Nonlinear Stability of Multibody Systems

by

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1.0 Abstract

The theory and methods of rigid body dynamics and nonlinear stabilization are discussed. The mathematical interest in the inherent instability of the intermediate axis motivates the study of stabilization of motion about this axis. Engineering applications to spin stabilized spacecraft systems require the craft model be quasi-rigid, and it is desired that the stability be global and asymptotic. A globally asymptotically stabilizing velocity field in the form of a nonlinear feedback control is developed and presented.

Thesis Supervisor: Charles P. Coleman
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4.0 Introduction

Nonlinear stabilization of intermediate axis rotation is not only of mathematical interest, but has considerable applications in engineering systems such as aerospace and underwater vehicles. Controlling the stability of the machines and objects which are used for industrial and scientific applications is clearly of deep commercial and academic interest. Furthermore, the implications in differential geometry and symmetry of mechanics are mathematically profound and challenging.

In the ideal world, energies and momenta of absolutely rigid rotating systems are completely conserved within the system. However, in the physical world, ignoring the internal energy dissipation of a spacecraft can lead to disastrous results. The unfortunately fate of the Explorer I in1958 is an example of the result of modeling the craft as an absolutely rigid body. The craft was intended to be in stable minor axis rotation, and theoretically, the minor axis rotation should have remained stable for all time. However, shortly after orbit insertion, Explorer I became unstable and the instability decayed into major axis rotation (flat spin). This behavior was unpredicted by the rigid body modeling, as small perturbations from minor axis rotation should be bound and should not lead to decay into flat spin. The accident investigation concluded that the overlooked internal energy dissipation was indeed responsible for the minor axis instability of Explorer I.

The study of stability of rotating bodies has important applications in space systems, as is seen by the satellites deployed by the space shuttle orbiter. When the satellites are ejected form the payload bay, they are normally spinning in a stable configuration. In May 1992, astronauts attempted to grab in space the Intelsat satellite to attach a rocket that would insert it into geosynchronous orbit. After two unsuccessful days of trying to attach the grappling fixture, the astronauts had to abort their mission because of the increased tumbling. Ground controllers took hours to re-stabilize the satellite using jet thrusters, and the satellite was left in a stable configuration of spinning slowly about its cylindrical symmetry axis (a principal axis). Finally, on the third day, three astronauts grabbed the slightly rotating satellite, stopped it, and put it into the payload bay where the rocket skirt was attached. The Intelsat satellite was finally successfully placed into orbit in time to broadcast the 1992 Barcelona Olympic summer games [MT95]..

4.1 Stability Concepts

Some concepts and definitions in stability [Mar92,SW91] are presented for clarification and consistency of notation.

Consider a dynamical system

$$\dot{x} = f(x) \tag{4.1}$$

where $x = (x^1,\ldots,x^n)$ and f is some smooth function of x.

Definition 1. ***Equilibria*** are points x_e such that $f(x_e) = 0$.

Definition 2. To determine ***stability***, find any solution to $\dot{x} = f(x)$ that starts near x_e and remains close to x_e for all future time.

Definition 3. The equilibrium point is ***asymptotically stable*** if it is stable, and in addition, there exists some $r > 0$ such that $\|x(0)\| < r$ implies that $x(t)$ goes to x_e as t approaches infinity.

Definition 4. If asymptotic stability holds for any initial states, the equilibrium point is said to be ***globally asymptotically stable***.

Examine the first variation

$$\dot{\xi} = D_x f(x_e)\xi \tag{4.2}$$

where

$$D_x f(x_e) = \left[\frac{\partial f^i}{\partial x^j}\right]_{x=x_e} \tag{4.3}$$

Definition 5. The system (4.2) is called the ***linearization*** of the original nonlinear system at the equilibrium point x_e.

Theorem 1. ***Lyapunov Theorem***.

- If the linearized system is strictly stable (i.e. if all eigenvalues of (4.3) are strictly in the left-half complex plane), then the equilibrium point is asymptotically stable for the actual nonlinear system.If the linearized system is unstable (i.e. if at least one eigenvalue of (4.3) is strictly in the right-half complex plane), then the equilibrium point is unstable for the actual nonlinear system.
- If the linearized system is marginally stable (i.e. if all eigenvalues of (4.3) are in the left-half complex plane, but at least one of them is on the $j\omega$ axis), the stability of the nonlinear system is inconclusive.

4.2 Stability of Rigid-Body Rotations

We now consider a rigid body undergoing force-free rotation about one of its principal axes and inquire whether such motion is stable as defined above.

Euler's equations for rigid body motion [Gol80,AM78] is given by

$$I_i \frac{d\omega_i}{dt} + \varepsilon_{ijk}\omega_j\omega_k I_k = N_i \tag{4.4}$$

where I is moment of inertia, ω is angular velocity, and N is torque.

For torque-free motion Euler's equations become

$$\begin{aligned} I_1 \dot{\omega}_1 - \omega_2\omega_3(I_2 - I_3) &= 0 \\ I_2 \dot{\omega}_2 - \omega_3\omega_1(I_3 - I_1) &= 0 \\ I_3 \dot{\omega}_3 - \omega_1\omega_2(I_1 - I_2) &= 0 \end{aligned} \tag{4.5}$$

We choose a general rigid body for which all principal axes of inertia are distinct, and label them $I_1 > I_2 > I_3$.

