is a connected 2-flexibility structure, and $EQ(\mathscr{A}^o)$ has at least one feasible solution $\mathbf{x}$. In that case $(\mathbf{x}, \mathbf{y}^*)$ is feasible for the linear program (P') defining $P^D(\mathscr{A}^o)$, and thus, we have $P^D(\mathscr{A}^o) \ge P^D(\mathscr{A}^*)$. And because $\mathscr{A}^o$ is a connected 2-flexibility structure, it must be a long chain and the proof is complete. $\square$

Finally, from Proposition 5.1 and Proposition 5.2, one may believe that when the supply chain cost is independent of the products, i.e., $p_{ijk} = p_{ij'k}$ for any i, k and any j, j', then there exists a long chain that is optimal among all 2-flexibility structures. Interestingly, this is not always true. For this purpose, we provide a counter-example with $n = n' = 4$,

$$p_{ijk} := \begin{cases} 1 & \text{if } i = 1 \text{ or } k = 1, 2 \\ 1 & \text{if } i = 2 \text{ or } k = 2, 3 \\ 1 & \text{if } i = 3 \text{ or } k = 3, 4 \\ 1 & \text{if } i = 4 \text{ or } k = 4, 1 \\ 2 & \text{otherwise,} \end{cases}$$

$d_{1k} = d_{2k} = d_{3k} = d_{4k} := 1$ for $1 \leq k \leq 4$, and $c_i := 2$ for all $i = 1, 2, 3, 4$. Then, let $\mathscr{A}^* = \mathscr{D}_4 \cup \{(a_1, b_3), (a_3, b_1), (a_2, b_4), (a_4, b_2)\}$ be the graph where both plants 1 and 3 have arcs to products 1 and 3, and both plants 2 and 4 have arcs to products 2 and 4. Then, $\mathscr{A}^*$ is a 2-flexible structure, and it is not difficult to check that no long chain achieves the same or lower cost than $\mathscr{A}^*$. Therefore, we have that when $p_{ijk} = p_{ij'k}$ for any i, k and any j, j', it is not always the case that there exists a long chain that generates the lowest costs among all 2-flexible structures.

When $p_{ijk} = p_{ij'k}$ for any i, k and any j, j', one can interpret p_{ijk} as the cost of shipping a unit of product from plant i to distribution center k. Thus, the assumption of $p_{ijk} = p_{ij'k}$ can be interpreted as having a model that attempts to minimize the firm's transportation costs. In the next section, we will perform a more detailed analysis for this particular case.

5.3 Transportation Costs

In this section, we restrict our attention to study the case where the transportation costs is proportional to the distances between plants and distribution centers. To gain insight on the flexibility structure that minimizes transportation cost, we study a stylized model where the demands of different products are equal at any distribution center (i.e. $d_{jk} = d_{j'k}$ for any j, j'.), plant i can always produce product i, (i.e.

$\mathscr{D}_n \subset \mathscr{A}$, for any $\mathscr{A}$ under consideration), and the capacity for each plant i is equal to the total demand for product i (i.e. $c_i = \sum_{k=1}^{n'} d_{ik}$).

Note that in this case, $c_i = c_{i'}$ for any $1 \le i, i' \le n$ and $\sum_{i=1}^{n} c_i = \sum_{j,k} d_{jk}$. Without loss of generality, we will assume $c_i = 1$ for all $1 \le i \le n$. We define $d_k := \sum_j d_{jk}$, then, $d_{jk} = \frac{1}{n} d_k$ for any $1 \le j \le n$. Finally, we assume that the unit transportation cost of shipping any product from plant i to distribution center k is equal to the Euclidean distance between plant i and distribution k.

5.3.1 1-Dimensional Case

First, in this subsection, we assume that the locations of all plants and all distribution centers lies on a 1-dimensional line. For each $1 \le i \le n$, $1 \le k \le n'$, we use $\mathbb{L}(a_i) \in \mathbb{R}$ to denote the location of plant i, $\mathbb{L}(b'_k) \in \mathbb{R}$ to denote the location of distribution center k. Without loss of generality, we assume that $\mathbb{L}(a_1) = 0$, $\mathbb{L}(a_1) \le \mathbb{L}(a_2) \le \mathbb{L}(a_3) \le ... \le \mathbb{L}(a_n)$, and $\mathbb{L}(b'_1) \le \mathbb{L}(b'_2) \le ... \le \mathbb{L}(b'_{n'})$. Note that the distance between plant i and distribution center k is simply $|\mathbb{L}(a_i) - \mathbb{L}(b'_k)|$. Thus, for a flexibility design $\mathscr{A}$, its transportation cost is represented by the following linear program.