In order to examine the stability properties of the rigid body, we let the body axes coincide with the principal axes, start the body rotating about the x_1 axis, the axis corresponding to the moment of inertia I_1, then introduce a small perturbation. Then, the angular velocity vector takes the form

$$\vec{\omega} = \omega_1 i_1 + \delta\omega_2 i_2 + \delta\omega_3 i_3$$

where $\delta\omega_2$ and $\delta\omega_3$ are quantities sufficiently small so that their product can be neglected in comparison to the all other quantities of interest.

The Euler's equations (4.5) become

$$(I_2 - I_3)\delta\omega_2\delta\omega_3 - I_1 \dot{\omega}_1 = 0$$

$$(I_3 - I_1)\delta\omega_3\omega_1 - I_2 \delta\dot{\omega}_2 = 0$$

$$(I_1 - I_2)\omega_1\delta\omega_2 - I_3 \delta\dot{\omega}_3 = 0$$

Solving for $\delta\omega_2$ yields

$$\delta\omega_2 = Ae^{i\Omega t} + Be^{-i\Omega t}$$

where

$$\Omega_1 \equiv \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}. \tag{4.6}$$

Considering rotations about x_2 and x_3 axes, expressions for Ω_2 and Ω_3 can be obtained by permutation of (4.6)

$$\Omega_1 \equiv \omega_1 \sqrt{\frac{(I_1 - I_3)(I_1 - I_2)}{I_2 I_3}}$$

$$\Omega_2 \equiv \omega_2 \sqrt{\frac{(I_2 - I_1)(I_2 - I_3)}{I_1 I_3}}$$

$$\Omega_3 \equiv \omega_3 \sqrt{\frac{(I_3 - I_2)(I_3 - I_1)}{I_1 I_2}}$$

Since we designated $I_1 > I_2 > I_3$, it is easily seen that Ω_1 and Ω_3 are real and Ω_2 is imaginary. Therefore, for rotations about x_1 and x_3, the perturbation produces steady oscillatory motion and rotation is stable. However, for rotation about x_2, the imaginary Ω_2 results in exponential increase in the perturbation. Rotation about the intermediate x_2 axis is unstable.

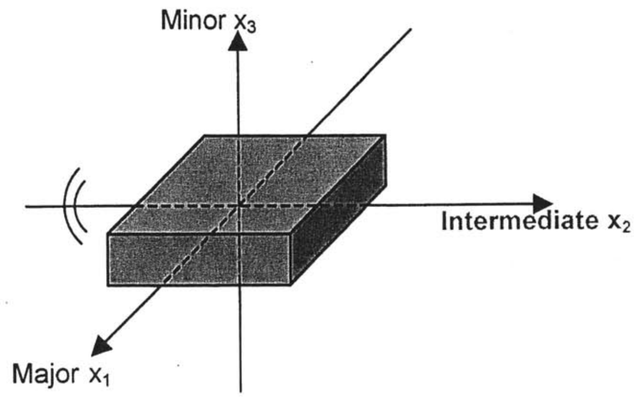

Figure 4.1 Check this shit out

4.3 Summary

We have introduced some basic definitions and concepts in stability. In addition, we have shown that rotation about the major and minor axes (Figure 1) of inertia are stable, and rotation about the intermediate axis (Figure 1) is unstable.

Realizing the limits of Theorem 1, a search for a conclusive method for deducing the nonlinear stability of marginally stable system is inevitable. In section 7.0, we introduce such a method called the Energy-Casimir method. First, we review the elements of Hamiltonian dynamics necessary for further analysis and formulation.

5.0 Hamiltonian Systems

In this section we present Hamiltonian systems and their integral invariants. A complete treatment of Hamiltonian Dynamics can be found in advanced mechanics texts such as [AM78, Gol80]. Then the Poisson bracket formulation of freely rotating rigid body is presented.

5.1 Hamiltonian Dynamics

Consider a general continuous Hamiltonian dynamical system

$$u_t = \vec{J}\frac{\delta H}{\delta u} \tag{5.1}$$

where $\vec{J}$ is skew-symmetric transformation from $\{u\}$ to $\{u\}$, satisfying

$$(u, \vec{J}v) = -(\vec{J}u, v) \tag{5.2}$$

where (,) is the Euclidean inner product on $\Re^3$.

This Hamiltonian dynamical system generally possesses the following three integral invariants:

1 The Hamiltonian H

Proposition 1. *H is an integral invariant of the Hamiltonian system.*

Proof: $$\frac{dH}{dt} = \left(\frac{\delta H}{\delta u}, u_t\right) = \left(\frac{\delta H}{\delta u}, \vec{J}\frac{\delta H}{\delta u}\right) = 0$$

which is true by the skew symmetry of $\vec{J}$.

2 Momentum invariants

If H is invariant under a spatial symmetry in x, then the associated ***momentum functional M*** is defined by

$$-u_x = \bar{J}\frac{\delta M}{\delta u} \tag{5.3}$$

Proposition 2. *M is conserved by the dynamics of the Hamiltonian system.*

Proof: $$\frac{dM}{dt} = \left(\frac{\delta M}{\delta u}, u_t\right) = \left(\frac{\delta M}{\delta u}, \bar{J}\frac{\delta H}{\delta u}\right) = -\left(\bar{J}\frac{\delta M}{\delta u}, \frac{\delta H}{\delta u}\right) = \left(u_x, \frac{\delta H}{\delta u}\right) = 0$$

3 Casimir invariants associated with the kernel of the operator $\bar{J}$.

Casimirs are defined to be the solutions of the equation

$$\bar{J}\frac{\delta C}{\delta u} = 0 \tag{5.4}$$

Proposition 3. *Casimirs are conserved by the dynamics of the Hamiltonian system.*