$$
\begin{aligned}
P^D(\mathscr{A}) = \min \sum_{i,j,k} & |\mathbb{L}(a_i) - \mathbb{L}(b'_k)| f_{ijk} \\
\text{s.t.} \quad & \sum_i f_{ijk} = \frac{d_k}{n}, \forall 1 \le j \le n, 1 \le k \le n', \\
& \sum_{j,k} f_{ijk} \le 1, \forall 1 \le i \le n, \\
& f_{ijk} \ge 0, \forall 1 \le i, j \le n, 1 \le k \le n', \\
& f_{ijk} = 0, \forall (a_i, b_j) \notin \mathscr{A}, \\
& \mathbf{f} \in \mathbb{R}^{n^2 n'},
\end{aligned}
$$

Next, we propose a linear program that is a relaxation of $P^D(\mathscr{A})$ for any flexibility structure $\mathscr{A} \in \mathbb{F}_2$.

$$\begin{aligned} \min \quad & \sum_{i,k} |\mathbb{L}(a_i) - \mathbb{L}(b'_k)| g_{ik} \qquad (RLP) \\ \text{s.t.} \quad & \sum_{i=1}^{n} g_{ik} = d_k, \forall 1 \le k \le n', \\ & \sum_{k=1}^{n'} g_{ik} = 1, \forall 1 \le i \le n, \\ & 0 \le g_{ik} \le \frac{2d_k}{n}, \forall 1 \le i \le n, 1 \le k \le n', \\ & \mathbf{g} \in \mathbb{R}^{nn'}. \end{aligned}$$

In the linear program defined by (RLP), g_{ik} can be interpreted as the volume of products transported from plant i to distribution center k. Note that for any $\mathscr{A} \in \mathbb{F}_2$, $\sum_{j:(a_i,b_j)\in\mathscr{A}} d_{jk} = \frac{2d_k}{n}$. Therefore, if we let $g_{ik} := \sum_j f_{ijk}$, then $\mathbf{g}$ is a feasible solution of (RLP), which implies that (RLP) is a relaxation of $P^D(\mathscr{A})$. In the rest of this section, we will characterize an optimal solution of (RLP), and the optimal solution will then be used to identify a long chain that has the same transportation cost as the objective value of (RLP).

Because we can always split a distribution center into two distribution centers without changing the structure of the optimization problem defining $P^D(\mathscr{A})$ and the optimization problem (RLP), we will assume without loss of generality, that there exist some t^l such that $\sum_{k=1}^{t^l} d_k = \frac{n}{2}$. The next proposition characterizes g^*_{1k}, where $\mathbf{g}^*$ is an optimal solution of (RLP).

Proposition 5.3. *Let t^l be the integer such that $\sum_{k=1}^{t^l} d_k = \frac{n}{2}$. Then, there exist an optimal solution $\mathbf{g}^*$ for (RLP) where*

$$g^*_{1k} = \begin{cases} \frac{2d_k}{n} & \textit{if } 1 \le k \le t^l \\ 0 & \textit{otherwise.} \end{cases} \tag{5.1}$$

Proof. We will use an algorithmic proof by constructing $\mathbf{g}^*$ starting from an optimal solution of (RLP), say, $\mathbf{g}$. Suppose that there exists some $1 \leq l \leq t^l$ such that $g_{1l} < \frac{2d_l}{n}$. Because $\sum_{k=1}^{n'} g_{1k} = 1$ there must exist some $h > t^l$ and $g_{1h} > 0$. Let $S = \{i \,|\, g_{il} > 0, i \neq 1\}$ and $T = \{i \,|\, g_{ih} < \frac{2d_h}{n}, i \neq 1\}$. Because $\sum_{i=1}^{n} g_{ik} = d_k$ for all k and $g_{ik} \leq \frac{2d_k}{n}$ for all i, k, we have that both $|S| \geq \frac{n}{2}$ and $|T| \geq \frac{n}{2}$. By the pigeonhole principle, we have that there exist some $i^* \in S \cap T$. Let $\boldsymbol{\tau}$ be the vector such that