Proof: $$\frac{dC}{dt} = \left(\frac{\delta C}{\delta u}, u_t\right) = \left(\frac{\delta C}{\delta u}, \bar{J}\frac{\delta H}{\delta u}\right) = -\left(\bar{J}\frac{\delta C}{\delta u}, \frac{\delta H}{\delta u}\right) = -\left(0, \frac{\delta H}{\delta u}\right) = 0$$

5.1.1 Poisson Brackets

The Hamiltonian system (5.1) can be represented by the Poisson bracket

$$\frac{dF}{dt} = \{F, H\} \tag{5.5}$$

where, the Poisson bracket { , } is defined by

$$\{F, G\} = \left(\frac{\delta F}{\delta u}, \bar{J}\frac{\delta G}{\delta u}\right) \tag{5.6}$$

The Casimir C was defined by equation (5.4). In the Poisson bracket formulation, a Casimir C is any functional which satisfies

$$\{C, G\} = 0 \tag{5.7}$$

for any function G defined on the phase space of the Hamiltonian system.

5.2 Rigid Body Poisson Bracket

We saw in the last section that Hamiltonian equations for the free rigid body can be written in Poisson bracket form

$$\dot{F} = \{F, H\} \tag{5.8}$$

The non-canonical Poisson bracket for the free rigid body is given by

$$\{F, H\}(\vec{m}) = (\nabla F, \vec{J}\nabla H) \tag{5.9}$$

where $\vec{m}$ is the vector of body angular momenta, $\vec{m} = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \end{bmatrix}$, and F is any function of $\vec{m}$.

As before (,) is the Euclidean inner product on $\Re^3$, and $\vec{J}$ is a skew symmetric operator satisfying (5.2).

H is the rigid body Hamiltonian is given by the function

$$H = \frac{1}{2}\left(\frac{m_1^{\ 2}}{I_1} + \frac{m_2^{\ 2}}{I_2} + \frac{m_3^{\ 2}}{I_3}\right) \tag{5.10}$$

and ∇ is the gradient operator is given by the expression

$$\nabla = \begin{bmatrix} \dfrac{\partial}{\partial m_1} \\ \dfrac{\partial}{\partial m_2} \\ \dfrac{\partial}{\partial m_3} \end{bmatrix}$$

For the rigid body, the skew symmetric operator is given by

$$\vec{J} = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \tag{5.11}$$

Because acting upon a vector with this operator is equivalent to taking the cross product between $\vec{m}$ and the vector,

$$\vec{J}\vec{v} = \begin{bmatrix} 0 & -m_3 & m_2 \\ m_3 & 0 & -m_1 \\ -m_2 & m_1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} -m_3 v_2 + m_2 v_3 \\ m_3 v_1 - m_1 v_3 \\ -m_2 v_1 + m_1 v_2 \end{bmatrix} = \vec{m} \times \vec{v} \tag{5.12}$$

the matrix representation (5.11) is often given the name ($\vec{m}\times$).

The velocity field of the body angular momenta is given by

$$\dot{\vec{m}} = \vec{J}\vec{\nabla}H \tag{5.13}$$

$$\vec{\nabla}H = \frac{1}{2} \begin{bmatrix} \dfrac{\partial}{\partial m_1} \\ \dfrac{\partial}{\partial m_2} \\ \dfrac{\partial}{\partial m_3} \end{bmatrix} \left(\frac{m_1^{\ 2}}{I_1} + \frac{m_2^{\ 2}}{I_2} + \frac{m_3^{\ 2}}{I_3} \right) = \vec{I}^{\,-1}\vec{m}$$

where I is the body frame inertia tensor containing the principal moments of inertia, I_1, I_2 and I_3.

$$\vec{I} = \begin{bmatrix} I_1 & 0 & 0 \\ 0 & I_2 & 0 \\ 0 & 0 & I_3 \end{bmatrix}$$

Using (5.11), the velocity field can also be written as

$$\dot{\vec{m}} = \vec{m} \times \left(\vec{I}^{-1}\vec{m}\right) \tag{5.14}$$

and the rigid body Euler's equations in terms of body angular momenta are recovered. Note that the equilibria of the velocity field correspond to principal axis rotations.

5.3 Summary

We have reviewed Hamiltonian dynamics and Poisson brackets and, with them, formulated the free rigid body dynamics. In the next section, we introduce double bracket energy sinks, which are useful for modeling quasi-rigid body dynamics. The double bracket energy sink will ultimately be used to develop a method for globally asymptotically stabilizing intermediate axis rotations.

6.0 Double Bracket Energy Sinks

Double bracket energy sinks are a large class of energy sinks derived from double bracket dynamical systems. Double bracket energy sinks are capable of modeling energy dissipation in both axisymmetric and non-axisymmetric systems[Col97].

6.1 Double Bracket Systems

Double bracket systems [BKMR94] are of the form:

$$\frac{dF}{dt} = \{F, G\}_{skew} + \{F, G\}_{symmetric} \tag{6.1}$$

The skew-symmetric bracket is the Poisson bracket defined in (5.6).

$$\{F, G\}_{skew} = \left(\frac{\delta F}{\delta u}, \bar{J} \frac{\delta G}{\delta u} \right) \tag{6.2}$$

The symmetric bracket is defined as

$$\{F, G\}_{symmetric} = \left(\frac{\delta F}{\delta u}, \bar{J} \bar{\alpha} \bar{J} \frac{\delta G}{\delta u} \right) \tag{6.3}$$

6.1.1 Energy-Casimir Modified Hamiltonian Dynamics

Recall from section 5.1 that the Hamiltonian system is given by

$$u_t = \bar{J} \frac{\delta H}{\delta u} \tag{5.1}$$

Now, consider the energy-Casimir modification [Col97] of the Hamiltonian system (5.1).

$$u_t = \bar{J} \frac{\delta H}{\delta u} + \bar{J} \bar{\alpha} \bar{J} \frac{\delta H}{\delta u} \tag{6.4}$$

where $\bar{\alpha}$ is a symmetric transformation with $(u, \bar{\alpha} u)$ of definite sign for all u.

The modified energy-Casimir dynamical system (6.4) possesses the following properties:

<u>Proposition 4</u> *The Casimir invariants of the original system (5.1) are also the Casimir invariants of the modified system (6.4).*

<u>Proof</u>: $$\frac{dC}{dt}=\left(\frac{\delta C}{\delta u},u_t\right)=\left(\frac{\delta C}{\delta u},\bar{J}\frac{\delta H}{\delta u}+\bar{J}\bar{\alpha}\bar{J}\frac{\delta H}{\delta u}\right)$$

$$=-\left(\bar{J}\frac{\delta C}{\delta u},\frac{\delta H}{\delta u}+\bar{\alpha}\bar{J}\frac{\delta H}{\delta u}\right)=-\left(0,\frac{\delta H}{\delta u}+\bar{\alpha}\bar{J}\frac{\delta H}{\delta u}\right)$$

$$=0$$

<u>Proposition 5</u> *The energy H of the modified system (6.4) increases or decreases monotonically depending on the sign of* $\bar{\alpha}$.

<u>Proof</u>: $$\frac{dH}{dt}=\left(\frac{\delta H}{\delta u},u_t\right)=\left(\frac{\delta H}{\delta u},\bar{J}\frac{\delta H}{\delta u}+\bar{J}\bar{\alpha}\bar{J}\frac{\delta H}{\delta u}\right)$$

$$=-\left(\bar{J}\frac{\delta H}{\delta u},\bar{\alpha}\bar{J}\frac{\delta H}{\delta u}\right)$$, which is of definite sign.

<u>Proposition 6</u> *The equilibria of the modified system (6.4) are those of the original system (5.1).*

<u>Proof</u>: From above, $\frac{dH}{dt}=-\left(\bar{J}\frac{\delta H}{\delta u},\bar{\alpha}\bar{J}\frac{\delta H}{\delta u}\right)$, is non-zero unless

$$\bar{J}\frac{\delta H}{\delta u}=0.$$

The energy-Casimir modified dynamics (6.4) belong to a class of double bracket systems[BKMR94].

6.1.2 Rigid Body Double Bracket

The rigid body Poisson bracket (5.8), (5.9) is skew symmetric.

$$\dot{F} = \{F, H\}_{skew} = (\nabla F, \vec{J}\nabla H) \tag{6.5}$$

From (6.3), the symmetric bracket can be written

$$\{F, H\}_{symmetric} = \left(\nabla F, \vec{J}\vec{\alpha}\vec{J}\nabla H\right) \tag{6.6}$$

where $\vec{\alpha}$ is a positive definite symmetric operator.

Define a double bracket dynamical system for the rigid body by the following equations:

$$\dot{F} = \{F, H\}_{skew} + \{F, H\}_{symmetric}$$

$$\dot{F} = \left(\nabla F, \vec{J}\nabla H\right) + \left(\nabla F, \vec{J}\vec{\alpha}\vec{J}\nabla H\right) \tag{6.7}$$

Notice that (6.6) corresponds to the additional term in the energy-Casimir modified Hamiltonian system. (6.6) will be referred to as energy-Casimir symmetric bracket.

6.2 Quasi-Rigid Bodies

The following dynamical properties are required for quasi-rigid bodies:

1. Kinetic energy is dissipated.
2. Equilibria of the rigid body system are preserved.
3. Angular momentum is conserved.

We will demonstrate that the rigid body double bracket fulfills these dynamical properties [Col97].

6.2.1 Kinetic Energy is Dissipated

The energy of the rigid body double bracket system is represented by the Hamiltonian function H. Using equation (6.6), we calculate the rate of change of H.

$$\begin{aligned}\dot{H} &= \left(\nabla H, \bar{J}\nabla H\right) + \left(\nabla H, \bar{J}\bar{\alpha}\bar{J}\nabla H\right) \\ &= \nabla H \cdot \left(\vec{m} \times \nabla H\right) - \left(\bar{J}\nabla H, \bar{\alpha}\bar{J}\nabla H\right) \\ &= -\left(\bar{J}\nabla H, \bar{\alpha}\bar{J}\nabla H\right).\end{aligned}$$

The quantity inside the parenthesis, $\left(\bar{J}\nabla H, \bar{\alpha}\bar{J}\nabla H\right)$, is positive when $\bar{J}\nabla H \neq 0$, in which case $\dot{H} < 0$ and energy is dissipated by the rigid body double bracket system.

6.2.2 Equilibria of the Rigid Body System are Preserved

The equilibria of the original Hamiltonian rigid body system (5.9) correspond to the equilibria of the velocity field $\bar{J}\nabla H$. (See equation (5.12).) This velocity field $\bar{J}\nabla H$ is only zero for principal axis rotations. There fore, the equilibria of the original rigid body Hamiltonian system (5.9) are those of the double bracket dynamical system (6.6).