$$\tau_{ik} := \begin{cases} 1 & \text{if } (i,k) = (1,l) \text{ or } (i^*,h) \\ -1 & \text{if } (i,k) = (1,h) \text{ or } (i^*,l) \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\boldsymbol{\tau}$ is a characteristic vector of an augmenting cycle of $\mathbf{g}$. Also, because $\mathbb{L}(a_1) \leq \mathbb{L}(a_{i*})$ and $\mathbb{L}(b'_l) \leq \mathbb{L}(b'_{t^l}) \leq \mathbb{L}(b'_h)$, we have

$$\begin{aligned} &|\mathbb{L}(a_1) - \mathbb{L}(b'_l)| + |\mathbb{L}(a_{i^*}) - \mathbb{L}(b'_h)| \\ \leq\ &|\mathbb{L}(a_1) - \mathbb{L}(b'_h)| + |\mathbb{L}(a_{i^*}) - \mathbb{L}(b'_l)| \end{aligned}$$

Hence, $\boldsymbol{\tau}$ represents an augmenting cycle with non-positive cost. Let ϵ be the maximum value such that $\mathbf{g} + \epsilon\boldsymbol{\tau}$ is feasible and we have that $\mathbf{g} + \epsilon\boldsymbol{\tau}$ is also an optimal solution of (RLP). Thus, we can reassign $\mathbf{g}$ to be $\mathbf{g} + \epsilon\boldsymbol{\tau}$, i.e., let $\mathbf{g} := \mathbf{g} + \epsilon\boldsymbol{\tau}$, and have $\mathbf{g}$ still be an optimal solution of the linear program (RLP).

We can continue to perform the pervious operation as long as there exist some $1 \leq l \leq t^l$ with $g_{1l} < \frac{2d_l}{n}$. Eventually, our procedure will terminate because each operations must find a unique augmenting vector $\boldsymbol{\tau}$. Hence, we eventually obtain an optimal solution $\mathbf{g}$ such that $g_{1k} = \frac{2d_k}{n}$ for any $1 \leq k \leq t^l$.

Finally, because $\sum_{k=1}^{n'} g^*_{1k} = 1$, we have $g^*_{1k} = 0$ for all $t^l + 1 \leq k \leq n$. Thus, $\mathbf{g}^*$ is an optimal solution of (RLP) satisfying Equation (5.1). □

With Proposition 5.3, we can now characterize an optimal solution of (RLP).

Theorem 5.1. *Let t^l be the integer such that $\sum_{k=1}^{t^l} d_k = \frac{n}{2}$. Then there exists an optimal solution $\mathbf{g}^*$ for (RLP) such that*

$$g^*_{ik} = \begin{cases} \frac{2d_k}{n} & \textit{if } 1 \le i \le \lfloor \frac{n}{2} \rfloor, 1 \le k \le t^l \\ \frac{2d_k}{n} & \textit{if } \lceil \frac{n}{2} \rceil + 1 \le i \le n, t^l + 1 \le k \le n' \\ \frac{d_k}{n} & \textit{if } n \textit{ is odd, } i = \frac{n+1}{2}, 1 \le k \le n' \\ 0 & \textit{otherwise.} \end{cases} \tag{5.2}$$

Proof. Similarly to Proposition 5.3, we start from a vector $\mathbf{g}'$ that is an optimal solution of (RLP) and provide an algorithm to construct $\mathbf{g}^*$ from $\mathbf{g}'$.

First, applying the algorithm from Proposition 5.3, we can construct an optimal solution $\mathbf{g}'$ such that g'_{1k} satisfies Equation (5.2) for every $1 \le k \le n'$. When we apply the algorithm symmetrically on g'_{nk}, it is easy to check that algorithm will not change g'_{1k} for any $1 \le k \le n'$. Therefore, after applying the algorithm from Proposition 5.3 on g'_{1k} and g'_{nk}, we end up with an optimal solution $\mathbf{g}'$ such that g'_{1k} and g'_{nk} satisfy Equation (5.2) for every $1 \le k \le n'$.