6.2.3 Angular Momentum is Conserved

Let $C(m)$ be a function of the magnitude of the angular momentum of the rigid body

$$C(m) = \frac{1}{2}\left(m_1^2 + m_2^2 + m_3^2\right).$$

Then,

$$\nabla C = \vec{m}.$$

Calculating the rate of change of C,

$$\begin{aligned}\dot{C} &= \left(\nabla C, \vec{J}\nabla H\right) + \left(\nabla C, \vec{J}\vec{\alpha}\vec{J}\nabla H\right) \\ &= \left(\vec{m}, \vec{J}\nabla H\right) + \left(\vec{m}, \vec{J}\vec{\alpha}\vec{J}\nabla H\right) \\ &= \vec{m}\cdot\left(\vec{m}\times\nabla H\right) + \vec{m}\cdot\left(\vec{m}\times(\vec{\alpha}\vec{J}\nabla H)\right) \\ &= 0.\end{aligned}$$

The magnitude of angular momentum is conserved by the rigid body double bracket system.

6.3 Double Bracket Energy Sinks

In the last section, it was shown that the rigid body double bracket system is obtained by adding the energy-Casimir symmetric bracket (6.6) to the rigid body Hamiltonian system (6.5).

$$\dot{F} = \{F, H\}_{skew} + \{F, H\}_{symmetric}$$

$$\dot{F} = \left(\nabla F, \vec{J}\nabla H\right) + \left(\nabla F, \vec{J}\vec{\alpha}\vec{J}\nabla H\right) \tag{6.7}$$

It was shown in section 6.2.1 that the rigid body double bracket system dissipates energy for non-principal axis rotations. In section 6.2.3, it was shown that this bracket also preserves the magnitude of angular momentum vector. The energy-Casimir symmetric bracket (6.6) contains the terms

$$\vec{J}\vec{\alpha}\vec{J}\nabla H \tag{6.8}$$

which are a valid ***energy sink***.

The Hamiltonian equations of motion for the free rigid body (5.13) can be written as

$$\dot{\vec{m}} = \vec{\vec{J}}\nabla H = (\vec{m}\times)\vec{I}^{-1}\vec{m} \tag{6.9}$$

Define

$$a_1 = \frac{(I_2 - I_3)}{I_2 I_3}$$

$$a_2 = \frac{(I_3 - I_1)}{I_1 I_3} \tag{6.10}$$

$$a_3 = \frac{(I_1 - I_2)}{I_1 I_2}$$

The equations of motion become

$$\dot{\vec{m}} = \begin{bmatrix} \dot{m}_1 \\ \dot{m}_2 \\ \dot{m}_3 \end{bmatrix} = \begin{bmatrix} a_1 m_2 m_3 \\ a_2 m_1 m_3 \\ a_3 m_1 m_2 \end{bmatrix} \tag{6.11}$$

Let the positive definite symmetric operator be given by

$$\vec{\alpha} = \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \alpha & 0 \\ 0 & 0 & \alpha \end{bmatrix} \tag{6.12}$$

where α is a positive definite number.

The double bracket energy sink $\vec{J}\vec{\alpha}\vec{J}\nabla H$ then becomes

$$\vec{J}\vec{\alpha}\vec{J}\nabla H = \alpha(\vec{m}\times)(\vec{m}\times)\vec{I}^{-1}\vec{m} \tag{6.13a}$$

$$\vec{J}\vec{\alpha}\vec{J}\nabla H = \alpha\begin{bmatrix} a_3 m_1 m_2^2 - a_2 m_1 m_3^2 \\ -a_3 m_1^2 m_2 + a_1 m_2 m_3^2 \\ a_2 m_1^2 m_3 - a_1 m_2^2 m_3 \end{bmatrix}. \tag{6.13b}$$

Evidently, this double bracket energy sink is cubic in the components of the body angular momentum vector.

The rigid body equations of motion (5.13) with the energy sink term (6.8) added is

$$\dot{\vec{m}} = \vec{J}\nabla H + \vec{J}\vec{\alpha}\vec{J}\nabla H \tag{6.14a}$$

$$\dot{\vec{m}} = (\vec{m}\times)\vec{I}^{-1}\vec{m} + \alpha(\vec{m}\times)(\vec{m}\times)\vec{I}^{-1}\vec{m} \tag{6.14b}$$

$$\dot{\vec{m}} = \begin{bmatrix} \dot{m}_1 \\ \dot{m}_2 \\ \dot{m}_3 \end{bmatrix} = \begin{bmatrix} a_1 m_2 m_3 \\ a_2 m_1 m_3 \\ a_3 m_1 m_2 \end{bmatrix} + \alpha\begin{bmatrix} a_3 m_1 m_2^2 - a_2 m_1 m_3^2 \\ -a_3 m_1^2 m_2 + a_1 m_2 m_3^2 \\ a_2 m_1^2 m_3 - a_1 m_2^2 m_3 \end{bmatrix} \tag{6.14c}$$

These are the equations which govern the dynamics of a quasi-rigid body as modeled by the double bracket energy sink.

From section 6.2.1, the expression for kinetic energy dissipation is given by

$$\dot{H} = -\left(\vec{J}\nabla H, \vec{\alpha}\vec{J}\nabla H\right)$$

$$\dot{H} = -\alpha\left(a_1^2 m_2^2 m_3^2 + a_2^2 m_1^2 m_3^2 + a_3^2 m_1^2 m_2^2\right) \tag{6.15}$$

This shows that for a given body configuration, the energy dissipation can be controlled by the parameter α. This parameter can be varied or chosen according to specific engineering models.

6.4 Summary

We have defined double bracket systems, and discussed its dynamical properties which were used to develop double bracket energy sinks. The energy-Casimir symmetric bracket was added to the original free rigid body Poisson bracket to form a rigid body double bracket.