Let $\mathbf{g}^s \in \mathbb{R}^{n'(n-2)}$ be the subvector of $\mathbf{g}'$ where $g^s_{ik} = g'_{ik}$ for $2 \le i \le n-1$, $1 \le k \le n'$. By optimality of $\mathbf{g}'$, $\mathbf{g}^s$ is the optimal solution for the optimization problem

$$\begin{aligned} \min \ & \sum_{i,k} t_{ik} g_{ik} \\ \text{s.t.} \quad & \sum_{i=2}^{n-1} g_{ik} = d_k - g'_{1k} - g'_{nk}, \forall\, 1 \le k \le n', \\ & \sum_{k=1}^{n'} g_{ik} = 1, \forall\, 2 \le i \le n-1, \\ & 0 \le g_{ik} \le \frac{2d_k}{n}, \forall, 2 \le i \le n-2, 1 \le k \le n', \\ & \mathbf{g} \in \mathbb{R}^{n'(n-2)} \end{aligned}$$

Note that for any $1 \le k \le n'$, $d_k - g'_{1k} - g'_{nk} = d_k - \frac{2d_k}{n} = \frac{(n-2)d_k}{n}$. Let $d^s_k = \frac{(n-2)d_k}{n}$ for

$1 \le k \le n'$, and we have that $\mathbf{g}^s$ is the optimal solution for the optimization problem

$$\begin{aligned} \min \ & \sum_{i,k} t_{ik} g_{ik} \\ \text{s.t.} \ & \sum_{i=2}^{n-1} g_{ik} = d_k^s, \forall\, 1 \le k \le n', \\ & \sum_{k=1}^{n'} g_{ik} = 1, \forall\, 2 \le i \le n-1, \\ & 0 \le g_{ik} \le \frac{2d_k^s}{n-2}, \forall\, 2 \le i \le n-1, 1 \le k \le n', \\ & \mathbf{g} \in \mathbb{R}^{n'(n-2)} \end{aligned}$$

But the optimization problem above has the exactly same structure as linear program (RLP) of a system with $n-2$ plants and products. Therefore, we can continuously apply the algorithm from Proposition 5.3 again, and construct $\mathbf{g}'$ such that g'_{1k}, g'_{2k}, g'_{n-1k}, and g'_{nk} all satisfy Equation (5.2) for every $1 \le k \le n'$. And by induction, we can eventually construct g'_{ik} that satisfy Equation (5.2), for all, $1 \le i \le n$ and for all $1 \le k \le n'$. Therefore, we have $\mathbf{g}' = \mathbf{g}^*$ and by construction, $\mathbf{g}^*$ is an optimal solution of (RLP). □

Having characterized $\mathbf{g}^*$ as an optimal solution of (RLP), we can now identify a flexibility structure $\mathscr{A}^*$ that is a long chain that is optimal among all 2-flexibility structures. Moreover, we will show that there exists an optimal solution $\mathbf{f}^*$ of the linear program defining $P^D(\mathscr{A})$, such that $g^*_{ik} = \sum_j f^*_{ijk}$.

Theorem 5.2. *If n is even, define*

$$\mathscr{A}^* := \mathscr{D}_n \cup \{(a_i, b_{n-i+1}), \forall 1 \le i \le \frac{n}{2}\} \cup \{(a_{n-i+1}, b_{i+1}), \forall 1 \le i \le \frac{n-2}{2}\} \cup \{(\frac{n+2}{2}, 1)\},$$

and if n is odd, define

$$\mathscr{A}^* := \mathscr{D}_n \cup \{(a_i, b_{n-i+1}), \forall 1 \le i \le \frac{n-1}{2}\} \cup \{(a_{n-i+1}, b_{i+1}), \forall 1 \le i \le \frac{n-1}{2}\} \cup \{(\frac{n+1}{2}, 1)\},$$

then $P^D(\mathscr{A}^*) = \min_{\mathscr{A}\in\mathbb{F}_2} P^D(\mathscr{A})$.

Proof. Let t^l be the integer such that $t = \sum_{k=1}^{t^l} d_k = n/2$. If n is even, let

$$f^*_{ijk} := \begin{cases} \frac{d_k}{n} & \text{if } (a_i, b_j) \in \mathscr{A}^*, 1 \le i \le \frac{n}{2}, 1 \le k \le t^l \\ \frac{d_k}{n} & \text{if } (a_i, b_j) \in \mathscr{A}^*, \frac{n+2}{2} \le i \le n, t^l + 1 \le k \le n' \\ 0 & \text{otherwise.} \end{cases}$$

and if n is odd, let

$$f^*_{ijk} := \begin{cases} \frac{d_k}{n} & \text{if } (a_i, b_j) \in \mathscr{A}^*, 1 \le i \le \frac{n-1}{2}, 1 \le k \le t^l \\ \frac{d_k}{n} & \text{if } (a_i, b_j) \in \mathscr{A}^*, \frac{n+3}{2} \le i \le n, t^l + 1 \le k \le n' \\ \frac{d_k}{n} & \text{if } i = \frac{n+1}{2}, j = \frac{n+1}{2}, 1 \le k \le t^l \\ \frac{d_k}{n} & \text{if } i = \frac{n+1}{2}, j = 1, t^l + 1 \le k \le n' \\ 0 & \text{otherwise.} \end{cases}$$