It was shown that the double bracket system preserves angular momenta of the original system, but dissipates energy at a specifiable rate α. Therefore, the energy-Casimir symmetric bracket was used as an energy sink, with which the dynamics of a quasi-rigid body was modeled. Later, in section 8.2, it will be shown that the energy sink (6.8) can be used as a nonlinear feedback control law to stabilize rotational motion.

Next, we discuss stabilization of rigid body dynamics by the Energy-Casimir method.

7.0 Energy-Casimir Method

In section 4.2, it was shown that intermediate axis rotation is inherently unstable. Also, according to Theorem 1, the linearized system is marginally stable, the stability of the nonlinear system is inconclusive. In this section we introduce the Energy-Casimir method [BM90] to state conclusively a nonlinear stabilization result – that the angular momentum equations of the rigid body can be stabilized about the intermediate axis by a single torque applied about the major or minor axis.

7.1 Control Torque Dynamics

The rigid body Euler's equations with a single torque about the minor axis are given by

$$\dot{\omega}_1 = \frac{I_2 - I_3}{I_1}\omega_2\omega_3$$

$$\dot{\omega}_2 = \frac{I_3 - I_1}{I_2}\omega_3\omega_1 \tag{7.1}$$

$$\dot{\omega}_3 = \frac{I_1 - I_2}{I_3}\omega_1\omega_2 + u$$

As before, $I_1 > I_2 > I_3$ and the control torque is given by

$$u = -k\frac{I_1 I_2}{I_3}\omega_1\omega_2 \tag{7.2}$$

For the controlled system, the constants of motion are

$$\begin{aligned} E_c &= \frac{1}{2}\left(I_1\omega_1^{\,2} + I_2\omega_2^{\,2} + I_3\omega_3^{\,2}\frac{a_3}{a_3 - k}\right) \\ M_c^{\,2} &= \frac{1}{2}\left(I_1^{\,2}\omega_1^{\,2} + I_2^{\,2}\omega_2^{\,2} + I_3^{\,2}\omega_3^{\,2}\frac{a_3}{a_3 - k}\right) \end{aligned} \tag{7.3}$$

where $a_3 = \dfrac{(I_1 - I_2)}{I_1 I_2}$.

Now perform the Legendre transform $\vec{m} = \dfrac{\partial E_c}{\partial \vec{\omega}}$.

Then the equations of motion become

$$\begin{aligned}\dot{m}_1 &= a_1 \frac{a_3 - k}{a_3} m_2 m_3 \\ \dot{m}_2 &= a_2 \frac{a_3 - k}{a_3} m_1 m_3 \\ \dot{m}_3 &= a_3 m_1 m_2\end{aligned} \tag{7.4}$$

where, as defined in(6.10),

$$a_1 = \frac{(I_2 - I_3)}{I_2 I_3} \qquad a_2 = \frac{(I_3 - I_1)}{I_1 I_3} \qquad a_3 = \frac{(I_1 - I_2)}{I_1 I_2} \tag{6.10}$$

The constants of motion (7.3) are now

$$\begin{aligned}H_1 &= \frac{1}{2}\left(\frac{m_1^{\ 2}}{I_1} + \frac{m_2^{\ 2}}{I_2} + \frac{m_3^{\ 2}}{I_3} \frac{a_3}{a_3 - k} \right) \\ M_1^{\ 2} &= \frac{1}{2}\left(m_1^{\ 2} + m_2^{\ 2} + m_3^{\ 2} \frac{a_3}{a_3 - k} \right)\end{aligned} \tag{7.5}$$

The free rigid body equations with $k = 0$ have relative equilibria when

$$\begin{aligned}(m_1, m_2, m_3) &= (M, 0, 0) \\ (m_1, m_2, m_3) &= (0, 0, M) \\ (m_1, m_2, m_3) &= (0, M, 0)\end{aligned}$$

The first two cases correspond to rotation about the major or minor axis and are well known to be nonlinearly stable. The last case corresponds to rotation about the intermediate axis, which we have shown to be unstable. The eigenvalues of the system linearized about $(0, M, 0)$ are given by the solutions of

$$\lambda\left(\lambda^2 - a_1 a_2 \frac{a_3 - k}{a_3} M^2\right) = 0 \tag{7.6}$$

For $k = 0$, there is one eigenvalue in the right half plane, and the system is unstable as expected. However, for $k > a_3$, there are two imaginary eigenvalues and one zero eigenvalue. The system is marginally stable and by Theorem 1, the nonlinear stability of the system cannot be deduced.

We now give a summary of the Energy-Casimir method for a finite dimensional system. (For the infinite dimensional case see [HMRW85].)

7.2 Energy Casimir Method

In this section, the Energy-Casimir method is presented in 4 steps. These steps are then applied to the rigid body system (7.4) introduced in the previous section.

Step1. Write the equations of motion in first order form $\dot{u} = F(u)$. Find a conserved function H.

Step 2. Find a family of constants of the motion for $\dot{u} = F(u)$.

Step 3. Find a Casimir Function C, such that $H + C$ has a critical point at the relative equilibrium of interest.

Step 4. Definiteness of the second variation of the Energy-Casimir function $H + C$ at the critical point is then sufficient to prove nonlinear stability.

The major task is to find the Casimir function, which was first defined in section 5.1. Casimir function C Poisson commutes with any function G defined on the phase space of the Hamiltonian system.

$$\{C, G\} = 0 \tag{5.7}$$

where $\{\ ,\ \}$ is the non-canonical Poisson bracket defined by (5.6)

$$\{F, G\} = \left(\frac{\delta F}{\delta u}, \bar{J} \frac{\delta G}{\delta u} \right) \tag{5.6}$$

where $\bar{J}$ is skew-symmetric transformation from $\{u\}$ to $\{u\}$, satisfying $(u, \bar{J}v) = -(\bar{J}u, v)$.