Then, it is straight forward to check that $\mathbf{f}^*$ is feasible for $P^D(\mathscr{A}^*)$. Moreover, if $\mathbf{g}^*$ satisfies Equation (5.2), then $\sum_j f^*_{ijk} = g^*_{ik}$ for any i, k. Because $P^D(\mathscr{A}^*) \le \sum_{i,j,k} |\mathbb{L}(a_i) - \mathbb{L}(b'_k)| f^*_{ijk} = \sum_{i,k} |\mathbb{L}(a_i) - \mathbb{L}(b'_k)| g^*_{ik} \le \min_{\mathscr{A}\in\mathbb{F}_2} P^D(\mathscr{A})$, we have that $P^D(\mathscr{A}^*) = \min_{\mathscr{A}\in\mathbb{F}_2} P^D(\mathscr{A})$. □

It is simple to check that $\mathscr{A}^*$ is connected and indeed a long chain. Thus, Theorem 5.2 proves that there exist a long chain that is minimizes transportation cost among 2-flexibility structures when all plants and distribution centers lie on a line.

Note that when plants and distribution centers are located at different regions, different long chains can have different transportation costs. As a result, even after a firm has has decided to implement a sparse flexibility structure like the long chain to better match demand and supply, the firm may want to implement a long chain that minimizes its transportation costs. Therefore, Theorem 5.2 provides a simple guideline on designing the long chain structure that minimizes the transportation costs.

Admittedly, placing all plants and distribution centers on a 1-dimensional line is a restrictive assumption, since the surface of earth is 2-dimensional. Next, we discuss the case where transportation costs is minimized when plants and distribution centers lie on a 2-D plane.

5.3.2 2-Dimensional Case

For this subsection, we assume that the locations of all plants and all distribution centers lies on a 2-dimensional plane. For each $1 \leq i \leq n$, $1 \leq k \leq n'$, we use $\mathbb{L}(a_i) \in \mathbb{R}^2$ to denote the location of plant i, $\mathbb{L}(b'_k) \in \mathbb{R}^2$ to denote the location of distribution center k. We use $\|\mathbb{L}(a_i) - \mathbb{L}(b'_k)\|$ to denote the Euclidean distance between plant i and distribution center k. For a flexibility design $\mathscr{A}$, its transportation cost is represented by the following linear program.

$$
\begin{aligned}
P^D(\mathscr{A}) = \min & \sum_{i,j,k} \|\mathbb{L}(a_i) - \mathbb{L}(b'_k)\| f_{ijk} \\
\text{s.t.} \quad & \sum_{i} f_{ijk} = \frac{d_k}{n}, \forall 1 \leq j \leq n, 1 \leq k \leq n', \\
& \sum_{j,k} f_{ijk} \leq 1, \forall 1 \leq i \leq n, \\
& f_{ijk} \geq 0, \forall 1 \leq i, j \leq n, 1 \leq k \leq n', \\
& f_{ijk} = 0, \forall (a_i, b_j) \notin \mathscr{A}, \\
& \mathbf{f} \in \mathbb{R}^{n^2 n'},
\end{aligned}
$$

Interestingly, on a 2-D plane, the long chain is not necessarily optimal. Consider an example with 4 plants and 4 products and 4 distribution centers. As shown in Figure 5-3, The 4 plants are located on each of the four corners of an unit square, while the 4 distributions are located on the midpoint of each edge of the square. In this case, it is easy to check that a short chains structure, e.g. $\mathscr{A}^* = \{(a_1, b_2), (a_2, b_1), (a_3, b_4), (a_4, b_3)\}$, achieves a transportation cost of 2, (i.e. $P^D(\mathscr{A}^*) = 2$), while any long chain structure has a transportation cost of at least $2 + \frac{\sqrt{5}-1}{4}$.

Despite the fact that long chain does not necessarily optimize the transportation

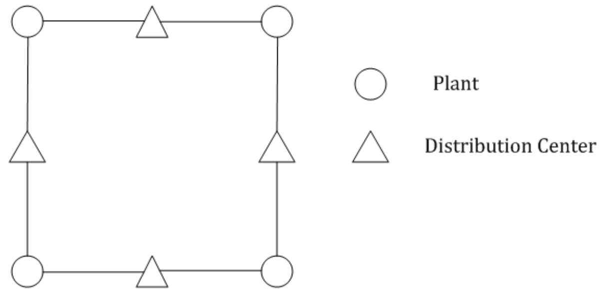


Figure 5-3: Counter Example

cost when plants and distribution centers lie on a 2-D plane, discussions in previous chapters of this thesis indicate that it is often in the firm's best interest to implement a long chain structure. Next, we state Algorithm 3, a heuristic that uses Theorem 5.2 to identify a long chain structure with low transportation costs on a 2-D plane.

Algorithm 3 Designing a near-optimal 2-flexibility structure $\mathscr{A}^*$ for $P^D(\mathscr{A}^*)$

---

procedure FIND ($\mathscr{A}^*$)

  Project all the plant/distribution center locations onto a line, such that $\mathbb{L}(a_i) \in \mathbb{R}$ for $1 \leq i \leq n$, and $\mathbb{L}(b'_k) \in \mathbb{R}$ for $1 \leq i \leq n'$.

  Relabel the plants and the distribution centers so that $\mathbb{L}(a_1) \leq \mathbb{L}(a_2) \leq \mathbb{L}(a_n)$ and $\mathbb{L}(b'_1) \leq \mathbb{L}(b'_2) \leq \mathbb{L}(b'_{n'})$.

  Define $\mathscr{A}^*$ as in the statement of Theorem 5.2.

  Return $\mathscr{A}^*$.

end procedure

In Algorithm 3, note that we did not specify to which line would we project the locations of all of our distribution centers and plants. Indeed, there are several options available. One option is to project the locations onto a line that minimizes the total distance between the line and all the plants and distribution centers. Another option is to randomly a line (i.e. x-axis) and its orthogonal (i.e. y-axis), and select the line that yields the best performance.

5.4 Conclusion

This chapter studies the effect and design of process flexibility with plants being required to satisfying deterministic demands at different distribution centers. We prove that when supply chain cost is either: (i) independent of distribution center, or (ii) independent of production plant, then there exists a long chain that minimizes the supply chain cost among all 2-flexibility structures. Interestingly, for the case when supply chain cost is independent of product, but not necessarily independent of plant or distribution center, we identify an example where there does not exist a long chain that minimizes the supply chain cost among all 2-flexibility structures.

Motivated by the counter example, we proceed to study process flexibility structures with only transportation cost. We prove that when there is only transportation cost, with all plants and all distribution centers lie on a line, there exists a long chain that is optimal among all 2-flexibility structures. Moreover, we propose Theorem 5.1, which identifies the structure of an optimal long chain. We note that while Theorem 5.1 does not extend to the case when plants and distribution centers lie 2-dimensional plane, it can be applied as a heuristic in identifying long chain that is effective for the purpose of reducing transportation costs.

Chapter 6

Conclusion

In this thesis, we have developed various theoretical results on the effectiveness of sparse process flexibility structures under different settings. Our analysis provides new theoretical support for implementing sparse process flexibility, as well as rigorous techniques for developing practical flexibility design guidelines. Moreover, our study has led to several exciting new research directions, which are discussed next.

The decomposition of the sales of the long chain, stated as Theorem 2.2 in Section 2.2, is a novel technique in studying the expected sales of process flexibility structures. In our analysis, the decomposition is applied to long chain and 2-flexibility structures. An interesting question is whether there exists a similar decomposition for general flexibility structures. A positive answer to this question may lead to breakthroughs in understanding more general flexibility structures, and in particular, structures with more than 2 degrees of flexibility. Another potential direction is to generalize the decomposition result to other problems with underlying network structure. Many network problems are known to be more difficult with the existence of cycles, e.g. Bayesian networks [Weiss and Freeman, 2001], and the decomposition technique may prove to be a useful tool for understanding those network problems with cycles.