Consider the Energy-Casimir function

$$H + C = H_1 + \phi(M_1^{\ 2}) \tag{7.7}$$

where ϕ is an arbitrarily smooth function.

According to <u>Step 4</u> above, the definiteness of the second variation $\delta^2[H_1 + \phi(M_1^{\ 2})]$ is sufficient to prove nonlinear stability.

The first variation is given by

$$\delta\left(H_1 + \phi(M_1^{\ 2})\right) =$$

$$\left(\frac{m_1^{\ 2}}{I_1} \delta m_1 + \frac{m_2^{\ 2}}{I_2} \delta m_2 + \frac{m_3^{\ 2}}{I_3} \frac{a_3 - k}{a_3} \delta m_3 \right) + \phi'(M_1^{\ 2}) \left(m_1 \delta m_1 + m_2 \delta m_2 + m_3 \delta m_3 \frac{a_3 - k}{a_3} \right)$$

which equals zero if

$$\frac{m_1}{I_1} + \phi' m_1 = 0,$$

$$\frac{m_2}{I_2} + \phi' m_2 = 0,$$

$$\frac{m_3}{I_3} \frac{a_3 - k}{a_3} + \phi' m_3 \frac{a_3}{a_3 - k} = 0.$$

At the relative equilibrium of interest $(m_1, m_2, m_3) = (0, M, 0)$, the above equations hold if

$\phi' = -\frac{1}{I_2}$.

Then, the second variation is given by

$$\delta^2\left(H_1 + \phi(M_1{}^2)\right) =$$

$$\frac{(\delta m_1)^2}{I_1} + \frac{(\delta m_2)^2}{I_2} + \frac{(\delta m_3)^2}{I_3}\frac{a_3 - k}{a_3} - \frac{1}{I_2}\left((\delta m_1)^2 + (\delta m_2)^2 + (\delta m_3)^2 \frac{a_3 - k}{a_3}\right) + \phi''(M_1{}^2)M_1{}^2(\delta m_2)^2$$

at the equilibrium of interest.

Recall that $I_1 > I_2 > I_3$, and $a_3 = \frac{(I_1 - I_2)}{I_1 I_2}$. Choose $\phi'' < 0$. Then, for k sufficiently large that $a_3 - k < 0$, the second variation is negative definite and we have nonlinear stability. A similar argument holds for a control torque about the major axis.

7.3 Summary

We have introduced the Energy-Casimir method and outlined the algorithm for deducing nonlinear stability. Using the Energy-Casimir method, we concluded that the angular momentum equations of the rigid body can be stabilized about the intermediate axis by a single torque applied about the major or minor axis.

However, this controlled system is not necessarily globally asymptotically stable. Any perturbation can cause the system to go into nutations. Although formally stable, it is natural that, in the engineering application we would seek to also ensure global asymptotic stability.

$$\dot{\vec{m}}_u = \vec{\vec{S}}\,\dot{\vec{m}} \tag{8.2}$$

where $\vec{\vec{S}}$ is defined as

$$\vec{\vec{S}} = \begin{bmatrix} S & 0 & 0 \\ 0 & S & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{8.3}$$

where $S = \dfrac{a_3 - k}{a_3}$.

8.2 Global Asymptotic Stabilization

We have already shown that this controlled dynamics (7.4) is nonlinearly stable. According to the calculation of the second variation of the Energy-Casimir function (7.7), the necessary gain condition for stabilization is $k > a_3$, where a_3 is given by (6.10). When this condition is met, the intermediate axis effectively turns into the major axis, thereby achieving stability.

The simple double bracket energy sink (6.8) can be used as a nonlinear feedback control law to stabilize major axis rotations. Because the energy sink seeks to dissipate kinetic energy, any perturbative or nutational motion will be damped out as specified by the parameter α (6.15).

Now, to make the system (7.4) globally asymptotically stable, we add the cubic double bracket energy sink, $\vec{J}\vec{\alpha}\vec{J}\nabla H_u$, where this time the Hamiltonian H_u is the controlled Hamiltonian dynamics given by (7.5)

$$H_u = \frac{1}{2}\left(\frac{m_1^{\,2}}{I_1} + \frac{m_2^{\,2}}{I_2} + \frac{m_3^{\,2}}{I_3}\frac{a_3}{a_3 - k} \right) \tag{8.4}$$

8.0 Global Asymptotic Stabilization of Intermediate Axis Rotation

In this section, we develop and present a velocity field in the form of a nonlinear feedback controller which globally asymptotically stabilizes rotation about the intermediate axis, using the all of the mathematical machinery we have developed in the preceding chapters.

8.1 Intermediate Axis Stabilization

In the previous section the controller torque of the form

$$u = -k\frac{I_1 I_2}{I_3}\omega_1\omega_2 \tag{7.2}$$

was added to the minor axis to stabilize intermediate axis rotation. To be consistent with the double bracket energy sink, we perform the Legendre transformation $\vec{m} = \frac{\partial E_c}{\partial \vec{\omega}}$.

From (7.4), the rigid body equations of motion with control torque u becomes

$$\dot{\vec{m}}_u = \begin{bmatrix} \dot{m}_1 \\ \dot{m}_2 \\ \dot{m}_3 \end{bmatrix} = \begin{bmatrix} a_1 \dfrac{a_3 - k}{a_3} m_2 m_3 \\ a_2 \dfrac{a_3 - k}{a_3} m_1 m_3 \\ a_3 m_1 m_2 \end{bmatrix}.$$

Recall that the velocity field for the original free rigid body Hamiltonian system (5.13) can be written

$$\dot{\vec{m}} = \vec{J}\vec{\nabla}H = (\vec{m}\times)\vec{I}^{-1}\vec{m} = \begin{bmatrix} a_1 m_2 m_3 \\ a_2 m_1 m_3 \\ a_3 m_1 m_2 \end{bmatrix} \tag{8.1}$$

The controlled dynamics $\dot{\vec{m}}_u$ can then be written

This makes , $\nabla H_u = \begin{bmatrix} \dfrac{m_1}{I_1} \\ \dfrac{m_2}{I_2} \\ \dfrac{m_3}{I_3}\dfrac{a_3}{a_3 - k} \end{bmatrix}$ which is not quite equal to $I^{-1}\vec{m}$.

For simplification and symmetry of notation, we define

$$I_3' = I_3 S$$

where $S = \dfrac{a_3 - k}{a_3}$, as previously. Then, equations (6.10) become

$$a_1' = \frac{(I_2 - I_3')}{I_2 I_3'}$$

$$a_2' = \frac{(I_3' - I_1)}{I_1 I_3'} \qquad (6.10')$$

and a_3 stays the same $$a_3 = \frac{(I_1 - I_2)}{I_1 I_2}$$

Then the nonlinearly globally asymptotically stable velocity field is

$$\dot{\vec{m}}_{NGAS} = \dot{\vec{m}}_u + \dot{\vec{m}}_{\sin k} = \vec{S}\vec{J}\nabla H + \vec{J}\vec{\alpha}\vec{J}\nabla H_u \qquad (8.4)$$

$$\dot{\vec{m}}_{NGAS} = \begin{bmatrix} S a_1 m_2 m_3 \\ S a_2 m_1 m_3 \\ a_3 m_1 m_2 \end{bmatrix} + \alpha \begin{bmatrix} a_3 m_1 m_2^2 - a_2' m_1 m_3^2 \\ a_1' m_2 m_3^2 - a_3 m_1^2 m_2 \\ a_2' m_1^2 m_3 - a_1' m_2^2 m_3 \end{bmatrix} \qquad (8.5)$$

8.3 Summary

We have taken the Energy-Casimir method and the quasi-rigid body modeling, and extracted two velocity field terms which both act as a nonlinear feedback control torques. These were then combined to globally asymptotically stabilize intermediate axis rotation.

9.0 Conclusion

We have introduced some basic definitions and concepts in stability, the elements of Hamiltonian dynamics and Poisson brackets to formulate free rigid body dynamics. We then introduced double bracket energy sinks, which were used to modeling quasi-rigid body dynamics. The double bracket energy sink was ultimately used to develop a method for globally asymptotically stabilizing intermediate axis rotations.

We have introduced the Energy-Casimir method and outlined the algorithm for deducing nonlinear stability. Using the Energy-Casimir method, we concluded that the angular momentum equations of the rigid body can be stabilized about the intermediate axis by a single torque applied about the major or minor axis.

We have taken the Energy-Casimir method and the quasi-rigid body modeling, and extracted two velocity field terms which both act as a nonlinear feedback control torques. These were then combined to globally asymptotically stabilize intermediate axis rotation. We hope that in the future we will be able to apply these nonlinear techniques to practical engineering systems.

10.0 References

[AM78] Abraham, Ralph, Marsden, Jerrold E. *Foundations of Mechanics*, 2nd Edition. Addison-Wesley Publishing Company, Reading, MA, USA, 1978

[BKMR96] Bloch, A.M., Krishnaprasad, P.S., Marsden, J.E., Ratiu, T.S. The Euler-Poincare equations and double bracket dissipation. *Communications in Mathematical Physics*, 175(1):1-42, January 1996.

[BM90] Bloch, Anthony M., Marsden, Jerrold E. Stabilization of rigid body dynamics by the Energy-Casimir method. *Systems & Controls Letters* 14(4):341-346, April 1990.

[Col97] Coleman, Charles P., *Geometric Mechanics and Control of Multibody Systems*. Department of Electrical Engineering and Computer Sciences, University of California, Berkely. 1997.

[Gol80] Goldstein, Herbert. *Classical Mechanics*, 2nd Edition. Addison-Wesley Publishing Company, Inc. 1980.

[HMRW85] Holm, D., Marsden, J.,Ratiu, T.,Weinstein, A. Nonlinear stability of fluid and plasma equilibria. *Physics Reports*, 123:11-16, 1985.

[Lik76] Likins, Peter W. The new generation of dynamic interaction problems. *Journal of the Astronautical sciences*, 27(2):103-113, 1979.

[Mar92] Marsden, Jerrold E. *Lectures on Mechanics*. London Mathematical Society lecture Notes Series. Cambridge University Press, Cambridge, England. 1992.

[MR94] Marsden, Jerrold E., Ratiu, Tudor S. An Introduction to Mechanics and Symmetry. Springer-Ver;ag, New York, NY, USA. 1994.

[MT95] Marion, Jerry B., Thornton, Stephen T. *Classical Dynamics of Particles and Systems*, 4^{th} Edition. Harcourt Brace & Company. 1995.

[SW91] Slotine, Jean-Jaques E., Li, Weiping. *Applied Nonlinear Control*. Prentice Hall, Upper Saddle River, NJ, USA.1991.

[van86] van der Schaft, A.J. Stabilization of Hamiltonian. *Nonlinear Analysis, Theory, Methods & Applications*. 10(10):1021-1035, 1986.