Our thesis has also developed tools to study the long chain and 2-flexibility structures under nonhomogeneous systems, that is, systems with unequal plant capacities and non-IID product demands. For example, if $c_i = \mathbb{E}[D_i]$, but $c_i \neq c_j$ for any i, j, then it is not known how to find the optimal long chain, or whether there exists a long chain that is optimal among all 2-flexibility structures. To study the nonhomogeneous systems, one can start from the generalized version of the supermodularity/decomposition results derived in Section 2.4. However, we note that the decomposition in nonhomogeneous systems can be much more complicated than the analysis of homogenous systems, as the homogenous assumption ensures that the expected sales of the long chain can be characterized by a simple and succinct formula.

Our investigation of the transportation model in Section 5.3 has also raised interesting open problems. One particular problem is to derive a performance guarantee for Algorithm 3 when plants and products lie on a 2-dimensional plane. Another problem is whether we can identify better heuristics than Algorithm 3. Moreover, when demands is uncertain, it is intuitive that process flexibility would improve the match between available capacity and uncertain demand, and reduce the transportation cost under deterministic demand. Therefore, analysis of an extended distribution system model with stochastic demand would improve our understanding of the ability of flexibility structures to achieve both benefits at the same time, and determine the flexibility structure that achieves the optimal balance between both benefits.

Appendix A

Proof for Corollary 4.4

First, we prove a lemma which is analogous to Lemma 4.1.

Lemma A.1. *For any for any flexibility structure* $\mathscr{A}$*, and any* $X \subseteq \{1, 2, ..., n\}$*. We have that*

$$R^s(\mathscr{A}, S) \leq \delta^T(\mathscr{A}, \boldsymbol{\mu}) + \min\{\sum_{j \in X} d_j | \mathbf{d} \in S\}, \text{ where } T = \sum_{j \in X} \mu_j.$$

Proof. Like the proof of Lemma 4.1, we can apply the max-flow min-cut theorem to show that for any flexibility structure $\mathscr{A}$, any $\mathbf{d} \in S$, we have

$$\begin{aligned} P(\mathbf{d}, \mathscr{A}) = \min & \sum_{i=1}^{m} c_i p_i + \sum_{j=1}^{n} q_j d_j \\ \text{s.t.} \quad & p_i + q_j \geq 1, \forall (i, j) \in I(\mathscr{A}) \\ & \mathbf{p} \in \{0, 1\}^m, \mathbf{q} \in \{0, 1\}^n. \end{aligned}$$

Let $\mathbf{p}'$, $\mathbf{q}'$ be the optimal solution of the binary program defining $\delta^T(\mathscr{A}, \boldsymbol{\mu})$. Let $X' = \{j | q'_j = 1\}$. Because S is min-consistent with $\boldsymbol{\mu}$ and $\sum_{j \in X'} \mu_j \leq T = \sum_{j \in X} \mu_j$, we can find $\mathbf{d}'$ such that $\sum_{j \in X'} d'_j \leq \min\{\sum_{j \in X} d_j | \mathbf{d} \in S\}$. Thus,

$$P(\mathbf{d}', \mathscr{A}) \leq \delta^T(\mathscr{A}, \boldsymbol{\mu}) + \sum_{j \in X'} d'_j \leq \delta^T(\mathscr{A}, \boldsymbol{\mu}) + \min\{\sum_{j \in X} d_j | \mathbf{d} \in S\}.$$

□

Now, we present the proof of Corollary 4.4.

Proof of Corollary 4.4. Let $\tau = \arg\min_{\mathbf{d}\in S} P(\mathscr{A}, \mathbf{d})$, apply max-flow min-cut theorem, we can find $X \subset \{1, 2, ..., n\}$ such that

$$R^s(\mathscr{A}_1, S) = \delta^T(\mathscr{A}_1) + \sum_{i\in X} \tau_i, \text{where } T = \sum_{i\in X} \mu_i.$$

Moreover, by minimality of τ, we have that $\sum_{i\in X} \tau_i = \min\{\sum_{j\in X} d_j|\mathbf{d} \in S\}$. Apply Lemma A.1, we have

$$R^s(\mathscr{A}_1, S) = \delta^T(\mathscr{A}_1) + \min\{\sum_{j\in X} d_j|\mathbf{d} \in S\} \geq \delta^T(\mathscr{A}_2) + \min\{\sum_{j\in X} d_j|\mathbf{d} \in S\} \geq R^s(\mathscr{A}_2, S).$$

□

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